Advanced Algorithms and Datastructures - Exam Notes

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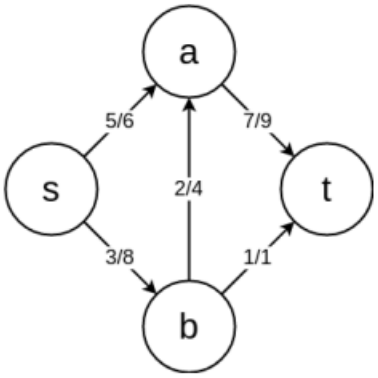
Max Flow

Disposition

1. Introduction
2. Introduction to
    - *Flow network*
    - *Residual network*
    - *Ford-Fulkerson + Edmonds-Karp*
3. Example of running the Edmonds-Karp Algorithm
4. Proof of the *Max-flow min-cut theorem*



Presentation

Hey guys. I will be talking about Max flow. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First I will be giving a brief introduction to the topic, then I will be running the Edmonds-Karp algorithm on the example at the bottom of the page. Lastly, I will be proving the so-called Max-Flow min-cut theorem.

In Max-flow we are given a ***flow network***, which is a directed graph where each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v) \geq 0$. We require that for each edge, the antiparallel edge does not exist. We distinguish two vertices; a ***source*** s and a ***sink*** t.

A ***flow*** in G is a function $f : V \times V \to \mathbb{R}$ that satisfies the following two properties:

1. ***Capacity constraint***: For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$
2. ***Flow conservation***; For all $u \in v - \{s, t\}$ we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v).$$

The goal is to maximize the value $|f|$ defined by

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

without violating any of the constraints.

To solve this problem we use a ***residual network***, which keeps track of where it is possible to add flow. This is induced by applying

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise.} \end{cases}$$

to each of the edges in G.

To solve this problem one can make use of the Ford-Fulkeson method. It iteratively increases the value of the flow. We start with $f(u, v) = 0$ for all $u, v \in V$. At each iteration, we increase the flow value in G by finding an augmenting path in an associated ***residual network*** G_f. Once we know the edges of an augmenting path p in G_f, we can define the ***residual capacity*** of the path p by

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

and the flow $f_p : V \times V \to \mathbb{R}$ in G_f by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \in p \\ 0 & \text{otherwise.} \end{cases}$$

We can then use this to augment the flow f by f_p to get closer to maximum, by

$$(f \uparrow f_p)(u, v) = \begin{cases} f(u, v) + f_p(u, v) - f_p(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We repeatedly change the flow until the residual network has no more augmenting paths. The max-flow min-cut theorem tells us, that upon termination, this process yields a maximum flow.

One implmentation of Ford-Fulkerson is the ***Edmonds-Karp Algorithm***, which in each iteration finds the augmenting path by using breadth-first search.

I will now run the algorithm on the example *Run Edmonds-Karp*... and since there are no augmenting paths, max-flow min-cut theorem tells us, that f is a maximum flow in G.

Now that I have shown how the Edmonds-Karp algorithm works, I will now prove the Max-Flow min-cut theorem. First, let's recap what the theorem tells us. The theorem says, that if f is a flow in a flow network $G = (V, E)$ with source s and a sink t, then the following cases are equivalent.

1. f is a maximum flow in G
2. The residual network G_f contains no augmenting paths
3. $|f| = c(s, T)$ for some cut (S, T) of G.

We can now prove the max-flow min-cut theorem. The proof works by proving that the cases imply each other. We start of by proving that the first case implies the second case:

> Suppose that f is a maximum flow in G but that G_f has an augmenting path p. Then the flow found by augmenting f by f_p is a flow in G with value strictly greater than $|f|$, contradicting the assumption that f is a maximum flow.

Now, let's prove that the second case implies the third case:

> Suppose that G_f has no augmenting path. Define
>
> $$S = \{v \in V : \text{ there exists a path from } s \text{ to } v \text{ in } G_f\}$$
>
> and $T = V - S$. The partition (S, T) is then a cut. Now, consider a pair of vertices $u \in S$ and $v \in T$. If $(u, v) \in E$, we must have $f(u, v) = c(u, v)$. If $(v, u) \in E$, we must have $f(v, u) = 0$. If neither (u, v) nor (v, u) is in E, then $f(u, v) = f(v, u) = 0$. We thus have
>
> $$\begin{aligned} f(S,T) &= \sum_{u \in S} \sum_{v \in t} f(u,v) - \sum_{u \in S} \sum_{v \in t} f(v,u) \\ &= \sum_{u \in S} \sum_{v \in T} c(u,v) - \sum_{u \in S} \sum_{v \in T} 0 \\ &= c(S,T). \end{aligned}$$
>
> From a lemma we have that $|f| = f(S, T)$. Thus, we have $|f| = f(S, T) = c(S, T)$.

Lastly, let's prove that the third case implies the first case:

> We have that $|f| \leq c(S, T)$ for all cuts (S, T). The condition $|f| = c(S, T)$ thus implies that f is a maximum flow.

Extra

A cut (S, T) is a partition of V into S and $T = V - S$, such that $s \in S$ and $t \in T$. The capacity of the cut $c(S, T)$ is then just simply the sum of the capacities of the edges going across the cut from S to T

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v),$$

and the ***net flow*** $f(S, T)$ across the cut is sum of the flows of the edges going across the cut from S to T minus the sum of the flows of the edges going across the cut from T to S

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u).$$

We also have, that $f(S,T) = |f|$.

Let $G = (V,E)$ be a flow network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the function $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.

Let $G = (V,E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Let f_p be defined as in equation (**??**), and suppose that we augment f by f_p. then the function $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$.

Questions they can ask

Why do we require that antiparallel edges do not exist?

I think it is because, that we in the residual network have antiparallel edges. If we then also had antiparallel edges in the flow network, something would probably screw up.

How do we handle antiparallel edges?

We split one of the edges up into two edges and introduce a new vertex, which connects the two new edges. The capacity of the two new edges is the same as the capacity of the original edge.

Why do we have flow going in the wrong direction in the residual network?

Sending flow back along an edge is equivalent to decreasing the flow on the edge, which is a necessary operation in many algorithms

Does the algorithm always terminate?

No. The Ford-Fulkerson method might fail if edge capacities are irrational numbers.

What is the running time of the algorithm?

This depends on how we find the augmenting path. If we assume that the capacities are integers, then Ford-Fulkerson has a running time of $O(E \cdot |f^*|)$, where f^* is a max flow. This is because the capacities are integers, we know Ford-Fulkerson increases the value by atleast 1 at each step, making the runtime be bounded by the max flow ($|f^*|$) and E is used for finding an augmenting path.

We can improve the bound on Ford-Fulkerson by finding the augmenting path with a breadth-first search. This has arunning time of $O(VE^2)$.

Linear Programming and Optimization

Program

1. Introduction
2. Introduction to
   - Various forms of LP
   - *Objective function*
   - *Constraints*
   - Algorithm for solving linear programming
3. Preparing and running SIMPLEX on example
4. Duality:
   - Motivation behind duality
   - Definition of *dual linear program*
   - Definition of *weak duality*
   - Proof of weak duality

$$\begin{array}{ll} \text{minimize} & -x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 = 6 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{array}$$

Presentation

Hey guys. I will be talking about Linear Programming. I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own program*. First I will be giving a brief introduction to the topic, then I will transform an example into standard form, which will then be transformed into slack form, which I then will run the SIMPLEX-algorithm on. Lastly, I will pove *weak duality*, which says, that the solution to the dual of a linear program is always an upper bound on the solution to the orignial linear program.

Linear programming is given in two forms; ***standard form*** and ***slack form***.

standard form has the following form

$$\sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, ..., m$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, ..., n$$

The first equation is the ***objective function***, which we wish to optimize with respect to the ***constraints***, which are the following lines, where the last constraint is required in standard form.

Slack form is very similar to standard form, however, only the nonnegativity constraints are the only inequality constraints, and the remaining constraints are equalities.

To solve a linear program we use the simplex algorithm, whose input is a linear program in ***slack form***.

Let's transform the example into standard and slack form *Write the example on the blackboard*. First off, we see that the example is a minimization problem instead of a maximization problem. This can easily be fixed by negating the coefficients in the objective function. Thus, by doing so we obtain the new objective function

$$x_1 + x_2.$$

Since the two linear programs have identical sets of feasible solutions and, for any feasible solution, the objective value the first linear program is the negative of the objective value in the second program, the two linear programs are equivalent.

