Dynamics of Moving Branes In AdS-S

Keegan Humphrey

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Abstract

In these notes we leverage Israel's Junction conditions to describe the dynamics of a codimension 1 surface moving in AdS-S. We extend the results of [1] to include both a scalar in the bulk, matter on the brane, and allow the brane to have a non-trivial profile. We find that the junction conditions determine the profile in terms of the energy density on the brane. This leads to the conclusion that there are no stable stationary points in the bulk for such a surface. As such we consider a scenario in which two branes are allowed to propagate in the bulk after emission from the horizon. We find that IR branes observers see a time independent and natural electroweak hierarchy provided the energy density on the branes is dark energy dominated, the UV brane truncates the space, and the branes have equal energy densities. We then go on to do a heuristic calculation of the emission rate of such a surface from the AdS-S black brane following a literature started by [2]. We find the hawking temperature is off by an order 1 factor, a problem which has been recorded in the literature.

1 IR Brane in AdS-S

In this section we compute the junction conditions for a brane propagating in AdS-S in the presence of the Energy Momentum Tensors (EMTs) we computed in appendix E.

1.1 Metric Parametrization

Consider the AdS_5 black brane metric. The metrics in $V_\pm$ (see Appendix A for definitions and notation) will be

$$\begin{aligned}
ds_\pm^2 =& dX^M dX^N g_{\pm,MN} && (1)\\
=& -g_\pm(p)dt^2 + \frac{dp^2}{g_\pm(p)} + \frac{p^2}{l^2} ds_3^2 && (2)\\
=& -g_\pm(p)dt^2 + \frac{dp^2}{g_\pm(p)} + \frac{p^2}{l^2} h_{ij} dx^i dx^j && (3)\\
& -g_\pm(r)dt^2 + \frac{dp^2}{g_\pm(p)} + \frac{p^2}{l^2}(dr^2 + S_k(r)^2 d\Omega^2) && (4)\\
g_\pm(p) \equiv& k + \frac{p^2}{l^2} - \frac{\mu_\pm}{p^2} && (5)
\end{aligned}$$

**** here should define $r \to \chi$ to distinguish between global radial cooridnate and the coordinate on the brane.

Where k takes on the values $0, -1, 1$ corresponding to flat, open or closed geometries respectively and we have defined

$$S_k(r) \equiv \begin{cases} \sinh r & k = -1 \\ r & k = 0 \\ \sin r & k = 1 \end{cases} \tag{6}$$

as in [3]. This family of solutions are the unique solutions to Einsteins equations with the isometries of FRW universes in the absence of the IR brane [1]. We can relate μ to the mass density of the black hole [4]

$$\mu_{\pm} = \frac{16\pi G M_{\pm}}{3\mathrm{Vol}(M_3)} = \frac{16\pi G \rho_{M\pm} l^3}{3} \equiv \frac{4\rho_{M\pm}}{\sigma_c} l^2 \tag{7}$$

Where M_3 is the manifold corresponding to a surface of fixed p and t and $\mathrm{Vol}(M_3) = \int_{M_3} dx^3 \sqrt{h}$. Here we can compare to conformal coordinates as $p = \frac{l^2}{z}$. And hence has Hawking temperature $T = \frac{1}{\pi z_h} = \frac{p_h}{\pi l^2}$ for a flat horizon geometry.

1.2 Basis and Induced Metric

Now consider the embedding of the IR brane in the bulk with an $SO(3)$ symmetry on the brane.

$$X^M = (T(\tau, r), R(\tau, r), 0, 0, P(\tau, r)) \tag{8}$$

This describes a congruence of spherical shells parametrized by r whose dynamics we will now determine using Israel's Junction conditions. We take as part of our our basis, the following two tangent vectors

$$u^M = \partial_\tau X^M = (\dot{T}, 0, 0, 0, \dot{P}) \tag{9}$$

$$e^M_{(r)} = \partial_r X^M = (T', R', 0, 0, P') \tag{10}$$

so we have imposed $\dot{R} = 0$ and denote $f' \equiv \partial_r f$. We want u to have unit time-like norm which determines it's first component and identifies τ with the proper time on the brane

$$-1 = \boldsymbol{u} \cdot \boldsymbol{u} \tag{11}$$

$$\implies \dot{T} = \frac{1}{g(p)} \sqrt{g(p) + \dot{P}^2} \tag{12}$$

$$\equiv \frac{\kappa_p}{g(p)} \tag{13}$$

The only constraints we have on the normal vector $\boldsymbol{n}$ are that it is orthogonal to the brane and that it has a space-like norm of 1. So we are free to take it to have 2 components

$$n^M = (n^t, 0, 0, 0, n^p) \tag{14}$$

Which from the above conditions are

$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \qquad \boldsymbol{n} \cdot \boldsymbol{n} = 1 \tag{15}$$

$$\implies n^t = \frac{\dot{P}}{g(p)} \tag{16}$$

$$n^p = \sqrt{g(p) + \dot{P}^2} \tag{17}$$

$$\equiv \kappa_p \tag{18}$$

So all that's left is to determine $\boldsymbol{e}_{(r)}$. We have for the induced metric

$$
\begin{aligned}
ds^2_{ind} =& g_{MN} e^M_{(\mu)} e^N_{(\nu)} dx^\mu dx^\nu && (19)\\
=& g_{MN} u^M u^N d\tau^2 + g_{MN} u^M e^N_{(i)} dx^i d\tau + g_{MN} e^M_{(i)} e^N_{(i)} dx^i dx^i && (20)\\
=& - d\tau^2 + \boldsymbol{u} \cdot \boldsymbol{e}_{(r)} dr d\tau + g_{MN} e^M_{(i)} e^N_{(i)} dx^i dx^i && (21)\\
& && (22)
\end{aligned}
$$

We choose the r components to match their bulk counterparts

$$
g_{MN} e^M_{(r)} e^N_{(r)} dr^2 = \frac{p^2}{l^2} dr^2 \tag{23}
$$

So we have

$$
\begin{aligned}
\boldsymbol{e}_{(r)} \cdot \boldsymbol{e}_{(r)} =& \frac{p^2}{l^2} && (24)\\
\boldsymbol{n} \cdot \boldsymbol{e}_{(r)} =& 0 && (25)
\end{aligned}
$$

Imposing these two conditions imply off diagonal elements in the induced metric

$$
\boldsymbol{u} \cdot \boldsymbol{e}_{(r)} = - \frac{P'}{\dot{P}} \tag{26}
$$

as well as the relation between R and r.

$$
R' = \sqrt{1 + \frac{l^2}{p^2} \frac{P'^2}{\dot{P}^2}} \tag{27}
$$

Notice that for a flat profile, $P' = 0$ so this identifies $R = r$ and makes the induced metric diagonal. So we find in all

$$
\begin{aligned}
u^M =& \partial_\tau X^M = \left(\frac{\kappa_p}{g(p)}, 0, 0, 0, \dot{P} \right) && (28)\\
e^M_{(r)} =& \left(\frac{\kappa_p P'}{g(p) \dot{P}}, R', 0, 0, P' \right) && (29)\\
n^M =& \left(\frac{\dot{P}}{g(p)}, 0, 0, 0, \kappa_p \right) && (30)
\end{aligned}
$$

We also allow the brane to inherit the angular coordinates of the bulk ($e^M_{(\Omega_i)} = \delta^M_{\Omega_i}$ which are clearly orthogonal to eachother and all other vectors in the basis), we then find the induced metric on Σ

$$
ds^2_\Sigma = - d\tau^2 - 2 \frac{P'}{\dot{P}} dr d\tau + \frac{P^2}{l^2} \left(dr^2 + S_k(R)^2 d\Omega^2 \right) \tag{31}
$$

Imposing that both sides of the Σ use this basis, this form of the induced metric will hold on both sides of the brane. Hence we have satisfied Israel's first junction condition. Notice that this reduces to the FRW metric in the limit of a flat brane profile ($P' = 0$).

$$
ds^2_\Sigma |_{P'=0} = - d\tau^2 + \frac{P^2}{l^2} \left(dr^2 + S_k(r)^2 d\Omega^2 \right) \tag{32}
$$

In which case we could relate the scale factor to P

$$a(\tau) = \frac{P(\tau)}{l} \tag{33}$$

where here, τ is identified with the cosmic rather than conformal time (which is usually denoted by τ).

1.3 Summary of Junction Conditions

For the action

$$\begin{aligned}
S =& S_{EH} + S_{GHY} + S_B + S_b + S_{PF} && (34)\\
=& \frac{1}{2\kappa}\int_V d^5X\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial V} d^4x\sqrt{-g_{\text{ind}}}K && (35)\\
&+ \int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi)\right) - \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}V_b(\phi) && (36)\\
&+ \frac{1}{2}\int_\Sigma d^4x\sqrt{g_{\text{ind}}}\left((\mathcal{P}+\rho)g^{\text{ind}}_{\mu\nu}u^\mu u^\nu - (\mathcal{P}-\rho)\right) && (37)
\end{aligned}$$

Israels Junction conditions impose the following (see appendices D-G)

$$\begin{aligned}
\dot{P}^2 =& -k + \frac{l^2}{P^2}\frac{\mu_{\text{avg}}}{l^2} + \frac{1}{16}\frac{l^6}{P^6}\frac{\sigma_c^2}{\sigma^2(\phi)}\frac{\mu_{\text{dif}}^2}{l^4} + \frac{P^2}{l^2}l^2H_\tau^2 && (38)\\
=& -2V(P) && (39)\\
-\partial_\tau V_b(\phi) =& \dot{\rho} + 3\frac{\dot{P}}{P}(\rho+\mathcal{P}) && (40)\\
-\partial_r V_b(\phi) =& \rho' + 3\frac{P'}{P}(\rho+\mathcal{P}) && (41)
\end{aligned}$$

As above, our basis also determines R' in terms of $\dot{P}$ and P'

$$R'^2 = 1 + \frac{l^2}{p^2}\frac{P'^2}{\dot{P}^2} = 1 - \frac{l^2}{p^2}\frac{P'^2}{2V(P)} \tag{42}$$

We find that it remains to determine P'. One can do this by computing an effective action and determining the equations of motion for P' derived from that action (see appendix H). We also collect the junction conditions for the scalar and from the normal components of Einstein's equations (see appendix C)

$$\begin{aligned}
\left[n^M\partial_M\phi\right] =& -V_b'(\phi) && (43)\\
S^\nu_{\mu;\nu} =& \partial_\mu V_b(\phi) && (44)\\
\frac{1}{2}\left[\partial_n\phi\partial_n\phi\right] =& -4\tilde{K}\sigma(\phi) && (45)
\end{aligned}$$

We also write down the velocity in global time (which are discontinuous across Σ)

$$\begin{aligned}
(\partial_t P)_\pm =& \frac{u^p}{u^t_\pm} = g_\pm(p)\frac{\dot{P}}{\sqrt{g_\pm(p)+\dot{P}^2}} && (46)\\
(\partial_t R)_\pm =& \frac{u^r}{u^t_\pm} = 0 && (47)
\end{aligned}$$

Notice that both vanish at the horizon.

1.4 Conditions for a static brane in the bulk

To have a static brane in the bulk at a position P_s, we need to have the following conditions met. We can understand this by making the analogy with a 0 energy 1D mechanics problem with a conserved energy $\frac{1}{2}\dot{P}^2 + V(P) = 0$.

$$\dot{P}|_{P_s} = 0 \qquad \ddot{P}|_{P_s} = 0 \tag{48}$$
$$\tag{49}$$

Using our above solution for $\dot{P}$, these imply

$$V(P_s) = 0 \qquad V'(P_s) = 0 \tag{50}$$

And finally, to have this be a stable configuration we need

$$V''(P_s) > 0 \tag{51}$$

Consider a shell in the probe brane limit (where $\rho_{\text{diff}} = 0$) moving in a flat horizon geometry (k=0)

$$-2V(P_s) = \frac{l^2}{P_s^2}\frac{\mu_{\text{avg}}}{l^2} + \frac{P_s^2}{l^2} l^2 H_\tau^2 = 0 \tag{52}$$
$$-2V'(P_s) = -2\frac{l^2}{P_s^3}\frac{\mu_{\text{avg}}}{l^2} + 2\frac{P_s}{l^2} l^2 H_\tau^2 = 0 \tag{53}$$

These equations clearly have no solution. Which implies that for a flat horizon geometry, in the probe brane limit, there are **no stationary points in the bulk**. This agrees with the conclusions of [1] which argue that there do not exist stable stationary points in the bulk for any value of k or $\mu_\pm$ save for the parameters used in the RS scenario, in which the energy density on the branes is tuned such that the potential vanishes identically.

