Satisfiability Checking - WS 2021/2022

Series 1

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Exercise 1

Let $AP = \{a, b\}$ be a set of propositions and let

$$\begin{aligned}
\varphi_1 &:= ((a \oplus \neg b) \to b) \vee (\neg a \leftrightarrow \neg b) \\
\varphi_2 &:= (((b \to \neg a) \oplus \neg b) \\
\varphi_3 &:= (\varphi_2 \wedge (a \vee \neg b))
\end{aligned}$$

be formulas over AP.

a) What are the truth tables for the above formulas?

b) What are $sat(\varphi_1)$, $sat(\varphi_2)$ and $sat(\varphi_3)$?

c) Which of the above formulas are satisfiable, which are unsatisfiable, and which are tautologies?

Solution:

a)

a	b	$a \oplus \neg b$	$(a \oplus \neg b) \to b$	$\neg a \leftrightarrow \neg b$	φ_1
0	0	1	0	1	1
0	1	0	1	0	1
1	0	0	1	0	1
1	1	1	1	1	1

a	b	$b \to \neg a$	$\neg b$	φ_2	$a \vee \neg b$	φ_3
0	0	1	1	0	1	0
0	1	1	0	1	0	0
1	0	1	1	0	1	0
1	1	0	0	0	1	0

b)
- $sat(\varphi_1) = Assign$
- $sat(\varphi_2) = \{\alpha\}$, with $\alpha(a) = 0$ and $\alpha(b) = 1$ and
- $sat(\varphi_3) = \emptyset$

c)
- Satisfiable: φ_1, φ_2
- Unsatisfiable: φ_3
- Tautology: φ_1

Exercise 2

Let $AP = \{a, b\}$ be a set of propositions and let $\alpha, \beta \in Assign$ with $\alpha(a) = 1$, $\alpha(b) = 1$ and $\beta(a) = 0$, $\beta(b) = 1$. Do the following hold?

1. $\alpha \models a \vee \neg b$
2. $\beta \not\models \neg a \wedge \neg b$
3. $\{\alpha, \beta\} \models a \wedge b$

4. $\{\alpha, \beta\} \models a \to b$
5. $a \vee b \models a \oplus b$
6. $sat(a \leftrightarrow b) \subseteq sat(a \to b)$

Solution:

1. $\alpha \models a \vee \neg b$ is true
2. $\beta \not\models \neg a \wedge \neg b$ is true
3. $\{\alpha, \beta\} \models a \wedge b$ is false
4. $\{\alpha, \beta\} \models a \to b$ is true
5. $a \vee b \models a \oplus b$ is false
6. $sat(a \leftrightarrow b) \subseteq sat(a \to b)$ is true

Exercise 3

Let $AP := \{a, b\}$ be a set of propositions and let $\varphi := (a \leftrightarrow b)$ be a formula over AP. Give a formula equivalent to φ that contains only propositions from AP and

1. the operators $\neg$ and $\wedge$,
2. the operators $\neg$ and $\vee$,
3. or the operator $\uparrow$ (called NAND).

(The binary operator $\uparrow$ has the following semantics: $\alpha \models (a \uparrow b) \leftrightarrow \alpha \models (\neg(a \wedge b))$ for all $a, b \in AP$ and $\alpha \in Assigns$.)

Solution:

1. Operators $\neg$ and $\wedge$:

$$\begin{aligned} & (a \leftrightarrow b) \\ \overset{1.}{\equiv} \quad & (a \to b) \wedge (b \to a) \\ \overset{2.}{\equiv} \quad & (\neg a \vee b) \wedge (\neg b \vee a) \\ \overset{3.}{\equiv} \quad & \neg(a \wedge \neg b) \wedge \neg(b \wedge \neg a) \end{aligned}$$

2. Operators $\neg$ and $\vee$:

$$\begin{aligned} & (a \leftrightarrow b) \\ \overset{1.-2.}{\equiv} \quad & (\neg a \vee b) \wedge (\neg b \vee a) \\ \equiv \quad & \neg(\neg(\neg a \vee b) \vee \neg(\neg b \vee a)) \end{aligned}$$

3. Operator $\uparrow$: We show that the operators $\neg$ and $\wedge$ can be expressed by $\uparrow$.

$$\begin{aligned} \neg a \quad & \equiv \quad (a \uparrow a) \\ (a \wedge b) \quad & \equiv \quad (a \uparrow b) \uparrow (a \uparrow b) \end{aligned}$$

Then:

$$
\begin{aligned}
& (a \leftrightarrow b) \\
\overset{1.-3.}{\equiv}\quad & \neg(a \land \neg b) \land \neg(b \land \neg a) \\
\equiv\quad & \neg(a \land (b \uparrow b)) \land \neg(b \land (a \uparrow a)) \\
\equiv\quad & (a \uparrow (b \uparrow b)) \land (b \uparrow (a \uparrow a)) \\
\equiv\quad & ((a \uparrow (b \uparrow b)) \uparrow (b \uparrow (a \uparrow a))) \uparrow ((a \uparrow (b \uparrow b)) \uparrow (b \uparrow (a \uparrow a)))
\end{aligned}
$$