Next, we see that x_2 does not have a nonnegativity constraint. We can fix this by replacing each occurence of x_2 by $x_2' - x_2''$ and add the nonnegativity constraint $x_2', x_2'' \geq 0$. To ease the readibility, I will instead let $x_2' = x_2$ and $x_2'' = x_3$. Thus, by doing so we receive the objective function $x_1 + 2x_2 - 2x_3$ the two constraints

$$x_1 + x_2 - x_3 = 6$$

and

$$x_1 - 2x_2 + 2x_3 \leq 4$$

and the nonnegativity constraint

$$x_1, x_2, x_3 \geq 0.$$

Any feasible solution $\hat{x}$ to the new linear program corresponds to a feasible solution $\bar{x}$ to the original linear program with $\hat{x}_j = \hat{x}_j' - \hat{x}_j''$ and with the same objective value. Also, any feasible solution $\hat{x}$ to the original linear program corresponds to a feasible solution $\hat{x}$ to the new linear program with $\hat{x}_j' = \bar{x}_j$ and $\hat{x}_j'' = 0$ if $\hat{x}_j \geq 0$, or with $\hat{x}_j'' = -\bar{x}_j$ and $\hat{x}_j' = 0$ if $\bar{x}_j < 0$. Thus, the two linear programs are equivalent.

We also see, that the first constraint has an equal sign instead of an $\geq$. We know that the equality only

holds if and only if both $\geq$ holds and $\leq$ holds. Thus, we can replace the equality constraint by the pair of inequality constraints that uses $\leq$ and $\geq$ instead. Thus, we replace the constraint

$$x_1 + x_2 - x_3 = 6$$

with the two constraints

$$x_1 + x_2 - x_3 \leq 6$$

and

$$x_1 + x_2 - x_3 \geq 6$$

Lastly, we see, that the constraint

$$x_1 + x_2 - x_3 \geq 6$$

has a greater-than-or-equal-to-sign instead of a less-than-or-equal-to-sign. This can easily be fixed by multiplying the constraint through by -1. Thus, we instead obtain

$$-x_1 - x_2 + x_3 \geq -6$$

In total we have

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 - 2x_3 \\ \text{subject to} & x_1 + x_2 - x_3 \leq 6 \\ & -x_1 - x_2 + x_3 \geq -6 \\ & x_1 - 2x_2 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The linear program is now in standard form. Let transform this further into slack form. This is rather easily done by introducing a slack variable for each constraint, which measures the difference in the left and right hand side. This is done by subtracting the left hand side on both sides and introducing a new nonnegativity constraint for each constraint. By doing so we get the linear programming

$$\begin{aligned} z &= x_1 + 2x_2 - 2x_3 \\ x_4 &= 6 - x_1 - x_2 + x_3 \\ x_5 &= -6 + x_1 + x_2 - x_3 \\ x_6 &= 4 - x_1 + 2x_2 - x_3 \end{aligned}$$

where we have omitted the nonnegativity constraints.
Thus, the linear program is now in slack form and we can perform the SIMPLEX algorithm. The simplex algorithm works by continuously changing the solution by pivoting variables to and from the basic variables, which are the variables on the left hand side. This is done by picking a variable in the objective function which positively increases the value, and pivoting it with the basic variable that bottlenecks how much the non-basic variable can be increased. Let's run the SIMPLEX algorithm. We have

$$\begin{aligned} z &= x_1 + 2x_2 - 2x_3 \\ x_4 &= 6 - x_1 - x_2 + x_3 \\ x_5 &= -6 + x_1 + x_2 - x_3 \\ x_6 &= 4 - x_1 + 2x_2 - x_3 \end{aligned}$$

We see, that x_1 and x_2 are the only variables that can increase the value of the objective function. We choose x_2 as the pivot element (**DETTE ER VIGTIGT FOR AT FÅ ALGORTIMEN TIL AT TERMINERE EFTER 1 ITERATION - ALT ANDET VIL FÅ DEN TIL AT KØRE LÆNGERE**). Next, we isolate x_2 in x_4, since x_4 is the only constraint with a negative coefficient for x_2. The value of x_2 is then inserted in the other constraints as well as the objective function, resulting in

$$\begin{aligned} z &= 12 - x_1 - 2x_4 \\ x_2 &= 6 - x_1 + x_3 - x_4 \\ x_5 &= -x_4 \\ x_6 &= 16 - 3x_1. \end{aligned}$$

We now stop, as no basic variable appear in the objective function with a positive coefficient. Our solution is this $(0, 6, 0, 0, 0, 16)$, or $(x_1, x_2) = (0, 6)$ for the original problem, with a value of 12. Now that we have run the algorithm, I will be proving *weak duality*. Weak duality is a weak version of something called *duality*, which is used to prove that a solution is optimal. Given a linear program in standard form we define the dual linear program as

$$\begin{aligned}
&\text{minimize} \\
&\qquad \sum_{i=1}^{m} b_i y_i \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{m} a_{ij} y_i \geq c_j && \text{for } j = 1, 2, ..., n \\
&\qquad y_i \geq 0 && \text{for } i = 1, 2, ..., m.
\end{aligned}$$

In weak duality we let $\bar{x}$ be a feasible solution to a primal, that is the "original", linear program in standard form and let $\bar{y}$ be any feasible solution to the corresponding linear program. Then, we have

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i.$$

because

$$\begin{aligned}
\sum_{j=1}^{n} c_j \bar{x}_j &\leq \sum_{j=1}^{m} \left(\sum_{i=1}^{m} a_{ij} \bar{y}_i \right) \bar{x}_j && \text{(Using } \sum_{i=1}^{m} a_{ij} y_i \geq c_j\text{)} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \bar{x}_j \right) \bar{y}_i \\
&\leq \sum_{i=1}^{m} b_i \bar{y}_i && \text{(Using } \sum_{j=1}^{n} a_{ij} x_j \leq b_i\text{)}
\end{aligned}$$

Extras

We say that two maximization linear programs L and L' are ***equivalent*** if for each feasible solution $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}'$ to L' with objective value z, and for each feasible solution $\bar{x}'$ to L' with objective value z, there is a corresponding feasible solution $\bar{x}$ to L with objective value z. A minimization linear program L and a maximization linear program L' are equivalent if for each feasible soltuion $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}'$ to L' with objective value $-z$, and for each feasible solution $\bar{x}'$ to L' with objective value z, there is a corresponding feasible solution $\bar{x}$ to L with objective value $-z$.

Corollary 29.9

Let $\bar{x}$be a feasible solution to aprimal linear program, and let $\bar{y}$ be a feasible solution to the corresponding dual linear program. If

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i,$$

then $\bar{x}$ and $\bar{y}$ are optimal solutions to the primal and dual linear programs, respectively.
Proof By the lemma of weak duality, the objective value of a feasible solution to the primal cannot exceed that of a feasible solution to the dual. The primal linear program is a maximization problem and the dual is a minimization problem. Thus, if feasible solutions $\bar{x}$ and $\bar{y}$ have the same objective value, neither can be improved.

Questions they can ask

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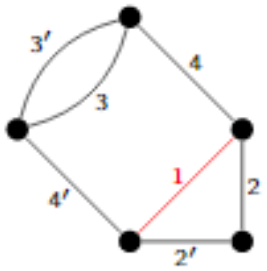
Randomized Algorithms

Disposition

1. Introduction
2. Quicksort:
   (a) Idea behind Quicksort
   (b) Example on running non-random Quicksort
   (c) Motivation behind randomness in Quicksort
   (d) Analysis of expected runtime of randomized quicksort
3. Min-cut:
   (a) Introduction to min-cut
   (b) Example on running the min-cut algorithm
4. Las Vegas Algorithms vs Monte Carlo Algorithms

$$[3, 5, 1, 2, 4]$$

NOTE: MÅSKE UDSKIFT LV vs MC MED BEVIS AF MIN-CUT's KØRETID?!?!?



Presentation

Hey guys. I will be talking about Randomized Algorithms. I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowd to *Show own program*. First, I will be giving a brief introduction to the topic. Then I will be talking about quicksort. Lastly, I will be giving a brief introduction to Las Vegas algorithms and Monte Carlo algorithms.

Quicksort works by recursively picking a pivot element and then group all elements smaller than the pivot element in one set and all of the elements greater than the pivot element in another set, making the pivot element be in its sorted index. Then the algorithm is performed on the two sets recursively.

Consider running the algorithm on the provided example. If we were to always pick the "median element" as the pivot element, this would result in a balanced search tree, so we might expect the number of comparisons to be something like $n \log n$.

However, if we instead were to pick an extreme element as the pivot, this would result in an extremely unbalanced binary search tree, so we might expect the number of comparisons to be someting like n^2.

Picking the pivot element thus have a great effect on the running time of the algorithm. For this reason, we can instead choose the pivot element uniformly at random. By doing so we can expect the number of comparisons to be $O(n \log n)$, which I will prove now.

