Advanced Algorithms and Datastructures - Exam Notes

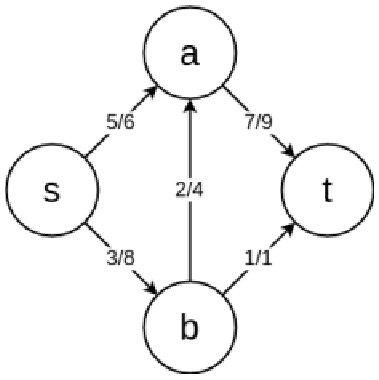
André O. Andersen

2021

Max Flow

Disposition

1. Introduction
2. Introduction to
    - *Flow network*
    - *Residual network*
    - *Ford-Fulkerson + Edmonds-Karp*
3. Example of running the Edmonds-Karp Algorithm
4. Proof of the *Max-flow min-cut theorem*



Presentation

Hey guys. I will be talking about Max flow. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First I will be giving a brief introduction to the topic, then I will be running the Edmonds-Karp algorithm on the example at the bottom of the page. Lastly, I will be proving the so-called Max-Flow min-cut theorem.

In Max-flow we are given a ***flow network***, which is a directed graph where each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v) \geq 0$. We require that for each edge, the antiparallel edge does not exist. We distinguish two vertices; a ***source*** s and a ***sink*** t.

A ***flow*** in G is a function $f : V \times V \to \mathbb{R}$ that satisfies the following two properties:

1. ***Capacity constraint***: For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$
2. ***Flow conservation***; For all $u \in v - \{s, t\}$ we require
$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v).$$

The goal is to maximize the value $|f|$ defined by

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

without violating any of the constraints.

To solve this problem we use a ***residual network***, which keeps track of where it is possible to add flow. This is induced by applying

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise.} \end{cases}$$

to each of the edges in G.

To solve this problem one can make use of the Ford-Fulkeson method. It iteratively increases the value of the flow. We start with $f(u, v) = 0$ for all $u, v \in V$. At each iteration, we increase the flow value in G by finding an augmenting path in an associated ***residual network*** G_f. Once we know the edges of an augmenting path p in G_f, we can define the ***residual capacity*** of the path p by

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

and the flow $f_p : V \times V \to \mathbb{R}$ in G_f by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \in p \\ 0 & \text{otherwise.} \end{cases}$$

We can then use this to augment the flow f by f_p to get closer to maximum, by

$$(f \uparrow f_p)(u, v) = \begin{cases} f(u, v) + f_p(u, v) - f_p(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We repeatedly change the flow until the residual network has no more augmenting paths. The max-flow min-cut theorem tells us, that upon termination, this process yields a maximum flow.

One implmentation of Ford-Fulkerson is the ***Edmonds-Karp Algorithm***, which in each iteration finds the augmenting path by using breadth-first search.

I will now run the algorithm on the example *Run Edmonds-Karp*... and since there are no augmenting paths, max-flow min-cut theorem tells us, that f is a maximum flow in G.

Now that I have shown how the Edmonds-Karp algorithm works, I will now prove the Max-Flow min-cut theorem. First, let's recap what the theorem tells us. The theorem says, that if f is a flow in a flow network $G = (V, E)$ with source s and a sink t, then the following cases are equivalent.

1. f is a maximum flow in G
2. The residual network G_f contains no augmenting paths
3. $|f| = c(s, T)$ for some cut (S, T) of G.

We can now prove the max-flow min-cut theorem. The proof works by proving that the cases imply each other. We start of by proving that the first case implies the second case:

> Suppose that f is a maximum flow in G but that G_f has an augmenting path p. Then the flow found by augmenting f by f_p is a flow in G with value strictly greater than $|f|$, contradicting the assumption that f is a maximum flow.

Now, let's prove that the second case implies the third case:

> Suppose that G_f has no augmenting path. Define
>
> $$S = \{v \in V : \text{ there exists a path from } s \text{ to } v \text{ in } G_f\}$$
>
> and $T = V - S$. The partition (S, T) is then a cut. Now, consider a pair of vertices $u \in S$ and $v \in T$. If $(u, v) \in E$, we must have $f(u, v) = c(u, v)$. If $(v, u) \in E$, we must have $f(v, u) = 0$. If neither (u, v) nor (v, u) is in E, then $f(u, v) = f(v, u) = 0$. We thus have
>
> $$\begin{aligned} f(S,T) &= \sum_{u \in S} \sum_{v \in t} f(u,v) - \sum_{u \in S} \sum_{v \in t} f(v,u) \\ &= \sum_{u \in S} \sum_{v \in T} c(u,v) - \sum_{u \in S} \sum_{v \in T} 0 \\ &= c(S,T). \end{aligned}$$
>
> From a lemma we have that $|f| = f(S, T)$. Thus, we have $|f| = f(S, T) = c(S, T)$.

Lastly, let's prove that the third case implies the first case:

> We have that $|f| \leq c(S, T)$ for all cuts (S, T). The condition $|f| = c(S, T)$ thus implies that f is a maximum flow.

Extra

A cut (S, T) is a partition of V into S and $T = V - S$, such that $s \in S$ and $t \in T$. The capacity of the cut $c(S, T)$ is then just simply the sum of the capacities of the edges going across the cut from S to T

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v),$$

and the ***net flow*** $f(S, T)$ across the cut is sum of the flows of the edges going across the cut from S to T minus the sum of the flows of the edges going across the cut from T to S

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u).$$

We also have, that $f(S,T) = |f|$.

Let $G = (V,E)$ be a flow network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the function $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.

Let $G = (V,E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Let f_p be defined as in equation (**??**), and suppose that we augment f by f_p. then the function $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$.

Questions they can ask

Why do we require that antiparallel edges do not exist?

I think it is because, that we in the residual network have antiparallel edges. If we then also had antiparallel edges in the flow network, something would probably screw up.

How do we handle antiparallel edges?

We split one of the edges up into two edges and introduce a new vertex, which connects the two new edges. The capacity of the two new edges is the same as the capacity of the original edge.

Why do we have flow going in the wrong direction in the residual network?

Sending flow back along an edge is equivalent to decreasing the flow on the edge, which is a necessary operation in many algorithms

Does the algorithm always terminate?

No. The Ford-Fulkerson method might fail if edge capacities are irrational numbers.

What is the running time of the algorithm?

This depends on how we find the augmenting path. If we assume that the capacities are integers, then Ford-Fulkerson has a running time of $O(E \cdot |f^*|)$, where f^* is a max flow. This is because the capacities are integers, we know Ford-Fulkerson increases the value by atleast 1 at each step, making the runtime be bounded by the max flow ($|f^*|$) and E is used for finding an augmenting path.

We can improve the bound on Ford-Fulkerson by finding the augmenting path with a breadth-first search. This has arunning time of $O(VE^2)$.

