aDM Energy Loss Explanatory Notes - From the Start

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Abstract

These notes aim to explain how to derive the transition rate and Energy Loss per unit length of a dark matter fermion, charged under a gauged U(1) which is kinetically mixed with the SM photon, due to interactions with SM material. We use Fermi's golden rule to derive an expression for the transition rate and energy loss in terms of the scattering (Coulomb) potential and the response of the material (the Dynamical Structure Function). We define the complex dielectric function (Longitudinal photon polarization) and show how it can be related to the dynamical structure function via the fluctuation dissipation theorem. We then compute the Structure function for the Free Electron Gas at finite temperature giving the energy loss per unit length for the dark matter particle. We show that throughout the earth, the degenerate limit is an excellent approximation. This limit yields the Lindhard Dielectric Function. We then extend this to the Mermin model which assumes the electrons collisions cause the excitations of the system to decay with a fixed lifetime towards the system's equilibrium configuration (the Random Time Approximation).

1 Energy Loss Per Unit Length - Master Formula

We are interested in the scattering of a single non-relativistic particle off of an unspecified system. In the absence of a perturbing interaction, the whole system is described by energy eigenstates of a free Hamiltonian

$$H_0|m\rangle = E_m|m\rangle \tag{1}$$

We then consider the tensor product of this system with a perturbing probe, an incoming DM particle in a momentum and spin eigenstate $|\boldsymbol{p}, \sigma\rangle$[1], and take the system to initially be in a Grand Canonical ensemble at finite temperature.

$$|i\rangle = \sum_m c_m|\boldsymbol{p}, \sigma\rangle \otimes |m\rangle \tag{2}$$

with c_m the amplitude for the microstate m in the ensemble. Following [1], [2] we can write the differential transition rate for the interaction using Fermi's golden rule as

$$\begin{aligned} dP =& \sum_f d\Gamma_{i\to f} && (3)\\ =& \sum_f |\langle f|H_{int}|i\rangle|^2 \frac{2\pi}{\hbar} D(E) && (4)\\ =& \frac{2\pi}{\hbar}\sum_{p'}\sum_{\sigma'}\sum_{m,n} P_m|\langle \boldsymbol{p'}, \sigma', n|H_{int}|\boldsymbol{p}, \sigma, m\rangle|^2 \delta(E_f^T - E_i^T) && (5)\end{aligned}$$

Where E^T denotes total energy of the combined system, V is the total system volume, $P_m = |c_m|^2$ is the m^{th} eigenvalue of the density operator for the ensemble given by $P_m = e^{-\beta(E_m-\mu)}Z^{-1}$, and $D(E)$ is

[1] bold faced characters denote 3-vectors

the density of states. We also neglect spin dependent interactions, namely we assume that H_{int} is equal to the identity when projected onto the space of spin states of the DM particle. This also leverages the fact that at leading order in the interaction, the state of the system is described by the eigenstates of the free Hamiltonian (the perturbation of the eigenstates in (30) appear at higher order in H_{int}). Hence

$$dP = \frac{2\pi}{\hbar} \sum_{p'} \sum_{m,n} P_m |\langle \boldsymbol{p}', n | H_{int} | \boldsymbol{p}, m \rangle|^2 \delta(E_f^T - E_i^T) \tag{6}$$

We can also write for the non-relativistic particle

$$\begin{aligned}
\hbar\omega_{\boldsymbol{q}} \equiv& E_i^\chi - E_f^\chi &(7)\\
=& \frac{1}{2} m_\chi v^2 - \frac{(m_\chi \boldsymbol{v} - \hbar \boldsymbol{q})^2}{2m_\chi} &(8)\\
=& \hbar \boldsymbol{q} \cdot \boldsymbol{v} - \frac{\hbar^2 q^2}{2m_\chi} &(9)\\
\implies E_f^T - E_i^T =& E_f - E_i - \hbar\omega_{\boldsymbol{q}} &(10)\\
=& E_n - E_m - \hbar\omega_{\boldsymbol{q}} &(11)
\end{aligned}$$

Where $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{q}$ with $\boldsymbol{q}$ the momentum transfer from the probe to the system, E^χ, E denote the Dark Matter and System energies respectively, and $E_{n,m}$ are the energies of states $|n, m\rangle$. We take the following fourier conventions appropriate for finite volume.

$$\begin{aligned}
\frac{1}{V}\int d^3x e^{iqx} e^{-ipx} =& \delta_{p,q} & \frac{1}{V}\sum_{\boldsymbol{q}} e^{-iq(x-x')} =& \delta(x - x') &(12)\\
\int d^3x e^{iqx} f(x) \equiv& f(q) & \implies \frac{1}{V}\sum_{\boldsymbol{q}} e^{-iqx} f(q) =& \sum_{\boldsymbol{q}} e^{-iqx} \frac{1}{V} \int d^3x' e^{iqx'} f(x') &(13)\\
&& =& \int d^3x' \delta(x - x') f(x') &(14)\\
&& =& f(x) &(15)
\end{aligned}$$

Note that these match the theoretical framework paper [1], but don't match [3] or Peskin and Schroeder. With these definitions, moving to the continuum proceeds as

$$\begin{aligned}
\frac{1}{V}\sum_{\boldsymbol{q}} \to& \int \frac{d^3q}{(2\pi)^3} &(16)\\
V\delta_{\boldsymbol{k},\boldsymbol{q}} \to& (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{q}) &(17)
\end{aligned}$$

We now take the interaction Hamiltonian to be a local potential for the interaction of the DM probe with the charges in the system, and factor the matrix element in the sum. In momentum space our Hamiltonian is [1]

$$\begin{aligned}
\langle \boldsymbol{p}', n | \hat{H}_{int} | \boldsymbol{p}, m \rangle =& \frac{1}{V} \int d^3x e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \langle \boldsymbol{x}', n | \hat{V}_{int}(\boldsymbol{x}) | \boldsymbol{x}, m \rangle &(18)\\
=& \frac{1}{V} \langle n | V_{int}(\boldsymbol{q}) | m \rangle &(19)\\
&&(20)
\end{aligned}$$

The Interaction Hamiltonian is the convolution of the charge density operator for the system with the Coulomb potential at the location of the DM probe.

$$
\begin{aligned}
V_{int}(\boldsymbol{q}) &= \int d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{x}} V_{int}(\boldsymbol{x}) && (21)\\
&= \int d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \int d^3x' V(\boldsymbol{x}-\boldsymbol{x}')\rho(\boldsymbol{x}') && (22)\\
&= \frac{1}{V^2}\int d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \int d^3x' \sum_{\boldsymbol{k}} V(k) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \sum_{\boldsymbol{p}} \rho_p e^{i\boldsymbol{p}\cdot\boldsymbol{x}'} && (23)\\
&= \frac{1}{V^2}\sum_{\boldsymbol{k}} V(k) \sum_{\boldsymbol{p}} \rho_p \int d^3x' e^{i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{x}'} \int d^3x e^{i(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{x}} && (24)\\
&= \sum_{\boldsymbol{k}} V(k) \sum_{\boldsymbol{p}} \rho_p \delta_{\boldsymbol{p},-\boldsymbol{k}}\delta_{\boldsymbol{q},\boldsymbol{k}} && (25)\\
&= \sum_{\boldsymbol{k}} V(k)\rho_{-k}\delta_{\boldsymbol{q},\boldsymbol{k}} && (26)\\
&= V(q)\rho_{-q} && (27)\\
&= \frac{4\pi\kappa e^2}{q^2}\rho_{-q} && (28)
\end{aligned}
$$

Note that since ρ has dimensions of $[L^{-3}]$, ρ_q is dimensionless, and hence $V_{int}(\boldsymbol{q})$ has dimensions $[L^3E]$. Here κ is defined to be the ratio of the Dark and SM charges times the dimensionless kinetic mixing between the SM and DM photon.

$$
\kappa \equiv \frac{e_D}{e}\epsilon_{mixing} \tag{29}
$$

Changing the momentum summation variable, the transition rate is then

$$
dP = \frac{1}{V^2}\sum_q V^2(q)\frac{2\pi}{\hbar}\sum_{m,n} P_m |\langle n|\rho_{-\boldsymbol{q}}|m\rangle|^2 \delta(E_n - E_m - \hbar\omega_{\boldsymbol{q}}) \tag{30}
$$

We now introduce an auxiliary variable for the energy transfer and define the Dynamical Structure Function (or Structure Factor)

$$
\begin{aligned}
dP &= \frac{1}{V^2}\sum_{\boldsymbol{q}} d\omega\delta(\omega-\omega_{\boldsymbol{q}}) V^2(q)\frac{2\pi}{\hbar}\sum_{m,n} P_m |\langle n|\rho_{-\boldsymbol{q}}|m\rangle|^2 \delta(E_n - E_m - \hbar\omega) && (31)\\
&\equiv \frac{1}{V}\sum_{\boldsymbol{q}} \frac{d\omega}{\hbar}\delta(\omega-\omega_{\boldsymbol{q}}) V^2(q) S(\boldsymbol{q},\omega) && (32)
\end{aligned}
$$

Where

$$
S(\boldsymbol{q},\omega) = \frac{2\pi}{V}\sum_{m,n} P_m |\langle n|\rho_{-\boldsymbol{q}}|m\rangle|^2 \delta(E_n - E_m - \hbar\omega) \tag{33}
$$

Where we use the energy labels f (i) and n (m) interchangeably. and We can now move to the contiuum in q, assuming an isotropic medium, to determine the transition rate as a function of ω

$$dP(\omega) = d\omega \int \frac{d^3q}{\hbar(2\pi)^3}\delta\left(\omega - \boldsymbol{q}\cdot\boldsymbol{v} + \frac{\hbar q^2}{2m_\chi}\right)V^2(q)S(q,\omega) \tag{34}$$

$$= d\omega \int \frac{dq\ q^2}{\hbar(2\pi)^3}V^2(q)S(q,\omega)\int d\Omega\delta\left(\omega - \boldsymbol{q}\cdot\boldsymbol{v} + \frac{\hbar q^2}{2m_\chi}\right) \tag{35}$$

$$= d\omega \int \frac{dq\ q^2}{\hbar(2\pi)^2}V^2(q)S(q,\omega)\int_{-1}^{1} du\delta\left(\omega - qvu + \frac{\hbar q^2}{2m_\chi}\right) \tag{36}$$

$$= d\omega \int \frac{dq\ q^2}{\hbar(2\pi)^2}V^2(q)S(q,\omega)\frac{1}{qv}\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{37}$$

***** compare to [4] A.8, there is a BE distribution enhancement for the mediator.

Where $\omega_\pm = qv \pm \frac{\hbar q^2}{2m_\chi}$. We can then use this to determine the *average* energy transferred from the probe to the system per unit length

$$\frac{dE}{dr}(v) = \frac{1}{v}\frac{dE}{dt}(v) = \frac{1}{v}\int dP(\omega)\hbar\omega = \int \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)\int d\omega\ \omega S(q,\omega)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{38}$$

Notice that the Structure Function depends only on the properties of the material, so the energy transferred per unit length scales quadratically with the kinetic mixing. Further the mass of the DM probe appears only in the Theta function. We can simplify this expression by restricting the domain of integration to the support of the theta functions.

$$\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \neq 0 \implies \qquad -\omega_+ \leq \omega \leq \omega_- \tag{39}$$

$$-qv - \frac{\hbar q^2}{2m_\chi} \leq \omega \leq qv - \frac{\hbar q^2}{2m_\chi} \tag{40}$$

So the average energy loss becomes

$$\frac{dE}{dr}(v) = \int_0^\infty \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)\int_{-\omega_+}^{\omega_-} d\omega\ \omega S(q,\omega) \tag{41}$$

We also note that the average energy loss (energy loss due to soft multiple scatters) is given by [5]

$$\frac{dE}{dr}(v) = \hbar n_T \int d\omega\ \omega \frac{d\sigma}{d\omega} \tag{42}$$

$$= \int \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)\int d\omega\ \omega S(q,\omega)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{43}$$

$$= \hbar n_T \int d\omega\ \omega \frac{1}{\hbar n_T}\int \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)S(q,\omega)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{44}$$

Where n_T is the target number density, and we have identified the recoil energy (energy transfered to the probe from the system) as $E_R = -\hbar\omega$ comparing to their notations. Hence we can identify

$$\frac{d\sigma}{dE_R}(v,\omega) = \frac{1}{\hbar}\frac{d\sigma}{d\omega} \tag{45}$$

$$= \frac{1}{n_T\hbar^2}\int \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)S(q,\omega)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{46}$$

$$= \frac{1}{vn_T\hbar}\frac{dP}{d\omega} \tag{47}$$

2 Dielectric Function and the Fluctuation Dissipation Theorem

As in classical EM, the complex dielectric function characterizes the modification of the potential due to the motion of free charges in the material in response to an applied field (in this case from the charged DM particle). One might expect that the modification of the potential (through the dielectric function) would be related to the transition rate (through the structure function). And one would be right! We now follow [6] and derive the precise relationship between the Complex Dielectric Function and the Dynamical Structure Function. This is the well known general fluctuation dissipation theorem, which follows from the optical theorem. We then evaluate the dielectric function in a free electron gas, and the Mermin model as first approximations to the energy loss of our kinetically mixed Dark Matter particle.

2.1 Validity of this approach to our problem

Our expression for the energy loss is only the average. This assumes that the energy loss is small compared to the energy of the particle, so that it can be treated as a classical particle of constant velocity driving the system with a fixed frequency. This is valid in the regime where multiple soft scatters dominate. This approach is taken in [7] [3] where it is quoted as the condition that the particle be heavy and fast compared to the charges in the material, keeping in mind a proton or ion.

Note however, that our expression for the transition rate as a function of ω is valid everywhere in parameter space provided the interaction is weak enough that it can be treated at leading order in perturbation theory (with Fermi's golden rule). So in regimes where soft scattering fails to dominate, we can use the transition rate as a PDF with which to investigate hard scattering, and understand the regime where our average energy loss is an accurate description of the behaviour of the DM particle. One can also adopt a unified approach of the hard and soft scattering regimes using for example [8].

Our approach of using the longitudinal polarization also neglects the transverse components of the polarization. Which is valid for our purposes since we assume non-relativistic DM and the leading term in the non-relativistic limit is the longitudinal component (transverse is surpressed by v/c) [4].

2.2 RPA vs Hartree Fock Approximations

Here we follow [3] section 3-5. We consider a DM probe with coupling κe and number density ρ_k^{DM} inducing a response in the SM material. It's field induces a scalar potential which is the sum of the potential due to the DM external probe and that induced by the polarization of the material in response to the probe. As in [7], we take a gauge such that the vector potential has no longitudinal component so that the longitudinal field is determined by the scalar potential alone.

$$\phi(k,\omega) = \phi_{ext}(k,\omega) + \phi_{pol}(k,\omega) = \frac{\phi_{ext}(k,\omega)}{\epsilon} \tag{48}$$

$$\phi_{pol} = -\frac{4\pi e}{k^2}\Delta\langle\hat{\rho}_{-k}\rangle \tag{49}$$

Note that following [4] this neglects vacuum loop and DM in medium contributions. As we will see later, once DM accumulates in the earth they will also contribute to the polarization. So the response of the system is to screen the charge. This gives the following expressions for the dielectric function

$$\frac{1}{\epsilon} = 1 + \frac{\phi_{pol}}{\phi_{ext}} = 1 - \frac{4\pi e}{k^2}\frac{\Delta\langle\hat{\rho}_{-k}\rangle}{\phi_{ext}} \tag{50}$$

$$\epsilon = 1 + \frac{4\pi e}{k^2}\frac{\Delta\langle\hat{\rho}_{-k}\rangle}{\phi} \tag{51}$$

$$\tag{52}$$

Now consider the external potential from the dark matter probe which drives the system at a frequency ω. So that $\rho_{ext}(t) = \rho_k^{DM} e^{-i\omega t}$, and has no other t dependence. This assumes that DM is in a, possible off shell, momentum eigenstate. Using (21) we find

$$\phi_{ext}(k,\omega) = -\frac{4\pi\kappa e}{k^2}\rho_{ext} \tag{53}$$

Combining this with the previous expression yields precisely what one would find for the macroscopic dielectric derived by fourier transforming Poisson's equations. In the Hartree Fock Approximation, one carries out the calculation of the dielectric function evaluating $\Delta\langle\hat{\rho}_{-k}\rangle$ at first order in Linear Response Theory (Time Dependent Perturbation Theory). As we will see in the next section, one finds

$$\Delta\langle\hat{\rho}_{-k}\rangle_{HF} = 4\pi\alpha_0(k,\omega)\kappa\rho_{ext} = 4\pi\alpha_0(k,\omega)\left(\frac{k^2}{4\pi e}\right)\phi_{ext}(k,\omega) \tag{54}$$

Where $4\pi\alpha_0(k,\omega)$ is the free electron polarizability [3]. So that in the Hartree Fock approximation, the dielectric function is

$$\frac{1}{\epsilon_{HF}} = 1 - \frac{4\pi e}{k^2}\frac{\Delta\langle\hat{\rho}_{-k}\rangle_{HF}}{\phi_{ext}} = 1 - 4\pi\alpha_0(k,\omega) \tag{55}$$

The Random Phase Approximation instead assumes that the material responds both to the polarized and external potentials.

$$\Delta\langle\hat{\rho}_{-k}\rangle_{RPA} = 4\pi\alpha_0(k,\omega)\left(\frac{k^2}{4\pi e}\right)\phi(k,\omega) \tag{56}$$

Notice that ϕ depends on ϵ, which in this scheme will be self consistently solved for. So using (49) the dielectric function is given by

$$\epsilon_{RPA} = 1 + \frac{4\pi e}{k^2}\frac{\Delta\langle\hat{\rho}_{-k}\rangle_{RPA}}{\phi} = 1 + 4\pi\alpha_0(k,\omega) \tag{57}$$

$$\implies \frac{1}{\epsilon_{RPA}} = \frac{1}{1 + 4\pi\alpha_0(k,\omega)} \tag{58}$$

So the RPA is analagous to 1PI resummation in field theory, where a first order result is resummed to determine a non-linear response. In fact this result can be also be derived with Feynman Diagrams by resumming loops as in field theory [3].

In the next section we derive the Fluctuation Dissipation Theorem in the Hartree Fock Approximation, allowing us to connect the dielectric function to the structure function, and hence also our energy loss formula.

2.3 Linear Response and Fluctuation Dissipation Theorem in the Hartree Fock Approximation

We begin by determining the perturbing Hamiltonian describing the interaction of the Probe and the Material. Convolving again our potentials and densities, we find for the potential

$$
\begin{aligned}
-e\int d^3x\ \phi_{ext}(r)\hat{\rho}(r) =& \frac{-e}{V^2}\int d^3x \sum_k e^{-ikx}\phi_{ext}\sum_q e^{-iqx}\hat{\rho}_q && (59)\\
=& \frac{4\pi\kappa e^2}{V^2}\int d^3x \sum_k e^{-ikx}\frac{\rho_{ext}}{k^2}\sum_q e^{-iqx}\hat{\rho}_q && (60)\\
=& \frac{4\pi\kappa e^2}{V}\sum_k \frac{\rho_{ext}}{k^2}\sum_q \hat{\rho}_q\delta_{-k,q} && (61)\\
=& \sum_k \frac{4\pi\kappa e^2}{Vk^2}\hat{\rho}_{-k}\rho_k^{DM}e^{-i\omega t} && (62)
\end{aligned}
$$

Since the dark matter particle is in a momentum eigenstate, ρ_k^{DM} will vanish for all but one eigenvalue of momentum. So we can drop the sum. Note also that $\hat{\rho}_{-k}$ is the only operator in this expression, all other factors are macroscopic quantities. We introduce a regulator which turns on the interaction adiabatically. This assumes a causal / retarded response as is familiar from QFT. We take $\omega \to \omega \pm i\eta$ for infinitesimal η so that the interaction decays exponentially as $t \to -\infty$.

$$
\hat{V}_1(t) = \frac{4\pi\kappa e^2}{Vk^2}[\hat{\rho}_{-k}\rho_k^{DM}e^{-i\omega t} + \text{h.c}]e^{\eta t} \tag{63}
$$

Where we symmetrize the dark matter density appropriately for a Hermitian operator, matching precisely [6] (noting that they use conventions where $V \equiv 1$). To describe the state of the material we use exact many body eigenstates in the interaction picture.

$$
\begin{aligned}
|m\rangle_I =& e^{iH_0t/\hbar}|m(t)\rangle_S && (64)\\
\hat{V}_I(t) =& e^{iH_0t/\hbar}\hat{V}_1(t)e^{-iH_0t/\hbar} && (65)
\end{aligned}
$$

Where $|m(t)\rangle_S$ is the corresponding Schrodinger picture ket. And H_0 is the free Hamiltonian. We then introduce our interaction picture potential $\hat{V}_I(t)$ and consider the time evolved state using Dyson's formula for the time evolution operator $\mathcal{U}$.

$$
\begin{aligned}
i\hbar\partial_t|m(t)\rangle_I =& \hat{V}_I(t)|m(t)\rangle && (66)\\
\implies |m(t)\rangle_I =& \mathcal{U}(t,-\infty)|m\rangle && (67)\\
=& T\left\{\exp\left(-\frac{i}{\hbar}\int_{-\infty}^t dt'\hat{V}_I(t')\right)\right\}|m\rangle && (68)\\
\approx& \left(1-\frac{i}{\hbar}\int_{-\infty}^t dt'\hat{V}_I(t')\right)|m\rangle && (69)
\end{aligned}
$$

Where $|m\rangle \equiv |m(t\to-\infty)\rangle_I = |m\rangle_S$ is an eigenstate of the free Hamiltonian, and we work to first order in the SM-DM coupling. We now evaluate the expectation value of the charge density fluctuation in the material in a finite temperature ensemble of such time evolved states.

$$
\begin{aligned}
\langle\hat{\rho}_k(t)\rangle =& \sum_m P_m\langle m(t)|_I\hat{\rho}_{k,I}(t)|m(t)\rangle_I && (70)\\
=& \sum_m P_m\langle m|\left(1+\frac{i}{\hbar}\int_{-\infty}^t dt'\hat{V}_I(t')\right)\hat{\rho}_{k,I}(t)\left(1-\frac{i}{\hbar}\int_{-\infty}^t dt'\hat{V}_I(t')\right)|m\rangle && (71)\\
\implies \Delta\langle\hat{\rho}_k(t)\rangle \equiv& \langle\hat{\rho}_k(t)\rangle - 1 = \frac{i}{\hbar}\sum_m P_m\int_{-\infty}^t dt'\langle m|\left[\hat{V}_I(t'),\ \hat{\rho}_{k,I}(t)\right]|m\rangle && (72)\\
& && (73)
\end{aligned}
$$

where E_m is the energy of the m^{th} many body eigenstate, and $\rho_{k,I}$ is the interaction picture number density operator as defined analagously to (65). We now use the fact that our exact many body asymptotic states (the unspecified states which exactly describe the system at $t = -\infty$) are eigenstates of the free Hamiltonian (1)

$$\langle m|\hat{V}_I(t')\hat{\rho}_{k,I}(t)|m\rangle = \sum_n \langle m|\hat{V}_I(t')|n\rangle\langle n|\hat{\rho}_{k,I}(t)|m\rangle \tag{74}$$

$$= \sum_n \langle m|e^{i(H_0 t')/\hbar}\hat{V}e^{i\eta t'}e^{-i(H_0 t')/\hbar}|n\rangle\langle n|e^{i(H_0 t)/\hbar}\hat{\rho}_k e^{-i(H_0 t)/\hbar}|m\rangle \tag{75}$$

$$= \sum_n e^{i(E_n - E_m)(t-t')/\hbar}e^{\eta t'}V_{mn}(\rho_k)_{nm} \tag{76}$$

$$\equiv \sum_n e^{iE_{nm}(t-t')/\hbar}e^{\eta t'}V_{mn}(\rho_k)_{nm} \tag{77}$$

Where we have defined $E_{nm} = E_n - E_m$ in the last line. Notice that we can drop the Hermitian conjugate term in the commutator since $\hat{\rho}$ has real eigenvalues $\hat{\rho}^\dagger_{-k} = \hat{\rho}_k$ and clearly $\hat{\rho}_k$ commutes with itself. So our matrix elements are

$$V_{nm} = \frac{4\pi e^2\kappa}{Vk^2}\langle m|\rho_k^{DM}\hat{\rho}_{-k}e^{-i\omega t'}|n\rangle \tag{78}$$

$$\implies V_{nm}(\rho_k)_{nm} = \frac{4\pi e^2\kappa}{Vk^2}|(\rho_k)_{nm}|^2\rho_k^{DM}e^{-i\omega t'} \tag{79}$$

We can now use the fact given on pg 125 of [3] that $\hat{\rho}_k(t) = \hat{\rho}_k e^{-i\omega t}$ at linear order.

$$\implies \Delta\langle\hat{\rho}_k\rangle = \frac{i}{\hbar}\sum_m P_m \int_{-\infty}^t dt' e^{i\omega t}\langle m|\left[\hat{V}_I(t'),\ \hat{\rho}_k\right]|m\rangle \tag{80}$$

$$= \frac{i}{\hbar}\sum_{mn} P_m \int_{-\infty}^t dt' e^{i\omega t}\Big(\langle m|\hat{V}_I(t')|n\rangle\langle n|\hat{\rho}_k|m\rangle - \langle m|\ \hat{\rho}_k|n\rangle\langle n|\hat{V}_I(t')|m\rangle\Big) \tag{81}$$

$$= \frac{i}{\hbar}\sum_{mn} P_m \int_{-\infty}^t dt' e^{i\omega t}\Big(\langle m|\hat{V}_I(t')|n\rangle\langle n|\hat{\rho}_k|m\rangle - \langle n|\hat{V}_I(t')|m\rangle\langle m|\hat{\rho}_k|n\rangle\Big) \tag{82}$$

$$= \frac{i}{\hbar}\sum_{mn} P_m \frac{4\pi e^2\kappa}{Vk^2}\int_{-\infty}^t dt' e^{\eta t'}\Big(|(\rho_k)_{nm}|^2\rho_k^{DM}e^{-i\omega(t'-t)}e^{-iE_{nm}(t'-t)/\hbar} - (n \leftrightarrow m)\Big) \tag{83}$$

$$= \frac{i}{\hbar}\sum_{mn} P_m \frac{4\pi e^2\kappa}{Vk^2}\int dt' e^{-i\omega(t'-t)}e^{\eta t'}\Theta(t-t')\Big(|(\rho_k)_{nm}|^2\rho_k^{DM}e^{-iE_{nm}(t'-t)/\hbar} - (n \leftrightarrow m)\Big) \tag{84}$$

$$= \frac{i}{\hbar}\sum_{mn} P_m \frac{4\pi e^2\kappa}{Vk^2}\Big(|(\rho_k)_{nm}|^2\rho_k^{DM}\int dt'\Theta(t-t')e^{-i(E_{nm}+\hbar\omega+i\eta)(t'-t)/\hbar} - (n \leftrightarrow m)\Big) \tag{85}$$

$$= \frac{i}{\hbar}\sum_{mn} P_m \frac{4\pi e^2\kappa}{Vk^2}\rho_k^{DM}\left[\frac{|(\rho_k)_{nm}|^2}{-i(\omega + E_{nm}/\hbar + i\eta)} - (n \leftrightarrow m)\right] \tag{86}$$

$$= -\sum_{mn} P_m \frac{4\pi e^2\kappa}{Vk^2}\rho_k^{DM}\left[\frac{|(\rho_k)_{nm}|^2}{\hbar\omega + E_{nm} + i\eta} - (n \leftrightarrow m)\right] \tag{87}$$