We start off by letting $S_{(i)}$ denote the ith smallest element in the sorted array S, for $1 \leq i \leq n$. Then, we define

$$X_{ij} = \begin{cases} 1 & \text{if } S_{(i)} \text{ and } S_{(j)} \text{ are compared in an execution} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the total number of comparisons is

$$\sum_{i<j} X_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}.$$

Then, we are interested in the expected number of comparisons

$$\mathbb{E}\left[\sum_{i<j} X_{ij}\right] = \sum_{i<j} \mathbb{E}\left[X_{ij}\right].$$

Let p_{ij} denote the probability that $S_{(i)}$ and $S_{(j)}$ are compared in an execution. Since X_{ij} only assumes the values 0 and 1 we have

$$\mathbb{E}[X_{ij}] = p_{ij} \cdot 1 + (1 - p_{ij}) \cdot 0 = p_{ij}.$$

Each recursive call returns some sublist $[S_{(a)}, ..., S_{(b)}]$. Let $x = S_{(c)}$ be the pivot. Suppose $a \leq i < j \leq b$. If $c < i$ or $c > j$, then $S_{(i)}$ and $S_{(j)}$ are not compared now, but are in the same subtree and might be compared later. If $i < c < j$, then $S_{(i)}$ and $S_{(j)}$ are never compared. If $c = i$ or $c = j$ then $S_{(i)}$ and $S_{(j)}$ are compared once, right now. Thus, $S_{(i)}$ and $S_{(j)}$ are compared iff $S_{(i)}$ or $S_{(j)}$ is first of $S_{(s)}, ..., S_{(j)}$ to be chosen as pivot. Hence why p_{ij} is the conditional probability of picking $S_{(i)}$ or $S_{(j)}$ given that the pivot is picked uniformly at random in $S_{(i)}, S_{(i+1)}, ..., S_{(j)}$

$$p_{ij} = \mathbb{P}\left[c \in \{i, j\} | c \in \{i, i+1, ..., j\}\right] = \frac{2}{|\{i, i+1, ..., j\}|} = \frac{2}{j - i + 1}.$$

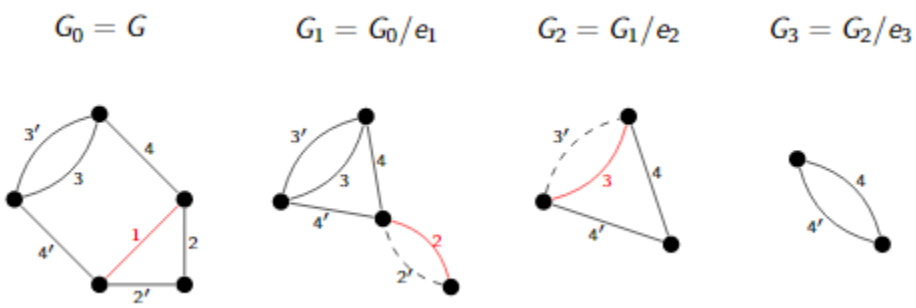
We thus have

$$
\begin{aligned}
\mathbb{E}\left[\sum_{i<j} X_{ij}\right] &= \sum_{i<j} \mathbb{E}\left[X_{ij}\right] && \text{(Linearity of expectation)} \\
&= \sum_{i<j} p_{ij} \\
&= \sum_{i<j} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} && \left(\text{Using } \sum_{i<j} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n}\right) \\
&= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k} \\
&< \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{2}{k} && \text{(Adding more positive terms increases total value)} \\
&= 2n \sum_{k=2}^{n} \frac{1}{k} \\
&= 2n \left(\left(\sum_{k=1}^{n} \frac{1}{k}\right) - 1\right) \\
&= 2n(H_n - 1) && \left(\text{Using } H_n = \sum_{k=1}^{n} \frac{1}{k}\right) \\
&\leq 2n \int_1^n \frac{1}{x} dx = 2n \ln n = O(n \log n) && \left(\text{Using } H_n - 1 \leq \int_1^n \frac{1}{x} dx = \ln n\right)
\end{aligned}
$$

Now, onto min-cut. In min-cut we G be a connected, undirected graph with n vertices. A *cut* in G is a set of edges whose removalresults in G being broken into two or more components. The goal is to find a cut with minimal cardinality.

The algorithm works by repeatedly picking an edge uniformly at random and merging the two vertices at its end-points. Edges between vertices that are merged are removed, so that there are never any self-loops.

By running this algorithm on the example we potentially get the following Thus, the edges 4 and $4'$ is



a min-cut.

Lastly, let us talk about Las Vegas and Monte Carlo algorithms. A Las Vegas algorithm is a randomized algorithm that always return the correct output, and the randomness lies in its running time. A Monte

Carlo algorithm on the other hand is a randomized algorithm that does not necessarily return a correct output, however, we can run it multiple times to get a correct output.

Extras

Lower-bound of finding min-cut

The min-cut algorithm has a probability of $\frac{2}{n(n-1)}$ of finding a specific min-cut.
For any vertex v in a multigraph G, the *neighborhood* of v, denoted $\Gamma(v)$, is the set of vertices of G that are adjacent to v. The *degree* of v, denoted $d(v)$, is the number of edges incident on v. For a set S of vertices of G, the neighborhood of S, denoted $\Gamma(S)$, is the union of the neighborhoods of the constituent vertices.

Let k be the min-cut size and $G_i = (V_i, E_i)$ be the graph G after i iterations, which has $n_i = n - i$ vertices. We fix our attention on a particular min-cut C with k edges. Then, G_i has at least $n_i|C|/2$ edges because

$$\frac{1}{2}\sum_{v\in V_i} d_i(v) \geq \frac{1}{2}n_i|C| \quad \text{(Using } d_i(v) \geq |C|\text{)}.$$

We will bound from below the probability that no edge of C is ever contracted during an execution of the algorithm.

Let ϵ_i denote the event of not picking an edge of C at the ith step, for $1 \leq i \leq n-2$. Let $p_i = \mathbb{P}\left[\epsilon_i | \cap_{j=i}^{i-1} \epsilon_j\right]$. The probability of picking an edge of C in the ith iteration, given that no edge of C has been picked in a previous iteration, is thus

$$\begin{aligned} 1 - p_i &= \mathbb{P}\left[\epsilon_i | \cap_{j=i}^{i-1} \epsilon_j\right] \\ &= \mathbb{P}\left[e \in E_{i-1} \text{ is in } C | \cap_{j=1}^{i-1} \epsilon_j\right] \\ &= \frac{|C|}{|E_{i-1}|} \leq \frac{|C|}{\frac{1}{2}n_{i-1}|C|} \qquad = \frac{2}{n_{i-1}} = \frac{2}{n-(i-1)} p_i \geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}. \quad \Rightarrow \end{aligned}$$

Thus, the probability of finding C can then by found by

$$\begin{aligned} \mathbb{P}\left[C \text{is found}\right] &= \prod_{i=1}^{n-2} p_i \\ &\geq \prod_{i=1}^{n-2} \frac{n-1-i}{n+1-i} \\ &= \frac{n-2}{n}\frac{n-3}{n-1}\frac{n-4}{n-2}\cdots\frac{3}{5}\frac{2}{4}\frac{1}{3} \\ &= \frac{2}{n(n-1)}. \end{aligned}$$

Suppose we were to repeat the min-cut algorithm $n^2/2$ times, making independent random choices each time. The probability that a min-cut is not found in any of the $n^2/2$ attempts is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2/2} < \frac{1}{e}.$$

Questions they can ask

What is the probability of the min-cut algorithm succeeding?

The probability of finding a specific min-cut is greater than or equal to

$$\frac{2}{n(n-1)}.$$

If we were to repeat the algorithm x amount of times, which probability would we get for a specific x?

Suppose we were to repeat the min-cut algorithm $n^2/2$ times, making independent random choices each time. The probability that a min-cut is not found in any of the $n^2/2$ attempts is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2/2} < \frac{1}{e}.$$

Hashing

Disposition

1. Introduction
2. Introduction to various terms:
   (a) Random hash functions
   (b) *(Strongly) universal hashing*
   (c) *C-approximately (strongly) universal hashing*
   (d) *Multiply-mod-prime*
   (e) *Multiply-shift*
3. Hash tables with chaining:
   (a) Introduction
   (b) Proof of expected number of elements in $L[h(x)]$
4. Signatures:
   (a) Introduction
   (b) Proof of upperbound on probability of collision

Presentation

Hey guys. I will be talking about Hashing. I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowd to *Show own program*. First, I will be giving a brief introduction to the topic. Then I will be talking about hash tables with chaining. Lastly, I will be talking about signatures using hashing.

