Dynamics of Moving Branes In the Presence of a Bulk Scalar

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Abstract

In these notes we leverage Israel's Junction conditions to describe the dynamics of a codimension 1 surface moving in AdS-S using Einstein's equations. We extend the results of [1] to include both a scalar in the bulk and allow the brane to have a non-trivial profile. We find that the junction conditions determine the profile in terms of the potential of the scalar. We then go on to do a heuristic calculation of the emission rate of such a surface from the AdS-S black brane following a literature started by [2]. We find the hawking temperature is off by a factor of 2, a problem which has been recorded in the literature.

1 Preliminaries

Here we simply summarize and restate important results from [3] which will be central to our analysis. Refer to the appendices for some mathematical results that are used throughout.

1.1 Israel's Junction Conditions

Here we state Israel's Junction conditions which will determine dynamics of surface layers. We consider a codimension 1 surface layer Σ which divides a spacetime manifold V into $V_\pm$. We will find the junction conditions are analgous to what we'd expect; that the metric at the surface layer is continuous and that the singularities that appear in the EFE from the surface match. One of the benefit's of [3] approach was to do this in a coordinate independent way. See [4] for a particularily clear exposition of the conditions.

1.1.1 Israel's First Junction Condition

Consistency requires that the induced metric on Σ is well defined. Hence we need that the metrics induced on Σ by the metrics in $V_\pm$ are equal. This is **Israel's First Junction Condition**.

$$ds_+^2|_\Sigma = ds_-^2|_\Sigma \tag{1}$$

This also ensures that the Ricci tensor is a well defined distribution.

1.1.2 Israel's Second Junction Condition

Consider the discontinuity in the extrinsic curvature (again refer to the appendices as needed)

$$\gamma_{\mu\nu} = K^+_{\mu\nu}|_\Sigma - K^-_{\mu\nu}|_\Sigma \tag{2}$$

We define the symmetric 2 tensor S on Σ by the Lanczos equations

$$\gamma_{\mu\nu} - g_{\mu\nu}\gamma = -\kappa S_{\mu\nu} \tag{3}$$

$$\implies \gamma - D_\Sigma \gamma = -\kappa S \tag{4}$$

$$\implies \gamma_{\mu\nu} = -\kappa\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1} g_{\mu\nu} S\right) \tag{5}$$

Where D_Σ is the dimension of the Σ and $\kappa = 8\pi G$. S will be identified with the surface energy momentum tensor for Σ. From the Lanczos equations and the Einstein equations (assuming $\Lambda = 0$) we have

$$G_{MN}e^M_{(\mu)}e^N_{(\nu)} = G_{\mu\nu} - n^J\partial_J(K_{\mu\nu} - g_{\mu\nu}K) - KK_{\mu\nu} + \frac{1}{2}\left(K_{\mu\nu}K^{\mu\nu} + K^2\right) \tag{6}$$
$$\equiv - n^J\partial_J(K_{\mu\nu} - g_{\mu\nu}K) + Z_{\mu\nu} \tag{7}$$
$$= \kappa T_{MN}e^M_{(\mu)}e^N_{(\nu)} \equiv \kappa T_{\mu\nu} \tag{8}$$
$$\tag{9}$$

We integrate this equation along a curve C through Σ from V_- to V_+. We do this along a curve parameterized by τ with Σ located at $\tau = 0$ and orthogonal to the brane. We also take the limit that the length of the curve goes to 0.

$$\kappa\int_C d\tau T_{\mu\nu} = - \int_C d\tau n^J\partial_J(K_{\mu\nu} - g_{\mu\nu}K) + \int_C d\tau Z_{\mu\nu} \tag{10}$$
$$= - \int_C d\tau\partial_\tau(K_{\mu\nu} - g_{\mu\nu}K) + \int_C d\tau Z_{\mu\nu} \tag{11}$$
$$= - (K_{\mu\nu} - g_{\mu\nu}K)_{\partial C} + \int_C d\tau Z_{\mu\nu} \tag{12}$$
$$= - (\gamma_{\mu\nu} - g_{\mu\nu}\gamma) + \int_C d\tau Z_{\mu\nu} \tag{13}$$
$$\tag{14}$$

*** this can be cleaned up and made coordinate independent but is fine for now.

Following [3] we make a heuristic argument by assuming that $Z_{\mu\nu}$ is finite on C so that

$$\int_C d\tau T_{\mu\nu} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\tau T_{\mu\nu} = -\frac{1}{\kappa}(\gamma_{\mu\nu} - g_{\mu\nu}\gamma) = S_{\mu\nu} \tag{15}$$

Hence for an energy momentum tensor with a singular contribution on Σ

$$S_{\mu\nu} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\tau\delta(\tau)S_{\mu\nu} \tag{16}$$
$$= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\tau T_{\mu\nu} \tag{17}$$

*** alternatively we could start by deriving the energy momentum tensors, and show that only the singular parts contribute to S.

From which we see that satisfying the Lanczos equations ensure that the singularities in the tangent components of the Einstein tensor from the presence of Σ cancel. And so given a solution to EFE in $V_\pm$, the tangent components of EFE will be satisfied. This is **Israel's second Junction Condition**. We can write this in a coordinate independent way as

$$T_{\mu\nu} = \delta_\Sigma(X)S_{\mu\nu} + (\text{regular}) \tag{18}$$

(see appendices) And so in the bulk we find

$$T_{MN} = \delta_\Sigma(X)S_{\mu\nu}e^{(\mu)}_M e^{(\nu)}_N + (\text{regular}) \tag{19}$$

This agrees precisely with [5] when one takes gaussian normal cooridnates (in which case $\delta_\Sigma(X) = \delta(\eta)$ with eta the normal direction which is 0 on the brane). One can follow the elegant analysis in [4] to see that these two conditions are necessary and sufficient to ensure that the EFEs are satisfied across a singular surface Σ.

1.2 Scalar Junction Conditions

Now consider the case of a scalar in the bulk with an interaction term on the brane.

$$S_\phi = -\int_\Sigma d^4x\sqrt{-g_{ind}}V_b(\phi) + \int_V d^5X\sqrt{-g}\left(\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi)\right) \tag{20}$$

$$= \int_V d^5X\sqrt{-g}\left(g^{MN}\partial_M\phi\partial_N\phi - V_B(\phi) - \delta_\Sigma(X)V_b(\phi)\right) \tag{21}$$

This gives EOM

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\phi\right) = -V_B'(\phi) - \delta_\Sigma(X)V_b'(\phi) \tag{22}$$

$$\phi(X) = \Theta_+(X)\phi_+ + \Theta_-(X)\phi_- \tag{23}$$

(see appendices) Where $\phi_\pm$ are solutions to the EOM in $V_\pm$. The EOM can only be satisfied at Σ if the singularities cancel. Only if ϕ is continuous can they cancel. Hence we have that

$$\partial_M\phi(X) = \Theta_+(X)\partial_M\phi_+ + \Theta_-(X)\partial_M\phi_- \tag{24}$$

$$g^{MN}\partial_N\partial_M\phi(X) = \delta_\Sigma(X)g^{MN}n_N\partial_M\left(\phi_+ - \phi_-\right) + (\text{regular}) \tag{25}$$

Where we have used the fact that $C_\pm(X)$ have opposite signed normal components at Σ. This is the only other singular contribution to the EOM. So to satisfy the EOM it must be that

$$n^M\partial_M\left(\phi_+ - \phi_-\right)|_\Sigma = \left[n^M\partial_M\phi\right] = -V_b'(\phi)|_\Sigma \tag{26}$$

2 Energy Momentum Sources

We now consider EMT sources from the scalar and from the tension on Σ.