Linear Programming and Optimization

Program

1. Introduction
2. Introduction to
   - Various forms of LP
   - *Objective function*
   - *Constraints*
   - Algorithm for solving linear programming
3. Preparing and running SIMPLEX on example
4. Duality:
   - Motivation behind duality
   - Definition of *dual linear program*
   - Definition of *weak duality*
   - Proof of weak duality

$$\begin{array}{ll} \text{minimize} & -x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 = 6 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{array}$$

Presentation

Hey guys. I will be talking about Linear Programming. I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own program*. First I will be giving a brief introduction to the topic, then I will transform an example into standard form, which will then be transformed into slack form, which I then will run the SIMPLEX-algorithm on. Lastly, I will pove *weak duality*, which says, that the solution to the dual of a linear program is always an upper bound on the solution to the orignial linear program.

Linear programming is given in two forms; ***standard form*** and ***slack form***.

standard form has the following form

$$\sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, ..., m$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, ..., n$$

The first equation is the ***objective function***, which we wish to optimize with respect to the ***constraints***, which are the following lines, where the last constraint is required in standard form.

Slack form is very similar to standard form, however, only the nonnegativity constraints are the only inequality constraints, and the remaining constraints are equalities.

To solve a linear program we use the simplex algorithm, whose input is a linear program in ***slack form***.

Let's transform the example into standard and slack form *Write the example on the blackboard*. First off, we see that the example is a minimization problem instead of a maximization problem. This can easily be fixed by negating the coefficients in the objective function. Thus, by doing so we obtain the new objective function

$$x_1 + x_2.$$

Since the two linear programs have identical sets of feasible solutions and, for any feasible solution, the objective value the first linear program is the negative of the objective value in the second program, the two linear programs are equivalent.

Next, we see that x_2 does not have a nonnegativity constraint. We can fix this by replacing each occurence of x_2 by $x'_2 - x''_2$ and add the nonnegativity constraint $x'_2, x''_2 \geq 0$. To ease the readibility, I will instead let $x'_2 = x_2$ and $x''_2 = x_3$. Thus, by doing so we receive the objective function $x_1 + 2x_2 - 2x_3$ the two constraints

$$x_1 + x_2 - x_3 = 6$$

and

$$x_1 - 2x_2 + 2x_3 \leq 4$$

and the nonnegativity constraint

$$x_1, x_2, x_3 \geq 0.$$

Any feasible solution $\hat{x}$ to the new linear program corresponds to a feasible solution $\bar{x}$ to the original linear program with $\hat{x}_j = \hat{x}'_j - \hat{x}''_j$ and with the same objective value. Also, any feasible solution $\hat{x}$ to the original linear program corresponds to a feasible solution $\hat{x}$ to the new linear program with $\hat{x}'_j = \bar{x}_j$ and $\hat{x}''_j = 0$ if $\hat{x}_j \geq 0$, or with $\hat{x}''_j = -\bar{x}_j$ and $\hat{x}'_j = 0$ if $\bar{x}_j < 0$. Thus, the two linear programs are equivalent.

We also see, that the first constraint has an equal sign instead of an $\geq$. We know that the equality only holds if and only if both $\geq$ holds and $\leq$ holds. Thus, we can replace the equality constraint by the pair of inequality constraints that uses $\leq$ and $\geq$ instead. Thus, we replace the constraint

$$x_1 + x_2 - x_3 = 6$$

with the two constraints

$$x_1 + x_2 - x_3 \leq 6$$

and

$$x_1 + x_2 - x_3 \geq 6$$

Lastly, we see, that the constraint

$$x_1 + x_2 - x_3 \geq 6$$

has a greater-than-or-equal-to-sign instead of a less-than-or-equal-to-sign. This can easily be fixed by multiplying the constraint through by -1. Thus, we instead obtain

$$-x_1 - x_2 + x_3 \geq -6$$

In total we have

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 - 2x_3 \\ \text{subject to} & x_1 + x_2 - x_3 \leq 6 \\ & -x_1 - x_2 + x_3 \geq -6 \\ & x_1 - 2x_2 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The linear program is now in standard form. Let transform this further into slack form. This is rather easily done by introducing a slack variable for each constraint, which measures the difference in the left and right hand side. This is done by subtracting the left hand side on both sides and introducing a new nonnegativity constraint for each constraint. By doing so we get the linear programming

$$\begin{aligned} &z = x_1 + 2x_2 - 2x_3 \\ &x_4 = 6 - x_1 - x_2 + x_3 \\ &x_5 = -6 + x_1 + x_2 - x_3 \\ &x_6 = 4 - x_1 + 2x_2 - x_3 \end{aligned}$$

where we have omitted the nonnegativity constraints.
Thus, the linear program is now in slack form and we can perform the SIMPLEX algorithm. The simplex algorithm works by continuously changing the solution by pivoting variables to and from the basic variables, which are the variables on the left hand side. This is done by picking a variable in the objective function which positively increases the value, and pivoting it with the basic variable that bottlenecks how much the non-basic variable can be increased. Let's run the SIMPLEX algorithm. We have

$$\begin{aligned} &z = x_1 + 2x_2 - 2x_3 \\ &x_4 = 6 - x_1 - x_2 + x_3 \\ &x_5 = -6 + x_1 + x_2 - x_3 \\ &x_6 = 4 - x_1 + 2x_2 - x_3 \end{aligned}$$

We see, that x_1 and x_2 are the only variables that can increase the value of the objective function. We choose x_2 as the pivot element (**DETTE ER VIGTIGT FOR AT FÅ ALGORTIMEN TIL AT TERMINERE EFTER 1 ITERATION - ALT ANDET VIL FÅ DEN TIL AT KØRE LÆNGERE**). Next, we isolate x_2 in x_4, since x_4 is the only constraint with a negative coefficient for x_2. The value of x_2 is then inserted in the other constraints as well as the objective function, resulting in

$$\begin{aligned} &z = 12 - x_1 - 2x_4 \\ &x_2 = 6 - x_1 + x_3 - x_4 \\ &x_5 = -x_4 \\ &x_6 = 16 - 3x_1. \end{aligned}$$

We now stop, as no basic variable appear in the objective function with a positive coefficient. Our solution is this $(0, 6, 0, 0, 0, 16)$, or $(x_1, x_2) = (0, 6)$ for the original problem, with a value of 12. Now that we have run the algorithm, I will be proving *weak duality.* Weak duality is a weak version of something called *duality*, which is used to prove that a solution is optimal. Given a linear program in standard form we define the dual linear program as

minimize

$$\sum_{i=1}^{m} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad \text{for } j = 1, 2, ..., n$$

$$y_i \geq 0 \qquad \text{for } i = 1, 2, ..., m.$$

In weak duality we let $\bar{x}$ be a feasible solution to a primal, that is the "original", linear program in standard form and let $\bar{y}$ be any feasible solution to the corresponding linear program. Then, we have