In hashing we have a universe U of keys, that we wish to map randomly to a range $[m] = \{0, ..., m-1\}$ of hash values. A random hash function $h : U \to [m]$ is thus a randomly chosen function that maps from to $U \to [m]$.

We then have, that a random hash function $h : U \to [m]$ is *universal* if, for all $x \neq y \in U$, we have $\mathbb{P}[h(x) = h(y)] \leq \frac{1}{m}$. It is *strongly universal* if for all $x \neq y$ and $q, r \in [m]$ we have $\mathbb{P}[h(x) = q \wedge h(y) = r] = \frac{1}{m^2}$.

Similarly, a random hash function $h : U \to [m]$ is *c-approximately universal* if for all $x \neq y \in U$, we have $\mathbb{P}[h(x) = h(y)] \leq \frac{c}{m}$. It is *c-approximately strongly universal*, if for all $x \neq y \in U$ and $q, r \in [m]$, we have $\mathbb{P}[h(x) = q \wedge h(y) = r] \leq \frac{c^2}{m^2}$.

Two common random hash function are *Multiply-mod-prime* and *Multiply-shift*. In Multiply-mod-prime, we let $U = [u]$ and $m < u$, pick a prime $p \geq u$ and choose $a, b \in [p]$ independently and uniformly at random. Then we have the 2-approximately strongly universal hash function

$$h^m_{a,b}(x) = ((ax + b) \bmod p) \bmod m.$$

For Multiply-shift we let $U = [2^w]$, $m = 2^l$ and choose $a \in [2^w]$ uniformly at random. Then we have the 2-approximately universal hash function

$$h_a(x) = \left\lfloor \frac{(ax) \bmod 2^w}{2^{2-l}} \right\rfloor.$$

In hash tables with chaining, we have a set $S \subseteq U$ of keys that we wish to store so that we can expect to find any key from S in constant time.

We start off by letting $n = |S|$ and $m \geq n$. We then pick a universal hash function $h : U \to [m]$, and create an array L of m lists, so that for $i \in [m]$, $L[i]$ is the list of keyts that hash to i. To find out if a key $x \in U$ is in S, we only have to check if x is in the list $L[h(x)]$, which can be done in time proportional to $1 + |L[h(x)]|$, where the 1 stands for the constant time it takes to look up the list even if it turns out to be empty. We expect that the length of $L[h(x)]$ is less than or equal to 1, which I will prove now.

To prove this, we assume that $x \notin S$, since we want the case where we look through all of $L[h(x)]$ and not just stop once we find the element in L. Then we wish to prove that

$$\mathbb{E}[|L[h(x)]|] \leq 1.$$

We then have

$$
\begin{aligned}
\mathbb{E}[|L[h(x)]|] &= \mathbb{E}\left[|\{y \in S | h(y) = h(x)\}|\right] && \text{by definition we have } L[i] = \{y \in S | h(y) = i\} \\
&= \mathbb{E}\left[\sum_{y \in S} [h(y) = h(x)]\right] \\
&= \sum_{y \in S} \mathbb{E}\left[[h(y) = h(x)]\right] && \text{Using linearity of expectation} \\
&= \sum_{y \in S} \mathbb{P}[h(x) = h(x)] && [h(x) = h(x)] \text{ can be distributed as a bernoulli variable} \\
&\leq |S| \frac{1}{m} && h \text{ is a universal hash function, so we have } \mathbb{P}[h(x) = h(y)] \leq \frac{1}{m}. \\
&= \frac{n}{m} && \text{we use } |S| = n \\
&\leq 1 && \text{we use } m > n
\end{aligned}
$$

So we have, that we can expect to find out if a key is in S in constant time. Not onto signatures.

In the problem with signature, we wish to assign a unique signature, $s(x)$, to each $x \in S \subseteq U$. Thus, we want $s(x) \neq s(y)$ for all distinct keys $x, y \in S$. To do this, we pick a universal hash function $s : U \to [n^3]$. The probability of an error is thus

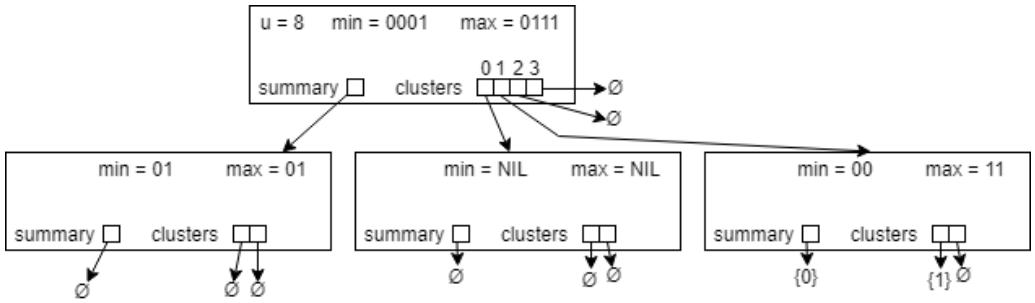
$$
\begin{aligned}
\mathbb{P}[\exists \{x, y\} \subseteq S : s(x) = s(y)] &\leq \sum_{\{x,y\} \subseteq S} \mathbb{P}[s(x) = s(y)] \\
&\leq \frac{\binom{n}{2}}{n^3} < \frac{1}{2n}.
\end{aligned}
$$

Where, we in the first inequality use union bound: the probability that at least one of multiple events happen is at most the sum of their probabilities. Thus, the probability of an error is upperbounded by $\frac{1}{2n}$.

Van Emde Boas Trees

Disposition

- Introduction
- Introduction to the structure of van Emde Boas Trees
- Example on running `Member`(4), `Insert`(6) and `Predecessor`(4) on the example.
- Proof of Running-time



$$\{1, 4, 5, 7\} = \{0001_2, 0100_2, 0101_2, 0111_2\}$$

Presentation

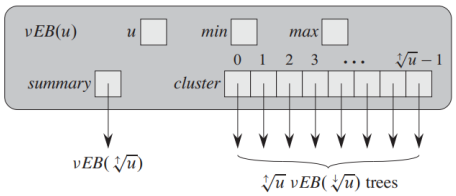
Hey guys. I will be talking about van Emde Boas Trees . I have here a program of the things I will go through *Hand out program*. As you can see, mine is identical to yours and it does not contain any information that it is not allowd to *Show own program*. First, I will be giving a brief introduction to the topic. Then I will be talking about the structure of the tree. This is followed by me running various procedures on the data structure. Lastly, I will be giving a proof of the running time of the various procedures.

Van Emde Boas Trees store integers in the range $[0, u-1]$, where u is an exact power of 2, without any duplicates.

The van Emde Boas Tree is a recursive data structure. Each node with $u > 2$, has the following structure

1. A field u denoting the universe size u
2. A field min that stores the minimum element in the vEB tree
3. A field max that stores the maximum element in the vEB tree
4. A field $Summary$ that points to a summary node, that shows shows whether other clusters contains any elements.
5. An array $cluster$ of pointers that points to sub-vEB trees, each of size $\sqrt[\downarrow]{u}$.

If $u = 2$, then it only has u, min and max as its fields.



For vEB trees we have the following equations which are important for the sturcutre

$$\sqrt[\uparrow]{u} = 2^{\lceil (\lg u)/2 \rceil}$$

and

$$\sqrt[\downarrow]{u} = 2^{\lfloor (\lg n)/2 \rfloor}.$$

Then we define the following functions

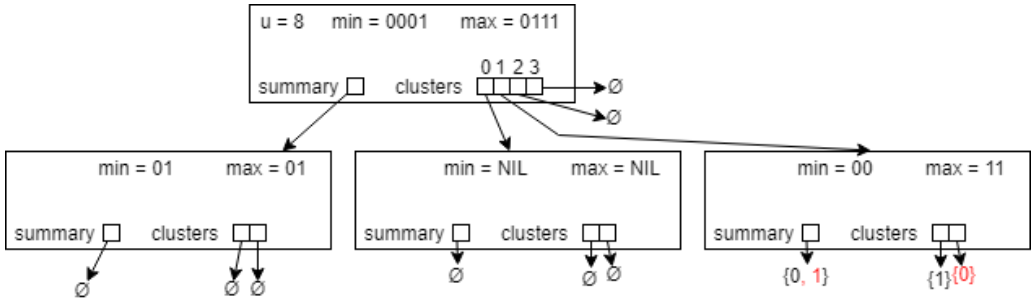
1. $high(x) = \left\lfloor \frac{x}{\sqrt[\downarrow]{u}} \right\rfloor$: The high half of x's bits
2. $low(x) = x \bmod \sqrt[\downarrow]{u}$: The low half of x's bits
3. $index(x, y) = x\sqrt[\downarrow]{u} + y$: The value of element at cluster x index y.

