The Radion in AdS-S

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February 2025

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1 Introduction

Stabilizing the RS scenario naturally (up to CC problem) has been looked at through many lenses. In [1] the junction conditions of branes in AdS (and other) backgrounds turned their dynamics into a 1D mechanics problem. In AdS the motion of branes was shown to be free. In [2] bulk matter content was added which introduced a potential for the brane separation naturally stabilizing it. In [3] this analysis was extended to include the back reaction of the scalar on the geometry. Then in [4] the perturbative stability was investigated in detail and the separation was given explicit dynamics.

All of these analyses are focused primarily on AdS. This is the setting of the original RS scenario. However, in AdS-S the story changes. In AdS separation of the branes is ensured to be a massless mode since translation along the orthogonal direction corresponds to a (conformal) Killing vector. But this symmetry is broken by the presence of the horizon[1]. So that in AdS-S the analysis of [1] shows that the background alone cannot be used to ensure the brane position is stationary. One can however show that there is a solution for a brane in the presence of a scalar which can back react on the geometry. However since the conformal symmetry is now broken, it is not clear that an analysis along the lines of [2] will reveal the only, or even the dominant contribution to the effective potential for the brane separation. Hence, the stability of such a stationary brane solution is difficult to analyse.

As such a detailed analysis which derives an explicit effective field theory for the brane separation in AdS-S along the lines of [4] is needed. In the AdS analysis, this field (the radion) was shown to be massless in the absence of bulk matter content. Which is to be expected given the conformal symmetry. However, in the AdS-S we will show that the radion is no longer massless in the absence of bulk matter content. Since the conformal symmetry is broken, the radion receives a mass which is proportional to the parameter controlling the position of the horizon. We find that the **radion mass is tachyonic** which is consistent with the findings of [1]. One may hope that in the presence of bulk matter content and a back reaction, that the radion mass can become positive.

2 Linear Perturbations of AdS-S in a brane world

For the rest of these notes, we will primarily be concerned with extending the results of [4] to AdS-S. Their analysis revealed that the perturbative stability of an RS type braneworld scenario is understood best through a metric where the branes reside at fixed coordinate and with the proper distance in the AdS metric taken from $\eta \to \eta + \epsilon(x)e^{2k\eta}$, with ϵ a 4D field describing the physical separation of the branes. Their analysis hinges on simultaneously solving the Field Equations in the bulk and junction conditions at the branes such that this field ϵ can be identified.

In what follows will write down the gauge fixed and diagonalized field equations and junction conditions for the perturbation. We will do this for an important class of metrics. We will then find that in AdS-S the field equations are not separable in this form and the radion has picked up a mass.

It remains to finish the analysis and solve these equations. We will likely work perturbatively in the far from horizon limit. Once this is complete we will look to derive a 4D effective theory and the coupling to bulk matter along the lines of [5].

2.1 Gauge Transformations of a Linearly Perturbed Gaussian Normal Metric

We begin by considering a stationary Gaussian Normal metric of the form

$$ds^2 = g^0_{\mu\nu}(\eta)dx^\mu dx^\nu - d\eta^2 \tag{1}$$

[1] By which we mean that there is no longer a conformal killing vector proportional to ∂_z for fixed horizon position. There is of course still a conformal symmetry involving a rescaling of the horizon position.

the AdS and AdS-S metrics are included in this class of metrics (as well as back reacted solutions involving stationary bulk scalars). This form of the metric is especially convenient for solving the junction conditions at each brane. We now consider a linear perturbation which preserves the GN gauge conditions

$$g_{\eta\eta} = -1 \tag{2}$$
$$g_{\mu\eta} = 0 \tag{3}$$

The perturbed metric is[2]

$$ds^2 = g_{MN}(X)dX^M dX^N \tag{4}$$
$$= (g^0_{\mu\nu}(\eta) + h_{\mu\nu}(x,\eta))dx^\mu dx^\nu - d\eta^2 \tag{5}$$

Our goal, as in [4], will be to identify the scalar degree of freedom corresponding to the relative motion of the branes. We now consider the small coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu \tag{6}$$
$$\eta \to \eta' = \eta + \xi^\eta \tag{7}$$

This induces a change in the metric

$$g_{MN}(X) \to g_{MN}(X) + 2\nabla_{(M}\xi_{N)} \tag{8}$$
$$= g_{MN}(X) + \xi^L \partial_L g^0_{MN}(\eta) + 2g^0_{J(M}\partial_{N)}\xi^J \tag{9}$$
$$\equiv g_{MN}(\eta) + \delta g_{MN}(X) \tag{10}$$

The GN conditions (2) imply

$$0 = \delta g_{\eta\eta}(X) = \delta g_{\mu\eta}(X) \tag{11}$$
$$\Longrightarrow \xi^\eta(X) = \xi^\eta(x) \equiv \epsilon^\eta(x) \tag{12}$$
$$\xi^\nu(X) = \partial_\mu \epsilon^\eta(x) \int d\eta\; g^{0,\nu\mu} + \epsilon^\nu(x) \tag{13}$$

Notice that the η dependence of ξ^M is now completely fixed. With these gauge conditions imposed, we can then determine the metric transformation δg in terms of $\epsilon^M(x)$

$$\delta g_{\mu\nu}(X) = \xi^L \partial_L g^0_{\mu\nu}(\eta) + 2g^0_{J(\mu}\partial_{\nu)}\xi^J \tag{14}$$
$$= \epsilon^\eta(x)\partial_\eta g^0_{\mu\nu} + 2g^0_{\sigma(\mu}\partial_{\nu)}\epsilon^\sigma(x) + 2\left(\int d\eta g^{0,\sigma\rho}\right) g^0_{\sigma(\mu}\partial_{\nu)}\partial_\rho \epsilon^\eta(x) \tag{15}$$

This matches on to [4] in the case that $g^0_{\mu\nu} = g^{AdS}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} = e^{-2k\eta}\eta_{\mu\nu}$

$$\delta g^{AdS}_{\mu\nu}(X) = -2ka^2\epsilon^\eta(x)\eta_{\mu\nu} + 2a^2\partial_{(\nu}\epsilon_{\nu)}(x) + \frac{1}{k}\partial_\mu\partial_\nu\epsilon^\eta(x) \tag{16}$$

[2]Our index conventions are summarized here. Lowercase Greek are 4D indices, and uppercase Latin are 5D indices.

2.2 Gauge Fixed Field Equations

The field equations for h are given in an appendix for a metric of the form (1). However, the general equations are a nasty set of coupled PDEs. So a critical piece of the analysis is to appropriately fix the gauge such that the field equations are diagonalized.

