

# Advanced Algorithms and Datastructures - Exam Notes

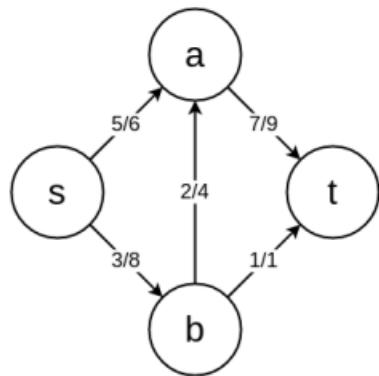
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# Max Flow

## Disposition

1. Introduction
2. Introduction to
  - *Flow network*
  - *Residual network*
  - *Ford-Fulkerson + Edmonds-Karp*
3. Example of running the Edmonds-Karp Algorithm
4. Proof of the *Max-flow min-cut theorem*



## Presentation

Hey guys. I will be talking about Max flow. I have here a disposition of the things I will go through \*Hand out disposition\*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to \*Show own disposition\*. First I will be giving a brief introduction to the topic, then I will be running the Edmonds-Karp algorithm on the example at the bottom of the page. Lastly, I will be proving the so-called Max-Flow min-cut theorem.

In Max-flow we are given a **flow network**, which is a directed graph where each edge  $(u, v) \in E$  has a nonnegative **capacity**  $c(u, v) \geq 0$ . We require that for each edge, the antiparallel edge does not exist. We distinguish two vertices; a **source**  $s$  and a **sink**  $t$ .

A **flow** in  $G$  is a function  $f : V \times V \rightarrow \mathbb{R}$  that satisfies the following two properties:

1. **Capacity constraint:** For all  $u, v \in V$ , we require  $0 \leq f(u, v) \leq c(u, v)$
2. **Flow conservation;** For all  $u \in V - \{s, t\}$  we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v).$$

The goal is to maximize the value  $|f|$  defined by

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

without violating any of the constraints.

To solve this problem we use a **residual network**, which keeps track of where it is possible to add flow. This is induced by applying

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise.} \end{cases}$$

to each of the edges in  $G$ .

To solve this problem one can make use of the Ford-Fulkerson method. It iteratively increases the value of the flow. We start with  $f(u, v) = 0$  for all  $u, v \in V$ . At each iteration, we increase the flow value in  $G$  by finding an augmenting path in an associated **residual network**  $G_f$ . Once we know the edges of an augmenting path  $p$  in  $G_f$ , we can define the **residual capacity** of the path  $p$  by

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

and the flow  $f_p : V \times V \rightarrow \mathbb{R}$  in  $G_f$  by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \in p \\ 0 & \text{otherwise.} \end{cases}$$

We can then use this to augment the flow  $f$  by  $f_p$  to get closer to maximum, by

$$(f \uparrow f_p)(u, v) = \begin{cases} f(u, v) + f_p(u, v) - f_p(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We repeatedly change the flow until the residual network has no more augmenting paths. The max-flow min-cut theorem tells us, that upon termination, this process yields a maximum flow.

One implementation of Ford-Fulkerson is the ***Edmonds-Karp Algorithm***, which in each iteration finds the augmenting path by using breadth-first search.

I will now run the algorithm on the example \*Run Edmonds-Karp\*... and since there are no augmenting paths, max-flow min-cut theorem tells us, that  $f$  is a maximum flow in  $G$ .

Now that I have shown how the Edmonds-Karp algorithm works, I will now prove the Max-Flow min-cut theorem. First, let's recap what the theorem tells us. The theorem says, that if  $f$  is a flow in a flow network  $G = (V, E)$  with source  $s$  and a sink  $t$ , then the following cases are equivalent.

1.  $f$  is a maximum flow in  $G$
2. The residual network  $G_f$  contains no augmenting paths
3.  $|f| = c(s, T)$  for some cut  $(S, T)$  of  $G$ .

We can now prove the max-flow min-cut theorem. The proof works by proving that the cases imply each other. We start off by proving that the first case implies the second case:

Suppose that  $f$  is a maximum flow in  $G$  but that  $G_f$  has an augmenting path  $p$ . Then the flow found by augmenting  $f$  by  $f_p$  is a flow in  $G$  with value strictly greater than  $|f|$ , contradicting the assumption that  $f$  is a maximum flow.