We then have, that each element x is stored in its corresponding cluster $high(x)$, except for elements that appear in min and max - this varies from implementation to implementation though, as some only excludes min or max, however, I have chosen to exclude both, as this was done in our lecture.

Now that we have introduced the structure of the tree, lets run some procedures on the example *Draw example on blackboard*. Let's start by calling `Member`(5), which simply finds out whether 5 is in the tree. First off, we have, that $5 = 0101_2$, hence why $high(5) = 01_2$ and $low(5) = 01_2$. Due to $high(5) = 01_2 = 1$, we look in cluster 1 for element $low(5) = 01_5$. We see that this does not appear in the min nor max, so we

find $high(01_5) = 0$ and $low(01_5) = 1$. We then look in cluster 0 for element 1, which appears, so 5 is in the tree and the algorithm returns `TRUE`.

Now, lets insert $6 = 0110_2$ into the tree. When doing insert we assume that the element to insert is not already in the tree. First we see, that the tree is not empty, because min and max are not equal to NIL. Then, we see, that $min < 0110_2 < max$, so we shall not swap 0110_2 with min nor max. Then we have $high(6) = 01_2$ and $low(6) = 10_2$. We then pass 10_2 to cluster $01_2 = 1$. We again have that $min < 10_2 < max$, so we should not swap 10_2 with min nor max. Then, we have $high(10_2) = 1$ and $low(10_2) = 0$. Since cluster 1 is empty we simply insert element 0 and update the summary such that it contains 1 aswell, since cluster 1 now contains an element, resulting in the tree Now, lets find the predecessor of $4 = 0100_2$, which is the great-



est element smaller than 4, which is stored in the tree. We have $high(x) = 01_2 = 1$ and $low(x) = 00_2 = 0$. We then check if cluster 1 is not empty and whether the minimum of cluster 1 is less than 4 (which would imply that the predecessor is in the cluster), which is false. We then check if the the summary is empty or if 1 is less-than-or-equal to the minimum of the summary, which is true, since we have $01_2 \leq 01_2$. We then return the minimum of the summary, $01_2 = 1$.

Now that we have performed the procedures on the example, lets find out what the worst case runtime of the procedures is. The recursive procedures that implement the operations all have running time characterized by the recurrence

$$T(u) \leq T(\sqrt[\uparrow]{u}) + O(1)$$

where the $O(1)$ comes from calls to $high(x)$ and $low(x)$. If we let $m = \lg u$, we can rewrite this to

$$T(2^m) \leq T(2^{\lfloor \frac{m}{2} \rfloor}) + O(1).$$

Noting that $\lfloor m/2 \rfloor \leq 2m/3$ for all $m \geq 2$, we have

$$T(2^m) \leq T(2^{2m/3}) + O(1).$$

Letting $S(m) = T(2^m)$, we can rewrite this to

$$S(m) \leq S(2m/3) + O(1)$$

which, by case 2 of the master method, has the solution $S(m) = O(\log m) = O(\log \log u)$. Since each procedure makes at most 1 recursive call, each procedure has a worst case running time of $O(\log \log u)$.

Questions they can ask

Extras

NP-Completeness

Disposition

1. Introduction
2. Introduction to various terms:
   (a) *Abstract problems*
   (b) *Encoding*
   (c) *Accepting/Rejection* and *Deciding*
   (d) *Verification*
   (e) *Reducibility*
   (f) `P`, `NP` and `NPC` (and `NP-hard`)
3. Introduction to the ham-cycle problem
4. Introduction to the traveling-salesman problem (TSP)
5. Proof of TSP being NP-complete.

Presentation

Hey guys. I will be talking about NP-Completeness. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First I will be giving this introduction, then I will be giving a brief introduction to various terms that I will be using, then I will be giving an introduction to two problems that are NP-complete, and lastly I will be proving that one of those problems is NP-complete.

An abstract problem is a binary relation on a set I of a problem ***instance*** (or input) and a set of problem ***solutions*** - (each instance can have multiple solutions - think of a graph with multiple shortest paths). In this topic we are only concerned with ***deicsion problems***: those having a yes/no solution. Thus, we can view an abstract decision problem as a function that maps the instance set I to the solution set $\{0,1\}$.

An ***encoding*** of a set of abstract objects is a mapping to the set of binary strings. Thus, an algorithm that solves some abstract decision problem actually takes an encoding of a problem instance as input. We say that an algorithm solves a concrete problem in time $O(T(n))$ if, when it is provided a problem instance i of length $n = |i|$, the algorithm can produce the solution in $O(T(n))$ time. I will be using the notation $\langle x \rangle$ to denote the encoding of an instance x of a problem.

We say that an algorithm A ***accepts*** a string $x \in \{0,1\}^*$ if, given input x, the output of the algorithm $A(x) \in \{0,1\}$ is 1. The language ***accpeted*** by an algorithm A is thus the set of strings $L = \{x \in \{0,1\}^* : A(x) = 1\}$.

However, there might strings that never terminates, thus A neither accepts nor rejects them. For this reason we have the term ***decide***. We say that, if additional all strings not in L are rejected by A, that is $A(x) = 0$ for all $x \in \{0,1\}^* \backslash L$, then L is ***decided*** by A. We further say, that L is ***decided by*** A ***in polynomial time if*** A ***decides*** L ***and runs in polynomial time on all strings***.

We define a ***verfirication algorithm*** as being a two-argument algorithm A, where one argument is an ordinary input string x and the other is a binary string y called a ***certificate***. A two-argument algorithm A ***verifies*** an input string x if there exists a certificate y such that $A(x, y) = 1$. The ***language verified*** by a verification algorithm A is

$$L = \{x \in \{0,1\}^* : \text{ there exists } y \in \{0,1\}^* \text{ such that } A(x,y) = 1\}.$$

Now onto reducability, which is a method for proving that a problem is in a specific complexity class. We say that the language L_1 is polynomial-time ***reducible*** to language L_2 if there is a polynomial-time computible function $f : \{0,1\}^* \to \{0,1\}^*$ such that for all $x \in \{0,1\}^*$ we have

$$x \in L_1 \Leftrightarrow f(x) \in L_2.$$

In this case, we write $L_1 \leq_P L_2$. This means, that if $L_2 \in P$, then must L_1 also be in P, since we can just transform any instance of L_1 into an instance of L_2 in polynomial time and then solve this new instance with a polynomial-time algorithm for L_2.

We can now define the various complexity classes. The complexity class P is the set of concrete decision problems that are polynomial-time solvable:

$$P = \{L \subseteq \{0,1\}^* : \text{ there exists and algorithm } A \text{ that decides } L \text{ in polynomial time}\}.$$

A language L belongs to ***NP*** iff there exist a two-input polynomial-time algorithm A and a constant c such that

$$L = \{x \in \{0,1\}^* : \text{ there exists a certificate } y \text{ with } |y| = O(|x|^c) \text{ such that } A(x,y) = 1\}.$$

A language $L \subseteq \{0,1\}^*$ is ***NP-complete*** if

1. $L \in NP$, and
2. $L' \leq_p L$ for every $L' \in NP$

If a language L satisfies property 2, but not necessarily property 1, we say that L is ***NP-hard***.

Now that I have introduced the various terms, let's get onto some problems. We start off with the Ham-cycle problem. In the ham-cycle problem we are given an undirected graph $G = V, E$. The Ham-cycle problem thus asks, whether G has a hamiltonian cycle - that is, a simple cycle that contains each vertex in V, or more precisely

$$HAM - CYCLE = \{\langle G \rangle : G \text{ has a hamiltonian cycle}\}.$$

I will not be giving a proof of this, however, this problem is NP-complete.

Now onto the ***Traveling-salesman problem*** also known as ***TSP***. In TSP we are given an complete graph $G = (V, E)$ and each edge (i, j) has some cost $c(i, j) \geq 0$ assigned to it. The goal is thus to find a cycle that visits every vertex and returns back to the start vertex, which in total should be of minimal cost. The formal language for the corresponding decision problem is

$$\begin{aligned} TSP = \{\langle G, c, k \rangle : & G = (V, E) \text{ is a complete graph} \\ & c \text{ is a function from } V \times V \to \mathbb{N} \\ & k \in \mathbb{N} \\ & G \text{ has a traveling-salesman tour with cost at most } k.\} \end{aligned}$$

This problem is also NP-complete, which I will be proving now.

