**Modeling Transmission Lines: Numerical Solutions to the Telegrapher's Equations**

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## Modeling Transmission Lines: Numerical Solutions to the Telegrapher's Equations

Since the dawn of the 20th century, electricity has been the foundation of society. To distribute this ever-vital resource, transmission lines exist everywhere. Mathematically, transmission lines can be modeled by the telegrapher's equations, two coupled first-order partial differential equations that when solved, yield functions for the voltage and current of the transmission line for any point or time in the domain. When written in the time domain, Ellingson (n.d., p. 56) shows that the system of equations is the following:

$$-\frac{\partial}{\partial x}V(x,t) = RI(x,t) + L\frac{\partial}{\partial t}I(x,t) \tag{1}$$

$$-\frac{\partial}{\partial x}I(x,t) = GV(x,t) + C\frac{\partial}{\partial t}V(x,t) \tag{2}$$

Besides the partial derivative terms, these equations also contain primary line characteristics, *R*, *L*, *G*, and *C*, physical properties of a transmission line. *R* is the resistance of the transmission line per unit length, *L* is the inductance of the transmission line per unit length, *G* is the conductance of the transmission line per unit length, and *C* is the capacitance of the transmission line per unit length. While these equations are correct, their coupled nature restricts their usability, and developing an analytical solution for all possible initial boundary value problems is unrealistic. Therefore, to create a working model of transmission lines, this paper formulates a numerical scheme that provides an accurate, stable, and efficient solution to the decoupled telegrapher's equations and then applies it to a toy problem.

### Method Formulation

To formulate a numerical scheme for the telegrapher's equations, they need to be decoupled, discretized, and analyzed. Among the formulas and schemes discussed in this section, although voltage is used as the function, the results hold for current as well.

**Equation Decoupling**

To decouple the voltage and current partial differential equations, a straightforward process is employed. To start, the partial derivative with respect to x will be taken on (1) and the partial derivative with respect to t will be taken on (2), producing the following equations:

$$-\frac{\partial^2}{\partial x^2}V(x,t) = R\frac{\partial}{\partial x}I(x,t) + L\frac{\partial^2}{\partial t\partial x}I(x,t)$$

$$-\frac{\partial^2}{\partial x\partial t}I(x,t) = G\frac{\partial}{\partial t}V(x,t) + C\frac{\partial^2}{\partial t^2}V(x,t)$$

Next, the latter equation is substituted into the former along with (2), resulting in this equation:

$$-\frac{\partial^2}{\partial x^2}V(x,t) = -R\left[GV(x,t) + C\frac{\partial}{\partial t}V(x,t)\right] - L\left[G\frac{\partial}{\partial t}V(x,t) + C\frac{\partial^2}{\partial t^2}V(x,t)\right]$$

After simplifying the equation by dividing by -1, grouping like terms, and moving all the second partial derivatives to the left side, a decoupled voltage equation is yielded:

$$\frac{\partial^2}{\partial x^2}V(x,t) - LC\frac{\partial^2}{\partial t^2}V(x,t) = (RC + GL)\frac{\partial}{\partial t}V(x,t) + GRV(x,t) \tag{3}$$

A similar procedure can be taken to decouple the current partial differential equation, yielding (3), except with *I(x,t)* as the function. Notably, for (3), if the transmission line is taken to be lossless (the resistance and conductance are zero) the first partial derivative and the regular voltage terms disappear, resulting in the wave equation. Regardless, using the decoupled partial differential equation, (3), a numerical scheme can be formulated for the voltage or current in a transmission line.

**Method Derivation**

To derive a numerical method that models the voltage or current in a transmission line, (3) will be discretized with finite differences. Throughout the process, the following notation will be used:

$$\alpha = LC,\ \beta = (RC + GL),\ \gamma = GR,\ V_m^n = V(t_n, x_m)$$

For the discretization, centered-time centered-space will be used for the second-order derivatives and centered-time will be used for the first-order derivative. The formulas for these discretizations and their respective errors are shown here:

$$\frac{\partial^2}{\partial x^2}V(x,t) = \frac{V_{m+1}^n - 2V_m^n + V_{m-1}^n}{h^2} + O(h^2)$$

$$\frac{\partial^2}{\partial t^2}V(x,t) = \frac{V_m^{n+1} - 2V_m^n + V_m^{n-1}}{k^2} + O(k^2)$$

$$\frac{\partial}{\partial t}V(x,t) = \frac{V_m^{n+1} - V_m^{n-1}}{2k} + O(k^2)$$

To construct the method, the first step is to substitute the new notation and the discretizations into (3).