2.1 Energy Momentum Tensor of a Moving Brane

We note that the action for our brane tension is given by the Nambu Goto Action.

$$S = -\rho\int_\Sigma d^4x\sqrt{-g_{\text{ind}}} \tag{27}$$

Where

$$g^{\mu\nu}_{ind}e^M_{(\mu)}e^N_{(\nu)} = g^{MN} - \epsilon(\boldsymbol{n})n^Mn^N \tag{28}$$

Which gives the usual expression as

$$g^{\mu\nu}_{ind}e^M_{(\mu)}e^N_{(\nu)}e^{(\alpha)}_Me^{(\beta)}_N = g^{\alpha\beta}_{ind} \tag{29}$$

$$= (g^{MN} - \epsilon(\boldsymbol{n})n^Mn^N)e^{(\alpha)}_Me^{(\beta)}_N \tag{30}$$

$$= g^{MN}e^{(\alpha)}_Me^{(\beta)}_N \tag{31}$$

$$= g^{MN}\frac{\partial x^\alpha}{\partial X^M}\frac{\partial x^\beta}{\partial X^N} \tag{32}$$

Using orthogonality of $\boldsymbol{e}_{(\mu)}$ and $\boldsymbol{n}$. Varying wrt the bulk metric we have

$$\delta S = -\rho \int_\Sigma d^4x \delta\sqrt{-g_{\text{ind}}} \tag{33}$$

$$=\rho \int_\Sigma d^4x \frac{1}{2\sqrt{-g_{\text{ind}}}} g_{\text{ind}} g_{\text{ind}}^{\alpha\beta} \delta g_{\text{ind},\alpha\beta} \tag{34}$$

$$= -\rho \int_\Sigma d^4x \frac{1}{2}\sqrt{-g_{\text{ind}}} g_{\text{ind}}^{\alpha\beta} e^M_{(\alpha)} e^N_{(\beta)} \delta g_{MN} \tag{35}$$

$$= -\rho \int_\Sigma d^4x \frac{1}{2}\sqrt{-g_{\text{ind}}} g_{\text{ind}}^{\alpha\beta} e^M_{(\alpha)} e^N_{(\beta)} \delta g_{MN} \tag{36}$$

$$= -\frac{\rho}{2} \int_V d^5X \sqrt{-g}\delta_\Sigma(X) g_{\text{ind}}^{\alpha\beta} e^M_{(\alpha)} e^N_{(\beta)} \delta g_{MN} \tag{37}$$

$$= -\frac{1}{2} \int_V d^5X \sqrt{-g} T^{MN} \delta g_{MN} \tag{38}$$

So we find the energy momentum tensor is

$$T^{MN} = \delta_\Sigma(X) \rho g_{\text{ind}}^{\alpha\beta} e^M_{(\alpha)} e^N_{(\beta)} \tag{39}$$

From which we can read off from above that

$$S^{\mu\nu} = \rho g_{\text{ind}}^{\mu\nu} \tag{40}$$

$$= \rho(g^{MN} - \epsilon(\boldsymbol{n}) n^M n^N) e^{(\mu)}_M e^{(\nu)}_N \tag{41}$$

$$= \rho g^{MN} e^{(\mu)}_M e^{(\nu)}_N \tag{42}$$

$$\tag{43}$$

Note that this is the stress energy tensor for vacuum energy [6].

2.2 Scalar Potential on the Brane

We can notice that a potential on Σ that depends only on ϕ will be independent of the metric, and hence will be given exactly as above with the simple replacement $\rho \to V_b(\phi)$

$$T_b^{MN} = \delta_\Sigma(X) V_b(\phi) g_{\text{ind}}^{\alpha\beta} e^M_{(\alpha)} e^N_{(\beta)} \tag{44}$$

which gives

$$S_b^{\mu\nu} = V_b(\phi) g_{\text{ind}}^{\mu\nu} \tag{45}$$

$$= V_b(\phi) g^{MN} e^{(\mu)}_M e^{(\nu)}_N \tag{46}$$

$$\tag{47}$$

For later convenience we defin

$$\sigma(\phi) = \rho + V_b(\phi) \tag{48}$$

2.3 Scalar in the Bulk

The bulk contribution to the scalar energy momentum tensor is given by the usual expression

$$T_{B,MN} = \partial_M\phi\partial_N\phi - g_{MN}\mathcal{L}_B \tag{49}$$

$$= \partial_M\phi\partial_N\phi - g_{MN}\left(\frac{1}{2}g^{JK}\partial_J\phi\partial_K\phi - V_B(\phi)\right) \tag{50}$$

$$\tag{51}$$

We showed above that $\partial_M\phi$ is regular at Σ, hence this expression will be as well. So $T_{B,MN}$ will not contribute to the surface energy momentum tensor. The EFEs do however relate the discontinuity in T to a source current for S.

$$j_\mu \equiv [T_{Mn}]e^M_{(\mu)} = -S^\nu_{\mu;\nu} \tag{52}$$

Both the other contributions to the stress energy tensor are proportional to $\delta_\Sigma(X)$ and so have vanishing bracket. Which implies

$$-S^\nu_{\mu;\nu} = \left[e^M_{(\mu)}n^N\left(\frac{1}{2}\partial_M\phi\partial_N\phi - g_{MN}\left(\frac{1}{2}g^{JK}\partial_J\phi\partial_K\phi - V_B(\phi)\right)\right)\right] \tag{53}$$

$$= \left[e^M_{(\mu)}n^N\frac{1}{2}\partial_M\phi\partial_N\phi\right] \tag{54}$$

using orthogonality. We also notice that for ϕ to have a well defined value on the brane it must be that

$$\left[e^M_{(\mu)}\partial_M\phi\right] = [\partial_\mu\phi] = 0 \tag{55}$$

Which implies we can write

$$S^\nu_{\mu;\nu} = -\, e^M_{(\mu)}\partial_M\phi\left[n^N\partial_N\phi\right] = \partial_\mu\phi V_b'(\phi)|_\Sigma \tag{56}$$

Where we have used the scalar junction conditions in the last equality. Hence for $S^{\mu\nu} = \sigma(\phi)g^{\mu\nu}_{ind}$ we have

$$S^\nu_{\mu;\nu} = \delta^\nu_\mu\partial_\nu\sigma(\phi) = \partial_\mu\phi V_b'(\phi) \tag{57}$$

where we have used metric compatibility. Hence, the surface energy momentum tensor is conserved in Σ provided the scalar junction conditions are satisfied.

**** see [4] (3.44) this is one of the constraint equations we need to solve. So this is good, but we also need to satisfy the T_{nn} constraint.

3 IR Brane in AdS-S

3.1 Metric Parametrization

Consider the AdS_5 black brane metric. The metrics in $V_\pm$ will be

$$
\begin{aligned}
ds_{\pm}^2 =& dX^M dX^N g_{\pm,MN} && (58)\\
=& -g_{\pm}(p)dt^2 + \frac{dp^2}{g_{\pm}(p)} + \frac{p^2}{l^2} ds_{H,d-1}^2 && (59)\\
& -g_{\pm}(r)dt^2 + \frac{dp^2}{g_{\pm}(p)} + \frac{p^2}{l^2}(dr^2 + r^2 d\Omega^2) && (60)\\
g_{\pm}(p) \equiv& \frac{p^2}{l^2} - \frac{2\mu_{\pm}}{p^2} && (61)
\end{aligned}
$$

Where

$$
2\mu_{\pm} = \frac{4\pi G \rho_{M\pm} l^3}{3} \equiv \frac{\rho_{M\pm}}{\sigma_c} l^2 \tag{62}
$$

Where ρ_M is the energy density of the black brane. Here we can compare to conformal coordinates as $p = \frac{l^2}{z}$. And hence has Hawking temperature $T = \frac{1}{\pi z_h} = \frac{p_h}{\pi l^2}$. We consider a geometry where the IR brane is located outside the horizon and between the horizon and the UV brane (a UV cutoff at z=1).

