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Exercise Sheet 2

Mathematics of Data Science

Presented in Tutorial: 27.10.2021

Topics: Norms, Singular Value Decomposition

This sheet contains theoretical exercises.

- All exercises are presented in the tutorial each Wednesday at 10:30.
- Next Wednesday, 27.10.2021, the first test counting towards 50% the goal is available at Dynexite. You can find the test by following the link in our Moodle course-room.

https://moodle.rwth-aachen.de/course/view.php?id=17730&section=0

Exercise 1

Given a matrix $A \in \mathbb{R}^{m\times n}$, prove that

a) $\sigma_1^2, \dots, \sigma_r^2$ are the non-zero eigenvalues of $A^\top A$ and $AA^\top$ if and only if $\sigma_1, \dots, \sigma_r$ are the singular values of A,

b) a vector $u_i \in \mathbb{R}^m$ is a left singular vector of A with corresponding singular value σ_i if and only if u_i is an normalized eigenvector of $AA^\top$ corresponding to the eigenvalue σ_i^2, for each $i = 1, \dots, r$,

c) a vector $v_i \in \mathbb{R}^n$ is a right singular vector of A with corresponding singular value σ_i if and only if v_i is an normalized eigenvector of $A^\top A$ corresponding to the eigenvalue σ_i^2, for each $i = 1, \dots, r$.

Hint: You may want to consult Corollary 3.6. in the lecture notes.

Exercise 2

Given $A \in \mathbb{R}^{2\times 3}$ with

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix},$$

a) calculate a reduced singular value decomposition of A, i.e. calculate orthogonal matrices $U \in \mathbb{R}^{2\times r}$ and $V \in \mathbb{R}^{3\times r}$ and a diagonal matrix $D \in \mathbb{R}^{r\times r}$, where $r = \operatorname{rank}(A)$, such that $A = UDV^\top$,

b) calculate a singular value decomposition of A, i.e. calculate orthogonal matrices $U \in \mathbb{R}^{2\times 2}$ and $V \in \mathbb{R}^{3\times 3}$ and a diagonal matrix $D \in \mathbb{R}^{2\times 3}$ such that $A = UDV^\top$,

c) sketch the 1-dimensional subspace W which minimizes the sum of squared distances of the columns of A to W. Draw the orthogonal projections of the columns of A onto W.

Exercise 3

Given matrices $A, B \in \mathbb{R}^{m\times n}$, prove

a) $\|A\|_{l^2\to l^2} \le \|A\|_F \le \sqrt{\min\{n,m\}}\|A\|_{l^2\to l^2}$,

b) $\sigma_{\max}(A) = \max_{\|x\|_2=1} \|Ax\|_2$ and $\sigma_{\min}(A) = \min_{\|x\|_2=1} \|Ax\|_2$,

c) $|\sigma_{\max}(A) - \sigma_{\max}(B)| \le \|A - B\|_F$ and $|\sigma_{\min}(A) - \sigma_{\min}(B)| \le \|A - B\|_F$

where $\sigma_{\max}(\cdot)$ is the largest and $\sigma_{\min}(\cdot)$ is the smallest singular value of the matrix. Note that $\sigma_{\min}(\cdot)$ maybe 0 here.

Exercise 4

Let $A \in \mathbb{R}^{m \times n}$ denote a matrix of rank r and let A_k denote its rank k-truncation for $k < r$. Let $\sigma_1 \geq \ldots \geq \sigma_r$ denote the singular values of A. Then,

$$\|A - A_k\|_{\ell^2 \to \ell^2} \leq \sigma_{k+1} \, .$$