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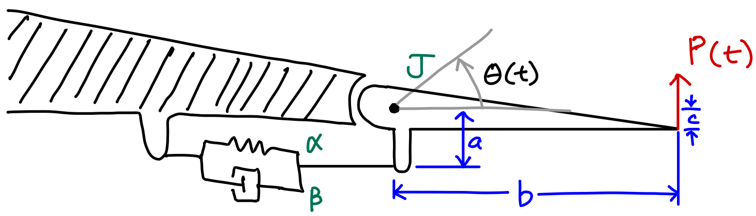
Mechanical Vibrations

1. Vibration of Single-Degree-of-Freedom Systems

1.1. Equation of Motion

- Consider the control surface shown below. The control surface rotates with an angle $\theta(t)$ and has a moment of inertia of J about the pivot. The actuator has an equivalent stiffness and damping of α and β, respectively. The actuator acts at a distance of a away from the pivot. Lastly, a force $P(t)$ acts on the control surface at a distance of b from the pivot.

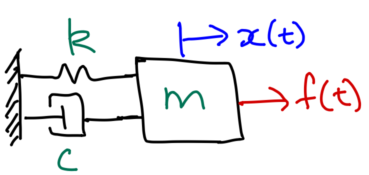


- Assuming small motion, the equation governing the motion of the control surface is found using Newton's method to be:

$$\underbrace{J\ddot{\theta}}_{\text{Inertial Force}} + \underbrace{a^2\beta\dot{\theta}}_{\text{Dissipative Force}} + \underbrace{a^2\alpha\theta}_{\text{Conservative/elastic force}} = \overbrace{bP(t)}^{\text{External force}}. \tag{1}$$

- The resulting equation of motion (EOM) is a second-order, ordinary differential equation (ODE) with constant coefficients.

- The EOM for the control surface has the same form as the EOM for a harmonic oscillator:



$$\begin{array}{ccccccccc} m & \ddot{x} & + & c & \dot{x} & + & k & x & = f(t). \\ \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ J & \ddot{\theta} & & a^2\beta & \dot{\theta} & & a^2\alpha & \theta & bP(t) \end{array} \tag{2}$$

- Note that both Eqs. 1 and 2 are linear because we only have $\ddot{x}$, $\dot{x}$, and x by themselves. There are no mixed terms ($x\dot{x}$, $\ddot{x}\dot{x}$, etc.), no powers (x^3, $\dot{x}^2$, etc.), and no functions of $\ddot{x}$, $\dot{x}$, or x ($\sin(x)$, $e^{\dot{x}}$, etc.). What about tx?

1.2. Damped, Unforced Oscillations

- We consider the EOM for a harmonic oscillator without external forcing:

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{4}$$

with initial conditions (ICs) of $x(0) = x_0$ and $\dot{x}(0) = v_0$.

- Dividing Eq. 4 by m, we can place the EOM into the classical form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \tag{5}$$

where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency and ζ is the damping ratio:

$$2\zeta\omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{k/m}} = \frac{c}{2\sqrt{km}} \Longrightarrow \boxed{\zeta = \frac{c}{2\sqrt{km}}} \tag{6}$$

- Note that the entire behavior of the oscillator is characterized entirely by only two constants: ω_n and ζ.

- We solve Eq. 5 by providing an ansatz:

$$\text{Ansatz: } x(t) = Ae^{\lambda t} \Rightarrow \dot{x}(t) = \lambda Ae^{\lambda t} \Rightarrow \ddot{x}(t) = \lambda^2 Ae^{\lambda t} \tag{7}$$

- Plugging in the ansatz into Eq. 5, we get

$$[\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2]A\underbrace{e^{\lambda t}}_{\text{Can't be zero } \forall t} = 0. \tag{8}$$

Then either $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ or $A = 0$. But $A = 0$ is the trivial solution, such that

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \rightarrow \text{Characteristic equation} \tag{9}$$

We solve Eq. 9 using the quadratic formula:

$$\lambda = \frac{1}{2}\left(-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}\right) = \cancel{\tfrac{1}{2}}\left(-\cancel{2}\zeta\omega_n \pm \sqrt{\cancel{4}\omega_n^2\zeta^2 - \cancel{4}\omega_n^2}\right)$$

$$\boxed{\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}} \tag{10}$$

- Clearly, the response (solution) depends on the value of the damping ratio, ζ:

Case 1: $\zeta = 0 \Rightarrow$ Undamped
Case 2: $0 < \zeta < 1 \Rightarrow$ Underdamped
Case 3: $\zeta = 1 \Rightarrow$ Critically damped
Case 4: $\zeta > 1 \Rightarrow$ Overdamped

- We will focus on Case 2 here: $0 < \zeta < 1$. In this case,

$$\zeta^2 - 1 < 0 \Rightarrow \sqrt{\zeta^2 - 1} = j\sqrt{1 - \zeta^2}, \quad j = \sqrt{-1}. \tag{11}$$

Then,

$$\lambda = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}. \tag{12}$$

- Let $\omega_d = \omega_n\sqrt{1 - \zeta^2} \equiv$ Damped natural frequency, then

