Strange Properties of IRC Safe Metrics? - Continued

Keegan Humphrey

February 2025

Here we look to determine for what choices of 1 dimensional metrics $g(x,x')$ does the smeared distribution

$$p(x|\epsilon) = \int dx' \Theta(\epsilon^2 - g(x,x')) \tag{1}$$

scale with the dimension of the space over which the metric is defined for small ϵ. This integral measures the size of the interval between points separated by a distance ϵ as measured by the metric g. We would expect that (at least for small ϵ) the scaling with ϵ would be linear. However, Andrew has shown that for the SEMD metric this property does not hold. The question arises: for what choice of metric do we have

$$\lim_{\epsilon\to 0} \partial_\epsilon p(x|\epsilon) = \mathcal{O}(\epsilon^0) \tag{2}$$

We can determine the answer to this question by direct computation as follows.

$$\begin{aligned}
\partial_\epsilon p(x|\epsilon) &= 2\epsilon \int dx' \delta(\epsilon^2 - g(x,x')) && (3)\\
&= 2\epsilon \int dx' \sum_i \frac{\delta(x' - \tilde{g}_i(x,\epsilon))}{|\partial_{x'} g(x,x')|} && (4)\\
&= 2\epsilon \int dx' \sum_i \frac{\delta(x' - \tilde{g}_i(x,\epsilon))}{|\partial_{x'} g(x,\tilde{g}_i(x,\epsilon))|} && (5)
\end{aligned}$$

Where the sum is over the solutions $\tilde{g}_i(x,\epsilon)$ which are defined as [1]

$$\epsilon^2 = g(x,\tilde{g}_i(x,\epsilon)) \tag{6}$$

So we see that (2) holds precisely when

$$\lim_{\epsilon\to 0} \partial_{x'} g(x,\tilde{g}_i(x,\epsilon)) = \mathcal{O}(\epsilon^1) \tag{7}$$

This condition can be made sharp given the definition of $\tilde{g}$

$$\begin{aligned}
2\epsilon &= \partial_\epsilon g(x,\tilde{g}_i(x,\epsilon)) && (8)\\
&= \partial_{x'} g(x,\tilde{g}_i(x,\epsilon)) \partial_\epsilon \tilde{g}_i(x,\epsilon) && (9)
\end{aligned}$$

Hence (2) holds exactly when

[1] Note that even if (1) is written with the argument $\epsilon - \sqrt{g(x,x')}$, the following equation must still be satisfied, so the argument that follows still holds.

$$\lim_{\epsilon \to 0} \partial_\epsilon \tilde{g}_i(x, \epsilon) = \mathcal{O}(\epsilon^0) \tag{10}$$

The question then arises for which choices of metric does $\tilde{g}$ satisfy this condition? This question can be best answered by use of the Lagrange Inversion Theorem [2]. This theorem states that for g of the form [3]

$$\epsilon^2 = g(x, x') = \sum_{j=0}^{\infty} b_j(x)(\xi(x, x') - 1)^j \tag{11}$$

$$\xi(x, x') \in \left\{ \frac{x'}{x}, \frac{x}{x'} \right\} \tag{12}$$

with $b_1 \neq 0$ we can write the solution for ξ in the form of a series

$$\xi = 1 + \sum_{k=1}^{\infty} a_k (\epsilon^2 - g(x, x))^k \tag{13}$$

$$a_k = \frac{1}{k!} \lim_{\zeta \to x} \frac{d^{k-1}}{d\zeta^{k-1}} \left[\frac{\xi(x, \zeta) - 1}{g(x, \zeta) - g(x, x)} \right]^k, \ k \geq 1 \tag{14}$$

And since $g(x, x) = 0$ (which in this form implies $b_0 = 0$) by the metric conditions this becomes

$$\xi = 1 + \sum_{k=1}^{\infty} a_k \epsilon^{2k} \tag{15}$$

$$a_k = \frac{1}{k!} \lim_{\zeta \to x} \frac{d^{k-1}}{d\zeta^{k-1}} \left[\frac{\xi(x, \zeta) - 1}{g(x, \zeta)} \right]^k, \ k \geq 1 \tag{16}$$