2 A 2 Brane Scenario for a Realistic and Natural Cosmology

Since we cannot construct a stable minimum in the bulk. We consider an alternative scenario in which we allow the IR and UV branes to propagate into the bulk. As such bubbles which nucleate from a flat horizon at different times would not coallesce into a single brane to complete the confinement deconfinement phase transition. So we are forced to consider instead a compact horizon (one with the geometry of a 3-sphere) in which the nucleation can proceed through a single emission event in finite time. Fortunately, the emission of a thin spherical shell from a spherical black hole, including back reaction effects, has been calculated in the context of S-wave hawking radiation (see eg. [2] and [5]).

The utility and motivation of this construction is from the RS scenario in which the ratio of positions of an IR and UV brane along the extra dimension warps down the planck scale to explain the planck electroweak hierarchy observed on the IR brane. As such we must look for solutions in which there are two emitted branes and the ratio of their positions is fixed throughout the motion of the two emitted branes. We will find that this can be accomplished for a dark energy dominated universe provided the UV brane truncates the spacetime and the two branes have energy densities of the same magnitude. This gives us a non-compact extra dimension consistent with Newtonian gravity as described in [6].

2.1 Summary of Spherical Profile Geometry and Dynamics

Given that our emitted shell respects the isometries of the closed AdS-S geometry (the 5D metric above with $k = 1$), we know that the metric on either side, without the need for the probe brane limit, is given by

$$ds_{\pm}^2 = -g_{\pm}(r)dt^2 + \frac{dp^2}{g_{\pm}(p)} + \frac{p^2}{l^2}d\Omega_3^2 \tag{54}$$

$$g_{\pm}(p) \equiv 1 + \frac{p^2}{l^2} - \frac{\mu_{\pm}}{p^2} \tag{55}$$

when we neglect the back reaction of the bulk scalar. Our induced metric becomes that of a closed FRW universe.

$$ds_{\Sigma}^2|_{P'=0} = -d\tau^2 + \frac{P^2}{l^2}d\Omega_3^2 \tag{56}$$

We again write down the junction conditions for the trivial bubble profile.

$$\dot{P}^2 = -1 + \frac{l^2}{P^2}\frac{\mu_{\text{avg}}}{\sigma_c} + \frac{1}{16}\frac{l^6}{P^6}\frac{\mu_{\text{diff}}^2}{\sigma^2(\phi)} + \frac{P^2}{l^2}l^2H_\tau^2 \tag{57}$$

$$= -2V(P) \tag{58}$$

$$l^2H_\tau^2 = \frac{\sigma^2(\phi)}{\sigma_c^2} - 1 \tag{59}$$

$$-\partial_\tau V_b(\phi) = \dot{\rho} + 3\frac{\dot{P}}{P}(\rho + \mathcal{P}) \tag{60}$$

Where again $\sigma(\phi)$ is the energy density on the brane, and σ_c is the critical energy density to which the tension is tuned in the RS scenario. Notice that the r continuity equation was only imposed for $P' \neq 0$. As such in the absence of the profile it does not provide a constraint. And for the scalar and from Einsteins equations we have

$$\left[n^M\partial_M\phi\right] = -V_b'(\phi) \tag{61}$$

$$S_{\mu;\nu}^\nu = \partial_\mu V_b(\phi) \tag{62}$$

$$\frac{1}{2}\left[\partial_n\phi\partial_n\phi\right] = -4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{63}$$

The second of these equations implies a consistency condition which must be compared to the above expressions from the junction conditions. In the absence of a profile, the surface energy momentum tensor is

$$S_\nu^\mu = (\rho + \mathcal{P})\delta_\nu^\tau\delta_\tau^\mu + (V_b(\phi) - \mathcal{P})\delta_\nu^\mu \tag{64}$$

Since $u^\mu = \delta_\tau^\mu$. We now compute the τ equation (others vanish by isometry) and determine the consistency condition. As shown in eg [7] pg 74, the intrinsic covariant derivative is taken with the connection compatible with the induced metric. As such this is precisely given by the continuity equation of the normal FRW matter sources. Since we effectively have 2 components, we must add the dark energy contribution of $V_b(\phi)$ and the perfect fluid contribution of eg [3] (1.3.98).

$$S_{\tau;\nu}^\nu = \dot{\rho} + 3\frac{\dot{P}}{P}(\rho + \mathcal{P}) + \partial_\tau V_b(\phi) \tag{65}$$

$$= \partial_\tau V_b(\phi) \tag{66}$$

Comparing to the above junction condition, this implies that the two components are seperately conserved. Namely, even in the presence of matter to couple to, the value of the scalar on the brane must be constant throughout the motion. This then implies that the $T_{n\mu}$ components of the bulk energy momentum tensor are continuous, and that the surface energy momentum tensor is conserved.

$$0 =\partial_\tau V_b(\phi) \tag{67}$$
$$0 =\dot{\rho} + 3\frac{\dot{P}}{P}(\rho + \mathcal{P}) \tag{68}$$

We also write down the velocity in global time

$$(\partial_t P)_\pm =\frac{u^p}{u^t_\pm} = g_\pm(p)\frac{\dot{P}}{\sqrt{g_\pm(p) + \dot{P}^2}} \tag{69}$$

2.2 2-Brane Dynamics - Arbitrary equation of state - small ρ expansion plus σ_c

It remains to show that the planck electroweak hierarchy is time independent. As in [8] this hierarhcy will be determined by the ratio of positions along the extra dimensions. Hence we must show that this ratio is independent of time. We take the IR and UV branes to be situated at P_{IR} and P_{UV} respectively. We imagine that the branes underwent a period of inflation after which $P_{IR}, P_{UV} \gg l$ exponentially supressing the curvature and horizon terms in the dynamical junction equation[1].

$$\dot{P}^2 \to P^2 H_\tau^2 \tag{70}$$

This also solves the flatness and horizon problems. So the metric on the branes is flat and our bulk near the branes is simply AdS_5.

$$ds^2_\pm =\frac{p^2}{l^2}\eta_{\mu\nu}dX^\mu dX^\nu + \frac{l^2}{p^2}dp^2 \tag{71}$$
$$\tag{72}$$

We consider orbifolding at the position of the UV brane (or just cutting off the space there) so that it acts as a UV cutoff[2]. We are interested in the planck electroweak hierarchy so we must determine the effective 4D gravitational action.

$$M^3 \int_V d^5X\sqrt{-g}R \supset M_4^2 \int_\Sigma d^4x\sqrt{-g_{\mathrm{ind}}}R_4 \tag{73}$$

However, because Σ is not stationary and as we have shown in detail above, the observers on Σ will see an FRW universe the integral over which will not factor from the integral over the bulk coordinate p. So we look now to parametrize the bulk using our normal coordinate η which will allow us to find the effective 4D action. We parametrize the family of possible IR brane trajectories by η such that $n^M = \frac{dX^M}{d\eta}$ and so our bulk metric is now given by

$$ds^2 = -d\tau^2 + \frac{p^2}{l^2}d\vec{x}^2 + d\eta^2 \tag{74}$$

[1]It would be interesting to imagine a scenario where initially the IR brane is empty, giving it a constant dark energy due to it's tension. Were there a way to exit inflation as the IR brane approaches the UV brane (possibly as a result of fields localized near it) then this could lead to inflation without an inflaton

[2]The analysis of the junction conditions is unaffected by the change in sign which results from orbifolding at the position of one of the branes as shown in [1].

So along a curve with tangent vector $\boldsymbol{n}$ we have

$$n^M = \frac{dX^M}{d\eta} = \left(\frac{\dot{P}}{g(P)}, 0, 0, 0, \kappa_p\right) \tag{75}$$

We now imagine there is matter on the brane with equation of state $\omega = \frac{\mathcal{P}}{\rho}$ as well as a critical dark energy component such that $|\sigma(\phi)| = |\rho + \sigma_c| > \sigma_c$ and $|\rho| \ll \sigma_c$ as in [1]. Then we have

$$\begin{aligned} l^2 H_\tau^2 =& \frac{\sigma^2(\phi)}{\sigma_c^2} - 1 && (76)\\ =& \left(\frac{\rho}{\sigma_c} + 1\right)^2 - 1 && (77)\\ & && (78) \end{aligned}$$

Since we are near the boundary, we have using the Friedmann equation

$$\begin{aligned} \frac{dP}{d\eta} =& \kappa_p && (79)\\ =& \sqrt{\frac{P^2}{l^2} + \dot{P}^2} && (80)\\ =& \frac{P}{l}\sqrt{1 + l^2 H_\tau^2} && (81)\\ =& \frac{P}{l}\left(1 + \frac{\rho}{\sigma_c}\right) && (82) \end{aligned}$$

Since we have assumed $\rho < \sigma_c$. Hence we find that for SEC satisfying matter at leading order in the small energy density expansion

$$\begin{aligned} \frac{dP}{d\eta} \to& \frac{P}{l} && (83)\\ \implies P(\tau, \eta) =& P_\eta(\tau) e^{l^{-1}(\eta - \eta_{IR})} && (84) \end{aligned}$$

Where we have included η_{IR} so that P_η is the value of P at the IR brane. This corresponds to a choice of coordinates. This approximation will improve with proper time as the energy density dilutes. If instead we are in a dark energy dominated phase, then we can relax the small energy density expansion and write

$$P(\tau, \eta) \to P_\eta(\tau) e^{l^{-1}\left|1 + \frac{\rho_\Lambda}{\sigma_c}\right|(\eta - \eta_{IR})} \tag{85}$$

So we define

$$H_\eta = \begin{cases} l^{-1} & \text{Satisfies SEC} \\ l^{-1}\left|1 + \frac{\rho_\Lambda}{\sigma_c}\right| & \text{Dark Energy Dominated} \end{cases} \tag{86}$$

We can now determine the evolution of the scale factor with proper time.

$$\left(\frac{\dot{P}}{P}\right)^2 = H_\tau^2 \tag{87}$$

$$= \frac{1}{l^2}\left[\left(\frac{\rho}{\sigma_c}+1\right)^2 - 1\right] \tag{88}$$

$$\rightarrow \begin{cases} \frac{2}{l^2}\frac{\rho}{\sigma_c} \propto (P)^{-3(1+\omega)} & \text{Satisfies SEC} \\ \frac{1}{l^2}\left[\left(\frac{\rho_\Lambda}{\sigma_c}+1\right)^2 - 1\right] & \text{Dark Energy Dominated} \end{cases} \tag{89}$$

So we find for $P(\tau, \eta)$

$$\Longrightarrow P(\tau,\eta) = \begin{cases} \left(\tau\sqrt{\frac{9}{2}\frac{l\rho_M}{\sigma_c}} + P_0^{3/2}\right)^{2/3} & MD \\ \left(\tau\sqrt{\frac{8l^2\rho_R}{\sigma_c}} + P_0^2\right)^{1/2} & RD \\ P_0 e^{\tau H_\tau} & \Lambda D \end{cases} \tag{90}$$

Here P_0 is determined by the initial condition $P(\tau = 0) = P_0$. We must now look for differences in η which are, or at least to a very good approximation, proper time independent given our above small energy density approximations. To find η_{UV}, we simply note that after flowing from η_{UV} to η_{IR}, P_{IR} will equal the value of the P_{UV} after flowing along η. This gives for dark energy domination the condition

$$P_{IR}(\tau, \eta_{UV}) = P_{UV}(\tau) \tag{91}$$

$$\Longrightarrow P_{\eta,IR}(\tau)e^{(\eta_{UV}-\eta_{IR})H_\eta} = P_{\eta,UV}(\tau) \tag{92}$$

$$\Longrightarrow e^{(\eta_{UV}-\eta_{IR})H_\eta} = \frac{P_{\eta,UV}(\tau)}{P_{\eta,IR}(\tau)} \tag{93}$$

$$= \begin{cases} \dfrac{\left(\tau\sqrt{\frac{9}{2}\frac{l\rho_{M,UV}}{\sigma_c}} + P_{0,UV}^{3/2}\right)^{2/3}}{\left(\tau\sqrt{\frac{9}{2}\frac{l\rho_{M,IR}}{\sigma_c}} + P_{0,IR}^{3/2}\right)^{2/3}} & MD \\ \dfrac{\left(\tau\sqrt{\frac{8l^2\rho_{R,UV}}{\sigma_c}} + P_{0,UV}^2\right)^{1/2}}{\left(\tau\sqrt{\frac{8l^2\rho_{R,IR}}{\sigma_c}} + P_{0,IR}^2\right)^{1/2}} & RD \\ \frac{P_{0,UV}}{P_{0,IR}} e^{\tau(H_{\tau,UV}-H_{\tau,IR})} & \Lambda D \end{cases} \tag{94}$$