Where we remember that as usual that there is an implicit limit on η so that only it's sign is and reality are relevant. So we find

$$1+\frac{\Delta\langle\hat{\rho}_k(t)\rangle}{\kappa\rho_{ext}(t)} = 1+\frac{\Delta\langle\hat{\rho}_k\rangle}{\kappa\rho_k^{DM}} \tag{88}$$

$$= 1-\frac{1}{\kappa\rho_k^{DM}}\sum_{mn}P_m\frac{4\pi e^2\kappa}{Vk^2}\rho_k^{DM}\left[\frac{|(\rho_k)_{nm}|^2}{\hbar\omega+E_{nm}+i\eta}-(n\leftrightarrow m)\right] \tag{89}$$

$$= 1-\sum_{mn}P_m\frac{4\pi e^2}{Vk^2}\left[\frac{|(\rho_k)_{nm}|^2}{\hbar\omega+E_{nm}+i\eta}-(n\leftrightarrow m)\right] \tag{90}$$

And we arrive at the dielectric function recalling that it is defined with the opposite signed momentum.

$$\frac{1}{\epsilon(k,\omega)} = 1+\frac{\Delta\langle\hat{\rho}_{-k}\rangle}{\kappa\rho_k^{DM}} \tag{91}$$

$$= 1-\sum_{mn}P_m\frac{4\pi e^2}{Vk^2}\left[\frac{|(\rho_{-k})_{nm}|^2}{\hbar\omega+E_{nm}+i\eta}-(n\leftrightarrow m)\right] \tag{92}$$

Which agrees with [6] in the limit $\beta\to\infty$ which restricts m to be the ground state (Note that they take units such that $V=1$ in some sections). Interestingly, this doesn't include the kinetic mixing, which is physically reasonable. The dielectric constant is a property of the material and hence independent of the charge of the probe. We can also define the conventional linear response function as

$$-\chi_0 \equiv -\frac{k^2}{4\pi e^2}4\pi\alpha_0 = \frac{k^2}{4\pi e^2}\left(\frac{1}{\epsilon}-1\right) \tag{93}$$

$$= -\frac{1}{V}\sum_{mn}P_m\left[\frac{|(\rho_{-k})_{nm}|^2}{\hbar\omega+E_{nm}+i\eta}-(n\leftrightarrow m)\right] \tag{94}$$

$$\tag{95}$$

We also give the ground state 0 temperature limit for later use at this stage. In this limit

$$\lim_{\beta\to\infty}P_m = \lim_{\beta\to\infty}\frac{e^{-\beta E_m}}{Z} = \delta_{m,0} \tag{96}$$

$$\lim_{\beta\to\infty}-\chi_0 = -\frac{1}{V}\sum_{n}\left[\frac{|(\rho_{-k})_{n0}|^2}{\hbar\omega+E_{n0}+i\eta}-(n\leftrightarrow 0)\right] \tag{97}$$

$$\tag{98}$$

Where $m=0$ denotes the ground state of the system. And using the argument given in note 28a of [3] on pg 164 this implies

$$\lim_{\beta\to\infty}-\chi_0 = -\frac{1}{V}\sum_{n}|(\rho_{-k})_{n0}|^2\left[\frac{1}{\hbar\omega+E_{n0}+i\eta}-\frac{1}{\hbar\omega-E_{n0}+i\eta}\right] \tag{99}$$

Which agrees with eqn (3-110a) on pg 126 of the same reference. Now we use this to relate the Complex Dielectric Function to the the Structure Function, a special case of the fluctuation dissipation theorem.

$$-\mathrm{Im}(\chi_0) = -\frac{1}{V}\sum_{mn} P_m \ \mathrm{Im}\left[\frac{|(\rho_{-k})_{nm}|^2}{\hbar\omega + E_{nm} + i\eta} - (n \leftrightarrow m)\right] \tag{100}$$

$$= -\frac{1}{V}\sum_{mn}(P_m - P_n) \ \mathrm{Im}\left[\frac{|(\rho_{-k})_{nm}|^2}{\hbar\omega + E_{nm} + i\eta}\right] \tag{101}$$

$$=\frac{\pi}{V}\sum_{mn}(P_m - P_n)|(\rho_{-k})_{nm}|^2 \ \delta(\hbar\omega + E_{nm}) \tag{102}$$

$$= -(1 - e^{-\beta\omega})\frac{\pi}{V}\sum_{mn} P_m|(\rho_{-k})_{nm}|^2 \ \delta(\hbar\omega + E_{nm}) \tag{103}$$

The prefactor is enforced by the delta function, and says the process transfering energy to the probe is suppressed by a Boltzmann factor. Where we recall that $P_m = e^{-\beta(E_m-\mu)}Z^{-1}$ appropriate for a grand canonical ensemble and have used the Cauchy-Dirac relation

$$\frac{1}{\hbar\omega + E_{nm} \mp i\eta} = \mathcal{P}\frac{1}{\hbar\omega + E_{nm}} \pm i\pi\delta(\hbar\omega + E_{nm}) \tag{104}$$

We also notice an important property of the Linear Response and Complex Dielectric Functions

$$\mathrm{Im}\epsilon(k,\omega) = -\mathrm{Im}\epsilon(k,-\omega) \tag{105}$$

The real part is even in ω. Which gives

$$\mathrm{Im}(\chi_0) = \frac{k^2}{4\pi e^2}\mathrm{Im}\left(\frac{-1}{\epsilon}\right) =(1 - e^{-\beta\hbar\omega}) \ \frac{\pi}{V}\sum_{mn} P_m|(\rho_{-k})_{nm}|^2 \ \delta(\hbar\omega + E_{mn}) \tag{106}$$

$$=(1 - e^{-\beta\hbar\omega}) \ \frac{\pi}{V}\sum_{mn} P_m|(\rho_{-k})_{nm}|^2 \ \delta(\hbar\omega - (E_f - E_i)) \tag{107}$$

$$=\frac{1}{2}(1 - e^{-\beta\hbar\omega})S(k,\omega) \tag{108}$$

Where we have recalled the definition (33). So we can find the Dynamical Structure Function by determining the Complex Dielectric function through the Fluctuation Dissipation Theorem relating structure and Linear Response Functions. Note that given our choice of units for S, this differs from some other common statements of the fluctuation dissipation theorem by a factor of $\hbar^{-1}$ on the RHS.

$$S(\vec{k},\omega) = \frac{2}{1 - e^{-\beta\hbar\omega}}\frac{k^2}{4\pi e^2}\mathrm{Im}\left(\frac{-1}{\epsilon}\right) = \frac{2}{1 - e^{-\beta\hbar\omega}}\mathrm{Im}\left(\chi_0\right) \tag{109}$$

Which agrees precisely with [9]. As has been pointed out in eg. [2], [6] and [10], the Complex Dielectric Function can also be determined directly from energy loss experiments with SM probes for a limited range of kinematics. This expression allows us to write our energy loss formula as

$$\frac{dE}{dr}(v) = \int_0^\infty \frac{dq}{(2\pi)^2}\frac{1}{v^2}\frac{4\pi\kappa^2 e^2}{q}\int_{-\omega_+}^{\omega_-} d\omega \ \frac{2\omega}{1 - e^{-\beta\hbar\omega}}\mathrm{Im}\left(\frac{-1}{\epsilon}\right) \tag{110}$$

$$=\frac{\kappa^2}{v^2}\int_0^\infty \frac{dq \ q}{(2\pi)^2}\int_{-\omega_+}^{\omega_-} d\omega \ \frac{-2\omega}{1 - e^{-\beta\hbar\omega}}\mathrm{Im}\left(\frac{4\pi e^2}{q^2\epsilon}\right) \tag{111}$$

So the energy loss is a functional of the imaginary part of the screened potential. This matches on to eqn (3-114) on pg 128 of [3]. We also have

$$\frac{d\sigma}{dE_R}(v,\omega) = \frac{2}{n_T v^2}\frac{1}{1 - e^{-\beta\hbar\omega}}\int \frac{dq}{(2\pi)^2}\frac{4\pi\kappa^2 e^2}{q}\mathrm{Im}\left(\frac{-1}{\epsilon}\right)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{112}$$

2.3.1 Fluctuation Dissipation Theorem and Energy Loss in RPA

Remarkably, but not unexpectedly, our expression for the fluctuation dissipation theorem still holds. This is no doubt the magic of the optical theorem at work! From (58) and (95) we have

$$
\begin{aligned}
S(\vec{k},\omega) =& \frac{2}{1-e^{-\beta\omega}}\frac{k^2}{4\pi e^2}\mathrm{Im}\left(\frac{-1}{\epsilon^{RPA}}\right) && (113)\\
=& \frac{2}{1-e^{-\beta\hbar\omega}}\frac{1}{|\epsilon^{RPA}|^2}\mathrm{Im}\left(-\chi_0^{\star}\right) && (114)\\
=& \frac{2}{1-e^{-\beta\hbar\omega}}\frac{1}{|\epsilon^{RPA}|^2}\mathrm{Im}\left(\chi_0\right) && (115)\\
=& \frac{2}{1-e^{-\beta\hbar\omega}}\frac{k^2}{4\pi e^2}\frac{1}{|\epsilon^{RPA}|^2}\mathrm{Im}\left(\frac{-1}{\epsilon^{HF}}\right) && (116)
\end{aligned}
$$

So we find that the optical theorem gives the leading order result with 2 factors of ϵ^{RPA} appropriate for the quadratic dependence on the potential in the structure function. In other words, the RPA screens the potentials appearing in the transition rate and energy loss formula! Which precisely mirrors the screening of the interaction potential in the Linear Response calculation of the RPA dielectric.

2.4 Lindhard Dielectric Function - Free Electron Gas Approximation

In this section we evaluate the Complex Dielectric Function for a free electron gas with no band structure. In crystalline solids one can find similar response functions [11] [12]. Similarly there also exist extensions to materials in which the band structure is important [9] [13] [14] [15]. We first evaluate the matrix element squared in the Dielectric Function at finite temperature using an occupation number basis. We then take the 0 temperature limit following [16] to match on to the canonical Lindhard function [7] found throughout the literature.

This is a valid approximation for dense materials where the ions are close enough together to be treated as a homogeneous background (so the ionic potentials are irrelevant) [3] and with low electron collision rates (so that interactions can be neglected) as we will soon see. Another possible modification is to account for the repulsion of short range same spin partners leads to a realistic dielectric function [3]. For ferromagnets (like iron at low temperatures) one must also account for the existence of spin wave excitations, however for our problem we can likely ignore this phenomenon since the majority of the iron in the earth is at temperatures high compared to it's currie temperature [3].

In this note we will extend the analysis to include electron collisions using the Random Time Approximation. We will not investigate any of the other extensions of the RPA or Free Electron Gas approximations. Resorting instead to estimates of their potential relevance using comparisons with available data.

2.4.1 Evaluation of Matrix Element Squared in RPA

To evaluate the matrix element we take an occupation number basis and write our number density operator in terms of particle creation and annihilation operators. Our density fluctuation operator is from eqn A-25 of [3]

$$
\hat{\rho}_k = \sum_{\boldsymbol{p}} a^{\dagger}_{p-k}a_p = \sum_{\boldsymbol{p}} a^{\dagger}_p a_{p+k} \tag{117}
$$

(118)

The operator in the summand corresponds to destroying a particle in the material of momentum p and creating one of momentum $p-k$. So we interpret $\hat{\rho}_{-k}$ as annihilating particles at momenta p and creating one of momentum $p+k$, precisely how one would interpret a momentum transfer of k to a particular electron in the system. These turn out to correspond to *particle hole excitations* [3]. Where an electron in the material polarizes to expose a patch of the ion background creating a small dipole which tends to screen an applied field. Note that since we are neglecting spin dependent interactions for the time being, this treats the electrons as spin scalars. We will simply include a spin degeneracy factor at the end of the calculation in lieu of dealing with the spin structure of the operators.

We now evaluate our matrix element squared

$$|(\rho_{-k})_{mn}|^2 = \langle m|\hat{\rho}_k|n\rangle\langle n|\hat{\rho}_{-k}|m\rangle \tag{119}$$

$$= \sum_{\boldsymbol{p},\boldsymbol{p}'} \langle m|a^\dagger_{p'-k}a_{p'}|n\rangle\langle n|a^\dagger_p a_{p-k}|m\rangle \tag{120}$$

$$= \sum_{\boldsymbol{p},\boldsymbol{p}'} \langle m|a^\dagger_{p'}a_{p'+k}|n\rangle\langle n|a^\dagger_{p+k}a_p|m\rangle \tag{121}$$

Using again $\hat{\rho}^\dagger_k = \hat{\rho}_{-k}$. Next we note that if we take an occupation number basis, that our creation and annihilation operators transition between orthonormal eigenstates of the system. Hence the matix elements are only non-vanishing if

$$a^\dagger_{p'}a_{p'+k}|n\rangle = (\langle\, n|a^\dagger_{p+k}a_p)^\dagger \tag{122}$$

$$= a^\dagger_p a_{p+k}|n\rangle \tag{123}$$

$$= |m\rangle \tag{124}$$

Using orthonormality, this implies

$$\langle m|a^\dagger_{p'}a_{p'+k}|n\rangle\langle n|a^\dagger_{p+k}a_p|m\rangle = \delta_{p',p}\langle m|a^\dagger_p a_{p+k}|n\rangle\langle n|a^\dagger_{p+k}a_p|m\rangle \tag{125}$$

So our matrix element becomes

$$|(\rho_{-k})_{mn}|^2 = \sum_{\boldsymbol{p}'}\sum_{\boldsymbol{p}} \delta_{p',p}\langle m|a^\dagger_p a_{p+k}|n\rangle\langle n|a^\dagger_{p+k}a_p|m\rangle \tag{126}$$

$$= \sum_{\boldsymbol{p}} \langle m|a^\dagger_p a_{p+k}|n\rangle\langle n|a^\dagger_{p+k}a_p|m\rangle \tag{127}$$

We now argue that E_{mn} is independent of $m,\ n$. The free Hamiltonian for the system is simply the sum of the kinetic energies of each electron in the free gas. This neglects the contribution of the external potential.

$$H_0 = \sum_{\boldsymbol{q}} E_q a^\dagger_q a_q \tag{128}$$

$$E_q = \frac{\hbar^2 q^2}{2m_e} \tag{129}$$

We also write down our anti-commutation relations

$$\{a_k,\ a^\dagger_{k'}\} = \delta_{\boldsymbol{k},\boldsymbol{k}'} \tag{130}$$

With all others vanishing. We now calculate E_{nm} using (124).

$$\begin{aligned}
E_m|m\rangle =&H_0|m\rangle && (131)\\
=&H_0 a_p^\dagger a_{p+k}|n\rangle && (132)\\
=&\left(\left[H_0,\ a_p^\dagger a_{p+k}\right] + a_p^\dagger a_{p+k}E_n\right)|n\rangle && (133)\\
=&\left[H_0,\ a_p^\dagger a_{p+k}\right]|n\rangle + E_n|n\rangle && (134)\\
& && (135)
\end{aligned}$$

So the quantity of interest is

$$\left[H_0,\ a_p^\dagger a_{p+k}\right] = \sum_{\boldsymbol{q}} E_q \left[a_q^\dagger a_q, a_p^\dagger a_{p+k}\right] \tag{136}$$

$$\tag{137}$$

Using the anti-commutation relations we have

$$\left[a_q^\dagger a_q, a_p^\dagger a_{p+k}\right] = a_q^\dagger a_q a_p^\dagger a_{p+k} - a_p^\dagger a_{p+k} a_q^\dagger a_q \tag{138}$$

Now consider the first term

$$\begin{aligned}
a_q^\dagger a_q a_p^\dagger a_{p+k} =& a_q^\dagger \left(\delta_{\boldsymbol{p},\boldsymbol{q}} - a_p^\dagger a_q\right) a_{p+k} && (139)\\
=&\delta_{\boldsymbol{p},\boldsymbol{q}} a_q^\dagger a_{p+k} - (-1)^2 a_p^\dagger a_q^\dagger a_{p+k} a_q && (140)\\
=&\delta_{\boldsymbol{p},\boldsymbol{q}} a_q^\dagger a_{p+k} - a_p^\dagger \left(\delta_{\boldsymbol{p+k},\boldsymbol{q}} - a_{p+k} a_q^\dagger\right) a_q && (141)\\
=&\delta_{\boldsymbol{p},\boldsymbol{q}} a_q^\dagger a_{p+k} - \delta_{\boldsymbol{p+k},\boldsymbol{q}} a_p^\dagger a_q + a_p^\dagger a_{p+k} a_q^\dagger a_q && (142)\\
& && (143)
\end{aligned}$$

The last term cancels in the commutator, leading to

$$\begin{aligned}
\left[H_0,\ a_p^\dagger a_{p+k}\right] =& \sum_{\boldsymbol{q}} E_q \left(\delta_{\boldsymbol{p},\boldsymbol{q}} a_q^\dagger a_{p+k} - \delta_{\boldsymbol{p+k},\boldsymbol{q}} a_p^\dagger a_q\right) && (144)\\
=& \left(E_p - E_{p+k}\right) a_p^\dagger a_{p+k} && (145)\\
& && (146)
\end{aligned}$$

So that we have finally

$$\begin{aligned}
-E_{nm}|m\rangle =& \left(E_p - E_{p+k}\right) a_p^\dagger a_{p+k}|n\rangle && (147)\\
=& \left(E_p - E_{p+k}\right)|m\rangle && (148)\\
\equiv& -\Delta E|m\rangle && (149)\\
& && (150)
\end{aligned}$$

Which, as expected, is independent of the details of the states $m,\ n$. In the RPA, the charges in the material (Free Electron Gas of conduction electrons) respond to the screened potential. As such, the perturbing Hamiltonian for the interaction (63) receives an additional factor of ϵ^{-1}.

$$\hat{V}_1^{RPA}(t) = \frac{1}{\epsilon^{RPA}} \frac{4\pi\kappa e^2}{Vk^2} [\hat{\rho}_{-k}\rho_k^{DM} e^{-i\omega t} + \text{h.c}] e^{\eta t} \tag{151}$$

Hence in the RPA, (92) becomes

$$\frac{k^2}{4\pi e^2}\left(\frac{1}{\epsilon^{RPA}}-1\right)=-\frac{1}{\epsilon^{RPA}}\sum_{mn}P_m\left[\frac{|(\rho_{-k})_{nm}|^2}{\hbar\omega+E_{nm}+i\eta}-(n\leftrightarrow m)\right] \tag{152}$$

$$=-\frac{1}{\epsilon^{RPA}}\sum_{mn}(P_m-P_n)\frac{|(\rho_{-k})_{nm}|^2}{\hbar\omega+\Delta E+i\eta} \tag{153}$$

$$=-\frac{1}{\epsilon^{RPA}}\left(\sum_m P_m\sum_n|(\rho_{-k})_{nm}|^2\right)\frac{(1-e^{-\beta\Delta E})}{\hbar\omega+\Delta E+i\eta} \tag{154}$$

Where we have relabelled n, m in the second line. So our matrix element squared is the only part of the expression that depends on the final state of the system. From (127) we have

$$\sum_n|(\rho_{-k})_{mn}|^2=\sum_{\boldsymbol{p}}\langle m|a_p^\dagger a_{p+k}\left(\sum_n|n\rangle\langle n|\right)a_{p+k}^\dagger a_p|m\rangle \tag{155}$$

$$=\sum_{\boldsymbol{p}}\langle m|a_p^\dagger a_{p+k}a_{p+k}^\dagger a_p|m\rangle \tag{156}$$

$$=\sum_{\boldsymbol{p}}\langle m|a_p^\dagger\left(1-a_{p+k}^\dagger a_{p+k}\right)a_p|m\rangle \tag{157}$$

$$\tag{158}$$

Using the fact that the many body eigenstates form a complete set of states.

$$a_{p+k}^\dagger a_{p+k}a_p=-a_{p+k}^\dagger a_p a_{p+k} \tag{159}$$

$$=a_p a_{p+k}^\dagger a_{p+k} \tag{160}$$

Where in the last line, we have used the fact that the first line vanishes for $k=0$ by Fermi statistics. Also, as pointed out in [3] Appendix A, such a term would correspond to a homogeneous contribution the electron density in the material, and hence could be absorbed into a redefinition of the homogeneous background.

$$\sum_m P_m\sum_n|(\rho_k)_{mn}|^2=\sum_m P_m\sum_{\boldsymbol{p}}\langle m|a_p^\dagger a_p\left(1-a_{p+k}^\dagger a_{p+k}\right)|m\rangle \tag{161}$$

$$=\sum_{\boldsymbol{p}}\left(\langle n_{E_p}\rangle-\langle n_{E_p}n_{E_{p+k}}\rangle\right) \tag{162}$$

$$=\sum_{\boldsymbol{p}}f(E_p)\left(1-f(E_{p+k})\right) \tag{163}$$

Where $f(E)$ is the Fermi distribution (average occupation number) [17].

$$f(E_p)=\langle n_{E_p}\rangle=\sum_m P_m\langle m|a_p^\dagger a_p|m\rangle=\frac{1}{e^{\beta(E_p-\mu)}+1} \tag{164}$$

Where n_E is the occupation of energy level E. We now derive the second line in (163). From [17] pg 148 we have

$$Z = \sum_{N}^{\infty} \sum_{n_E=0}^{1} e^{-\beta(\sum_E n_E E - N\mu)} \tag{165}$$

$$\zeta = \ln Z = \sum_E \ln(1 + e^{-\beta(E-\mu)}) \tag{166}$$

$$\implies \langle n_E \rangle = \frac{-1}{\beta} \frac{1}{Z} \frac{\partial Z}{\partial E} \tag{167}$$

$$\implies \langle n_{E_1} n_{E_2} \rangle = \left(\frac{-1}{\beta}\right)^2 \frac{1}{Z} \frac{\partial^2 Z}{\partial E_1 \partial E_2} \tag{168}$$

Where the derivatives are taken with other energies held constant (which we leave implicit).

$$\begin{aligned}
\left(\frac{-1}{\beta}\right)^2 \frac{\partial^2 \zeta}{\partial E_1 \partial E_2} &= \left(\frac{-1}{\beta}\right)^2 \frac{\partial}{\partial E_2} \left(\frac{1}{Z} \frac{\partial Z}{\partial E_1}\right) && (169)\\
&= \left(\frac{-1}{\beta}\right)^2 \left(\frac{1}{Z} \frac{\partial^2 Z}{\partial E_1 \partial E_2} - \frac{1}{Z^2} \frac{\partial Z}{\partial E_2} \frac{\partial Z}{\partial E_1}\right) && (170)\\
&= \langle n_{E_1} n_{E_2} \rangle - \langle n_{E_1} \rangle \langle n_{E_2} \rangle && (171)\\
\left(\frac{-1}{\beta}\right)^2 \frac{\partial^2 \zeta}{\partial E_1 \partial E_2} &= \left(\frac{-1}{\beta}\right)^2 \frac{\partial^2}{\partial E_1 \partial E_2} \sum_E \ln(1 + e^{-\beta(E-\mu)}) && (172)\\
&= \left(\frac{-1}{\beta}\right) \frac{\partial}{\partial E_2} \frac{1}{1 + e^{\beta(E_1-\mu)}} && (173)\\
&= 0 && (174)\\
\implies \langle n_{E_1} n_{E_2} \rangle &= \langle n_{E_1} \rangle \langle n_{E_2} \rangle && (175)
\end{aligned}$$

Since the energies are independent variables by assumption. From which we find the result used above

$$\sum_m P_m \langle m | a_p^\dagger a_p a_{p+k}^\dagger a_{p+k} | m \rangle = \langle n_{E_p} n_{E_{p+k}} \rangle = \langle n_{E_p} \rangle \langle n_{E_{p+k}} \rangle = f(E_p) f(E_{p+k}) \tag{176}$$

Where we have again excluded the case of $k = 0$. This gives for the dielectric function in the RPA from (154)

$$\begin{aligned}
\frac{k^2}{4\pi e^2} \left(\frac{1}{\epsilon^{RPA}} - 1\right) &= -\frac{1}{\epsilon^{RPA}} \frac{1}{V} \sum_{\boldsymbol{p}} f(E_p) \left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta \Delta E})}{\hbar\omega + \Delta E + i\eta} && (177)\\
\implies \epsilon^{RPA} &= 1 + \frac{4\pi e^2}{k^2} \frac{1}{V} \sum_{\boldsymbol{p}} f(E_p) \left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta \Delta E})}{\hbar\omega + \Delta E + i\eta} && (178)\\
& && (179)
\end{aligned}$$

At this point we simply introduce a spin degeneracy factor of $g_s = 2$ assuming there are no spin dependent interactions.

$$\implies \epsilon^{RPA} = 1 + \frac{4\pi e^2}{k^2} \frac{g_s}{V} \sum_{\boldsymbol{p}} f(E_p) \left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta \Delta E})}{\hbar\omega + \Delta E + i\eta} \tag{180}$$

We also find from (95) that our results allow us to identify as

$$\chi_0 = \frac{g_s}{V} \sum_{\boldsymbol{p}} f(E_p)\left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta \Delta E})}{\hbar\omega + \Delta E + i\eta} \tag{181}$$

Which matches on to 2-144 pg 137 of [3] in the 0 temperature limit. Note that in this limit, the summand simply sums over the states in the system with p below the Fermi surface (occupied), which have $p+k$ above the Fermi surface (unoccupied). Going back to the continuum

$$\epsilon^{RPA} = 1 + \frac{4\pi e^2}{k^2}\chi_0 \tag{182}$$

$$\chi_0 = g_s \int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta \Delta E})}{\hbar\omega + \Delta E + i\eta} \tag{183}$$

Which is consistent for both the RPA and Hartree Fock expressions we found for the dielectric function.