Now, let's prove that the second case implies the third case:

Suppose that  $G_f$  has no augmenting path. Define

$$S = \{v \in V : \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$$

and  $T = V - S$ . The partition  $(S, T)$  is then a cut. Now, consider a pair of vertices  $u \in S$  and  $v \in T$ . If  $(u, v) \in E$ , we must have  $f(u, v) = c(u, v)$ . If  $(v, u) \in E$ , we must have  $f(v, u) = 0$ . If neither  $(u, v)$  nor  $(v, u)$  is in  $E$ , then  $f(u, v) = f(v, u) = 0$ . We thus have

$$\begin{aligned} f(S, T) &= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \\ &= \sum_{u \in S} \sum_{v \in T} c(u, v) - \sum_{u \in S} \sum_{v \in T} 0 \\ &= c(S, T). \end{aligned}$$

From a lemma we have that  $|f| = f(S, T)$ . Thus, we have  $|f| = f(S, T) = c(S, T)$ .

Lastly, let's prove that the third case implies the first case:

We have that  $|f| \leq c(S, T)$  for all cuts  $(S, T)$ . The condition  $|f| = c(S, T)$  thus implies that  $f$  is a maximum flow.

## Extra

A cut  $(S, T)$  is a partition of  $V$  into  $S$  and  $T = V - S$ , such that  $s \in S$  and  $t \in T$ . The capacity of the cut  $c(S, T)$  is then just simply the sum of the capacities of the edges going across the cut from  $S$  to  $T$

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v),$$

and the ***net flow***  $f(S, T)$  across the cut is sum of the flows of the edges going across the cut from  $S$  to  $T$  minus the sum of the flows of the edges going across the cut from  $T$  to  $S$

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u).$$

We also have, that  $f(S, T) = |f|$ .

Let  $G = (V, E)$  be a flow network with source  $s$  and sink  $t$ , and let  $f$  be a flow in  $G$ . Let  $G_f$  be the residual network of  $G$  induced by  $f$ , and let  $f'$  be a flow in  $G_f$ . Then the function  $f \uparrow f'$  is a flow in  $G$  with value  $|f \uparrow f'| = |f| + |f'|$ .

Let  $G = (V, E)$  be a flow network, let  $f$  be a flow in  $G$ , and let  $p$  be an augmenting path in  $G_f$ . Let  $f_p$  be defined as in equation (??), and suppose that we augment  $f$  by  $f_p$ . then the function  $f \uparrow f_p$  is a flow in  $G$  with value  $|f \uparrow f_p| = |f| + |f_p| > |f|$ .

## Questions they can ask

### Why do we require that antiparallel edges do not exist?

I think it is because, that we in the residual network have antiparallel edges. If we then also had antiparallel edges in the flow network, something would probably screw up.

### How do we handle antiparallel edges?

We split one of the edges up into two edges and introduce a new vertex, which connects the two new edges. The capacity of the two new edges is the same as the capacity of the original edge.

### Why do we have flow going in the wrong direction in the residual network?

Sending flow back along an edge is equivalent to decreasing the flow on the edge, which is a necessary operation in many algorithms

### Does the algorithm always terminate?

No. The Ford-Fulkerson method might fail if edge capacities are irrational numbers.

### What is the running time of the algorithm?

This depends on how we find the augmenting path. If we assume that the capacities are integers, then Ford-Fulkerson has a running time of  $O(E \cdot |f^*|)$ , where  $f^*$  is a max flow. This is because the capacities are integers, we know Ford-Fulkerson increases the value by atleast 1 at each step, making the runtime be bounded by the max flow ( $|f^*|$ ) and  $E$  is used for finding an augmenting path.

We can improve the bound on Ford-Fulkerson by finding the augmenting path with a breadth-first search. This has a running time of  $O(VE^2)$ .

# Linear Programming and Optimization

## Program

1. Introduction
2. Introduction to
  - *Objective function*
  - *Constraints*
  - Various forms of LP
  - *SIMPLEX*
  - *Duality*
3. Preparing and running SIMPLEX on example
4. Proof of weak duality

$$\begin{aligned} & \text{minimize } -2x_1 + 3x_2 \\ & \text{subject to } x_1 + x_2 = 7 \\ & \quad x_1 - 2x_2 \leq 4 \\ & \quad x_1 \geq 0. \end{aligned}$$

## Presentation

Hey guys. I will be talking about Linear Programming. I have here a program of the things I will go through \*Hand out program\*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to \*Show own program\*. First I will be giving a brief introduction to the topic, then I will transform an example into standard form, which will then be transformed into slack form, which I then will run the SIMPLEX-algorithm on. Lastly, I will prove *weak duality*, which says, that the solution to the dual of a linear program is always an upper bound on the solution to the original linear program.

