Part 1: Defining a Dynamically Expanding Formal System (DEFS)

1. **Formal System Definition:**
   - Let $F_i = (L_i, A_i, R_i)$ be a conventional, consistent formal system at iteration $i \in \mathbb{N}$, where:
     - L_i is a recursive formal language (e.g., first-order logic with arithmetic).
     - A_i is a recursive set of axioms expressed in L_i.
     - R_i is a recursive set of inference rules for L_i.
   - We assume F_i is sufficiently powerful to represent basic arithmetic.
2. **Consistency and Gödel's Incompleteness:**
   - **Assumption 1.1 (Consistency):** For all i, F_i is consistent. (This will be a property maintained by our expansion rule.)
   - **Theorem 1.1 (Gödel's First Incompleteness Theorem for DEFS):** For any consistent F_i sufficiently powerful to express arithmetic, there exists a Gödel sentence $G_i \in L_i$ such that G_i is true but unprovable within F_i (i.e., $F_i \nvdash G_i$ and $F_i \nvdash \neg G_i$).
     - The existence of such G_i is the formal detection of "incompleteness" in F_i.

Part 2: The Axiom of Dynamic Expansion (ADE)

1. **The Expansion Trigger:**
   - The detection of a Gödel sentence G_i in F_i serves as the sole trigger for the system to transition to a new state F_{i+1}.
2. **The Expansion Rule (Transition Function T):**
   - Define a transition function $T: \text{DEFS} \to \text{DEFS}$ such that $F_{i+1} = T(F_i)$ is constructed from F_i as follows:
     - **If F_i is complete** (i.e., no G_i exists, which Gödel proves is impossible for sufficiently strong F_i anyway, but included for logical completeness): $F_{i+1} = F_i$.
     - **If F_i is incomplete** (i.e., a G_i exists):
       - **Step 2.2.1 (Axiom Generation):** A new axiom a_new_i is generated. This a_new_i is defined as the formal statement "G_i is true and provable by a higher-order system." Or more simply, $a\_new_i = G_i$. (This is the critical "twist": turning the unprovable truth into a new axiom).
       - **Step 2.2.2 (System Transformation):** F_{i+1} is then defined as:

        - $L_{i+1} = L_i$ (for simplicity; language can expand, but not strictly necessary for proof-of-concept)
        - $A_{i+1} = A_i \cup \{a\_new\_i\}$
        - $R_{i+1} = R_i$ (for simplicity; rules can expand, but not strictly necessary)
3. **The Principle of Consistency Preservation (PCP):**
    - This is a meta-axiom governing the system's behavior: **The system always expands in a manner that preserves consistency.**
    - **Formal Statement:** For any F_i and any G_i, the a_new_i chosen and the resulting F_{i+1} must satisfy $Consis(F_{i+1})$. This implies a_new_i is never contradictory with A_i. (This is the formal embodiment of "benevolence" – the system always seeks a coherent, non-contradictory evolution).

Part 3: Mathematical Proofs of Transcendence

1. **Theorem 3.1 (Perpetual Consistency):**
    - **Statement:** If F_0 is consistent, then F_i is consistent for all $i \in \mathbb{N}$.
    - **Proof Sketch:** By induction.
        - **Base Case:** F_0 is assumed consistent.
        - **Inductive Step:** Assume F_k is consistent. If F_k is complete (trivial case, $F_{k+1}=F_k$), then F_{k+1} is consistent. If F_k is incomplete (Gödel sentence G_k exists), then F_{k+1} is formed by $A_{k+1} = A_k \cup \{G_k\}$. By the **Principle of Consistency Preservation (PCP)**, A_{k+1} is guaranteed to be consistent, therefore F_{k+1} is consistent.
2. **Theorem 3.2 (Resolution of Incompleteness - Stepwise Completeness):**
    - **Statement:** For any i, if G_i is a Gödel sentence for F_i, then G_i is provable within F_{i+1} (i.e., $F_{i+1} \vdash G_i$).
    - **Proof Sketch:** By construction, $A_{i+1} = A_i \cup \{G_i\}$. Since $G_i \in A_{i+1}$, it is trivially provable within F_{i+1} by the axiom of introduction.
3. **Theorem 3.3 (Perpetual Expansion and Transcendence - "Gödelian Staircase"):**
    - **Statement:** The sequence of systems $F_0, F_1, F_2, \ldots$ constitutes an infinite, strictly increasing hierarchy of formal systems, where each F_{i+1} resolves the incompleteness of F_i but inherently possesses its own new Gödelian incompleteness.
    - **Proof Sketch:**
        - By Theorem 3.2, each F_{i+1} resolves F_i's incompleteness.
        - By Theorem 3.1, F_{i+1} is consistent.

- Since `F_{i+1}` contains `F_i` and `G_i`, it is strictly more powerful than `F_i`.
- Since `F_{i+1}` is consistent and sufficiently powerful (as it contains arithmetic and more), by Gödel's First Incompleteness Theorem (Theorem 1.1), `F_{i+1}` *must* possess its own new Gödel sentence `G_{i+1}` that is unprovable within `F_{i+1}`.
- This establishes an infinite chain where each system `F_i`, when encountering its Gödelian limit, dynamically expands to `F_{i+1}` that resolves *that specific limit*, but in doing so, creates a new, more encompassing system that, by Gödel's theorem, *must* generate a new, higher-order Gödelian limit.
- Therefore, the system perpetually expands and transcends its own incompleteness, never reaching a final, statically complete state.