Exercise 3

a) 4) $\Rightarrow$ 5)

Proof: G is connected and $|E| = n-1$.

$\Rightarrow$ G is connected and acyclic.

Since G is connected, every pair of vertices is connected by at least one path. If two different paths exist between u and v, then the union of these paths is a closed walk, then it contains a cycle, a contradiction.

So, for every $u, v \in V$, $u \neq v$, there exists a single unique $[u,v]$-path.

b) 5) $\Rightarrow$ 6)

Proof: If G contains a cycle, all vertices on the cycle are connected by at least two paths, thus G is acyclic.

Since G contains a $[u,v]$-path P, $P \cup \{u,v\}$

is a cycle. If $G+\{u,v\}$ contains further cycles, then G already had a cycle and there would be more than one $[u,v]$-path exist, a contradiction.

So, G is acyclic and if an arc $\{u,v\}$ with $u,v \in V$ and $\{u,v\} \notin E$ is added to G there exists exactly one circle.

c) $6) \Rightarrow 7)$

Proof: G is acyclic, then G is a forest.

Assume G is not connected, i.e. G is a group of unconnected trees.

Then adding an arc between the unconnected trees won't cause a cycle, a contradiction.

So, G is conneceted.

Then with G acyclic and connected.

Assume one edge $e \in E$, the subgraph $G-e=(V, E \setminus \{e\})$ is still conneceted, say by path P.

Then there are two paths between the end vertices of e: $\{e\}$ and P. leads to G is cyclic.

A contradiction. So 6) $\Rightarrow$ 7)

d) 7) $\Rightarrow$ 4)

G is connected and for an edge $e \in E$, the subgraph $G-e = (V, E \setminus \{e\})$ is not connected.

Assume G is cyclic, then for one pair of vertices $(u,v) \in E$ in G, we can find at least two paths connecting them.

As we remove one of the path (u,v), there's still another path connected them, $G-e$ is still connected, a contradiction. So G is connected and acyclic

$\Rightarrow$ G is connected and $|E| = n-1$ 7) $\Rightarrow$ 4)

Exercise 4

Let v be an arbitary vertex of G, since G is a tree, G is connected, and $\deg(v) \geq 1$.

Assume $\deg(v) = 1$. Then v has a neighbor $w \in V$

If $\deg(w) = 1$, we have found two leaves (vertices with degree 1) v, w.

If $\deg(w) > 1$, we have another vertex $u \neq v$ adjacent to w. For vertex u,

if $\deg(u) = 1$, we found two leaves v, u.

otherwise, we can find another neighbor $\neq w$

Since a tree doesn't contain a cycle, we can not return to previous vertex.

Since the graph only has a finite number of vetices, we have to end at a leaf

If $\deg(v) \geq 2$, we can proceed similarly with every its neighbors, and can find at lease 2 leaves