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i.$$

because

$$\begin{aligned}
\sum_{j=1}^{n} c_j \bar{x}_j &\leq \sum_{j=1}^{m} \left(\sum_{i=1}^{m} a_{ij} \bar{y}_i \right) \bar{x}_j && (\text{Using } \sum_{i=1}^{m} a_{ij} y_i \geq c_j) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \bar{x}_j \right) \bar{y}_i \\
&\leq \sum_{i=1}^{m} b_i \bar{y}_i && (\text{Using } \sum_{j=1}^{n} a_{ij} x_j \leq b_i)
\end{aligned}$$

Extras

We say that two maximization linear programs L and L' are ***equivalent*** if for each feasible solution $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}'$ to L' with objective value z, and for each feasible solution $\bar{x}'$ to L' with objective value z, there is a corresponding feasible solution $\bar{x}$ to L with objective value z. A minimization linear program L and a maximization linear program L' are equivalent if for each feasible soltuion $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}'$ to L' with objective value $-z$, and for each feasible solution $\bar{x}'$ to L' with objective value z, there is a corresponding feasible solution $\bar{x}$ to L with objective value $-z$.

Corollary 29.9

Let $\bar{x}$be a feasible solution to aprimal linear program, and let $\bar{y}$ be a feasible solution to the corresponding dual linear program. If

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i,$$

then $\bar{x}$ and $\bar{y}$ are optimal solutions to the primal and dual linear programs, respectively.
Proof By the lemma of weak duality, the objective value of a feasible solution to the primal cannot exceed that of a feasible solution to the dual. The primal linear program is a maximization problem and the dual is a minimization problem. Thus, if feasible solutions $\bar{x}$ and $\bar{y}$ have the same objective value, neither can be improved.

Questions they can ask

ER DER NOGET HER?

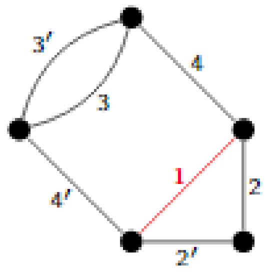
Randomized Algorithms

Disposition

1. Introduction
2. Quicksort:
   (a) Idea behind Quicksort
   (b) Example on running non-random Quicksort
   (c) Motivation behind randomness in Quicksort
   (d) Analysis of expected runtime of randomized quicksort
3. Min-cut:
   (a) Introduction to min-cut
   (b) Example on running the min-cut algorithm
4. Las Vegas Algorithms vs Monte Carlo Algorithms

$$[3, 5, 1, 2, 4]$$

NOTE: MÅSKE UDSKIFT LV vs MC MED BEVIS AF MIN-CUT's KØRETID?!?!?



Presentation

Hey guys. I will be talking about Randomized Algorithms. I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowd to *Show own program*. First, I will be giving a brief introduction to the topic. Then I will be talking about quicksort. Lastly, I will be giving a brief introduction to Las Vegas algorithms and Monte Carlo algorithms.

Quicksort works by recursively picking a pivot element and then group all elements smaller than the pivot element in one set and all of the elements greater than the pivot element in another set, making the pivot element be in its sorted index. Then the algorithm is performed on the two sets recursively.

Consider running the algorithm on the provided example. If we were to always pick the "median element" as the pivot element, this would result in a balanced search tree, so we might expect the number of comparisons to be something like $n \log n$.

However, if we instead were to pick an extreme element as the pivot, this would result in an extremely unbalanced binary search tree, so we might expect the number of comparisons to be someting like n^2.

Picking the pivot element thus have a great effect on the running time of the algorithm. For this reason, we can instead choose the pivot element uniformly at random. By doing so we can expect the number of comparisons to be $O(n \log n)$, which I will prove now.

We start off by letting $S_{(i)}$ denote the ith smallest element in the sorted array S, for $1 \leq i \leq n$. Then, we define

$$X_{ij} = \begin{cases} 1 & \text{if } S_{(i)} \text{ and } S_{(j)} \text{ are compared in an execution} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the total number of comparisons is

$$\sum_{i<j} X_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}.$$

Then, we are interested in the expected number of comparisons

$$\mathbb{E}\left[\sum_{i<j} X_{ij}\right] = \sum_{i<j} \mathbb{E}\left[X_{ij}\right].$$

Let p_{ij} denote the probability that $S_{(i)}$ and $S_{(j)}$ are compared in an execution. Since X_{ij} only assumes the values 0 and 1 we have

$$\mathbb{E}[X_{ij}] = p_{ij} \cdot 1 + (1 - p_{ij}) \cdot 0 = p_{ij}.$$

Each recursive call returns some sublist $[S_{(a)}, ..., S_{(b)}]$. Let $x = S_{(c)}$ be the pivot. Suppose $a \leq i < j \leq b$. If $c < i$ or $c > j$, then $S_{(i)}$ and $S_{(j)}$ are not compared now, but are in the same subtree and might be compared later. If $i < c < j$, then $S_{(i)}$ and $S_{(j)}$ are never compared. If $c = i$ or $c = j$ then $S_{(i)}$ and $S_{(j)}$ are compared once, right now. Thus, $S_{(i)}$ and $S_{(j)}$ are compared iff $S_{(i)}$ or $S_{(j)}$ is first of $S_{(s)}, ..., S_{(j)}$ to be chosen as pivot. Hence why p_{ij} is the conditional probability of picking $S_{(i)}$ or $S_{(j)}$ given that the pivot is picked uniformly at random in $S_{(i)}, S_{(i+1)}, ..., S_{(j)}$

$$p_{ij} = \mathbb{P}\left[c \in \{i, j\} | c \in \{i, i+1, ..., j\}\right] = \frac{2}{|\{i, i+1, ..., j\}|} = \frac{2}{j - i + 1}.$$

