MAD 2019/2020 Exam Answers

Exam Number: 18

January 13, 2020

Exercise 1

We are given the following $N = 8$ 2-dimensional points:

coordinate x	0.1	0.5	1.1	-0.5	1.3	0.2	-0.1	1
coordinate y	1	1	2	0.2	-0.1	-0.1	-1.5	-2.5

First of we need to make each row have zero mean. This is done, by first computing $\bar{x}$ and $\bar{y}$:

$$\bar{x} = \frac{1}{8}\sum_{n=1}^{8} x_n = 0.45, \qquad \bar{y} = \frac{1}{8}\sum_{n=1}^{8} y_n = 0$$

Which gives us the mean point of data, $(0.45, 0)$. $\bar{x}$ and $\bar{y}$ are then subtracted from respectively coordinate x and coordinate y, which results in the following rows, with zero mean:

coordinate x	-0.35	0.05	0.65	-0.95	0.85	-0.25	-0.55	0.55
coordinate y	1	1	2	0.2	-0.1	-0.1	-1.5	-2.5

From this we can extract the following matrix:

$$\boldsymbol{S} = \begin{bmatrix} -0.35 & 1 \\ 0.05 & 1 \\ 0.65 & 2 \\ -0.95 & 0.2 \\ 0.85 & -0.1 \\ -0.25 & -0.1 \\ -0.55 & -1.5 \\ 0.55 & -2.5 \end{bmatrix}$$

Which we use to compute the covariance matric $\boldsymbol{C}$:

$$\boldsymbol{C} = \tfrac{1}{N} \cdot \boldsymbol{S}^T \cdot \boldsymbol{S} =$$

$$\tfrac{1}{8} \cdot \begin{bmatrix} -0.35 & 0.05 & 0.65 & -0.95 & 0.85 & -0.25 & -0.55 & 0.55 \\ 1 & 1 & 2 & 0.2 & -0.1 & -0.1 & -1.5 & -2.5 \end{bmatrix} \cdot \begin{bmatrix} -0.35 & 1 \\ 0.05 & 1 \\ 0.65 & 2 \\ -0.95 & 0.2 \\ 0.85 & -0.1 \\ -0.25 & -0.1 \\ -0.55 & -1.5 \\ 0.55 & -2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.355 & 0.025 \\ 0.025 & 1.82 \end{bmatrix}$$

Now that we have found the covariance matrix $\boldsymbol{C}$, we can find the eigenvalues. First we need to compute $\boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2$:

$$\boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2 = \begin{bmatrix} 0.355 & 0.025 \\ 0.025 & 1.82 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0.355 - \lambda & 0.025 \\ 0.025 & 1.82 - \lambda \end{bmatrix}$$

Which we need to find the determinant of:

$$\begin{aligned} det(\boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2) &= \begin{vmatrix} 0.355 - \lambda & 0.025 \\ 0.025 & 1.82 - \lambda \end{vmatrix} \\ &= (0.355 - \lambda) \cdot (1.82 - \lambda) - 0.025 * 0.025 \\ &= 0.355 \cdot 1.82 - \lambda \cdot 1.82 + \lambda^2 - 0.025 \cdot 0.025 - 0.355 \cdot \lambda \\ &\approx \lambda^2 - 2.175\lambda + 0.6455 \end{aligned}$$

And now solve for λ:

$$\lambda^2 - 2.175\lambda + 0.6455 = 0$$

$\Leftrightarrow$

$$\lambda \ \frac{-(-2.175) \pm \sqrt{(-2.175)^2 - 4 \cdot 1 \cdot 0.6455}}{2 \cdot 1} \approx \begin{cases} 1.820 \\ 0.355 \end{cases}$$

Thus, we have found the two eigenvalues $\lambda_1 = 1.820$ and $\lambda_2 = 0.355$.

Now, for finding the associated eigenvectors, we first let $\lambda = \lambda_1$ and find $\boldsymbol{B} = \boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2$:

$$\begin{aligned} \boldsymbol{B} = \boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2 &= \begin{bmatrix} 0.355 - 1.820 & 0.025 \\ 0.025 & 1.82 - 1.820 \end{bmatrix} \\ &= \begin{bmatrix} -1.4658 & 0.025 \\ 0.025 & 0 \end{bmatrix} \end{aligned}$$

Now, we put $\boldsymbol{Bx} = \boldsymbol{0}$ on row echelon form

$$\boldsymbol{Bx} = \boldsymbol{0}$$

$\Rightarrow$

$$\left[\begin{array}{cc|c} -1.4658 & 0.025 & 0 \\ 0.025 & 0 & 0 \end{array}\right]$$