So we see that only for dark energy domination can we have a time independent hierarchy, and only if we set $H_{\tau,UV} - H_{\tau,IR}$ corresponding to equal energy densities (up to a sign) on the branes. For the other types of domination, we find that η_{UV} has proper time dependence. Further, we cannot simply take the limit of large P_0 as this will be washed out at late times, and will lead to linear scaling with τ before then. We can now investigate the 4D effective planck mass given the known scaling of P with η. Switching to conformal time on Σ ($d\tau = \frac{P}{l}d\tau_c$) we can follow [8] to find the hierarchy as seen on the IR brane

$$
\begin{aligned}
M^3\int_V d^5X\sqrt{-g}R \supset & M^3\int_{\Sigma_{IR}} d^4x\int d\eta\sqrt{-g_{\text{ind}}}R && (95)\\
=&M^3\int_{\Sigma_{IR}} d^4x\int d\eta\sqrt{-\bar{g}}\frac{P^2}{l^2}\bar{R} && (96)\\
=&M^3\int_{\Sigma_{IR}} d^4x\int d\eta\sqrt{-\bar{g}}\frac{P_\eta^2(\tau)e^{2(\eta-\eta_{IR})H_\eta}}{l^2}\bar{R} && (97)\\
=&M^3\int_{\Sigma_{IR}} d^4x\sqrt{-\bar{g}}\frac{e^{2(\eta_{UV}-\eta_{IR})H_\eta}}{2H_\eta}\frac{P_\eta^2(\tau)}{l^2}\bar{R} && (98)\\
=&M^3\int_{\Sigma_{IR}} d^4x\sqrt{-g_{\text{ind}}}\frac{e^{2(\eta_{UV}-\eta_{IR})H_\eta}}{2H_\eta}R_4 && (99)
\end{aligned}
$$

Where we note that the lower limit of integration over the normal direction occurs at $P = 0$ which implies the lower η boundary is $-\infty$[3]. During dark energy domination, this gives for the planck electroweak hierarchy

$$
\begin{aligned}
M_4^2 =&M^3\frac{e^{2(\eta_{UV}-\eta_{IR})H_\eta}}{2H_\eta} && (100)\\
\approx&M^3l^{-1}e^{2(\eta_{UV}-\eta_{IR})l^{-1}} && (101)
\end{aligned}
$$

As in the original RS proposal, we now have a natural hierarchy for

$$
(\eta_{UV}-\eta_{IR})H_\eta = \mathcal{O}(10) \tag{102}
$$

We also note that there is no need to consider the warping of fields localized to IR brane. The warp factor would now be interpretted as the scale factor of IR brane observer, which matches the metric appearing the effective 4D gravitational action.

$$
S_H = \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\left(D_\mu HD^\mu H^\dagger - (|H|^2 - v^2)^2\right) \tag{103}
$$

2.3 Summary of Approximations

Here we briefly summarize the approximations made in the above argument.

- We assumed there was a period of inflation, so that we could work near the boundary (large $\frac{P}{l}$) and neglect the affects of the black hole and positive curvature of the branes.
- We worked at leading order in the limit of small of $\frac{\rho}{\sigma_c}$ and added a critical dark energy contribution to get something resembling a standard cosmology (as in [1])
- We assumed that the UV brane cutsoff the spacetime (perhaps as a result of orbifolding)
- For the dark energy dominated scenario, we assumed that the energy densities of the two branes were identical, leading to a common scaling with τ, and them being part of the same family of solutions.
- We neglected any backreaction on the metric from any bulk EMT sources
- We neglected any fluctuations of the metric a la 4D graviton

[3]This agrees with [8] when one tunes to the critical energy density on the branes, truncates the space at the IR brane, includes a factor of 2 from integrating over the orbifolded S^1 rather than an interval, and sets $\eta_{IR} = 0$

3 Tunneling Rate

We now look to leverage the calculation carried out in [2] for the emission rate of a thin shell from the horizon of a black hole. This calculation leverages the WKB approximation, this is valid since for a spherical shell emitted from a spherical horizon the system reduces to a 1D problem for the dynamics of the shell. They also managed to argue for the effect of the back reaction on the horizon.

3.1 Barrier Transmission

We omit the derivation of the matching conditions need at the classical turning points where the WKB approximation fails and simply write down the final expression for the tunneling probability through a classically forbidden region (which is wide compard to the deBroglie wavelength of the particle) after matching. This approximation is valid to and from points that are far separated from the turning points on either side of the classically forbidden region. As Coleman said, "every child knows that..."

$$|P_T|^2 = \exp\left(-2\mathrm{Im}\int_{\Sigma_f} dx p(x)\right) = \exp\left(-2\mathrm{Im}S\right) \tag{104}$$

Where Σ_f is the classically forbidden region of the potential and $p^2(x) = 2m(E - V(x))$ is the classical momentum of the particle.

In [2], the interpretation is a spherical shell initially behind the horizon at a position $p_h(\rho_M) - \epsilon$ falls away from the horizon towards the singularity. As it does so the horizon contracts to a position $p_h(\rho_M - \rho_\omega)$ with the particle ending up just outside the horizon at some position $p_h(\rho_M - \rho_\omega) + \epsilon$. This calculation has been carried out in AdS-S in Painleve coordinates in [9] for s-wave massless radiation. Here we use our solution to the junction conditions to carry out this calculation for a massive brane in AdS Schwarzschild coordinates.

For the tunneling of a thin shell from the inside to the outisde of the horizion, causing the black hole to change from energy density ρ_M to $\rho_M - \rho_\omega$, the classically forbidden region for the particle is the region swept out by the horizon as it contracts $\Sigma_f = [p_h(\rho_M - \rho_\omega), p_h(\rho_M)]$. So the action receives an imaginary contribution

$$\begin{aligned}\mathrm{Im}S =& \mathrm{Im}\int_{p_{in}}^{p_{out}} dp p_p & (105)\\ =& \mathrm{Im}\int_{\Sigma_s} d^3x \int_{p_{in}}^{p_{out}} dp P_p & (106)\\ =& \mathrm{Im}\int_{\Sigma_s} d^3x \int_{p_{in}}^{p_{out}} dp \int_0^{P_z} dP'_p & (107)\end{aligned}$$

Where p_{out} and p_{in} are coordinates the horizon before and after emission respectively with $p_{out} < p_{in}$ (so are the boundary of the classically forbidden region that the shell traverses), x parametrizes the space orthogonal to the p direction, P_p is the canonically conjugate momentum density to the position p of the horizon (where we all other momenta must vanish by 3 dimensional isotropy), Σ_s a spatial slice of Σ at fixed time, and ρ_M is the Hamiltonian density for the horizon. We can write, using $\rho'_M = \rho_M - \rho'_\omega$ (where ρ_ω is the Hamiltonian density of the shell) giving $d\rho'_M = -d\rho'_\omega$.

$$\begin{aligned}\mathrm{Im}S =& \mathrm{Im}\int_{\Sigma_s} d^3x \int_{p_{in}}^{p_{out}} dp \int_0^{P_z} dP'_p & (108)\\ =& \mathrm{Im}\int_{\Sigma_S} d^3x \int_{p_{in}}^{p_{out}} dp \int_{\rho_M}^{\rho_M-\rho_\omega} \frac{d\rho'_M}{\partial_t P} & (109)\\ =& -\mathrm{Im}\int_{\Sigma_S} d^3x \int_{p_{in}}^{p_{out}} dp \int_0^{\rho_\omega} \frac{d\rho'_\omega}{\partial_t P} & (110)\end{aligned}$$

In the last line we have used the Hamiltonian equations of motion to change the measure.

$$\partial_t P = \frac{d\rho_M}{dP_p} \tag{111}$$

$$\implies d\rho_M = \frac{d\rho_M}{dP_p} dP_p = \partial_t P dP_p \tag{112}$$

$$\implies dP'_p = \frac{d\rho'_M}{\partial_t P} \tag{113}$$

[10] gives an argument that observers between the horizon and brane will see a metric for a black hole with energy density $\rho_M - \rho_\omega$, and those on the other side of the brane will see a metric with energy density ρ_M.

3.2 Brane Emission Rate

We recall that

$$\mu_\pm = \frac{4\rho_M}{l^2 \sigma_c} \equiv \frac{\rho_M}{A} \tag{114}$$

$$g_\pm(p_h) = 1 - \frac{\mu_\pm}{p_h^2} + \frac{p_h^2}{l^2} = 0 \tag{115}$$

$$\implies p_h^2 = \frac{l^2}{2}\left(-1 + \sqrt{1 + 4\frac{\mu_\pm^2}{l^2}}\right) \tag{116}$$

So we can write

$$P_z = -\int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \int_0^{\rho_\omega} \frac{d\rho'_\omega}{\partial_t P} \tag{117}$$

$$= -\int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \int_0^{\rho_\omega} d\rho'_\omega \frac{1}{g(p)} \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \tag{118}$$

$$= -\int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \int_0^{\rho_\omega} d\rho'_\omega \frac{p^2}{p^2 - \mu + \frac{p^4}{l^2}} \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \tag{119}$$

$$= -\int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \int_0^{\rho_\omega} d\rho'_\omega \frac{A p^2}{A(p^2 + \frac{p^4}{l^2}) - \rho_M + \rho'_\omega} \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \tag{120}$$

$$\tag{121}$$

Where we have used the expression we derived above for $\partial_t P$. As we argued above, since $\dot{P}$ is non-vanishing at the horizon, the integrand has a simple pole at the horizon. Further, the p integral will always be over the simple pole. Hence we evaluate the imaginary part using the prescription $\rho'_\omega \to \rho'_\omega - i\epsilon$

$$\text{Im}P_z = - \text{Im} \int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \int_0^{\rho_\omega} d\rho'_\omega \frac{Ap^2}{A(p^2 + \frac{p^4}{l^2}) - \rho_M + \rho'_\omega - i\epsilon} \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \tag{122}$$

$$= - \int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \int_0^{\rho_\omega} d\rho'_\omega \pi \delta \left(A(p^2 + \frac{p^4}{l^2}) - \rho_M + \rho'_\omega \right) Ap^2 \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \tag{123}$$

$$= - \pi \int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \left[Ap^2 \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \right]_{\rho'_\omega = -A(p^2 + \frac{p^4}{l^2}) + \rho_M} \tag{124}$$

Where we've used the Cauchy Dirac relation. We can reduce this expression further as

$$g(p)|_{\rho'_\omega = -A(p^2 + \frac{p^4}{l^2}) + \rho_M} = \left[1 - \frac{\rho_M - \rho'_\omega}{Ap^2} + \frac{p^2}{l^2} \right]_{\rho'_\omega = -A(p^2 + \frac{p^4}{l^2}) + \rho_M} = 0 \tag{125}$$

$$\tag{126}$$

So we can write the above expression as

$$\text{Im}P_z = - \pi \int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp \left[Ap^2 \frac{\sqrt{g(p) + \dot{P}^2}}{\dot{P}} \right]_{\rho'_\omega = -A(p^2 + \frac{p^4}{l^2}) + \rho_M} \tag{127}$$

$$= - \text{sgn}(\dot{P}) \pi \int_{p_h(\rho_M)}^{p_h(\rho_M - \rho_\omega)} dp A p^2 \tag{128}$$

$$= - \text{sgn}(\dot{P}) \frac{\pi}{3} A \left[p_h^3(\rho_M) - p_h^3(\rho_M - \rho_\omega) \right] \tag{129}$$

Hence the tunneling Probability is given by

$$|P_t|^2 = \exp \left\{ \text{sgn}(\dot{P}) \text{Vol}(\Sigma_s) \frac{2\pi}{3} A \left[p_h^3(\rho_M) - p_h^3(\rho_M - \rho_\omega) \right] \right\} \tag{130}$$

We then take the negative sign. Which corresponds to motion towards the horizon as in [2]. Interestingly however in the low energy limit

$$|P_t|^2 = \exp \left\{ - \text{Vol}(\Sigma_s) \frac{2\pi}{3} A \left[p_h^3(\rho_M) - p_h^3(\rho_M - \rho_\omega) \right] \right\} \tag{131}$$

$$\to \exp \left\{ - \text{Vol}(\Sigma_s) \frac{\pi}{\sqrt{2}} \frac{l p_h(\rho_M)}{\sqrt{1 + 4\frac{\mu^2}{l^2}}} \rho_\omega \right\} \tag{132}$$

$$= \exp \left\{ - \frac{1}{2\sqrt{2}} \frac{2\pi p_h(\rho_M)}{\sqrt{1 + 4\frac{\mu^2}{l^2}}} \omega \right\} \tag{133}$$

[9] uses AdS Painleve coordinates, and find a value for the Hawking temperature for radiation of

$$\beta = \frac{2\pi p_h(\rho_M)}{\sqrt{1 + 4\frac{\mu^2}{l^2}}} \tag{134}$$

Corresponding to the surface gravity on the horizon. So we find we are off by a factor of $2\sqrt{2}$ from the expected Hawking temperature. This can likely be attributed to an issue with canonical invariance in this approach which was pointed out in eg. [11] [5]. However, we will compute the transition rate in another formalism to ensure we have the correct result.