$$\frac{V_{m+1}^n - 2V_m^n + V_{m-1}^n}{h^2} - \alpha\frac{V_m^{n+1} - 2V_m^n + V_m^{n-1}}{k^2} = \beta\frac{V_m^{n+1} - V_m^{n-1}}{2k} + \gamma V_m^n$$

From here, the $V_m^{n+1}$ terms are moved to the left side and simplified while the rest of the terms are sent to the right side.

$$\left(-\frac{\alpha}{k^2} - \frac{\beta}{2k}\right)V_m^{n+1} = \alpha\frac{-2V_m^n + V_m^{n-1}}{k^2} - \beta\frac{V_m^{n-1}}{2k} + \gamma V_m^n - \frac{V_{m+1}^n - 2V_m^n + V_{m-1}^n}{h^2}$$

Next, both sides are divided by the coefficient of $V_m^{n+1}$ and the like terms on the right side of the equation are grouped.

$$V_m^{n+1} = \left(\frac{1}{\frac{\alpha}{k^2} + \frac{\beta}{2k}}\right)\left[\left(\frac{2\alpha}{k^2} - \gamma - \frac{2}{h^2}\right)V_m^n + \left(\frac{\beta}{2k} - \frac{\alpha}{k^2}\right)V_m^{n-1} + \left(\frac{1}{h^2}\right)(V_{m+1}^n + V_{m-1}^n)\right]$$

To reach the final method, the notation in the equation is replaced by the original notation, yielding the final numerical method:

$$V_m^{n+1} = \left(\frac{1}{\frac{LC}{k^2} + \frac{RC+GL}{2k}}\right)\left[\left(\frac{2LC}{k^2} - GR - \frac{2}{h^2}\right)V_m^n + \left(\frac{RC+GL}{2k} - \frac{LC}{k^2}\right)V_m^{n-1} + \left(\frac{1}{h^2}\right)(V_{m+1}^n + V_{m-1}^n)\right] \quad (4)$$

Since second-order finite differences were used to discretize the decoupled partial differential equation, (4) will be second-order accurate for the spatial and temporal step size. With this numerical method, an approximate solution to the telegrapher's equations can be iteratively found, with space step size *h* and time step size *k*.

**Stability Analysis**

With the scheme of (4), the telegrapher's equations can be solved numerically, however, its stability must be analyzed to understand the method's limitations. To this end, Von Neumann stability analysis will be employed. To simplify the process, the following notation will be used:

$$\delta = \frac{1}{\frac{LC}{k^2} + \frac{RC + GL}{2k}}\ ,\ \varepsilon = \left(\frac{2LC}{k^2} - GR - \frac{2}{h^2}\right),\ \mu = \left(\frac{RC + GL}{2k} - \frac{LC}{k^2}\right),\ \rho = \frac{1}{h^2}$$

Using this new notation, (4) becomes the following:

$$V_m^{n+1} = \delta[\varepsilon V_m^n + \mu V_m^{n-1} + \rho(V_{m+1}^n + V_{m-1}^n)]$$

To perform Von Neumann stability analysis, the following discrete Fourier Transform will represent each voltage:

$$V_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\omega} \hat{V}^n(\omega) d\omega$$

Applying this to the numerical scheme and combining the integrands yields the following equation:

$$V_m^{n+1} = \frac{\delta}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \varepsilon e^{imh\omega} \hat{V}^n(\omega) + \mu e^{imh\omega} \hat{V}^{n-1}(\omega) + \rho e^{i(m+1)h\omega} \hat{V}^n(\omega) + \rho e^{i(m-1)h\omega} \hat{V}^n(\omega)\, d\omega$$

From here, the equation is simplified by factoring out one of the complex exponentials and grouping the like terms in the integrand.

$$V_m^{n+1} = \frac{\delta}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\omega} \left[\left(\varepsilon + \rho e^{ih\omega} + \rho e^{-ih\omega}\right) \hat{V}^n(\omega) + \mu \hat{V}^{n-1}(\omega)\right] d\omega$$

Next, by generalizing and simplifying the expression, a simple difference equation is produced.

$$\hat{V}^{n+1} = \delta\left(\varepsilon + \rho e^{ih\omega} + \rho e^{-ih\omega}\right) \hat{V}^n(\omega) + \delta\mu \hat{V}^{n-1}(\omega)$$

Using the properties of complex exponentials, the equation is simplified further.

$$\hat{V}^{n+1} = \delta[\varepsilon + 2\rho \cos(h\omega)] \hat{V}^n(\omega) + \delta\mu \hat{V}^{n-1}(\omega)$$

In this difference equation, multiple time steps are used to calculate the next. Due to this, determining the stability criteria will require the system to be converted into a matrix equation in the form of $\hat{V}^{n+1} = A\hat{V}^{n}$. Performing this task produces the following matrix equation:

$$\hat{V}^{n+1} = \begin{bmatrix} 0 & 1 \\ \delta\mu & \delta\varepsilon + 2\delta\rho\cos(h\omega) \end{bmatrix} \begin{bmatrix} \hat{V}^{n-1} \\ \hat{V}^{n} \end{bmatrix}$$

From here, the stability criteria of the numerical scheme are determined by the eigenvalues of the system's *A* matrix, which will now be calculated.