2.4.2 Screening in the Long Wavelength Limit and Plasmons

We now consider the long wavelength limit of the dielectric function to determine the screening of the coulomb potential for a static impurity.

$$\frac{V(k)}{\epsilon^{RPA}} = \frac{V(k)}{1 + \frac{4\pi e^2}{k^2}\chi_0} \tag{184}$$

$$\chi_0 = g_s \int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta \Delta E})}{\hbar\omega + \Delta E + i\eta} \tag{185}$$

$$\Delta E = E_{p+k} - E_p = \frac{\hbar^2 k^2 + 2\hbar \boldsymbol{k}\cdot\boldsymbol{p}}{2m} \tag{186}$$

We take the long wavelength limit with $\omega = 0$. This corresponds to the screening of a point charge impurity in the material [3] (pg 146).

$$\phi(\boldsymbol{r}) = \frac{1}{V}\sum_{\boldsymbol{k}} \frac{4\pi\kappa e^2}{k^2 \epsilon(k,0)} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{187}$$

Since $\mp\omega_\pm = -qv + \mathcal{O}(q^2)$ this limit is also is a good approximation to determine the infrared behaviour of the energy loss (especially for highly a non-relativistic dark matter particle).

$$\frac{1}{g_s}\lim_{k\to 0}\chi_0(k,0) = \lim_{k\to 0}\int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_{p+k})\right)\frac{(1 - e^{-\beta\Delta E})}{\Delta E + i\eta} \tag{188}$$

$$= \lim_{k\to 0}\int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_p)\right)\frac{\beta\Delta E}{\Delta E + i\eta} \tag{189}$$

$$= \beta\int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_p)\right) \tag{190}$$

$$\tag{191}$$

We now evaluate this expression by the generating functional method, or differentiating under the integral.

$$
\begin{aligned}
\frac{1}{g_s}\lim_{k\to 0}\chi_0(k,0) =&\beta\int\frac{d^3p}{(2\pi)^3}f(E_p)\left(1-f(E_p)\right) && (192)\\
=&\beta\int\frac{d^3p}{(2\pi)^3}\frac{e^{\beta(E_p-\mu)}}{(e^{\beta(E_p-\mu)}+1)^2} && (193)\\
=&\frac{\partial}{\partial\mu}\int\frac{d^3p}{(2\pi)^3}\frac{1}{e^{\beta(E_p-\mu)}+1} && (194)\\
=&\frac{\partial}{\partial\mu}\int\frac{d^3p}{(2\pi)^3}f(E_p) && (195)\\
& && (196)
\end{aligned}
$$

But this is just the Sommerfeld integral for the number density of electrons in a gas of free fermions [17].

$$
\begin{aligned}
n =&g_s\int\frac{d^3p}{(2\pi)^3}f(E_p) && (197)\\
=&\frac{g_s 4\pi\sqrt{2}m^{3/2}}{(2\pi\hbar)^3}\int_0^\infty\frac{dE\ \sqrt{E}}{e^{\beta(E-\mu)}+1} && (198)\\
\equiv&A\int_0^\infty\frac{dE\ \sqrt{E}}{e^{\beta(E-\mu)}+1} && (199)
\end{aligned}
$$

Where g_s is the spin degeneracy of the fermion. So we find

$$
\lim_{k\to 0}\chi_0(k,0) =\frac{\partial}{\partial\mu}n(\beta,\mu) \qquad (200)
$$

Giving for the screened potential

$$
\frac{V(k)}{\epsilon^{RPA}}=\frac{4\pi\kappa e^2}{k^2+4\pi e^2\frac{\partial}{\partial\mu}n(\beta,\mu)}\implies V(r)=\frac{\kappa e^2}{r}e^{-k_{TF}r} \qquad (201)
$$

Where we have defined the Thomas-Fermi screening momentum

$$
k_{TF}^2\equiv 4\pi e^2\lim_{k\to 0}\chi_0(k,0)=4\pi e^2\frac{\partial}{\partial\mu}n \qquad (202)
$$

When the screening momentum is non-zero, this tells us that the photon picks up an effective in-medium mass [2]. The longitudinal mode that arises from this effective mass can be treated as a new set of (quasi) particles in the material. In the case of a single component free electron gas, this quasi particle is called a *plasmon* [3]. As we will soon see, these quasi particles can be resonantly produced leading to a qualitatively different material response than one might expect [3]. Notice also that at low temperatures compared to the Fermi Energy [17] pg 237

$$
\begin{aligned}
n =&\frac{2A}{3}\mu^{3/2}(1+\mathcal{O}(\beta\mu)^{-2}) && (203)\\
\implies\ \frac{\partial}{\partial\mu}n =&A\sqrt{\mu}(1+\mathcal{O}(\beta\mu)^{-2}) && (204)\\
\mu =&E_F(1+\mathcal{O}(\beta\mu)^{-2}) && (205)
\end{aligned}
$$

[2]this situation also arises for hot plasmas in thermal QFT [18]

[3]also since the analogue of RPA in QFT is 1PI resummation, the plasmon analogue in QFT seems to be the Landau pole where effective charge diverges

Where $E_F = \frac{\hbar^2}{2m}(3\pi^2 n_e)^{2/3}$ and n_e is the number density of conduction electrons [17] pg 235. So we find for k_{TF} at low temperatures

$$k_{TF}^2 = 4\pi e^2 \frac{3n}{2E_F} \ , \beta\mu \gg 1 \tag{206}$$

Which matches on precisely to the value given in [6]. In the case of high temperature ($\beta\mu \gg 1$) ie. the classical limit the Grand Canonical partition function is given by [17]

$$Z = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_c = \sum_{N=0}^{\infty} \frac{e^{\beta\mu N} Z_1^N}{N!} = \exp\left(Z_1 e^{\beta\mu}\right) \tag{207}$$

Where Z_c is the Canonical partition function for an ideal gas, and Z_1 is the corresponding single particle partition function. So we have for the number density

$$n = \frac{1}{\beta V}\frac{\partial}{\partial\mu}\zeta = \frac{1}{\beta V}\frac{\partial}{\partial\mu}\ln Z = \frac{Z_1}{V}e^{\beta\mu} \tag{208}$$

So we have for k_{TF} at high temperatures

$$k_{TF}^2 = 4\pi e^2 \frac{\partial}{\partial\mu} n = 4\pi e^2 n\beta \ , \beta\mu \ll 1 \tag{209}$$

Which is the Debeye screening parameter!

3 Evaluation of Linear Response Function for the FEG

We look to evaluate the linear response function for the Free Electron Gas at finite temperature. In doing so we look to match on to the Lindhard Functions [7].

$$\frac{1}{g_s}\chi_0 = \int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta\Delta E})}{\hbar\omega + \Delta E + i\eta} \tag{210}$$

$$\Delta E = E_{p+k} - E_p = \frac{\hbar^2 k^2 + 2\hbar \boldsymbol{k}\cdot\boldsymbol{p}}{2m} \tag{211}$$

We first simplify the integrand

$$\frac{1}{g_s}\chi_0 = \int \frac{d^3p}{(2\pi)^3} f(E_p)\left(1 - f(E_{p+k})\right) \frac{(1 - e^{-\beta\Delta E})}{\hbar\omega + \Delta E + i\eta} \tag{212}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left[\frac{f(E_p)(1 - f(E_{p+k}))}{\hbar\omega + \Delta E + i\eta} - \frac{f(E_{p+k})(1 - f(E_p))}{\hbar\omega + \Delta E + i\eta}\right] \tag{213}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{f(E_p) - f(E_{p+k})}{\hbar\omega + \Delta E + i\eta} \tag{214}$$

In the second line we can see that this is consistent with detailed balance, both the forward and backward rates / probabilities are added and their ratio is the appropriate Boltzmann factor. Where we have used

$$f(E_p)(1 - f(E_{p+k}))e^{-\beta\Delta E} = \frac{1}{e^{\beta(E_p-\mu)} + 1} \frac{e^{\beta(E_{p+k}-\mu)}}{e^{\beta(E_{p+k}-\mu)} + 1} e^{-\beta(E_{p+k}-E_p)} \tag{215}$$

$$= \frac{e^{\beta(E_p-\mu)}}{e^{\beta(E_p-\mu)} + 1} \frac{1}{e^{\beta(E_{p+k}-\mu)} + 1} \tag{216}$$

$$= f(E_{p+k})(1 - f(E_p)) \tag{217}$$

We can match on to [3] eqn 3-163 on pg 142 by redefining $p \to -p-k$ and using the even parity of E_p

$$\frac{1}{g_s}\chi_0 = \int \frac{d^3p}{(2\pi)^3}\frac{f(E_p)-f(E_{p+k})}{\hbar\omega+(E_{p+k}-E_p)+i\eta} \tag{218}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{f(E_{-p-k})-f(E_{-p})}{\hbar\omega+(E_{-p}-E_{-p-k})+i\eta} \tag{219}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{f(E_{p+k})-f(E_p)}{\hbar\omega+(E_p-E_{p+k})+i\eta} \tag{220}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{f(E_{p+k})-f(E_p)}{\hbar\omega-\Delta E+i\eta} \tag{221}$$

$$= \frac{k^2}{4\pi e^2}4\pi\alpha_0 \tag{222}$$

Where we can match onto their definition of α_0 (the polarizability) through the RPA definition of the dielectric function in [3]. This can also be conveniently written as

$$\frac{1}{g_s}\chi_0 = \int \frac{d^3p}{(2\pi)^3}\left[\frac{f(E_{p+k})}{\hbar\omega-(E_{p+k}-E_p)+i\eta} - \frac{f(E_p)}{\hbar\omega-(E_{p+k}-E_p)+i\eta}\right] \tag{223}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{f(E_p)}{\hbar\omega-(E_p-E_{p-k})+i\eta} - \frac{f(E_p)}{\hbar\omega-(E_{p+k}-E_p)+i\eta}\right] \tag{224}$$

$$= \int \frac{d^3p}{(2\pi)^3}f(E_p)\left[\frac{1}{\hbar\omega-(E_p-E_{p-k})+i\eta} - \frac{1}{\hbar\omega-(E_{p+k}-E_p)+i\eta}\right] \tag{225}$$

We now write the angular dependence in the denominator in a convenient explicit way.

$$\hbar\omega-(E_p-E_{p-k}) = \hbar\omega - \left(\frac{\hbar^2p^2}{2m} - \frac{\hbar^2(p-k)^2}{2m}\right) \tag{226}$$

$$= \hbar\omega - \frac{\hbar^2}{2m}\left(2pku-k^2\right) \tag{227}$$

$$= \frac{\hbar^2k}{m}\left(\frac{\omega m}{\hbar k}+\frac{k}{2}-pu\right) \tag{228}$$

$$\hbar\omega-(E_{p+k}-E_p) = \hbar\omega - \left(\frac{\hbar^2(p+k)^2}{2m} - \frac{\hbar^2p^2}{2m}\right) \tag{229}$$

$$= \hbar\omega - \frac{\hbar^2}{2m}\left(2pku+k^2\right) \tag{230}$$

$$= \frac{\hbar^2k}{m}\left(\frac{\omega m}{\hbar k}-\frac{k}{2}-pu\right) \tag{231}$$

Defining $p_\pm = \frac{\omega m}{\hbar k} \pm \frac{k}{2}$ we can then write the linear response function as follows

$$\frac{1}{g_s}\chi_0 = \frac{2\pi m}{\hbar^2 k}\int_0^\infty \frac{dp\ p^2}{(2\pi)^3}f(E_p)\int_{-1}^1 dy\left[\frac{1}{p_+-py+i\eta} - \frac{1}{p_--py+i\eta}\right] \tag{232}$$

We proceed to evaluate the real and imaginary parts of the linear response function independently using the Kramers Kronig relations valid for any function that is analytic in the upper half ω plane and vanishes faster than $|\omega|^{-1}$ as $\omega \to \infty$.

$$\mathrm{Re}\chi_0(k,\omega) = \frac{1}{\pi}\mathcal{P}\int_{\mathbb{R}}\frac{d\omega'}{\omega'-\omega}\mathrm{Im}\chi_0(k,\omega') \tag{233}$$

$$\mathrm{Im}\chi_0(k,\omega) = \frac{-1}{\pi}\mathcal{P}\int_{\mathbb{R}}\frac{d\omega'}{\omega'-\omega}\mathrm{Re}\chi_0(k,\omega') \tag{234}$$

Using the Cauchy Dirac relation (104) we have for the imaginary part

$$\mathrm{Im}\chi_0 = -\pi g_s \int \frac{d^3p}{(2\pi)^3}[f(E_{p+k}) - f(E_p)]\delta(\hbar\omega - \Delta E) \tag{235}$$

and for the real part, using Mathematica

$$\begin{aligned}\mathrm{Re}\chi_0(k,\omega) =& - g_s \int \frac{d^3p}{(2\pi)^3}[f(E_{p+k}) - f(E_p)]\mathcal{P}\int_{\mathbb{R}} \frac{d\omega'}{\omega' - \omega}\delta(\hbar\omega' - \Delta E) && (236)\\ =& g_s \int \frac{d^3p}{(2\pi)^3}\frac{f(E_{p+k}) - f(E_p)}{\hbar\omega - \Delta E} && (237)\end{aligned}$$

Which, as we just showed, leads to

$$\mathrm{Re}\chi_0 = g_s \frac{2\pi m}{\hbar^2 k}\int_0^\infty \frac{dp\ p^2}{(2\pi)^3}f(E_p)\int_{-1}^1 dy\left[\frac{1}{p_+ - py} - \frac{1}{p_- - py}\right] \tag{238}$$

Our friend Mathematica tells us that

$$\int_{-1}^1 dy \frac{1}{p_\pm - py} = \frac{1}{p}\ln\left|\frac{p_\pm + p}{p_\pm - p}\right| \tag{239}$$

So our response function is reduced to a single integral over p

$$\mathrm{Re}\chi_0 = g_s \frac{2\pi m}{\hbar^2 k}\int_0^\infty \frac{dp\ p}{(2\pi)^3}f(E_p)\left(\ln\left|\frac{p_+ + p}{p_+ - p}\right| - \ln\left|\frac{p_- + p}{p_- - p}\right|\right) \tag{240}$$

We now change to dimensionless variables $p \to k_F\zeta$

$$\begin{aligned}\mathrm{Re}\chi_0 =& g_s \frac{2\pi m}{\hbar^2 k}\int_0^\infty \frac{dp\ p}{(2\pi)^3}f(E_p)\left(\ln\left|\frac{p_+ + p}{p_+ - p}\right| - \ln\left|\frac{p_- + p}{p_- - p}\right|\right) && (241)\\ =& g_s \frac{2\pi m k_F^2}{\hbar^2 k}\int_0^\infty \frac{d\zeta\ \zeta}{(2\pi)^3}f(E_{k_F\zeta})\left(\ln\left|\frac{\frac{p_+}{k_F} + \zeta}{\frac{p_+}{k_F} - \zeta}\right| - \ln\left|\frac{\frac{p_-}{k_F} + \zeta}{\frac{p_-}{k_F} - \zeta}\right|\right) && (242)\end{aligned}$$

We simplify by defining the following dimensionless variables

$$\begin{aligned}\frac{p_\pm}{k_F} =& \frac{\omega m}{\hbar k_F k} \pm \frac{k}{2k_F} && (243)\\ =& \frac{\omega}{v_F k} \pm \frac{k}{2k_F} && (244)\\ \equiv& u' \pm z && (245)\\ E_{k_F\zeta} =& \frac{\hbar^2 k_F^2 \zeta^2}{2m} && (246)\\ =& E_F\zeta^2 && (247)\end{aligned}$$

Which gives for the real part in our dimensionless variables

$$\mathrm{Re}\chi_0 = g_s \frac{\pi m}{\hbar^2(2\pi)^3}\frac{k_F}{z}\int_0^\infty \frac{d\zeta\ \zeta}{1 + e^{\beta\mu(\zeta^2 - 1)}}\left(\ln\left|\frac{u' + z + \zeta}{u' + z - \zeta}\right| - \ln\left|\frac{u' - z + \zeta}{u' - z - \zeta}\right|\right) \tag{248}$$

Where we have worked to lowest order in $(\beta E_F)^{-1}$ to take $\mu = E_F$ (which as we will soon see is valid throughout the earth for iron). The imaginary part of the linear response function becomes

$$\begin{aligned}
\mathrm{Im}\chi_0 =& g_s \mathrm{Im} \int \frac{d^3p}{(2\pi)^3} f(E_p) \left[\frac{1}{\hbar\omega - (E_p - E_{p-k}) + i\eta} - \frac{1}{\hbar\omega - (E_{p+k} - E_p) + i\eta} \right] && (249)\\
=& - \pi g_s \int \frac{d^3p}{(2\pi)^3} f(E_p) \left[\delta(\hbar\omega - (E_p - E_{p-k})) - \delta(\hbar\omega - (E_{p+k} - E_p)) \right] && (250)\\
=& - 2\pi^2 g_s \int_{\mathbb{R}^+} \frac{dp\ p^2}{(2\pi)^3} f(E_p) \int_{-1}^{1} dy \left[\delta\left(\frac{\hbar^2 k}{m}(p_+ - py) \right) - \delta\left(\frac{\hbar^2 k}{m}(p_- - py) \right) \right] && (251)
\end{aligned}$$

The angular integrals are non-vanishing exactly when $p > |p_\pm|$ so we have

$$\int_{-1}^{1} du \delta\left(\frac{\hbar^2 k}{m}(p_\pm - py) \right) = \frac{m}{\hbar^2 kp} \Theta(p - |p_\pm|) \tag{252}$$

The imaginary part of the linear response function is then

$$\begin{aligned}
\mathrm{Im}\chi_0 =& - g_s \frac{2\pi^2 m}{\hbar^2 k} \left(\int_{|p_+|}^{\infty} - \int_{|p_-|}^{\infty} \right) \frac{dp\ p}{(2\pi)^3} f(E_p) && (253)\\
=& g_s \frac{2\pi^2 m}{\hbar^2 k} \left(\int_{|p_-|}^{\infty} - \int_{|p_+|}^{\infty} \right) \frac{dp\ p}{(2\pi)^3} f(E_p) && (254)\\
=& g_s \frac{2\pi^2 m}{\hbar^2 k} \int_{|p_-|}^{|p_+|} \frac{dp\ p}{(2\pi)^3} f(E_p) && (255)\\
=& g_s \frac{\pi^2 m}{\hbar^2} \frac{k_F}{z} \int_{|u'-z|}^{|u'+z|} \frac{d\zeta\ \zeta}{(2\pi)^3} \frac{1}{1 + e^{\beta\mu(\zeta^2 - 1)}} && (256)
\end{aligned}$$

3.1 Physical Scales and 0 Temperature Limit

In the model of a free electron gas, there are two relevant physical scales. The temperature β^{-1} and the chemical potential μ. We are concerned with energy loss in the earth, for which the dominant material contribution to the energy loss will be from iron. So we must consider ambient conditions to conditions in the earth's core, for which the temperature falls in the interval

$$\beta^{-1} \in (290,\ 5470)\ K\ k_B = (0.025,\ 0.5)\ eV \tag{257}$$

The density of valence electrons in the Earth are determined by the mass density, and properties of iron. We find for the iron in the earth

$$n_{iron} \in (1.1,\ 1.9)\ 10^{12} eV^3 \tag{258}$$

So we find for the Fermi energy of the free electron gas of valence electrons.

$$E_F \in (1.1, 1.5) keV \tag{259}$$

Since from (205) the chemical potential varies as $\mu = E_F(1+\mathcal{O}(\beta E_F)^{-2})$, we can take $\mu \equiv E_F$ throughout our calculations. Further, we will be able to expand in large $\beta\mu$ to a good approximation. Specifically we have

$$\beta\mu \approx \beta E_F \in (3.2\times 10^3,\ 4.2\times 10^4) \tag{260}$$

Where the upper boundary corresponds to the Earth's crust, and the lower boundary corresponds to the core. The only dimensionful quantity in the integrals for the real or imaginary parts appear as $\beta\mu$. Further, this is the only temperature dependence of the response function. So to an excellent approximation for iron in the earth it is valid to take the limit $\beta\mu\to\infty$. Since $\zeta > 0$ we then find

$$\lim_{\beta\mu\to\infty}\frac{1}{1+e^{\beta\mu(\zeta^2-1)}} = \Theta(1-\zeta) \tag{261}$$

Working in this limit we can then write for the real part

$$\begin{aligned}
\mathrm{Re}\chi_0 =& g_s\frac{\pi m}{\hbar^2(2\pi)^3}\frac{k_F}{z}\int_0^1 d\zeta\ \zeta\left(\ln\left|\frac{u'+z+\zeta}{u'+z-\zeta}\right| - \ln\left|\frac{u'-z+\zeta}{u'-z-\zeta}\right|\right) && (262)\\
=& g_s\frac{\pi m}{\hbar^2(2\pi)^3}\frac{k_F}{z}\left(2z+\frac{1}{2}(1-(u'+z)^2)\ln\left|\frac{1+u'+z}{1-(u'+z)}\right| - \frac{1}{2}(1-(u'-z)^2)\ln\left|\frac{1+u'-z}{1-(u'-z)}\right|\right) && (263)\\
=& g_s\frac{mk_F}{2\pi^2\hbar^2}\left(\frac{1}{2}+\frac{1}{8z}(1-(u'+z)^2)\ln\left|\frac{u'+z+1}{u'+z-1}\right| + \frac{1}{8z}(1-(u'-z)^2)\ln\left|\frac{z-u'+1}{z-u'-1}\right|\right) && (264)\\
\equiv& \frac{mk_F}{2\pi^2\hbar^2}f_1(u',z) && (265)
\end{aligned}$$

and for the imaginary part in this limit we find

$$\begin{aligned}
\mathrm{Im}\chi_0 =& g_s\pi\frac{\pi m}{\hbar^2(2\pi)^3}\frac{k_F}{z}\int_{|u'-z|}^{|u'+z|} d\zeta\ \zeta\Theta(1-\zeta) && (266)\\
=& g_s\pi\frac{\pi m}{\hbar^2(2\pi)^3}\frac{k_F}{z}\begin{cases} 2u'z, & u'+z<1\\ \frac{1}{2}\left(1-(u'-z)^2\right), & |u'-z|<1<u'+z\\ 0, & 1<|u'-z|\end{cases} && (267)\\
=& g_s\frac{mk_F}{2\pi^2\hbar^2}\begin{cases} \frac{\pi}{2}u', & u'+z<1\\ \frac{\pi}{8z}\left(1-(u'-z)^2\right), & |u'-z|<1<u'+z\\ 0, & 1<|u'-z|\end{cases} && (268)\\
\equiv& g_s\frac{mk_F}{2\pi^2\hbar^2}f_2(u',z) && (269)
\end{aligned}$$

Notice also that $\mathrm{Re}\chi_0$ is even and $\mathrm{Im}\chi_0$ is odd in u' (equivalently in ω). So we can define $u\equiv|u'|$ and find

$$\begin{aligned}
\mathrm{Re}\chi_0(u,z) =& \mathrm{Re}\chi_0(u',z) && (270)\\
\mathrm{Im}\chi_0(u,z) =& \mathrm{sgn}(u')\mathrm{Im}\chi_0(u',z) && (271)
\end{aligned}$$

So we find for the Lindhard dielectric function

$$
\begin{aligned}
\epsilon_{Lin}(u', z) =& 1 + \frac{4\pi e^2}{k^2}\chi_0 && (272)\\
=& 1 + \frac{4\pi e^2}{k^2}\frac{g_s m k_F}{2\pi^2\hbar^2}\left(f_1(u', z) + i f_2(u', z)\right) && (273)\\
=& 1 + \frac{g_s e^2}{2\pi\hbar v_F}\frac{1}{z^2}\left(f_1(u', z) + i f_2(u', z)\right) && (274)\\
\equiv& 1 + \frac{g_s \chi^2}{2z^2}\left(f_1(u', z) + i f_2(u', z)\right) && (275)\\
=& 1 + \frac{\chi^2}{z^2}\left(f_1(u', z) + i f_2(u', z)\right) && (276)
\end{aligned}
$$

Which matches precisely onto [19] and [7]. It is also worth noting that using this function in the energy loss in the limit of $v \gg v_F$ and $m_\chi \gg m$ we find the Bethe formula for the energy loss of swift ions [19] ie. Rutherford Scattering with IR and UV cutoffs in the momentum integral.

3.1.1 Lindhard Function and Energy Loss Region of Integration

As we showed above the Lindhard Function is given by

$$
\epsilon_{Lin}(\omega, k) = 1 + \frac{\chi^2}{z^2}\left[f_1\left(u', z\right) + i f_2\left(u', z\right)\right] \tag{277}
$$

$$
z = \frac{k}{2k_F} \qquad u' = \frac{\omega}{k v_F} \qquad u = |u'| \qquad \chi^2 = \frac{e^2}{\pi\hbar v_F} \tag{278}
$$

$$
\begin{aligned}
f_1(u, z) =& \frac{1}{2} + \frac{1}{8z}[g(z-u) + g(z+u)] && (279)\\
f_2(u, z) =& \begin{cases} \frac{\pi}{2}u, & z+u<1 \\ \frac{\pi}{8z}(1-(z-u)^2), & |z-u|<1<z+u \\ 0, \quad |z-u|>1 \end{cases} && (280)\\
g(x) =& (1-x^2)\ln\left|\frac{1+x}{1-x}\right| && (281)
\end{aligned}
$$

From this expression we find for the energy loss

$$
\begin{aligned}
\lim_{\beta\to\infty}\frac{dE}{dr}(v) =& \lim_{\beta\to\infty}\int_0^\infty \frac{dq}{(2\pi)^2}\frac{1}{v^2}\frac{4\pi\kappa^2 e^2}{q}\int_{-\omega_+}^{\omega_-} d\omega\ \frac{2\omega}{1-e^{-\beta\omega}}\mathrm{Im}\left(\frac{-1}{\epsilon^{RPA}}\right) && (282)\\
=& \frac{1}{v^2}\frac{8\pi\kappa^2 e^2}{(2\pi)^2}\int_0^\infty \frac{dq}{q}\int_{-\omega_+}^{\omega_-} d\omega\ \omega\mathrm{Im}\left(\frac{-1}{\epsilon_{Lin}}\right) && (283)\\
& && (284)
\end{aligned}
$$

**** should change the limit to $\beta\mu$ and keep the factor of $(1 - e^{-\beta\hbar\omega})$? In that case we would have $\hbar\omega = \hbar u k v_F = \hbar 2 u z k_F v_F = 4uzE_F$

$$
\begin{aligned}
\frac{1}{1-e^{-\beta\omega\hbar}} =& \frac{1}{1-e^{-\beta 4uzE_F}} && (285)\\
\approx& 1 + \Theta(1-\beta 4uzE_F)\left(\frac{1}{1-e^{-\beta 4uzE_F}} - 1\right) && (286)
\end{aligned}
$$

$$\lim_{\beta\to\infty}\frac{dE}{dr}(v) = \lim_{\beta\to\infty}\int_0^\infty \frac{dq}{(2\pi)^2}\frac{1}{v^2}\frac{4\pi\kappa^2e^2}{q}\int_{-\omega_+}^{\omega_-} d\omega\ \frac{2\omega}{1-e^{-\beta\omega}}\text{Im}\left(\frac{-1}{\epsilon^{RPA}}\right) \tag{287}$$

$$\neq \frac{1}{v^2}\frac{8\pi\kappa^2e^2}{(2\pi)^2}\int_0^\infty \frac{dq}{q}\int_{-\omega_+}^{\omega_-} d\omega\ \omega\text{Im}\left(\frac{-1}{\epsilon_{Lin}}\right) \tag{288}$$

$$\tag{289}$$

as limit should be $\lim_{\beta E_F\to\infty}$, and $\beta\hbar\omega$ is not necessarily large in this limit. What we should have is

$$\lim_{\beta E_F\to\infty}\frac{dE}{dr}(v) = \frac{1}{v^2}\frac{8\pi\kappa^2e^2}{(2\pi)^2}\int_0^\infty \frac{dq}{q}\int_{-\omega_+}^{\omega_-} d\omega\ \frac{\omega}{1-e^{-\beta\omega}}\text{Im}\left(\frac{-1}{\epsilon_{Lin}}\right) \tag{290}$$

$$\tag{291}$$

*****

And

$$\lim_{\beta\to\infty}\frac{d\sigma}{dE_R}(v,\omega) = \frac{1}{n_Tv^2}\frac{8\pi\kappa^2e^2}{(2\pi)^2}\int_0^\infty \frac{dq}{q}\text{Im}\left(\frac{-1}{\epsilon_{Lin}}\right)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{292}$$

Where again $\omega_\pm = qv \pm \frac{\hbar q^2}{2m_\chi}$. Note that the discrepancy between the limits of integration in the ω integral compared to [19] arise because they took the limit of a heavy charged particle when evaluating the angular delta function in (37). Setting the q^2 term to 0 there yields the same result. Changing to $u,\ z$ variables.

$$dz = \frac{dq}{2k_F} \qquad du = \frac{d\omega}{qv_F} \tag{293}$$

$$d\omega\ |\omega| = du\ u(qv_F)^2 = du\ u(2zk_Fv_F)^2 \tag{294}$$

$$u'_\pm \equiv \frac{\omega_\pm}{qv_F} = \frac{v}{v_F} \pm \frac{\hbar q}{2m_\chi v_F} \tag{295}$$

$$= \frac{v}{v_F} \pm \frac{\hbar z k_F}{m_\chi v_F} = \frac{v}{v_F} \pm z\frac{m}{m_\chi} \tag{296}$$

$$\implies u_\pm \equiv \frac{|\omega_\pm|}{qv_F} = \left|\frac{v}{v_F} \pm \frac{\hbar q}{2m_\chi v_F}\right| \tag{297}$$

$$= \left|\frac{v}{v_F} \pm \frac{\hbar z k_F}{m_\chi v_F}\right| = \left|\frac{v}{v_F} \pm z\frac{m}{m_\chi}\right| \tag{298}$$

We note that for $\omega < 0$

$$\omega\text{Im}\left(\frac{-1}{\epsilon_{Lin}(q,\omega)}\right) = -|\omega|\text{Im}\left(\frac{-1}{\epsilon_{Lin}(q,-|\omega|)}\right) \tag{299}$$

$$= |\omega|\text{Im}\left(\frac{-1}{\epsilon_{Lin}(q,|\omega|)}\right) \tag{300}$$

where we've used (105). And clearly, this also holds for $\omega > 0$. Using the even parity of the integrand and Figure 1 we have

$$\int_0^\infty dz\int_{-u'_+}^{u'_-} du = \int_{A\cup B\setminus C} dz\ du \tag{301}$$

$$= \int_0^\infty dz\int_0^{u_+} du + \int_0^{z_t} dz\int_0^{u_-} du - \int_{z_t}^\infty dz\int_0^{u_-} du \tag{302}$$

$$\equiv \int_{\Sigma_{uz}} dzdu \tag{303}$$

Notice also that in the limit $\frac{m_\chi}{m} \to \infty$ regions A and B in Figure 1 are equal and region C vanishes, reproducing the bounds given by [19] for the energy loss of a heavy charged particle.