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4 Appendices

4.1 A - Geometry of Co-Dimension 1 Surfaces

Consider a spacetime manifold V with a codimension 1 submanifold Σ with a normal vector $\boldsymbol{n}$ which may be either spacelike or timelike, and whose norm has magnitude 1.

$$\boldsymbol{n} \cdot \boldsymbol{n} = \epsilon(\boldsymbol{n}) = \pm 1 \tag{135}$$

The extrinsic curvature is given by the projection of the derivative of $\boldsymbol{n}$, a vector normal to the surface, onto the surface with tangent vectors $\boldsymbol{e}_{(\mu)}$. We are concerned with co-dimension 1 surfaces, so these vectors span our spacetime.

$$\partial_\mu \boldsymbol{n} = K^\lambda_\mu \boldsymbol{e}_{(\lambda)} \tag{136}$$

Giving

$$\begin{aligned} K_{\mu\nu} =& K^\lambda_\mu \boldsymbol{e}_{(\nu)} \cdot \boldsymbol{e}_{(\lambda)} && (137)\\ =& \boldsymbol{e}_{(\nu)} \cdot \partial_\mu \boldsymbol{n} && (138)\\ =& g_{\nu\lambda} \nabla_\mu n^\lambda && (139)\\ =& \nabla_\mu n_\nu && (140) \end{aligned}$$

Since $g_{\mu\nu} = \boldsymbol{e}_{(\mu)} \cdot \boldsymbol{e}_{(\nu)}$ and using normality. From this we see that [12] since the vectors are orthogonal

$$\begin{aligned} K_{\mu\nu} =& \boldsymbol{e}_{(\nu)} \cdot \partial_\mu \boldsymbol{n} && (141)\\ =& -\boldsymbol{n} \cdot \partial_\mu \boldsymbol{e}_{(\nu)} && (142)\\ =& -\boldsymbol{n} \cdot \partial_\nu \boldsymbol{e}_{(\mu)} && (143)\\ =& K_{\nu\mu} && (144) \end{aligned}$$

Where we've used

$$e^M_{(\mu)} = \partial_\mu X^M \tag{145}$$

and the symmetry of mixed partials to swap indices in second to last line. So we see that K is symmetric. Then consider a field in Σ

$$A_\mu = \boldsymbol{A} \cdot \boldsymbol{e}_{(\mu)}, \qquad \boldsymbol{A} = A^i \boldsymbol{e}_{(i)} \tag{146}$$

Then the components of the covariant derivative projected onto the tangent space of Σ are given by

$$\begin{aligned} \partial_\nu A_\mu =& \partial_\nu (\boldsymbol{A} \cdot \boldsymbol{e}_{(\mu)}) && (147)\\ =& \boldsymbol{e}_{(\mu)} \cdot \partial_\nu \boldsymbol{A} + \boldsymbol{A} \cdot \partial_\nu \boldsymbol{e}_{(\mu)} && (148)\\ \implies A_{\mu;\nu} \equiv& \boldsymbol{e}_{(\mu)} \cdot \partial_\nu \boldsymbol{A} && (149)\\ =& \partial_\nu A_\mu - \boldsymbol{A} \cdot \partial_\nu \boldsymbol{e}_{(\mu)} && (150)\\ =& \partial_\nu A_\mu - A^\lambda \boldsymbol{e}_{(\lambda)} \cdot \partial_\nu \boldsymbol{e}_{(\mu)} && (151)\\ =& \partial_\nu A_\mu - A^\lambda \Gamma_{\lambda,\nu\mu} && (152)\\ =& \partial_\nu A_\mu - A_\lambda \Gamma^\lambda_{\nu\mu} && (153) \end{aligned}$$

Further, we find the Gauss-Weingarten equations

$$\begin{aligned} \partial_\mu \boldsymbol{e}_{(\nu)} =& \boldsymbol{n}(\boldsymbol{n} \cdot \partial_\mu \boldsymbol{e}_{(\nu)}) + \boldsymbol{e}_{(\lambda)} (\boldsymbol{e}^{(\lambda)} \cdot \partial_\mu \boldsymbol{e}_{(\nu)}) && (154)\\ =& -\boldsymbol{n}\epsilon(\boldsymbol{n}) K_{\mu\nu} + \boldsymbol{e}_{(\lambda)} \Gamma^\lambda_{\mu\nu} && (155)\\ & && (156) \end{aligned}$$

Where we have used the fact that Σ is codimension 1 and so the union of it with the set of tangent vectors span V. And introduced the cotangent basis vectors $\boldsymbol{e}^{(\mu)}$. Hence

$$\partial_\mu \boldsymbol{A} = \boldsymbol{e}_{(\nu)} A^\nu{}_{;\mu} - \boldsymbol{n}\epsilon(\boldsymbol{n}) A^\nu K_{\mu\nu} \tag{157}$$

We will not derive it, but will simply note that from these relations one can derive as [12] did the Gauss Codazzi equations. From these equations and noting that the metric induced on Σ ($g^{\mu\nu}$) by g^{MN} is given by

$$g^{\mu\nu} e^M_{(\mu)} e^N_{(\nu)} = g^{MN} - \epsilon(\boldsymbol{n}) n^M n^N \tag{158}$$

they go on to compute the Einstein tensor [4]

$$\begin{aligned} -2\epsilon(\boldsymbol{n}) G_{MN} n^M n^N =& R_\Sigma - \epsilon(\boldsymbol{n})(K_{\mu\nu}K^{\mu\nu} - K^2) && (159)\\ G_{MN} e^M_{(\mu)} n^N =& K_\mu{}^\nu{}_{;\nu} - K_{;\mu} && (160)\\ G_{MN} e^M_{(\mu)} e^N_{(\nu)} =& G^\Sigma_{\mu\nu} - n^J \partial_J (K_{\mu\nu} - g_{\mu\nu} K) - K K_{\mu\nu} + \frac{1}{2}\left(K_{\mu\nu}K^{\mu\nu} + K^2\right) && (161) \end{aligned}$$

Where $K = K_{\mu\nu} g^{\mu\nu}$. They then go on to note that given a 2 tensor $S^{\mu\nu}$ defined on Σ one can associate a discontinuous tensor in V as

$$S^{MN} = S^{\mu\nu} e^M_{(\mu)} e^N_{(\nu)} \text{ on } \Sigma, \qquad S^{MN} = 0 \text{ off } \Sigma \tag{162}$$

from which we find

$$\nabla_M S^{NM} = e^N_{(\mu)} S^{\mu\nu}_{;\nu} - \epsilon(\boldsymbol{n}) S^{\mu\nu} K_{\mu\nu} n^N \tag{163}$$

where $S^{\mu\nu}_{;\lambda}$ denotes the covariant derivative of $S^{\mu\nu}$ in Σ.

4.2 B - An Aside on Integration

Here we construct objects which allow us to conveniently work in a coordinate independent way. These objects were introduced formally in [7]. We introduce a function from coordinates in Σ to bulk coordinates in V.

$$X = X_\Sigma(x) \tag{164}$$

Define

$$\delta_\Sigma(X) = \int_\Sigma d^4x \sqrt{-g_{\text{ind}}} \frac{1}{\sqrt{-g}} \delta^5(X - X_\Sigma(x)) \tag{165}$$

This expression is a scalar under diffeomorphoisms which follows from the fact that $d^5X\delta^5(X)$ is, which in turn implies $\frac{1}{\sqrt{-g}}\delta^5(X)$ is as well. This object will allow us to change between integration over V and integration over Σ. Notice

$$\begin{aligned} \int_V d^5X \sqrt{-g}\delta_\Sigma(X) h(X) &= \int_V d^5X \sqrt{-g}\ h(X) \int_\Sigma d^4x \sqrt{-g_{\text{ind}}} \frac{1}{\sqrt{-g}} \delta^5(X - X_\Sigma(x)) && (166)\\ &= \int_\Sigma d^4x \sqrt{-g_{\text{ind}}} \int_V d^5X h(X) \delta^5(X - X_\Sigma(x)) && (167)\\ &= \int_\Sigma d^4x \sqrt{-g_{\text{ind}}} h(X_\Sigma(x)) && (168) \end{aligned}$$

[4] The last equation we take from [13]

We now formally define a distribution $\Theta_\pm(X)$ as in [7] so that

$$\Theta_\pm(X)|_{V_\pm} = 1 \tag{169}$$
$$\Theta_\pm(X)|_{V_\mp} = 0 \tag{170}$$
$$\partial_M \Theta_\pm(X) = n_M \delta_\Sigma(X) \tag{171}$$

We can construct an object with these properties by integrating over δ_Σ along a curve $C_\pm(X)$ with tangent vector $\boldsymbol{n}$, and endpoints X and somewhere in $V_\mp$ along the curve.

$$\Theta_\pm(X) = \int_{C_\pm(X)} d\eta \delta_\Sigma(X(\eta)) \tag{172}$$

This then satisfies the two first properties, as the integral will evaluate to 1 if the endpoints of $C_\pm(X)$ are on opposite sides of Σ, and 0 otherwise. So we only need to check the third property.

$$\begin{aligned}
\partial_M \Theta_\pm(X) =& \partial_M \int_{C_\pm(X)} d\eta \delta_\Sigma(X(\eta)) && (173)\\
=& \left[\delta_\Sigma(X(\eta)) \partial_M C_\pm(X)\right]_{\partial C} && (174)\\
=& \delta_\Sigma(X) n_M && (175)
\end{aligned}$$

Where we have used the fact that n_M is tangent to $C_\pm(X)$ and the fact that the endpoint in $V_\mp$ is independent of X.