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ \delta\mu & \delta\varepsilon + 2\delta\rho\cos(h\omega) - \lambda \end{bmatrix}$$

$$\det(A - I) = (-\lambda)[\delta\varepsilon + 2\delta\rho\cos(h\omega) - \lambda] - \delta\mu = 0$$

$$0 = \lambda^2 + \delta[-\varepsilon - 2\rho\cos(h\omega)]\lambda - \delta\mu$$

Since this expression is a polynomial of order two, the quadratic formula is used to determine the eigenvalues. At this point, the notation will be modified further, with $h\omega = \theta \in [0, \pi]$.

$$\lambda = \frac{\delta[\varepsilon + 2\rho\cos(\theta)]}{2} \pm \frac{\sqrt{[-\delta\varepsilon - 2\delta\rho\cos(\theta)]^2 + 4\delta\mu}}{2}$$

For the system to be stable, $0 \leq |\lambda| \leq 1$. After applying this condition, substituting in the primary line characteristics, and simplifying the expression, an inequality for the stability of (4) is shown to be:

$$0 \leq \left| \left(\frac{1}{\frac{LC}{k^2} + \frac{RC + GL}{2k}}\right)\left[\left(\frac{2LC}{k^2} - GR - \frac{2}{h^2}\right) + \frac{2}{h^2}\cos(\theta)\right] \pm \sqrt{\left(\frac{1}{\frac{LC}{k^2} + \frac{RC + GL}{2k}}\right)^2 \left[\left(\frac{2LC}{k^2} - GR - \frac{2}{h^2}\right) + \frac{2}{h^2}\cos(\theta)\right]^2 + 4\left(\frac{1}{\frac{LC}{k^2} + \frac{RC + GL}{2k}}\right)\left(\frac{RC + GL}{2k} - \frac{LC}{k^2}\right)} \right| \leq 2 \qquad (5)$$

for all possible $\theta$. Interestingly, the appearance of *k* in the denominator of (5) results in the method becoming unstable for small *k*, a severe limitation on the numerical scheme's use cases. Overall, (4) is conditionally stable, dependent on the inequality given by (5).

## Results

Using the numerical scheme derived, (4), and the stability criteria developed. (5), a toy problem will be solved.

**Toy Problem**

For a toy problem, let there exist a transmission line of a length of 0.05 meters, which will be observed for 10 seconds, governed by the following primary line characteristics: $R = 500\frac{\Omega}{m},\ L = 1\frac{H}{m}, C = 0.1\frac{F}{m}\ , G = 1\frac{S}{m}$. Initially, the transmission line will contain no voltage nor a change in voltage. At the left end of the transmission line there is a power source producing a voltage of $sin(\pi t)\ V$ and on the right end there is a voltage sink. For this problem, its mathematical representation is the following:

$$\frac{\partial^2}{\partial x^2}V - 0.1\frac{\partial^2}{\partial t^2}V = 51\frac{\partial}{\partial t}V + 500\,V,\ \ 0 < x < 0.05\ ,\ 0 < t \leq 10$$

$$V(x,0) = 0,\ \frac{\partial}{\partial t}V(x,0) = 0,\ 0 \leq x \leq 0.05$$

$$V(0,t) =\ sin(\pi t)\ V,\ \ V(0.05,t) = 0,\ 0 < t \leq 10$$

Using the above information, (4) can be used to predict the voltage across the transmission line at the times within the domain. For this method, a space step of *h = 0.01* m and a time step of *k = 0.001* s will be used, given that they meet the stability criteria mandated by (5).

**Numerical Solution**

Through the use of (4) and MATLAB, the toy problem can be solved numerically. After iterating the system over the entire time domain, a solution was constructed.

