Eksamensnoter - Divide and Conquer

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2.3 Designing algorithms

2.3.1 The divide-and-conquer approach

- The divide-and-conquer paradigm involves three steps at each level of the recursion:
  - **Divide** the problem into a number of subproblems that are smaller instances of the same problem
  - **Conquer** the subproblems by solving them recursively. If the subproblem sizes are small enough, however, just solve the subroblems in a straightforward manner
  - **Combine** the solutions to the subproblems into the solution for the original problem
- The *merge sort* algorithm closely follows the divide-and-conquer paradigm:
  - **Divide:** Divide the n-element sequence to be sorted into two subsequences of $n/2$ elements each
  - **Conquer:** Sort the two subsequences recursively using merge sort
  - **Combine:** Merge the two sorted subsequences to produce the sorted answer

2.3.2 Analyzing divide-and-conquer algorithms

- When an algorithm contains a recursive call to itself, we can often describe its running time by a *recurrence equation* or *recurrence*, which describes the overall running time on a problem of size n in terms of the running time on smaller inputs. When can then use mathematical tools to solve the recurrence and provide bounds on the performance of the algorithm
- We let $T(n)$ be the running time on a problem of size n. If the problem size is small enough, say $n \leq c$ for some constant c, the straightforward solution takes $\Theta(1)$. Suppose that our division of the problem yields a subproblems, each of which is $1/b$ tbe size of the original. It takes time $T(n/b)$ to solve one subproblem of size n/b, and so it takes time $aT(n/b)$ to solve a of them. If we take $D(n)$ time to divide the problem into subproblems and $C(n)$ time to combine the solutions to the subproblems into the solution to the original problem, we get the recurrence

$$T(n) = \begin{cases} \Theta(1) & in \leq c \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

Analysis of merge sort

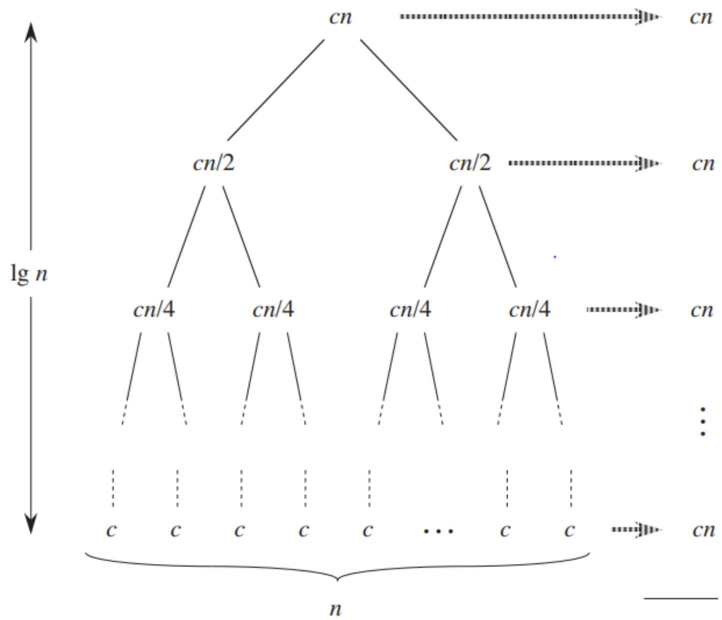
- We reason as follows to set up the recurrence for $T(n)$, the worst-case running time of merge sort on n numbers. Merge sort on just one element takes constant time. When we have $n > 1$ elements ,we break down the running time as follows:
  - **Divide:** The divide step just computes the middle of the subarray, which takes constant time. Thus, $D(n) = \Theta(1)$
  - **Conquer:** We recursively solve two subproblems, each of size $n/2$, which contributes $2T(n/2)$ to the running time
  - **Combine:** Merging n-elements takes $\Theta(n)$, and so $C(n) = \Theta(n)$
- Letting $D(n) + C(n) = \Theta(1) + \Theta(n) = \Theta(n)$ and adding this to the $2T(n/2)$ term from the "conquer" step gives the recurrence for the worst-case running time $T(n)$ of merge sort:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

- We can use an *recursion tree* to understand why $T(n) = \Theta(n \lg n)$. This is done, by first rewritting $T(n)$:

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

  where c represents the time requiered to solve problems of size 1 as well as the time per array element of the divide and combine steps (it is unlikely that the same constant represents both. We can get around this by letting c be the larger/lesser of these times and understanding that our recurrence gives an upper/lower bound on the running time).



(d)

Total: $cn \lg n + cn$

Next, we add the costs across each level of the tree. In general, the level i below the top has 2^i nodes, each contributing a cost of $c(n/2^i)$, so that the ith level below the top has total cost $2^i c(n/2^i) = cn$.
The toal number of levels of the recursion tree is $\lg n + 1$, where n is the number of leaves, corresponding to the input size. An informal inductive argument justifies this claim:

- The base case occurs when $n = 1$, in which case the tree has only one level. Since $\lg 1 = 0$, we have that $\lg n + 1$ gives the correct number of levels.
- Assume, that the number of levels of a recursion tree with 2^i leaves is $\lg 2^i + 1 = i + 1$. Because we are assuming that the input size is a power of 2, the next input size to consider is 2^{i+1}. A tree with $n = 2^{i+1}$ leaves has one more level than a tree with 2^i leaves, and so the total number of levels is $(i + 1) + 1 = \lg 2^{i+1} + 1$.

To compute the total cost represented by the recurrence $T(n)$, we add up the costs of all the levels; the tree has $\lg n + 1$ levels, each costing cn, for a total of $cn(\lg n + 1) = cn \lg n + cn = \Theta(n \lg n)$.