We first show that TSP belongs to NP. Given an instance of the problem, we use as a certificate the sequence of n vertices in the tour. The verification algorithm checks that this sequence contains each vertex exactly once, sums up the edge costs, and checks whether the sum is at most k, which all in all can be done in polynomial time.
To prove that TSP is NP-hard, we show that $HAM - CYCLE \leq_p TSP$. Let $G = (V, E)$ be an isntance of $HAM - CYCLE$. We construct an instance of TSP as follows. We form the complete graph $G' = (V, E')$, where $E' = \{(i, j) : i, j \in V \text{ and } i \neq j\}$, and we define the cost function c by

$$c(i, j) = \begin{cases} 0 & \text{if } (i, j) \in E \\ 1 & \text{if } (i, j) \notin E. \end{cases}$$

The instance of TSP is then $\langle G', c, 0 \rangle$, which we can easily create in polynomial time.
We now show that graph G has a hamiltonian cycle iff G' has a tour of cost at most $+$. Suppose that graph G has a hamiltonian cycle h. each edge in h belongs to E and thus has cost 0 in G'. Thus, h is a tour in G' with cost 0. Conversely, suppose that graph G' has a tour h' of cost at most 0. Since the costs of the edges in E' are 0 and 1, the cost of tour h' is exactly 0 and each edge on the tour must have cost 0. Therefore, h' constains only edges in E. We conclude that h' is a hamiltonian cycle in graph G.

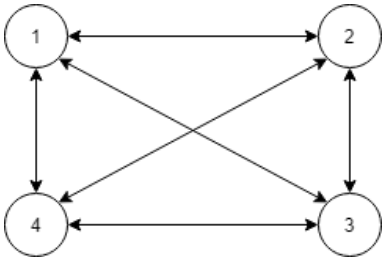
Extra

Questions they can ask

Exact Exponential Algorithms and Parameterized Complexity

Disposition

1. Introduction
2. *Exact exponential algorithms* vs *Parameterized algorithms*
3. *Travelling salesman problem*:
   (a) Introduction with example
   (b) Naive approach + running time analysis
   (c) Dynamic programming - example of running + running time analysis
4. *Bar fight prevention problem*
   (a) Introduction with example
   (b) Naive approach + running time analysis
   (c) Kernelization - example of running + running time analysis
   (d) Bounded Search tree - example of running + running time analysis
5. *Fixed Parameter Tractable* vs *Slice-wise polynomial* problems



$$c = \begin{array}{c} \begin{array}{cccc} & \text{T} & \text{o} & \\ 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{bmatrix} \end{array} \begin{array}{cc} & \\ 1 & \text{F} \\ 2 & \text{r} \\ 3 & \text{o} \\ 4 & \text{m} \end{array}$$

Presentation

Hey guys. I will be talking about Exact Exponential Algorithms and Parameterized Complexity. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First, I will be talking about exact exponential algorithms and parameterized algorithms. Then, I will be talking about the Travelling salesman problem. This is followed by the Bar fight prevention problem. Lastly, I will be talking about Fixed Parameter Tractable and Slice-wise polynomial problems.

First off, what is an *Exact exponential algorithm* and a *Parameterized algorithm*? Well, an Exact exponential algorithm is an exponential algorithm that for all instances find an exact solution. On the other hand, a Parameterized algorithm is an algorithm that for instances with some small fixed values of parameter find an exact solution in polynomial time. Lastly, there are also ***approximation algorithms*** that in polynomial time find an approximation to a solution of all instances, however, I will not be talking about those.

Now that I have introduced the algorithms, I will be talking about some examples. Lets start with the NP-Complete problem called "the Traveling salesman problem", also known as ***TSP***, which we will be solving using an exact exponential algorithm. In TSP we are given an complete graph $G = (V, E)$ and each edge (i, j) has some cost $c(i, j) \geq 0$ assigned to it. The goal is thus to find a cycle that visits every vertex and returns back to the start vertex, which in total should be of minimal cost.

How would one solve this problem? Well, the naive approach is to just generate all possible solutions and then just use the minimum as solution. This would require $n!$ steps, which of course is very bad. Instead, one can make use of dynamic programming for a faster algorithm. We start off by defining

$$OPT[S, c_i] = \text{"path of minimum length that starts in } c_1\text{, visits all of } S \text{ once, and end in } c_i$$

for all $S \subseteq \{c_2, ..., c_n\}$ and $c_i \in S$. Then

$$\min\{\text{OPT}[\{c_2, ..., c_n\}, c_i] + d(c_i, c_1) | c_i \in \{c_2, ..., c_n\}\}$$

is the length of the minimal tour. Thus, we have

$$OPT[S, c_i] = \begin{cases} d(c_1, c_i) & \text{if } S = \{c_i\} \\ \min\{OPT[S \backslash \{c_i\}, c_k] + d(c_k, c_i) | c_k \in S \backslash \{c_i\}\} & \text{if } \{c_i\} \subset S. \end{cases}$$

Now that I have introduced the problem and how to solve the problem using dynamic programming, let's have a look at the example *Write example on blackboard*. First we start at the bottom of the recursion tree. Thus, for each vertex that is not 1, we find the cost of traveling to 1. In the following $OPT(a, b)$ means, that we are currently at vertex a, and we have not visited the vertices in the set b. Thus, we have

$$OPT(2, \emptyset) = 5$$

$$Opt(3, \emptyset) = 6$$

$$OPT(4, \emptyset) = 8$$

Now, we just use these to move up the recursion tree, resulting in

$$OPT(2, \{3\}) = OPT(3, \emptyset) + c(2, 3) = 6 + 9 = 15$$

$$OPT(2, \{4\}) = OPT(4, \emptyset) + c(2, 4) = 8 + 10 = 18$$

$$OPT(3, \{2\}) = OPT(2, \emptyset) + c(3, 2) = 5 + 13 = 18$$

$$OPT(3, \{4\}) = OPT(4, \emptyset) + c(3, 4) = 8 + 12 = 20$$

$$OPT(4, \{2\}) = OPT(2, \emptyset) + c(4, 2) = 5 + 8 = 13$$

$$OPT(4, \{3\}) = OPT(3, \emptyset) + c(4, 3) = 6 + 9 = 15.$$

From now on, we actually need to use the min-function, as we have multiple paths that we can take.

$$OPT(2, \{3, 4\}) = \min\{OPT(3, \{4\} + c(2, 3), OPT(4, \{3\} + c(2, 4)\} = \min\{20 + 9, 15 + 10\} = 25$$

$$OPT(3, \{2, 4\}) = \min\{OPT(2, \{4\} + c(3, 2), OPT(4, \{2\} + c(3, 4)\} = \min\{18 + 13, 13 + 12\} = 25$$

$$OPT(4, \{2, 3\}) = \min\{OPT(2, \{3\} + c(4, 2), OPT(3, \{2\} + c(4, 3)\} = \min\{15 + 8, 18 + 9\} = 23$$

Lastly, we have

$$OPT(1, \{2, 3, 4\}) = \min\{OPT(2, \{4, 3\} + c(1, 2), OPT(3, \{2, 4\}) + c(1, 3), OPT(4, \{2, 3\} + c(1, 4)\}$$

$$= \min\{25 + 10, 25 + 15, 23 + 20\} = 35.$$

Now, we need to find the path that actually leads to this. To do so, we just simply follow the elements, that meet the min's. By doing so we find the path $(1, 2, 4, 3)$. This algorithm using dynamic programming has a runtime of $O(n^2 \cdot 2^n)$, which is far better than the previous algorithm's runtime of $O(n!)$.

Now, let's get into the ***Bar fight prevention problem***, which we will be solving using a parameterized algorithm. In the Bar fight prevention problem, a bouncer has to block at most k of n people at the door to prevent fights. The problems then asks who should be blocked such that this is possible?

We can view this problem as a graph $G = (V, E)$ of people, where each vertex $v \in V$ is a person, and each edge $(i, j) \in E$ between to people $i, j \in V$ means that person i and j will start a fight if they are at the bar at the same time. The goal is thus to figure out if we have an integer k and a ***vertex cover*** C, that is a set of vertices that inclues at least on endpoint of every edge of G, such that $|C| \leq k$. This problem is NP-complete. To solve it, one could use the naive approach of just bruce-forcing it by trying all 2^n subsets of vertices, which runs in $O(2^k)$. However, there is a smarter approach.

First off, let's let $d(v)$ for all $v \in V$ be the degree of v, let k be a counter for how many people are yet allowed to block at the door and G be a graph of all of the people we have not yet decided wether they should be blocked or not. We then use the following 3 ideas:

1. we have, that if $d(v) = 0$, then we can let v in and drop v from G, since v does not have any conflicts
2. if $d(v) \geq k + 1$, then we decreaste k by 1, remove v from G and and reject v, since if were not to do this, then we would have to reject all of v's $k + 1$, or more, neighbors instead, which would contradict our goal of rejecting at most k people.
3. If a vertice $v \in V$ only has one neighbor $w \in V \backslash v$, then we remove v, w from G, decrease K by 1, accept v and reject w. This is because we know, that w cannot have less neighbors than v and we cannot accept both v and w.