3.2 Basis and Induced Metric

Now consider the embedding of the IR brane in the bulk.

$$
X^M = (T(\tau, r), R(\tau, r), 0, 0, P(\tau, r)) \tag{63}
$$

This describes a congruence of spherical shells whose dynamics we will now determine using Israel's Junction conditions. We take as part of our our basis, the following two tangent vectors

$$
\begin{aligned}
u^M =& \partial_\tau X^M = (\dot{T}, \dot{R}, 0, 0, \dot{P}) && (64)\\
e_{(r)}^M =& (e_{(r)}^t, e_{(r)}^r, 0, 0, e_{(r)}^p) && (65)
\end{aligned}
$$

we want u to have unit time-like norm which determines it's first component

$$
\begin{aligned}
-1 =& \boldsymbol{u} \cdot \boldsymbol{u} && (66)\\
=& -g(p)(u^t)^2 + \frac{(u^p)^2}{g(p)} + \frac{p^2}{l^2}(u^r)^2 && (67)\\
\implies u^t =& \sqrt{\frac{1}{g(p)} \left(1 + \frac{p^2}{l^2}\dot{R}^2 + \frac{\dot{P}^2}{g(p)}\right)} && (68)\\
=& \frac{1}{g(p)} \sqrt{\left(1 + \frac{p^2}{l^2}\dot{R}^2\right) g(p) + \dot{P}^2} && (69)\\
\equiv& \frac{\kappa_p}{g(p)} && (70)
\end{aligned}
$$

The only constraints we have on the normal vector $\boldsymbol{n}$ are that it is orthogonal to the brane and that it has a space-like norm of 1. So we are free to take it to have 2 components

$$
n^M = (n^t, 0, 0, 0, n^p) \tag{71}
$$

Which from the above conditions are

$$
\begin{aligned}
\boldsymbol{n}\cdot\boldsymbol{u} =&0 \qquad \boldsymbol{n}\cdot\boldsymbol{n}=1 &(72)\\
\implies n^t =&\frac{1}{g(p)}\frac{\dot{P}}{\sqrt{1+\frac{p^2}{l^2}\dot{R}}} &(73)\\
\equiv&\frac{\dot{P}}{g(p)\kappa_r} &(74)\\
n^p =&\sqrt{g(p)+\frac{\dot{P}^2}{1+\frac{p^2}{l^2}\dot{R}^2}} &(75)\\
=&\frac{\kappa_p}{\sqrt{1+\frac{p^2}{l^2}\dot{R}^2}} &(76)\\
=&\frac{\kappa_p}{\kappa_r} &(77)
\end{aligned}
$$

So all that's left is to determine $e_{(r)}$. We have for the induced metric

$$
\begin{aligned}
ds^2_{ind} =&g_{MN}e^M_{(\mu)}e^N_{(\nu)}dx^\mu dx^\nu &(78)\\
=&g_{MN}u^M u^N d\tau^2 + g_{MN}e^M_{(i)}e^N_{(i)}dx^i dx^i &(79)\\
=&-d\tau^2 + g_{MN}e^M_{(i)}e^N_{(i)}dx^i dx^i &(80)\\
& &(81)
\end{aligned}
$$

We choose the r components to match their bulk counterparts

$$g_{MN}e^M_{(r)}e^N_{(r)}dr^2 = \frac{P^2}{l^2}dr^2 \tag{82}$$

So we have

$$
\begin{aligned}
\boldsymbol{e}_{(r)}\cdot\boldsymbol{e}_{(r)} =&\frac{P^2}{l^2} &(83)\\
\boldsymbol{n}\cdot\boldsymbol{e}_{(r)} =&0 &(84)
\end{aligned}
$$

And we take

$$\boldsymbol{u}\cdot\boldsymbol{e}_{(r)} =0 \tag{85}$$

This determines the components of $\boldsymbol{e}_{(r)}$ (3 equations on 3 components). We won't need to find them explicitly, but this shows that they can be found. And allowing the brane to inheret the angular coordinates of the bulk, we then find the induced metric

$$
\begin{aligned}
ds^2_{ind} =& -d\tau^2 + \frac{p^2}{l^2}\left(dr^2 + r^2 d\Omega^2\right) &(86)\\
ds^2_{\Sigma} =& -d\tau^2 + \frac{P^2(r,\tau)}{l^2}\left(dr^2 + R^2(r,\tau)d\Omega^2\right) &(87)
\end{aligned}
$$

Since [4] tells us that both sides of the Σ agree on the basis, this will hold on both sides of the brane. Hence we have satisfied Israel's first junction condition. Notice that the induced metric is the flat FRW metric. We could then relate the scale factor to P

$$a(\tau) = \frac{P(\tau)}{l} \tag{88}$$

where here, τ is identified with the cosmic rather than conformal time (which is usually denoted by τ).

3.3 Israel's Second Junction Condition

For the extrinsic curvature we have

$$\begin{aligned} K_{\mu\nu} =& n_{\nu;\mu} &(89)\\ =& \partial_\mu n_\nu - \Gamma^M_{\mu\nu} n_M &(90)\\ =& \partial_\mu n_\nu + \frac{1}{2} n^M \partial_M g_{\mu\nu} &(91) \end{aligned}$$

3.3.1 Extrinsic Curvature

Noticing that the metric is time independent (off of Σ) and n is independent of the angular coordinates we have

$$\begin{aligned} K_{\Omega_i\Omega_j} =& n^p \frac{1}{2} \partial_p g_{\Omega_i\Omega_j} &(92)\\ =& n^p \frac{1}{p} g_{\Omega_i\Omega_j} &(93) \end{aligned}$$

So the spatial components of the second junction condition (Lanczos equation) is

$$\begin{aligned} [K_{\Omega_i\Omega_j}] =& -\kappa(S_{\Omega_i\Omega_j} - \frac{1}{3} g_{\Omega_i\Omega_j} \mathrm{Tr} S) &(94)\\ =& -\kappa\sigma(\phi)(g_{\Omega_i\Omega_j} - \frac{1}{3} g_{\Omega_i\Omega_j} \mathrm{Tr} g) &(95)\\ =& \frac{\kappa}{3}\sigma(\phi) g_{\Omega_i\Omega_j} &(96)\\ n^p_+ - n^p_- = \left[\frac{\kappa_p}{\kappa_r}\right] =& \frac{\kappa}{3} P\sigma(\phi) \equiv 2\frac{P}{l}\frac{\sigma(\phi)}{\sigma_c} &(97)\\ & &(98) \end{aligned}$$

This can be written in the form

$$\begin{aligned} \implies \frac{\dot{P}^2}{1 + \frac{P^2}{l^2}\dot{R}^2} = \frac{\dot{P}^2}{\kappa_r^2} =& \frac{2\mu_+ + 2\mu_-}{2P^2} + \frac{l^2(\mu_+ - \mu_-)^2\sigma_c^2}{4P^6\sigma^2(\phi)} - \frac{P^2}{l^2}\left(1 - \frac{\sigma^2(\phi)}{\sigma_c^2}\right) &(99)\\ \equiv& \frac{l^2}{P^2}\frac{\rho_+ + \rho_-}{2\sigma_c} + \frac{l^6}{P^6}\frac{(\rho_+ - \rho_-)^2}{16\sigma^2(\phi)} - \frac{P^2}{l^2} l^2 H_p^2 &(100)\\ \equiv& -2V(P) &(101) \end{aligned}$$

Using (62). Note also that We are looking to create a stable minimum in the bulk near the AdS boundary (so for large p). And in this (and small $\dot{R}$) limit the H_p term dominates and is precisely Hubble on the brane (ie. energy density for a flat spacetime geometry on the brane).

