Mathematics of Data Science
Chapter I: Examples of Mathematics in Data Science

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- Development and understanding of methods of data science often requires significant amount of mathematics
- It is often crucial to prove that algorithms in data science work (under suitable conditions), not just to rely on numerical tests
- It is also crucial to understand (prove) limits of methods, i.e., to understand when algorithms fail.

Mathematical fields in data science

Methods of data science require mathematical tools from various fields

- Linear algebra
- Analysis
- Probability theory
- Statistics
- Optimization
- Discrete mathematics (discrete optimization, graph theory)
- Numerical analysis
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We will only cover a small part in this course. More material will be provided in specialized courses.

Example 1 – Regression

Predict weight of a person from sex, age and height!

Person	1	2	$\cdots$	p
x_1: Sex	1 (female)	-1 (male)	$\cdots$	1
x_2: Age (years)	25	37	$\cdots$	45
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Analysis of accuracy of prediction: Probability Theory and Statistics

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Method: Set $X = (x^1|x^2|\cdots|x^p) \in \mathbb{R}^{n\times p}$ and represent W as the range of a matrix $U \in \mathbb{R}^{n\times m}$.
Find minimizer of optimization problem

$$\min_{U\in\mathbb{R}^{n\times m}} \sum_{j=1}^{p} \|x^j - UU^T x^j\|_2^2$$

Minimizer can be computed using the singular value decomposition (SVD) of X (linear algebra)

Example 3 – Supervised Learning

Given an image, automatically determine whether it contains a cat or not!

Supervised Learning

Mathematical problem: Given pairs (x^j, y^j),$j = 1, \ldots, p$ of (training) input data with $x^j \in \mathbb{R}^p$ and $y^j \in \mathbb{R}$, or $y^j \in \{-1, 1\}$ find a function $h : \mathbb{R}^p \to \mathbb{R}$ that accurately predicts labels y of future data x.

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Given a loss function $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ (e.g. $\ell(y, z) = (y - z)^2$) and a set $\mathcal{H}$ of possible hypothesis functions $h : \mathbb{R}^p \to \mathbb{R}$ find minimizer of optimization problem

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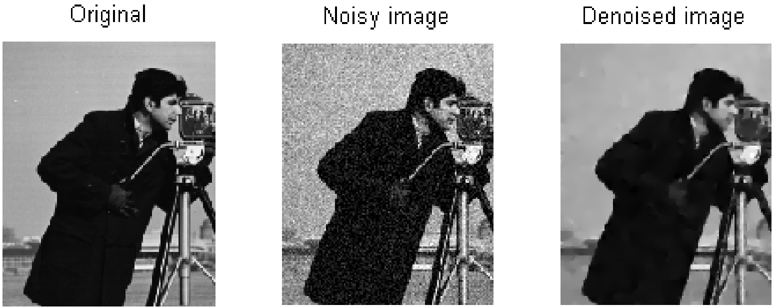
Analysis of prediction error requires techniques from probability theory.

Example for $\mathcal{H}$ (deep learning):
Deep neural networks of a prescribed structure (parametrization).

Details in course:
Mathematical Foundations of Machine Learning

Example 4 - Image Denoising



Reconstruct original image $u \in \mathbb{R}^{m \times n}$ from measured noisy image $\widetilde{u} = u + e$, where $e \in \mathbb{R}^{m \times n}$ represents noise!

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Original

Noisy image

Denoised image

Reconstruct original image $u \in \mathbb{R}^{m \times n}$ from measured noisy image $\widetilde{u} = u + e$, where $e \in \mathbb{R}^{m \times n}$ represents noise!

Variational approach: compute minimizer $\hat{u}$ of optimization program

$$\min_{v \in \mathbb{R}^{m \times n}} \|\widetilde{u} - v\| + \lambda R(v),$$

where $R : \mathbb{R}^{m \times n} \to \mathbb{R}$ is a regularization term, representing prior assumptions on u such as smoothness, sparsity etc.

Example 5 - Analog-to-Digital Conversion

For a digital representation, a continuous time signal (e.g. music) $f : \mathbb{R} \to \mathbb{R}$ is sampled (and quantized) at discrete times, e.g., for $B > 0$,

$$y_j = f\left(\frac{j}{2B}\right), \quad j \in \mathbb{Z}.$$

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Fourier transform of f:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}dt.$$

If f is square-integrable and such that $\hat{f}(\xi) = 0$ for $|\xi| \geq B$ (f belongs to the Paley-Wiener space), then f can be reconstructed exactly via the sampling series

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Mathematical field: Fourier analysis

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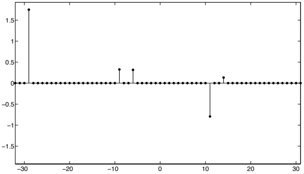
Johnson-Lindenstrauss Lemma: If $P \in \mathbb{R}^{m \times n}$ is chosen at random (e.g. as Gaussian random matrix), then the inequality holds with probability at least $1 - \delta$ if

$$m \geq C\epsilon^{-2} \log(p/\delta).$$

Mathematical tools: Probability Theory in High Dimensions

Example 7 - Signal Reconstruction (Compressed Sensing)



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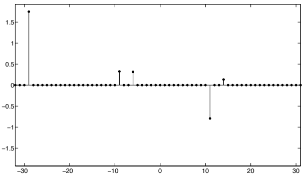

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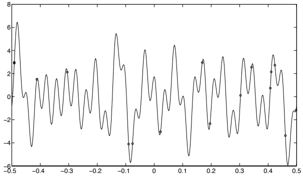

Time-Domain Signal with 16 Samples

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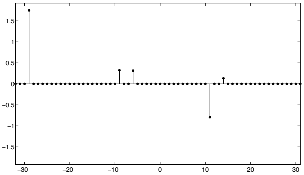

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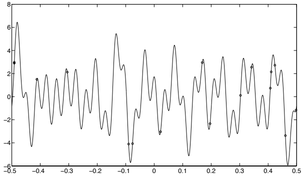

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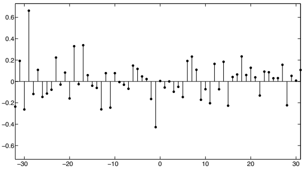

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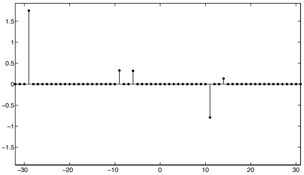

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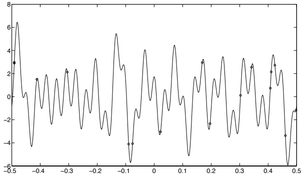

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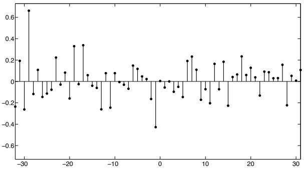

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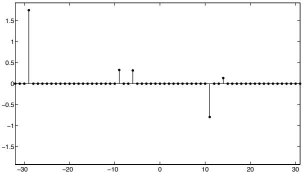

Reconstruction via ℓ_1-minimization

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Details: Course on Compressive Sensing.