Recording agreement

- The lectures will be recorded.
- You are not obligated neither to enable video or audio nor to share other personal data as real names or photos of yourself.
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- Note that the breakout rooms are not recorded.

Satisfiability Checking

Propositional logic on examples

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Satisfiability with truth table

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

a	b	c	$\neg(a \rightarrow (b \vee \neg c))$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Satisfiability with semantical algorithm

$$\begin{array}{lcl}
\text{Eval}(\alpha, p) & = & \alpha(p) \\
\text{Eval}(\alpha, \neg A) & = & \neg\text{Eval}(\alpha, A) \\
\text{Eval}(\alpha, A \vee B) & = & \text{Eval}(\alpha, A) \vee \text{Eval}(\alpha, B) \\
\text{Eval}(\alpha, A \wedge B) & = & \text{Eval}(\alpha, A) \wedge \text{Eval}(\alpha, B) \\
\text{Eval}(\alpha, A \rightarrow B) & = & \text{Eval}(\alpha, \neg A) \vee \text{Eval}(\alpha, B) \\
\text{Eval}(\alpha, A \leftrightarrow B) & = & \text{Eval}(\alpha, A \rightarrow B) \wedge \text{Eval}(\alpha, A \leftarrow B)
\end{array}$$

Satisfiability with semantical algorithm

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

- $\alpha : a = 0, b = 0, c = 0$:
- $\text{Eval}(\alpha, \neg(a \rightarrow (b \vee \neg c))) = \neg\text{Eval}(\alpha, a \rightarrow (b \vee \neg c))$

  $\quad\text{Eval}(\alpha, a \rightarrow (b \vee \neg c)) = \text{Eval}(\alpha, \neg a) \vee \text{Eval}(\alpha, b \vee \neg c)$

  $\quad\quad\text{Eval}(\alpha, \neg a) = \neg\text{Eval}(\alpha, a) = 1$

  $\quad\quad\text{Eval}(\alpha, b \vee \neg c) = \text{Eval}(\alpha, b) \vee \text{Eval}(\alpha, \neg c))$

  $\quad\quad\quad\text{Eval}(\alpha, b) = 0$

  $\quad\quad\quad\text{Eval}(\alpha, \neg c) = \neg\text{Eval}(\alpha, c) = 1$

  $\quad\quad\text{Eval}(\alpha, b \vee \neg c) = 1$

  $\quad\text{Eval}(\alpha, a \rightarrow (b \vee \neg c)) = 1$

  $\text{Eval}(\alpha, \neg(a \rightarrow (b \vee \neg c))) = 0$

NNF conversion

Only operators $\neg, \vee, \wedge$, negation only in front of atomic propositions.

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

$$\begin{array}{rl} & \neg(a \rightarrow (b \vee \neg c)) \\ = & \neg(\neg a \vee (b \vee \neg c)) \\ = & \neg(\neg a \vee b \vee \neg c) \\ = & a \wedge \neg b \wedge c \end{array}$$

CNF conversion: The exponential way

CNF: $\bigwedge_{i=1,\ldots,n} \bigvee_{j=1,\ldots,m} l_{ij}$

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$$
\begin{aligned}
& (a \wedge b) \vee (\neg c \wedge (d \vee e)) \\
= \;& (a \vee (\neg c \wedge (d \vee e))) \wedge (b \vee (\neg c \wedge (d \vee e))) \\
= \;& (a \vee (\neg c \wedge d) \vee (\neg c \wedge e)) \wedge (b \vee (\neg c \wedge d) \vee (\neg c \wedge e)) \\
= \;& (a \vee \neg c \vee (\neg c \wedge e)) \wedge (a \vee d \vee (\neg c \wedge e)) \wedge \\
& (b \vee \neg c \vee (\neg c \wedge e)) \wedge (b \vee d \vee (\neg c \wedge e)) \\
= \;& (a \vee \neg c \vee \neg c) \wedge (a \vee \neg c \vee e) \wedge (a \vee d \vee \neg c) \wedge (a \vee d \vee e) \wedge \\
& (b \vee \neg c \vee \neg c) \wedge (b \vee \neg c \vee e) \wedge (b \vee d \vee \neg c) \wedge (b \vee d \vee e)
\end{aligned}
$$

CNF conversion: Tseitin's encoding

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

- $a_1 \leftrightarrow (a_2 \vee a_3)$
- $a_2 \leftrightarrow (a \wedge b)$
- $a_3 \leftrightarrow (\neg c \wedge a_4)$
- $a_4 \leftrightarrow (d \vee e)$
- a_1