In the appendices, we've shown that in addition to $0 = h_{\eta\eta} = h_{\mu\eta}$, it is consistent to impose

$$0 = h = g^{0,\mu\nu} h_{\mu\nu} \tag{17}$$
$$0 = \Gamma^{\nu}_{\mu\eta} h^{\mu}_{\nu} \propto h_{\nu\beta} \partial_\eta g^{0,\nu\beta} = -g^{0,\nu\beta} \partial_\eta h_{\nu\beta} \tag{18}$$
$$0 = g^{0,\mu\nu} \partial_\mu h_{\sigma\nu} \tag{19}$$

Notice that in AdS the second condition is degenerate with the first since the metric is an eigentensor of ∂_η. In that case these gauge conditions reduce to 4D transverse traceless gauge. Using these gauge conditions and the perturbed Ricci Tensor in GN gauge 3.2.5 the field equations reduce to [3]

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\partial_\eta^2 h_{\mu\nu} - \frac{1}{2} g^{0,\sigma\rho} \partial_\sigma \partial_\rho h_{\mu\nu} + \frac{1}{4}\left(g^{0,\sigma\rho} (\partial_\eta g^0_{\sigma\rho} \partial_\eta h_{\mu\nu} - 4 \partial_\eta g^0_{\sigma(\nu|} \partial_\eta h_{|\mu)\rho}) + 2 \partial_\eta g^0_{\sigma\nu} h^{\sigma\rho} \partial_\eta g^0_{\mu\rho} \right) \tag{20}$$

Note that if we take g^0 to be diagonal, then this equation will be diagonal. For the AdS-S metric (72) mathematica gives

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\partial_\eta^2 h_{\mu\nu} - \frac{1}{2} g^{0,\sigma\rho} \partial_\sigma \partial_\rho h_{\mu\nu} + 2k^2 \mathrm{th}(\eta) h_{\mu\nu} \tag{21}$$
$$0 = R^{(1)}_{\mu\nu} - 4k^2 h_{\mu\nu} \tag{22}$$

Unfortunately, because g^0 is not proportional to the identity and is η dependent, this doesn't separate. In AdS with these coordinates the radion decouples from graviton, ie one can write the solution as $\chi_{\mu\nu}(x) r(\eta)$. But here this will only be true at a given order in the far from horizon limit. Since $\mathrm{th}(\eta) \to 1$ in the AdS limit, this equation implies the AdS expression in [4] [4].

2.3 Junction Conditions

Recall from [6] that Israel's second junction condition at a codimension-1 surface Σ is given by

$$[K_{\mu\nu}]_\Sigma = -8\pi G \left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1} g_{\mu\nu} S \right) \tag{23}$$

Where $[f]_\Sigma \equiv f(\eta_\Sigma + \epsilon) - f(\eta_\Sigma - \epsilon)$ and

$$T_{\mu\nu} = \delta_\Sigma(X) S_{\mu\nu} + \text{(regular)} \tag{24}$$

defines S as the singular contribution to the bulk energy momentum tensor due to the energy and momentum on Σ, and

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_{\boldsymbol{n}} g_{\mu\nu} \tag{25}$$
$$= \boldsymbol{e}_\nu \cdot \partial_\mu \boldsymbol{n} \tag{26}$$
$$= \nabla_\mu n_\nu \tag{27}$$

[3] For homogeneous and isotropic g^0, the initial data constraints are trivially satisfied by the above gauge conditions.

[4] Noting that equation (11) written in [4] is missing a factor of $\frac{1}{2}$

With $\boldsymbol{n}$ a vector normal to Σ with a unit space-like norm, and $\mathcal{L}$ denoting a Lie derivative. As pointed out in eg. [7], with Gaussian Normal coordinates and a coordinate basis for the tangent space of Σ we have[5]

$$n_\nu = \delta^\eta_\nu \tag{28}$$
$$\implies \nabla_\mu n_\nu = -\Gamma^\eta_{\mu\nu} = -\frac{1}{2}\partial_\eta g_{\mu\nu} \tag{29}$$

2.3.1 Junction Conditions in AdS-S

We now consider the junction conditions for an arbitrary fluid content on the branes in the case of branes which either are or are not orbifold fixed points

$$[K_{\mu\nu}]_\Sigma = [-\frac{1}{2}\partial_\eta g_{\mu\nu}]_\Sigma \tag{30}$$
$$= -\frac{1}{2}\begin{cases} 2\partial_\eta g_{\mu\nu} & \text{, Fixed point} \\ \partial_\eta g^+_{\mu\nu} - \partial_\eta g^-_{\mu\nu} & \text{, Not a fixed point} \end{cases} \tag{31}$$

For now we'll focus on the fixed point case, since it is simpler. However we must always keep it in the back of our minds that for a Randall Lykken type scenario, one of our branes will not be a fixed point. Then we take an arbitrary $\gamma_{\mu\nu}$ of the form for a perfect fluid which is homogeneous and isotropic at fixed η

$$\gamma_{\mu\nu} = -\kappa[A(\eta)g_{\mu\nu} + B(\eta)u_\mu u_\nu] \tag{32}$$
$$= -\kappa\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1}g_{\mu\nu}S\right) \tag{33}$$

Now let's think about the 0th order junction condition. For an orbifold fixed point

$$\partial_\eta g^0_{\mu\nu} = \kappa[A(\eta_\Sigma)g^0_{\mu\nu} + B(\eta_\Sigma)u_\mu u_\nu] \tag{34}$$

Where we make explicit the fact that the coefficients may depend on the position of the brane, but that this is assumed to be fixed. Since g^0 is not an eigentensor of ∂_η we must have that $A(\eta) \neq 0$ and $B(\eta) \neq 0$. Specifically we can write

$$\partial_\eta g^0_{\mu\nu} = \partial_\eta\left(2\text{ch}\eta_{\mu\nu} - \frac{2\mu}{\text{ch}}\delta^0_\mu\delta^0_\nu\right) \tag{35}$$
$$= -2k\text{th}\left(2\text{ch}\eta_{\mu\nu} + \frac{2\mu}{\text{ch}}\delta^0_\mu\delta^0_\nu\right) \tag{36}$$
$$= -2k\text{th}\left(g^0_{\mu\nu} + \frac{4\mu}{\text{ch}}\delta^0_\mu\delta^0_\nu\right) \tag{37}$$
$$= -2k\text{th}\left(g^0_{\mu\nu} + \frac{4\mu}{\text{ch}}\frac{1}{g^2_{00}}u_\mu u_\nu\right) \tag{38}$$

Where we've assumed g^0 is diagonal and taken $u^\mu = \delta^\mu_0$. Notice also that in the limit $\mu \to 0$, $2\text{ch} \to a(\eta)$ and $\text{th} \to 1$ so that this is consistent with the AdS result.

$$A = -\frac{2k}{\kappa}\text{th}\Big|_\Sigma \tag{39}$$
$$B = -\frac{2k}{\kappa}\text{th}\frac{4\mu}{\text{ch}}\frac{1}{g^2_{00}}\Big|_\Sigma = -\frac{2k}{\kappa}\frac{\mu}{\text{sh}^3}\Big|_\Sigma \tag{40}$$

[5]Note that they had opposite metric sign convention, which accounts for the difference in sign in the last line.