4.3 C - Einstein's Equations a la Israel

We now look to consider the normal components of EFEs in the presence of a bulk EMT. From [7], these will impose constraints on the induced metric and extrinsic curvature. On either side of Σ

$$\begin{aligned}
\kappa T^\pm_{MN} n^M n^N =& G^\pm_{MN} n^M n^N = \frac{-2}{\epsilon(\boldsymbol{n})} \left[R_\Sigma - \epsilon(\boldsymbol{n})(K^\pm_{\mu\nu} K^{\pm\mu\nu} - K^{\pm 2})\right] && (176)\\
\kappa T^\pm_{MN} e^M_{(\mu)} n^N =& G^\pm_{MN} e^M_{(\mu)} n^N = K^{\pm\nu}_{\mu\ ;\nu} - K^\pm_{;\mu} && (177)
\end{aligned}$$

Where we have used the fact that Israel's first junction condition implies that observers in $V_\pm$ agree on R_Σ, the intrinsic curvature in Σ. Adding and subtracting the second of these equations we find

$$\begin{aligned}
\kappa [T_{MN} n^N] e^M_{(\mu)} =& \gamma^{\pm\nu}_{\mu\ ;\nu} - \gamma^\pm_{;\mu} = (\gamma^\nu_\mu - \delta^\nu_\mu \gamma)_{;\nu} = -\kappa S^\nu_{\mu;\nu} && (178)\\
\implies [T_{MN} n^N] e^M_{(\mu)} =& - S^\nu_{\mu;\nu} && (179)
\end{aligned}$$

Where $[f] \equiv f^+ - f^-$. Adding them we find

$$\tilde{T}_{Mn} e^M_{(\mu)} = \tilde{K}_\mu{}^\nu{}_{;\nu} - \tilde{K}_{;\mu} \tag{180}$$

Where $\tilde{f} \equiv \frac{1}{2}(f^+ + f^-)$ and $f_n \equiv f_M n^M$. Doing the same with the first equation we have

$$\begin{aligned}
\kappa [T^\pm_{nn}] =& 2\left[K_{\mu\nu} K^{\mu\nu} - K^2\right] && (181)\\
=& 2\left(K^+_{\mu\nu} K^{+\mu\nu} - K^{+2} - (K^-_{\mu\nu} K^{-\mu\nu} - K^{-2})\right) && (182)
\end{aligned}$$

Now notice

$$\begin{aligned}
\tilde{K}_{\mu\nu}S^{\mu\nu} =& \frac{-1}{2\kappa}(K^{+}_{\mu\nu} + K^{-}_{\mu\nu})(\gamma^{\mu\nu} - g^{\mu\nu}\gamma) && (183)\\
=& \frac{-1}{2\kappa}(K^{+}_{\mu\nu}K^{+\mu\nu} - K^{-}_{\mu\nu}K^{-\mu\nu} - K^{+2} + K^{-2}) && (184)\\
=& \frac{-1}{2\kappa}\left[K_{\mu\nu}K^{\mu\nu} - K^2\right] && (185)
\end{aligned}$$

Where we have used again Israel's first junction condition to equate the induced metric on either side of Σ. So we have

$$[T^{\pm}_{nn}] = -4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{186}$$

Finally, adding we have

$$\kappa\tilde{T}_{MN}n^M n^N = \frac{-2}{\epsilon(\boldsymbol{n})}R_\Sigma - (K^{+}_{\mu\nu}K^{+\mu\nu} - K^{+2}) - (K^{-}_{\mu\nu}K^{-\mu\nu} - K^{-2}) \tag{187}$$

Drudging through the details and using

$$K^{\pm}_{\mu\nu} = \tilde{K}_{\mu\nu} \pm \frac{1}{2}\gamma_{\mu\nu} \tag{188}$$

we have

$$\begin{aligned}
K^{\pm}_{\mu\nu}K^{\pm\mu\nu} =& \left(\tilde{K}_{\mu\nu} \pm \frac{1}{2}\gamma_{\mu\nu}\right)\left(\tilde{K}^{\mu\nu} \pm \frac{1}{2}\gamma^{\mu\nu}\right) && (189)\\
=& \tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} \pm \gamma_{\mu\nu}\tilde{K}^{\mu\nu} + \frac{1}{4}\gamma_{\mu\nu}\gamma^{\mu\nu} && (190)\\
K^{\pm 2} =& \left(\tilde{K} \pm \frac{1}{2}\gamma\right)^2 && (191)\\
=& \tilde{K}^2 \pm K\gamma + \frac{1}{4}\gamma^2 && (192)
\end{aligned}$$

Hence

$$\kappa\tilde{T}_{MN}n^M n^N = \frac{-2}{\epsilon(\boldsymbol{n})}R_\Sigma - 2\left(\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} + \frac{1}{4}\gamma_{\mu\nu}\gamma^{\mu\nu} - \tilde{K}^2 - \frac{1}{4}\gamma^2\right) \tag{193}$$

Which can be put in terms of S by using

$$\begin{aligned}
\gamma_{\mu\nu} =& -\kappa\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1}g_{\mu\nu}S\right) && (194)\\
\implies \gamma_{\mu\nu}\gamma^{\mu\nu} =& \kappa^2\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1}g_{\mu\nu}S\right)\left(S^{\mu\nu} - \frac{1}{D_\Sigma - 1}g^{\mu\nu}S\right) && (195)\\
=& \kappa^2\left(S_{\mu\nu}S^{\mu\nu} - \frac{2}{D_\Sigma - 1}S^2 + \frac{D_\Sigma}{(D_\Sigma - 1)^2}S^2\right) && (196)\\
=& \kappa^2\left[S_{\mu\nu}S^{\mu\nu} + \left(\frac{1}{(D_\Sigma - 1)^2} - \frac{1}{D_\Sigma - 1}\right)S^2\right] && (197)\\
\implies \gamma^2 =& \kappa^2\left(S - \frac{D_\Sigma}{D_\Sigma - 1}S\right)^2 = \kappa^2\frac{S^2}{(D_\Sigma - 1)^2} && (198)\\
\implies \gamma_{\mu\nu}\gamma^{\mu\nu} - \gamma^2 =& \kappa^2\left(S_{\mu\nu}S^{\mu\nu} - \frac{1}{D_\Sigma - 1}S^2\right) && (199)
\end{aligned}$$

Hence

$$-\frac{\kappa}{2}\tilde{T}_{MN}n^M n^N - \frac{1}{\epsilon(\boldsymbol{n})}R_\Sigma =(\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} - \tilde{K}^2) + \frac{\kappa^2}{4}\left(S_{\mu\nu}S^{\mu\nu} - \frac{1}{D_\Sigma - 1}S^2\right) \tag{200}$$

So our conditions are

$$[T_{Mn}]e^M_{(\mu)} =- S^\nu_{\mu;\nu} \tag{201}$$
$$\tilde{T}_{Mn}e^M_{(\mu)} =\tilde{K}_\mu{}^\nu{}_{;\nu} - \tilde{K}_{;\mu} \tag{202}$$
$$[T^\pm_{nn}] =- 4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{203}$$
$$-\frac{\kappa}{2}\tilde{T}_{nn} - \frac{1}{\epsilon(\boldsymbol{n})}R_\Sigma =(\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} - \tilde{K}^2) + \frac{\kappa^2}{4}\left(S_{\mu\nu}S^{\mu\nu} - \frac{1}{D_\Sigma - 1}S^2\right) \tag{204}$$

Only the bracket expressions will impose consraints, as the other two will be satisfied provided we have solutions to the bulk EFE on either side of Σ.

4.4 D - Junction Conditions

Here we state Israel's Junction conditions which will determine dynamics of surface layers. We consider a codimension 1 surface layer Σ which divides a spacetime manifold V into $V_\pm$. We will find the junction conditions are analgous to what we'd expect; that the metric at the surface layer is continuous and that the singularities that appear in the EFE from the surface match. One of the benefit's of [12] approach was to do this in a coordinate independent way. See [7] for a particularily clear exposition of the conditions.

4.4.1 Israel's First Junction Condition

Consistency requires that the induced metric on Σ is well defined. Hence we need that the metrics induced on Σ by the metrics in $V_\pm$ are equal. This is **Israel's First Junction Condition**.

$$ds^2_+|_\Sigma = ds^2_-|_\Sigma \tag{205}$$

This also ensures that the Ricci tensor is a well defined distribution.

4.4.2 Israel's Second Junction Condition

Consider the discontinuity in the extrinsic curvature (again refer to the appendices as needed)

$$\gamma_{\mu\nu} = K^+_{\mu\nu}|_\Sigma - K^-_{\mu\nu}|_\Sigma \tag{206}$$

We define the symmetric 2 tensor S on Σ by the Lanczos equations

$$\gamma_{\mu\nu} - g_{\mu\nu}\gamma =- \kappa S_{\mu\nu} \tag{207}$$
$$\implies \gamma - D_\Sigma\gamma =- \kappa S \tag{208}$$
$$\implies \gamma_{\mu\nu} =- \kappa\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1}g_{\mu\nu}S\right) \tag{209}$$

Where D_Σ is the dimension of the Σ and $\kappa = 8\pi G$. S will be identified with the surface energy momentum tensor for Σ. From the Lanczos equations and the Einstein equations (the CC is irrelevant for this discussion) we have

$$G_{MN}e^M_{(\mu)}e^N_{(\nu)} = G_{\mu\nu} - n^J\partial_J(K_{\mu\nu} - g_{\mu\nu}K) - KK_{\mu\nu} + \frac{1}{2}\left(K_{\mu\nu}K^{\mu\nu} + K^2\right) \tag{210}$$

$$\equiv - n^J\partial_J(K_{\mu\nu} - g_{\mu\nu}K) + Z_{\mu\nu} \tag{211}$$

$$= \kappa T_{MN}e^M_{(\mu)}e^N_{(\nu)} = \kappa T_{\mu\nu} \tag{212}$$

$$\tag{213}$$

We integrate this equation along a curve C through Σ from V_- to V_+. We do this along a curve C with tangent vector $\boldsymbol{n} = n^M\partial_M = \frac{dX^M}{d\eta}\partial_M = \frac{d}{d\eta}$. So that $d\eta = dX^M n_M$. η is a gaussian normal coordinate given that $\boldsymbol{n}$ is orthogonal to the rest of the basis and has unit norm. We denote the value of η on Σ by η_Σ.

$$\kappa\int_C d\eta T_{\mu\nu} = -\int_C d\eta\ n^J\partial_J(K_{\mu\nu} - g_{\mu\nu}K) + \int_C d\eta Z_{\mu\nu} \tag{214}$$

$$= -\int_C d\eta\ \partial_\eta(K_{\mu\nu} - g_{\mu\nu}K) + \int_C d\eta Z_{\mu\nu} \tag{215}$$

$$= -(K_{\mu\nu} - g_{\mu\nu}K)_{\partial C} + \int_C d\eta Z_{\mu\nu} \tag{216}$$

We then take the limit that the length of the curve goes to 0. Following [12] we make a heuristic argument by assuming that $Z_{\mu\nu}$ is finite on C so that

$$\lim_{|C|\to 0}\int_C d\eta T_{\mu\nu} = \lim_{\epsilon\to 0}\int_{\eta_\Sigma-\epsilon}^{\eta_\Sigma+\epsilon} d\eta T_{\mu\nu} = -\frac{1}{\kappa}\lim_{|C|\to 0}(K_{\mu\nu} - g_{\mu\nu}K)_{\partial C} = -\frac{1}{\kappa}(\gamma_{\mu\nu} - g_{\mu\nu}\gamma) = S_{\mu\nu} \tag{217}$$

Hence for an energy momentum tensor with a singular contribution on Σ[5]

$$S_{\mu\nu} = \lim_{\epsilon\to 0}\int_{\eta_\Sigma-\epsilon}^{\eta_\Sigma+\epsilon} d\eta\ \delta_\Sigma(X(\eta))S_{\mu\nu} \tag{218}$$

$$= \lim_{\epsilon\to 0}\int_{\eta_\Sigma-\epsilon}^{\eta_\Sigma+\epsilon} d\eta T_{\mu\nu} \tag{219}$$

From which we see that satisfying the Lanczos equations ensure that the singularities in the tangent components of the Einstein tensor from the presence of Σ cancel. And so given a solution to EFE in $V_\pm$, the tangent components of EFE will be satisfied. This is **Israel's second Junction Condition**. And we interpret S as the singular contribution to the energy momentum tensor on Σ.

$$T_{\mu\nu} = \delta_\Sigma(X)S_{\mu\nu} + \text{(regular)} \tag{220}$$

This agrees precisely with [13] since $\delta_\Sigma(X(\eta)) = \delta(\eta - \eta_\Sigma)$. And so in the bulk we find

$$T_{MN} = \delta_\Sigma(X)S_{\mu\nu}e^{(\mu)}_M e^{(\nu)}_N + \text{(regular)} \tag{221}$$

One can follow the elegant analysis in [7] or [12] to see that these two conditions, along with the constraints from the normal components of Einsteins equations (see appendix D), are necessary and sufficient to ensure that the EFEs are satisfied across a singular surface Σ.