3.3.2 Temporal component

Here we follow [7]. The proper time component of the extrinsic curvature evaluated at the position of the brane is

$$
\begin{aligned}
K_{\tau\tau} =& u^N u^M \nabla_M n_N && (102)\\
=& -n_N u^M \nabla_M u^N - n_N u^N \nabla_M u^M && (103)\\
=& -n_N \nabla_\tau u^N && (104)\\
\equiv& -n_N a^N && (105)
\end{aligned}
$$

Where a is the covariant accelaration of the brane. $u^2 = -1 \implies u_M a^M = 0$ so we have[1]

$$
\begin{aligned}
0 =& g_{tt} u^t a^t + g_{rr} u^r a^r + g_{pp} u^p a^p && (106)\\
=& -g(p) \frac{\kappa_p}{g(p)} a^t + \frac{p^2}{l^2} \dot{R} a^r + \frac{1}{g(p)} \dot{P} a^p && (107)\\
\implies a^t =& \frac{-1}{\kappa_p} \left(\frac{p^2}{l^2} \dot{R} a^r + \frac{1}{g(p)} \dot{P} a^p \right) && (108)
\end{aligned}
$$

So we have

$$
\begin{aligned}
K_{\tau\tau} =& -g_{tt} n^t a^t - g_{pp} n^p a^p && (109)\\
=& g(p) \frac{\dot{P}}{\kappa_r g(p)} \frac{1}{\kappa_p} \left(\frac{p^2}{l^2} \dot{R} a^r + \frac{1}{g(p)} \dot{P} a^p \right) - \frac{\kappa_p}{\kappa_r} \frac{1}{g(p)} a^p && (110)\\
=& \frac{1}{\kappa_p \kappa_r} \left(\frac{p^2}{l^2} \dot{R} \dot{P} a^r + \frac{1}{g(p)} \dot{P}^2 a^p - \kappa_p^2 \frac{1}{g(p)} a^p \right) && (111)\\
=& \frac{1}{\kappa_p \kappa_r} \left(\frac{p^2}{l^2} \dot{R} \dot{P} a^r - \kappa_r^2 a^p \right) && (112)\\
=& -a^p \frac{\kappa_r}{\kappa_p} + a^r \frac{1}{\kappa_p \kappa_r} \frac{p^2}{l^2} \dot{R} \dot{P} && (113)
\end{aligned}
$$

We can also get expressions for the acceleration by evaluateing the covariant derivatives

$$
\begin{aligned}
a^r =& \nabla_\tau \dot{R} && (114)\\
=& \ddot{R} + \Gamma^r_{MN} u^M u^N && (115)\\
=& \ddot{R} + \frac{2}{p} \dot{P} \dot{R} && (116)\\
a^p =& \nabla_\tau \dot{P} && (117)\\
=& \ddot{P} + \Gamma^p_{MN} u^M u^N && (118)\\
=& \ddot{P} + \frac{1}{2} \kappa_r^2 g'(p) - g(p) \dot{R}^2 \frac{p}{l^2} && (119)
\end{aligned}
$$

We find

$$
\begin{aligned}
K_{\tau\tau} =& -a^p \frac{\kappa_r}{\kappa_p} + a^r \frac{1}{\kappa_p \kappa_r} \frac{p^2}{l^2} \dot{R} \dot{P} && (120)\\
=& -\left(\ddot{P} + \frac{1}{2} \kappa_r^2 g'(p) - g(p) \dot{R}^2 \frac{p}{l^2} \right) \frac{\kappa_r}{\kappa_p} + \left(\ddot{R} + \frac{2}{p} \dot{P} \dot{R} \right) \frac{1}{\kappa_p \kappa_r} \frac{p^2}{l^2} \dot{R} \dot{P} && (121)\\
& && (122)
\end{aligned}
$$

[1]note that for this section p is used interchangeably with P

We can also see that

$$\dot{\kappa}_r = \frac{p^2}{l^2}\frac{\dot{R}}{\kappa_r}\left(\ddot{R} + \frac{\dot{R}\dot{P}}{p}\right) \tag{123}$$

$$\dot{\kappa}_p = \frac{1}{2\kappa_p}\left(2\dot{P}\ddot{P} + \partial_\tau(\kappa_r^2 g(p))\right) \tag{124}$$

$$= \frac{\dot{P}}{\kappa_p}\left(\ddot{P} + \kappa_r^2\frac{1}{2\dot{P}}\partial_\tau g(p) + \frac{1}{2\dot{P}}g(p)\partial_\tau\kappa_r^2\right) \tag{125}$$

$$= \frac{\dot{P}}{\kappa_p}\left(\ddot{P} + \kappa_r^2\frac{1}{2}g'(p) + \frac{1}{2\dot{P}}g(p)\partial_\tau\kappa_r^2\right) \tag{126}$$

We can write this as

$$\ddot{R} = \frac{\kappa_r\dot{\kappa}_r}{\dot{R}}\frac{l^2}{p^2} - \frac{\dot{R}\dot{P}}{p} \tag{127}$$

$$\ddot{P} = \frac{\kappa_p\dot{\kappa}_p}{\dot{P}} - \kappa_r^2\frac{1}{2}g'(p) - \frac{1}{2\dot{P}}g(p)\partial_\tau\kappa_r^2 \tag{128}$$

So we have

$$a^r = \ddot{R} + \frac{2}{p}\dot{P}\dot{R} \tag{129}$$

$$= \frac{\kappa_r\dot{\kappa}_r}{\dot{R}}\frac{l^2}{p^2} + \frac{\dot{P}\dot{R}}{p} \tag{130}$$

$$a^p = \ddot{P} + \frac{1}{2}\kappa_r^2 g'(p) - g(p)\dot{R}^2\frac{p}{l^2} \tag{131}$$

$$= \frac{\kappa_p\dot{\kappa}_p}{\dot{P}} - \kappa_r^2\frac{1}{2}g'(p) - \frac{1}{2\dot{P}}g(p)\partial_\tau\kappa_r^2 + \frac{1}{2}\kappa_r^2 g'(p) - g(p)\dot{R}^2\frac{p}{l^2} \tag{132}$$

$$= \frac{\kappa_p\dot{\kappa}_p}{\dot{P}} - \frac{1}{2\dot{P}}g(p)\partial_\tau\kappa_r^2 - g(p)\dot{R}^2\frac{p}{l^2} \tag{133}$$

Hence, substituting into $K_{\tau\tau}$ we can simplify to

$$\implies K_{\tau\tau} = \frac{\kappa_p\dot{\kappa}_r - \kappa_r\dot{\kappa}_p}{\dot{P}} + \partial_p\kappa_r \tag{134}$$

We are assuming that the only discontinuity between $V_\pm$ is arises from the mass density of the black hole. Namely, in $g_\pm(p)$. Since κ_r is independent of $g_\pm(p)$ it (and it's p derivative) must be continuous across the brane. Hence we have

$$\left[\frac{\kappa_p\dot{\kappa}_r - \kappa_r\dot{\kappa}_p}{\dot{P}}\right] = \frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{135}$$

$$\to \kappa_r^2\partial_\tau\left[\frac{\kappa_p}{\kappa_r}\right] = \dot{P}\frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{136}$$

$$\implies [\dot{n}^p] = \frac{\dot{P}}{\kappa_r^2}\frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{137}$$

Since they agree on $\dot{P}$. And this goes to the expression given in [7] for $\dot{R} \to 0$. We have $\dot{P}$ in terms of $\dot{R}$ and p, so in principle we can solve for $\dot{R}$. Note however that the spatial components of the extrinsic curvature also give an expression for the LHS of this equality

$$[n^p] = p\frac{2}{l}\frac{\sigma(\phi)}{\sigma_c} \tag{138}$$
$$\implies [\dot{n}^p] = \dot{P}\frac{2}{l}\frac{\partial_p(P\sigma(\phi))}{\sigma_c} \tag{139}$$