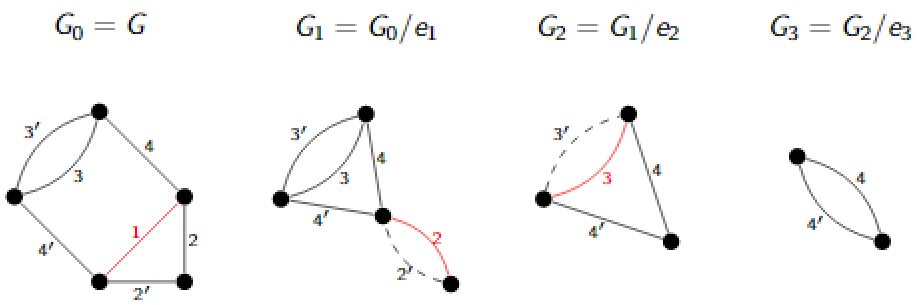
We thus have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i<j} X_{ij}\right] &= \sum_{i<j} \mathbb{E}\left[X_{ij}\right] && \text{(Linearity of expectation)}\\
&= \sum_{i<j} p_{ij}\\
&= \sum_{i<j} \frac{2}{j-i+1}\\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} && \left(\text{Using} \sum_{i<j} = \sum_{i=1}^{n-1} \sum_{j=i+1}\right)\\
&= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k}\\
&< \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{2}{k} && \text{(Adding more positive terms increases total value)}\\
&= 2n \sum_{k=2}^{n} \frac{1}{k}\\
&= 2n \left(\left(\sum_{k=1}^{n} \frac{1}{k} \right) - 1 \right)\\
&= 2n(H_n - 1) && \left(\text{Using } H_n = \sum_{k=1}^{n} \frac{1}{k}\right)\\
&\leq 2n \int_1^n \frac{1}{x} dx = 2n \ln n = O(n \log n) && \left(\text{Using } H_n - 1 \leq \int_1^n \frac{1}{x} dx = \ln n\right)
\end{aligned}$$

Now, onto min-cut. In min-cut we G be a connected, undirected graph with n vertices. A *cut* in G is a set of edges whose removalresults in G being broken into two or more components. The goal is to find a cut with minimal cardinality.

The algorithm works by repeatedly picking an edge uniformly at random and merging the two vertices at its end-points. Edges between vertices that are merged are removed, so that there are never any self-loops.

By running this algorithm on the example we potentially get the following Thus, the edges 4 and $4'$ is



a min-cut.

Lastly, let us talk about Las Vegas and Monte Carlo algorithms. A Las Vegas algorithm is a randomized algorithm that always return the correct output, and the randomness lies in its running time. A Monte

Carlo algorithm on the other hand is a randomized algorithm that does not necessarily return a correct output, however, we can run it multiple times to get a correct output.

Extras

Lower-bound of finding min-cut

The min-cut algorithm has a probability of $\frac{2}{n(n-1)}$ of finding a specific min-cut.
For any vertex v in a multigraph G, the *neighborhood* of v, denoted $\Gamma(v)$, is the set of vertices of G that are adjacent to v. The *degree* of v, denoted $d(v)$, is the number of edges incident on v. For a set S of vertices of G, the neighborhood of S, denoted $\Gamma(S)$, is the union of the neighborhoods of the constituent vertices.

Let k be the min-cut size and $G_i = (V_i, E_i)$ be the graph G after i iterations, which has $n_i = n - i$ vertices. We fix our attention on a particular min-cut C with k edges. Then, G_i has at least $n_i|C|/2$ edges because

$$\frac{1}{2}\sum_{v\in V_i} d_i(v) \geq \frac{1}{2}n_i|C| \quad (\text{Using } d_i(v) \geq |C|).$$

We will bound from below the probability that no edge of C is ever contracted during an execution of the algorithm.

Let ϵ_i denote the event of not picking an edge of C at the ith step, for $1 \leq i \leq n-2$. Let $p_i = \mathbb{P}\left[\epsilon_i | \cap_{j=i}^{i-1} \epsilon_j\right]$. The probability of picking an edge of C in the ith iteration, given that no edge of C has been picked in a previous iteration, is thus

$$\begin{aligned}
1 - p_i &= \mathbb{P}\left[\epsilon_i | \cap_{j=i}^{i-1} \epsilon_j\right] \\
&= \mathbb{P}\left[e \in E_{i-1} \text{ is in } C | \cap_{j=1}^{i-1} \epsilon_j\right] \\
&= \frac{|C|}{|E_{i-1}|} \leq \frac{|C|}{\frac{1}{2}n_{i-1}|C|} \qquad = \frac{2}{n_{i-1}} = \frac{2}{n-(i-1)} p_i \geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}. \quad \Rightarrow
\end{aligned}$$

Thus, the probability of finding C can then by found by

$$\begin{aligned}
\mathbb{P}\left[C \text{is found}\right] &= \prod_{i=1}^{n-2} p_i \\
&\geq \prod_{i=1}^{n-2} \frac{n-1-i}{n+1-i} \\
&= \frac{n-2}{n}\frac{n-3}{n-1}\frac{n-4}{n-2}\cdots\frac{3}{5}\frac{2}{4}\frac{1}{3} \\
&= \frac{2}{n(n-1)}.
\end{aligned}$$

Suppose we were to repeat the min-cut algorithm $n^2/2$ times, making independent random choices each time. The probability that a min-cut is not found in any of the $n^2/2$ attempts is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2/2} < \frac{1}{e}.$$

Questions they can ask

What is the probability of the min-cut algorithm succeeding?

The probability of finding a specific min-cut is greater than or equal to

$$\frac{2}{n(n-1)}.$$

If we were to repeat the algorithm x amount of times, which probability would we get for a specific x?

Suppose we were to repeat the min-cut algorithm $n^2/2$ times, making independent random choices each time. The probability that a min-cut is not found in any of the $n^2/2$ attempts is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2/2} < \frac{1}{e}.$$

Hashing

Disposition

1. Introduction
2. Introduction to various terms:
   (a) Random hash functions
   (b) *(Strongly) universal hashing*
   (c) *C-approximately (strongly) universal hashing*
   (d) *Multiply-mod-prime*
   (e) *Multiply-shift*
3. Hash tables with chaining:
   (a) Introduction
   (b) Proof of expected number of elements in $L[h(x)]$
4. Signatures:
   (a) Introduction
   (b) Proof of upperbound on probability of collision

Presentation

Hey guys. I will be talking about Hashing. I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowd to *Show own program*. First, I will be giving a brief introduction to the topic. Then I will be talking about hash tables with chaining. Lastly, I will be talking about signatures using hashing.

In hashing we have a universe U of keys, that we wish to map randomly to a range $[m] = \{0, ..., m-1\}$ of hash values. A random hash function $h : U \to [m]$ is thus a randomly chosen function that maps from to $U \to [m]$.

We then have, that a random hash function $h : U \to [m]$ is *universal* if, for all $x \neq y \in U$, we have $\mathbb{P}[h(x) = h(y)] \leq \frac{1}{m}$. It is *strongly universal* if for all $x \neq y$ and $q, r \in [m]$ we have $\mathbb{P}[h(x) = q \wedge h(y) = r] = \frac{1}{m^2}$.

Similarly, a random hash function $h : U \to [m]$ is *c-approximately universal* if for all $x \neq y \in U$, we have $\mathbb{P}[h(x) = h(y)] \leq \frac{c}{m}$. It is *c-approximately strongly universal*, if for all $x \neq y \in U$ and $q, r \in [m]$, we have $\mathbb{P}[h(x) = q \wedge h(y) = r] \leq \frac{c^2}{m^2}$.