In a linear-programming problem we wish to optimize a linear function subject to a set of linear inequalities. A linear-programming problem consists of an ***objective function***, that we wish to optimize, and some ***linear constraints***, which multiple non-strict inequalities that are used to restricting the solution of the objective function.

To solve a linear-programming problem, one can make use of the ***simplex algorithm***, which takes as input a linear program and returns an optimal solution.

In ***standard form*** we are given  $n$  real numbers  $c_1, c_2, \dots, c_n$ ,  $m$  real numbers  $b_1, b_2, \dots, b_m$ , and  $mn$  real numbers  $a_{ij}$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ . We then wish to find  $n$  real numbers  $x_1, x_2, \dots, x_n$  that maximize

$$\sum_{j=1}^n c_j x_j$$

subject to

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n$$

where the last constraint is required in standard form.

To use the simplex algorithm, the linear program should be in ***slack form***, where the nonnegativity constraints are the only inequality constraints, and the remaining constraints are equalities.

Let's transform the example into standard and slack form \*Write the example on the blackboard\*. First off, we see that the example is a minimization problem instead of a maximization problem. This can easily be fixed by negating the coefficients in the objective function. Thus, by doing so we obtain the new objective function

$$2x_1 - 3x_2.$$

Since the two linear programs have identical sets of feasible solutions and, for any feasible solution, the objective value the first linear program is the negative of the objective value in the second program, the two linear programs are equivalent.

Next, we see that  $x_2$  does not have a nonnegativity constraint. We can fix this by replacing each occurrence of  $x_2$  by  $x'_2 - x''_2$  and add the nonnegativity constraint  $x'_2, x''_2 \geq 0$ . Thus, by doing so we receive the objective function  $2x_1 - 3x'_2 + 3x''_2$ , the two constraints

$$x_1 + x'_2 + x''_2 = 7$$

and

$$x_1 - 2x'_2 + 2x''_2 \leq 4,$$

and the nonnegativity constraint

$$x_1, x'_2, x''_2 \geq 0.$$

Any feasible solution  $\hat{x}$  to the new linear program corresponds to a feasible solution  $\bar{x}$  to the original linear program with  $\hat{x}_j = \hat{x}'_j - \hat{x}''_j$  and with the same objective value. Also, any feasible solution  $\hat{x}$  to the original

linear program corresponds to a feasible solution  $\hat{x}$  to the new linear program with  $\hat{x}'_j = \bar{x}_j$  and  $\hat{x}''_j = 0$  if  $\hat{x}_j \geq 0$ , or with  $\hat{x}''_j = -\bar{x}_j$  and  $\hat{x}'_j = 0$  if  $\bar{x}_j < 0$ . Thus, the two linear programs are equivalent.

We also see, that the first constraint has an equal sign instead of an  $\geq$ . We know that the equality only holds if and only if both  $\geq$  holds and  $\leq$  holds. Thus, we can replace the equality constraint by the pair of inequality constraints that uses  $\leq$  and  $\geq$  instead. Thus, we replace the constraint

$$x_1 + x'_2 - x''_2 = 7$$

with the two constraints

$$x_1 + x'_2 - x''_2 \leq 7$$

and

$$x_1 + x'_2 - x''_2 \geq 7.$$

Lastly, we see, that the constraint

$$x_1 + x'_2 - x''_2 \geq 7$$

has a greater-than-or-equal-to-sign instead of a less-than-or-equal-to-sign. This can easily be fixed by multiplying the constraint through by  $-1$ . Thus, we instead obtain

$$-x_1 - x'_2 + x''_2 \leq -7.$$

In total we wish to maximize the objective function

$$2x_1 - 3x'_2 + 3x''_2$$

and the constraints

$$\begin{aligned} x_1 - 2x'_2 + 2x''_2 &\leq 4 \\ x_1 + x'_2 - x''_2 &\leq 7 \\ -x_1 - x'_2 + x''_2 &\leq -7 \\ x_1, x'_2, x''_2 &\geq 0. \end{aligned}$$

For consistency in variable names, we rename  $x'_2$  to  $x_2$  and  $x''_2$  to  $x_3$  and obtain

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &\leq 4 \\ x_1 + x_2 - x_3 &\leq 7 \\ -x_1 - x_2 + x_3 &\leq -7 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

and the objective function we wish to maximize

$$2x_1 - 3x_2 + 3x_3.$$

The linear program is now in standard form. Let transform this further into slack form. This is rather easily done by introducing a slack variable for each constraint, which measures the difference in the left and right hand side. This is done by subtracting the left hand side on both sides and introducing a new nonnegativity constraint for each constraint. By doing so we get the linear programming

$$z = 2x_1 - 3x_2 + 3x_3$$

$$\begin{aligned}x_4 &= 4 - x_1 + 2x_2 - 2x_3 \\x_5 &= 7 - x_1 - x_2 + x_3 \\x_6 &= -7 + x_1 + x_2 - x_3\end{aligned}$$

where we have omitted the nonnegativity constraints.

