MAD 2019/2020 Exam Answers

Exam Number: 18

January 13, 2020

Exercise 1

I find the task of this exercise a little ambiguous, as I am not sure if I am supposed to (only) compute the mean point of the data, the two eigenvalues and the two eigenvectors or if I am also supposed to convert the dimensionality. To be on the safe side, I will also convert the dimensionality, however, if I was not supposed to do this, please just overlook that part of my answer.

We are given the following $N = 8$ 2-dimensional points:

coordinate x	0.1	0.5	1.1	-0.5	1.3	0.2	-0.1	1
coordinate y	1	1	2	0.2	-0.1	-0.1	-1.5	-2.5

First of we need to make each row have zero mean. This is done, by first computing $\bar{x}$ and $\bar{y}$:

$$\bar{x} = \frac{1}{8}\sum_{n=1}^{8} x_n = 0.45, \qquad \bar{y} = \frac{1}{8}\sum_{n=1}^{8} y_n = 0$$

Which gives us the mean point of data, $(0.45, 0)$. $\bar{x}$ and $\bar{y}$ are then subtracted from respectively coordinate x and coordinate y, which results in the following rows, with zero mean:

coordinate x	-0.35	0.05	0.65	-0.95	0.85	-0.25	-0.55	0.55
coordinate y	1	1	2	0.2	-0.1	-0.1	-1.5	-2.5

From this we can extract the following matrix:

$$\boldsymbol{S} = \begin{bmatrix} -0.35 & 1 \\ 0.05 & 1 \\ 0.65 & 2 \\ -0.95 & 0.2 \\ 0.85 & -0.1 \\ -0.25 & -0.1 \\ -0.55 & -1.5 \\ 0.55 & -2.5 \end{bmatrix}$$

Which we use to compute the covariance matric $\boldsymbol{C}$:

$$\boldsymbol{C} = \tfrac{1}{N} \cdot \boldsymbol{S}^T \cdot \boldsymbol{S} =$$

$$\tfrac{1}{8} \cdot \begin{bmatrix} -0.35 & 0.05 & 0.65 & -0.95 & 0.85 & -0.25 & -0.55 & 0.55 \\ 1 & 1 & 2 & 0.2 & -0.1 & -0.1 & -1.5 & -2.5 \end{bmatrix} \cdot \begin{bmatrix} -0.35 & 1 \\ 0.05 & 1 \\ 0.65 & 2 \\ -0.95 & 0.2 \\ 0.85 & -0.1 \\ -0.25 & -0.1 \\ -0.55 & -1.5 \\ 0.55 & -2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.355 & 0.025 \\ 0.025 & 1.82 \end{bmatrix}$$

Now that we have found the covariance matrix $\boldsymbol{C}$, we can find the eigenvalues. First we need to compute $\boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2$:

$$\boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2 = \begin{bmatrix} 0.355 & 0.025 \\ 0.025 & 1.82 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0.355 - \lambda & 0.025 \\ 0.025 & 1.82 - \lambda \end{bmatrix}$$

Which we need to find the determinant of:

$$det(\boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2) = \begin{vmatrix} 0.355 - \lambda & 0.025 \\ 0.025 & 1.82 - \lambda \end{vmatrix}$$
$$= (0.355 - \lambda) \cdot (1.82 - \lambda) - 0.025 * 0.025$$
$$= 0.355 \cdot 1.82 - \lambda \cdot 1.82 + \lambda^2 - 0.025 \cdot 0.025 - 0.355 \cdot \lambda$$
$$\approx \lambda^2 - 2.175\lambda + 0.6455$$

And now solve for λ:

$$\lambda^2 - 2.175\lambda + 0.6455 = 0$$

$\Leftrightarrow$

$$\lambda \frac{-(-2.175) \pm \sqrt{(-2.175)^2 - 4 \cdot 1 \cdot 0.6455}}{2 \cdot 1} \approx \begin{cases} 1.820 \\ 0.355 \end{cases}$$

Thus, we have found the two eigenvalues $\lambda_1 = 1.820$ and $\lambda_2 = 0.355$.

Now, for finding the associated eigenvectors, we first let $\lambda = \lambda_1$ and find $\boldsymbol{B} = \boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2$:

$$\boldsymbol{B} = \boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2 = \begin{bmatrix} 0.355 - 1.820 & 0.025 \\ 0.025 & 1.82 - 1.820 \end{bmatrix}$$
$$= \begin{bmatrix} -1.4658 & 0.025 \\ 0.025 & 0 \end{bmatrix}$$

Now, we put $\boldsymbol{Bx} = \boldsymbol{0}$ on row echelon form

$$\boldsymbol{Bx} = \boldsymbol{0}$$

$\Rightarrow$

$$\left[\begin{array}{cc|c} -1.4658 & 0.025 & 0 \\ 0.025 & 0 & 0 \end{array}\right]$$