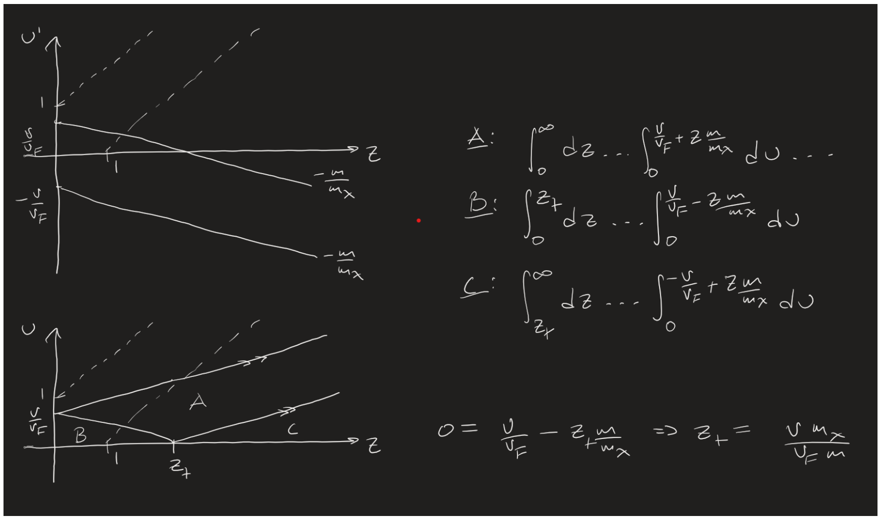

Figure 1: Diagram of region of integration in u' and u variables. The dashed lines correspond to the boundary of the particle hole regime, outside of which the imaginary part of the Lindhard dielectric vanishes.

Hence, we find for the energy loss

$$\frac{dE}{dr}(v) = \frac{1}{v^2}\frac{8\pi\kappa^2 e^2}{(2\pi)^2}\int_{\Sigma_{uz}} \frac{dz}{z}(du\ u)(2zk_F v_F)^2 \mathrm{Im}\left(\frac{-1}{\epsilon_{Lin}}\right) \tag{304}$$

$$= \frac{(k_F v_F)^2}{v^2}\frac{8\kappa^2 e^2}{\pi}\int_{\Sigma_{uz}} (dz\ z)(du\ u)\ \mathrm{Im}\left(\frac{-1}{\epsilon_{Lin}}\right) \tag{305}$$

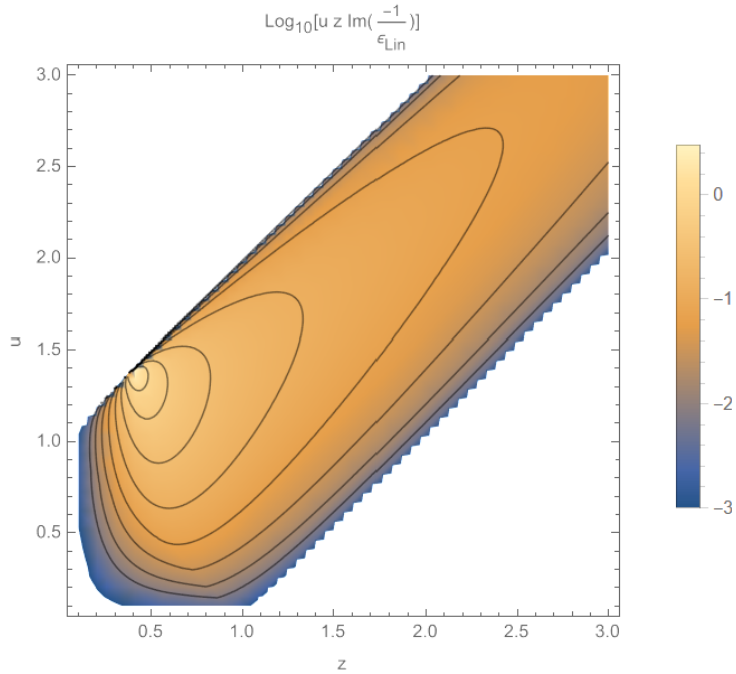

Figure 2: Lindhard Dielectric Energy Loss Integrand

3.2 κ_{min} for Capture in the Earth

3.3 Plasmon Resonance Avoidance

For a range of kinematics of the scattering DM particle, the energy loss will get contributions from the resonance curve corresponding to plasmons in the material. This region is determined by the equation

$$\epsilon^{Lin}(k,\omega) = 0 \tag{306}$$

This appears along a curve where $|z-u| > 1$ and so f_2 vanishes. This region can be avoided by expanding around the point where it meets the non-singular region and using equipartition rules to relate it to particle-hole (non-singular) parts of the integral [7]. We look to avoid this task for the time being and consider instead kinematics for which we can avoid the resonance curve. The point $(u,z) = (u_c, u_c - 1)$ where the curve meets the particle hole regime [7] is given by [3]

$$u_c \approx \frac{\omega_c}{k_c v_F} = \frac{1}{k_c v_F}\left(k_c v_F + \frac{\hbar k_c^2}{2m v_F}\right) = 1 + \frac{\hbar k_c}{2m v_F} \tag{307}$$

$$k_c \approx \frac{\omega_p}{v_F} \qquad \omega_p^2 = \frac{4\pi e^2 n_e}{m} \tag{308}$$

We look now to choose kinematics such that the boundary that the resonance curve meets is away from this meeting point.

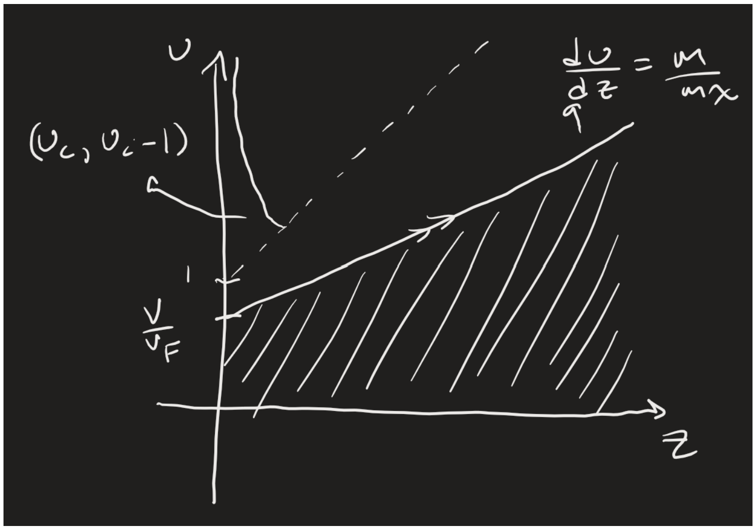

Figure 3: Diagram of region of integration (hatched region) and plasmon line. The dashed line corresponds to a boundary of the particle-hole regime

The boundary of the region of integration which could meet the plasmon line is given by

$$u_+(z) = \left|\frac{v}{v_F} + z\frac{m}{m_\chi}\right| = \frac{v}{v_F} + z\frac{m}{m_\chi} \tag{309}$$

The condition for avoidance of the plamson line is then

$$u_c > u_+(u_c - 1) \tag{310}$$
$$u_c > \frac{v}{v_F} + (u_c - 1)\frac{m}{m_\chi} \tag{311}$$
$$u_c\left(1 - \frac{m}{m_\chi}\right) > \frac{v}{v_F} - \frac{m}{m_\chi} \tag{312}$$
$$\tag{313}$$

This condition can be examined through the sign of x given by

$$x = u_c\left(1 - \frac{m}{m_\chi}\right) + \frac{m}{m_\chi} - \frac{v}{v_F} \tag{314}$$

For $x > 0$, the condition is satisfied and we avoid the plasmon resonance curve. Otherwise the region of integration crosses the plasmon resonance. For a given mass, the contour determining v that avoid the plasmon line is

$$\frac{v}{v_F} = u_c\left(1 - \frac{m}{m_\chi}\right) + \frac{m}{m_\chi} \tag{315}$$
$$= u_c + \frac{m}{m_\chi}(1 - u_c) \tag{316}$$

Hence the minimum value of m_χ that can avoid the plasmon line is given by

$$0 = u_c + \frac{m}{m_\chi}(1 - u_c) \tag{317}$$
$$\implies \frac{m}{m_\chi} = \frac{u_c}{u_c - 1} \tag{318}$$

Note that the contribution from plamson resonances can also be calculated using equipartition rules that relate the value of the integral along the plasmon line to values in the particle hole regime at fixed u using an equipartition rule derived in [19]. We do not undertake this task.

3.4 Energy Loss Earth's Core - 0 Temperature FEG

As a toy model, we conservatively treat the Earth as a solid ball of 0 Temperature gas of electrons with the same density as the number density of valence electrons in the Earth's iron core. Based on the above analysis we begin first with a coarse grid of 6 masses and 4 velocities at the surface of the Earth which avoid the plasmon resonances.

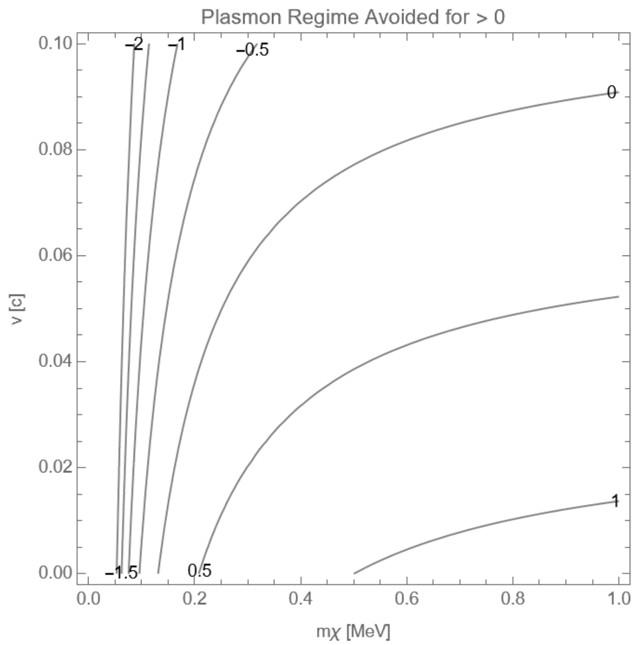

Figure 4: Contours for avoidance of the plasmon resonances given by (314)

We choose a kinematic boundary near the $x = 0$ contour.

$$m_\chi \in \{5 \times 10^5, 2.5 \times 10^6, 2.5 \times 10^7, 5 \times 10^8, 2.5 \times 10^9, 2.5^{10}\}\ eV \tag{319}$$
$$v \in \{2.5 \times 10^{-5}, 2.5 \times 10^{-4}, 2.5 \times 10^{-3}, 2.5 \times 10^{-2}\}\ c \tag{320}$$

Where the velocities correspond to the values at the *surface* of the Earth. Numerically integrating the (437) in Mathematica we find for the Energy Loss in units of eVm^{-1}

$m_\chi[eV]\backslash v[c]$	2.5×10^{-5}	2.5×10^{-4}	2.5×10^{-3}	2.5×10^{-2}
2.5×10^5	2.8×10^{17}	2.8×10^{16}	2.8×10^{15}	2.8×10^{14}
2.5×10^6	8.5×10^{15}	8.5×10^{14}	8.6×10^{13}	2.3×10^{13}
2.5×10^7	7.7×10^{13}	7.8×10^{12}	$2. \times 10^{12}$	1.3×10^{13}
2.5×10^8	7.8×10^{11}	$2. \times 10^{11}$	1.3×10^{12}	1.3×10^{13}
2.5×10^9	$2. \times 10^{10}$	1.3×10^{11}	1.3×10^{12}	6.4×10^{12}
2.5×10^{10}	1.3×10^{10}	1.3×10^{11}	6.3×10^{11}	6.4×10^{12}

(321)

****** need to understand why this varies linearly in v for large m they mention it in lindhard equipartition paper [19]

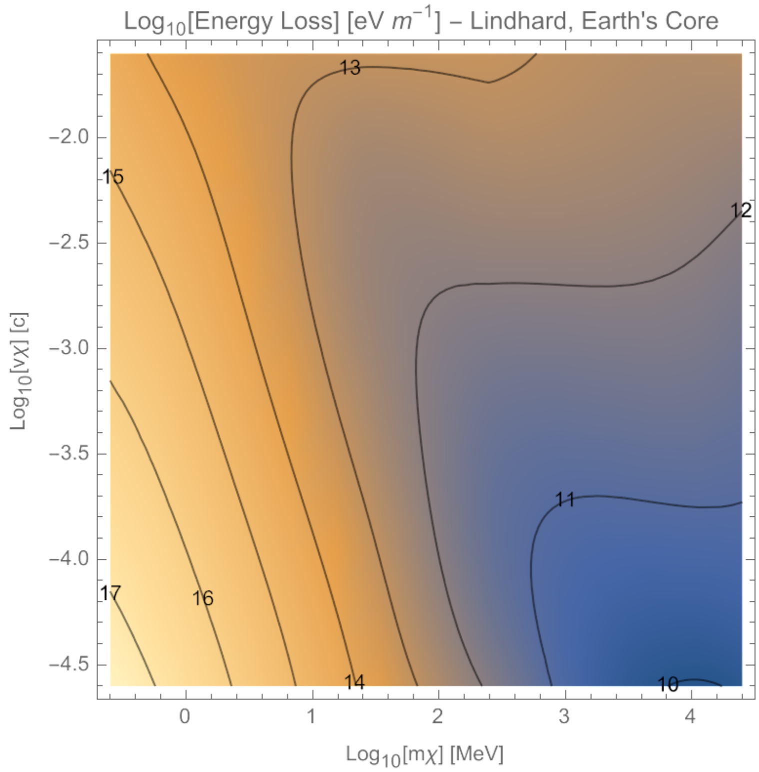

Figure 5: Lindhard Dielectric Energy Loss Integrand

3.5 κ_{min} for Capture in the Earth

We look to compute the minimum value of the kinetic mixing for the capture length of DM is less than the diameter of the Earth. We note that considering non-zero impact parameter will only increase the necessary kinetic mixing. This is determined by the condition that particle losses all it's initial kinetic energy as it traverses the earth. For mixings larger than this the particle will not be able to escape to infinity and be bound to the Earth.

$$\frac{1}{2}m_\chi v_i^2 = \int_0^{2R} dr \frac{dE}{dr}[v(r)] \tag{322}$$

Where v_i is the initial velocity of the particle and R is the radius of the Earth. We can solve this for $v(r)$ as an ODE and then vary κ so that the zero of $v(r)$ coincides with the diameter of the Earth. Our outline of the algorithm to determine κ_{min} is as follows

1. Calculate the energy loss (with κ set to 1) over a grid of m_χ and v $(m_{min}, m_{max}) \times (v_{min}, v_{max})$
2. Determine an interpolating function $f(m_\chi, v)$ over this grid (we take interpolation of order 1 in Mathematica to ensure postitivity and prevent overshooting)
3. Solve the following ODE at fixed κ on the interval $(0, 2R)$

$$\frac{dE(r,\kappa)}{dr} = -\kappa^2 f\left(m_\chi, \sqrt{\max\left[\frac{2}{m_\chi}E(r,\kappa), v_{min}^2\right]}\right) \tag{323}$$

$$E(0,\kappa) = \frac{1}{2}m_\chi v^2 \tag{324}$$

we note that imposing the restriction on the interpolating function domain in the ODE is a necessary but uncontrolled approximation. To avoid this issue one must choose a grid that extend to sufficiently low v_{min} so that the root is found before the restriction is enforced.

4. Find the root in κ of $E(2R, \kappa) - E_{escape}$. This zero corresponds to κ_{min}

This is a ham handed version of the Shooting Method to solve our Boundary Value Problem. This yield the following table for the minimum mixing needed for capture. [4]

$m_\chi[eV]\backslash v[c]$	1.3×10^{-5}	2.1×10^{-4}	2.5×10^{-3}	2.5×10^{-2}
2.5×10^{5}	1.1×10^{-14}	1.2×10^{-13}	3.8×10^{-12}	1.2×10^{-10}
2.5×10^{6}	$2. \times 10^{-13}$	2.2×10^{-12}	$7. \times 10^{-11}$	1.5×10^{-9}
2.5×10^{7}	6.6×10^{-12}	7.2×10^{-11}	1.6×10^{-9}	$1. \times 10^{-8}$
2.5×10^{8}	2.3×10^{-10}	1.7×10^{-9}	9.5×10^{-9}	$3. \times 10^{-8}$
2.5×10^{9}	3.2×10^{-9}	9.2×10^{-9}	3.1×10^{-8}	1.2×10^{-7}
2.5×10^{10}	1.2×10^{-8}	2.9×10^{-8}	1.2×10^{-7}	4.6×10^{-7}

(325)

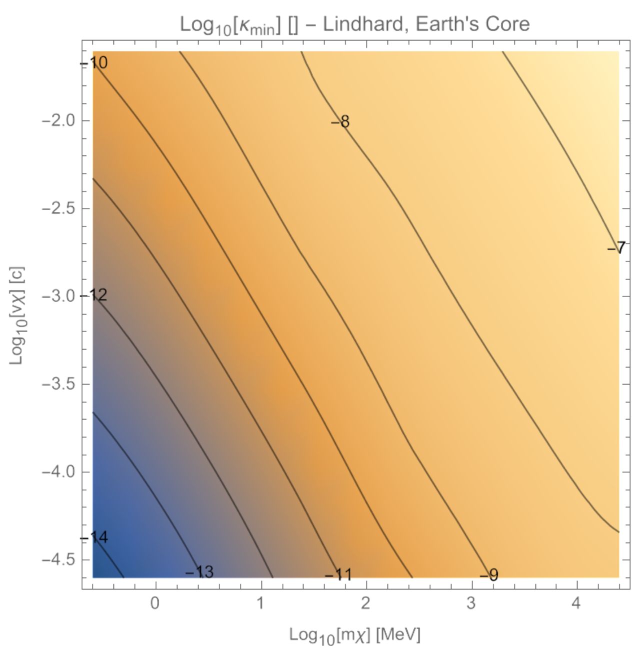

Figure 6: Lindhard Dielectric κ_{min}

4 Dielectric Function in the Random Time Approximation

Instead of using the equipartition rule [19] to extend the region of integration and include the plasmon line, we switch to Mermin dielectric which is non-singular. This is also more realistic as it accounts for the decay of the plasmon excitations. Further, this lifetime is sensitive to the properties of density and temperature of the material.

The Mermin dielectric function [20] extends the Lindhard Dielectric to include a finite width for the plasmons. [21] derives the mermin dielectric function. We reproduce their results here for completeness and later use.

[4]Both the κ_{min} and energy loss tables were generated using version 5 of the Mathematica code AdM capture

4.1 Mermin Dielectric Derivation using Quantum Vlasov equation in the RTA - Ignoring DM Self Interaction

To derive the Mermin dielectric, we start with the quantum boltzmann equation for the single particle reduced number density operator of species α in the material $\hat{f}_\alpha$. This is the Heisenberg equation of motion in the mean field approximation [22].

$$i\hbar\frac{d\hat{f}_\alpha}{dt} = i\hbar\frac{\partial\hat{f}_\alpha}{\partial t} - [\hat{H}_\alpha, \hat{f}_\alpha] = \hat{I}_\alpha \tag{326}$$

Where $\hat{I}_\alpha$ describes the collisions rate of change of single particle reduced density operator analagous to a collision integral in classical Boltzmann equations. Taking the expectation value we find

$$\begin{aligned} I_{k',k,\alpha} =& i\hbar\frac{\partial f_{k',k,\alpha}}{\partial t} - \langle k'|[\hat{H}_{\alpha,0} + \hat{U}^{eff}_\alpha, \hat{f}_\alpha]|k\rangle && (327)\\ =& i\hbar\frac{\partial f_{k',k,\alpha}}{\partial t} - (E_{k',\alpha} - E_{k,\alpha})f_{k',k} - \langle k'|[\hat{U}^{eff}_\alpha, \hat{f}_\alpha]|k\rangle && (328)\end{aligned}$$

**** compare to references, they have factors of number density here which eventually cancel out

***** also go back to IQM notes. How do we interpret off diagonal terms in the reduced density operator

where we have defined $\langle k'|\hat{\mathcal{O}}|k\rangle = \mathcal{O}_{k',k}$, $\hat{H}_{0,\alpha}$ is the equilibrium hamiltonian, and $\hat{U}^{eff}_\alpha$ is the interparticle interaction potential. Note that the Hamiltonian needn't be a simple kinetic term. For example we can take

$$\hat{H}_{\alpha,0} = \frac{\hbar^2\hat{k}^2}{2m_\alpha} + m_\alpha\phi_{grav}(r) + e_\alpha\phi_{earth}(r) \tag{329}$$

**** need to include ambient earth here since it contributes to the equlibrium configuration [23]. That leads to macroscopic charge separation. The dielectric only cares about individual charges and their behaviour in material and the resultant response

-¿ assuming the state is an eigenstate of the grav potential (or say it is an c-number vev with externally varying parameter), then it can be absorbed into the chamical potential and ignored.

For the interaction potential, we define

$$\begin{aligned} \hat{U}^{eff}_\alpha =& e_\alpha(\phi_{ext} + \phi_{ind}) = e_\alpha\frac{\phi_{ext}}{\epsilon^M} && (330)\\ \phi_{ext} =& V_{ext}(q)\rho_{ext} && (331)\\ \phi_{ind} =& V_{ext}(q)\rho_q && (332)\\ \rho_q =& \sum_\beta e_\beta F_\beta(q) n^{(1)}_{q,\beta}(\omega) && (333)\\ \implies \frac{\phi_{ext}}{\epsilon^M} =& \phi_{ext} + V_{ext}(q)\rho_q && (334)\\ \implies \epsilon^M =& 1 - \frac{\epsilon^M}{\rho_{ext}}\sum_\beta e_\beta n^{(1)}_{q,\beta}(\omega) && (335)\end{aligned}$$

***** should insert argument in onenote following [4]. Diagramatically, the induced polarization (and screening) shouldn't depend on the mixing matrix

Where $n^{(1)}_{q,\beta}(\omega)$ is the induced number density due to the perturbing field, we have defined the Mermin dieletric, and ρ_q is the total induced charge. Notice also that we have included a form factor in the effective charge density which will allow us to consider nuclear and atomic scattering (the result for the dielectric is unchanged if we instead include it in the effective charge density). Here $n^{(1)}_{q,\beta}(\omega)$ describes the number of effective atoms / nuclei / charges induced by the interaction, and $e_\beta F_\beta(q)$ describes the effective charge of each scattering particle. For concreteness, we ignore DM self interactions for the time being. So we are essentially computing the following diagram

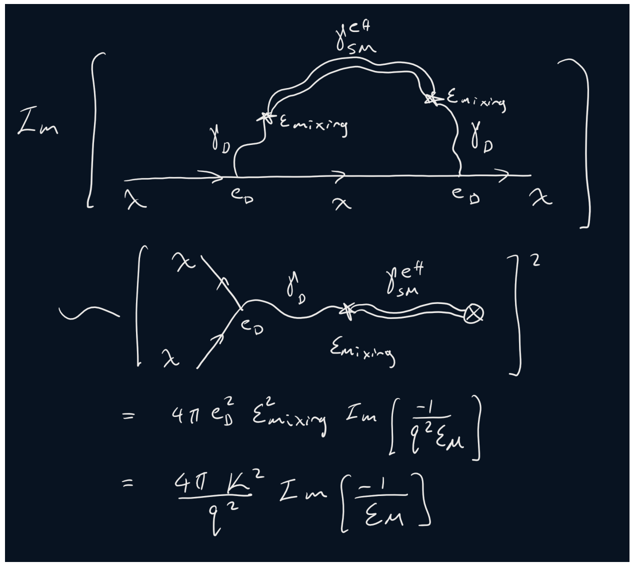

Figure 7: Diagram Governing Multispecies DM-SM interactions

We take for our equilibrium solution (in the absence of collisions or our effective interaciton potential)

$$f^{(0)}_{k',k,\alpha} = \delta_{k',k} f^{(0)}_{k,\alpha} = \delta_{k',k} \frac{1}{e^{\beta(E_{k,\alpha}-\mu)} + 1} \tag{336}$$

Where $E_{k,\alpha}$ is an eigenvalue of the equilibrium Hamiltonian. Note however that none of what follows depends on the details of the equilibirum distribtuion (aside from it's time independence and the fact that it is diagonal in momentum). We have also defined the external potential

$$\phi_{ext} = \frac{4\pi\rho_{ext}}{\hat{k}^2} \tag{337}$$

**** need to think about this (see comment under equilibrium hamiltonian and citation)

Where $\phi_{earth}(r)$ is a c-number with r a locally constant external parameter. Following [21] and [22] we also take the Relaxation Time Approximation (RTA) in which we assume the deviation of the single particle reduced density operator from equilibrium are small. This is valid at late times in the non-equilibrium interaction [22]. This gives

$$\frac{df_{k',k,\alpha}}{dt} \approx I^{RTA}[f_{k',k,\alpha}] = -\frac{1}{\tau}(f_{k',k,\alpha} - f^{EQ}_{k',k,\alpha}) \tag{338}$$

where $f^{(0)}_{\alpha}$ is the equilibrium distribution of species α.

$$\frac{df_{k,\alpha}}{dt} = -\frac{1}{\tau}(f_{k,\alpha} - f^{(0)}_{k,\alpha}) \ , \qquad f_{k,\alpha}(t_0) = f^{t_0}_{k,\alpha} \tag{339}$$

$$\implies f_{k,\alpha}(t) = f^{t_0}_{k,\alpha} e^{-(t-t_0)/\tau} + f^{(0)}_{k,\alpha}(1 - e^{-(t-t_0)/\tau}), \ \forall t > t_0 \tag{340}$$

$$\tag{341}$$

Where we have defined $\nu = \tau^{-1}$. We now linearize the equation by considering a weak external potential.