By doing this we end up with $|V| \leq k^2$, because

$$|V| = \sum_{v \in V} 1 = \frac{1}{2} \sum_{v \in V} 2 \leq \frac{1}{2} d(v) \leq k^2.$$

We can the brute-force the remaining graph by trying all $\binom{k^2}{k}$ subsets of k people.

Therefore, instead of using brute force to go through an enormous search space, we use simple observations to reduce the search space to a manageable size. This algorithmic technique, using reduction rules to decrease the size of the instance, is called *kernelization*. The general idea is to use the parameter k to quickly reduce to a smaller subproblem of the same type, whose size ideally depends only on k and not on n.

Thus, we have shown taht if there is a solution for a given k and a given graph G with n vertices and

m edges, then we can find a kernel H with at most k^2 vertices. Furthermore, such a kernel can be found in $O(m+n)$ time, and checking if a given subset of size at most k is a solution can be done in $O(k^2)$ time. Thus, for any fixed k, the total running time of this algorithm is

$$O(m+n+\binom{k^2}{k}k^2).$$

This can be improved further by using a Bounded Search Tree. The important observation here is, that for each edge $(u,v) \in E$, at least one of u, v must be rejected. The idea is then to pick an arbitrary edge (u,v), reject u and run the algorithm recursively to check whether the remaining conflicts can be resolved by rejecting at most $k-1$ guests. If this succeeds you already have a solution. If it fails, then move u back into the undecided list, move v to the reject list and run the algorithm recursively to check whether the remaining conflicts can be resolved by tejecting at most $k-$ additional guests. If this recursive call also fails to find a solution, then you can be sure that there is no way to avoid a fight by rejecting at most k guests.

This procedure results in a recursion tree of depth k, thus there is a total of 2^k recursive calls. If we start by rejecting all vertices of degree $d(v) \geq k+1$, the resulting graph has at most $|E| = \frac{1}{2}\sum_{v\in V} d(v) \leq \frac{1}{2}nk$ edges, so constructing each recursive call can be done in $O(nk)$ time. The total running time is then $O(m+nk\cdot 2^k)$.

An important feature of the Bar Fight Prevention problem is the existence of the parameter k. The problem of finding the minimum k that works is NP-complete, but for any fixed constant k we have just seen two linear-time algorithms. We say the problem is *parameterized by the parameter k*.

A parameterized problem is *Fixed Parameter Tractable (FPT)* if it has an algorithm with running time $f(k)\cdot n^c$ for some function f and some constant $c \in \mathbb{R}$

A parameterized problem is *Slice-wise Polynomial (XP)* if it has an algorithm with running time $f(k)\cdot n^{g(k)}$ for some function f, g. We have $FPT \subset XP$, because we can just let $g(k) = c$.

Approximation Algorithms

Disposition

1. Introduction
2. Motivation behind approximation algorithms
3. Definition of the *approximation ratio*, a $\rho(n)$-*approximation algorithm* and a *randomized* $\rho(n)$-*approximation algorithm*.
4. The Vertex-cover problem
   (a) Introduction
   (b) Proof that `APPROX-VERTEX-COVER` is a 2-approximation algorithm
5. MAX-3-CNF
   (a) Introduction
   (b) Proof that the randomized algorithm for `MAX-3-CNF` is a randomized 8/7-approximation algorithm

Presentation

Hey guys. I will be talking about approximation algorithms. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First, I will be giving a brief motivation behind approximation algorithms. This is then followed by an definition of som terms that I will be using. Lastly, I will be covering two different approximation algorithms.

So, why do we care about approximation algorithms? Well, in many application we want a solution that runs in polynomial time, however, this is not always possible. However, instead an approximation of the actual solution is fine, which is often possible to find in polynomial time, which is why we sometimes decide just to use an approximation algorithm instead.

We say that an algorithm for a problem has an ***approximation ratio*** of $\rho(n)$ if, for any input of size n, the cost C of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n). \tag{1}$$

If an algorithm achieves an approximation ratio of $\rho(n)$, we call it a $\rho(n)$***-approximation algorithm***.

Likewise, we say that a randomized algorithm for a problem has an ***approximation ratio*** of $\rho(n)$ if, for any input of size n, the expected cost C of the solution procuded by the randomized algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

We call a randomized algorithm that achieves an approximation ratio of $\rho(n)$ a ***randomized*** $\rho(n)$***-approximation algorithm***

Now that I have introduced some basic terms, let's get into the examples. Let's start with the ***vertex cover problem***. A ***vertex cover*** of an unidrected graph $G = (V, E)$ is a subset $V' \subseteq V$, such that if (u, v) is an edge of G, then either $u \in V'$ or $v \in V'$ (or both). The goal of the vertex cover problem is thus to find a vertex cover of minimum size in a given undirected graph. Finding such a graph is NP-complete, however, we can use an approximation algortihm to find a vertex cover whose size is no more than twice the size of the optimal vertex cover, using a running time of $O(V + E)$, using adjacency lists to represent E. The algorithm works by continuously picking an arbitrary edge (u, v), saving u and v in a set C, and removing (u, v) and all other edges incident on either u or v from G. This is done untill G has no edges left.

Let's look at an example of this

Now that we have the algorithm introduced, let's prove, that the returned vertex cover is at most twice at big as the optimal vertex cover.

The set C of vertices that is returned by the algorithm is a vertex cover, since the algorithm loops until every edge in E has been covered by some vertex in C. To see that the algorithm returns a vertex cover that is at most twice the size of an optimal cover, let A denote the set of edges that are picked. In order to cover the edges in A, the optimal cover C^* must include at least one endpoint of each edge in A. No two edges in A share an endpoint, since once an edge is picked all other edges that are incident on it sendpoints are deleted from E. Thus, no two edges in A are covered by the same vertex from C^*, and we have

$$|C^*| \geq |A|.$$

Each iteration picks an edge for which neither of its endpoints is already in C, yielding an exact upper bound on the size of the vertex cover returned

$$|C| = 2|A|.$$

Thus, we have

$$|C| = 2|A| \leq 2|C^*|.$$

Thus, we have, that the algorithm is a 2-approximation algorithm.

Now that we have solved a problem using a non-random approximation algorithm, let's instead solve a problem using a random approximation algorithm. The problem we will solve is the ***MAX-3-CNF satisfiability problem***. Let me start off by explaining the problem. In 3-CNF-SAT, we are asked whether a given boolean formula in 3-CNF is satisfiable. We say that a boolean formula is in ***3-CNF*** if it is expressed as an AND of clauses, each of which is the OR of one or more literals, where each clause has exactly three distinct literals and a ***literal*** is a variable or its negation. For the boolean formula to be satisfiable, there must exist an assignment of the variables so that every clause evaluates to 1, which is NP-complete. In the approximation problem ***MAX-3-CNF satisfiability***, we instead wish to find an assignment of the variables that satisfies as many clauses as possible. The input to MAX-3-CNF satisfiability is thus a boolean formula of n variables and m clauses, and the goal is goal is to return an assignment of the variables that maximizes the number of clauses evaluating to 1.

To solve this problem by randomly setting each variable to 1 with probability 1/2 and to 0 with probability 1/2, which yields a randomized 8/7-approximation algorithm. We require each clause to consist of exactly three distinct literals, and we assume that no clause contains both a variable and its negation. I will now prove, that this randomized algorithm is a randomized 8/7-approximation algorithm.

Suppose that we have independently set each variable to 1 with probability 1/2 and to 0 with probability 1/2. For $i = 1, 2, ..., m$, we define the indicator random variable

$$Y_i = I\{\text{clause } i \text{ is satisfied}\}$$

so that $Y_i = 1$ as long as we have set at least one of the literals in the ith clause to 1. Since no literal appears more than once in the same clause, and since we have assumed that no variable and its negation appear in the smae clause, the settings of the three literals in each clause are independent. A clause is not satisfied only if all three of its literals are set to 0, and so $Pr\{\text{clause } i \text{ is not satisfied}\} = (1/2)^3 = 1/8$. Thus, we have $Pr\{\text{clause } i \text{ is satisfied}\} = 1 - 1/8 = 7/8$, and $E[Y_i] = 7/8$. Let Y be the number of satisfied clauses overall, so that $Y = Y_1 + Y_2 + ... + Y_m$. Then, we have

$$E[Y] = E\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} E[Y_i] = \sum_{i=1}^{m} 7/8 = 7m/8.$$

Clearly, m is an upper bound on the number of satisfied clauses, and hence why the approximation ratio is at most $m/(7m/8) = 8/7$.

We thus have, that the approximation algorithm is a 8/7-approximation algorithm.