Hence, to consistently have a solution to the Junction conditions it must be that for $\dot{P} \neq 0$

$$\frac{\sigma(\phi)}{\kappa_r^2} = \partial_p(P\sigma(\phi)) \tag{140}$$
$$\implies \kappa_r^2 = \frac{\sigma(\phi)}{P\sigma'(\phi) + \sigma(\phi)} \tag{141}$$
$$= (P\partial_p \ln \sigma(\phi) + 1)^{-1} \tag{142}$$
$$\dot{R}^2 = \frac{l^2}{p^2}\left(\frac{\sigma(\phi)}{p\sigma'(\phi) + \sigma(\phi)} - 1\right) \tag{143}$$
$$= \frac{l^2}{p^2}\left(\frac{P\sigma'(\phi)}{p\sigma'(\phi) + \sigma(\phi)}\right) \tag{144}$$
$$= \frac{l^2}{p^2}\left(\frac{P\partial_p \ln \sigma(\phi)}{P\partial_p \ln \sigma(\phi) + 1}\right) \tag{145}$$

This implies, $\dot{R} = 0 \iff (\sigma'(\phi) = 0 \text{ or } \dot{P} = 0)$. We define for convenience $\tilde{\sigma}(\phi) = P\partial_p \ln \sigma(\phi)$. So we can write

$$\dot{P}^2 = -2V(P)\kappa_r^2 = -2\frac{V(P)}{\tilde{\sigma}(\phi) + 1} \equiv -2\tilde{V}(P) \tag{146}$$
$$\dot{R}^2 = \frac{l^2}{P^2}\left(\frac{\tilde{\sigma}(\phi)}{\tilde{\sigma}(\phi) + 1}\right) \tag{147}$$

Provided $\tilde{\sigma}(\phi) \neq 0$. Hence we find the profile to be

$$\partial_R P = \frac{\dot{P}}{\dot{R}} \tag{148}$$
$$= \frac{P^2}{l^2}\sqrt{\frac{-2V(P)}{\tilde{\sigma}(\phi)}} \tag{149}$$
$$\implies R(P) - R_0 = \int_{P_0}^{P} \frac{dP'\ P'^2}{l^2}\sqrt{\frac{\tilde{\sigma}(\phi)}{-2V(P')}} \tag{150}$$

Which defines $P(r)$ implicitly. And the speeds in global time

$$(\partial_t P)_\pm = \frac{u^p}{u^t_\pm} = g_\pm(P)\frac{\dot{P}}{\sqrt{\kappa_r^2 g_\pm(P) + \dot{P}^2}} \tag{151}$$
$$(\partial_t P)_\pm = \frac{u^r}{u^t_\pm} = g_\pm(P)\frac{\dot{R}}{\sqrt{\kappa_r^2 g_\pm(P) + \dot{P}^2}} \tag{152}$$

Notice that both vanish at the horizon.

3.4 Conditions for a static brane in the bulk

To have a static brane in the bulk at a position P_s, we need to have the following conditions met.

$$\dot{P}|_{P_s} = 0 \qquad \ddot{P}|_{P_s} = 0 \tag{153}$$

$$\tag{154}$$

Using our above solution for $\dot{P}$, these imply

$$\tilde{V}(P_s) = 0 \qquad \tilde{V}'(P_s) = 0 \tag{155}$$

And finally, to have this be a stable configuration we need

$$\tilde{V}''(P_s) > 0 \tag{156}$$

Notice the first condition is a little unusual. This condition is required because the Junction conditions give what is in effect an equation expressing conservation of energy. One is not free to vary the effective energy of the system as in normal particle mechanics. As pointed out in [1], this is exactly analagous to the Friedmann equations from cosmology. The effective energy, in this case, corresponds to the curvature of the horizon. So for a flat horizon the effective energy of the system is necessarily 0. This in turn will likely imply a high degree of tuning to have a stable minimum in the bulk, analagous to trying to organize the universe's energy content to have a vanishing hubble.

3.5 Effective Action

The junction conditions give equations which are of the form of energy conservation. It is very tempting to construct 1D effective actions for P

$$S_{P,eff} = \int d\tau \left(\frac{1}{2}\dot{P}^2 - \tilde{V}(P)\right) \tag{157}$$

As we showed above, R is then determined by P. This may allow us to compute the tunneling rate through a potential barrier using standard quantum mechanics methods.

3.6 Tunneling Rate

3.6.1 Barrier Transmission

We omit the derivation of the matching conditions need at the classical turning points where the WKB approximation fails and simply write down the final expression for the tunneling probability through a classically forbidden region (which is wide compard to the deBroglie wavelength of the particle) after matching. This approximation is valid to and from points that are far separated from the turning points on either side of the classically forbidden region.

$$|P_T|^2 = \exp\left(-2\text{Im}\int_{\Sigma_f} dxp(x)\right) \tag{158}$$

Where Σ_f is the classically forbidden region of the potential and $p^2(x) = 2m(E - V(x))$ is the classical momentum of the particle.

In [2], the interpretation is a particle initially behind the horizon at a position $z_h(\rho_M) - \epsilon$ falls away from the horizon towards the singularity. As it does so the horizon contracts to a position $z_h(\rho_M - \rho_\omega)$ with the particle ending up just outside the horizon at some position $z_h(\rho_M-\rho_\omega)+\epsilon$. The classically forbidden region is the position of the horizon which over the course of the tunneling is given by $\Sigma_f=[z_h(\rho_M),z_h(\rho_M-\rho_\omega)]$. This is the starting point of the calculation which we now undertake with our adaptation. In particular $z_{in}=z_h(\rho_M)$ and $z_{out}=z_h(\rho_M-\rho_\omega)$

For the tunneling of a plane wave of massless radiation from the inside to the outisde of the horizion, causing the black brane to change from energy density ρ_M to $\rho_M-\rho_\omega$, the classically forbidden region for the particle is the region swept out by the horizon as it contracts $\Sigma_f=[z_h(\rho_M),z_h(\rho_M-\rho_\omega)]$. So the action receives an imaginary contribution

$$
\begin{aligned}
\mathrm{Im}S =&\mathrm{Im}\int_{z_{in}}^{z_{out}} dz p_z && (159)\\
=&\mathrm{Im}\int_{\Sigma_s} d^3x \int_{z_{in}}^{z_{out}} dz P_z && (160)\\
=&\mathrm{Im}\int_{\Sigma_s} d^3x \int_{z_{in}}^{z_{out}} dz \int_0^{P_z} dP'_z && (161)\\
=&\mathrm{Im}\int_{\Sigma_s} d^3x \int_{z_{in}}^{z_{out}} dz \int_0^{\rho_\omega} \frac{d\rho'_\omega}{\dot{z}} && (162)
\end{aligned}
$$

Where z_{out} and z_{in} are coordinates the horizon before and after emission respectively with $z_{in}<z_{out}$ (so are the boundary of the classically forbidden region that the particle traverses), x parametrizes the space orthogonal to the z direction, P_z is the canonically conjugate momentum density to z of the radiation (where we all other momenta must vanish by 3 dimensional isotropy, and we ignore possible radial momentum), Σ_s the shell region, and ω is the Hamiltonian density for the particle. In the last line we have used the Hamiltonian equations of motion to change the measure.

$$
\begin{aligned}
\dot{z} =&\frac{d\rho_\omega}{dP_z} && (163)\\
\implies d\rho_\omega =&\frac{d\rho_\omega}{dP_z}dP_z=\dot{z}dP_z && (164)\\
\implies dP'_z =&\frac{d\rho'_\omega}{\dot{z}} && (165)
\end{aligned}
$$

We assume that the argument given by [8] holds also for the AdS_5 black brane (which we look to check in later sections). This argument states that the shell of energy density ρ_ω propagates in an black brane background with energy density $\rho_M-\rho_\omega$.

