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Chair of Mathematics of Information processing
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13.10.2021

Exercise Sheet 1

Mathematics of Data Science

Presented in Tutorial: 20.10.2021

Topics: Eigenvectors, norms, inner products, orthogonal projections

On each sheet you can find theoretical exercises concerning the course material.

- All exercises are presented in the tutorial each Wednesday at 10:30.

Further information you can find in the course-room

https://moodle.rwth-aachen.de/course/view.php?id=17730&section=0

Exercise 1

Given a $\mathbb{R}$-vector space V, a mapping $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called a *symmetric inner product* if the following properties hold:

a) For all $v \in V$ we have $(v, v) \geq 0$.

b) The property $(v, v) = 0$ implies that $v = 0$.

c) For all $\lambda \in \mathbb{R}$ and all $u, v, w \in V$ we have $(v + \lambda u, w) = (v, w) + \lambda(u, w)$.

d) For all $u, v \in V$ we have $(u, v) = (v, u)$.

Prove the following statements:

a) The mapping $\|\cdot\| : V \to \mathbb{R}$ with

$$\|v\| := \sqrt{(v, v)}$$

is a norm on V if $(\cdot,\cdot)$ is an symmetric inner product on V.

b) The Frobenius scalar product defined by

$$\langle A, B\rangle_F := \operatorname{tr}\left(AB^T\right)$$

is a symmetric inner product on $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ and the associated norm is the Frobenius norm

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}.$$

Exercise 2

Given $A \in \mathbb{R}^{m\times n}$, $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^m, \|\cdot\|_Y)$

a) prove that

$$\|A\|_{X\to Y} := \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}$$

is a norm on $\mathbb{R}^{m\times n}$,

b) prove that

$$\|A\|_{l^2\to l^2} = \max_{j=1,\dots,n} \sqrt{\lambda_j\left(A^*A\right)}$$

where $\lambda_j\left(A^*A\right)$ is the j-th eigenvalue of A^*A.

Exercise 3
Given the matrix $A \in \mathbb{R}^{3\times 3}$ where

$$A = \begin{pmatrix} 7 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 7 \end{pmatrix}$$

calculate a diagonal matrix $D \in \mathbb{R}^{3\times 3}$ and an orthogonal matrix $V \in \mathbb{R}^{3\times 3}$ such that $A = VDV^\top$.

Exercise 4
Let $W \subset \mathbb{R}^m$ be a linear subspace of dimension k with orthonormal basis $w_1, \dots, w_k \in \mathbb{R}^m$ and let $u \in \mathbb{R}^m$. Prove the following statements:

a) The minimizer $\hat{w}$ of $\min_{w\in W} \|u - w\|_2$ exists, is unique and is given by $\hat{w} = \sum_{j=1}^k \langle u, w_j\rangle w_j$.

b) The difference vector $u - \hat{w}$ is orthogonal to every vector $w \in W$.

c) It holds $\|\hat{w}\|_2^2 = \sum_{j=1}^k \langle u, w_j\rangle^2$.

d) It holds $\|u - \hat{w}\|_2^2 = \|u\|_2^2 - \sum_{j=1}^k \langle u, w_j\rangle^2$.