Two common random hash function are *Multiply-mod-prime* and *Multiply-shift*. In Multiply-mod-prime, we let $U = [u]$ and $m < u$, pick a prime $p \geq u$ and choose $a, b \in [p]$ independently and uniformly at random. Then we have the 2-approximately strongly universal hash function

$$h^m_{a,b}(x) = ((ax + b) \bmod p) \bmod m.$$

For Multiply-shift we let $U = [2^w]$, $m = 2^l$ and choose $a \in [2^w]$ uniformly at random. Then we have the 2-approximately universal hash function

$$h_a(x) = \left\lfloor \frac{(ax) \bmod 2^w}{2^{2-l}} \right\rfloor.$$

In hash tables with chaining, we have a set $S \subseteq U$ of keys that we wish to store so that we can expect to find any key from S in constant time.

We start off by letting $n = |S|$ and $m \geq n$. We then pick a universal hash function $h : U \to [m]$, and create an array L of m lists, so that for $i \in [m]$, $L[i]$ is the list of keyts that hash to i. To find out if a key $x \in U$ is in S, we only have to check if x is in the list $L[h(x)]$, which can be done in time proportional to $1 + |L[h(x)]|$, where the 1 stands for the constant time it takes to look up the list even if it turns out to be empty. We expect that the length of $L[h(x)]$ is less than or equal to 1, which I will prove now.

To prove this, we assume that $x \notin S$, since we want the case where we look through all of $L[h(x)]$ and not just stop once we find the element in L. Then we wish to prove that

$$\mathbb{E}[|L[h(x)]|] \leq 1.$$

We then have

$$\begin{aligned}
\mathbb{E}[|L[h(x)]|] &= \mathbb{E}\left[|\{y \in S | h(y) = h(x)\}|\right] && \text{by definition we have } L[i] = \{y \in S | h(y) = i\} \\
&= \mathbb{E}\left[\sum_{y \in S} [h(y) = h(x)]\right] \\
&= \sum_{y \in S} \mathbb{E}\left[[h(y) = h(x)]\right] && \text{Using linearity of expectation} \\
&= \sum_{y \in S} \mathbb{P}[h(x) = h(x)] && [h(x) = h(x)] \text{ can be distributed as a bernoulli variable} \\
&\leq |S|\frac{1}{m} && h \text{ is a universal hash function, so we have } \mathbb{P}[h(x) = h(y)] \leq \frac{1}{m}. \\
&= \frac{n}{m} && \text{we use } |S| = n \\
&\leq 1 && \text{we use } m > n
\end{aligned}$$

So we have, that we can expect to find out if a key is in S in constant time. Not onto signatures.

In the problem with signature, we wish to assign a unique signature, $s(x)$, to each $x \in S \subseteq U$. Thus, we want $s(x) \neq s(y)$ for all distinct keys $x, y \in S$. To do this, we pick a universal hash function $s : U \to [n^3]$. The probability of an error is thus

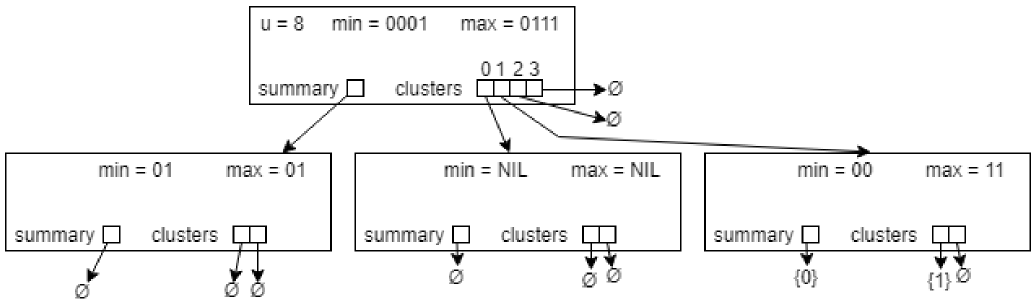
$$\begin{aligned}
\mathbb{P}[\exists \{x, y\} \subseteq S : s(x) = s(y)] &\leq \sum_{\{x,y\} \subseteq S} \mathbb{P}[s(x) = s(y)] \\
&\leq \frac{\binom{n}{2}}{n^3} < \frac{1}{2n}.
\end{aligned}$$

Where, we in the first inequality use union bound: the probability that at least one of multiple events happen is at most the sum of their probabilities. Thus, the probability of an error is upperbounded by $\frac{1}{2n}$.

Van Emde Boas Trees

Disposition

- Introduction
- Introduction to the structure of van Emde Boas Trees
- Example on running `Member`(4), `Insert`(6) and `Predecessor`(4) on the example.
- Proof of Running-time



$$\{1,4,5,7\} = \{0001_2, 0100_2, 0101_2, 0111_2\}$$

Presentation

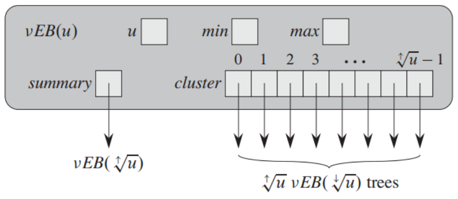
Hey guys. I will be talking about van Emde Boas Trees . I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowd to *Show own program*. First, I will be giving a brief introduction to the topic. Then I will be talking about the structure of the tree. This is followed by me running various procedures on the data structure. Lastly, I will be giving a proof of the running time of the various procedures.

Van Emde Boas Trees store integers in the range $[0, u-1]$, where u is an exact power of 2, without any duplicates.

The van Emde Boas Tree is a recursive data structure. Each node with $u > 2$, has the following structure

1. A field u denoting the universe size u
2. A field *min* that stores the minimum element in the vEB tree
3. A field *max* that stores the maximum element in the vEB tree
4. A field *Summary* that points to a summary node, that shows shows whether other clusters contains any elements.
5. An array *cluster* of pointers that points to sub-vEB trees, each of size $\sqrt[\downarrow]{u}$.