3.7 Brane Emission Corrected - global time

We recall that [2]

$$
z_h^{-4}=\frac{\rho_M}{A}=\frac{4\rho_M}{l^4\sigma_c} \qquad (166)
$$

Hence we can write, using $\rho'_M=\rho_M-\rho'_\omega$

[2]These calculations were done before the full profile computation, and so are in conformal coordinates. It's on my TODOs to swap coordinates.

$$
\begin{aligned}
P_z &= - \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} \int_0^{\rho_\omega} \frac{dz_0 d\rho_\omega}{\partial_t z} &(167)\\
&= - \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \int_0^{\rho_\omega} d\rho'_\omega \frac{1}{f(z)} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} &(168)\\
&= - \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \int_0^{\rho_\omega} d\rho'_\omega \frac{1}{1 - \frac{z_0^4}{A}(\rho_M - \rho'_\omega)} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} &(169)\\
&= - \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \int_0^{\rho_\omega} d\rho'_\omega \frac{1}{\frac{A}{z_0^4} - \rho_M + \rho'_\omega} \frac{A}{z_0^4} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} &(170)
\end{aligned}
$$

As we argued above, the integrand has a simple pole at the horizon. Since $\dot{z}$ is non-vanishing there. Further, the z_0 integral will always be over the simple pole. Hence we evaluate the imaginary part using the prescription $\rho'_\omega \to \rho'_\omega - i\epsilon$

$$
\begin{aligned}
\text{Im}P_z &= - \text{Im} \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \int_0^{\rho_\omega} d\rho'_\omega \frac{1}{\frac{A}{z_0^4} - \rho_M + \rho'_\omega - i\epsilon} \frac{A}{z_0^4} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} &(171)\\
&= - \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \int_0^{\rho_\omega} d\rho'_\omega \pi \delta \left(\frac{A}{z_0^4} - \rho_M + \rho'_\omega \right) \frac{A}{z_0^4} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} &(172)\\
&= - \pi \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \left[\frac{A}{z_0^4} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} \right]_{\rho'_\omega = \rho_M - \frac{A}{z_0^4}} &(173)
\end{aligned}
$$

Where we've used the Cauchy Dirac relation. We can reduce this expression further as

$$
f(z_0)|_{\rho'_\omega = \rho_M - \frac{A}{z_0^4}} = \left[1 - \frac{z_0^4}{A}(\rho_M - \rho'_\omega) \right]_{\rho'_\omega = \rho_M - \frac{A}{z_0^4}} = 0 \qquad (174)
$$

$$
(175)
$$

So we can write the above expression as

$$
\begin{aligned}
\text{Im}P_z &= - \pi \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \left[\frac{A}{z_0^4} \frac{\sqrt{\frac{z^2}{l^2} f(z) + \dot{z}^2}}{\dot{z}} \right]_{\rho'_\omega = \rho_M - \frac{A}{z_0^4}} &(176)\\
&= - \text{sgn}(\dot{z}) \pi \int_{z_h(\rho_M)}^{z_h(\rho_M-\rho_\omega)} dz_0 \frac{A}{z_0^4} &(177)\\
&= - \text{sgn}(\dot{z}) \frac{\pi}{3} A \left[\frac{1}{z_h^3(\rho_M)} - \frac{1}{z_h^3(\rho_M - \rho_\omega)} \right] &(178)\\
&= - \text{sgn}(\dot{z}) \frac{\pi}{3} A \frac{1}{z_h^3(\rho_M)} \left[1 - \left(1 - \frac{\rho_\omega}{\rho_M} \right)^{3/4} \right] &(179)\\
&= - \text{sgn}(\dot{z}) \frac{\pi}{3} z_h \rho_M \left[1 - \left(1 - \frac{\rho_\omega}{\rho_M} \right)^{3/4} \right] &(180)
\end{aligned}
$$

Hence the tunneling Probability is given by

$$|P_t|^2 = \exp\left\{ \operatorname{sgn}(\dot{z}) \mathrm{Vol}(\Sigma_s) \frac{2\pi}{3} z_h \rho_M \left[1 - \left(1 - \frac{\rho_\omega}{\rho_M} \right)^{3/4} \right] \right\} \tag{181}$$

We then take the negative sign. Which corresponds to motion away from the horizon. Interestingly however in the low energy limit

$$|P_t|^2 \to \exp\left(-\frac{\pi}{2} z_h \omega \right) \tag{182}$$

Which yields a temperature

$$\beta = \frac{\pi}{2} z_h \tag{183}$$

This is off by a factor of 2 from the expected Hawking temperature. This can likely be attributed to an issue with canonical invariance in this approach which was pointed out in eg. [9] [7].

4 A 2 Brane Scenario for a Realistic and Natural Cosmology

If we find that we cannot construct a stable minimum in the bulk, an alternative idea is to allow the IR brane to propagate and construct a realistic cosmology. In order for this construction to be of any use, we would then need to have another brane with a stable seperation from the first which would allow us to construct our Planck EW hierarchy. Stabilizing the seperation of the branes would be the main challenge of the construction.

4.1 Seperation Stabilization

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5 Appendices

5.1 A - Geometry of Co-Dimension 1 Surfaces

Consider a spacetime manifold V with a codimension 1 submanifold Σ with a normal vector $\boldsymbol{n}$ which may be either spacelike or timelike, and whose norm has magnitude 1.

$$\boldsymbol{n}\cdot\boldsymbol{n}=\epsilon(\boldsymbol{n})=\pm 1 \tag{184}$$

The extrinsic curvature is given by the projection of the derivative of $\boldsymbol{n}$, a vector normal to the surface, onto the surface with tangent vectors $\boldsymbol{e}_{(\mu)}$. We are concerned with co-dimension 1 surfaces, so these vectors span our spacetime.

$$\partial_\mu\boldsymbol{n}=K_\mu^\lambda\boldsymbol{e}_{(\lambda)} \tag{185}$$

Giving

$$\begin{aligned}
K_{\mu\nu}=&K_\mu^\lambda\boldsymbol{e}_{(\nu)}\cdot\boldsymbol{e}_{(\lambda)} &(186)\\
=&\boldsymbol{e}_{(\nu)}\cdot\partial_\mu\boldsymbol{n} &(187)\\
=&g_{\nu\lambda}\nabla_\mu n^\lambda &(188)\\
=&\nabla_\mu n_\nu &(189)
\end{aligned}$$

Since $g_{\mu\nu}=\boldsymbol{e}_{(\mu)}\cdot\boldsymbol{e}_{(\nu)}$ and using normality. From this we see that [3] since the vectors are orthogonal

$$\begin{aligned}
K_{\mu\nu}=&\boldsymbol{e}_{(\nu)}\cdot\partial_\mu\boldsymbol{n} &(190)\\
=&-\boldsymbol{n}\cdot\partial_\mu\boldsymbol{e}_{(\nu)} &(191)\\
=&-\boldsymbol{n}\cdot\partial_\nu\boldsymbol{e}_{(\mu)} &(192)\\
=&K_{\nu\mu} &(193)
\end{aligned}$$

Where we've used

$$e^M_{(\mu)}=\partial_\mu X^M \tag{194}$$

and the symmetry of mixed partials to swap indices in second to last line. So we see that K is symmetric. Then consider a field in Σ

$$A_\mu=\boldsymbol{A}\cdot\boldsymbol{e}_{(\mu)},\qquad \boldsymbol{A}=A^i\boldsymbol{e}_{(i)} \tag{195}$$

