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Satisfiability Checking
02 Propositional logic I

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Abstract syntax of well-formed propositional formulae:

$$\varphi \;:=\; a \;\mid\; (\neg\varphi) \;\mid\; (\varphi \wedge \varphi)$$

where AP is a set of (atomic) propositions (Boolean variables) and $a \in AP$.
We write *PropForm* for the set of all propositional logic formulae.

Syntactic sugar:

$$\begin{array}{rcl}
\bot & := & (a \wedge \neg a) \\
\top & := & (a \vee \neg a) \\
(\varphi_1 \vee \varphi_2) & := & \neg((\neg\varphi_1) \wedge (\neg\varphi_2)) \\
(\varphi_1 \rightarrow \varphi_2) & := & ((\neg\varphi_1) \vee \varphi_2) \\
(\varphi_1 \leftrightarrow \varphi_2) & := & ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)) \\
(\varphi_1 \bigoplus \varphi_2) & := & (\varphi_1 \leftrightarrow (\neg\varphi_2))
\end{array}$$

Formulae

- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \land (b \land c))$
  - $(a \rightarrow (b \rightarrow c))$
- We omit parentheses whenever we may restore them through operator precedence:

  binds stronger

  $$\xleftarrow{\qquad\qquad\qquad\qquad}$$

  $$\neg \quad \land \quad \lor \quad \rightarrow \quad \leftrightarrow$$

- We will also use the "big" Boolean notation, e.g.

$$\bigwedge_{i=1}^{5} x_i \qquad \textit{is defined as} \qquad x_1 \land x_2 \land x_3 \land x_4 \land x_5 \;.$$

Semantics: Assignments

Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
- The interpretation assigns Boolean values to the variables:

$$\alpha : AP \rightarrow \{0, 1\}$$

We call these special interpretations assignments and use *Assign* to denote the set of all assignments.
Example: $AP = \{a, b\}, \alpha(a) = 0, \alpha(b) = 1$

Equivalently, we can see an assignment α as a set of variables ($\alpha \in 2^{AP}$), defining the variables from the set to be true and the others false.
Example: $AP = \{a, b\}, \alpha = \{b\}$

An assignment can also be seen as being of type $\alpha \in \{0, 1\}^{AP}$, if we have an order on the propositions.
Example: $AP = \{a, b\}, \alpha = 01$

Only the projected assignment matters...

- Let $\alpha_1, \alpha_2 \in \mathit{Assign}$ and $\varphi \in \mathit{PropForm}$.
- Let $AP(\varphi)$ be the atomic propositions in φ.
- Clearly $AP(\varphi) \subseteq AP$.
- Lemma: if $\alpha_1|_{AP(\varphi)} = \alpha_2|_{AP(\varphi)}$, then

Projection

$$(\alpha_1 \text{ satisfies } \varphi) \quad \text{iff} \quad (\alpha_2 \text{ satisfies } \varphi)$$

- We will assume, for simplicity, that $AP = AP(\varphi)$.

Semantics I: Truth tables

- Truth tables define the semantics (=meaning) of the operators. They can be used to define the semantics of formulae inductively over their structure.
- Convention: 0= false, 1= true

a⊕b = (¬a ∧ b) ∨ (a ∧¬b

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

Q: How many binary operators can we define that have different semantics?
A: 16

Semantics I: Example

- Let φ be defined as $(a \vee (b \rightarrow c))$.
- Let $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ be an assignment with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.
- Q: Does α satisfy φ?
- A1: Compute with truth table:

a	b	c	$b \rightarrow c$	$a \vee (b \rightarrow c)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	1
1	1	1	1	1

Semantics II: Satisfaction relation

满足符

Satisfaction relation: $\models \subseteq \textit{Assign} \times \textit{PropForm}$

Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

- α satisfies φ or
- φ holds for α or
- α is a model of φ.

$\models$ is defined recursively:

$$
\begin{array}{lll}
\alpha \models p & \textit{iff} & \alpha(p) = \textit{true} \\
\alpha \models \neg\varphi & \textit{iff} & \alpha \not\models \varphi \\
\alpha \models \varphi_1 \wedge \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ and } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \vee \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ or } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \rightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ implies } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \leftrightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ iff } \alpha \models \varphi_2
\end{array}
$$

Note: More elegant but semantically equivalent to truth tables.

Semantics II: Example

- Let φ be defined as $(a \vee (b \rightarrow c))$.
- Let $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ be an assignment with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

- Q: Does α satisfy φ?