$$f_{k,k',\alpha} = f^{(0)}_{k,k',\alpha} + f^{(1)}_{k,k',\alpha} + O(U^{eff}_{k',k,\alpha})^2 \tag{342}$$

We use the superscript numbers to denote the order in expansion of the perturbing potential. Since the leading order distribution is diagonal, the commutator vanishes and we find in the RTA. The linear term in (328) is given by

$$i\hbar \frac{\partial f^{(1)}_{k',k,\alpha}}{\partial t} - (E_{k',\alpha} - E_{k,\alpha}) f^{(1)}_{k',k} - \hat{U}^{eff}_{k',k,\alpha}(\hat{f}^{(0)}_{k,\alpha} - \hat{f}^{(0)}_{k',\alpha}) = I^{(1)}_{k',k,\alpha} \tag{343}$$

We now make the assumption that the plasma is locally homogenous. Namely that the large scale macroscopic fields due to captured DM charges deeper in the earth vary over much longer length scales than any of the relevant length scales in the plasma. We define

$$Q = \frac{k+k'}{2} \qquad q = k' - k \tag{344}$$

$$k' = Q + \frac{q}{2} \qquad k = Q - \frac{q}{2} \tag{345}$$

As argued in [21] the fourier transformed and homogeneous potential is insensitive to Q. So we have, and using notation $Q \pm q/2 \to \pm$ (so that $k' \to +$ and $k \to -$)

$$i\hbar \frac{\partial f^{(1)}_{+,-,\alpha}}{\partial t} - (E_{+,\alpha} - E_{-,\alpha}) f^{(1)}_{+,-} - U^{eff}_{q,\alpha}(f^{(0)}_{-,\alpha} - f^{(0)}_{+,\alpha}) = I^{(1)}_{+,-,\alpha} \tag{346}$$

Since the collision term relaxes the distribution to it's equilbrium, in the limit $t \to \infty$ we have $I^{(1)}_{+,-,\alpha} \to 0$ so we can write (with superscript ∞ denoting long time limit)

$$0 = (E_{+,\alpha} - E_{-,\alpha}) f^{(1)\infty}_{+,-,\alpha} + U^{eff,\infty}_{q,\alpha}(f^{(0)}_{-,\alpha} - f^{(0)}_{+,\alpha}) \tag{347}$$

$$f^{(1)\infty}_{+,-,\alpha} = U^{eff,\infty}_{q,\alpha} \frac{f^{(0)}_{+,\alpha} - f^{(0)}_{-,\alpha}}{E_{+,\alpha} - E_{-,\alpha}}, \qquad \omega = 0 \tag{348}$$

Where we have used the fact that to drop the time derivative term

$$\lim_{t\to\infty} f_{k',k,\alpha} = \delta_{k',k} f^{(0)}_{k,\alpha} \tag{349}$$

Fourier transforming in time

$$f^{(1)}_{+,-,\alpha}(t) = \int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{-i\omega t} f^{(1)}_{+,-,\alpha}(\omega) \tag{350}$$

and using the fact that since $f^{(0)}$ solves the leading term, and $f^{(1)}$ asymptotes to $f^{(1)\infty}$

$$I^{(1)}_{+,-,\alpha}(\omega) = -i\hbar\nu \left(f^{(1)}_{+,-,\alpha}(\omega) - U^{eff(1)\infty}_{q,\alpha} \frac{f^{(0)}_{+,\alpha} - f^{(0)}_{-,\alpha}}{E_{+,\alpha} - E_{-,\alpha}} \right) \tag{351}$$

$$\hbar\omega f^{(1)}_{+,-,\alpha} - (E_{+,\alpha} - E_{-,\alpha}) f^{(1)}_{+,-,\alpha} - \hat{U}^{eff(1)}_{q,\alpha}(f^{(0)}_{-,\alpha} - f^{(0)}_{+,\alpha}) = I^{(1)}_{+,-,\alpha}(\omega) \tag{352}$$

Solving for the linear correction to the distribution we have

$$f^{(1)}_{+,-,\alpha}(\omega) = \left(\hat{U}^{eff(1)}_{q,\alpha} - i\hbar\nu \frac{U^{eff(1)\infty}_{q,\alpha}}{E_{+,\alpha} - E_{-,\alpha}}\right) \frac{f^{(0)}_{-,\alpha} - f^{(0)}_{+,\alpha}}{\hbar\hat{\omega} - (E_{+,\alpha} - E_{-,\alpha})} \tag{353}$$

Where we have defined $\hat{\omega} = \omega + i\nu$. We now impose continuity to solve for $\hat{U}^{eff(1)\infty}_{q,\alpha}$ which contains implicit dependence on the distribution function.

$$\frac{\partial n(\vec{r},t)}{\partial t} + \nabla \boldsymbol{j}(\vec{r},t) = 0 \tag{354}$$

We also expand the number density and current in the perturbing potential. Since the equilibrium distribution is locally homogeneous and time independent, the derivatives in the continuity equation vanish. So we are left with, after fourier transforming

$$\omega n^{(1)}_{q,\alpha}(\omega) = \boldsymbol{q} \cdot \boldsymbol{j}^{(1)}_{q,\alpha}(\omega) \tag{355}$$

here they are defined by

$$n^{(1)}_{q,\alpha}(\omega) = g_s \int \frac{d^3Q}{(2\pi)^3} f^{(1)}_{Q+\frac{q}{2},Q-\frac{q}{2},\alpha}(\omega) \tag{356}$$

$$\boldsymbol{j}^{(1)}_{q,\alpha}(\omega) = g_s \int \frac{d^3Q}{(2\pi)^3} \frac{\hbar \boldsymbol{Q}}{m_\alpha} f^{(1)}_{Q+\frac{q}{2},Q-\frac{q}{2},\alpha}(\omega) \tag{357}$$

**** why integral over Q instead of q?
where we have included a factor g_s for the spin degeneracy. Defining

$$\Pi^{RPA}_{n\alpha} = g_s \int \frac{d^3Q}{(2\pi)^3} \left(\frac{\hbar \boldsymbol{Q}}{m_\alpha}\right)^n \frac{f^{(0)}_{-,\alpha} - f^{(0)}_{+,\alpha}}{\hbar\hat{\omega} - (E_{+,\alpha} - E_{-,\alpha})} \tag{358}$$

$$\Pi_{\nu n\alpha} = g_s \int \frac{d^3Q}{(2\pi)^3} \left(\frac{\hbar \boldsymbol{Q}}{m_\alpha}\right)^n \frac{1}{E_{+,\alpha} - E_{-,\alpha}} \times \frac{f^{(0)}_{-,\alpha} - f^{(0)}_{+,\alpha}}{\hbar\hat{\omega} - (E_{+,\alpha} - E_{-,\alpha})} \tag{359}$$

(360)

The continuity equation can be written in terms of the polarizations, their moments, and potentials using (353)

$$n^{(1)}_{q,\alpha}(\omega) = U^{eff(1)}_{q,\alpha} \Pi^{RPA}_{0\alpha} - i\hbar\nu U^{eff(1)\infty}_{q,\alpha} \Pi_{\nu 0\alpha} \tag{361}$$

$$\boldsymbol{j}^{(1)}_{q,\alpha}(\omega) = U^{eff(1)}_{q,\alpha} \Pi^{RPA}_{1\alpha} - i\hbar\nu U^{eff(1)\infty}_{q,\alpha} \Pi_{\nu 1\alpha} \tag{362}$$

Notice that $\Pi^{RPA}_{0\alpha}(q,\hat{\omega}) = -\chi_0(q,\hat{\omega})$. From this definition and using

$$E_{+,\alpha} - E_{-,\alpha} = \frac{\hbar^2(Q+\frac{q}{2})^2}{2m_\alpha} - \frac{\hbar^2(Q-\frac{q}{2})^2}{2m_\alpha} = \frac{\hbar^2 \boldsymbol{Q}\cdot\boldsymbol{q}}{m_\alpha} \tag{363}$$

we can also derive the soon to be useful identity

$$\hbar\boldsymbol{q}\cdot\Pi^{RPA}_{1\alpha} = \hbar\boldsymbol{q}\cdot g_s\int\frac{d^3Q}{(2\pi)^3}\frac{\hbar\boldsymbol{Q}}{m_\alpha}\frac{f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha}}{\hbar\hat{\omega}-(E_{+,\alpha}-E_{-,\alpha})} \tag{364}$$

$$= g_s\int\frac{d^3Q}{(2\pi)^3}(f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha})\frac{E_{+,\alpha}-E_{-,\alpha}}{\hbar\hat{\omega}-(E_{+,\alpha}-E_{-,\alpha})} \tag{365}$$

$$= g_s\int\frac{d^3Q}{(2\pi)^3}(f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha})\left(\frac{\hbar\hat{\omega}}{\hbar\hat{\omega}-(E_{+,\alpha}-E_{-,\alpha})}-1\right) \tag{366}$$

$$= \hbar\hat{\omega}\Pi^{RPA}_{0\alpha} - g_s\int\frac{d^3Q}{(2\pi)^3}(f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha}) \tag{367}$$

$$= \hbar\hat{\omega}\Pi^{RPA}_{0\alpha} \tag{368}$$

Notice that this implies the RPA approximation also solves the continuity equation (362) in the limit that $\nu\to 0$. Namely, the RPA conserves charge and is the collisionless limit of the RTA. Following identical logic we also have

$$\hbar\boldsymbol{q}\cdot\Pi_{\nu1\alpha} = \hbar\boldsymbol{q}\cdot g_s\int\frac{d^3Q}{(2\pi)^3}\frac{\hbar\boldsymbol{Q}}{m_\alpha}\frac{1}{E_{+,\alpha}-E_{-,\alpha}}\times\frac{f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha}}{\hbar\hat{\omega}-(E_{+,\alpha}-E_{-,\alpha})} \tag{369}$$

$$= \hbar\hat{\omega}\Pi_{\nu0\alpha} - g_s\int\frac{d^3Q}{(2\pi)^3}\frac{f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha}}{E_{+,\alpha}-E_{-,\alpha}} \tag{370}$$

$$= \hbar\hat{\omega}\Pi_{\nu0\alpha}(q,\hat{\omega}) + \Pi^{RPA}_{0\alpha}(q,0) \tag{371}$$

Notice we also have from this

$$\hbar\boldsymbol{q}\cdot\Pi_{\nu1\alpha} = \hbar\boldsymbol{q}\cdot g_s\int\frac{d^3Q}{(2\pi)^3}\frac{\hbar\boldsymbol{Q}}{m_\alpha}\frac{1}{E_{+,\alpha}-E_{-,\alpha}}\times\frac{f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha}}{\hbar\hat{\omega}-(E_{+,\alpha}-E_{-,\alpha})} \tag{372}$$

$$= g_s\int\frac{d^3Q}{(2\pi)^3}\frac{f^{(0)}_{-,\alpha}-f^{(0)}_{+,\alpha}}{\hbar\hat{\omega}-(E_{+,\alpha}-E_{-,\alpha})} \tag{373}$$

$$= \Pi^{RPA}_{0\alpha}(q,\hat{\omega}) \tag{374}$$

Combining the two we find

$$\Pi^{RPA}_{0\alpha}(q,\hat{\omega}) = \hbar\hat{\omega}\Pi_{\nu0\alpha}(q,\hat{\omega}) + \Pi^{RPA}_{0\alpha}(q,0) \tag{375}$$

$$\implies \Pi_{\nu0\alpha}(q,\hat{\omega}) = \frac{1}{\hbar\hat{\omega}}\left[\Pi^{RPA}_{0\alpha}(q,\hat{\omega}) - \Pi^{RPA}_{0\alpha}(q,0)\right] \tag{376}$$

We can now solve for the asymptotic effective potential

$$\hbar\boldsymbol{q}\cdot\boldsymbol{j}^{(1)}_{q,\alpha}(\omega) = U^{eff(1)}_{q,\alpha}\hbar\boldsymbol{q}\cdot\Pi^{RPA}_{1\alpha} - i\hbar\nu U^{eff(1)\infty}_{q,\alpha}\hbar\boldsymbol{q}\cdot\Pi_{\nu1\alpha} \tag{377}$$

$$= U^{eff(1)}_{q,\alpha}\hbar\hat{\omega}\Pi^{RPA}_{0\alpha}(q,\hat{\omega}) - i\hbar\nu U^{eff(1)\infty}_{q,\alpha}\left[\hbar\hat{\omega}\Pi_{\nu0\alpha}(q,\hat{\omega}) + \Pi^{RPA}_{0\alpha}(q,0)\right] \tag{378}$$

$$= \hbar\omega n^{(1)}_{q,\alpha}(\omega) + i\hbar\nu\left[U^{eff(1)\infty}_{q,\alpha}\hbar\omega\Pi_{\nu0\alpha} + U^{eff(1)}_{q,\alpha}\Pi^{RPA}_{0\alpha}\right] \tag{379}$$

$$- i\hbar\nu U^{eff(1)\infty}_{q,\alpha}\left[\hbar\hat{\omega}\Pi_{\nu0\alpha}(q,\hat{\omega}) + \Pi^{RPA}_{0\alpha}(q,0)\right] \tag{380}$$

$$= \hbar\omega n^{(1)}_{q,\alpha}(\omega) + i\hbar\nu\left[U^{eff(1)}_{q,\alpha}\Pi^{RPA}_{0\alpha} - U^{eff(1)\infty}_{q,\alpha}\left(i\hbar\nu\Pi_{\nu0\alpha}(q,\hat{\omega}) + \Pi^{RPA}_{0\alpha}(q,0)\right)\right] \tag{381}$$

Where we have used the following expression in the third line

$$\omega n^{(1)}_{q,\alpha}(\omega) = U^{eff(1)}_{q,\alpha}\omega\Pi^{RPA}_{0\alpha} - i\hbar\nu U^{eff(1)\infty}_{q,\alpha}\omega\Pi_{\nu 0\alpha} \tag{382}$$

$$= U^{eff(1)}_{q,\alpha}\hat{\omega}\Pi^{RPA}_{0\alpha} - i\hbar\nu\left[U^{eff(1)\infty}_{q,\alpha}\omega\Pi_{\nu 0\alpha} + U^{eff(1)}_{q,\alpha}\Pi^{RPA}_{0\alpha}\right] \tag{383}$$

Enforcing that the continuity equation is satisfied imposes a condition on $U^{eff(1)\infty}_{q,\alpha}$. Namely

$$U^{eff(1)\infty}_{q,\alpha} = U^{eff(1)}_{q,\alpha}\frac{\Pi^{RPA}_{0\alpha}}{i\hbar\nu\Pi_{\nu 0\alpha}(q,\hat{\omega}) + \Pi^{RPA}_{0\alpha}(q,0)} \tag{384}$$

From this we can solve for the number density perturbation

$$n^{(1)}_{q,\alpha}(\omega) = U^{eff(1)}_{q,\alpha}\Pi^{RPA}_{0\alpha} - i\hbar\nu U^{eff(1)\infty}_{q,\alpha}\Pi_{\nu 0\alpha} \tag{385}$$

$$= U^{eff(1)}_{q,\alpha}\Pi^{RPA}_{0\alpha} - i\hbar\nu U^{eff(1)}_{q,\alpha}\frac{\Pi^{RPA}_{0\alpha}}{i\hbar\nu\Pi_{\nu 0\alpha} + \Pi^{RPA}_{0\alpha}(q,0)}\Pi_{\nu 0\alpha} \tag{386}$$

$$= U^{eff(1)}_{q,\alpha}\Pi^{RPA}_{0\alpha}\left(1 - i\hbar\nu\frac{\Pi_{\nu 0\alpha}}{i\hbar\nu\Pi_{\nu 0\alpha} + \Pi^{RPA}_{0\alpha}(q,0)}\right) \tag{387}$$

$$= U^{eff(1)}_{q,\alpha}\frac{\Pi^{RPA}_{0\alpha}(q,\hat{\omega})}{1 + i\hbar\nu\frac{\Pi_{\nu 0\alpha}(q,\hat{\omega})}{\Pi^{RPA}_{0\alpha}(q,0)}} \tag{388}$$

$$\equiv U^{eff(1)}_{q,\alpha}\Pi^{M}_{0\alpha}(q,\hat{\omega}) \tag{389}$$

$$= U^{eff(1)\infty}_{q,\alpha}(\omega)\Pi^{RPA}_{0\alpha}(q,0) \tag{390}$$

Where we have defined the Mermin polarization and used (384) in the last line. Using (376) we can simplify as

$$1 + i\hbar\nu\frac{\Pi_{\nu 0\alpha}(q,\hat{\omega})}{\Pi^{RPA}_{0\alpha}(q,0)} = 1 + i\frac{\hbar\nu}{\hbar\hat{\omega}}\frac{\Pi^{RPA}_{0\alpha}(q,\hat{\omega}) - \Pi^{RPA}_{0\alpha}(q,0)}{\Pi^{RPA}_{0\alpha}(q,0)} \tag{391}$$

$$= \frac{1}{\hbar\hat{\omega}}\left(\hbar\hat{\omega} + i\hbar\nu\frac{\Pi^{RPA}_{0\alpha}(q,\hat{\omega})}{\Pi^{RPA}_{0\alpha}(q,0)} - i\hbar\nu\right) \tag{392}$$

$$\implies \Pi^{M}_{0\alpha}(q,\hat{\omega}) = \frac{\hat{\omega}\Pi^{RPA}_{0\alpha}(q,\hat{\omega})}{\omega + i\nu\frac{\Pi^{RPA}_{0\alpha}(q,\hat{\omega})}{\Pi^{RPA}_{0\alpha}(q,0)}} \tag{393}$$

$$= -\frac{\hat{\omega}\chi_{0\alpha}(q,\hat{\omega})}{\omega + i\nu\frac{\chi_{0\alpha}(q,\hat{\omega})}{\chi_{0\alpha}(q,0)}} \tag{394}$$

From this and (335) we can derive the dielectric

$$\implies \epsilon^M = 1 - \frac{\epsilon^M}{\rho_{ext}}\sum_\beta e_\beta n^{(1)}_{q,\beta}(\omega) \tag{395}$$

$$= 1 - \frac{\epsilon^M}{\rho_{ext}}\sum_\beta e_\beta U^{eff(1)}_{q,\beta}\Pi^{M}_{0\beta}(q,\hat{\omega}) \tag{396}$$

$$= 1 - V_{ext}(q)\sum_\beta e^2_\beta F_\beta(q)\Pi^{M}_{0\beta}(q,\hat{\omega}) \tag{397}$$

4.2 Evaluation of the Mermin Dielectric in the degenerate limit

From (183) and (397) we have

$$\epsilon^M(k,\omega) = 1 + \frac{(\omega+i\nu)[\epsilon_{RPA}(k,\omega+i\nu)-1]}{\omega + i\nu\frac{[\epsilon_{RPA}(k,\omega+i\nu)-1]}{[\epsilon_{RPA}(k,0)-1]}} \tag{398}$$

which agrees with [24]. This corresponds to a single species (the SM electrons) with $V_{ext}(q) = \frac{4\pi}{q^2}$ in (397). Here ν is the collision rates of the electrons which is assumed to be independent of ω, k and can be parametrized by n, T [24]. This is valid in the kinetic stage of relaxation processes [22] pg 121. Notice that $\lim_{\nu\to 0}\epsilon_M = \epsilon_{RPA}$. So we must evaluate

$$\begin{aligned}
\chi_0(k,\omega+i\nu) =& g_s\int \frac{d^3p}{(2\pi)^3}\frac{f(E_{p+k})-f(E_p)}{\hbar\omega - \Delta E + i\hbar\nu} && (399)\\
=& g_s\int \frac{d^3p}{(2\pi)^3}\left[\frac{f(E_{p+k})}{\hbar\omega-(E_{p+k}-E_p)+i\hbar\nu} - \frac{f(E_p)}{\hbar\omega-(E_{p+k}-E_p)+i\hbar\nu}\right] && (400)\\
=& g_s\int \frac{d^3p}{(2\pi)^3}\left[\frac{f(E_{p})}{\hbar\omega-(E_{p}-E_{p-k})+i\hbar\nu} - \frac{f(E_p)}{\hbar\omega-(E_{p+k}-E_p)+i\hbar\nu}\right] && (401)\\
=& g_s\int \frac{d^3p}{(2\pi)^3}f(E_p)\left[\frac{1}{\hbar\omega-(E_{p}-E_{p-k})+i\hbar\nu} - \frac{1}{\hbar\omega-(E_{p+k}-E_p)+i\hbar\nu}\right] && (402)\\
=& g_S\int_0^\infty \frac{dp\; p^2}{(2\pi)^2}f(E_p)\int_{-1}^1 dy\left[\frac{1}{\frac{\hbar^2 k}{m}(p_+-py)+i\hbar\nu} - \frac{1}{\frac{\hbar^2 k}{m}(p_--py)+i\hbar\nu}\right] && (403)\\
=& g_s\frac{m}{\hbar^2 k(2\pi)^2}\int_0^\infty dp\; p^2 f(E_p)\int_{-1}^1 dy\left[\frac{1}{p_+-py+ik_Fu_\nu} - \frac{1}{p_--py+ik_Fu_\nu}\right] && (404)\\
=& g_s\frac{m}{\hbar^2 k_F^2(2\pi)^2}\frac{1}{2z}\int_0^\infty dp\; p^2 f(E_p)\int_{-1}^1 dy\left[\frac{1}{u'+z-\frac{py}{k_F}+iu_\nu} - \frac{1}{u'-z-\frac{py}{k_F}+iu_\nu}\right] && (405)\\
=& g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^\infty d\zeta\; \zeta^2 f(E_{k_F\zeta})\int_{-1}^1 dy\left[\frac{1}{u'+z-\zeta y+iu_\nu} - \frac{1}{u'-z-\zeta y+iu_\nu}\right] && (406)
\end{aligned}$$

Where we have defined $u_\nu \equiv \frac{\nu}{v_F k}$. We again consider the limit of large $\beta\mu$ so we find

$$\lim_{\beta\mu\to\infty}\chi_0(k,\omega+i\nu) = g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^1 d\zeta\;\zeta^2\int_{-1}^1 dy\left[\frac{1}{u'+z-\zeta y+iu_\nu} - \frac{1}{u'-z-\zeta y+iu_\nu}\right] \tag{407}$$

We can divide this expression into it's real and imaginary parts as

$$\frac{1}{u'\pm z-\zeta y+iu_\nu} = \frac{u'\pm z-\zeta y}{(u'\pm z-\zeta y)^2+u_\nu^2} - i\frac{u_\nu}{(u'\pm z-\zeta y)^2+u_\nu^2} \tag{408}$$

The angular integral yields

$$\int_{-1}^1 dy\frac{1}{a-\zeta y+iu_\nu} = \frac{1}{\zeta}\left[\frac{1}{2}\ln\left|\frac{(a+\zeta)^2+u_\nu^2}{(a-\zeta)^2+u_\nu^2}\right| - i\left(\mathrm{Arctan}\left(\frac{a+\zeta}{u_\nu}\right) - \mathrm{Arctan}\left(\frac{a-\zeta}{u_\nu}\right)\right)\right] \tag{409}$$

$$\tag{410}$$

So we find

$$\lim_{\beta\mu\to\infty}\mathrm{Re}\chi_0(k,\omega+i\nu) = g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\left[\frac{1}{2}\int_0^1 d\zeta\;\zeta\ln\left|\frac{(a+\zeta)^2+u_\nu^2}{(a-\zeta)^2+u_\nu^2}\right|\right]_{a=u'-z}^{u'+z} \tag{411}$$

$$\lim_{\beta\mu\to\infty}\mathrm{Im}\chi_0(k,\omega+i\nu) = g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\left[\left[-\int_0^1 d\zeta\;\zeta\mathrm{Arctan}\left(\frac{a+y\zeta}{u_\nu}\right)\right]_{y=-1}^{1}\right]_{a=u'-z}^{u'+z} \tag{412}$$

For the real part

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 d\zeta\; \zeta \ln\left|\frac{(a+\zeta)^2+u_\nu^2}{(a-\zeta)^2+u_\nu^2}\right| && (413)\\
=&a - au_\nu\left(\text{Arctan}\left[\frac{1-a}{u_\nu}\right] + \text{Arctan}\left[\frac{1+a}{u_\nu}\right]\right) + \frac{1}{2}(1-a^2+u_\nu^2)\text{Arctanh}\left[\frac{2a}{1+a^2+u^2}\right] && (414)\\
=&a - au_\nu\left(\text{Arctan}\left[\frac{1-a}{u_\nu}\right] + \text{Arctan}\left[\frac{1+a}{u_\nu}\right]\right) + \frac{1}{2}(1-a^2+u_\nu^2)\frac{1}{2}\ln\left[\frac{(1+a)^2+u_\nu^2}{(1-a)^2+u_\nu^2}\right] && (415)\\
\equiv&a + au_\nu h_1(a,u_\nu) + \frac{1}{2}(1-a^2+u_\nu^2)h_2(a,u_\nu) && (416)
\end{aligned}
$$

Which has odd parity. We can write for the imaginary part

$$
\begin{aligned}
&\left[-\int_0^1 d\zeta\; \zeta\text{Arctan}\left(\frac{a+y\zeta}{u_\nu}\right)\right]_{y=-1}^{1} && (417)\\
=&-\int_0^1 d\zeta\; \zeta\left(\text{Arctan}\left(\frac{a+\zeta}{u_\nu}\right) - \text{Arctan}\left(\frac{a-\zeta}{u_\nu}\right)\right) && (418)\\
=&\int_0^1 d\zeta\; \zeta\text{Arctan}\left(\frac{|a|-\zeta}{u_\nu}\right) - \int_0^1 d\zeta\; \zeta\text{Arctan}\left(\frac{|a|+\zeta}{u_\nu}\right) && (419)\\
=&\int_0^{-1} d\zeta'\; \zeta'\text{Arctan}\left(\frac{|a|+\zeta'}{u_\nu}\right) - \int_0^1 d\zeta\; \zeta\text{Arctan}\left(\frac{|a|+\zeta}{u_\nu}\right) && (420)\\
=&-\int_{-1}^1 d\zeta\; \zeta\text{Arctan}\left(\frac{|a|+\zeta}{u_\nu}\right) && (421)\\
=&-\int_{|a|-1}^{|a|+1} d\zeta\; (\zeta-|a|)\text{Arctan}\left(\frac{\zeta}{u_\nu}\right) && (422)\\
=&u_\nu - \frac{1}{2}(1-a^2+u_\nu^2)\left(\text{Arctan}\left[\frac{1-|a|}{u_\nu}\right] + \text{Arctan}\left[\frac{1+|a|}{u_\nu}\right]\right) + |a|u_\nu\frac{1}{2}\ln\left[\frac{(|a|-1)^2+u_\nu^2}{(|a|+1)^2+u_\nu^2}\right] && (423)\\
=&u_\nu - \frac{1}{2}(1-a^2+u_\nu^2)\left(\text{Arctan}\left[\frac{1-a}{u_\nu}\right] + \text{Arctan}\left[\frac{1+a}{u_\nu}\right]\right) - au_\nu\frac{1}{2}\ln\left[\frac{(a+1)^2+u_\nu^2}{(a-1)^2+u_\nu^2}\right] && (424)\\
=&u_\nu + \frac{1}{2}(1-a^2+u_\nu^2)h_1(a,u_\nu) - au_\nu h_2(a,u_\nu) && (425)
\end{aligned}
$$

Which has even parity. Where we have used the fact that $\text{Arctan}(x)$ is odd in x, so that the integrand is even in a. So we can write

$$
\begin{aligned}
\lim_{\beta\mu\to\infty}\chi_0(k,\omega+i\nu) \equiv& g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\left[\mathcal{Z}(a,u_\nu)\right]_{a=u'-z}^{u'+z} && (426)\\
=& g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\left[\mathcal{Z}(u'+z,u_\nu) - \mathcal{Z}(u'-z,u_\nu)\right] && (427)
\end{aligned}
$$

Where we have defined

$$
\mathcal{Z}(a,u_\nu) = \begin{bmatrix}1 & i\end{bmatrix}\left(\begin{bmatrix}a\\ u_\nu\end{bmatrix} + \begin{bmatrix}au_\nu & \frac{1}{2}(1-a^2+u_\nu^2)\\ \frac{1}{2}(1-a^2+u_\nu^2) & -au_\nu\end{bmatrix}\begin{bmatrix}h_1(a,u_\nu)\\ h_2(a,u_\nu)\end{bmatrix}\right) \tag{428}
$$

Note that because of the integration limits, the real and imaginary parts of χ_0 have the opposite parity in u' as $\mathcal{Z}$ does in a. So the imaginary part of χ_0 has odd u', ω parity and the real part has even parity, exactly as expected. As we defined in the Lindhard dielectric function derivation

$$\frac{4\pi e^2}{k^2}\frac{g_s m k_F}{2\pi^2\hbar^2} = \frac{\chi^2}{z^2} \tag{429}$$

So we find

$$\lim_{\beta\mu\to\infty} \epsilon_{RPA}(k,\omega+i\nu) - 1 = \frac{4\pi e^2}{k^2}\lim_{\beta\mu\to\infty}\chi_0(k,\omega+i\nu) \tag{430}$$

$$= \frac{\chi^2}{z^2}\frac{1}{4z}\left[\mathcal{Z}(u'+z,u_\nu) - \mathcal{Z}(u'-z,u_\nu)\right] \tag{431}$$

$$= \lim_{\beta\mu\to\infty}\epsilon_{RPA}(u',z,u_\nu) - 1 \tag{432}$$

**** take $\nu \to 0$ limit to show it matches on to Lindhard function

Which allows us to use the Mermin dielectric function to compute the energy loss.

