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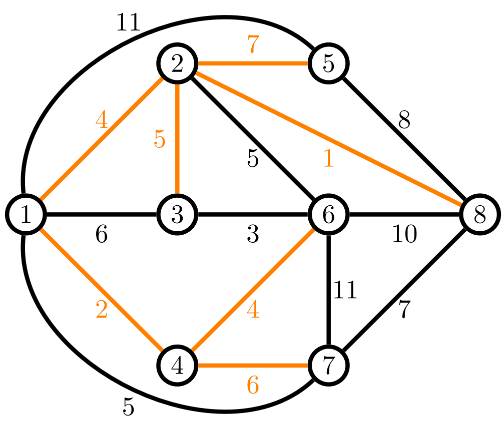
Combinatorial Optimization
Teaching and Reasearch Area

Exercise Sheet 2

Submission until: 27.10.2021 8:00 o'clock

Exercise 1 **1 Points**

Let $G = (V, E)$ be the following graph. The orange edges F form a spanning tree $T = (V, F)$.

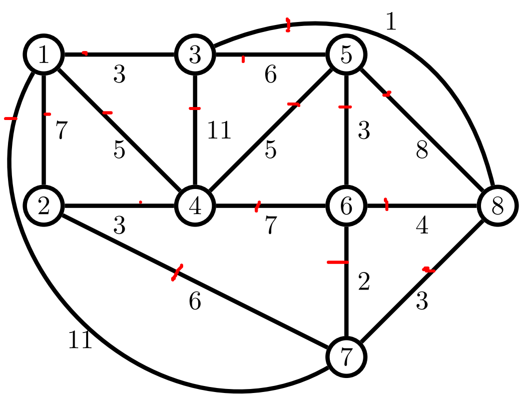


a) For every minimum spanning tree, it holds true that $F \cup e$ contains exactly one circle $C \subseteq E$ for any arbitrary edge $e \in E \setminus F$. Proof that T is not minimal if, for such an edge e, there exists an edge $f \in F \cap C$ with $c(f) > c(e)$.

b) Use a. to proof that the orange tree T is not minimal.

Exercise 2 **1.5 Points**

Let $G = (V, E)$ be the following undirected graph with non-negative edge weights and use Kruskal's algorithm to compute a minimum spanning tree of G.



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Exercise 3 **2.5 Points**

Let $G = (V, E)$ be an undirected graph, $c : E \to \mathbb{R}_{\geq 0}$ be a non-negative cost function, and $T = (V, F)$ be a minimum spanning tree of G with respect to c. Further, let $X \subsetneq V$ be a proper subset of the node set.

a) Is T then always a minimum spanning tree of G with respect to

$$c^i : E \to \mathbb{R}_{\geq 0},\ e \mapsto (c(e))^2 \quad ?$$

b) Is T then always a minimum spanning tree of G with respect to

$$c^{ii} : E \to \mathbb{R}_{\geq 0},\ e \mapsto \begin{cases} 2c(e) & \text{if} e = \{v, w\} \text{ with } v \in X, w \in V \backslash X \\ c(e) & \text{otherwise} \end{cases} \quad ?$$

c) Is T then always a minimum spanning tree of G with respect to

$$c^{iii} : E \to \mathbb{R}_{\geq 0},\ e \mapsto \begin{cases} c(e) + 2 & \text{if } e = \{v, w\} \text{ if } v, w \in X \\ c(e) & \text{otherwise} \end{cases} \quad ?$$

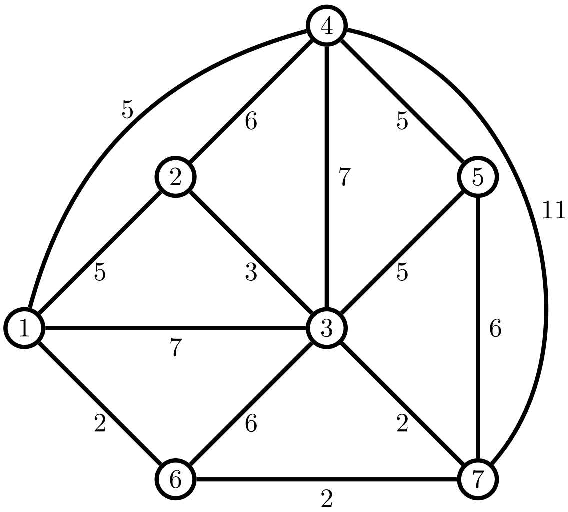
d) Given a non-negative function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, formulate a condition for f that is as concise and as sharp as possible, so that T is always a minimum spanning tree of G with respect to

$$c^{iv} : E \to \mathbb{R}_{\geq 0},\ e \mapsto f(c(e)).$$

Proof the sharpness, i.e., show that T is not always a minimum spanning tree with respect to c^{iv} for functions f that do not satisfy the condition (counter example).

Exercise 4 **1.5 Points**

Let $G = (V, E)$ be the following undirected graph with non-negative edge weights.



Use Prim's algorithm to compute a minimum spanning tree $T = (V, F)$ of G, starting at node number 1.

Exercise 5 **1 Points**

Can you use Prim's algorithm to compute a *maximum* spanning tree of G? Proof your claim.

Exercise 6 **2.5 Points**

a) Consider the pseudocode of Prim's algorithm provided in the lecture. Proof that this implementation uses a total number of operations in the size of $|V|^3$. Remark therefore that the total number of edges $|E|$ of a complete graph is $|E| = \frac{|V|(|V|-1)}{2}$, i.e., assume that $|E| \sim |V|^2$.

b) Can you outline an implementation of Prim's algorithm that uses only a total number of operations in the size of $|V|^2$?