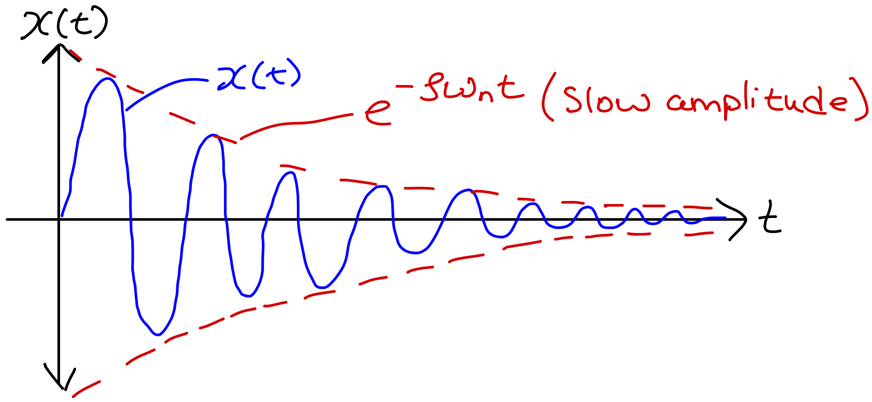
$$\lambda = -\zeta\omega_n \pm j\omega_d. \tag{13}$$

- The solution to Eq. 5 is then

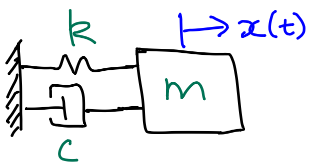
$$x(t) = e^{-\zeta\omega_n t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right) \tag{14}$$

- Using Euler's formula and the ICs, we get

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos(\omega_d t) + \frac{V_0 + \zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\right]} \tag{15}$$



Example 1



Given: A damped oscillator is released from equilibrium with a velocity of 0.05 m/s. The oscillator has a stiffness of 256 N/m, a mass of 1 kg, and a damping coefficient of 4 Ns/m.

Find: The maximum displacement that the oscillator achieves.

Solution

1. Start by determining the initial conditions, then plug them into Eq. 15 to find the displacement.

Oscillator starts at equilibrium: $x(0) = x_0 = 0$

Released with velocity of 0.05 m/s: $\dot{x}(0) = V_0 = 0.05\ \text{m/s}$

$$\Rightarrow \quad x(t) = e^{-\zeta\omega_n t}\left[\cancel{x_0\cos(\omega_d t)}^{\,\to 0} + \frac{V_0 + \cancel{\zeta\omega_n x_0}^{\,\to 0}}{\omega_d}\sin(\omega_d t)\right]$$

$$\Rightarrow \quad x(t) = \frac{V_0}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t)$$

2. The displacement is maximized whenever the velocity is equal to zero. Compute the velocity analytically. You will need the product rule.

$$x(t) = \frac{V_0}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

$$\Rightarrow \dot{x}(t) = -\underbrace{\frac{V_0 \zeta \omega_n}{\omega_d}} e^{-\zeta\omega_n t} \sin(\omega_d t) + \frac{V_0 \cancel{\omega_d}}{\cancel{\omega_d}} e^{-\zeta\omega_n t} \cos(\omega_d t)$$

$$\frac{V_0 \zeta \omega_n}{\omega_n \sqrt{1-\zeta^2}} = \frac{V_0 \zeta}{\sqrt{1-\zeta^2}} = V_0 \gamma, \quad \gamma = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\dot{x}(t) = V_0 e^{-\zeta\omega_n t}\left[\cos(\omega_d t) - \gamma \sin(\omega_d t)\right]$$

3. The maximum displacement occurs when the velocity is zero for the first time. Let this time be t^*. Set $\dot{x}(t^*) = 0$, then solve for t^* analytically, then numerically.

$$\dot{x}(t^*) = \underbrace{V_0 e^{-\zeta\omega_n t^*}}_{\substack{\to 0 \text{ as } t \to \infty \\ t^* < \infty}} \left[\cos(\omega_d t^*) - \gamma \sin(\omega_d t^*)\right] = 0 \Rightarrow \cos(\omega_d t^*) - \gamma \sin(\omega_d t^*) = 0$$

$$\Rightarrow \cos(\omega_d t^*) = \gamma \sin(\omega_d t^*) \Rightarrow \frac{\sin(\omega_d t^*)}{\cos(\omega_d t^*)} = \gamma^{-1}$$

$$\Rightarrow \tan(\omega_d t^*) = \gamma^{-1} \Rightarrow \omega_d t^* = \operatorname{atan}(\gamma^{-1}) \Rightarrow \boxed{t^* = \frac{1}{\omega_d} \operatorname{atan}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$

Need to find ζ and ω_d: $\zeta = \frac{c}{2\sqrt{km}} = \frac{4}{2\sqrt{(256)(1)}} = \frac{4}{32} = \frac{1}{8} \Rightarrow \zeta = 0.125.$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{256}{1}} \frac{\text{rad}}{\text{s}} \Rightarrow \omega_n = 16 \frac{\text{rad}}{\text{s}}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 16\sqrt{1-(0.125)^2} \frac{\text{rad}}{\text{s}} \Rightarrow \quad \omega_d = 15.875 \frac{\text{rad}}{\text{s}}$$

$$t^* = \frac{1}{15.875} \operatorname{atan}\left(\frac{\sqrt{1-(0.125)^2}}{0.125}\right) \text{s} = 0.091 \text{ s} \qquad \boxed{t^* = 0.091 \text{ s}}$$

4. Using the numerical value for t^*, compute the maximum displacement

$$x(t^*) = \frac{V_0}{\omega_d} e^{-\zeta\omega_n t^*} \sin(\omega_d t^*) = \frac{0.05}{15.875} e^{-(0.125)(16)(0.091)} \sin(15.875(0.091))$$

$$x(t^*) = 0.0026 \text{ m}$$

$$\boxed{\text{Maximum displacement: } x(t^*) = 0.0026 \text{ m}}$$