[5] see appendix B for the definition of δ_Σ

4.4.3 Scalar Junction Conditions

Now consider the case of a scalar in the bulk with an interaction term on the brane.

$$S_\phi = -\int_\Sigma d^4x\sqrt{-g_{ind}}V_b(\phi) + \int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi)\right) \tag{222}$$

$$= \int_V d^5X\sqrt{-g}\left(g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi) - \delta_\Sigma(X)V_b(\phi)\right) \tag{223}$$

This gives EOM

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\phi\right) = -V_B'(\phi) - \delta_\Sigma(X)V_b'(\phi) \tag{224}$$

$$\phi(X) = \Theta_+(X)\phi_+ + \Theta_-(X)\phi_- \tag{225}$$

(see appendix B) Where $\phi_\pm$ are solutions to the EOM in $V_\pm$. The EOM can only be satisfied at Σ if the singularities cancel. Only if ϕ is continuous can they cancel. Hence we have that

$$\partial_M\phi(X) = \Theta_+(X)\partial_M\phi_+ + \Theta_-(X)\partial_M\phi_- \tag{226}$$

$$g^{MN}\partial_N\partial_M\phi(X) = \delta_\Sigma(X)g^{MN}n_N\partial_M\left(\phi_+ - \phi_-\right) + (\text{regular}) \tag{227}$$

Where we have used the fact that $C_\pm(X)$ have opposite signed normal components at Σ. This is the only other singular contribution to the EOM. So to satisfy the EOM it must be that

$$n^M\partial_M\left(\phi_+ - \phi_-\right)|_\Sigma = \left[n^M\partial_M\phi\right] = -V_b'(\phi)|_\Sigma \tag{228}$$

4.5 E - Action and Energy Momentum Sources

We consider the system described by the following action.

$$S = S_{EH} + S_{GHY} + S_B + S_b + S_{PF} \tag{229}$$

$$= \frac{1}{2\kappa}\int_V d^5X\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial V} d^4x\sqrt{-g_{\text{ind}}}K \tag{230}$$

$$+ \int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi)\right) - \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}V_b(\phi) \tag{231}$$

$$+ \frac{1}{2}\int_\Sigma d^4x\sqrt{g_{\text{ind}}}\left((\mathcal{P}+\rho)g^{\text{ind}}_{\mu\nu}u^\mu u^\nu - (\mathcal{P}-\rho)\right) \tag{232}$$

Where $K \equiv K_{\mu\nu}g^{\mu\nu}_{\text{ind}}$. These are the Einstein-Hilbert and Gibbons-Hawking-York actions as well as the action for a bulk scalar with a potential on the surface and the action for a perfect fluid (with pressure $\mathcal{P}$, energy density ρ, and 4-velocity u of an observer relative to the fluid) parametrizing matter content on the surface [14]. [6] We will be interested in AdS-S solutions in which we neglect the back reaction of the scalar on the metric. So going forward we will put the metric on shell. We consider the energy momentum contribution of each of the actions in turn.

[6] We will use capital latin letters to denote 5D indices and lowercase greek letters to denote 4D coordinates on the surface. We also use X to denote bulk coordinates and x to denote surface coordinates.

4.5.1 Perfect fluid

Consider the action for a perfect fluid on the surface

$$S_{PF} = \frac{1}{2}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\left((\mathcal{P}+\rho)g^{\text{ind}}_{\mu\nu}u^\mu u^\nu - (\mathcal{P}-\rho)\right) = \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\mathcal{L}_{PF} \tag{233}$$

Where

$$g^{\mu\nu}_{ind}e^M_{(\mu)}e^N_{(\nu)} = g^{MN} - \epsilon(\boldsymbol{n})n^M n^N \tag{234}$$

Which gives the usual expression as

$$\begin{aligned} g^{\mu\nu}_{ind}e^M_{(\mu)}e^N_{(\nu)}e^{(\alpha)}_M e^{(\beta)}_N =& g^{\alpha\beta}_{ind} & (235)\\ =&(g^{MN} - \epsilon(\boldsymbol{n})n^M n^N)e^{(\alpha)}_M e^{(\beta)}_N & (236)\\ =&g^{MN}e^{(\alpha)}_M e^{(\beta)}_N & (237)\\ =&g^{MN}\frac{\partial x^\alpha}{\partial X^M}\frac{\partial x^\beta}{\partial X^N} & (238) \end{aligned}$$

Using orthogonality of $\boldsymbol{e}_{(\mu)}$ and $\boldsymbol{n}$. Varying wrt the bulk metric we have

$$\begin{aligned} \delta S =& \int_\Sigma d^4x\left(\delta\sqrt{-g_{\text{ind}}}\mathcal{L}_{PF} + \sqrt{-g_{\text{ind}}}\delta\mathcal{L}_{PF}\right) & (239)\\ =&\int_\Sigma d^4x\left(\frac{1}{2\sqrt{-g_{\text{ind}}}}(-g_{\text{ind}})g^{\alpha\beta}_{\text{ind}}\delta g^{\text{ind}}_{\alpha\beta}\mathcal{L}_{PF} + \frac{1}{2}\sqrt{-g_{\text{ind}}}(\mathcal{P}+\rho)u^\alpha u^\beta\delta g^{\text{ind}}_{\alpha\beta}\right) & (240)\\ =&\frac{1}{2}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\left(g^{\alpha\beta}_{\text{ind}}\mathcal{L}_{PF} + (\mathcal{P}+\rho)u^\alpha u^\beta\right)e^M_{(\alpha)}e^N_{(\beta)}\delta g_{MN} & (241)\\ =&\frac{1}{2}\int_V d^5X\sqrt{-g}\delta_\Sigma(X)\left(g^{\alpha\beta}_{\text{ind}}\mathcal{L}_{PF} + (\mathcal{P}+\rho)u^\alpha u^\beta\right)e^M_{(\alpha)}e^N_{(\beta)}\delta g_{MN} & (242)\\ =&-\frac{1}{2}\int_V d^5X\sqrt{-g}T^{MN}_{PF}\delta g_{MN} & (243) \end{aligned}$$

Where we have defined $\delta_\Sigma(X)$ in the appendix. So we find the energy momentum tensor is

$$\begin{aligned} T^{MN}_{PF} =&\delta_\Sigma(X)\left(g^{\alpha\beta}_{\text{ind}}\mathcal{L}_{PF} + (\mathcal{P}+\rho)u^\alpha u^\beta\right)e^M_{(\alpha)}e^N_{(\beta)} & (244)\\ =&\delta_\Sigma(X)\left(-\mathcal{P}g^{\alpha\beta}_{\text{ind}} + (\mathcal{P}+\rho)u^\alpha u^\beta\right)e^M_{(\alpha)}e^N_{(\beta)} & (245)\\ \equiv&\delta_\Sigma(X)S^{\alpha\beta}_{PF}e^M_{(\alpha)}e^N_{(\beta)} & (246) \end{aligned}$$

Where we have put the 4-velocity on shell in the second line [7].

4.5.2 Scalar Potential on the Brane

We can notice that a potential on Σ that depends only on ϕ will be independent of the metric, and hence will be given exactly as above by taking the equation of state $\mathcal{P} = -\rho$ and taking $\rho \to V_b(\phi)$. This identifies the potential as a contribution to the dark energy on the brane.

$$T^{MN}_b = \delta_\Sigma(X)V_b(\phi)g^{\alpha\beta}_{\text{ind}}e^M_{(\alpha)}e^N_{(\beta)} \tag{247}$$

[7] Being flippant with einbeins.

which gives

$$
\begin{aligned}
S_b^{\mu\nu} =& V_b(\phi) g_{\text{ind}}^{\mu\nu} && (248)\\
=& V_b(\phi) g^{MN} e_M^{(\mu)} e_N^{(\nu)} && (249)\\
& && (250)
\end{aligned}
$$

So our total surface energy momentum tensor is given by

$$
S^{\alpha\beta} = (V_b(\phi) - \mathcal{P}) g_{\text{ind}}^{\alpha\beta} - (\mathcal{P} + \rho) u^\alpha u^\beta \tag{251}
$$

This surface energy momentum tensor has trace (WRT the induced metric)

$$
S = (\mathcal{P} + \rho) + 4(V_b(\phi) - \mathcal{P}) = 4V_b(\phi) - 3\mathcal{P} + \rho \tag{252}
$$

So we have

$$
\begin{aligned}
-\kappa(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu} S) =& -\kappa\left((V_b(\phi) - \mathcal{P}) g_{\mu\nu} - (\mathcal{P} + \rho) u_\mu u_\nu - \frac{1}{3} g_{\mu\nu} S\right) && (253)\\
=& -\kappa\left(\left(V_b(\phi) - \mathcal{P} - \frac{4V_b(\phi) - 3\mathcal{P} + \rho}{3}\right) g_{\mu\nu} - (\mathcal{P} + \rho) u_\mu u_\nu\right) && (254)\\
=& \frac{\kappa}{3}\left(V_b(\phi) + \rho\right) g_{\mu\nu} + \kappa(\mathcal{P} + \rho) u_\mu u_\nu && (255)\\
\equiv& \frac{\kappa}{3} \sigma(\phi) g_{\mu\nu} + \kappa \sigma_u u_\mu u_\nu && (256)\\
& && (257)
\end{aligned}
$$

Defining $\frac{2}{l\sigma_c} \equiv \frac{\kappa}{3}$ we have for the second Junction condition

$$
[K_{\mu\nu}] = \frac{2}{l} \frac{\sigma(\phi)}{\sigma_c} g_{\mu\nu} + 3 \frac{2}{l} \frac{\sigma_u}{\sigma_c} u_\mu u_\nu \tag{258}
$$

4.5.3 Scalar in the Bulk

The bulk contribution to the scalar energy momentum tensor is given by the usual expression

$$
\begin{aligned}
T_{B,MN} =& \partial_M \phi \partial_N \phi - g_{MN} \mathcal{L}_B && (259)\\
=& \partial_M \phi \partial_N \phi - g_{MN}\left(\frac{1}{2} g^{JK} \partial_J \phi \partial_K \phi - V_B(\phi)\right) && (260)\\
& && (261)
\end{aligned}
$$

We showed above that $\partial_M \phi$ is regular at Σ, hence this expression will be as well. So $T_{B,MN}$ will not contribute to the surface energy momentum tensor. The EFEs do however relate the discontinuity in T to a source current for S.

$$
j_\mu \equiv [T_{Mn}] e_{(\mu)}^M = -S_{\mu;\nu}^\nu \tag{262}
$$

Both the other contributions to the stress energy tensor are proportional to $\delta_\Sigma(X)$ and so have vanishing bracket. Which implies

$$-S^{\nu}_{\mu;\nu} = \left[e^M_{(\mu)} n^N \left(\frac{1}{2}\partial_M\phi\partial_N\phi - g_{MN}\left(\frac{1}{2}g^{JK}\partial_J\phi\partial_K\phi - V_B(\phi)\right)\right)\right] \tag{263}$$

$$= \left[e^M_{(\mu)} n^N \frac{1}{2}\partial_M\phi\partial_N\phi\right] \tag{264}$$

using orthogonality. We also notice that for ϕ to have a well defined value on the brane it must be that

$$\left[e^M_{(\mu)}\partial_M\phi\right] = [\partial_\mu\phi] = 0 \tag{265}$$

Which implies we can write

$$S^{\nu}_{\mu;\nu} = - e^M_{(\mu)}\partial_M\phi \left[n^N\partial_N\phi\right] = \partial_\mu\phi V_b'(\phi) \tag{266}$$

$$= \partial_\mu V_b(\phi) \tag{267}$$

Where we have used the scalar junction conditions in the last equality. We also have another constraint that must be satisfied.

$$[T_{nn}] = - 4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{268}$$

$$\tag{269}$$

Again noting that the surface contributions won't contribute to the bracket, we have

$$[T_{nn}] = [T_{B,nn}] \tag{270}$$

$$= \left[\partial_n\phi\partial_n\phi - \epsilon(\boldsymbol{n})\left(\frac{1}{2}\left(g^{\mu\nu}_{\text{ind}}e^J_{(\mu)}e^K_{(\nu)} + \epsilon(\boldsymbol{n})n^Jn^K\right)\partial_J\phi\partial_K\phi - V_B(\phi)\right)\right] \tag{271}$$

$$= \left[\frac{1}{2}\partial_n\phi\partial_n\phi - \epsilon(\boldsymbol{n})\left(\frac{1}{2}g^{\mu\nu}_{\text{ind}}\partial_\mu\phi\partial_\nu\phi - V_B(\phi)\right)\right] \tag{272}$$

Where we have used (234). The term in parentheses is 0 by the scalar junction conditions and (265). So we are left with

$$[T_{nn}] = \frac{1}{2}\left[\partial_n\phi\partial_n\phi\right] = -4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{273}$$

Where $\tilde{K} = \tilde{K}_{\mu\nu}g^{\mu\nu}_{\text{ind}}$.