And we have at linear order

$$\partial_\eta h_{\mu\nu} = \kappa \left[A h_{\mu\nu} + 2B h_{\sigma(\mu} u_{\nu)} u^\sigma \right] \tag{41}$$

Where, naturally, $u_\mu = g^0_{\mu\nu} u^\nu$. However this form of the junction condition is inconvenient given that it is not diagonal. So instead consider the Junction conditions for h with raised indices.

$$K^{\mu\nu} = -\frac{1}{2} g^{\mu\sigma} g^{\nu\rho} \partial_\eta g_{\sigma\rho} = \frac{1}{2} \partial_\eta g^{\mu\nu} \tag{42}$$

Now with upper indices, the u^μ are independent of the metric and hence 0th order in h which implies we now have

$$\begin{aligned} \partial_\eta g^{\mu\nu} =& -\kappa [A g^{\mu\nu} + B u^\mu u^\nu] && (43) \\ -\partial_\eta h^{\mu\nu} =& \kappa A h^{\mu\nu} && (44) \\ =& -2k \mathrm{th} h^{\mu\nu} \Big|_\Sigma && (45) \end{aligned}$$

since $g^{\mu\nu} = g^{0,\mu\nu} - h^{\mu\nu}$.

2.3.2 The Radion Mass

As explained in [4], the field equations are gauge invariant but critically the junction conditions are not. As such we take our gauge fixed metric perturbation to be $\tilde{h}$ (ie. which satisfies the conditions in 2.2) and consider the GN consistent η translation of this metric perturbation.

$$\begin{aligned} h_{\mu\nu} =& \tilde{h}_{\mu\nu} + \delta g^\eta_{\mu\nu} && (46) \\ \equiv& \tilde{h}_{\mu\nu} + \epsilon^\eta(x) \partial_\eta g^0_{\mu\nu} + 2 \left(\int d\eta g^{0,\sigma\rho} \right) g^0_{\sigma(\mu} \partial_{\nu)} \partial_\rho \epsilon^\eta(x) && (47) \\ & && (48) \end{aligned}$$

like the field equations, the junction conditions will likely need to be solved perturbatively. However, as in [4], we can take the trace of the junction condition to determine the mass of the field $\epsilon^\eta(x)$ (which will be proportional to the radion). From our gauge conditions $\tilde{h} = g^{0,\mu\nu} \partial_\eta \tilde{h}_{\mu\nu} = 0$ so that

$$\begin{aligned} g^0_{\mu\nu} \partial_\eta h^{\mu\nu} =& -\kappa A h && (49) \\ -\kappa A \delta g^\eta =& g^0_{\mu\nu} \partial_\eta \delta g^{\eta,\mu\nu} && (50) \\ =& \partial_\eta \delta g^\eta - \delta g^{\eta,\mu\nu} \partial_\eta g^0_{\mu\nu} && (51) \\ =& \partial_\eta \delta g^\eta - \delta g^{\eta,\mu\nu} \kappa [A g^0_{\mu\nu} + B u_\mu u_\nu] && (52) \\ \implies \partial_\eta \delta g^\eta =& \kappa B u_\mu u_\nu \delta g^{\eta,\mu\nu} && (53) \end{aligned}$$

And as we showed in the apendix

$$\begin{aligned} \partial_\eta \delta g^\eta =& 2 g^{0,\mu\nu} \partial_\mu \partial_\nu \epsilon^\eta + \epsilon^\eta \partial_\eta (g^{0,\mu\nu} \partial_\eta g^0_{\mu\nu}) && (54) \\ & && (55) \end{aligned}$$

So we find the consistency of our gauge conditions with the junction condition implies

$$g^{0,\mu\nu} \partial_\mu \partial_\nu \epsilon^\eta + \frac{1}{2} \epsilon^\eta \partial_\eta (g^{0,\mu\nu} \partial_\eta g^0_{\mu\nu}) - \frac{1}{2} \kappa B u_\mu u_\nu \delta g^{\eta,\mu\nu} = 0 \tag{56}$$

Where

$$u_\mu u_\nu \delta g^{\eta,\mu\nu} = u^\mu u^\nu \delta g^\eta_{\mu\nu} \tag{57}$$

$$= \epsilon^\eta(x) u^\mu u^\nu \partial_\eta g^0_{\mu\nu} + 2\left(\int d\eta g^{0,\sigma\rho}\right) u^\mu u^\nu g^0_{\sigma(\mu}\partial_{\nu)}\partial_\rho \epsilon^\eta(x) \tag{58}$$

for diagonal g^0 this equation will contribute to the coefficient of the second time derivative of ϵ^η, so that this is a massive wave equation for ϵ^η. As alluded to in [1], to an observer on the branes, who will see other waves (eg. photons) propagating on the branes according to the differential operator $g^{0,\mu\nu}\partial_\mu\partial_\nu$, the radion will accordingly appear to propagate with a different speed of light. Notice, as usual, that the equation reduces to $\Box\epsilon^\eta = 0$ in AdS. In AdS-S when the novel terms are non-vanishing, we see that the radion has acquired a non-zero mass which will be dependent on the horizon temperature. The mass of this *un-normalized* field is

$$m^2 = \frac{1}{2}\left(\partial_\eta(g^{0,\mu\nu}\partial_\eta g^0_{\mu\nu}) - \kappa B u^\mu u^\nu \partial_\eta g^0_{\mu\nu}\right) \tag{59}$$

$$= -\frac{1}{2}\frac{8k^2\mu}{\mathrm{sh}^2}\left(1 + \frac{\mu}{\mathrm{ch}^2}\right) \tag{60}$$

$$= -16k^2\xi + \mathcal{O}(\xi^2) \tag{61}$$

$$\approx -\left(4k\sqrt{\mu}e^{2\frac{\eta}{l}}\right)^2 \tag{62}$$

$$= -\left(4ke^{2\frac{\eta-\eta_h}{l}}\right)^2 \tag{63}$$

So the radion mass is **tachyonic** in AdS-S with no bulk matter. This gives further credence to the analyis of [1] which found that one cannot have a stable brane insertion in an AdS-S background. However, we can also notice that (up to changes in the normalization) the radion mass will be suppressed by the ξ. As such if we are far from the horizon, then the matter content in the bulk and resulting back reaction may be enough to change the sign of the radion mass and stabilize the brane separation.