Thus, the linear program is now in slack form and we can perform the SIMPLEX algorithm. The simplex algorithm works by continuously changing the solution by pivoting variables to and from the basic variables, which are the variables on the left hand side. This is done by picking a variable in the objective function which positively increases the value, and pivoting it with the basic variable that bottlenecks how much the non-basic variable can be increased. Let's run the SIMPLEX algorithm. We have

$$z = 2x_1 - 3x_2 + 3x_3$$

$$\begin{aligned}x_4 &= 4 - x_1 + 2x_2 - 2x_3 \\x_5 &= 7 - x_1 - x_2 + x_3 \\x_6 &= -7 + x_1 + x_2 - x_3\end{aligned}$$

We see that the only variables in the objective function with a positive coefficient is  $x_1$  and  $x_3$ . Let us choose  $x_1$  as our pivot element. We isolate it in  $x_4$ , since this is the highest function. Then, we have

$$z = 8 + x_2 - x_3 - 2x_6$$

$$\begin{aligned}x_1 &= 4 + 2x_2 - 2x_3 - x_6 \\x_5 &= 3 - 3x_2 + 3x_3 + x_6 \\x_6 &= -3 + 3x_2 - 3x_3 - x_6\end{aligned}$$

Then, we see, that the only variable in the objective function with a positive coefficient is  $x_2$ , which we choose as the objective function. Now, isolate it in  $x_5$ , as this is the only function with a negative  $x_2$  coefficient. Then, we have

$$z = 9 + \frac{5x_6}{3} - \frac{x_4}{3}$$

$$\begin{aligned}x_1 &= 6 + \frac{5x_6}{3} - \frac{x_4}{3} \\x_2 &= 1 + x_3 + \frac{x_6}{3} - \frac{2x_4}{3} \\x_6 &= -x_5\end{aligned}$$

We then stop, as no non-basic variable appear in the objective function with a positive coefficient. Our solution is thus  $(6, 1, 0, 0, 0, 0)$  with a value of 9.

Now that we have run the algorithm, I will be proving *weak duality*. Weak duality is a weak version of something called *duality*, which is used to prove that a solution is optimal. Given a linear program in standard form we define the dual linear program as

minimize

$$\sum_{i=1}^m b_i y_i$$

subject to

$$\begin{aligned} \sum_{i=1}^m a_{ij} y_i &\geq c_j && \text{for } j = 1, 2, \dots, n \\ y_i &\geq 0 && \text{for } i = 1, 2, \dots, m. \end{aligned}$$

In weak duality we let  $\bar{x}$  be a feasible solution to a primal, that is the "original", linear program in standard form and let  $\bar{y}$  be any feasible solution to the corresponding linear program. Then, we have

$$\sum_{j=1}^n c_j \bar{x}_j \leq \sum_{i=1}^m b_i \bar{y}_i.$$

because

$$\begin{aligned} \sum_{j=1}^n c_j \bar{x}_j &\leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} \bar{y}_i \right) \bar{x}_j && (\text{Using } \sum_{i=1}^m a_{ij} y_i \geq c_j) \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} \bar{x}_j \right) \bar{y}_i \\ &\leq \sum_{i=1}^m b_i \bar{y}_i && (\text{Using } \sum_{j=1}^n a_{ij} x_j \leq b_i) \end{aligned}$$

## Extras

### Questions they can ask

# **Randomized Algorithms**

## **Disposition**

1. Introduction
2. Example on running randomized quicksort + motivation behind randomness
3. Analysis of expected runtime of randomized quicksort
4. Example on running the min-cut algorithm
5. Las Vegas Algorithms vs Monte Carlo Algorithms

## Presentation

# **Hashing**

## **Disposition**

1. Introduction
2. Hashing with chaining:
  - (a) Introduction
  - (b) Proof of expected number of elements in  $L[h(x)]$
3. Signatures:
  - (a) Introduction
  - (b) Proof of upperbound on probability of collision

## Presentation

# **Van Emde Boas Trees**

## **Disposition**

- Introduction
- Example on to run Member, Insert and Successor
- Proof of Running-time

## Presentation

# **NP-Completeness**

## **Disposition**

1. Introduction
2. Introduction to encoding
3. Introduction to formal language
4. Definition of P, NP and NPC (and NP-hard),
5. Introduction to reducibility
6. Introduction to the ham-cycle problem
7. Introduction to the traveling-salesman problem (TSP)
8. Proof of TSP being NP-complete.