If $u = 2$, then it only has u, *min* and *max* as its fields.


vEB(u)
u
min
max
0 1 2 3 ... $\sqrt[\uparrow]{u}-1$
summary
cluster
vEB($\sqrt[\downarrow]{u}$)
$\sqrt[\uparrow]{u}$ vEB($\sqrt[\downarrow]{u}$) trees

For vEB trees we have the following equations which are important for the sturcutre

$$\sqrt[\uparrow]{u} = 2^{\lceil (\lg u)/2 \rceil}$$

and

$$\sqrt[\downarrow]{u} = 2^{\lfloor (\lg n)/2 \rfloor}.$$

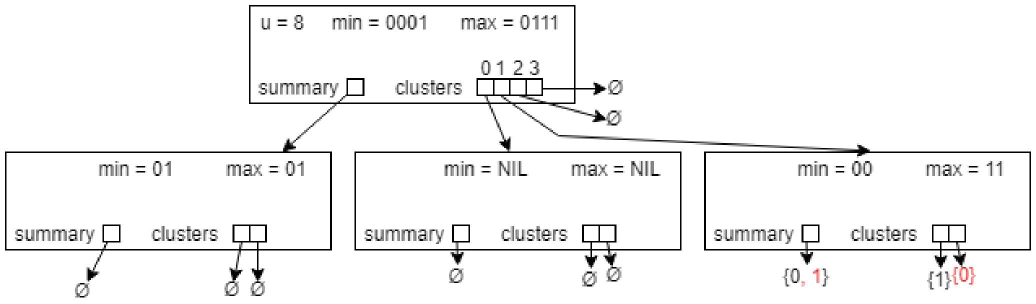
Then we define the following functions

1. $high(x) = \left\lfloor \frac{x}{\sqrt[\downarrow]{u}} \right\rfloor$: The high half of x's bits
2. $low(x) = x \bmod \sqrt[\downarrow]{u}$: The low half of x's bits
3. $index(x, y) = x\sqrt[\downarrow]{u} + y$: The value of element at cluster x index y.

We then have, that each element x is stored in its corresponding cluster $high(x)$, except for elements that appear in *min* and *max* - this varies from implementation to implementation though, as some only excludes *min* or *max*, however, I have chosen to exclude both, as this was done in our lecture.

Now that we have introduced the structure of the tree, lets run some procedures on the example *Draw example on blackboard*. Let's start by calling `Member`(5), which simply finds out whether 5 is in the tree. First off, we have, that $5 = 0101_2$, hence why $high(5) = 01_2$ and $low(5) = 01_2$. Due to $high(5) = 01_2 = 1$, we look in cluster 1 for element $low(5) = 01_5$. We see that this does not appear in the *min* nor *max*, so we find $high(01_5) = 0$ and $low(01_5) = 1$. We then look in cluster 0 for element 1, which appears, so 5 is in the tree and the algorithm returns `TRUE`.

Now, lets insert $6 = 0110_2$ into the tree. When doing insert we assume that the element to insert is not already in the tree. First we see, that the tree is not empty, because *min* and *max* are not equal to NIL. Then, we see, that $min < 0110_2 < max$, so we shall not swap 0110_2 with *min* nor *max*. Then we have $high(6) = 01_2$ and $low(6) = 10_2$. We then pass 10_2 to cluster $01_2 = 1$. We again have that $min < 10_2 < max$, so we should not swap 10_2 with *min* nor *max*. Then, we have $high(10_2) = 1$ and $low(10_2) = 0$. Since cluster 1 is empty we simply insert element 0 and update the summary such that it contains 1 aswell, since cluster 1 now contains an element, resulting in the tree Now, lets find the predecessor of $4 = 0100_2$, which is the great-



est element smaller than 4, which is stored in the tree. We have $high(x) = 01_2 = 1$ and $low(x) = 00_2 = 0$. We then check if cluster 1 is not empty and whether the minimum of cluster 1 is less than 4 (which would imply that the predecessor is in the cluster), which is false. We then check if the the summary is empty or if 1 is less-than-or-equal to the minimum of the summary, which is true, since we have $01_2 \leq 01_2$. We then return the minimum of the summary, $01_2 = 1$.

Now that we have performed the procedures on the example, lets find out what the worst case runtime of the procedures is. The recursive procedures that implement the operations all have running time characterized by the recurrence

$$T(u) \leq T(\sqrt[\uparrow]{u}) + O(1)$$

where the $O(1)$ comes from calls to $high(x)$ and $low(x)$. If we let $m = \lg u$, we can rewrite this to

$$T(2^m) \leq T(2^{\lfloor \frac{m}{2} \rfloor}) + O(1).$$

Noting that $\lfloor m/2 \rfloor \leq 2m/3$ for all $m \geq 2$, we have

$$T(2^m) \leq T(2^{2m/3}) + O(1).$$

Letting $S(m) = T(2^m)$, we can rewrite this to

$$S(m) \leq S(2m/3) + O(1)$$

which, by case 2 of the master method, has the solution $S(m) = O(\log m) = O(\log \log u)$. Since each procedure makes at most 1 recursive call, each procedure has a worst case running time of $O(\log \log u)$.

Questions they can ask

Extras

NP-Completeness

Disposition

1. Introduction
2. Introduction to various terms:
   (a) *Abstract problems*
   (b) *Encoding*
   (c) *Accepting/Rejection* and *Deciding*
   (d) *Verification*
   (e) *Reducibility*
   (f) `P`, `NP` and `NPC` (and `NP-hard`)
3. Introduction to the ham-cycle problem
4. Introduction to the traveling-salesman problem (TSP)
5. Proof of TSP being NP-complete.

Presentation

Hey guys. I will be talking about NP-Completeness. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First I will be giving this introduction, then I will be giving a brief introduction to various terms that I will be using, then I will be giving an introduction to two problems that are NP-complete, and lastly I will be proving that one of those problems is NP-complete.

An abstract problem is a binary relation on a set I of a problem ***instance*** (or input) and a set of problem ***solutions*** - (each instance can have multiple solutions - think of a graph with multiple shortest paths). In this topic we are only concerned with ***deicsion problems***: those having a yes/no solution. Thus, we can view an abstract decision problem as a function that maps the instance set I to the solution set $\{0, 1\}$.

An ***encoding*** of a set of abstract objects is a mapping to the set of binary strings. Thus, an algorithm that solves some abstract decision problem actually takes an encoding of a problem instance as input. We say that an algorithm solves a concrete problem in time $O(T(n))$ if, when it is provided a problem instance i of length $n = |i|$, the algorithm can produce the solution in $O(T(n))$ time. I will be using the notation $\langle x \rangle$ to denote the encoding of an instance x of a problem.

We say that an algorithm A ***accepts*** a string $x \in \{0, 1\}^*$ if, given input x, the output of the algorithm $A(x) \in \{0, 1\}$ is 1. The language ***accpeted*** by an algorithm A is thus the set of strings $L = \{x \in \{0, 1\}^* : A(x) = 1\}$.

However, there might strings that never terminates, thus A neither accepts nor rejects them. For this reason we have the term ***decide***. We say that, if additional all strings not in L are rejected by A, that is $A(x) = 0$ for all $x \in \{0, 1\}^* \backslash L$, then L is ***decided*** by A. We further say, that L is ***decided by A in polynomial time if A decides L and runs in polynomial time on all strings***.