Polygon Triangulation

Disposition

1. Introduction
2. Introduction to the Art Gallary Problem.
3. The 3-coloring approach
   (a) Introduction
   (b) Runtime analysis
4. Monotone pieces + triangulation
   (a) Motivation
   (b) *Split*, *merge*, *start*, end and regular vertices
   (c) Runtime analysis
5. Conclusion on the Art Gallary Problem:
   (a) Final runtime
   (b) Example on running the algorithm

Presentation

Hey guys. I will be talking about polygon triangulation. I have here a disposition of the things I will go through *Hand out disposition*. As you can see, mine is identical to yours and it does not contain any information that it is not allowed to *Show own disposition*. First, I will be giving this brief introduction. Then, I will be giving an introduction to the problem we are trying to solve. This is followed by me talking about the optimal approach in worst case. Lastly, I will be presenting another linear approach.

The problem we are trying to solve is the so-called *Art Gallary problem*, which asks "how many cameras do we need to guard a given gallary and where do we decide to place them?". We model a gallary as a polygonal region in the plane. We further restrict ourselves to polygons that are *simple polygons*, that is, a polygon of a single chain that does not intersect itself and without any holes.

In the 3-coloring apprach we let $T_{\mathcal{P}}$ be a triangulation of the simple polygon $\mathcal{P}$, select a subset of vertices of $\mathcal{P}$, such that any triangle of $T_{\mathcal{P}}$ has at least one selected vertex, and then we place the cameras at the selected vertices.

We do the triangulation, since triangles are easy to guard. We place the cameras at vertices, since a vertex can be incident to many triangles, and a camera at that vertex then guards all of them.

To find the subset we assign each vertex of $\mathcal{P}$ a color: white, grey or black. The coloring will be such that any vertices connected by an edge or a diagonal have different colors. this is called a *3-coloring* of a triangulated polygon. In a 3-coloring of a triangulated polygon, every triangle has a white, black and gray vertex. Hence, if we place cameras at all gray vertics, say, we have guarded the whole polygon. By choosing the smallest color class to place the cameras, we guard $\mathcal{P}$ using at most $\lfloor \frac{n}{3} \rfloor$ cameras. We can compute the set of at most $\lfloor n/3 \rfloor$ camera positions in linear time by using depth first search on the dual graph to compute a 3-coloring and take the smallest color class to place the cameras.

But we need a fast algortihm for triangulating a simple polygon. For convex polygons we can just pick one vertex and draw diagonals from this vertex to all other vertices except it neighbors. For non-convex polygons we decompose the polygon into so-called y-monotone pieces, and then triangulate the pieces. A simple polygon is *y-monotone* iff any horizontal line intersects it in a connected set or not at all.

We can partition a polygon into monotone pieces by removing all *turn vertices*, that is vertices where there is a change in the direction of the vertical walk from the top of the polygon to the bottom of the polygon. There are 4 types of turn vertices:

1. *Start vertex*: If the two neighbors lie below it and the interior angle is less than π.
2. *Split vertex*: If the two neighbors lie below it and the interior angle is greater than π
3. *End vertex*: If its two neighbors lie above it and the interior angle is less than π.
4. *Merge vertex*: If its two neighbors lie above it and the interior angle is greater than π.

Lastly, there are also the *Regular vertices*, which are vertices where one of its neighbors is above it and the other neighbor is below it.

Now that I have introduced the various types of vertices, let's look at how we can remove each of the different types of turn vertices.

- Split vertex: We use a plane sweep method. The algorithm moves a line downward over the plane. The goal of the sweep is to add diagonals from each split vertex to a vertex lying above it. For a split vertex v_i let e_j be the edge immediately to the left of v_i on the sweep line, and let e_k be the edge immediately to the right of v_i on the sweep line. Then we can always conenct v_i to the lowest vertex in between e_j and e_k, and above v_i. If there is no such vertex then we can connect v_i to the upper endpoint of e_j or to the upper endpoint of e_k. This vertex is called the *helper* of e_j and is denoted by $helper(e_j)$.

- Merge vertex: Now we know how to get rid of split vertices. What about merge vertices. Suppose the sweep line reaches a merge vertex v_i. let e_j and e_k be the edge simmediately to the right and left of v_i on the sweepline. Observe that v_i becomes the new helper of e_j when we reach it. We would like to connect v_i to the highest vertex below the sweep line in between e_j and e_k. We do not know the highest vertex below the sweep line when we reach v_i. But it is easy to find later on: when we reach a vertex v_m, that replaces v_i as the helper of e_j, then this is the vertex we are looking for. So whenever we replace the helper of some edge, we check whether the old helper is amerge vertex and, if so, we add the diagonal between the old helper and the new one. This diagonal is always added when the new helper is a split vertex, to get rid of the split vertex. It can also happen that the helper of e_j is not replaced anymore below v_i. In this case we can connect v_i to the lower endpoint of e_j.

We store the event points in an event queue. The event queue is a priority queue, where the priority of a vertex is its y-coordinate. Thus, to make the polygon into monotone pieces we continuously remove the vertex with the highest priority from the event queue and handle the vertex depending on its type. What is the running time of this algorithm? The algorithm is upperbounded by the priority queue which handles events in $O(\log n)$ time. Thus, the total algorithm runs in $O(n \log n)$.

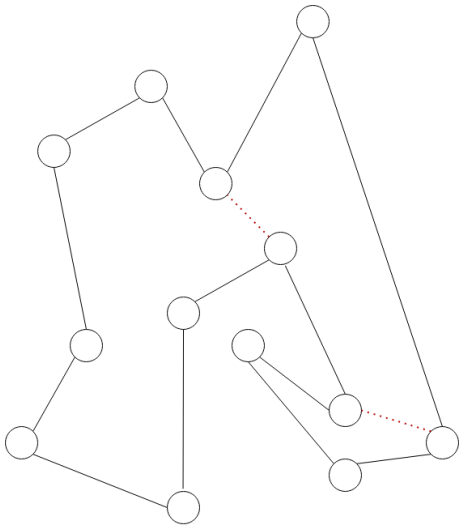
Now that we know how to partition a simple polygon into y-monotone pieces, we should take a look at how to triangulate the monotone polygons. We let $\mathcal{P}$ be an y-monotone polygon with n vertices. We assume that $\mathcal{P}$ is *strictly y-monotone*, that is, it does not contain horizontal edges.

The triangulation algorithm handles the vertices in order of decreasing y-coordinate. The algorithm starts off by initializing a stack and pushing the first two vertices onto the stack. The more recently a vertex has been encountered, the high on the stack it is. The algorithm works by continuously taking the next vertex v and add as many diagonals from this vertex to vertices on the stack as possible, while popping the vertices from the stack. This is the followed by pushing v onto the stack.

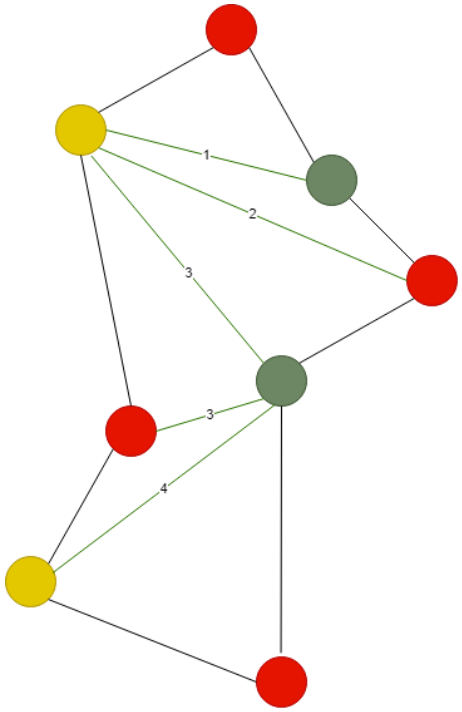
I will not be going into too much details with the running time of the algortihm, but the algorithm simply runs through the sequence of vertices of the polygon, making the algorithm have a running time of $O(n)$.

Thus, we can triangulate a simple polygon with n vertices in $O(n \log n)$ time. Remember, we can use the 3-coloring approach to place cameras in linear time, so in total we can solve the Art Gallary problem in $O(n \log n)$ time with $\lfloor n/3 \rfloor$ cameras.

Now that I have talked about the algorithms, let me run a short example. We are given the following example *Write example on black board*, which we first off split into monotone pieces. By doing so we get the following result Now, we should triangulize the polygon. However, due to time limits I will only be using



the big figure on the left. By triangulizing this pieces and coloring the nodes we get We thus have 4 red



nodes, 2 yellow nodes and 2 green nodes. Thus, optimal solution for this pieces would be to place cameras at each othe yellow nodes or at each of the green nodes.