Then the components of the covariant derivative projected onto the tangent space of Σ are given by

$$\begin{aligned}
\partial_\nu A_\mu=&\partial_\nu(\boldsymbol{A}\cdot\boldsymbol{e}_{(\mu)}) &(196)\\
=&\boldsymbol{e}_{(\mu)}\cdot\partial_\nu\boldsymbol{A}+\boldsymbol{A}\cdot\partial_\nu\boldsymbol{e}_{(\mu)} &(197)\\
\implies A_{\mu;\nu}\equiv&\boldsymbol{e}_{(\mu)}\cdot\partial_\nu\boldsymbol{A} &(198)\\
=&\partial_\nu A_\mu-\boldsymbol{A}\cdot\partial_\nu\boldsymbol{e}_{(\mu)} &(199)\\
=&\partial_\nu A_\mu-A^\lambda\boldsymbol{e}_{(\lambda)}\cdot\partial_\nu\boldsymbol{e}_{(\mu)} &(200)\\
=&\partial_\nu A_\mu-A^\lambda\Gamma_{\lambda,\nu\mu} &(201)\\
=&\partial_\nu A_\mu-A_\lambda\Gamma^\lambda_{\nu\mu} &(202)
\end{aligned}$$

Further, we find the Gauss-Weingarten equations

$$\partial_\mu \boldsymbol{e}_{(\nu)} = \boldsymbol{n}(\boldsymbol{n} \cdot \partial_\mu \boldsymbol{e}_{(\nu)}) + \boldsymbol{e}_{(\lambda)}(\boldsymbol{e}^{(\lambda)} \cdot \partial_\mu \boldsymbol{e}_{(\nu)}) \tag{203}$$

$$= -\boldsymbol{n}\epsilon(\boldsymbol{n})K_{\mu\nu} + \boldsymbol{e}_{(\lambda)}\Gamma^\lambda_{\mu\nu} \tag{204}$$

$$\tag{205}$$

Where we have used the fact that Σ is codimension 1 and so the union of it with the set of tangent vectors span V. And introduced the cotangent basis vectors $\boldsymbol{e}^{(\mu)}$. Hence

$$\partial_\mu \boldsymbol{A} = \boldsymbol{e}_{(\nu)} A^\nu{}_{;\mu} - \boldsymbol{n}\epsilon(\boldsymbol{n}) A^\nu K_{\mu\nu} \tag{206}$$

We will not derive it, but will simply note that from these relations one can derive as [3] did the Gauss Codazzi equations. From these equations and noting that the metric induced on Σ ($g^{\mu\nu}$) by g^{MN} is given by

$$g^{\mu\nu} e^M_{(\mu)} e^N_{(\nu)} = g^{MN} - \epsilon(\boldsymbol{n}) n^M n^N \tag{207}$$

they go on to compute the Einstein tensor [3]

$$-2\epsilon(\boldsymbol{n}) G_{MN} n^M n^N = R_\Sigma - \epsilon(\boldsymbol{n})(K_{\mu\nu}K^{\mu\nu} - K^2) \tag{208}$$

$$G_{MN} e^M_{(\mu)} n^N = K_\mu{}^\nu{}_{;\nu} - K_{;\mu} \tag{209}$$

$$G_{MN} e^M_{(\mu)} e^N_{(\nu)} = G^\Sigma_{\mu\nu} - n^J \partial_J (K_{\mu\nu} - g_{\mu\nu} K) - K K_{\mu\nu} + \frac{1}{2}\left(K_{\mu\nu}K^{\mu\nu} + K^2\right) \tag{210}$$

Where $K = K_{\mu\nu} g^{\mu\nu}$. They then go on to note that given a 2 tensor $S^{\mu\nu}$ defined on Σ one can associate a discontinuous tensor in V as

$$S^{MN} = S^{\mu\nu} e^M_{(\mu)} e^N_{(\nu)} \text{ on } \Sigma, \qquad S^{MN} = 0 \text{ off } \Sigma \tag{211}$$

from which we find

$$\nabla_M S^{NM} = e^N_{(\mu)} S^{\mu\nu}_{;\nu} - \epsilon(\boldsymbol{n}) S^{\mu\nu} K_{\mu\nu} n^N \tag{212}$$

where $S^{\mu\nu}_{;\lambda}$ denotes the covariant derivative of $S^{\mu\nu}$ in Σ.

5.2 B - An Aside on Integration

Here we construct objects which allow us to conveniently work in a coordinate independent way. These objects were introduced formally in [4]. We introduce a function from coordinates in Σ to bulk coordinates in V.

$$X = X_\Sigma(x) \tag{213}$$

Define

$$\delta_\Sigma(X) = \int_\Sigma d^4x \sqrt{-g_{\text{ind}}} \frac{1}{\sqrt{-g}} \delta^5(X - X_\Sigma(x)) \tag{214}$$

[3] The last equation we take from [5]

This expression is a scalar under diffeomorphoisms which follows from the fact that $d^5X\delta^5(X)$ is, which in turn implies $\frac{1}{\sqrt{-g}}\delta^5(X)$ is as well. This object will allow us to change between integration over V and integration over Σ. Notice

$$\int_V d^5X\sqrt{-g}\delta_\Sigma(X)h(X) = \int_V d^5X\sqrt{-g}\ h(X)\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\frac{1}{\sqrt{-g}}\delta^5(X - X_\Sigma(x)) \tag{215}$$

$$= \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\int_V d^5Xh(X)\delta^5(X - X_\Sigma(x)) \tag{216}$$

$$= \int_\Sigma d^4x\sqrt{-g_{\text{ind}}}h(X_\Sigma(x)) \tag{217}$$

We now look to define a scalar function $\Theta_\pm(X)$ so that

$$\partial_M\Theta_\pm(X) = n_M\delta_\Sigma(X) \tag{218}$$

$$\Theta_\pm(X)|_{V_\pm} = 1 \tag{219}$$

$$\Theta_\pm(X)|_{V_\mp} = 0 \tag{220}$$

To determine $\Theta_\pm$ we integrate over a contour $C_\pm(X)$ with endpoints at X and in $V_\pm$ and parametrized as $\gamma(\tau)$ where $d\gamma = dX^M n_M = \boldsymbol{n}$

$$\int_{C_\pm(X)} dX'^M\partial_M\Theta_\pm(X) = \Theta_\pm(X)|_{\partial C_\pm(X)} \tag{221}$$

$$= \int_{C_\pm(X)} dX'^M n_M\delta_\Sigma(X') \tag{222}$$

$$= \int_{C_\pm(X)} d\gamma\delta_\Sigma(\gamma) \tag{223}$$

$$= \int_{C_\pm(X)} d\tau\gamma'(\tau)\int_\Sigma d^4x\sqrt{-g_{\text{ind}}}\frac{1}{\sqrt{-g}}\delta^5(\gamma(\tau) - X_\Sigma(x)) \tag{224}$$

$$= \int_{C_\pm(X)\times\Sigma} \frac{d\tau\gamma'(\tau)\wedge d^4x\sqrt{-g_{\text{ind}}}}{\sqrt{-g}}\delta^5(\gamma(\tau) - X_\Sigma(x)) \tag{225}$$

Noting that $d\gamma\wedge d^4x\sqrt{-g_{\text{ind}}} = \boldsymbol{n}\wedge\boldsymbol{e}^{(0)}\wedge\boldsymbol{e}^{(1)}\wedge\boldsymbol{e}^{(2)}\wedge\boldsymbol{e}^{(3)} = d^5X\sqrt{-g}$, we see that if $C_\pm(X)$ intersects Σ, then $\Theta_\pm(X)|_{\partial C_\pm(X)}$ is 1 otherwise it is 0. So taking

$$\Theta_\pm(X) = \int_{C_\pm(X)} d\gamma\delta_\Sigma(\gamma) \tag{226}$$

satisfies the above equations (evaluating $\Theta_\pm$ on the lower boundary of the contour gives 0 / integration over a set of measure 0), and gives desired properties for $\Theta_\pm$.