The final needed ingredient is a parametrization for the inverse lifetime. Which is given by [24], [25] and holds for $0.01 \le x \le 100$ with arbitrary degeneracy

$$\nu = \frac{\nu_0}{\sqrt{1+0.2T/T_F}} = \frac{\nu_0}{\sqrt{1+\frac{0.2}{\beta E_F}}} \tag{433}$$

$$\nu_0 = J(y)\frac{3}{2\hbar m c^2\beta^2}\sqrt{\frac{\alpha x^3}{\pi^3(1+x^2)^{5/2}}} \tag{434}$$

$$\omega_p \equiv \sqrt{\frac{4\pi e^2 n_e}{m(1+x^2)^{1/2}}} \tag{435}$$

Where $x \equiv \frac{v_F}{c}$, and $\beta_p\hbar\omega_p \equiv 1$ and $y \equiv \frac{\sqrt{3}T_p}{T}$ and

$$J(y) = \left[\frac{y^3}{3(1+0.07414y)^3}\ln\left(\frac{2.810}{y} - \frac{0.810x^2}{y(1+x^2)} + 1\right) + \frac{\pi^5}{6}\frac{y^4}{(13.91+y)^4}\right]\left(1+\frac{6}{5x^2}+\frac{2}{5x^4}\right) \tag{436}$$

Our energy loss integral variable change for the Lindhard Dielectric depended only on it's parity. So we can write down immediately in the degenerate limit

$$\frac{dE}{dr}(v) = \frac{(k_F v_F)^2}{v^2}\frac{8\kappa^2 e^2}{\pi}\int_{\Sigma_{uz}}(dz\ z)(du\ u)\ \mathrm{Im}\left(\frac{-1}{\epsilon_M}\right) \tag{437}$$

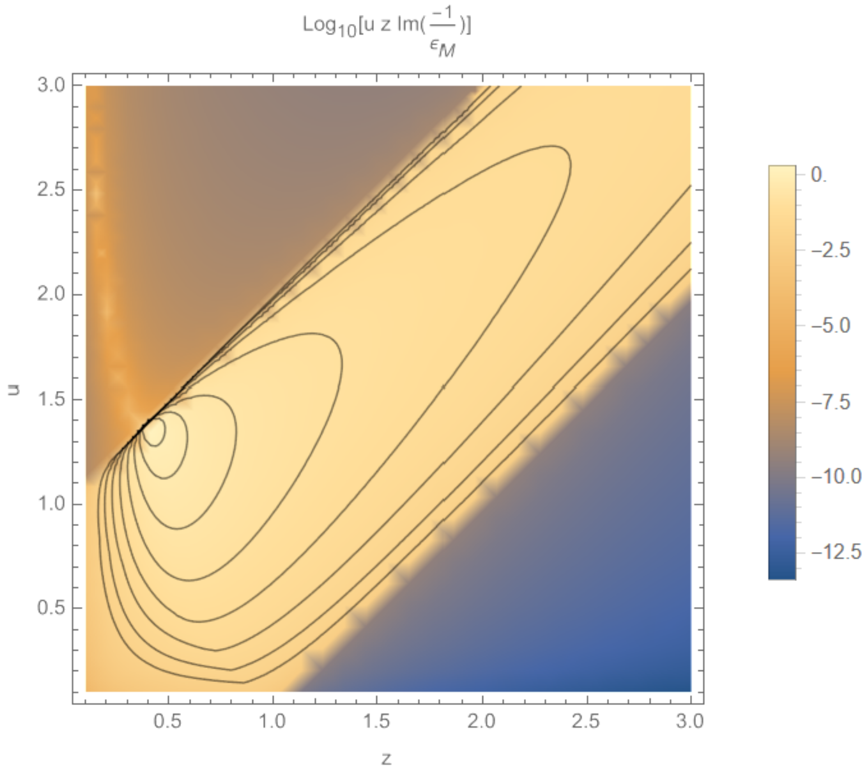

Figure 8: Mermin Dielectric Energy Loss Integrand

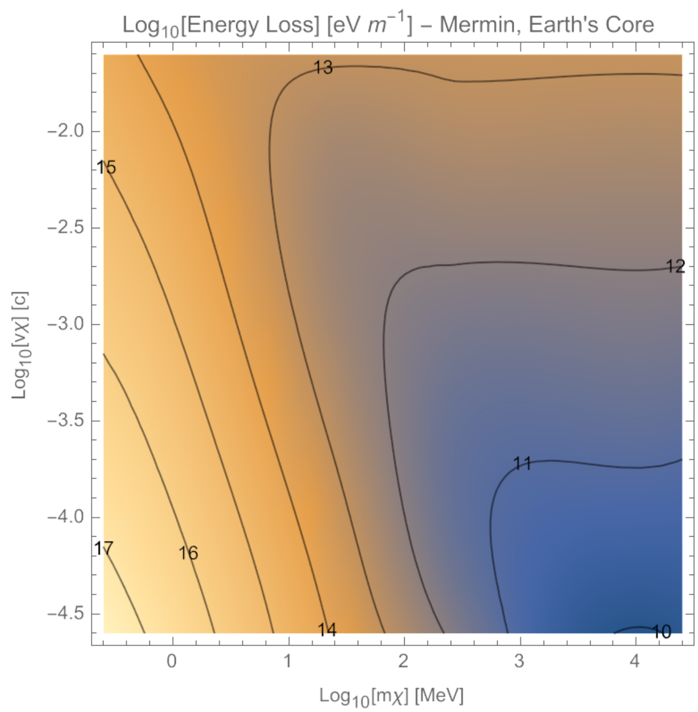

Figure 9: Mermin Energy Loss per unit length

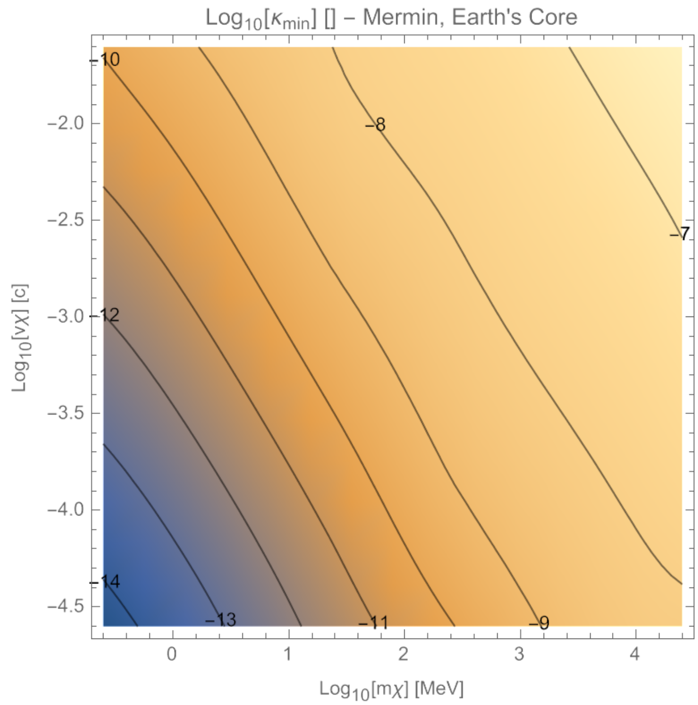

Figure 10: Mermin Energy Loss κ_{min}

5 Inner Shell Electrons and Silicate Models for the Mantle

Following [26], we look to model the electron shell structure of SiO_2 and MgO by fitting to optical data. Optical data allows one to empirically determine the loss function (minus the imaginary part of the inverse dielectric) in the limit of 0 momentum transfer. One can then perform a fit of a linear combination of dielectrics in the optical limit to arrive at an empircal dielectric valid at finite momentum tranfer by relaxing the limit. This is known as the Shellwise Local Plasma Approximation, where each electron shell is treated as an independent plasma modelled by (in our case) a Mermin dielectric in the degenerate limit.

$$\mathrm{Im}\left[\frac{-1}{\epsilon_M(q,\omega)}\right] = \sum_i A_i(q)\mathrm{Im}\left[\frac{-1}{\epsilon_M(q,\omega,n_i,\nu_i)}\right]\Theta(\omega-\omega_{\mathrm{edge},i}) \tag{438}$$

Where $\omega_{\mathrm{edge},i}$ is used to truncate the low energy behaviour of the dielectric, so that inner shells are probed only at high enough energies to penetrate into the molecule or atom. The coefficients are determined from the optical fit [14] and by ensuring the Θ function truncation preserves charge conservation (the sum rules). One must also impose $\sum_i A_i = 1$ for the Kramers-Kronig relations to hold (analyticity) [27].

$$A_i(q) = A_i(0)\frac{\int_0^\infty d\omega\ \omega\ \mathrm{Im}\left[\frac{-1}{\epsilon_M(0,\omega,n_i,\nu_i)}\right]\Theta(\omega-\omega_{\mathrm{edge},i})}{\int_0^\infty d\omega\ \omega\ \mathrm{Im}\left[\frac{-1}{\epsilon_M(q,\omega,n_i,\nu_i)}\right]\Theta(\omega-\omega_{\mathrm{edge},i})} \tag{439}$$

Note that when the truncations are removed ($\omega_{\mathrm{edge},i} \to 0$) the sum rule [3]

$$\int_0^\infty d\omega\ \omega\ \mathrm{Im}\left[\frac{-1}{\epsilon_M(q,\omega,n_i,\nu_i)}\right] = \frac{\pi}{2}\omega_p^2 \tag{440}$$

(where $\omega_p^2 \equiv \frac{4\pi n e^2}{m}$ is the plasma frequency) ensures that the coefficients are independent of momentum. In the optical and degenerate limits the Mermin dielectric becomes

$$\lim_{q \to 0} \epsilon_M(q, \omega, \nu) = 1 + i \frac{4(k_F v_F \chi)^2}{3\omega(\nu - i\omega)} \tag{441}$$

$$= 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)} \tag{442}$$

Where we have used $(k_F v_F \chi)^2 = \frac{3\pi e^2 n}{m} = \frac{3}{4}\omega_p^2$. This is precisely the Drude dielectric [28], which conveniently allows us to compare to any Drude model optical data available. This gives for the loss function

$$\lim_{q \to 0} \text{Im}\left[\frac{-1}{\epsilon_M(q, \omega, \nu)}\right] = \frac{\nu\omega\omega_p^2}{\nu^2\omega^2 + \left(\omega_p^2 - \omega^2\right)^2} \tag{443}$$

Notice that in the limit of vanishing ν, this expression vanishes, meaning that one cannot use the Lindhard Dielectric to fit to optical data. We can also see that in this limit at $\omega = \omega_p$ real part of the dielectric also vanishes, and so this corresponds to the position of the optical plasmon pole for the Lindhard Function. This function is analogous to a Lorentzian, in that it has a single peak, which is located at

$$\omega_{extr}^2 = \frac{1}{6}\left(2\omega_p^2 - \nu^2 + \sqrt{\left(2\omega_p^2 - \nu^2\right)^2 + 12\omega_p^4}\right) \tag{444}$$

When fitting, we look to fix the locations of the peaks by finding the local extrema in ω of the optical data. This reduces the number of DOFs per dielectric from 3 to 2 and determines the number of dielectrics used in the fit. We can do this explicitly by solving for ω_p in terms of the position of the peak and ν

$$\omega_p^2 = \sqrt{\nu^2\omega_{extr}^2 + 4\omega_{extr}^4} - \omega_{extr}^2 \tag{445}$$

**** may also want to fix the height of the peak, with A_i so we are only solving for the width with ν

Which, is non-negative for all real ν and ω_{extr}. For now, we ignore truncations, so we look to fit

$$\text{Im}\left[\frac{-1}{\epsilon_M(0, \omega)}\right] = \sum_i A_i(0)\text{Im}\left[\frac{-1}{\epsilon_M(0, \omega, \omega_i, \nu_i)}\right] \tag{446}$$

$$= \sum_i A_i(0) \frac{\nu_i\omega\omega_i^2}{\nu_i^2\omega^2 + \left(\omega_i^2 - \omega^2\right)^2} \tag{447}$$

to optical data from [29]. Where we have defined $\omega_i \equiv \omega_p(\nu_i, \omega_{extr,i})$. We also note that the above limit yields exactly the Drude dielectric, and hence our use of the data in [29] (which uses a Drude model) is perfectly appropriate. We also note that the Earth's Mantle composition is given in [30]. For simplicity, and to find an order 1 estimate, we only consider SiO_2 and MgO which compose 44.7 and 38.7 percent of the Mantle respectively.

**** add in figures

We also use iron optical data from [31] and compare to our analytic formula

For completeness, we also include some additional useful limits of the Mermin Dielectric. The following large u (ω) limit is useful for cutting off the integral in the denominator in (439) for numerical stability.

$$\lim_{u \to \infty} \epsilon_M(u, z, u_\nu) = 1 - \frac{\chi^2}{3z^2u^2}\left(1 - i\frac{u_\nu}{u}\right) + \mathcal{O}(u^{-4}) \tag{448}$$

Hence we have

$$\lim_{u\to\infty} \mathrm{Im}\left[\frac{-1}{\epsilon_M(u,z,u_\nu)}\right] = \frac{u_\nu \chi^2}{3u^3 z^2} + \mathcal{O}(u^{-4}) \tag{449}$$

We also note that the low energy limit of the Optical Mermin Function is given by

$$\lim_{\omega\to 0}\lim_{q\to 0} \mathrm{Im}\left[\frac{-1}{\epsilon_M(q,\omega,\nu)}\right] = \lim_{\omega\to 0} \frac{\nu\omega\omega_p^2}{\nu^2\omega^2 + \left(\omega_p^2 - \omega^2\right)^2} \tag{450}$$

$$= \frac{\nu\omega}{\omega_p^2} + \mathcal{O}(\omega^2) \tag{451}$$

Hence the low energy scaling of optical data (where we expect our conduction electron approximation to be valid) is determined by the ratio of the collision rate to the plasmon frequency.

6 Nuclear and Atomic Interactions

We now consider the effect of including additional species, namely atomic and nuclear interactinos within a particular scattering species. We model the aditional species as a free gas as we did with electrons. This is valid at energies for which we can neglect the lattice structure of the material. From (397) we can write

$$\implies \epsilon^M = 1 - \frac{4\pi e^2}{q^2}\left(\Pi^M_{0e}(q,\hat\omega) + F_I(q)\Pi^M_{0I}(q,\hat\omega)\right) \tag{452}$$

Where e indexes electrons and I indexes ions (note we have absorbed the charge of the ion into the form factor). We also recall

$$\implies \Pi^M_{0\alpha}(q,\hat\omega) = -\frac{\hat\omega\chi_{0\alpha}(q,\hat\omega)}{\omega + i\nu\frac{\chi_{0\alpha}(q,\hat\omega)}{\chi_{0\alpha}(q,0)}} \tag{453}$$

$$\tag{454}$$

**** also see `https://doi.org/10.1103/PhysRevE.91.023102` for discussion of exact thermal RPA in appendix

Hence, given an Ionic form factor, we can readily and consistently determine the total scattering with the SM material.

**** notice that because the ions are so much heavier, we probably will take the non-degenerate limit for the ions

Going back to (412) and instead taking the non-degnerate limit ($\beta\mu \to 0$)

$$\lim_{\beta\mu\to 0} \frac{1}{1+e^{\beta(E_F\zeta^2-\mu)}} = e^{-\beta E_F\zeta^2} \tag{455}$$

we have

$$\lim_{\beta\mu\to\infty} \mathrm{Re}\chi_0(k,\omega+i\nu) = g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\left[\frac{1}{2}\int_0^\infty d\zeta\ \zeta e^{-\beta E_F\zeta^2}\ln\left|\frac{(a+\zeta)^2+u_\nu^2}{(a-\zeta)^2+u_\nu^2}\right|\right]_{a=u'-z}^{u'+z} \tag{456}$$

$$\lim_{\beta\mu\to\infty} \mathrm{Im}\chi_0(k,\omega+i\nu) = g_s\frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\left[\left[-\int_0^\infty d\zeta\ \zeta e^{-\beta E_F\zeta^2}\mathrm{Arctan}\left(\frac{a+y\zeta}{u_\nu}\right)\right]_{y=-1}^{1}\right]_{a=u'-z}^{u'+z} \tag{457}$$

*****
We have from (406)

$$\lim_{\beta\mu\to 0} \chi_0(k,\omega+i\nu) \tag{458}$$

$$= \lim_{\beta\mu\to 0} g_s \frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^\infty d\zeta\ \zeta^2 f(E_{k_F\zeta})\int_{-1}^1 dy\left[\frac{1}{u'+z-\zeta y+iu_\nu}-\frac{1}{u'-z-\zeta y+iu_\nu}\right] \tag{459}$$

$$=g_s \frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^\infty d\zeta\ \zeta^2 e^{-\beta E_F\zeta^2}\int_{-1}^1 dy\left[\frac{1}{u'+z-\zeta y+iu_\nu}-\frac{1}{u'-z-\zeta y+iu_\nu}\right] \tag{460}$$

Noticing that the integrand vanishes on the boundary, we can integrate by parts.

$$\lim_{\beta\mu\to 0} \chi_0(k,\omega+i\nu) \tag{461}$$

$$=g_s \frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^\infty d\zeta\ \frac{-1}{2\beta E_F}e^{-\beta E_F\zeta^2}\partial_\zeta\left\{\int_{-1}^1 \zeta dy\left[\frac{1}{u'+z-\zeta y+iu_\nu}-\frac{1}{u'-z-\zeta y+iu_\nu}\right]\right\} \tag{462}$$

$$=g_s \frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^\infty d\zeta\ \frac{-1}{2\beta E_F}e^{-\beta E_F\zeta^2}\partial_\zeta\left\{\int_{-\zeta}^\zeta dw\left[\frac{1}{u'+z-w+iu_\nu}-\frac{1}{u'-z-w+iu_\nu}\right]\right\} \tag{463}$$

$$=g_s \frac{m}{\hbar^2(2\pi)^2}\frac{k_F}{2z}\int_0^\infty d\zeta\ \frac{-1}{2\beta E_F}e^{-\beta E_F\zeta^2}\left[\frac{1}{u'+z-w+iu_\nu}-\frac{1}{u'-z-w+iu_\nu}\right]_{w=-\zeta}^{\zeta} \tag{464}$$

*** we should do it carefully, but since ζ and y appear together in the denominator of the angular integrand, we will essentially do an integral, and then differentiate the result when we integrate by parts. The result of which will just be to knock down two factors of ζ. So we will end up with something like a complex Hilbert transform of the gaussian. ie.

$$\propto \int_0^\infty d\zeta\frac{e^{-\beta E_F\zeta^2}}{u\pm z\pm\zeta+iu_\nu} \tag{465}$$

$$=-\int_0^{-\infty} d\zeta\frac{e^{-\beta E_F\zeta^2}}{u\pm z\mp\zeta+iu_\nu} \tag{466}$$

$$=\int_{-\infty}^0 d\zeta\frac{e^{-\beta E_F\zeta^2}}{u\pm z\mp\zeta+iu_\nu} \tag{467}$$

$$\tag{468}$$

$$=\int_0^\infty d\zeta\ \left[\frac{e^{-\beta E_F\zeta^2}}{u'+z-wiu_\nu}\right]_{w=-\zeta}^{\zeta} \tag{469}$$

$$=\int_0^\infty d\zeta\ \left[\frac{e^{-\beta E_F\zeta^2}}{u'+z-\zeta+iu_\nu}-\frac{e^{-\beta E_F\zeta^2}}{u'+z+\zeta+iu_\nu}\right] \tag{470}$$

$$=\int_{-\infty}^0 d\zeta\ \left[\frac{e^{-\beta E_F\zeta^2}}{u'+z-w+iu_\nu}\right]_{w=\zeta}^{-\zeta} \tag{471}$$

this has analytic solutions according to mathematica.
**** see also `https://arxiv.org/abs/1412.5705`
I think we should also be able to evaluate this as a contour integral.

[32] considers how to include both ionic and electronic responses consistently in the sun. This treats the ions in a free gas as well. So we can take this and include form factors in the potential (eg. in paper Rebecca recommended which has nuclear form factors [33] - also see `http://lampx.tugraz.at/~hadley/ss1/crystaldiffraction/atomicformfactors/formfactors.php` which gives for iron and shows how to get (static) structure factor from it) to get a realistic treatment. It only neglects shell and lattice effects. Could also use [34] which gives more sophisticated form factors. Also can include any of these in the below toy model as they do in the above url - we can also include a delta function to make it dynamical as we have above.

In neither electron or ions have we included ionization effects. Can see slpa paper [15] which discusses including ionization for electrons

also see `http://www.astro.spbu.ru/JPDOC/f-dbase.html` for database of optical constants

6.1 Form Factors

7 Energy Loss Per Unit Length - Nuclear Scattering

Here, we adapt the above derivation of the mean Energy Loss in terms of the Structure Function to use with Atomic and Nuclear form factors. We begin with (30) which gives the transition rate of an arbitrary system in response to a couloumbic perturbing probe.

$$dP = \frac{1}{V^2}\sum_q V^2(q)\frac{2\pi}{\hbar}\sum_{m,n} P_m|\langle n|\rho_{-\boldsymbol{q}}|m\rangle|^2\delta(E_n - E_m - \hbar\omega_{\boldsymbol{q}}) \tag{472}$$

As we showed above, this leads to

$$\frac{dE}{dr}(v) = \int_0^\infty \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)\int_{-\omega_+}^{\omega_-} d\omega\ \omega S(q,\omega) \tag{473}$$

$$S(\boldsymbol{q},\omega) = \frac{2\pi}{V}\sum_{m,n} P_m|\langle n|\rho_{-\boldsymbol{q}}|m\rangle|^2\delta(E_n - E_m - \hbar\omega) \tag{474}$$

Where $\omega_\pm = qv \pm \frac{\hbar q^2}{2m_\chi}$

7.1 Finite Temperature

We also now take the non-degenerate limit and assume that the nuclear charge distribution of the atom is unaffected by the DM interaction. We ignore the lattice structure of the material, taking expectation with momentum centre of mass eigenstates of the atom (producted with a state describing it's charge distribution in it's galilean rest frame). So the structure function becomes (using the orthogonality of the momentum states)

$$S(\boldsymbol{q},\omega) = \frac{2\pi}{V}\sum_p n_N\frac{V}{Z_1}e^{-\beta\frac{\hbar^2p^2}{2m_N}}|\langle p+q, Z_{nuc}|\rho_{-\boldsymbol{q}}|p, Z_{nuc}\rangle|^2\delta\left(\frac{\hbar^2(p+q)^2}{2m_N} - \frac{\hbar^2p^2}{2m_N} - \hbar\omega\right) \tag{475}$$

Where we have taken the probability according to our non-degenerate (classical) limit, and defined n_N as the number density of nuclei. We are assuming a non-interacting gas of nuclei, so we can use the classical ideal gas, reducing it to a single particle distribution of momenta to give (see eg. [17])

$$\frac{1}{V}\sum_m P_m \tag{476}$$
$$=\frac{1}{V}\sum_p n_N \frac{V}{Z_1} e^{-\beta\frac{\hbar^2 p^2}{2m_N}} \tag{477}$$
$$=\frac{1}{V}\sum_p n_N \frac{(2\pi\hbar)^3}{(2\pi m_N T)^{3/2}} e^{-\beta\frac{\hbar^2 p^2}{2m_N}} \tag{478}$$

where m_N is the mass of the nucleus and Z_1 is the single particle classical ideal gas partition function. We have made the assumption that all momentum transfer goes to the kinetic energy of the centre of mass of the atom, so that charge structure is unperturbed. So the squared matrix element is nothing other than the effective charge of the nucleus as a function of the momentum transfer. Using again (17), (37), and assuming a spherically symmetric nuclear charge distribution, we then find

$$S(q,\omega) = Z_{eff}^2(q)\frac{V}{Z_1}\int \frac{d^3p}{(2\pi)^2} e^{-\beta\frac{\hbar^2 p^2}{2m_N}} \delta\left(\frac{\hbar^2(p+q)^2}{2m_N} - \frac{\hbar^2 p^2}{2m_N} - \hbar\omega\right) \tag{479}$$
$$= Z_{eff}^2(q)\frac{V}{Z_1}\int \frac{dp}{(2\pi)^2}\ p^2 e^{-\beta\frac{\hbar^2 p^2}{2m_N}} \int d\Omega\ \delta\left(\frac{\hbar^2 q^2}{2m_N} - \frac{\hbar^2 \vec{p}\cdot\vec{q}}{m_N} - \hbar\omega\right) \tag{480}$$
$$= Z_{eff}^2(q)\frac{V}{Z_1}\int \frac{dp}{2\pi}\ p^2 e^{-\beta\frac{\hbar^2 p^2}{2m_N}} \frac{m_N}{\hbar^2 qp}\Theta(\omega_-^N - \omega)\Theta(\omega_+^N + \omega) \tag{481}$$
$$= \frac{Z_{eff}^2(q)}{2\pi}\frac{V}{Z_1}\frac{m_N}{\hbar^2 q}\int_{p_-}^{-p_+} \frac{dp}{2\pi}\ p e^{-\beta\frac{\hbar^2 p^2}{2m_N}} \tag{482}$$
$$= \frac{Z_{eff}^2(q)}{2\pi q}\frac{V}{Z_1}\frac{m_N}{\hbar^2 q}\frac{m_N}{\beta\hbar^2}\left[-e^{-\beta\frac{\hbar^2 p^2}{2m_N}}\right]_{p_-}^{-p_+} \tag{483}$$
$$= \frac{Z_{eff}^2(q)}{2\pi q}\frac{(2\pi\hbar)^3}{(2\pi m_N T)^{3/2}}\frac{m_N^2}{\beta\hbar^4}\left[-e^{-\beta\frac{\hbar^2 p^2}{2m_N}}\right]_{p_-}^{-p_+} \tag{484}$$
$$= \frac{Z_{eff}^2(q)}{2\pi q\hbar}(2\pi)^{3/2}\sqrt{m_N\beta}\left[-e^{-\beta\frac{\hbar^2 p^2}{2m_N}}\right]_{p_-}^{-p_+} \tag{485}$$