  A2: Compute with the satisfaction relation:

$$
\begin{array}{ll}
 & \alpha \models (a \vee (b \rightarrow c)) \\
\textit{iff} & \alpha \models a \textit{ or } \alpha \models (b \rightarrow c) \\
\textit{iff} & \alpha \models a \textit{ or } (\alpha \models b \textit{ implies } \alpha \models c) \\
\textit{iff} & 0 \textit{ or } (0 \textit{ implies } 1) \\
\textit{iff} & 0 \textit{ or } 1 \\
\textit{iff} & 1
\end{array}
$$

- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \to \{0,1\}$ is a model of a propositional logic formula $\varphi \in \mathit{PropForm}$:

```
Eval(α, φ) {
   if φ ≡ a return α(a);
   if φ ≡ (¬φ1) return not Eval(α, φ1);
   if φ ≡ (φ1 op φ2)
         return Eval(α, φ1) ⟦op⟧ Eval(α, φ2);
}
```

- Equivalent to the $\models$ relation, but from the algorithmic view.
- Q: Complexity? A: Polynomial (time and space).

Semantics III: Example

- Recall our example
  - $\varphi = (a \vee (b \to c))$
  - $\alpha : \{a, b, c\} \to \{0, 1\}$ with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

- $$\begin{aligned} \mathit{Eval}(\alpha,\, \varphi) \;=\; & \mathit{Eval}(\alpha,\, a) \textit{ or } \mathit{Eval}(\alpha,\, b \to c) = \\ & 0 \textit{ or } (\mathrm{Eval}(\alpha,\, b) \textit{ implies } \mathrm{Eval}(\alpha,\, c)) = \\ & 0 \textit{ or } (0 \textit{ implies } 1) = \\ & 0 \textit{ or } 1 = \\ & 1 \end{aligned}$$

- Hence, $\alpha \models \varphi$.

Satisfying assignments

- Intuition: each formula specifies a set of assignments satisfying it.
- Remember: *Assign* denotes the set of all assignments.
- Function $\mathit{sat} : \mathit{PropForm} \to 2^{\mathit{Assign}}$

  (a formula $\to$ set of its satisfying assignments)
- Recursive definition:

$$\begin{array}{lcl}
\mathit{sat}(a) & = & \{\alpha \mid \alpha(a) = 1\}, \quad a \in AP \\
\mathit{sat}(\neg\varphi_1) & = & \mathit{Assign} \setminus \mathit{sat}(\varphi_1) \\
\mathit{sat}(\varphi_1 \wedge \varphi_2) & = & \mathit{sat}(\varphi_1) \cap \mathit{sat}(\varphi_2) \\
\mathit{sat}(\varphi_1 \vee \varphi_2) & = & \mathit{sat}(\varphi_1) \cup \mathit{sat}(\varphi_2) \\
\mathit{sat}(\varphi_1 \to \varphi_2) & = & (\mathit{Assign} \setminus \mathit{sat}(\varphi_1)) \cup \mathit{sat}(\varphi_2)
\end{array}$$

- For $\varphi \in \mathit{PropForm}$ and $\alpha \in \mathit{Assign}$ it holds that

$$\alpha \models \varphi \qquad \mathit{iff} \qquad \alpha \in \mathit{sat}(\varphi)$$

Satisfying assignments: Example

$$
\begin{aligned}
& sat(a \vee (b \to c)) && = \\
& sat(a) \cup sat(b \to c) && = \\
& sat(a) \cup ((Assign \setminus sat(b)) \cup sat(c)) && = \\
& \{\alpha \in Assign \mid \alpha(a) = 1\} \cup \\
& \{\alpha \in Assign \mid \alpha(b) = 0\} \cup \\
& \{\alpha \in Assign \mid \alpha(c) = 1\} && = \\
& \{\alpha \in Assign \mid \alpha(a) = 1 \text{ or } \alpha(b) = 0 \text{ or } \alpha(c) = 1\}
\end{aligned}
$$

Extensions of $\models$

- We define $\models\ \subseteq\ 2^{Assign} \times PropForm$ by

$$T \models \varphi \text{ iff } T \subseteq sat(\varphi)$$

for formulae $\varphi \in PropForm$ and assignment sets $T \subseteq 2^{Assign}$.