3 Appendices

3.1 AdS-S Metric Parametrization

In this section, we give the AdS-S metric in Gaussian Normal Coordinates and define our far from horizon expansion parameter. We start from the metric used in eg [8] and [1]. We then rescale μ and p so that both are dimensionless ($\mu \to l^2\mu$ and $p \to lp$ with $k = l^{-1}$) and change our metric signature to mostly minus as in [4].

$$ds^2 = \left(\frac{p^2}{l^2} - \frac{\mu}{p^2}\right) dt^2 - \frac{p^2}{l^2} d\vec{x}^2 - \frac{dp^2}{\frac{p^2}{l^2} - \frac{\mu}{p^2}} \tag{64}$$

$$\to \left(p^2 - \frac{\mu}{p^2}\right) dt^2 - p^2 d\vec{x}^2 - \frac{l^2 dp^2}{p^2 - \frac{\mu}{p^2}} \tag{65}$$

We look for a coordinate η such that

$$l\partial_\eta p = - \sqrt{p^2 - \frac{\mu}{p^2}} \tag{66}$$

$$\implies p^2(\eta) = \frac{1}{2}\left(e^{-\frac{2}{l}(\eta-\eta_0)} + e^{\frac{2}{l}(\eta-\eta_0)}\mu\right) \tag{67}$$

$$\implies \eta = \eta_0 - \frac{l}{2}\ln\left(p^2 + \sqrt{p^4 - \mu}\right) \tag{68}$$

Notice that we have chosen the sign of η so that it increases towards the horizon (away from the UV brane). We can also note that our coordinate η, the proper distance from the UV brane, now has implicit dependence on the horizon temperature. We take $\eta_0 = 0$. It is convenient to define

$$\text{ch} \equiv \sqrt{\mu}\cosh(-\ln\sqrt{\xi}) = \frac{\sqrt{\mu}}{2\sqrt{\xi}}(1+\xi) \tag{69}$$

$$\text{sh} \equiv \sqrt{\mu}\sinh(-\ln\sqrt{\xi}) = \frac{\sqrt{\mu}}{2\sqrt{\xi}}(1-\xi) \tag{70}$$

$$\text{th} \equiv \frac{\text{sh}}{\text{ch}} = \tanh(-\ln\sqrt{\xi}) = \frac{1-\xi}{1+\xi} \tag{71}$$

Where we have defined $\xi \equiv \mu e^{4\frac{\eta}{l}} = e^{4\frac{\eta-\eta_h}{l}}$ with η_h the proper distance from the UV brane to the horizon. The parameter ξ then defines the far from horizon limit (and corresponds to the coordinate z used in [8]). So that $\text{ch} = p^2$. With these definitions it is clear that our metric becomes

$$ds^2 = 2\left(\frac{\text{sh}^2}{\text{ch}} dt^2 - \text{ch} d\vec{x}^2\right) - d\eta^2 \tag{72}$$

Where we have also rescaled $x \to 2x$ so that in the limit $\xi \to 0$ this metric goes to the AdS metric. It is also convenient to note the identities

$$\mu = \text{ch}^2 - \text{sh}^2 \tag{73}$$

$$\text{ch}' = -2k\text{sh} \tag{74}$$

$$\text{sh}' = -2k\text{ch} \tag{75}$$

3.2 Perturbed Einstein Field Equations

3.2.1 Connection

The connection which is compatible with g^0 only has the following non-vanishing components

$$\Gamma^{0,\eta}_{\mu\nu} = \frac{1}{2}\partial_\eta g^0_{\mu\nu} \tag{76}$$
$$\Gamma^{0,\sigma}_{\eta\mu} = \frac{1}{2}g^{0,\sigma\nu}\partial_\eta g^0_{\mu\nu} \tag{77}$$

Similarily, at first order in h, the connection compatible with g has non-vanishing components

$$\Gamma^{1,\eta}_{\mu\nu} = \frac{1}{2}\partial_\eta h_{\mu\nu} \tag{78}$$
$$\Gamma^{1,\sigma}_{\eta\mu} = \frac{1}{2}g^{0,\sigma\nu}\partial_\eta h_{\mu\nu} - \frac{1}{2}h^{\sigma\nu}\partial_\eta g^0_{\mu\nu} \tag{79}$$
$$\Gamma^{1,\sigma}_{\mu\nu} = g^{0,\sigma\rho}\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}g^{0,\sigma\rho}\partial_\rho h_{\mu\nu} \tag{80}$$

3.2.2 Riemann Tensor

The Riemann and Ricci Tensors are given by

$$R^K_{MNJ} = 2\Gamma^L_{J[M}\Gamma^K_{N]L} - 2\partial_{[M}\Gamma^K_{N]J} \tag{81}$$
$$R_{MJ} = R^N_{MNJ} \tag{82}$$
$$= 2\Gamma^L_{J[M}\Gamma^N_{N]L} - 2\partial_{[M}\Gamma^N_{N]J} \tag{83}$$
$$\implies R^{(1)}_{MJ} = 2\Gamma^{0,L}_{J[M}\Gamma^{1,N}_{N]L} + 2\Gamma^{1,L}_{J[M}\Gamma^{0,N}_{N]L} - 2\partial_{[M}\Gamma^{1,N}_{N]J} \tag{84}$$

3.2.3 $\eta\eta$ component of the Ricci Tensor

From the previous subsection we can compute the Ricci tensor component by component at linear order in h

$$R^{(1)}_{\eta\eta} = 2\Gamma^{0,L}_{\eta[\eta}\Gamma^{1,N}_{N]L} + 2\Gamma^{1,L}_{\eta[\eta}\Gamma^{0,N}_{N]L} - 2\partial_{[\eta}\Gamma^{1,N}_{N]\eta} \tag{85}$$
$$= -\frac{1}{2}g^{0,\nu\rho}\partial_\eta g^0_{\mu\rho}\left(g^{0,\mu\alpha}\partial_\eta h_{\nu\alpha} - h^{\mu\alpha}\partial_\eta g^0_{\nu\alpha}\right) - \frac{1}{2}\partial_\eta\left(g^{0,\mu\nu}\partial_\eta h_{\mu\nu} - h^{\mu\nu}\partial_\eta g^0_{\mu\nu}\right) \tag{86}$$

We next note the identity which we will continually make use of

$$0 = \partial_\eta \delta^\beta_\alpha \tag{87}$$
$$\implies \partial_\eta g^{0,\mu\beta} = -g^{0,\mu\alpha}g^{0,\beta\nu}\partial_\eta g^0_{\alpha\nu} \tag{88}$$
$$\tag{89}$$

Which allows us to write

$$R^{(1)}_{\eta\eta} = -\frac{1}{2}h_{\mu\nu}\left(\partial^2_\eta g^{0,\mu\nu} - g^0_{\sigma\rho}\partial_\eta g^{0,\sigma\mu}\partial_\eta g^{0,\rho\nu}\right) - \frac{1}{2}\partial_\eta\left(g^{0,\mu\nu}\partial_\eta h_{\mu\nu}\right) \tag{90}$$

The term proportional to $h_{\mu\nu}$ vanishes if g^0 is diagonal and has components proportional to $e^{c\eta}$ for some c. Ie. in the case that $g^0_{\mu\nu} = a^2\eta_{\mu\nu} = e^{-2k\eta}\eta_{\mu\nu}$ we then find

$$
\begin{aligned}
R^{(1)}_{\eta\eta} =& -\frac{1}{2}h\left(\partial_\eta^2 a^{-2} - a^2(\partial_\eta a^{-2})^2\right) - \frac{1}{2}\partial_\eta\left(a^{-2}\partial_\eta h\right) && (91)\\
=& -\frac{1}{2}\partial_\eta\left(a^{-2}\partial_\eta h\right) && (92)
\end{aligned}
$$

Where *only for AdS* when comparing to [4] do we make use of the definition $h \equiv \eta^{\mu\nu}h_{\mu\nu}$ (otherwise $h \equiv g^{0,\mu\nu}h_{\mu\nu}$) and we note explicitly that $h^{\mu\nu}\eta_{\mu\nu} = h_{\beta\alpha}g^{\beta\nu}g^{\alpha\mu}\eta_{\mu\nu} = a^{-4}h$. Which is precisely what was found in [4] (up to boundary terms).