**Figure 1**

*Plot of Toy Problem Solution*

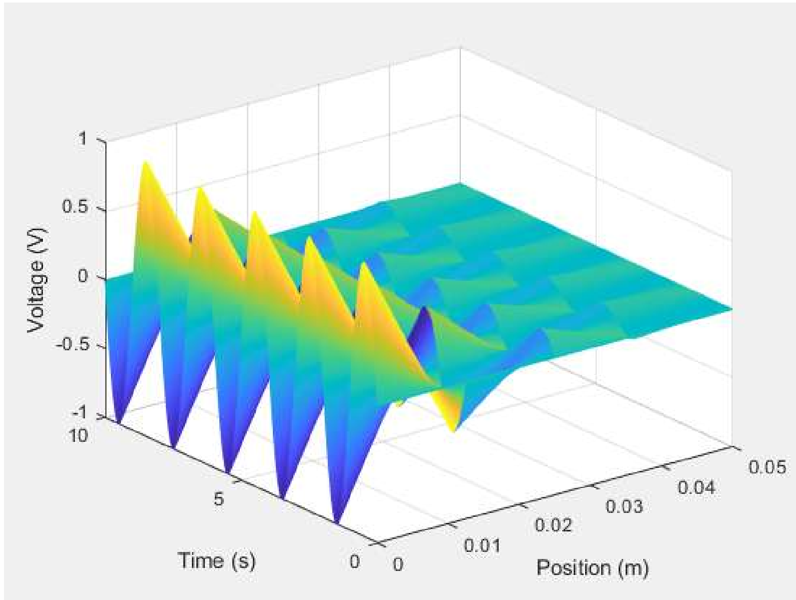
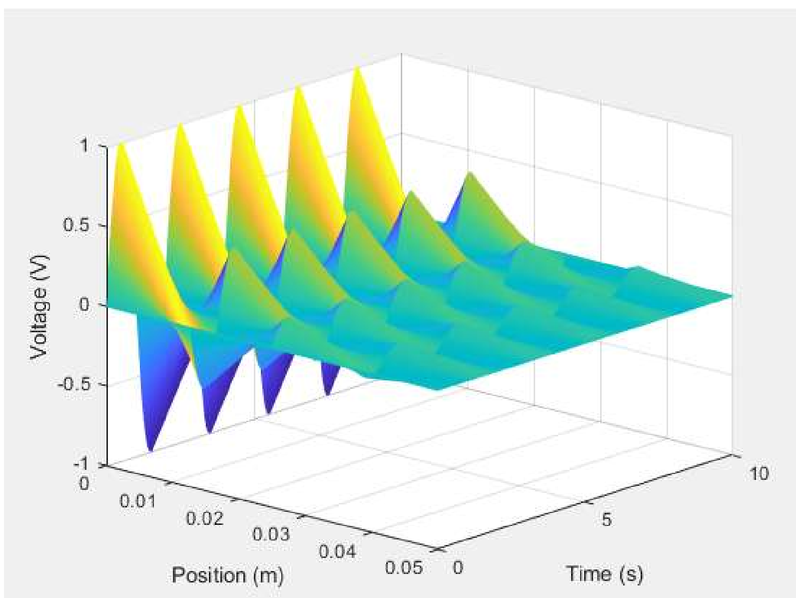



**Figure 2**

*Second Plot of Toy Problem Solution*




In Figures 1 and 2, the solution to the toy problem is plotted over the domain. Given the shape of the curve, the validity of the numerical solution is affirmed. Over the spatial and temporal intervals, the input voltage at the left end of the transmission line decays exponentially, an expected result given the overwhelming resistance of the transmission line in question. While an explicit analytical solution to the toy problem was not formed, the error of the approximation is expected to be second order. In addition to providing an accurate solution to the initial boundary value problem, the method has other merits. For this initial boundary value problem, the solution is stable by requirements of (5), as the largest eigenvalue is 0.9901 and satisfies the inequality $0 \leq |1.9802| \leq 2$. In terms of efficiency, the numerical scheme is calculated in *O(nm)* time, where n is the number of spatial steps and m is the number of temporal steps. For the toy problem, the average computation time for the numerical solution when computed 50,000 times is 895.93 microseconds. Overall, for the toy problem examined, the numerical scheme yields an accurate, efficient, and stable solution.

**Conclusion**

Within this paper, an accurate, efficient, and conditionally stable numerical scheme for the telegrapher's equations was developed. From the toy problem examined, it is shown that the method provided consistent results with the physical system of a transmission line. While the numerical scheme can model transmission lines, given the extremely stringent stability criteria outlined by (5), the application of it to real-world power systems is limited. For example, given that the frequency of AC current in the United States is 60 Hz, an extremely small time step would be required to track the current, however, a small time step results in the numerical scheme becoming unstable. Due to this, the method's application is limited to those that permit large temporal steps. In the future, investigating other discretizations for the telegrapher's equations may yield a more stable method for modeling transmission lines.

**References**

Ellingson, S. W. (n.d.). Transmission lines. In *Electromagnetics I* (pp. 48-97). Virginia Tech. https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Electro-Optics/Book%3A_Electromagnetics_I_(Ellingson)/03%3A_Transmission_Lines