Which is done by adding row 1 times 0.0171 to row 2. Thus, we get the following augmented matrix:

$$\left[\begin{array}{cc|c} -1.4658 & 0.025 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we have the following equation, where we isolate x_1:

$$-1.4658 \cdot x_1 + 0.025 \cdot x_2 = 0$$

$\Leftrightarrow$

$$0.025 \cdot x_2 = 1.4658 \cdot x_1$$

$\Leftrightarrow$

$$0.171 \cdot x_2 = x_1$$

From this point, we can choosing any integer for x_2 and get the eigenvector for the eigenvalue 1.820. I am here choosing $x_2 = 1$:

$$0.171 \cdot 1 = x_1$$

$\Leftrightarrow$

$$0.171 = x_1$$

Thus, the eigenvector for the eigenvalue 1.820 is:

$$\begin{bmatrix} 0.171 \\ 1 \end{bmatrix}$$

Now, we let $\lambda = \lambda_2$ and follow the same procedure:

$$\begin{aligned} \boldsymbol{B} = \boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2 &= \begin{bmatrix} 0.355 - 0.355 & 0.025 \\ 0.025 & 1.82 - 0.355 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0.025 \\ 0.025 & 1.465 \end{bmatrix} \end{aligned}$$

$$\boldsymbol{Bx} = \boldsymbol{0}$$

$\Rightarrow$

$$\left[\begin{array}{cc|c} 0 & 0.025 & 0 \\ 0.025 & 1.465 & 0 \end{array}\right]$$

To put the matrix on row echelon form, we simply swap row 1 and row 2 and optain the following matrix

$$\left[\begin{array}{cc|c} 0.025 & 1.465 & 0 \\ 0 & 0.025 & 0 \end{array}\right]$$

Thus, we have the following equation, where x_1 needs to be isolated:

$$0.025 \cdot x_1 + 1.465 \cdot x_2 = 0$$

$\Leftrightarrow$

$$1.465 \cdot x_2 = -0.025 \cdot x_1$$

$\Leftrightarrow$

$$-58.6 \cdot x_2 = x_1$$

Again, here it is possible to choose any value for x_2, I will let $x_2 = 1$ and get the following:

$$-58.6 \cdot 1 = x_1$$

$\Leftrightarrow$

$$-58.6 = x_1$$

Thus, for the eigenvalue 0.355, we have the eigenvector

$$\begin{bmatrix} -58.6 \\ 1 \end{bmatrix}$$

Thus, sorting the eigenvalues in descending order and the eigenvectors accordingly, we get the following matrix of the eigenvectors:

$$\boldsymbol{P} = \begin{bmatrix} 0.171 & -58.6 \\ 1 & 1 \end{bmatrix}$$

The dimensionality of the given data can now be converted by multiplying the zero-mean matrix, $\boldsymbol{S}$ with $\boldsymbol{P}$:

$$\begin{bmatrix} -0.35 & 1 \\ 0.05 & 1 \\ 0.65 & 2 \\ -0.95 & 0.2 \\ 0.85 & -0.1 \\ -0.25 & -0.1 \\ -0.55 & -1.5 \\ 0.55 & -2.5 \end{bmatrix} \cdot \begin{bmatrix} 0.171 & -58.6 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.9402 & 21.51 \\ 1.0086 & -1.93 \\ 2.1112 & -36.09 \\ 0.03755 & 55.87 \\ 0.04535 & -49.91 \\ -0.1428 & 14.55 \\ -1.5941 & 30.73 \\ -2.406 & -34.73 \end{bmatrix}$$

Exercise 2

First, we compute $Gini(T_{org}, \lambda)$:

$$Gini(T_{org}, \lambda) = \frac{17}{28} \cdot gini(T_{first}) + \frac{11}{28} \cdot gini(T_{second})$$

$$= \frac{17}{28}\left(1 - \left(\frac{6}{17}\right)^2 - \left(\frac{1}{17}^2\right) - \left(\frac{10}{17}\right)^2\right) + \frac{11}{28}\left(1 - \left(\frac{1}{11}\right)^2 - \left(\frac{7}{11}\right)^2 - \left(\frac{3}{11}\right)^2\right)$$

$$= 0.5206$$

Next, we compute $Gain(T_{orig}, \lambda)$:

$$Gain(T_{orig}, \lambda) = H(7, 8, 13) - \frac{17}{28} \cdot H(6, 1, 10) - \frac{11}{28} \cdot H(1, 7, 3)$$

$$= \left(-\frac{13}{28} \cdot \log_2\left(\frac{13}{28}\right) - \frac{8}{28} \cdot \log_2\left(\frac{8}{28}\right) - \frac{7}{28} \cdot \left(\frac{7}{28}\right)\right)$$

$$-\frac{17}{28}\left(-\frac{10}{17} \cdot \log_2\left(\frac{10}{17}\right) - \frac{6}{17} \cdot \log_2\left(\frac{6}{17}\right) - \frac{1}{17} \cdot \left(\frac{1}{17}\right)\right)$$

$$-\frac{11}{28}\left(-\frac{7}{11} \cdot \log_2\left(\frac{7}{11}\right) - \frac{3}{11} \cdot \log_2\left(\frac{3}{11}\right) - \frac{1}{11} \cdot \log_2\left(\frac{1}{11}\right)\right)$$

$$= 0.3015$$

Thus, it has been computed, that $Gini(T_{org}, \lambda) = 0.5206$ and $Gain(T_{orig}, \lambda) = 0.3015$