4.6 F - Computing the Extrinsic Curvature

We now use the parametrization and metric defined in the body of the text to compute the extrinsic curvature, we notice the following fact

$$\epsilon(\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{n} \tag{274}$$

$$\implies 0 = \frac{1}{2}\partial_M(\boldsymbol{n}\cdot\boldsymbol{n}) \tag{275}$$

$$= n^N\nabla_M n_N \tag{276}$$

$$= n^J(\partial_M n_J - \Gamma^N_{MJ}n_N) \tag{277}$$

$$= n^t(\partial_M n_t - \Gamma^N_{Mt}n_N) + n^p(\partial_M n_p - \Gamma^N_{Mp}n_N) \tag{278}$$

$$\implies \nabla_M n_t = -\frac{n^p}{n^t}\nabla_M n_p \tag{279}$$

where we have used the fact that n^M has no radial or angular components. Alternatively we can write

$$\nabla_M n^t = -\frac{n_p}{n_t}\nabla_M n^p \tag{280}$$

Which then allows us to write

$$K_{\mu\nu} = e^M_{(\mu)} e^N_{(\nu)} \nabla_M n_N \tag{281}$$
$$= e^t_{(\nu)}\nabla_\mu n_t + e^r_{(\nu)}\nabla_\mu n_r + e^p_{(\nu)}\nabla_\mu n_p + e^\theta_{(\nu)}\nabla_\mu n_\theta + e^\phi_{(\nu)}\nabla_\mu n_\phi \tag{282}$$
$$= \left(-e^t_{(\nu)}\frac{n^p}{n^t} + e^p_{(\nu)}\right)\nabla_\mu n_p + e^r_{(\nu)}\nabla_\mu n_r + e^\theta_{(\nu)}\nabla_\mu n_\theta + e^\phi_{(\nu)}\nabla_\mu n_\phi \tag{283}$$
$$= \left(-e^t_{(\nu)}\frac{n^p}{n^t} + e^p_{(\nu)}\right)\left(\partial_\mu n_p - e^M_{(\mu)}\Gamma^N_{Mp} n_N\right) + e^r_{(\nu)}\nabla_\mu n_r + e^\theta_{(\nu)}\nabla_\mu n_\theta + e^\phi_{(\nu)}\nabla_\mu n_\phi \tag{284}$$
$$\tag{285}$$

Since $\Gamma^N_{\Omega_i p} = 0$ for all angular variables Ω_i, and $\Gamma^N_{\Omega_i\Omega_j}$ is only non-vanishing for $i = j$ and $N = p$, We have that the non vanishing components of $K_{\mu\nu}$ are

$$K_{\mu\nu} = \begin{pmatrix} K_{\tau\tau} & K_{\tau r} & 0 & 0 \\ K_{r\tau} & K_{rr} & 0 & 0 \\ 0 & 0 & K_{\theta\theta} & 0 \\ 0 & 0 & 0 & K_{\phi\phi} \end{pmatrix}_{\mu\nu} \tag{286}$$

With $K_{\Omega_i\Omega_j} = -\Gamma^p_{\Omega_i\Omega_j} n_p = \frac{1}{2} n^p \partial_p g_{\Omega_i\Omega_j}$. We can then compute

$$-u^t\frac{n^p}{n^t} + u^p = -\frac{g(p)}{\dot{P}} \tag{287}$$
$$-e^t_{(r)}\frac{n^p}{n^t} + e^p_{(r)} = -\frac{g(p)}{\dot{P}}\frac{P'}{\dot{P}} \tag{288}$$

We also have

$$-u^M\Gamma^N_{Mp} n_N = \kappa_p \dot{P}\frac{g'(p)}{g^2(p)} \tag{289}$$
$$-e^M_{(r)}\Gamma^N_{Mp} n_N = \kappa_p \dot{P}\frac{g'(p)}{g^2(p)}\frac{P'}{\dot{P}} \tag{290}$$

We also have that

$$e^r_{(\nu)}\nabla_\mu n_r = e^r_{(\nu)} e^M_{(\mu)}(\partial_M n_r - \Gamma^N_{Mr} n_N) \tag{291}$$
$$= -e^r_{(\nu)} e^M_{(\mu)}\Gamma^N_{MJ} n_N \tag{292}$$
$$\tag{293}$$

since $n_r = 0$ (recall the indices of n are raised and lowered with the bulk metric, which is diagonal). Only the r basis vector has non-vanishing r components. Hence

$$\Gamma^N_{Mr} n_N = -\delta^r_M \frac{p}{l^2}\kappa_p = -\delta^r_M \frac{p}{l^2} n^p \tag{294}$$
$$\implies e^r_{(\nu)}\nabla_\mu n_r = e^r_{(\nu)} e^r_{(\mu)}\frac{p}{l^2} n^p \tag{295}$$

4.6.1 Angular components

From above, we have simply that the angular components of the extrinsic curvature are

$$
\begin{aligned}
K_{\Omega_i\Omega_j} =& n^p \frac{1}{2}\partial_p g_{\Omega_i\Omega_j} && (296)\\
=& n^p \frac{1}{p} g_{\Omega_i\Omega_j} && (297)\\
\implies [K_{\Omega_i\Omega_j}] =& [n^p]\frac{1}{p} && (298)
\end{aligned}
$$

4.6.2 $\tau\tau$ Component

Notice that

$$
\begin{aligned}
\partial_\tau n_p =& \partial_\tau (g_{pp} n^p) && (299)\\
=& \partial_\tau \left(\frac{1}{g(p)} n^p\right) && (300)\\
=& \frac{1}{g(p)}\partial_\tau n^p - n^p \frac{\dot{P} g'(p)}{g^2(p)} && (301)\\
=& \frac{1}{g(p)}\partial_\tau \left(\kappa_p\right) - \kappa_p \frac{\dot{P} g'(p)}{g^2(p)} && (302)
\end{aligned}
$$

Which then allows us to find the simple expression

$$
\begin{aligned}
K_{\tau\tau} &= \left(-u^t \frac{n^p}{n^t} + u^p\right)\nabla_\tau n_p + e^r_{(\tau)}\nabla_\tau n_r && (303)\\
&= \left(-u^t \frac{n^p}{n^t} + u^p\right)\left(\partial_\tau n_p - u^M \Gamma^N_{Mp} n_N\right) + e^r_{(\tau)}\nabla_\tau n_r && (304)\\
&= \left(-\frac{g(p)}{\dot{P}}\right)\left(\partial_\tau n_p + \kappa_p \frac{\dot{P} g'(p)}{g^2(p)}\right) + u^r u^r \frac{p}{l^2} n^p && (305)\\
&= \left(-\frac{g(p)}{\dot{P}}\right)\frac{1}{g(p)}\partial_\tau n^p && (306)\\
&= -\frac{1}{\dot{P}}\dot{n}^p && (307)\\
\implies [K_{\tau\tau}] &= -\frac{1}{\dot{P}}[\dot{n}^p] && (308)
\end{aligned}
$$

4.6.3 τr components

$$
\begin{aligned}
K_{r\tau} &= \left(-e^t_{(r)}\frac{n^p}{n^t} + e^p_{(r)}\right)\nabla_\tau n_p + e^r_{(r)}\nabla_\tau n_r && (309)\\
&= -\frac{P'}{\dot{P}}\frac{1}{\dot{P}}\dot{n}^p + e^r_{(r)} u^r \frac{p}{l^2} n^p && (310)\\
&= -\frac{P'}{\dot{P}^2}\dot{n}^p && (311)\\
\implies [K_{r\tau}] &= -\frac{P'}{\dot{P}^2}[\dot{n}^p] && (312)
\end{aligned}
$$

4.6.4 rr component

Where we have used the above result computed from the $\tau\tau$ component of the Lanczos equation.And as we have shown in the appendix, K is symmetric, so this also determines the $K_{\tau r}$ component. Now for the rr component

$$K_{rr} = \left(-e^t_{(r)}\frac{n^p}{n^t} + e^p_{(r)}\right)\nabla_r n_p + e^r_{(r)}\nabla_r n_r \tag{313}$$

$$= \left(-e^t_{(r)}\frac{n^p}{n^t} + e^p_{(r)}\right)\left(\partial_r n_p - e^M_{(r)}\Gamma^N_{Mp}n_N\right) + e^r_{(r)}e^r_{(r)}\frac{p}{l^2}n^p \tag{314}$$

$$= -\frac{g(p)}{\dot{P}}\frac{P'}{\dot{P}}\left(\partial_r n_p + \kappa_p\frac{\dot{P}g'(p)}{g^2(p)}\frac{P'}{\dot{P}}\right) + R'^2\frac{p}{l^2}n^p \tag{315}$$

As for the time component, we have

$$\partial_r n_p = \partial_r(g_{pp}n^p) \tag{316}$$

$$= \partial_r\left(\frac{1}{g(p)}n^p\right) \tag{317}$$

$$= \frac{1}{g(p)}\partial_r n^p - n^p\frac{P'g'(p)}{g^2(p)} \tag{318}$$

$$= \frac{1}{g(p)}\partial_r\left(\kappa_p\right) - n^p\frac{\dot{P}g'(p)}{g^2(p)}\frac{P'}{\dot{P}} \tag{319}$$

Which gives

$$K_{rr} = -\frac{g(p)}{\dot{P}}\frac{P'}{\dot{P}}\left(\partial_r n_p + \kappa_p\frac{\dot{P}g'(p)}{g^2(p)}\frac{P'}{\dot{P}}\right) + R'^2\frac{p}{l^2}n^p \tag{320}$$

$$= -\frac{1}{\dot{P}}\frac{P'}{\dot{P}}\partial_r n^p + R'^2\frac{p}{l^2}n^p \tag{321}$$

$$\implies [K_{rr}] = -\frac{1}{\dot{P}}\frac{P'}{\dot{P}}\left[\partial_r n^p\right] + R'^2\frac{p}{l^2}[n^p] \tag{322}$$

4.7 G - Computing the Junction Conditions

We now see that for a brane which has a non-trivial profile and is moving in the bulk, we have the following discontinuity in the extrinsic curvature at the brane

$$[K_{\mu\nu}] = \begin{pmatrix} -\frac{1}{\dot{P}}[\dot{n}^p] & -\frac{P'}{\dot{P}^2}[\dot{n}^p] & 0 & 0 \\ -\frac{P'}{\dot{P}^2}[\dot{n}^p] & -\frac{P'}{\dot{P}^2}[n^{p\prime}] + R'^2\frac{p}{l^2}[n^p] & 0 & 0 \\ 0 & 0 & \frac{1}{P}[n^p]g_{\theta\theta} & 0 \\ 0 & 0 & 0 & \frac{1}{P}[n^p]g_{\phi\phi} \end{pmatrix}_{\mu\nu} \tag{323}$$

We also note that we are using comoving spatial coordinates on the brane, so that the four velocity u_μ appearing in the perfect fluid action is precisely $u_M e^M_{(\mu)} = \boldsymbol{u}\cdot\boldsymbol{e}_{(\mu)}$. So in the absence of a profile, our coordinates correspond to the rest frame of the fluid. We also then have (using $\boldsymbol{u}\cdot\boldsymbol{e}_{(\mu)} = -\frac{P'}{\dot{P}}$ and $\boldsymbol{u}\cdot\boldsymbol{u} = -1$)

$$u_\mu u_\nu = \begin{pmatrix} 1 & \frac{P'}{\dot{P}} & 0 & 0 \\ \frac{P'}{\dot{P}} & \frac{P'^2}{\dot{P}^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} \tag{324}$$

the induced metric is given by

$$g^{\text{ind}}_{\mu\nu} = \begin{pmatrix} -1 & -\frac{P'}{\dot{P}} & 0 & 0 \\ -\frac{P'}{\dot{P}} & \frac{P^2}{l^2} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ 0 & 0 & 0 & g_{\phi\phi} \end{pmatrix}_{\mu\nu} \tag{325}$$

We can now evaluate the junction equations by recalling

$$[K_{\mu\nu}] = \frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} g_{\mu\nu} + 3\frac{2}{l}\frac{\sigma_u}{\sigma_c} u_\mu u_\nu \tag{326}$$

4.7.1 Angular

The angular equations for the second junction condition are

$$[K_{\Omega_i\Omega_j}] = [n^p]\frac{1}{P} g_{\Omega_i\Omega_j} = \frac{2\sigma(\phi)}{l\sigma_c} g_{\Omega_i\Omega_j} \tag{327}$$

$$\implies [n^p] = [\kappa_p] = 2\frac{P}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{328}$$

Which tells us the angular equations are degenerate. Solving for $\dot{P}$ we find

$$\dot{P}^2 = -k + \frac{\mu_+ + \mu_-}{2P^2} + \frac{l^2\sigma_c^2}{16P^6\sigma^2(\phi)}(\mu_+ - \mu_-)^2 + \frac{P^2}{l^2}\left(\frac{\sigma^2(\phi)}{\sigma_c^2} - 1\right) \tag{329}$$

$$\equiv -k + \frac{l^2}{P^2}\frac{\mu_{\text{avg}}}{l^2} + \frac{1}{16}\frac{l^6}{P^6}\frac{\sigma_c^2}{\sigma^2(\phi)}\frac{\mu_{\text{dif}}^2}{l^4} + \frac{P^2}{l^2} l^2 H_\tau^2 \tag{330}$$

$$\equiv -2V(P) \tag{331}$$

Notice also that for large $\frac{P}{l}$ and P independent H_τ

$$\left(\frac{\dot{P}}{P}\right)^2 = H_\tau^2 - \frac{k}{P^2} + \mathcal{O}\left(\frac{l}{P}\right)^4 \tag{332}$$

Which is precisely one of the Friedmann equations [3].