5.3 C - Einstein's Equations a la Israel

We now look consider the normal components of EFEs in the presence of a bulk EMT. From [4], these will impose constraints on the induced metric and extrinsic curvature. On either side of Σ

$$\kappa T^\pm_{MN}n^Mn^N = G^\pm_{MN}n^Mn^N = \frac{-2}{\epsilon(\boldsymbol{n})}\left[R_\Sigma - \epsilon(\boldsymbol{n})(K^\pm_{\mu\nu}K^{\pm\mu\nu} - K^{\pm 2})\right] \tag{227}$$

$$\kappa T^\pm_{MN}e^M_{(\mu)}n^N = G^\pm_{MN}e^M_{(\mu)}n^N = K^{\pm\nu}_{\mu\ ;\nu} - K^\pm_{;\mu} \tag{228}$$

Where we have used the fact that Israel's first junction condition implies that observers in $V_\pm$ agree on R_Σ, the intrinsic curvature in Σ. Adding and subtracting the second of these equations we find

$$\kappa[T_{MN}n^N]e^M_{(\mu)} = \gamma^{\pm\nu}_{\mu\ ;\nu} - \gamma^{\pm}_{;\mu} = (\gamma^\nu_\mu - \delta^\nu_\mu\gamma)_{;\nu} = -\kappa S^\nu_{\mu;\nu} \tag{229}$$

$$\implies [T_{MN}n^N]e^M_{(\mu)} = -S^\nu_{\mu;\nu} \tag{230}$$

Where $[f] \equiv f^+ - f^-$. Adding them we find

$$\tilde{T}_{Mn}e^M_{(\mu)} = \tilde{K}_\mu{}^\nu{}_{;\nu} - \tilde{K}_{;\mu} \tag{231}$$

Where $\tilde{f} \equiv \frac{1}{2}(f^+ + f^-)$ and $f_n \equiv f_M n^M$. Doing the same with the first equation we have

$$\kappa[T^\pm_{nn}] = 2\left[K_{\mu\nu}K^{\mu\nu} - K^2\right] \tag{232}$$

$$= 2\left(K^+_{\mu\nu}K^{+\mu\nu} - K^{+2} - (K^-_{\mu\nu}K^{-\mu\nu} - K^{-2})\right) \tag{233}$$

Now notice

$$\tilde{K}_{\mu\nu}S^{\mu\nu} = \frac{-1}{2\kappa}(K^+_{\mu\nu} + K^-_{\mu\nu})(\gamma^{\mu\nu} - g^{\mu\nu}\gamma) \tag{234}$$

$$= \frac{-1}{2\kappa}(K^+_{\mu\nu}K^{+\mu\nu} - K^-_{\mu\nu}K^{-\mu\nu} - K^{+2} + K^{-2}) \tag{235}$$

$$= \frac{-1}{2\kappa}\left[K_{\mu\nu}K^{\mu\nu} - K^2\right] \tag{236}$$

Where we have used again Israel's first junction condition to equate the induced metric on either side of Σ. So we have

$$[T^\pm_{nn}] = -4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{237}$$

Finally, adding we have

$$\kappa\tilde{T}_{MN}n^M n^N = \frac{-2}{\epsilon(\boldsymbol{n})}R_\Sigma - (K^+_{\mu\nu}K^{+\mu\nu} - K^{+2}) - (K^-_{\mu\nu}K^{-\mu\nu} - K^{-2}) \tag{238}$$

Drudging through the details and using

$$K^\pm_{\mu\nu} = \tilde{K}_{\mu\nu} \pm \frac{1}{2}\gamma_{\mu\nu} \tag{239}$$

we have

$$K^\pm_{\mu\nu}K^{\pm\mu\nu} = \left(\tilde{K}_{\mu\nu} \pm \frac{1}{2}\gamma_{\mu\nu}\right)\left(\tilde{K}^{\mu\nu} \pm \frac{1}{2}\gamma^{\mu\nu}\right) \tag{240}$$

$$= \tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} \pm \gamma_{\mu\nu}\tilde{K}^{\mu\nu} + \frac{1}{4}\gamma_{\mu\nu}\gamma^{\mu\nu} \tag{241}$$

$$K^{\pm 2} = \left(\tilde{K} \pm \frac{1}{2}\gamma\right)^2 \tag{242}$$

$$= \tilde{K}^2 \pm K\gamma + \frac{1}{4}\gamma^2 \tag{243}$$

Hence

$$\kappa\tilde{T}_{MN}n^M n^N = \frac{-2}{\epsilon(\boldsymbol{n})}R_\Sigma - 2\left(\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} + \frac{1}{4}\gamma_{\mu\nu}\gamma^{\mu\nu} - \tilde{K}^2 - \frac{1}{4}\gamma^2\right) \tag{244}$$

Which can be put in terms of S by using

$$\gamma_{\mu\nu} = -\kappa\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1}g_{\mu\nu}S\right) \tag{245}$$

$$\implies \gamma_{\mu\nu}\gamma^{\mu\nu} = \kappa^2\left(S_{\mu\nu} - \frac{1}{D_\Sigma - 1}g_{\mu\nu}S\right)\left(S^{\mu\nu} - \frac{1}{D_\Sigma - 1}g^{\mu\nu}S\right) \tag{246}$$

$$= \kappa^2\left(S_{\mu\nu}S^{\mu\nu} - \frac{2}{D_\Sigma - 1}S^2 + \frac{D_\Sigma}{(D_\Sigma - 1)^2}S^2\right) \tag{247}$$

$$= \kappa^2\left[S_{\mu\nu}S^{\mu\nu} + \left(\frac{1}{(D_\Sigma - 1)^2} - \frac{1}{D_\Sigma - 1}\right)S^2\right] \tag{248}$$

$$\implies \gamma^2 = \kappa^2\left(S - \frac{D_\Sigma}{D_\Sigma - 1}S\right)^2 = \kappa^2\frac{S^2}{(D_\Sigma - 1)^2} \tag{249}$$

$$\implies \gamma_{\mu\nu}\gamma^{\mu\nu} - \gamma^2 = \kappa^2\left(S_{\mu\nu}S^{\mu\nu} - \frac{1}{D_\Sigma - 1}S^2\right) \tag{250}$$

Hence

$$-\frac{\kappa}{2}\tilde{T}_{MN}n^M n^N - \frac{1}{\epsilon(\boldsymbol{n})}R_\Sigma = (\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} - \tilde{K}^2) + \frac{\kappa^2}{4}\left(S_{\mu\nu}S^{\mu\nu} - \frac{1}{D_\Sigma - 1}S^2\right) \tag{251}$$

So our conditions are

$$[T_{Mn}]e^M_{(\mu)} = -S_{\mu;\nu}^{\nu} \tag{252}$$

$$\tilde{T}_{Mn}e^M_{(\mu)} = \tilde{K}_\mu{}^\nu{}_{;\nu} - \tilde{K}_{;\mu} \tag{253}$$

$$[T^\pm_{nn}] = -4\tilde{K}_{\mu\nu}S^{\mu\nu} \tag{254}$$

$$-\frac{\kappa}{2}\tilde{T}_{nn} - \frac{1}{\epsilon(\boldsymbol{n})}R_\Sigma = (\tilde{K}_{\mu\nu}\tilde{K}^{\mu\nu} - \tilde{K}^2) + \frac{\kappa^2}{4}\left(S_{\mu\nu}S^{\mu\nu} - \frac{1}{D_\Sigma - 1}S^2\right) \tag{255}$$

Only the bracket expressions will impose consraints, as the other two will be satisfied provided we have solutions to the bulk EFE on either side of Σ.