3.2.4 $\mu\eta$ component of the Ricci Tensor

We also have

$$
R^{(1)}_{\mu\eta} = g^{0,\nu\alpha}\partial_\eta g^0_{[\mu|\alpha}\left(g^{0,\sigma\rho}\partial_{(\sigma]}h_{\nu)\rho} - \frac{1}{2}g^{0,\sigma\rho}\partial_\rho h_{|\sigma]\nu}\right) - \partial_{[\mu|}\partial_\eta\left(g^{0,\nu\sigma}h_{|\nu]\sigma}\right) \tag{93}
$$

Next we note that if

$$
\Gamma^{0,\nu}_{\eta\mu} \propto g^{0,\nu\alpha}\partial_\eta g^0_{\mu\alpha} \propto \delta^\nu_\mu \tag{94}
$$

Then

$$
\begin{aligned}
\Gamma^{0,\nu}_{\eta[\mu}\Gamma^{1,\sigma}_{\sigma]\nu} \propto& \delta^\nu_{[\mu}\left(g^{0,\sigma\rho}\partial_{(\sigma]}h_{\nu)\rho} - \frac{1}{2}g^{0,\sigma\rho}\partial_\rho h_{|\sigma]\nu}\right) && (95)\\
=& \left(g^{0,\sigma\rho}\partial_{([\sigma}h_{\nu])\rho} - \frac{1}{2}g^{0,\sigma\rho}\partial_\rho h_{[\sigma\nu]}\right) && (96)\\
=& 0 && (97)
\end{aligned}
$$

The condition (94) is a simple condition to determine

$$
\begin{aligned}
\frac{1}{2}g^{0,\nu\alpha}\partial_\eta g^0_{\mu\alpha} \propto& \delta^\nu_\mu && (98)\\
\implies \partial_\eta g^0_{\mu\alpha} \propto& g^0_{\mu\alpha} && (99)
\end{aligned}
$$

So as for the $\eta\eta$ condition, we find that the non-vanishing components of g^0 must be exponential in η just as is true for the AdS metric. So for the AdS metric we find

$$
\begin{aligned}
0 =& R^{(1)}_{\mu\eta} && (100)\\
=& -2\partial_{[\mu}\Gamma^{1,\nu}_{\nu]\eta} && (101)\\
=& \frac{1}{2}\partial_\eta\left[g^{0,\nu\sigma}\left(\partial_\nu h_{\mu\sigma} - \partial_\mu h_{\nu\sigma}\right)\right] && (102)\\
=& \frac{1}{2}\partial_\eta\left[a^{-2}\left(\eta^{\nu\sigma}\partial_\nu h_{\mu\sigma} - \partial_\mu h\right)\right] && (103)
\end{aligned}
$$

Which precisely agrees with [4].

3.2.5 $\mu\nu$ component of the Ricci Tensor

The 4D components of the Riemann tensor, though they are the least compact, can still be handled in the same way.

$$R^{(1)}_{\mu\nu} = 2\Gamma^{0,L}_{\nu[\mu}\Gamma^{1,N}_{N]L} + 2\Gamma^{1,L}_{\nu[\mu}\Gamma^{0,N}_{N]L} - 2\partial_{[\mu}\Gamma^{1,N}_{N]\nu} \tag{104}$$

$$= 2\Gamma^{0,\eta}_{\nu[\mu}\Gamma^{1,\sigma}_{\sigma]\eta} + 2\Gamma^{1,\eta}_{\nu[\mu}\Gamma^{0,\sigma}_{\sigma]\eta} - \Gamma^{0,\sigma}_{\nu\eta}\Gamma^{1,\eta}_{\mu\sigma} - \Gamma^{1,\sigma}_{\nu\eta}\Gamma^{0,\eta}_{\mu\sigma} - 2\partial_{[\mu}\Gamma^{1,\sigma}_{\sigma]\nu} + \partial_\eta\Gamma^{1,\eta}_{\mu\nu} \tag{105}$$

So we have

$$\partial_\eta\Gamma^{1,\eta}_{\mu\nu} - 2\partial_{[\mu}\Gamma^{1,\sigma}_{\sigma]\nu} = \frac{1}{2}\partial^2_\eta h_{\mu\nu} + \frac{1}{2}g^{0,\sigma\rho}\left(2\partial_\sigma\partial_{(\nu}h_{\mu)\rho} - \partial_\sigma\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\nu h_{\sigma\rho}\right) \tag{106}$$

So both derivative terms agree with [4]. In total for the quadratic terms

$$2\Gamma^{1,\eta}_{\nu[\mu}\Gamma^{0,\sigma}_{\sigma]\eta} + 2\Gamma^{0,\eta}_{\nu[\mu}\Gamma^{1,\sigma}_{\sigma]\eta} - \Gamma^{0,\sigma}_{\nu\eta}\Gamma^{1,\eta}_{\mu\sigma} - \Gamma^{1,\sigma}_{\nu\eta}\Gamma^{0,\eta}_{\mu\sigma} \tag{107}$$

$$= \frac{1}{4}\left(\partial_\eta g^0_{\mu\nu}g^{0,\sigma\rho}\partial_\eta h_{\sigma\rho} + g^{0,\sigma\rho}(\partial_\eta g^0_{\sigma\rho}\partial_\eta h_{\mu\nu} - 4\partial_\eta g^0_{\sigma(\nu|}\partial_\eta h_{|\mu)\rho}) - \partial_\eta g^0_{\mu\nu}h^{\sigma\rho}\partial_\eta g^0_{\sigma\rho} + 2\partial_\eta g^0_{\sigma\nu}h^{\sigma\rho}\partial_\eta g^0_{\mu\rho}\right) \tag{108}$$

In AdS we then have

$$2\Gamma^{1,\eta}_{\nu[\mu}\Gamma^{0,\sigma}_{\sigma]\eta} + 2\Gamma^{0,\eta}_{\nu[\mu}\Gamma^{1,\sigma}_{\sigma]\eta} - \Gamma^{0,\sigma}_{\nu\eta}\Gamma^{1,\eta}_{\mu\sigma} - \Gamma^{1,\sigma}_{\nu\eta}\Gamma^{0,\eta}_{\mu\sigma} \tag{109}$$

$$= \frac{1}{4}\left(-2k\eta_{\mu\nu}\partial_\eta h - 4k^2\eta_{\mu\nu}h + 8k^2 h_{\mu\nu}\right) \tag{110}$$

So we find in total

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\partial^2_\eta h_{\mu\nu} + 2k^2 h_{\mu\nu} - \left(\frac{k}{2}\partial_\eta h + k^2 h\right)\eta_{\mu\nu} + \frac{1}{2a^2}\left(2\eta^{\sigma\rho}\partial_\sigma\partial_{(\nu}h_{\mu)\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \tag{111}$$

Which precisely agrees with [4].