From the definition of ξ we can write

$$\tilde{g}(x, \epsilon) \in \left\{ \frac{x}{\xi}, x\xi \right\} \tag{17}$$

and since ξ is polynomial in ϵ^2 we have in both cases

$$\lim_{\epsilon \to 0} \partial_\epsilon \tilde{g}(x, \epsilon) = \mathcal{O}(\epsilon^1) \tag{18}$$

So that a metric of the above form for which $b_1(x) \neq 0$ will not satisfy the condition (2). Notice that in the case of a fixed jet mass, the SEMD metric takes the form

$$g_{SEMD}(x, x') \propto -(\bar{\xi}(x, x') - 1) \tag{19}$$

$$\bar{\xi}(x, x') = \min \left(\frac{x}{x'}, \frac{x'}{x} \right) \in [0, 1] \tag{20}$$

[2]We found details of this theorem in the online Encyclopedia of Mathematics (`https://encyclopediaofmath.org/wiki/Inversion_of_a_series#References`), but we haven't been able to track down a primary source yet.

[3]We have introduced this function ξ so that we can work with metrics (like the SEMD metric for $x < x'$ $1 - \frac{x}{x'}$) which may not have a Taylor series in x' about x. But with the metric defined in this way, we can work with instead with the series in ξ, covering both the SEMD like cases as well as any function that can be Taylor expanded in x' about x. And, though we haven't attempted to prove it yet, it seems reasonable that something close to this form of the metric may work for general metrics which are non-singular given the condition $g(x, x) = 0$.

So that $b_1 \neq 0$ and it does not satisfy the condition (2) as was found in Andrew's notes.

The next natural question is: for what forms of the metric do we satisfy the expected scaling condition. The answer is again given by a series inversion as described in the above source. For a metric of the form

$$\epsilon^2 = g(x,x') = \sum_{j=m}^{\infty} b_j(x)(\xi(x,x') - 1)^j, \; m \geq 2, \; b_m \neq 0 \tag{21}$$

The solution for ξ to this equation is given by

$$\begin{aligned}\xi =& 1 + \sum_{k=1}^{\infty} a_k(x)(\epsilon^2 - g(x,x))^{\frac{k}{m}} && (22)\\ =& 1 + \sum_{k=1}^{\infty} a_k(x)\epsilon^{2\frac{k}{m}} && (23)\\ a_k =& \frac{1}{k!}\lim_{\zeta\to x}\frac{d^{k-1}}{d\zeta^{k-1}}\left[\frac{\xi(x,\zeta)-1}{g^{\frac{1}{m}}(x,\zeta)}\right]^k, \; k \geq 1 && (24)\end{aligned}$$

So we see that only if $m = 2$ will we have the leading non-trivial term be linear and

$$\xi = 1 + \sum_{k=1}^{\infty} a_k(x)\epsilon^k \tag{25}$$

(26)

and so, since ξ is now a polynomial in ϵ rather than ϵ^2 we have

$$\lim_{\epsilon\to 0} \partial_\epsilon \tilde{g}(x,\epsilon) = \mathcal{O}(\epsilon^0) \tag{27}$$

So (2) is satisfied and we find the expected scaling at small ϵ.

So we have shown that for any metric $g(x,x')$ which can be written in the form

$$g(x,x') = \sum_{j=1}^{\infty} b_j(x)\left(\frac{x}{x'} - 1\right)^j \tag{28}$$

or

$$g(x,x') = \sum_{j=1}^{\infty} c_j(x)\,(x' - x)^j \tag{29}$$

the condition (2) will be satisfied if and only if $b_1 = 0$ ($c_1 = 0$) and $b_2 \neq 0$ ($c_2 \neq 0$).

We can notice that both of the other metrics given in Andrew's notes have this form and so give the expected scaling.

$$\begin{aligned}g_{Euc}(z,z') =& (z' - z)^2 && (30)\\ g_{2D}(\phi,\phi') =& 2 - 2\cos(\phi - \phi') = (\phi - \phi')^2 + \mathcal{O}(\phi - \phi')^4 && (31)\end{aligned}$$

So we have been able to identify the scaling behavior (2) with a metric that is quadratic in the difference of it's arguments at leading order. However, the connection between metrics of this form and IRC safety, if one exists, must still be made and understood.