We define a ***verfirication algorithm*** as being a two-argument algorithm A, where one argument is an ordinary input string x and the other is a binary string y called a ***certificate***. A two-argument algorithm A ***verifies*** an input string x if there exists a certificate y such that $A(x, y) = 1$. The ***language verified*** by a verification algorithm A is

$$L = \{x \in \{0, 1\}^* : \text{ there exists } y \in \{0, 1\}^* \text{ such that } A(x, y) = 1\}.$$

Now onto reducability, which is a method for proving that a problem is in a specific complexity class. We say that the language L_1 is polynomial-time ***reducible*** to language L_2 if there is a polynomial-time computible function $f : \{0, 1\}^* \to \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$ we have

$$x \in L_1 \Leftrightarrow f(x) \in L_2.$$

In this case, we write $L_1 \leq_P L_2$. This means, that if $L_2 \in P$, then must L_1 also be in P, since we can just transform any instance of L_1 into an instance of L_2 in polynomial time and then solve this new instance with a polynomial-time algorithm for L_2.

We can now define the various complexity classes. The complexity class P is the set of concrete decision problems that are polynomial-time solvable:

$$P = \{L \subseteq \{0, 1\}^* : \text{ there exists and algorithm } A \text{ that decides } L \text{ in polynomial time}\}.$$

A language L belongs to ***NP*** iff there exist a two-input polynomial-time algorithm A and a constant c such that

$$L = \{x \in \{0, 1\}^* : \text{ there exists a certificate } y \text{ with } |y| = O(|x|^c) \text{ such that } A(x, y) = 1\}.$$

A language $L \subseteq \{0, 1\}^*$ is ***NP-complete*** if

1. $L \in NP$, and
2. $L' \leq_p L$ for every $L' \in NP$

If a language L satisfies property 2, but not necessarily property 1, we say that L is ***NP-hard***.

Now that I have introduced the various terms, let's get onto some problems. We start off with the Ham-cycle problem. In the ham-cycle problem we are given an undirected graph $G = V, E$. The Ham-cycle problem thus asks, whether G has a hamiltonian cycle - that is, a simple cycle that contains each vertex in V, or more precisely

$$HAM - CYCLE = \{\langle G \rangle : G \text{ has a hamiltonian cycle}\}.$$

I will not be giving a proof of this, however, this problem is NP-complete.

Now onto the ***Traveling-salesman problem*** also known as ***TSP***. In TSP we are given an complete graph $G = (V, E)$ and each edge (i, j) has some cost $c(i, j) \geq 0$ assigned to it. The goal is thus to find a cycle that visits every vertex and returns back to the start vertex, which in total should be of minimal cost. The formal language for the corresponding decision problem is

$$\begin{aligned} TSP = \{\langle G, c, k \rangle : & G = (V, E) \text{ is a complete graph} \\ & c \text{ is a function from } V \times V \to \mathbb{N} \\ & k \in \mathbb{N} \\ & G \text{ has a traveling-salesman tour with cost at most } k.\} \end{aligned}$$

This problem is also NP-complete, which I will be proving now.

We first show that TSP belongs to NP. Given an instance of the problem, we use as a certificate the sequence of n vertices in the tour. The verification algorithm checks that this sequence contains each vertex exactly once, sums up the edge costs, and checks whether the sum is at most k, which all in all can be done in polynomial time.
To prove that TSP is NP-hard, we show that $HAM - CYCLE \leq_p TSP$. Let $G = (V, E)$ be an isntance of $HAM - CYCLE$. We construct an instance of TSP as follows. We form the complete graph $G' = (V, E')$, where $E' = \{(i, j) : i, j \in V \text{ and } i \neq j\}$, and we define the cost function c by

$$c(i, j) = \begin{cases} 0 & \text{if } (i, j) \in E \\ 1 & \text{if } (i, j) \notin E. \end{cases}$$

The instance of TSP is then $\langle G', c, 0 \rangle$, which we can easily create in polynomial time.
We now show that graph G has a hamiltonian cycle iff G' has a tour of cost at most $+$. Suppose that graph G has a hamiltonian cycle h. each edge in h belongs to E and thus has cost 0 in G'. Thus, h is a tour in G' with cost 0. Conversely, suppose that graph G' has a tour h' of cost at most 0. Since the costs of the edges in E' are 0 and 1, the cost of tour h' is exactly 0 and each edge on the tour must have cost 0. Therefore, h' constains only edges in E. We conclude that h' is a hamiltonian cycle in graph G.

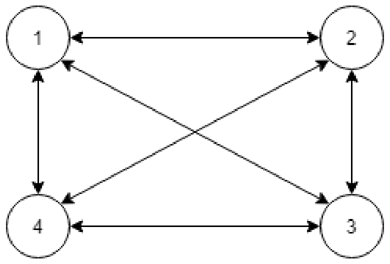
Extra

Questions they can ask

Exact Exponential Algorithms and Parameterized Complexity

Disposition

1. Introduction
2. *Exact exponential algorithms* vs *Parameterized algorithms*
3. *Travelling salesman problem*:
   (a) Introduction with example
   (b) Naive approach + running time analysis
   (c) Dynamic programming - example of running + running time analysis
4. *Bar fight prevention problem*
   (a) Introduction with example
   (b) Naive approach + running time analysis
   (c) Kernelization - example of running + running time analysis
5. *Fixed Parameter Tractable* vs *Slice-wise polynomial* problems



$$c = \begin{array}{cccccc} & \text{T} & \text{o} & & & \\ 1 & 2 & 3 & 4 & & \\ 0 & 10 & 15 & 20 & 1 & \text{F} \\ 5 & 0 & 9 & 10 & 2 & \text{r} \\ 6 & 13 & 0 & 12 & 3 & \text{o} \\ 8 & 8 & 9 & 0 & 4 & \text{m} \end{array}$$

Presentation

Hey guys. I will be talking about Exact Exponential Algorithms and Parameterized Complexity. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First, I will be talking about exact exponential algorithms and parameterized algorithms. Then, I will be talking about the Travelling salesman problem. This is followed by the Bar fight prevention problem. Lastly, I will be talking about Fixed Parameter Tractable and Slice-wise polynomial problems.

First off, what is an *Exact exponential algorithm* and a *Parameterized algorithm*? Well, an Exact exponential algorithm is an exponential algorithm that for all instances find an exact solution. On the other hand, a Parameterized algorithm is an algorithm that for instances with some small fixed values of parameter find an exact solution in polynomial time. Lastly, there are also ***approximation algorithms*** that in polynomial time find an approximation to a solution of all instances, however, I will not be talking about those.

Now that I have introduced the algorithms, I will be talking about some examples. Lets start with the NP-Complete problem called "the Traveling salesman problem", also known as ***TSP***, which we will be solving using an exact exponential algorithm. In TSP we are given an complete graph $G = (V, E)$ and each edge (i, j) has some cost $c(i, j) \geq 0$ assigned to it. The goal is thus to find a cycle that visits every vertex and returns back to the start vertex, which in total should be of minimal cost.