Where we have defined $\omega_\pm^N \equiv \frac{\hbar qp}{m_N} \pm \frac{\hbar q^2}{2m_N}$ and $p_\pm \equiv \frac{m_N\omega}{q\hbar} \pm \frac{q}{2}$. This leads to a mean energy loss

$$\frac{dE}{dr}(v) = \int_0^\infty \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)\int_{-\omega_+}^{\omega_-} d\omega\ \omega\frac{Z_{eff}^2(q)}{2\pi q\hbar}(2\pi)^{3/2}\sqrt{m_N\beta}\left[-e^{-\beta\frac{\hbar^2 p^2}{2m_N}}\right]_{p_-}^{-p_+} \tag{486}$$
$$= \frac{1}{v^2\hbar}\int_0^\infty \frac{dq}{(2\pi)^{3/2}} Z_{eff}^2(q)V^2(q)\sqrt{m_N\beta}\int_{-\omega_+}^{\omega_-} d\omega\ \omega\left[-e^{-\beta\frac{\hbar^2 p^2}{2m_N}}\right]_{p_-}^{-p_+} \tag{487}$$

7.2 0 Temperature

We also now take the non-degenerate limit and assume that the nuclear charge distribution of the atom is unaffected by the DM interaction. We ignore the lattice structure of the material, taking expectation with momentum centre of mass eigenstates of the atom (producted with a state describing it's charge distribution in it's galilean rest frame). We assume single particle scattering here. So the structure function becomes (using the orthogonality of the momentum states)

$$\lim_{T\to 0} S(\boldsymbol{q},\omega) = \lim_{T\to 0}\frac{2\pi}{V}\sum_{m,n} P_m |\langle n|\rho_{-\boldsymbol{q}}|m\rangle|^2 \delta(E_n - E_m - \hbar\omega) \tag{488}$$

$$= \frac{2\pi}{V}\sum_i |\langle q, Z_{nuc}|\rho_{-\boldsymbol{q}}|0, Z_{nuc}\rangle|^2 \delta\left(\frac{\hbar^2 q^2}{2m_N} - \hbar\omega\right) \tag{489}$$

$$= 2\pi n_N |\langle q, Z_{nuc}|\rho_{-\boldsymbol{q}}|0, Z_{nuc}\rangle|^2 \delta\left(\frac{\hbar^2 q^2}{2m_N} - \hbar\omega\right) \tag{490}$$

**** need to update discussions

Where we have assumed the nuclei are initially at rest. Where m_N is the mass of the nucleus. We have made the assumption that all momentum transfer goes to the kinetic energy of the centre of mass of the atom, so that charge structure is unperturbed. So the squared matrix element is nothing other than the effective charge of the nucleus as a function of the momentum transfer. Using again (17), (37), and assuming a spherically symmetric nuclear charge distribution, we then find

$$S(q,\omega) = 2\pi n_N Z_{eff}^2(q)\delta\left(\frac{\hbar^2 q^2}{2m_N} - \hbar\omega\right) \tag{491}$$

Where n_N is the number density of nuclei. This leads to a mean energy loss

$$\frac{dE}{dr}(v) = \int_0^\infty \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)\int_{-\omega_+}^{\omega_-} d\omega\ \omega 2\pi n_N Z_{eff}^2(q)\delta\left(\frac{\hbar^2 q^2}{2m_N} - \hbar\omega\right) \tag{492}$$

$$= \frac{n_N}{v^2}\int_0^\infty \frac{dq}{2\pi} q Z_{eff}^2(q) V^2(q)\int_{-\omega_+}^{\omega_-} d\omega\ \omega\delta\left(\frac{\hbar^2 q^2}{2m_N} - \hbar\omega\right) \tag{493}$$

$$= \frac{n_N}{v^2}\int_0^\infty \frac{dq}{2\pi} q Z_{eff}^2(q) V^2(q)\frac{q^2}{2m_N}\Theta\left(\frac{\hbar q^2}{2m_N} - \omega_-\right)\Theta\left(\frac{\hbar q^2}{2m_N} + \omega_+\right) \tag{494}$$

$$= \frac{n_N}{v^2}\int_0^\infty \frac{dq}{2\pi} q Z_{eff}^2(q) V^2(q)\frac{q^2}{2m_N}\Theta\left(\frac{\hbar q^2}{2m_N} - qv + \frac{\hbar q^2}{2m_\chi}\right)\Theta\left(\frac{\hbar q^2}{2m_N} + qv + \frac{\hbar q^2}{2m_\chi}\right) \tag{495}$$

$$= \frac{n_N}{v^2}\int_0^\infty \frac{dq}{2\pi} q Z_{eff}^2(q) V^2(q)\frac{q^2}{2m_N}\Theta\left(\frac{\hbar q^2}{2\mu} - qv\right)\Theta\left(\frac{\hbar q^2}{2\mu} + qv\right) \tag{496}$$

$$= \frac{n_N}{v^2}\int_0^{\frac{2\mu v}{\hbar}} \frac{dq}{2\pi} q Z_{eff}^2(q) V^2(q)\frac{q^2}{2m_N} \tag{497}$$

$$= \frac{n_N m_\chi m_N}{2E_\chi}\int_0^{\frac{\hbar^2}{2m_N}(\frac{2\mu v}{\hbar})^2} \frac{dE_R}{2\pi\hbar^2} Z_{eff}^2(q)\left(\frac{\kappa e^2}{q^2\epsilon_0}\right)^2\frac{E_R}{\hbar^2} \tag{498}$$

$$= \frac{n_N m_\chi m_N}{2E_\chi}\int_0^{\frac{\hbar^2}{2m_N}(\frac{2\mu v}{\hbar})^2} \frac{dE_R}{2\pi\hbar^2} Z_{eff}^2(\sqrt{2m_N E_R})\left(\frac{\kappa e^2\hbar^2}{2m_N E_R\epsilon_0}\right)^2\frac{E_R}{\hbar^2} \tag{499}$$

$$= \frac{n_N m_\chi m_N}{2E_\chi\hbar^4}\int_0^{\frac{\hbar^2}{2m_N}(\frac{2\mu v}{\hbar})^2} \frac{dE_R}{2\pi} Z_{eff}^2(\sqrt{2m_N E_R})\left(\frac{2\pi\kappa\alpha\hbar^3 c}{m_N E_R}\right)^2 E_R \tag{500}$$

$$= \frac{n_N m_\chi \pi\alpha^2\hbar^2 c^2\kappa^2}{m_N E_\chi}\int_0^{E_\chi 4\frac{\mu^2}{m_\chi m_N}} dE_R\frac{Z_{eff}^2(\sqrt{2m_N E_R})}{E_R} \tag{501}$$

Where $\frac{1}{\mu} \equiv \frac{1}{m_N} + \frac{1}{m_\chi}$ is the reduced mass between the DM particle and the nucleus / atom, and we have used the fact that the $q, v > 0$ in the last line. We can see that we exactly match [5] eqn (A.1) up to the inclusion of the form factor which regulates the IR singularity of the coulomb potential that they cutoff in an ad-hoc manner.

8 $\frac{d\sigma}{dE_R}$ calculation

$$\frac{d\sigma}{dE_R}(v,\omega) = \frac{1}{\hbar}\frac{d\sigma}{d\omega} \tag{502}$$
$$= \frac{1}{n_T\hbar^2}\int \frac{dq}{(2\pi)^2}\frac{q}{v^2}V^2(q)S(q,\omega)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{503}$$
$$= \frac{1}{vn_T\hbar}\frac{dP}{d\omega} \tag{504}$$

From above we have for $\omega > 0$

$$\frac{dP}{d\omega} = \frac{1}{V}\sum_{\boldsymbol{q}}\frac{1}{\hbar}\delta(\omega-\omega_{\boldsymbol{q}})V^2(q)S(\boldsymbol{q},\omega) \tag{505}$$
$$= \frac{1}{\hbar}\int \frac{dq}{(2\pi)^2}\frac{q}{v}V^2(q)S(q,\omega)\Theta(\omega_- - \omega)\Theta(\omega_+ + \omega) \tag{506}$$
$$= \frac{1}{\hbar}\int \frac{dq}{(2\pi)^2}\frac{q}{v}V^2(q)S(q,\omega)\Theta(q - q_-)\Theta(q_+ - q) \tag{507}$$
$$= \frac{1}{\hbar}\int_{q_-}^{q_+} \frac{dq}{(2\pi)^2}\frac{q}{v}V^2(q)S(q,\omega) \tag{508}$$

Where again $\omega_\pm = qv \pm \frac{\hbar q^2}{2m_\chi}$, we have used the fact that $\omega_+ + \omega > 0$ for $\omega > 0$, and we have defined $q_\pm = \frac{m_\chi v \pm \sqrt{2m_\chi\left(\frac{1}{2}m_\chi v^2 - \omega\hbar\right)}}{\hbar}$ (the solutions to $\omega_- = qv - \frac{\hbar q^2}{2m_\chi}$ which are valid for $E_\chi \geq \hbar\omega$). We can also note that this is a dimensionless quantity as it must be.

8.1 Nuclear Contribution

8.1.1 0 Temperature

For 0 Temperature Nuclear Scattering we then find case we then find.

$$\frac{dP}{d\omega} = \frac{1}{\hbar}\int_{q_-}^{q_+} \frac{dq}{(2\pi)^2}\frac{q}{v}V^2(q)2\pi n_N Z_{eff}^2(q)\delta\left(\frac{\hbar^2 q^2}{2m_N} - \hbar\omega\right) \tag{509}$$
$$= \left[\frac{1}{\hbar(2\pi)^2}\frac{q}{v}V^2(q)2\pi n_N Z_{eff}^2(q)\right]_{q=q_\omega}\frac{q_\omega}{2\omega\hbar}\Theta(q_+ - q_\omega)\Theta(q_\omega - q_-) \tag{510}$$

Where $q_\omega \equiv \sqrt{\frac{2m_N\omega}{\hbar}}$.
*** note also that this is still dimensionless

8.2 Electronic Contribution

*** note we are using SI below where we are using cgs above
We can now use (508) and the Fluctuation Dissipation theorem to determine the elctronic contribution to $\frac{d\sigma}{dE_R}$

$$\frac{dP}{d\omega} = \frac{1}{\hbar}\int_{q_-}^{q_+}\frac{dq}{(2\pi)^2}\frac{q}{v}V^2(q)S(q,\omega) \tag{511}$$

$$= \frac{\kappa}{\hbar}\int_{q_-}^{q_+}\frac{dq}{(2\pi)^2}\frac{q}{v}V(q)\frac{2}{1-e^{-\beta\hbar\omega}}\mathrm{Im}\left(\frac{-1}{\epsilon_M}\right) \tag{512}$$

$$= \frac{1}{\hbar}\int_{q_-}^{q_+}\frac{dq}{(2\pi)^2}\frac{q}{v}\frac{e^2\kappa^2}{q^2\epsilon_0}\frac{2}{1-e^{-\beta\hbar\omega}}\mathrm{Im}\left(\frac{-1}{\epsilon_M}\right) \tag{513}$$

$$= \frac{1}{v\hbar(2\pi)^2}\frac{e^2\kappa^2}{\epsilon_0}\frac{2}{1-e^{-\beta\hbar\omega}}\int_{z_-}^{z_+}\frac{dz}{z}\mathrm{Im}\left(\frac{-1}{\epsilon_M(z,u=\frac{\hbar\omega}{4E_F z})}\right) \tag{514}$$

Where as before $z = \frac{q}{2k_F}$, $u = \frac{\omega}{k_F q}$ and we have defined $z_\pm = \frac{m_\chi v \pm \sqrt{2m_\chi\left(\frac{1}{2}m_\chi v^2 - \omega\hbar\right)}}{2k_F\hbar}$. Using (504) we then find

$$\frac{d\sigma}{dE_R}(v,\omega) = \frac{1}{n_e v^2\hbar^2(2\pi)^2}\frac{e^2\kappa^2}{\epsilon_0}\frac{2}{1-e^{-\beta\hbar\omega}}\int_{z_-}^{z_+}\frac{dz}{z}\mathrm{Im}\left(\frac{-1}{\epsilon_M(z,u=\frac{\hbar\omega}{4E_F z})}\right) \tag{515}$$

9 DM self interaction and screening

10 Analytic Solutions to Vlasov Equation

[23] this seems ideal! For single component plasma we can use their results directly. For multicomponent it will be more complicated, but we can use this as equilibrium distribution and then include it in our RPA and RTA calculations of the DM contribution to screening.

11 conductivity in Lindhard model

See [6] for an expression for the conductivity in the RPA.

12 Capture Psuedo-Code

In this section we describe a numerical algorithm to calculate the capture rate of the in the earth.

12.1 Focusing and Impact Parameter

We first must compute the gravational / electromagnetic focussing of the incoming dark matter distribution as it propagates towards the earth. To do so we ignore the interactions among the incoming dark matter species, assuming that they are dominated by the electromangetic and gravitational charges in the earth. As such we can treat each incoming dark matter particle as an independent particle moving in a central potential given determined by the earth (away from it's boundary). Defining ϕ to be the angular coordinate in the plane orthongonal to it's conserved angular momentum (L) we find the effective Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m_\chi(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \tag{516}$$

$$= \frac{1}{2}m_\chi(\dot{r}^2 + \frac{L^2}{m_\chi^2 r^2}) - V(r) \tag{517}$$

$$\tag{518}$$

We also have that for an incoming velocity $\vec{v}$, at some radius r, making an angle θ with the displacement from the Earth's Centre.

$$
\begin{aligned}
L =& m_\chi r^2 \dot{\phi} && (519)\\
=& \sin(\theta) r v m_\chi && (520)\\
=& r v_\perp m_\chi && (521)
\end{aligned}
$$

Where $v_\perp$ is the velocity in the direction orthogonal to the radial component. The impact parameter b is also given by $b = r\sin\theta$. Hence we can write

$$
L = b v m_\chi \tag{522}
$$

Since this quantity is conserved, given some initial condition at a large radius r_∞ we have at any other radius r

$$
b_\infty v_\infty = b(r) v(r) \tag{523}
$$

The conserved energy is

$$
E = \frac{1}{2} m_\chi \left(\dot{r}^2 + \frac{L^2}{m_\chi^2 r^2} \right) + V(r) \tag{524}
$$

So using this and (521), we can determine the velocity as a function of radius

$$
\begin{aligned}
v_r \equiv& |\dot{r}| = \sqrt{\frac{2}{m_\chi}(E - V(r)) - \frac{L^2}{m_\chi^2 r^2}} && (525)\\
v_\perp =& \frac{L}{m_\chi r} && (526)
\end{aligned}
$$

The impact parameter is can then be determined as a function of r from the initial conditions using (523)

$$
\begin{aligned}
b(r) =& \frac{b_\infty v_\infty}{\sqrt{\frac{2}{m_\chi}(E - V(r))}} && (527)\\
=& \frac{b_\infty}{\sqrt{1 + \frac{V(r_\infty) - V(r)}{E_{K,\infty}}}} && (528)\\
& && (529)
\end{aligned}
$$

Where $E_{K,\infty} \equiv \frac{1}{2} m_\chi v_\infty^2$. Hence the condition for the DM particle to reach the surface is

$$
\begin{aligned}
r_E \geq& b(r_E) && (530)\\
r_E \geq& \frac{b_\infty}{\sqrt{1 + \frac{V(r_\infty) - V(r_E)}{E_{K,\infty}}}} && (531)
\end{aligned}
$$

We must also have that the radial kinetic energy is greater than 0 at this point (so that it will not be stopped by the angular momentum barrier) [5].

[5]For now we will not consider the second barrier which may appear in the potential from the presence of the screened Coulomb potential at the Earth (this will be valid in the regime of small captured charge)

$$0 \leq v_r^2 = \frac{2}{m_\chi}(E - V(r_E)) - \frac{L^2}{m_\chi^2 r_E^2} \tag{532}$$

$$E_{k,\infty} \geq \frac{L^2}{2m_\chi r_E^2} + [V(r_E) - V(r_\infty)] \tag{533}$$

$$E_{k,\infty} \geq \frac{m_\chi}{2} v_\infty^2 b_\infty^2 \frac{1}{r_E^2} + [V(r_E) - V(r_\infty)] \tag{534}$$

$$r_E^2 \geq \frac{b_\infty^2}{1 - \frac{V(r_E) - V(r_\infty)}{E_{k,\infty}}} \tag{535}$$

$$\tag{536}$$

Which is precisely the condition derived above. The condition for bounded motion is that $E < 0$. Which ocurrs when

$$E_{k,\infty} + V(r_\infty) < 0 \tag{537}$$

12.2 Δr and Interaction Regimes

We subdivide our parameter space into 3 different regimes based on the interaction strength of the DM with the Earth. We determine the length δr such that the particle looses a fraction y of it's kinetic energy as it passes through the Earth. We calculate this quantity at fixed direction through the Earth.

$$y(\Delta r) \equiv \frac{1}{E_{K,E}} \int_0^{\Delta r} dl \frac{dE(v_E, |\vec{x}(l)|)}{dl} \tag{538}$$

Where subscript E denotes a parameter evaluated as the DM particle reaches the Earth. Where $\partial_t \vec{x}(l) \equiv \vec{v}_E$. For a given Our subdivision is taken to be for $y = 10^{-2}$

1. $\Delta r < \frac{d_E}{500}$ - Strong Scattering Regime
2. $\frac{d_E}{500} < \Delta r < \frac{d_E}{10}$ - Intermediate Scattering Regime
3. $\frac{d_E}{10} < \Delta r$ - Weak Scattering Regime

Where d_E is the straight path length of the particle through the earth parallel to $\vec{v}_E$ and is given by $d_E = 2\sqrt{r_E^2 - b_E^2}$. Notice that the distance from the surface to the core is greater than the corresponding $\frac{d_E}{10}$. So in our two Layer Earth model, where the Energy Loss is constant in each layer, outside the Weak Scattering Regime we have

$$\Delta r = y E_{K,E} \left(\frac{dE(v_E, r_{mantle})}{dl} \right)^{-1} \tag{539}$$

We can determine if we are in the weak scattering regime if $\Delta r > l_{S \to M}$ where $l_{S \to M}$ is the distance from the surface to the mantle.In the Strong Scattering Regime, we consider the particle to be captured once it enters the Earth.

12.3 Weak Scattering Regime and Disipative Dynamics

In the Weak Scattering Regime, we solve the classical trajectory of the particle accounting for average dissipation along the path, then compute the probability for a hard scattering event that could lead to capture along the path. To find the Classical Trajectory including the energy loss, we consider the Lagrangian for the particle inside the Earth in Cartesian coordinates. The angular momentum is still conserved as the energy loss depends only on the radius in the Earth. As such, our dynamics are still effectively 2 Dimensional

$$\mathcal{L} =\frac{1}{2}m_\chi v^2 - V(r) + \int_0^t dt' \frac{dE(r,v)}{dt'} \tag{540}$$

Our model of the Earth divides it into two components (mantle and core) seperated by a barrier of negligible thickness. As such, momentarily ignoring boundary effects, the equations of motion of are given by

$$m\dot{v}^i = -\frac{x^i}{r}V'(r) - \frac{v^i}{v}\frac{d}{dv^i}\frac{dE(r,v)}{dt} \tag{541}$$

Our model involves a two component Earth with Energy Loss function near the boundaries r_b given by a function of the form

$$\frac{dE(r,v)}{dt} = \begin{cases} W_1(v), & r > r_b \\ W_2(v), & r < r_b \end{cases} \tag{542}$$

For some W_i. Hence we have

$$\frac{d}{dr}\frac{dE(r,v)}{dt} = (W_1(v) - W_2(v))\delta(r - r_b) \equiv \Delta W(v)\delta(r - r_b) \tag{543}$$

Hence the boundary term is given by

$$\frac{d\mathcal{L}}{dx^i} \supset \int_0^t dt' \frac{x^i}{r}\frac{d}{dr}\frac{dE(r,v)}{dt'} \tag{544}$$

$$= \int_0^t dt' \frac{x^i}{r}\Delta W(v)\delta(r - r_b) \tag{545}$$

Now take the particle to reach the boundary at a time $t = t_b$, then we find

$$\int_0^t dt' \frac{x^i}{r}\Delta W(v)\delta(r - r_b) = \int_0^t dt' \frac{x^i}{r}\frac{\Delta W}{v_r}\delta(t - t_b) \tag{546}$$

$$= \begin{cases} 0, & t < t_b \\ \Delta W(v_b)\frac{x_b^i}{r_b v_{r,b}}, & t > t_b \end{cases} \tag{547}$$

Where we have introduced the notation $f(t_b) \equiv f_b$. So we find that there is an additional radial force introduced to the dynamics as a result of the discontinuity, which is removed when the particle passes back over the boundary. Including this effect in the equations of motion we find

$$m\dot{v}^i = -\frac{x^i}{r}V'(r) - \frac{v^i}{v}\frac{d}{dv^i}\frac{dE(r,v)}{dt} - \frac{x_b^i}{r_b v_{r,b}}\frac{dE(r,v_b)}{dt} \tag{548}$$

$$= -\frac{x^i}{r}V'(r) - v^i\frac{d}{dv^i}\frac{dE(r,v)}{dl} - \frac{x_b^i}{r_b|\cos(\theta_r)|}\frac{dE(r,v_b)}{dl} \tag{549}$$

Where here r_b corresponds to the previous boundary passed by the particle, and θ_r is the angle of the particle velocity from the normal to the boundary ($v_{r,b} = \cos(\theta_r)v$). The Intermediate Scattering Regime is less trivial. We discuss it subsequently.

12.4 Intermediate Scattering Regime

For the Intermediate Scattering Regime (ISR) we numerically integrate using a Montecarlo. From the surface, we propagate as follows

1. Initialize the particle at $(r, \vec{v}, b) = (r_E, \vec{v}_E, b_E)$

2. Throw a random number $\xi_1 \in [0, 1)$ step the particle position by $\Delta\vec{x} = \hat{v}l$ (ie. update to the corresponding value r') where l is the Interaction length (see eg. [35])

$$l = \lambda \ln (1 - \xi_1)$$

$$\lambda^{-1} \equiv n_T \int dE_R \frac{d\sigma}{dE_R}$$

3. Update the particle velocity according to the change in potential

$$v \to v + v_{esc}(r') - v_{esc}(r)$$

4. Throw another random number ξ_2 and determine the interaction process (eg. nuclear or electronic, material etc.) [36]

$$\sum_i^n \frac{\sigma_i}{\sigma} \le \xi_2 < \sum_i^{n+1} \frac{\sigma_i}{\sigma}, \qquad \sigma = \sum_i^{N_{processes}} \sigma_i$$

5. Sample energy loss due to that process sampling from the distribution from either (see eg. [36])

   - **Transformation method** - determine the PDF for the energy loss by that process over the path length l as

   $$P_i(E_R) = n_T l \frac{d\sigma_i}{dE_R}$$

   Compute and invert the CDF as

   $$C_i(E_R) = \int_0^{E_R} dE'_R P_i(E'_R)$$

   Invert (possibly numerically) to find the EL as a function of another thrown PRN

   $$E_R = C_i^{-1}(\xi_3)$$

   - **Rejection method** - Given the known PDF $P_i(E_R)$ with maximum value M on the interval $E_R \in [0, E_K)$ (with E_K the kinetic energy of the DM particle), sample two random numbers ξ_4, ξ_5. We accept the value of

   $$E_R = \xi_4 E_K$$

   only if

   $$\xi_5 M \le P_i(E_R)$$

   Otherwise we reject the point and repeat the procedure.

6. For the case of a thermal target, use one of the above sampling techniques to determine the speed of the target particle v_T. For the targets we are interested in we have the PDFs

$$P(v_T) = \frac{1}{v_F} \Theta(v_F - v_T), \qquad \text{degenerate Fermions}$$

$$P(v_T) \propto v_T^2 e^{-\beta m_T v_T^2}, \qquad \text{non-degenerate / classical}$$

7. Uniformly sample the 2-sphere to find the velocity of the Target. (equivalently, just sample the angle θ_T it makes with $\vec{q}$)

8. Energy and momentum conservation in the interaction imposes

$$E_R = \frac{q^2}{2m_\chi} - \vec{q} \cdot \vec{v} = \frac{q^2}{2m_T} - \vec{q} \cdot \vec{v}_T$$

Compute the momentum transfer as

$$q = q_T \left(\cos\theta_T + \sqrt{\frac{2m_T E_R}{q_T^2} + \cos^2\theta_T} \right)$$

Where $q_T = m_T v_T$. And the scattering angle θ of the DM is given by

$$\theta = \arccos\left[\frac{1}{qv} \left(E_R - \frac{q^2}{2m_\chi} \right) \right]$$

9. we now uniformly sample the azimuthal angle and $\phi \in [0, 2\pi)$ and update the DM velocity as

$$\vec{v} \to \vec{v} + qR(\theta, \phi)\hat{v}$$

check if $E' = E - E_R < 0$. If so the particle is bound and considered captured. Else continue.

10. We repeat steps 2-10 until we either find that the particle has been captured (can use bound criteria, or thermalization criteria) or the particle exits the earth with a high enough velocity to escape to infinity.

$$v_r \geq v_{esc}(r_E)$$

**** could also do velocity is lower than that need to just escape the Earth.

13 Nuclear Structure Function Toy Model at 0 Temperature

**** NOTE TO DAVID: This section hasn't been updated since this note was started. I think it is qualitatively correct, but I wouldn't take it's results too seriously. We will need to revisit this once we have studied the nuclear and atomic form factors and start adding in nuclear interactions more carefully.

Here we follow [37] and write down a toy model for atomic nuclei in material at 0 Temperature. The Structure Function is

$$S(\mathbf{q},\omega) = \frac{2\pi}{V}\sum_f |\langle f|\sum_I f_I \exp i\mathbf{q}\cdot\mathbf{r}_I|\rangle i|^2 \delta(E_f - E_i - \omega) \tag{550}$$

This assumes spin independent interactions. In this paper they go on to expand the interaction picture operator in the matrix element.

$$\boldsymbol{r}_I = \boldsymbol{r}_{nj} + \boldsymbol{\phi}_{nj} = \mathbf{R}_n + \mathbf{R}_j^0 + \boldsymbol{\phi}_{nj} \tag{551}$$

Where the ion positions are $\boldsymbol{r}_I$, $\mathbf{R}_n$ is the position of the unit cell n, and $\mathbf{R}_j^0$ is the equilibrium position of ion j in the unit cell. The ion displacement $\boldsymbol{\phi}_{nj}$ is given by

$$\boldsymbol{\phi}_{nj} = \sum_{\nu,\mathbf{k}} \frac{1}{\sqrt{2N_{\text{cell}}M_j\omega_{\nu,k}}}\left[a^\dagger_{\nu,\boldsymbol{k}}\mathbf{e}^\star_{\nu,\mathbf{k},j}e^{-i\mathbf{k}\cdot r_{nj}} + \text{h.c.}\right] \tag{552}$$

Where N_{cell} is the number of ions per unit cell. Notice that in field theory, ie. going to the continuum, $\boldsymbol{\phi}_{nj}$ corresponds to the mode expansion of a real scalar vector field. In real, space these correspond to derivative interactions. Since they appear linearly in the matrix element, this corresponds to a tadpole interaction, the $\boldsymbol{\phi}_{nj}$ excitations are not conserved. This is in line with what we'd expect, since the phonons ($\boldsymbol{\phi}_{nj}$), are identified with the 3-generators of translational symmetry for each ion which are broken by the presence of a discrete lattice.