4.7.2 $\tau\tau$ Component

The $\tau\tau$ equation is given by

$$-\frac{1}{\dot{P}}[\dot{n}^p] = -\frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} + 3\frac{2}{l}\frac{\sigma_u}{\sigma_c} \tag{333}$$

$$\implies [\dot{n}^p] = \dot{P}\frac{2}{l}\frac{\sigma(\phi) - 3\sigma_u}{\sigma_c} \tag{334}$$

Where we have used $g_{\tau\tau} = -1$. Which agrees with [5] and [15] (the later mentions that the $\tau\tau$ components being the derivative of the spatial components is guaranteed by the bianchi identity). Notice that

$$[n^p] = P\frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{335}$$

$$\implies [\dot{n}^p] = \frac{2}{l}\frac{\partial_\tau(P\sigma(\phi))}{\sigma_c} \tag{336}$$

Hence, to consistently have a solution to the Junction conditions it must be that for $\dot{P} \neq 0$ (and since $p \neq 0$)

$$\dot{P}(\sigma(\phi) - 3\sigma_u) = \partial_\tau(P\sigma(\phi)) \tag{337}$$

$$\implies \partial_\tau \sigma(\phi) = -3\frac{\dot{P}}{P}\sigma_u \tag{338}$$

$$\implies \dot{\rho} + 3\frac{\dot{P}}{P}(\rho + \mathcal{P}) = -\partial_\tau V_b(\phi) \tag{339}$$

This is the equation of energy conservation for a perfect fluid in FRW with a source given by the time derivative of the scalar potential. This also implies that if the equation of state for the perfect fluid is $\mathcal{P} = -\rho$ and a potential for the bulk scalar is given on the brane, it's value at a point on the brane is constant along it's trajectory through the bulk.

4.7.3 τr Component

$$[K_{r\tau}] = \frac{2}{l}\frac{\sigma(\phi)}{\sigma_c}g_{\tau r} + 3\frac{2}{l}\frac{\sigma_u}{\sigma_c}u_r u_\tau \tag{340}$$

$$= -\frac{2}{l}\frac{\sigma(\phi) - 3\sigma_u}{\sigma_c}\frac{P'}{\dot{P}} \tag{341}$$

$$\implies -\frac{P'}{\dot{P}^2}[\dot{n}^p] = -\frac{2}{l}\frac{\sigma(\phi) - 3\sigma_u}{\sigma_c}\frac{P'}{\dot{P}} \tag{342}$$

$$\frac{1}{\dot{P}}[\dot{n}^p] = \frac{2}{l}\frac{\sigma(\phi) - 3\sigma_u}{\sigma_c} \tag{343}$$

Which is degenerate with our previous equation for the $\tau\tau$ component.

4.7.4 rr Component

From the Lanczos equations, we have

$$[K_{rr}] = -\frac{1}{\dot{P}}\frac{P'}{\dot{P}}[\partial_r n^p] + R'^2\frac{p}{l^2}[n^p] \tag{344}$$

$$= \frac{2}{l}\frac{\sigma(\phi)}{\sigma_c}g_{rr} + 3\frac{2}{l}\frac{\sigma_u}{\sigma_c}u_r u_r \tag{345}$$

$$= \frac{2}{l}\frac{\sigma(\phi)}{\sigma_c}\frac{P^2}{l^2} + 3\frac{2}{l}\frac{\sigma_u}{\sigma_c}\frac{P'^2}{\dot{P}^2} \tag{346}$$

And from the angular components

$$[n^p] = 2\frac{P}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{347}$$

$$\implies [\partial_r n^p] = \frac{2}{l\sigma_c}\partial_r(P\sigma(\phi)) = [n^p]\frac{P'}{P} + \frac{2P}{l\sigma_c}\partial_r\sigma(\phi) \tag{348}$$

Giving

$$
\begin{aligned}
\frac{P}{l^2}[n^p] + 3\frac{2}{l}\frac{\sigma_u}{\sigma_c}\frac{P'^2}{\dot{P}^2} =& -\frac{1}{\dot{P}}\frac{P'}{\dot{P}}\left[\partial_r n^p\right] + R'^2\frac{P}{l^2}[n^p] && (349)\\
3\frac{2}{l}\frac{\sigma_u}{\sigma_c}\frac{P'^2}{\dot{P}^2} =& -\frac{1}{\dot{P}}\frac{P'}{\dot{P}}\left[\partial_r n^p\right] + (R'^2-1)\frac{P}{l^2}[n^p] && (350)\\
=& -\frac{1}{\dot{P}}\frac{P'}{\dot{P}}\left([n^p]\frac{P'}{P} + \frac{2p}{l\sigma_c}\partial_r\sigma(\phi)\right) + \frac{l^2}{p^2}\frac{P'^2}{\dot{P}^2}\frac{P}{l^2}[n^p] && (351)\\
=& -\frac{P'}{\dot{P}^2}\frac{2P}{l\sigma_c}\partial_r\sigma(\phi) && (352)
\end{aligned}
$$

Hence for $P' \neq 0$ we must have

$$
\begin{aligned}
\partial_r\sigma(\phi) + 3\frac{P'}{P}\sigma_u =& 0 && (353)\\
\implies \rho' + 3\frac{P'}{P}(\rho + \mathcal{P}) =& -\partial_r V_b(\phi) && (354)
\end{aligned}
$$

This will no doubt correspond to conservation of the r components of the energy momentum tensor in FRW. This is the final Junction condition one needs to satisfy.

4.8 H - Effective Action and Profile

We now look to compute an effective action to determine the profile of the brane. The total action for our system is

$$
\begin{aligned}
S =& S_{EH} + S_{GHY} + S_B + S_{NG} && (355)\\
=& \frac{1}{2\kappa}\int_V d^5X\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial V} d^4x\sqrt{-g_{\text{ind}}}K && (356)\\
&+ \int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi)\right) - \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\sigma(\phi) && (357)\\
&+ \frac{1}{2}\int_\Sigma d^4x\sqrt{g_{\text{ind}}}\left((\mathcal{P}+\rho)g^{\text{ind}}_{\mu\nu}u^\mu u^\nu - (\mathcal{P}-\rho)\right) && (358)
\end{aligned}
$$

Where $K \equiv K_{\mu\nu}g^{\mu\nu}_{\text{ind}}$. We can then notice that on shell the Gibbons Hawking York term can be related to S_{NG}

$$
\begin{aligned}
\frac{1}{\kappa}\int_{\partial V} d^4x\sqrt{-g_{\text{ind}}}K =& \frac{1}{\kappa}\left(\int_{\partial V_+} - \int_{\partial V_-}\right) d^4x\sqrt{-g_{\text{ind}}}K && (359)\\
=& \frac{1}{\kappa}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}[K_{\mu\nu}]g^{\mu\nu}_{\text{ind}} && (360)\\
=& -\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\left(S_{\mu\nu} - \frac{1}{3}g_{\text{ind},\mu\nu}S\right)g^{\mu\nu}_{\text{ind}} && (361)\\
=& \frac{1}{3}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}S && (362)\\
=& \frac{4}{3}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\sigma(\phi) && (363)\\
=& \frac{4}{3}S_{NG} && (364)
\end{aligned}
$$

The perfect fluid action, on shell becomes

$$S_{PF} =\frac{1}{2}\int_\Sigma d^4x\sqrt{g_{\text{ind}}}\left((\mathcal{P}+\rho)g^{\text{ind}}_{\mu\nu}u^\mu u^\nu-(\mathcal{P}-\rho)\right) \tag{365}$$

$$=-\int_\Sigma d^4x\sqrt{g_{\text{ind}}}\mathcal{P} \tag{366}$$

We can also note that the AdS-S solutions have constant curvature Λ for any value of the Black Brane mass, which implies that on shell S_{EH} is a constant on shell which we neglect as it won't affect dynamics. This assumes the 0 backreaction limit for the both the scalar and the tension on the brane. In which case we have the onshell action

$$S=\int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi-V_B(\phi)\right)+\frac{1}{3}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}(\sigma(\phi)-3\mathcal{P}) \tag{367}$$

We also know that given our coordinate choice, we have

$$\sqrt{-g_{\text{ind}}}=R^2\cos\theta\frac{P^3}{l^3}\sqrt{1+\frac{l^2P'^2}{P^2\dot{P}^2}}=R^2\cos\theta\frac{P^3}{l^3}R' \tag{368}$$

Our solution assumes that ϕ is constant on Σ. Hence we have

$$S=\int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi-V_B(\phi)\right)+\frac{1}{3}\frac{4\pi}{l^3}\int_{\Sigma/S^2}d\tau dr\frac{P^3R^2}{P'}\sqrt{1+\frac{l^2P'^2}{P^2\dot{P}^2}}(\sigma(\phi)-3\mathcal{P}) \tag{369}$$

In order to vary the action and find the initial conditions for the profile of Σ, we must find solutions such that ϕ is constant on Σ. We note that the EOM for ϕ are (assuming the junction conditions have been satisfied)

$$\nabla^2\phi=-V_B'(\phi) \tag{370}$$

Integrating by parts and putting ϕ on it's EOM we have

$$S_B=\int_V d^5X\sqrt{-g}\left(-\frac{1}{2}\phi\nabla^2\phi-V_B(\phi)\right)+\frac{1}{2}\int_V d^5X\partial_M(\sqrt{-g}g^{MN}\phi\partial_N\phi) \tag{371}$$

$$=\int_V d^5X\sqrt{-g}\left(\frac{1}{2}\phi V_B'(\phi)-V_B(\phi)\right)+\frac{1}{2}\int_{\partial V}d^4x\sqrt{-g_{\text{ind}}}\phi n^M\partial_M\phi \tag{372}$$

$$=\int_V d^5X\sqrt{-g}\left(\frac{1}{2}\phi\partial_\phi-1\right)V_B(\phi)+\frac{1}{2}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\left[\phi n^M\partial_M\phi\right] \tag{373}$$

$$=\int_V d^5X\sqrt{-g}\left(\frac{1}{2}\phi\partial_\phi-1\right)V_B(\phi)-\frac{1}{2}\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}V_b'(\phi) \tag{374}$$

Where we have neglected contributions from the boundary of V that are unaffected by the dynamics of Σ, and we have used the scalar junction conditions in the last line. Again using the fact that ϕ is constant on Σ we have for the on shell action

$$S=\int_V d^5X\sqrt{-g}\left(\frac{1}{2}\phi\partial_\phi-1\right)V_B(\phi)+\frac{4\pi}{l^3}\int_{\Sigma/S^2}d\tau dr\frac{P^3R^2}{P'}\sqrt{1+\frac{l^2P'^2}{p^2\dot{P}^2}}\left(\frac{1}{3}\sigma(\phi)-\mathcal{P}-\frac{1}{2}V_b'(\phi)\right) \tag{375}$$

We can see that in the case of a quadratic potential (as for the canonical GW [16]), we end up with

$$S = \frac{4\pi}{l^3} \int_{\Sigma/S^2} d\tau dr P^3 R^2 \sqrt{1 + \frac{l^2 P'^2}{P^2 \dot{P}^2}} \left(\frac{1}{3}\sigma(\phi) - \mathcal{P} - \frac{1}{2} V_b'(\phi) \right) \tag{376}$$

We can now use the junction condition for $\dot{P}^2$ to turn this into an effective 1D problem.

$$S = \frac{4\pi}{l^3} \int_{\Sigma/S^2} d\tau dr P^3 R^2 \sqrt{1 - \frac{l^2 P'^2}{P^2 2V(P)}} \left(\frac{1}{3}\sigma(\phi) - \mathcal{P} - \frac{1}{2} V_b'(\phi) \right) \tag{377}$$

$$= \int_{\Sigma/S^2} d\tau dr \mathcal{L} \tag{378}$$

Note that this is a nasty Lagrangian since R is determined by an integral over P'.