How would one solve this problem? Well, the naive approach is to just generate all possible solutions and then just use the minimum as solution. This would require $n!$ steps, which of course is very bad. Instead, one can make use of dynamic programming for a faster algorithm. We start off by defining

$$OPT[S, c_i] = \text{"path of minimum length that starts in } c_1\text{, visits all of } S \text{ once, and end in } c_i$$

for all $S \subseteq \{c_2, ..., c_n\}$ and $c_i \in S$. Then

$$\min\{\text{OPT}[\{c_2, ..., c_n\}, c_i] + d(c_i, c_1) | c_i \in \{c_2, ..., c_n\}\}$$

is the length of the minimal tour. Thus, we have

$$OPT[S, c_i] = \begin{cases} d(c_1, c_i) & \text{if } S = \{c_i\} \\ \min\{OPT[S \backslash \{c_i\}, c_k] + d(c_k, c_i) | c_k \in S \backslash \{c_i\}\} & \text{if } \{c_i\} \subset S. \end{cases}$$

Now that I have introduced the problem and how to solve the problem using dynamic programming, let's have a look at the example *Write example on blackboard*. First we start at the bottom of the recursion tree. Thus, for each vertex that is not 1, we find the cost of traveling to 1. In the following $OPT(a, b)$ means, that we are currently at vertex a, and we have not visited the vertices in the set b. Thus, we have

$$OPT(2, \emptyset) = 5$$

$$Opt(3, \emptyset) = 6$$

$$OPT(4, \emptyset) = 8$$

Now, we just use these to move up the recursion tree, resulting in

$$OPT(2, \{3\}) = OPT(3, \emptyset) + c(2, 3) = 6 + 9 = 15$$

$$OPT(2, \{4\}) = OPT(4, \emptyset) + c(2, 4) = 8 + 10 = 18$$

$$OPT(3, \{2\}) = OPT(2, \emptyset) + c(3, 2) = 5 + 13 = 18$$

$$OPT(3, \{4\}) = OPT(4, \emptyset) + c(3, 4) = 8 + 12 = 20$$

$$OPT(4, \{2\}) = OPT(2, \emptyset) + c(4, 2) = 5 + 8 = 13$$

$$OPT(4,\{3\}) = OPT(3,\emptyset) + c(4,3) = 6 + 9 = 15.$$

From now on, we actually need to use the min-function, as we have multiple paths that we can take.

$$OPT(2,\{3,4\}) = \min\{OPT(3,\{4\} + c(2,3), OPT(4,\{3\} + c(2,4)\} = \min\{20+9, 15+10\} = 25$$

$$OPT(3,\{2,4\}) = \min\{OPT(2,\{4\} + c(3,2), OPT(4,\{2\} + c(3,4)\} = \min\{18+13, 13+12\} = 25$$

$$OPT(4,\{2,3\}) = \min\{OPT(2,\{3\} + c(4,2), OPT(3,\{2\} + c(4,3)\} = \min\{15+8, 18+9\} = 23$$

Lastly, we have

$$OPT(1,\{2,3,4\}) = \min\{OPT(2,\{4,3\} + c(1,2), OPT(3,\{2,4\}) + c(1,3), OPT(4,\{2,3\} + c(1,4)\}$$

$$= \min\{25+10, 25+15, 23+20\} = 35.$$

Now, we need to find the path that actually leads to this. To do so, we just simply follow the elements, that meet the min's. By doing so we find the bath $1, 2, 4, 3$.

Approximation Algorithms

Disposition

1. Introduction
2. Definition of the *approximation ratio*, a $\rho(n)$*-approximation algorithm* and a *randomized* $\rho(n)$*-approximation algorithm*.
3. The Vertex-cover problem
   (a) Introduction
   (b) Proof that `APPROX-VERTEX-COVER` is a 2-approximation algorithm
4. MAX-3-CNF
   (a) Introduction
   (b) Proof that the randomized algorithm for `MAX-3-CNF` is a randomized 8/7-approximation algorithm

Presentation

Definition of *approximation ratio*
We say that an algorithm for a problem has an ***approximation ratio*** of $\rho(n)$ if, for any input of size n, the cost C of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

Definition of $\rho(n)$-*approximation algorithm*
If an algorithm achieves an approximation ratio of $\rho(n)$, we call it a $\rho(n)$-***approximation algorithm***.

Definition of *randomized* $\rho(n)$-*approximation algorithm*
We say that a randomized algorithm for a problem has an ***approximation ratio*** of $\rho(n)$ if, for any input of size n, the expected cost C of the solution procuded by the randomized algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

We call a randomized algorithm that achieves an approximation ratio of $\rho(n)$ a ***randomized*** $\rho(n)$-***approximation algorithm***

Introduction to *vertex cover*
A ***vertex cover*** of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if (u, v) is an edge of G, then either $u \in V'$ or $v \in V'$ (or both). The size of a vertex cover is the number of vertices in it. The ***vertex-cover problem*** is to find a vertex cover of minimum size in a given undirected graph. We call such a vertex cover an ***optimal vertex cover***.

The set C of vertices that is returned by `APPROX-VERTEX-COVER` is a vertex cover, since the laogrithm loops until every edge in $G.E$ has been covered by some vertex in C.

Algorithm 1 APPROX-VERTEX-COVER

```
Require: Undirected graph $G$
1: $C = \emptyset$
2: $E' = G.E$
3: while $E' \neq \emptyset$ do
4:     let $(u, v)$ be an arbitrary edge of $E'$
5:     $C = C \cup \{u, v\}$
6:     remove from $E'$ edge $(u, v)$ and every edge incident on either $u$ or $v$
7: return $C$
```

Proof that `APPROX-VERTEX-COVER` is a 2-approximation algorithm
Let A denote the set of edges that line 4 picked. Not two edges in A share an endpoint. Thus no two edges in A are covered by the same vertex from an optimal cover C^*, and we have the lower bound

$$|C^*| \geq |A| \tag{1}$$

on the size of an optimal vertex cover. Since A consists of the edges between two vertices in C (and since all of the elements in C are unique), we have the (exact) upper bound on the size of the vertex cover returned

$$|C| = 2|A| \tag{2}$$

Combining equation (1) and (2), we obtain

$$|C| = 2|A| \leq 2|C^*|$$

Polygon Triangulation

Disposition

1. Introduction
2. The 3-coloring approach
    (a) Example on running the algorithm
    (b) Proving that the 3-coloring approach is optimal in worst case
3. Example on partitioning a polygon into monotone pieces + runtime analysis
4. Example on triangulating a monotone polygon + runtime analysis

Presentation