CNF conversion: Tseitin's encoding

- $h \leftrightarrow (p_1 \vee p_2)$
  $$\begin{aligned} &= (h \rightarrow (p_1 \vee p_2)) && \wedge && (h \leftarrow (p_1 \vee p_2)) \\ &= (\neg h \vee (p_1 \vee p_2)) && \wedge && (h \vee \neg(p_1 \vee p_2)) \\ &= (\neg h \vee p_1 \vee p_2) && \wedge && (h \vee (\neg p_1 \wedge \neg p_2)) \\ &= (\neg h \vee p_1 \vee p_2) && \wedge && (h \vee \neg p_1) \quad \wedge \quad (h \vee \neg p_2) \end{aligned}$$

- $h \leftrightarrow (p_1 \wedge p_2)$
  $$\begin{aligned} &= (h \rightarrow (p_1 \wedge p_2)) && \wedge && (h \leftarrow (p_1 \wedge p_2)) \\ &= (\neg h \vee (p_1 \wedge p_2)) && \wedge && (h \vee \neg(p_1 \wedge p_2)) \\ &= (\neg h \vee p_1) \quad \wedge \quad (\neg h \vee p_2) && \wedge && (h \vee \neg p_1 \vee \neg p_2) \end{aligned}$$

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$$\boxed{\begin{array}{l} h \leftrightarrow (p_1 \vee p_2) = (\neg h \vee p_1 \vee p_2) \wedge (h \vee \neg p_1) \wedge (h \vee \neg p_2) \\ h \leftrightarrow (p_1 \wedge p_2) = (\neg h \vee p_1) \wedge (\neg h \vee p_2) \wedge (h \vee \neg p_1 \vee \neg p_2) \end{array}}$$

$$\begin{array}{llll} a_1 \leftrightarrow (a_2 \vee a_3) & = (\neg a_1 \vee a_2 \vee a_3) & \wedge (a_1 \vee \neg a_2) & \wedge (a_1 \vee \neg a_3) \\ a_2 \leftrightarrow (a \wedge b) & = (\neg a_2 \vee a) & \wedge (\neg a_2 \vee b) & \wedge (a_2 \vee \neg a \vee \neg b) \\ a_3 \leftrightarrow (\neg c \wedge a_4) & = (\neg a_3 \vee \neg c) & \wedge (\neg a_3 \vee a_4) & \wedge (a_3 \vee c \vee \neg a_4) \\ a_4 \leftrightarrow (d \vee e) & = (\neg a_4 \vee d \vee e) & \wedge (a_4 \vee \neg d) & \wedge (a_4 \vee \neg e) \end{array}$$

CNF conversion: Tseitin's encoding

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$\text{CNF}(\phi) =$

$$\begin{array}{lclclcl}
(\neg a_1 \vee a_2 \vee a_3) & \wedge & (a_1 \vee \neg a_2) & \wedge & (a_1 \vee \neg a_3) & & \wedge \\
(\neg a_2 \vee a) & \wedge & (\neg a_2 \vee b) & \wedge & (a_2 \vee \neg a \vee \neg b) & & \wedge \\
(\neg a_3 \vee \neg c) & \wedge & (\neg a_3 \vee a_4) & \wedge & (a_3 \vee c \vee \neg a_4) & & \wedge \\
(\neg a_4 \vee d \vee e) & \wedge & (a_4 \vee \neg d) & \wedge & (a_4 \vee \neg e) & & \wedge \\
a_1 & & & & & &
\end{array}$$

Resolution

$$\frac{(l, l_1, \ldots, l_n) \quad (\neg l, l'_1, \ldots, l'_m)}{(l_1, \ldots, l_n, l'_1, \ldots, l'_m)}$$

Examples:

- $\frac{(a \vee b) \quad (\neg a \vee c)}{(b \vee c)}$
- $\frac{(a \vee b) \quad (\neg a \vee b)}{(b)}$
- $\frac{(a \vee b) \quad (\neg a \vee \neg b)}{(\texttt{true})}$
- $\frac{(a) \quad (\neg a)}{()}$

Resolution: Completeness

$$(a \vee P_1) \wedge \ldots \wedge (a \vee P_n) \wedge (\neg a \vee Q_1) \wedge \ldots (\neg a \vee Q_m) \wedge R$$
$$\Leftrightarrow$$
$$(P_1 \vee Q_1) \wedge \ldots \wedge (P_1 \vee Q_m) \wedge \ldots (P_n \vee Q_1) \wedge \ldots (P_n \vee Q_m) \wedge R$$

Similar: Quantifier elimination

$$\phi \Leftrightarrow \phi[\mathtt{true}/a] \vee \phi[\mathtt{false}/a]$$

Learning target

- Deepen the understanding of propositional logic syntax and semantics
- Be able to transform propositional logic formulas to NNF, DNF and CNF
- Be able to apply the resolution rule to derive new clauses