Examples: $\{\alpha \in Assign \mid \alpha(a) = \alpha(c) = 1\} \models a \lor (b \rightarrow c)$
$\{\alpha \in Assign \mid \alpha(x_1) = 1\} \models x_1 \lor x_2$

- We define $\models\ \subseteq\ PropForm \times PropForm$ by

$$\varphi_1 \models \varphi_2 \text{ iff } sat(\varphi_1) \subseteq sat(\varphi_2)$$

for formulae $\varphi_1, \varphi_2 \in PropForm$.

Examples: $a \land c \models a \lor (b \rightarrow c)$
$x_1 \models x_1 \lor x_2$

Short summary for propositional logic syntax and semantics

- Syntax of propositional formulae $\varphi \in \mathit{PropForm}$:

$$\varphi \ := \ \mathit{AP} \mid (\neg\varphi) \mid (\varphi \land \varphi)$$

- Semantics:
  - Assignments $\alpha \in \mathit{Assign}$:

$$\begin{array}{l} \alpha : \mathit{AP} \to \{0,1\} \\ \alpha \in 2^{\mathit{AP}} \\ \alpha \in \{0,1\}^{\mathit{AP}} \end{array}$$

  - Satisfaction relation:

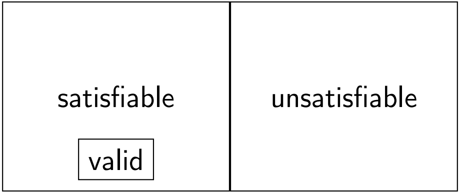
$$\begin{array}{lll} \models \ \subseteq \mathit{Assign} \times \mathit{PropForm} & , & (\text{e.g., } \alpha \models \varphi \ ) \\ \models \ \subseteq 2^{\mathit{Assign}} \times \mathit{PropForm} & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} \models \varphi \ ) \\ \models \ \subseteq \mathit{PropForm} \times \mathit{PropForm}, & & (\text{e.g., } \varphi_1 \models \varphi_2) \\ \mathit{sat} : \ \mathit{PropForm} \to 2^{\mathit{Assign}} & , & (\text{e.g., } \ \mathit{sat}(\varphi) \ ) \end{array}$$

Semantic classification of formulae

- A formula φ is called valid if $sat(\varphi) = Assign$.
  (Also called a tautology).

- A formula φ is called satisfiable if $sat(\varphi) \neq \emptyset$.

- A formula φ is called unsatisfiable if $sat(\varphi) = \emptyset$.
  (Also called a contradiction).



Some notations

- We can write:
  - $\models \varphi$ when φ is valid
  - $\not\models \varphi$ when φ is not valid
  - $\not\models \neg\varphi$ when φ is satisfiable
  - $\models \neg\varphi$ when φ is unsatisfiable

Examples

- $(x_1 \wedge x_2) \to (x_1 \vee x_2)$ is valid
- $(x_1 \vee x_2) \to x_1$ is satisfiable
- $(x_1 \wedge x_2) \wedge \neg x_1$ is unsatisfiable

Examples

- Here are some valid formulae:
  - $\models a \wedge 1 \leftrightarrow a$
  - $\models a \wedge 0 \leftrightarrow 0$
  - $\models \neg\neg a \leftrightarrow a$ (double-negation rule)
  - $\models a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c)$

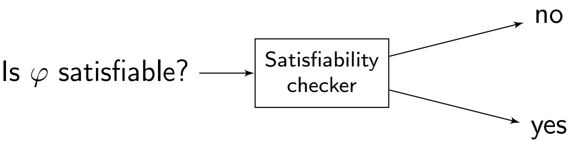
- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
  - $\models \neg(a \vee b) \leftrightarrow (\neg a \wedge \neg b)$

The satisfiability problem for propositional logic

- The satisfiability problem for propositional logic is as follows:

  Given an input propositional formula φ, decide whether φ is satisfiable.
- This problem is decidable but NP-complete.
- An algorithm that always terminates for each propositional logic formula with the correct answer is called a decision procedure for propositional logic.

Goal: Design and implement such a decision procedure:



Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

Learning target

- What are the rules to (syntactically) build propositional logic formulas?
- How to interpret propositional logic formulas...
  - ... using truth tables?
  - ... using the satisfaction relation?
  - ... algorithmically by the `Eval` function?
- How to compute the set of all satisfying assignments recursively by the `sat` function?
- When is a propositional logic formula valid, satisfiable or unsatisfiable?