**** not conserved is true, but they are not the goldstone mode. They have finite excitation energy. The goldstone mode is the acoustic phonons, which are not described by interactions with ions in a single cell.

*** it does however tell us that once we know the excitation energy we can write down a diagram for a fixed phonon source diagram were we to do it in QFT. Much like the famous QFT1 problem.

$$\begin{aligned} f_I e^{i\mathbf{q}\cdot\mathbf{r}_I} =& f_I e^{i\mathbf{q}\cdot\mathbf{r}_{nj}}e^{i\mathbf{q}\cdot\boldsymbol{\phi}_{nj}} && (553)\\ =& f_I e^{i\mathbf{q}\cdot\mathbf{r}_{nj}}e^{-W_j(\mathbf{q})}\prod_{\nu\mathbf{k}}\exp\left(\frac{i\mathbf{q}\cdot\mathbf{e}^\star_{\nu,\mathbf{k},j}}{\sqrt{2N_{\text{cell}}M_j\omega_{\nu,k}}}a^\dagger_{\nu,\boldsymbol{k}}\right)\times\text{h.c.} && (554)\end{aligned}$$

Where we BCH formula has been used, and the exponential factor is called the Debye-Waller factor which is defined as

$$W_j(\mathbf{q}) \equiv \frac{1}{2}\sum_{\nu,\mathbf{k}}\frac{|\mathbf{q}\cdot\mathbf{e}^\star_{\nu,\mathbf{k},j}|^2}{2N_{\text{cell}}M_j\omega_{\nu,k}} \tag{555}$$

We now specialize to the case of a simple harmonic oscillator potential to get an analytic expression for the form factor and determine the scaling with $\mathbf{q}$. This is a toy model can be used to describe a simple system of a diatomic molecule, as an intermediary system between free atoms and condensed matter systems, in the limit of small anharmonic corrections to the potential.

It can also be used for small momentum transfer in general for Recoil energies below $\mathcal{O}(10)$ eV, which is the energy to break molecular bonds or displace an ion in a crystal. This corresponds to intermediate DM masses.

This is for a Debeye model of a solid assuming contact interactions where particles scatter coherently off the nucleus and inner shell electrons (taking the nuclear form factor to 1 which (see above eqn 78) is valid for sub-GeV DM which has kinetic energies well below the scale at which we can resolve the incoherent structure of the nucleus).

The potential in this toy model is

$$\mathcal{V}(r) = \frac{1}{2}M\omega_0^2 r^2 \tag{556}$$

For an isotropic medium, this gives cononical harmonic oscillator creation and annihilation operators for excitations about the ground state of each ion and all $j,\ \nu$. (*** make contact with $\boldsymbol{\phi}$ here.

$$\exp\left(i\mathbf{q}\cdot\boldsymbol{\phi}_{nj}\right) = \exp\left[\frac{iq_i}{q_0}\left(a_i + a_i^\dagger\right)\right] \tag{557}$$

For all $n,\ j$. Where $q_0 = \sqrt{2M\omega_0}$.

$$\implies |f(n,q)|^2 = \frac{1}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2} \tag{558}$$

So the number of optical phonons is Poisson distributed. The structure function becomes

$$\begin{aligned} S(\mathbf{q},\omega) =& \frac{2\pi}{V}\sum_{I,n}|f(n,q)|^2\delta(E_f - E_i - \omega) && (559)\\ =& \frac{2\pi}{V}\sum_n \frac{N_T}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2}\delta(n\omega_0 - \omega) && (560)\\ & && (561)\end{aligned}$$

Giving energy loss

$$\begin{aligned} \frac{dE}{dr}(v) =& -N_T\int\frac{dq\ q}{v^2\hbar(2\pi)^2}V^2(q)\int d\omega\ \omega S(q,\omega)\Theta(v - v_{\min}) && (562)\\ =& -N_T\int\frac{dq\ q}{v^2\hbar(2\pi)^2}V^2(q)\int d\omega\ \omega\frac{2\pi}{V}\sum_n\frac{1}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2}\delta(n\omega_0 - \omega)\Theta(v - v_{\min}) && (563)\\ =& -\int\frac{dq\ q}{v^2\hbar(2\pi)^2}V^2(q)2\pi n_T\sum_n\frac{n\omega_0}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2}\Theta\left[v - \left(\frac{n\omega_0}{q} + \frac{q}{2m_\chi}\right)\right] && (564)\end{aligned}$$

We now define

$$\bar{n}_q \equiv \sum_n\frac{n}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2} = \frac{q^2}{q_0^2} = \frac{q^2}{2M\omega_0} = \frac{E_R}{\omega_0} \tag{565}$$

And make the replacement $n \to \bar{n}_q$ in the Θ function. As we will see, this let's us reproduce the results of the atomic dark matter capture appendix (*** cite). In the next section we discuss the implications and requirements of making this replacement.

$$\begin{aligned} \frac{dE}{dr}(v) \approx& -n_T\int\frac{dq\ q}{v^2\hbar 2\pi}V^2(q)\Theta\left[v - \left(\frac{\bar{n}_q\omega_0}{q} + \frac{q}{2m_\chi}\right)\right]\sum_n\frac{n\omega_0}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2} && (566)\\ =& n_T\int\frac{dq\ q}{v^2\hbar 2\pi}V^2(q)\Theta\left[v - \frac{q}{2}\left(\frac{1}{M} + \frac{1}{m_\chi}\right)\right]\omega_0\bar{n}_q && (567)\\ & && (568)\end{aligned}$$

Giving us

$$\frac{dE}{dr}(v) \approx -n_T \int \frac{dq\ q}{v^2\hbar 2\pi} V^2(q) E_R \Theta\left[v - \frac{q}{2\mu}\right] \tag{569}$$

$$\tag{570}$$

Where μ is the reduced mass between the Nuclei and Dark Matter. Using the expression for the Coulomb potential

(*** this is in natural units, restore c, $\hbar$ to be consistent with the rest of the calculation

$$V^2(q) = \left(\frac{2\pi\epsilon\alpha Z}{q^2}\right)^2 \tag{571}$$

We arrive at

$$\frac{dE}{dr}(v) \approx -n_T \int \frac{dq\ q}{v^2\hbar 2\pi}\left(\frac{2\pi\alpha\epsilon Z}{q^2}\right)^2 \frac{q^2}{2M}\Theta\left[v - \frac{q}{2\mu}\right] \tag{572}$$

$$= -\frac{\pi\alpha^2\epsilon^2 Z^2 n_T}{Mv^2\hbar}\int dq\frac{1}{q}\Theta\left[v - \frac{q}{2\mu}\right] \tag{573}$$

$$\tag{574}$$

We now impose an IR cutoff at the finite size of the atom (ie. neglect coherent scattering off the lattice and acoustic phonons) to make contact with the calculation done in the appendix.

$$q^{\min} = 2m_e\alpha\hbar \tag{575}$$

a characteristic size for the finite atom. In units natural units. To get

$$\frac{dE}{dr}(v) \approx -n_T\frac{\pi\alpha^2\epsilon^2 Z^2}{Mv^2\hbar}\ln\left(\frac{2v\mu}{E_R^{\min}}\right) \tag{576}$$

$$= -n_T\frac{\pi\alpha^2\epsilon^2 Z^2}{Mv^2\hbar}\ln\left(\frac{2v\mu}{2m_e\alpha\hbar}\right) \tag{577}$$

$$= -n_T\frac{2\pi\alpha^2\epsilon^2 Z^2}{Mv^2\hbar}\ln\left[\left(\frac{v\mu}{m_e\alpha\hbar}\right)^2\right] \tag{578}$$

Which agrees with the appendix calculation.

As mentioned in a footnote of the paper, this again does not include the cross terms that would arise from the electron structure of the atom.

From this calculation, we can enumerate the approximations and regime of validity of the Single particle Rutherford scattering calculation done in the appendix

*** read off from the above to show to David

13.1 Removing average n phonon approximation

Let's see what the affect of removing the average phonon excitation approximation used above. To do so we need to include the Θ function in the sum over the number of excited phonons.

$$\sum_n \frac{n}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2}\Theta\left[v - \left(\frac{n\omega_0}{q} + \frac{q}{2m_\chi}\right)\right] = \sum_n \frac{n}{n!}\left(\mu_q\right)^n e^{-\mu_q}\Theta\left[\frac{q}{\omega_0}\left(v - \frac{q}{2m_\chi}\right) - n\right] \tag{579}$$

$$= \sum_n \frac{n}{n!}\left(\mu_q\right)^n e^{-\mu_q}\Theta\left[n_{\max} - n\right] \tag{580}$$

$$\tag{581}$$

Where we've assumed that the atom is in the ground state so that ω_0 is positive (requires 0 temperature I think ***). We have defined

$$\mu_q = \frac{q^2}{q_0^2} \qquad\qquad n_{\max} = \frac{q}{\omega_0}\left(v - \frac{q}{2m_\chi}\right) \tag{582}$$

Leading to the the truncated Poisson

$$\sum_{n=0}^{\lfloor n_{\max}\rfloor} \frac{n}{n!}\left(\mu_q\right)^n e^{-\mu_q} = \mu_q e^{-\mu_q} \sum_{n=0}^{\lfloor n_{\max}\rfloor - 1} \frac{1}{n!}\left(\mu_q\right)^n \tag{583}$$

Meaning that our above approximation amounts to taking the limit that $n_{\max} \to \infty$. Other words taking $\omega_0 \to 0$ at finite $q,\ v$, which means physically that we are taking the 0 excitation energy limit of the phonons. One could have predicted this as phonons correspond to the energy cost for translation of the ion, so this corresponds to the free ion limit. This corresponds more precisely to

$$\mu_q << n_{\max} \tag{584}$$

**** also nbarq is equal to the mean muq

Equivalently the limit of large ion mass compared to the DM mass so that the Ion is not displaced significantly from it's equilbrium position

We must also take this limit such that we can make the replacement in the Θ function.

$$\bar{n}_q << n_{\max} \iff E_R = \frac{q^2}{2M} << q\left(v - \frac{q}{2m_\chi}\right) \tag{585}$$

is satisfied. Which is possible to only for the limit of large incoming velocity.

$$\bar{n}_q << n_{\max} \iff \frac{q}{2\mu} << v \tag{586}$$

This is the limit that the nucleus is heavy compared to the incoming dark matter. So the limit of a static target, where the nucleus is barely displaced from it's equilibrium and it costs very little to excite many phonons.

From this simple observation, one would predict generically that taking the 0 excitation energy limit of the phonons would recover free particle elastic scattering independently of the model of the material interactions. Further, this determines the regime where this approximation is no longer valid. For q comparable to the excitation energy of the phonons, the finite value of $n_{\max}$ can no longer be ignored, and one needs an accurate models to describe their excitation.

Our friend Mathematica tells us that this sum evaluates to

$$\begin{aligned}
&\frac{e^{-\mu_q}\mu_q}{\Gamma(1+\lfloor n_{\max}\rfloor)}\left(e^{\mu_q}\Gamma(1+\lfloor n_{\max}\rfloor,\mu_q) - \mu^{\lfloor n_{\max}\rfloor}\right) &(587)\\
=&\frac{\mu_q}{\Gamma(1+\lfloor n_{\max}\rfloor)}\left(\Gamma(1+\lfloor n_{\max}\rfloor,\mu_q) - e^{-\mu_q}\mu^{\lfloor n_{\max}\rfloor}\right) &(588)\\
=&\frac{\mu_q}{\Gamma(1+\lfloor n_{\max}\rfloor)}\left(\Gamma(1+\lfloor n_{\max}\rfloor) - \gamma(1+\lfloor n_{\max}\rfloor,\mu_q) - e^{-\mu_q}\mu^{\lfloor n_{\max}\rfloor}\right) &(589)\\
=&\mu_q\left(1 - \frac{\gamma(1+\lfloor n_{\max}\rfloor,\mu_q) + e^{-\mu_q}\mu^{\lfloor n_{\max}\rfloor}}{\Gamma(1+\lfloor n_{\max}\rfloor)}\right) &(590)\\
=&\mu_q\left(1 - \frac{\gamma(1+\lfloor n_{\max}\rfloor,\mu_q) + e^{-\mu_q}\mu^{\lfloor n_{\max}\rfloor}}{\lfloor n_{\max}\rfloor!}\right) &(591)\\
=&\mu_q\left(1 - P(\lfloor n_{\max}\rfloor,\mu_q) - \frac{\gamma(1+\lfloor n_{\max}\rfloor,\mu_q)}{\lfloor n_{\max}\rfloor!}\right) &(592)\\
\equiv&\mu_q C(\lfloor n_{\max}\rfloor,\mu_q) &(593)
\end{aligned}$$

Which would be quite challenging to integrate over analytically. However, it is worth noting that with μ_q fixed

$$\lim_{\lfloor n_{\max} \rfloor \to 0} \mu_q C(n_{\max}, \mu_q) = 0 \quad , \quad \lim_{\lfloor n_{\max} \rfloor \to \infty} \mu_q C(n_{\max}, \mu_q) = \mu_q \tag{594}$$

So, when properly evaluated, this factor would likely cut off our infrared singularity without needing to impose one ad-hoc. This makes intuitive sense. We have a discrete optical phonon spectrum. So below the excitation energy, no energy would be transfered to the system.

It is also important to know that both the probability and Upper Incomplete Gamma function are non-negative, so that for lower values of $\lfloor n_{\max} \rfloor / \mu_q$ we only get suppression of the energy loss. Which is obvious from looking at the above sum.

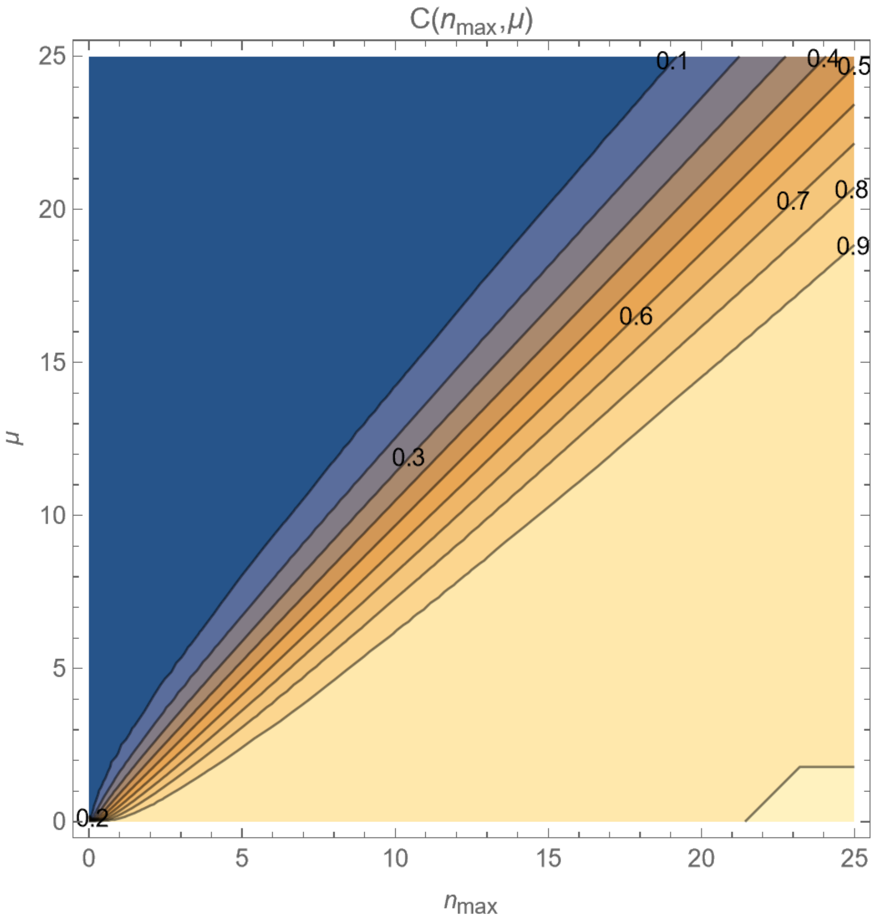

Figure 11: Numerical evaluation of $C(n_{\max}, \mu_q)$

14 Connection to Liquid Iron Paper

In this section we check that our above expressions for dynamical structure function reproduce (at least qualitatively) the liquid iron paper which makes the same computation in a full ab Initio DFT simulation.

***** write down structure functions (not stopping) for both phonons and electrons and work with thermal to cut off singularites etc. Also just add, ignore cross terms for now.

The structure function for nuclei we took was

$$S(\mathbf{q},\omega;\omega_0) = \frac{2\pi}{V}\sum_{I,n}|f(n,q)|^2\delta(E_f - E_i - \omega) \tag{595}$$

$$= \frac{2\pi}{V}\sum_n \frac{N_T}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2}\delta(n\omega_0 - \omega) \tag{596}$$

$$= 2\pi n_T \sum_n \frac{1}{n!}\left(\frac{q^2}{q_0^2}\right)^n e^{-q^2/q_0^2}\delta(n\omega_0 - \omega) \tag{597}$$

$$\tag{598}$$

We evaluate at $n = \frac{\omega}{\omega_0}$ and remove the delta function as an approximation to this function. At finite temperature these poles will be smoothed out. Here we only look to determine their qualitative placement, the correct value at the pole will require us to carry out the full finite temperature calculation.

$$S(\mathbf{q},\omega;\omega_0,q_0) = \frac{2\pi n_T}{\Gamma(\frac{\omega}{\omega_0}+1)}\left(\frac{q^2}{q_0^2}\right)^{\frac{\omega}{\omega_0}} e^{-q^2/q_0^2} \tag{599}$$

$$\tag{600}$$

****** for ω_0 we use the Debeye frequency of Iron.
For ω_0 we take the Debye frequency of iron.

$$\omega_0 = \omega_D = \frac{k\Theta_D}{\hbar} \tag{601}$$

The Debye temperature of iron is [6]

$$\Theta_D = 420K \tag{602}$$

We use the fluctuation dissipation theorem to get the Structure function from the Lindhard Dielectric function (at 0 Temperature).

$$\epsilon_{Lin}(\omega,k;\omega_p) = 1 + \frac{3\omega_p^2}{k^2 v_F^2}\left[f_1\left(\frac{\omega}{kv_F},\frac{k\hbar}{2mv_F}\right) + if_2\left(\frac{\omega}{kv_F},\frac{k\hbar}{2mv_F}\right)\right] \tag{603}$$

$$\implies S(\omega,k;\omega_p) = \frac{k^2}{2\pi\alpha_{em}}\text{Im}\left(\frac{1}{\epsilon_{Lin}(\omega,k;\omega_p)} - 1\right) \tag{604}$$

$$= \frac{k^2}{2\pi\alpha_{em}}\frac{\text{Im }\epsilon_{Lin}(\omega,k;\omega_p)}{|\epsilon_{Lin}(\omega,k;\omega_p)|^2} \tag{605}$$

$$\equiv \frac{k^2}{2\pi\alpha\hbar c}\frac{\text{Im }\epsilon_{Lin}(\omega,k;\omega_p)}{|\epsilon_{Lin}(\omega,k;\omega_p)|^2} \tag{606}$$

$$= \frac{k^2}{2\pi\alpha\hbar c}\text{Im}\left(1 + \frac{3\omega_p^2}{k^2 v_F^2}\left[f_1\left(\frac{\omega}{kv_F},\frac{k\hbar}{2mv_F}\right) + if_2\left(\frac{\omega}{kv_F},\frac{k\hbar}{2mv_F}\right)\right]\right)^{-1} \tag{607}$$

*** change sign here
Where we have defined $\alpha_{em} = \alpha\hbar c$ with α dimensionless EM coupling. Now we rescale S and $\omega,\ k$ to eV from SI.

[6] `https://materialsdata.nist.gov/handle/11256/32`

$$[k] = [m^{-1}] = \left[\frac{J}{\hbar c}\right] = \left[\frac{eV}{\hbar c}\frac{J}{eV}\right] \tag{608}$$

$$[\omega] = [s^{-1}] = \left[\frac{J}{\hbar}\right] = \left[\frac{eV}{\hbar}\frac{J}{eV}\right] \tag{609}$$

$$[S] = \left[\frac{k^2}{\hbar c}\right] = \left[\frac{m^{-2}}{Js\ ms^{-1}}\right] = \left[\frac{1}{m^3 J}\right] = \left[\frac{s^2}{m^5 kg}\right] = \left[\frac{J^3}{\hbar^3 c^3 J}\right] = \left[\frac{eV^2}{\hbar^3 c^3}\frac{J^2}{eV^2}\right] \tag{610}$$

$$\implies [eV^2] = \left[S\hbar^3 c^3 \frac{eV^2}{J^2}\right] \tag{611}$$

To compare to the Liquid Iron paper, we also need

$$[k] = [10^{-10} Ang^{-1}] = [m^{-1}] \tag{612}$$

$$[\omega] = [10^{-12} ps^{-1}] = [s^{-1}] \tag{613}$$

For the structure function, it is unclear what units are being used. An educated guess is the following

$$[S_{LI}] = [k^2] = [Ang^{-2}] \tag{614}$$

$$\implies [S] = \left[\frac{1}{m^3 J}\right] = \left[\frac{S_{LI}}{\hbar c} 10^{-20}\right] \tag{615}$$

Now we also convert the energy loss to eV

$$\left[\frac{dE}{dr}\right] = \left[\frac{J}{m}\right] = \left[\frac{eV^2}{\hbar c}\frac{J^2}{eV^2}\right] \tag{616}$$

where

$$f_2(u,z) = \begin{cases} \frac{\pi}{2}u, & z+u<1 \\ \frac{\pi}{8z}(1-(z-u)^2), & |z-u|<1<z+u \\ 0, \quad |z-u|>1 \end{cases} \tag{617}$$

$$f_1(u,z) = \frac{1}{2} + \frac{1}{8z}[g(z-u) + g(z+u)] \tag{618}$$

$$g(x) = (1-x^2)\ln\left|\frac{1+x}{1-x}\right| \tag{619}$$

$$\omega_p = \sqrt{\frac{4\pi\alpha_{em} n_e}{m_e}} = \sqrt{\frac{4\pi\alpha n_e \hbar c}{m_e}} \tag{620}$$

$$v_F = \hbar\left(\frac{3\pi\omega_p^2}{4\alpha_{em} m_e^2}\right)^{1/3} = \hbar\left(\frac{3\pi^2 n_e \hbar c}{4m_e^3 \hbar c}\right)^{1/3} = \frac{\hbar}{m_e}\left(\frac{3\pi^2}{4}n_e\right)^{1/3} \tag{621}$$

$$\tag{622}$$

***** need to check units, they look wrong here, they have mass dimension

We now look to compare to the liquid Iron paper. To do so we need to determine the Fermi Velocity and Plasmon Frequency. Namely, we need to determine the values of

$$\frac{1}{v_F}\left(\frac{\omega}{k} + \frac{k}{2m}\right) \tag{623}$$

$$\frac{1}{v_F}\left|\frac{\omega}{k} - \frac{k}{2m}\right| \tag{624}$$

$$\tag{625}$$

**** can also compare to hard sphere structure function given in https://doi.org/10.1103/PhysRevLett.92.185701 - fits well with experimental data!

The units used by [38] are Hartree atomic units defined by taking

$$\hbar = e = m_e = \frac{1}{4\pi\epsilon_0} = 1 \tag{626}$$

The fundamental scale becomes a unit of length, the Bohr radius

$$a_0 \approx 0.5\text{Å} \tag{627}$$

As such,

$$c = \frac{1}{\alpha a_0} = 137 a_0^{-1} \tag{628}$$

$$\implies \alpha_{em} = \alpha\hbar c = \frac{e^2}{(4\pi\epsilon_0)} = \alpha c = \frac{1}{a_0} \tag{629}$$

hence in Hartree atomic units the dimensionful constants in the Lindhard dielectric function become

$$\omega_p = \sqrt{\frac{4\pi\alpha_{em} n_e}{m_e}} = \sqrt{4\pi\frac{n_e}{a_0}} \tag{630}$$

$$v_F = \hbar\left(\frac{3\pi\omega_p^2}{4\alpha_{em} m_e^2}\right)^{1/3} = \left(\frac{3\pi^2}{4} n_e\right)^{1/3} \tag{631}$$

$$\implies \frac{\omega_p^2}{v_F^2} = \frac{4\pi n_e}{a_0\left(\frac{3\pi^2}{4} n_e\right)^{2/3}} = \left(\frac{4^5 n_e}{3^2\pi}\right)^{1/3}\frac{1}{a_0} \tag{632}$$

our structure function becomes

$$\implies S(\omega, k; \omega_p) = \frac{k^2}{2\pi\alpha\hbar c}\text{Im}\left(1 + \frac{3\omega_p^2}{k^2 v_F^2}\left[f_1\left(\frac{\omega}{k v_F}, \frac{k\hbar}{2m v_F}\right) + i f_2\left(\frac{\omega}{k v_F}, \frac{k\hbar}{2m v_F}\right)\right]\right)^{-1} \tag{633}$$

$$= \frac{k^2 a_0}{2\pi}\text{Im}\left(1 + \left(\frac{4^5 n_e}{3^2\pi}\right)^{2/3}\frac{1}{k^2 a_0^2}\left[f_1\left(\frac{\omega}{k v_F}, \frac{k}{2v_F}\right) + i f_2\left(\frac{\omega}{k v_F}, \frac{k}{2v_F}\right)\right]\right)^{-1} \tag{634}$$

$$\tag{635}$$

We must also convert momentum and frequency to atomic units [7]

$$[k] = [m^{-1}] = [a_0^{-1}] \tag{636}$$

$$[\omega] = [s^{-1}] = [E_h\hbar^{-1}] = [\alpha^2 m_e c^2\hbar^{-1}] = [a_0^{-2}] \tag{637}$$

$$1s = \frac{1}{2.42\times 10^{-17}} a.u. \tag{638}$$

[7] https://handwiki.org/wiki/Physics:Atomic_units

15 Kinetic Theory and Non-Equilibrium Velocity Distributions

See AdM capture one-note and dynamics directory for references

approach is to assume 0 thermal diffusion ie. gas at rest then compute distribution for hydrostatic equilibrium -¿ can also look for them from [25] which gives kinetic coefficients

should amount to non-relativistic vlasov poisson system including EM and gravity.

from this we should be able to solve for number densities and hence first order corrections to the equilibrium velocity distributions in the Enskog expansion.

****** XXX to the above

Mermin dielectric is derived from boltzman equations and collision integrals in a similar way. May be worth while to go through the derivation; once we understand it, it may be very straightforward to extend it to extract the non-equilibrium velocity distributions from this approach (see mermin section papers and pines)

yes! see eqn (29) of [21], they derive distribution as a function of ω in Enskog expansion with an arbitrary effective potential. We can use this, adding a fixed background gravitational potential to derive the 1st enskog correction hopefully

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