3.3 Fixing the Gauge

Here we follow [9] section 7.5 and chapter 10. By leveraging tools and ideas from the development of the initial value formation of General Relativity, we can show that the GN gauge is in general compatible with traceless and Harmonic Coordinates.

As it turns out we can ensure that the gauge choice holds into the bulk without first solving the constraints. The ingredients are; the untouched 4D gauge invariance, and the existence and uniqueness theorems outlined in [9] and [10]. And while this level of detail is not necessary in AdS (since it is straightforward to show that we can impose the gauge conditions in the bulk), it will be necessary for other spacetimes (ie. for AdS-S).

We will take for granted that the existence theorem laid out for space-like hyper-surfaces also apply to time-like hyper-surfaces. Specifically, that there exists a unique solution to a linear diagonal second order hyperbolic equation (in any derivative operator), given initial data (the field and it's normal derivative) on the time-like surface Σ throughout spacetime. See eg theorem 10.1.2 of [9].

3.3.1 Harmonic Ricci Tensor

As layed out in [9] chapter 10, the perturbed Ricci Tensor is

$$
\begin{aligned}
R_{AB} =& -2\nabla_{[A}\dot{\Gamma}^{C}_{C]B} && (112)\\
=& -\nabla_A\dot{\Gamma}^{C}_{CB} + \nabla_C\dot{\Gamma}^{C}_{AB} && (113)\\
=& -\frac{1}{2}g^{0,CD}\nabla_A\left(2\nabla_{(C}h_{B)D} - \nabla_D h_{CB}\right) + \frac{1}{2}g^{0,CD}\nabla_C\left(2\nabla_{(A}h_{B)D} - \nabla_D h_{AB}\right) && (114)\\
=& -\frac{1}{2}g^{0,CD}\nabla_A\nabla_B h_{CD} + g^{0,CD}\nabla_C\nabla_{(A}h_{B)D} - \frac{1}{2}g^{0,CD}\nabla_C\nabla_D h_{AB} && (115)\\
=& -\frac{1}{2}g^{0,CD}\nabla_A\nabla_B h_{CD} + \nabla_{(A}\nabla^{D}h_{B)D} + R^{C\ \ D}_{\ AB}h_{CD} - \frac{1}{2}g^{0,CD}\nabla_C\nabla_D h_{AB} && (116)\\
=& \nabla_{(A}\nabla^{D}\bar{h}_{B)D} + R^{C\ \ D}_{\ AB}h_{CD} - \frac{1}{2}g^{0,CD}\nabla_C\nabla_D h_{AB} && (117)
\end{aligned}
$$

Where we have defined

$$\bar{h}_{AB} = h_{AB} - \frac{1}{2}g_{AB}h \tag{118}$$

as in [9]. As in [10], we will refer to coordinates in which $\nabla^B\bar{h}_{AB} = 0$ as harmonic coordinates. Notice that in harmonic coordinates we have

$$R^{H}_{AB} = R^{C\ \ D}_{\ AB}h_{CD} - \frac{1}{2}\nabla^2 h_{AB} \tag{119}$$

So that we can always write

$$R_{AB} = R^{H}_{AB} + \nabla_{(A}\nabla^{D}\bar{h}_{B)D} \tag{120}$$

We can also note via the existence and uniqueness Theorem 10.1.2 of [9], that there always exists a solution for $h_{\mu\nu}$ to the following

$$R^{H}_{AB} = \kappa\left(T_{AB} - \frac{1}{D-2}g_{AB}T\right) \tag{121}$$

provided T_{AB} does not depend on the second derivatives of the metric. If we take T to be the energy momentum tensor for a bulk CC, then implies that there exists a solution to

$$G^{H}_{AB} \equiv R^{H}_{AB} - \frac{1}{2}g_{AB}R^{H} = -\Lambda h_{AB} \tag{122}$$

Our goal in the rest of the section will be to show that any solution to this equation can also be a solution to the EFEs given a judicious choice of the remaining gauge freedom.

3.3.2 Gauge Fixing Consistently with GN

We now look to show choose Harmonic coordinates on Σ, which are necessary for the analysis of [9], using the remaining 4D gauge freedom. Recall that after imposing the GN gauge, we still have full 4D gauge invariance.

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\epsilon_{\nu)} \tag{123}$$

As such, on a surface of fixed η (Σ) we can impose any chosen 4D gauge. We will look to impose the 5D harmonic gauge, but first we must first impose the condition

$$\partial_\eta h\Big|_\Sigma = 0 \tag{124}$$

We impose this condition first because under (123) we have $\partial_\eta h \to \partial_\eta h$. So to impose this condition we will use a 5D coordinate transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \epsilon^\eta(x)\partial_\eta g^0_{\mu\nu} + 2\left(\int d\eta g^{0,\sigma\rho}\right) g^0_{\sigma(\mu}\partial_{\nu)}\partial_\rho\epsilon^\eta(x) \tag{125}$$

Under which

$$\partial_\eta h \to \partial_\eta h + \epsilon^\eta\partial_\eta(g^{0,\mu\nu}\partial_\eta g^0_{\mu\nu}) + 2g^{0,\mu\nu}\partial_\mu\partial_\nu\epsilon^\eta \tag{126}$$

From which we see that there exists a unique solution to

$$0 = \partial_\eta h + \epsilon^\eta\partial_\eta(g^{0,\mu\nu}\partial_\eta g^0_{\mu\nu}) + 2g^{0,\mu\nu}\partial_\mu\partial_\nu\epsilon^\eta \tag{127}$$

on Σ by the usual existence and uniqueness theorem in [9] and [10]. So we can now move on to showing that we can also always impose

$$0 = \dot{H}_M \tag{128}$$
$$\equiv \nabla_J \bar{h}^J_M \tag{129}$$
$$= \frac{1}{\sqrt{g}}\partial_J\left(\sqrt{g}\bar{h}^J_M\right) - \Gamma^K_{JM}\bar{h}^J_K \tag{130}$$
$$= \delta^\mu_M\frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}\bar{h}^\nu_\mu\right) - \delta^\eta_M\Gamma^\rho_{\sigma\eta}\bar{h}^\sigma_\rho \tag{131}$$
$$= \delta^\mu_M\partial_\nu\bar{h}^\nu_\mu - \delta^\eta_M\Gamma^\rho_{\sigma\eta}\bar{h}^\sigma_\rho \tag{132}$$

on Σ. Where we use the same notation as [9]

$$\bar{h}^J_M \equiv h^J_M - \frac{1}{2}\delta^J_M h \tag{133}$$

So we will look to impose the conditions

$$0 = g^{0,\sigma\nu}\partial_\sigma h_{\mu\nu}\Big|_\Sigma \tag{134}$$
$$0 = \Gamma^\rho_{\sigma\eta}h^\sigma_\rho\Big|_\Sigma \tag{135}$$
$$0 = h\Big|_\Sigma \tag{136}$$

