Mathematical Model

1 Sets and Indices

I: Set of workpieces, $I = \{0, 1, \ldots, n, n+1\}$, where n is the total number of actual workpieces.

K: Set of stations, $K = \{0, 1, \ldots, 10\}$.

K_1: Subset of stations 1–5, $K_1 = \{1, \ldots, 5\}$.

K_2: Subset of stations 8–10, $K_2 = \{8, 9, 10\}$.

i, j: Indices for workpieces.

k: Index for stations.

m: Index for machines at Station 6, $m \in \{1, 2\}$.

2 Parameters

$P_{i,k}$: Processing time of workpiece i at station k (where $P_{i,6}$ is the time on machine m at Station 6).

$S_{i,j}$: Setup time required to switch from workpiece i to workpiece j on any machine.

M: A sufficiently large constant.

3 Decision Variables

$x_{i,j}$: Binary variable; 1 if workpiece i is the immediate predecessor of workpiece j at stations 1–5, 0 otherwise.

$y_{m,i,j}$: Binary variable; 1 if workpiece i is the immediate predecessor of workpiece j on machine m ($m = 1, 2$) at Station 6, 0 otherwise.

$z_{i,j}$: Binary variable; 1 if workpiece i is the immediate predecessor of workpiece j at stations 7–10, 0 otherwise.

$B_{i,6}$: Start time of workpiece i at Station 6.

$C_{i,k}$: Completion time of workpiece i at station k.

w_i: Auxiliary binary variable used to linearize conditional logic.

4 Model

$\min(obj) = \min C_{n+1,10}$

Initial Conditions

$$C_{i,0} = 0, \qquad \forall i \in I \tag{1}$$

$$C_{0,k} = 0, \qquad \forall k \in K \tag{2}$$

$$B_{0,k} = 0, \qquad \forall k \in K \tag{3}$$

Sequence Constraints (Stations 1–5)

$$\sum_{i=0}^{n} x_{i,j} = 1, \qquad \forall j \in I \setminus \{0\} \tag{4}$$

$$\sum_{j=1}^{n+1} x_{i,j} = 1, \qquad \forall i \in I \setminus \{n+1\} \tag{5}$$

$$x_{i,i} = 0, \qquad \forall i \in I \tag{6}$$

$$\sum_{i=0}^{n+1} x_{i,0} + \sum_{j=0}^{n+1} x_{n+1,j} = 0 \tag{7}$$

Sequence Constraints (Station 6 Machines)

$$\sum_{i=0}^{n} (y_{1,i,j} + y_{2,i,j}) = 1, \qquad \forall j \in I \setminus \{0, n+1\} \tag{8}$$

$$\sum_{j=1}^{n+1} (y_{1,i,j} + y_{2,i,j}) = 1, \qquad \forall i \in I \setminus \{0, n+1\} \tag{9}$$

$$y_{m,i,i} = 0, \qquad \forall i \in I, m \in \{1, 2\} \tag{10}$$

$$\sum_{i=0}^{n+1} y_{m,i,0} + \sum_{j=0}^{n+1} y_{m,n+1,j} = 0, \qquad \forall m \in \{1, 2\} \tag{11}$$

$$\sum_{i=0}^{n} y_{m,i,p} = \sum_{j=1}^{n+1} y_{m,p,j}, \qquad \forall p \in I \setminus \{0, n+1\}, m \in \{1, 2\} \tag{12}$$

$$\sum_{j=0}^{n+1} y_{m,0,j} = 1, \qquad \forall m \in \{1, 2\} \tag{13}$$

$$\sum_{i=0}^{n+1} y_{m,i,n+1} = 1, \qquad \forall m \in \{1, 2\} \tag{14}$$

Sequence Constraints (Stations 7–10)

$$\sum_{i=0}^{n} z_{i,j} = 1, \qquad \forall j \in I \setminus \{0\} \tag{15}$$

$$\sum_{j=1}^{n+1} z_{i,j} = 1, \qquad \forall i \in I \setminus \{n+1\} \tag{16}$$

$$z_{i,i} = 0, \qquad \forall i \in I \tag{17}$$

$$\sum_{i=0}^{n+1} z_{i,0} + \sum_{j=0}^{n+1} z_{n+1,j} = 0 \tag{18}$$

Completion Time and Start Time Constraints

$$C_{j,k} = \max\left(C_{j,k-1}, \sum_{i=0}^{n} x_{i,j} \cdot C_{i,k}\right) + \sum_{i=0}^{n} x_{i,j} \cdot S_{i,j} + P_{j,k}, \qquad \forall j \in I \setminus \{0\}, k \in K_1 \tag{19}$$

$$C_{j,k} = \max\left(C_{j,k-1}, \sum_{i=0}^{n} z_{i,j} \cdot C_{i,k}\right) + \sum_{i=0}^{n} z_{i,j} \cdot S_{i,j} + P_{j,k}, \qquad \forall j \in I \setminus \{0\}, k \in K_2 \tag{20}$$

$$C_{j,7} \geq \max\left(C_{j,6}, \sum_{i=0}^{n} z_{i,j} \cdot C_{i,7}\right) + \sum_{i=0}^{n} x_{i,j} \cdot S_{i,j} + P_{j,7}, \qquad \forall j \in I \setminus \{0\} \tag{21}$$

$$C_{j,6} = B_{j,6} + \sum_{i=0}^{n} (y_{1,i,j} + y_{2,i,j}) \cdot S_{i,j} + \sum_{i=0}^{n} (y_{1,i,j} + y_{2,i,j} \cdot 1.2) \cdot P_{i,6}, \qquad \forall j \in I \setminus \{0\} \tag{22}$$

$$B_{j,6} = \max\left(C_{j,5}, \sum_{i=0}^{n} (y_{1,i,j} + y_{2,i,j}) \cdot C_{i,6}\right), \qquad \forall j \in I \setminus \{0\} \tag{23}$$

$$B_{j,6} \geq \sum_{i=0}^{n} x_{i,j} B_{i,6}, \qquad \forall j \in I \setminus \{0, n+1\} \tag{24}$$

Linearized Logical Constraints

$$\sum_{i=0}^{n} x_{i,j} C_{i,6} - C_{j,5} + 0.00001 \leq M(1 - w_j), \qquad \forall j \in I \setminus \{0, n+1\} \tag{25}$$

$$\sum_{i=0}^{n} (y_{1,i,j} + y_{2,i,j}) \cdot C_{i,6} - \sum_{i=0}^{n} x_{i,j} C_{i,6} \leq M \cdot w_j, \qquad \forall j \in I \setminus \{0, n+1\} \tag{26}$$