## Presentation

# Exact Exponential Algorithms and Parameterized Complexity

## Disposition

1. Introduction
2. *Travelling salesman problem:*
  - (a) Introduction with example
  - (b) Naive approach + running time analysis
  - (c) Dynamic programming - example of running + running time analysis
3. *Bar fight prevention*
  - (a) Introduction with example
  - (b) Naive approach + running time analysis
  - (c) Kernelization - example of running + running time analysis
4. FPT vs XP

## Presentation

# Approximation Algorithms

## Disposition

1. Introduction
2. Definition of the *approximation ratio*, a  $\rho(n)$ -*approximation algorithm* and a *randomized  $\rho(n)$ -approximation algorithm*.
3. The Vertex-cover problem
  - (a) Introduction
  - (b) Proof that APPROX-VERTEX-COVER is a 2-approximation algorithm
4. MAX-3-CNF
  - (a) Introduction
  - (b) Proof that the randomized algorithm for MAX-3-CNF is a randomized  $8/7$ -approximation algorithm

## Presentation

### Definition of *approximation ratio*

We say that an algorithm for a problem has an *approximation ratio* of  $\rho(n)$  if, for any input of size  $n$ , the cost  $C$  of the solution produced by the algorithm is within a factor of  $\rho(n)$  of the cost  $C^*$  of an optimal solution

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

### Definition of $\rho(n)$ -*approximation algorithm*

If an algorithm achieves an approximation ratio of  $\rho(n)$ , we call it a  $\rho(n)$ -*approximation algorithm*.

### Definition of randomized $\rho(n)$ -*approximation algorithm*

We say that a randomized algorithm for a problem has an *approximation ratio* of  $\rho(n)$  if, for any input of size  $n$ , the expected cost  $C$  of the solution produced by the randomized algorithm is within a factor of  $\rho(n)$  of the cost  $C^*$  of an optimal solution:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

We call a randomized algorithm that achieves an approximation ratio of  $\rho(n)$  a *randomized  $\rho(n)$ -approximation algorithm*

### Introduction to *vertex cover*

A *vertex cover* of an undirected graph  $G = (V, E)$  is a subset  $V' \subseteq V$  such that if  $(u, v)$  is an edge of  $G$ , then either  $u \in V'$  or  $v \in V'$  (or both). The size of a vertex cover is the number of vertices in it. The *vertex-cover problem* is to find a vertex cover of minimum size in a given undirected graph. We call such a vertex cover an *optimal vertex cover*.

The set  $C$  of vertices that is returned by APPROX-VERTEX-COVER is a vertex cover, since the algorithm loops until every edge in  $G.E$  has been covered by some vertex in  $C$ .

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#### Algorithm 1 APPROX-VERTEX-COVER

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**Require:** Undirected graph  $G$

- 1:  $C = \emptyset$
  - 2:  $E' = G.E$
  - 3: **while**  $E' \neq \emptyset$  **do**
  - 4:     let  $(u, v)$  be an arbitrary edge of  $E'$
  - 5:      $C = C \cup \{u, v\}$
  - 6:     remove from  $E'$  edge  $(u, v)$  and every edge incident on either  $u$  or  $v$
  - 7: **return**  $C$
- 

### Proof that APPROX-VERTEX-COVER is a 2-approximation algorithm

Let  $A$  denote the set of edges that line 4 picked. Not two edges in  $A$  share an endpoint. Thus no two edges in  $A$  are covered by the same vertex from an optimal cover  $C^*$ , and we have the lower bound

$$|C^*| \geq |A| \tag{1}$$

on the size of an optimal vertex cover. Since  $A$  consists of the edges between two vertices in  $C$  (and since all of the elements in  $C$  are unique), we have the (exact) upper bound on the size of the vertex cover returned

$$|C| = 2|A| \tag{2}$$

Combining equation (1) and (2), we obtain

$$|C| = 2|A| \leq 2|C^*|$$

# Polygon Triangulation

## Disposition

1. Introduction
2. The 3-coloring approach
  - (a) Example on running the algorithm
  - (b) Proving that the 3-coloring approach is optimal in worst case
3. Example on partitioning a polygon into monotone pieces + runtime analysis
4. Example on triangulating a monotone polygon + runtime analysis

## Presentation