This is possible, since for the form of the metric we are considering $g^0_{\mu\nu} = g^0_{\mu\nu}(\eta)$, the metric can always be brought to the Minkowski metric on Σ via a diagonalization and constant (possibly η_Σ dependent) coordinate rescalings. As such the standard gauge fixing of perturbations on Minkowski space can be used here. The last two conditions can be imposed after the first via a gauge transformation of the above form provided

$$
\begin{aligned}
0 =& g^{0,\sigma\nu}\partial_\sigma\left(\partial_{(\mu}\epsilon_{\nu)} - \frac{1}{2}g_{\mu\nu}\partial_\lambda\epsilon^\lambda\right)\Big|_\Sigma && (137)\\
=& g^{0,\sigma\nu}\partial_\sigma\partial_\nu\epsilon_\mu\Big|_\Sigma && (138)\\
& && (139)
\end{aligned}
$$

As in [9], to impose our conditions, we take on an initial surface $t = t_0$ the following values for ϵ^μ and it's time derivative.

$$
\begin{aligned}
0 =& h + 2\partial_\mu\epsilon^\mu && (140)\\
0 =& \partial_t h + 2\partial_t\partial_\mu\epsilon^\mu && (141)\\
0 =& \Gamma^\rho_{\sigma\eta}\left(h^\sigma_\rho + 2\partial_\rho\epsilon^\sigma - \frac{1}{2}\delta^\sigma_\rho\partial_\mu\epsilon^\mu\right) && (142)\\
0 =& \Gamma^\rho_{\sigma\eta}\partial_t\left(h^\sigma_\rho + 2\partial_\rho\epsilon^\sigma - \frac{1}{2}\delta^\sigma_\rho\partial_\mu\epsilon^\mu\right) && (143)
\end{aligned}
$$

Where we have included the term $-\frac{1}{2}\delta^\sigma_\rho\partial_\mu\epsilon^\mu$ since h will already be set to 0. As such any part of $\Gamma^\rho_{\sigma\eta}$ that is $\propto \delta^\rho_\sigma$ will not contribute to either equation.

We then have

$$
\begin{aligned}
f =& h + 2\partial_\mu\epsilon^\mu && (144)\\
\implies g^{0,\sigma\rho}\partial_\sigma\partial_\rho f =& g^{0,\sigma\rho}\partial_\sigma\partial_\rho(h + 2\partial_\mu\epsilon^\mu) && (145)\\
=& 2\partial_\mu(g^{0,\sigma\rho}\partial_\sigma\partial_\rho\epsilon^\mu) && (146)\\
=& 0 && (147)
\end{aligned}
$$

With $\partial_t f = f = 0$ on the initial surface. Which then implies that $f = 0$ throughout Σ. The same holds for the other condition. So we have shown in all that

$$
\begin{aligned}
0 =& h\Big|_\Sigma && (148)\\
0 =& \partial_\eta h\Big|_\Sigma && (149)\\
0 =& \partial_\nu \bar{h}^\nu_\mu\Big|_\Sigma = \partial_\nu h^\nu_\mu\Big|_\Sigma && (150)\\
0 =& \Gamma^\rho_{\sigma\eta}\bar{h}^\sigma_\rho\Big|_\Sigma = \Gamma^\rho_{\sigma\eta}h^\sigma_\rho\Big|_\Sigma && (151)
\end{aligned}
$$

3.3.3 Initial Value Data and Harmonic Coordinates

We want to show that if

$$
\dot{H}_M\Big|_\Sigma = \nabla_N\bar{h}^N_M\Big|_\Sigma = 0 \qquad (152)
$$

and the initial data constraints are satisfied on Σ, then $R^H_{MN} = 0$ implies that the solution is also a solution to the EFEs in a neighborhood of Σ and $\dot{H}_M = 0$ in that same neighborhood.

Consider the perturbed Einstein Tensor, which can always be written

$$
G_{AB} = R^H_{AB} + \nabla_{(A}\dot{H}_{B)} - \frac{1}{2}\left(R^H + \nabla_C\dot{H}^C\right)g^0_{AB} \qquad (153)
$$

$$G_{AB} = R^H_{AB} + \nabla_{(A}\dot{H}_{B)} - \frac{1}{2}\left(R^H + \nabla_C \dot{H}^C\right) g^0_{AB} \tag{154}$$

Hence if we have $G^H_{AB} = R^H_{AB} - \frac{1}{2}R^H g^0_{AB} = -\Lambda h_{AB}$ in a neighborhood of Σ then we can write

$$G_{AB} = \nabla_{(A}\dot{H}_{B)} - \frac{1}{2}\nabla_C \dot{H}^C g^0_{AB} - \Lambda h_{AB} \tag{155}$$

the initial data constraints give (since for a bulk CC $T_{AB}n^B = h_{AB}n^B = 0$)

$$\begin{aligned} 0 =& G_{AB}n^B && (156)\\ =& \left(\nabla_{(A}\dot{H}_{B)} - \frac{1}{2}\nabla_C \dot{H}^C g^0_{AB}\right) n^B && (157)\\ =& \frac{1}{2}\left(n^B \nabla_A \dot{H}_B + n^B \nabla_B \dot{H}_A - \nabla_C \dot{H}^C n_A\right) && (158) \end{aligned}$$

To simplify this further, we use the fact that since $\dot{H}^M = 0$ on Σ, we know that $e^M_{(\mu)}\nabla_M H^N = 0$ on Σ. So we have

$$\begin{aligned} 0 =& G_{AB}n^B e^A_{(\alpha)} && (159)\\ =& \frac{1}{2} n^B e^A_{(\alpha)} \nabla_B \dot{H}_A && (160) \end{aligned}$$

using the orthogonality of $\boldsymbol{e}_{(\mu)}$ and $\boldsymbol{n}$. Further

$$\begin{aligned} 0 =& G_{AB}n^B n^A && (161)\\ =& \frac{1}{2}\left(n^B \nabla_A \dot{H}_B + n^B \nabla_B \dot{H}_A - \nabla_C \dot{H}^C n_A\right) n^A && (162)\\ =& \frac{1}{2}\left(2n^A n^B \nabla_A \dot{H}_B + n^A n^B \nabla_A \dot{H}_B + g^{ind,AB}\nabla_A \dot{H}_B\right) && (163)\\ =& \frac{3}{2} n^A n^B \nabla_A \dot{H}_B && (164) \end{aligned}$$

where we've used $\boldsymbol{n}\cdot\boldsymbol{n} = -1$ and $e^M_{(\mu)}\nabla_M H^N = 0$. So in all we have that given our assumptions on Σ we have

$$\begin{aligned} 0 =& n^B \nabla_B \dot{H}^A && (165)\\ =& n^B \partial_B \dot{H}^A && (166) \end{aligned}$$

since $\dot{H}^A = 0$ on Σ. Note also that this implies

$$0 = \mathcal{L}_{\boldsymbol{n}} H^M \tag{167}$$

Next consider the Bianchi identity

$$
\begin{aligned}
\nabla^A G_{AB} =& \nabla^A \left(\nabla_{(A} \dot{H}_{B)} - \frac{1}{2} \nabla_C \dot{H}^C g^0_{AB} \right) && (168)\\
=& \frac{1}{2} \left(\nabla^A \nabla_A \dot{H}_B + \nabla_A \nabla_B \dot{H}^A - \nabla_B \nabla_A \dot{H}^A \right) && (169)\\
=& \frac{1}{2} \left(\nabla^A \nabla_A \dot{H}_B - R_{BAC}{}^A \dot{H}^C \right) && (170)\\
=& \frac{1}{2} \left(\nabla^A \nabla_A \dot{H}_B - R^A{}_{BAC} \dot{H}^C \right) && (171)\\
=& \frac{1}{2} \left(\nabla^A \nabla_A \dot{H}_B - R_{BC} \dot{H}^C \right) && (172)\\
=& 0 && (173)
\end{aligned}
$$

So that the Bianchi identity implies $\dot{H}_B$ satisfies a linear diagonal hyperbolic equation, for which we know there exists a unique solution given initial data on a hyper-surface. Since we have shown $\dot{H}_B\Big|_\Sigma = n^A \nabla_A \dot{H}_B\Big|_\Sigma = 0$, we find that given our above assumptions, $\dot{H}_B = 0$ in the bulk as well. And further, that

$$
G_{AB} = G^H_{AB} = -\Lambda h_{AB} \tag{174}
$$

So that the solution is also a solution to the vacuum field equations.

3.3.4 Vanishing trace in the bulk

Finally, we want to have our gauge conditions are satisfied in the bulk as well. To do so we need to notice that in vacuum

$$
\begin{aligned}
\Lambda h \propto& R^H && (175)\\
=& g^{0,AB} R^C{}_{AB}{}^D h_{CD} - \frac{1}{2} \nabla^2 h && (176)
\end{aligned}
$$

And since we have made the gauge choices

$$
\begin{aligned}
0 =& \partial_\eta h\Big|_\Sigma && (177)\\
0 =& h\Big|_\Sigma && (178)
\end{aligned}
$$

our favourite existence and uniqueness theorem tells us that $h = 0$ in the bulk as well. Which then implies that since $\dot{H}_M = 0$ in the bulk, we have finally

$$
\begin{aligned}
0 =& h && (179)\\
0 =& \partial_\mu \bar{h}^\mu_\nu = \partial_\mu h^\mu_\nu && (180)\\
0 =& \Gamma^\rho_{\sigma\eta} \bar{h}^\sigma_\rho = \Gamma^\rho_{\sigma\eta} h^\sigma_\rho && (181)\\
& && (182)
\end{aligned}
$$

3.4 Gauge Fixed Initial Data Constraints

3.4.1 The $\mu\eta$ equations

Using the above gauge conditions we have

$$
\begin{aligned}
R^{(1)}_{\mu\eta} =& g^{0,\nu\alpha}\partial_\eta g^0_{[\mu|\alpha}\left(g^{0,\sigma\rho}\partial_{(\sigma]}h_{\nu)\rho} - \frac{1}{2}g^{0,\sigma\rho}\partial_\rho h_{|\sigma]\nu}\right) - \partial_{[\mu|}\partial_\eta\left(g^{0,\nu\sigma}h_{|\nu]\sigma}\right) && (183)\\
=& 0 && (184)\\
& && (185)
\end{aligned}
$$

So that this initial data constraint is trivially satisfied.

3.4.2 The $\eta\eta$ equation

Given the above gauge conditions we have

$$
\begin{aligned}
R^{(1)}_{\eta\eta} =& -\frac{1}{2}h_{\mu\nu}\left(\partial^2_\eta g^{0,\mu\nu} - g^0_{\sigma\rho}\partial_\eta g^{0,\sigma\mu}\partial_\eta g^{0,\rho\nu}\right) - \frac{1}{2}\partial_\eta\left(g^{0,\mu\nu}\partial_\eta h_{\mu\nu}\right) && (186)\\
=& -\frac{1}{2}h_{\mu\nu}\left(\partial^2_\eta g^{0,\mu\nu} - g^0_{\sigma\rho}\partial_\eta g^{0,\sigma\mu}\partial_\eta g^{0,\rho\nu}\right) && (187)
\end{aligned}
$$

Which can be rewritten

$$
\begin{aligned}
\partial^2_\eta g^{0,\mu\nu} - g^0_{\sigma\rho}\partial_\eta g^{0,\sigma\mu}\partial_\eta g^{0,\rho\nu} =& \partial^2_\eta g^{0,\mu\nu} - \partial_\eta\left(g^{0,\sigma\mu}g^0_{\sigma\rho}\partial_\eta g^{0,\rho\nu}\right) + g^{0,\sigma\mu}\partial_\eta\left(g^0_{\sigma\rho}\partial_\eta g^{0,\rho\nu}\right) && (188)\\
=& \partial^2_\eta g^{0,\mu\nu} - \partial^2_\eta g^{0,\mu\nu} + g^{0,\sigma\mu}\partial_\eta\left(g^0_{\sigma\rho}\partial_\eta g^{0,\rho\nu}\right) && (189)\\
=& -2g^{0,\sigma\mu}\partial_\eta\Gamma^\nu_{\sigma\eta} && (190)\\
& && (191)
\end{aligned}
$$

So we can write

$$
\begin{aligned}
R^{(1)}_{\eta\eta} =& h_{\mu\nu}g^{0,\sigma\mu}\partial_\eta\Gamma^\nu_{\sigma\eta} && (192)\\
=& -\frac{1}{2}h_{\mu\nu}g^{0,\sigma\mu}\partial_\eta\left(g^0_{\sigma\rho}\partial_\eta g^{0,\rho\nu}\right) && (193)
\end{aligned}
$$

While this may not hold for a general metric, if we impose homogeneity and isotropy (as in AdS-S and AdS) then there will be at most 2 independent scalars that can be built out of $h_{\mu\nu}$ using the metric and it's derivatives. Since our gauge conditions impose that both vanish, we then have that this will be some linear combination of these vanishing scalars, and so will also be satisfied. Hence under the assumption of homogeneity and isotropy on 4D slices of the bulk, this initial data constraint is satisfied.

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