Which is done by adding row 1 times 0.0171 to row 2. Thus, we get the following augmented matrix:

$$\left[\begin{array}{cc|c} -1.4658 & 0.025 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we have the following equation, where we isolate x_1:

$$-1.4658 \cdot x_1 + 0.025 \cdot x_2 = 0$$

$\Leftrightarrow$

$$0.025 \cdot x_2 = 1.4658 \cdot x_1$$

$\Leftrightarrow$

$$0.171 \cdot x_2 = x_1$$

From this point, we can choosing any integer for x_2 and get the eigenvector for the eigenvalue 1.820. I am here choosing $x_2 = 1$:

$$0.171 \cdot 1 = x_1$$

$\Leftrightarrow$

$$0.171 = x_1$$

Thus, the eigenvector for the eigenvalue 1.820 is:

$$\begin{bmatrix} 0.171 \\ 1 \end{bmatrix}$$

Now, we let $\lambda = \lambda_2$ and follow the same procedure:

$$\begin{aligned} \boldsymbol{B} = \boldsymbol{C} - \lambda \cdot \boldsymbol{I}_2 &= \begin{bmatrix} 0.355 - 0.355 & 0.025 \\ 0.025 & 1.82 - 0.355 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0.025 \\ 0.025 & 1.465 \end{bmatrix} \end{aligned}$$

$$\boldsymbol{Bx} = \boldsymbol{0}$$

$\Rightarrow$

$$\left[\begin{array}{cc|c} 0 & 0.025 & 0 \\ 0.025 & 1.465 & 0 \end{array}\right]$$

To put the matrix on row echelon form, we simply swap row 1 and row 2 and optain the following matrix

$$\left[\begin{array}{cc|c} 0.025 & 1.465 & 0 \\ 0 & 0.025 & 0 \end{array}\right]$$

Thus, we have the following equation, where x_1 needs to be isolated:

$$0.025 \cdot x_1 + 1.465 \cdot x_2 = 0$$

$\Leftrightarrow$

$$1.465 \cdot x_2 = -0.025 \cdot x_1$$

$\Leftrightarrow$

$$-58.6 \cdot x_2 = x_1$$

Again, here it is possible to choose any value for x_2, I will let $x_2 = 1$ and get the following:

$$-58.6 \cdot 1 = x_1$$

$\Leftrightarrow$

$$-58.6 = x_1$$

Thus, for the eigenvalue 0.355, we have the eigenvector

$$\begin{bmatrix} -58.6 \\ 1 \end{bmatrix}$$

Thus, it has been computed, that we have the mean point of data at $(0.45, 0)$, the eigenvalues 1.820 and 0.355 and the following eigenvectors

$$\begin{bmatrix} 0.171 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -58.6 \\ 1 \end{bmatrix}$$

Exercise 2

First, we compute $Gini(T_{org}, \lambda)$:

$$Gini(T_{org}, \lambda) = \frac{17}{28} \cdot gini(T_{first}) + \frac{11}{28} \cdot gini(T_{second})$$

$$= \frac{17}{28}\left(1 - \left(\frac{6}{17}\right)^2 - \left(\frac{1}{17}\right)^2 - \left(\frac{10}{17}\right)^2\right) + \frac{11}{28}\left(1 - \left(\frac{1}{11}\right)^2 - \left(\frac{7}{11}\right)^2 - \left(\frac{3}{11}\right)^2\right)$$

$$= 0.5206$$

Next, we compute $Gain(T_{orig}, \lambda)$:

$$Gain(T_{orig}, \lambda) = H(7, 8, 13) - \frac{17}{28} \cdot H(6, 1, 10) - \frac{11}{28} \cdot H(1, 7, 3)$$

$$= \left(-\frac{13}{28} \cdot \log_2\left(\frac{13}{28}\right) - \frac{8}{28} \cdot \log_2\left(\frac{8}{28}\right) - \frac{7}{28} \cdot \log_2\left(\frac{7}{28}\right)\right)$$

$$-\frac{17}{28}\left(-\frac{10}{17} \cdot \log_2\left(\frac{10}{17}\right) - \frac{6}{17} \cdot \log_2\left(\frac{6}{17}\right) - \frac{1}{17} \cdot \log_2\left(\frac{1}{17}\right)\right)$$

$$-\frac{11}{28}\left(-\frac{7}{11} \cdot \log_2\left(\frac{7}{11}\right) - \frac{3}{11} \cdot \log_2\left(\frac{3}{11}\right) - \frac{1}{11} \cdot \log_2\left(\frac{1}{11}\right)\right)$$

$$= 0.3015$$

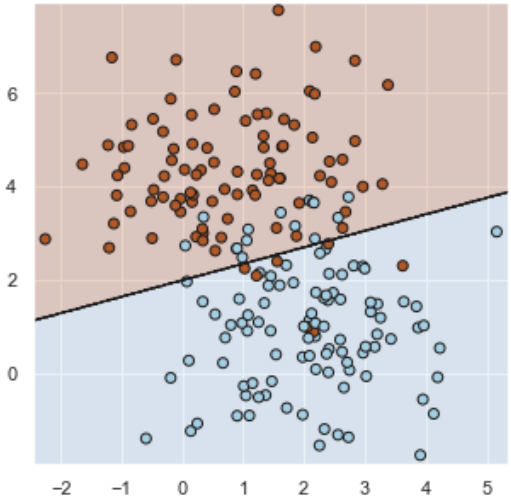
Thus, it has been computed, that $Gini(T_{org}, \lambda) = 0.5206$ and $Gain(T_{orig}, \lambda) = 0.3015$

Exercise 3

With my implementation, I have the following performance of the cross-validation:

C	Accuracy
0.001	0.9225
0.01	0.92125
0.1	0.9274999999999999
1	0.9237500000000001
10	0.9237500000000001
100	0.9237500000000001

And the following decision boundary for the optimal svm classifier:



Exercise 4

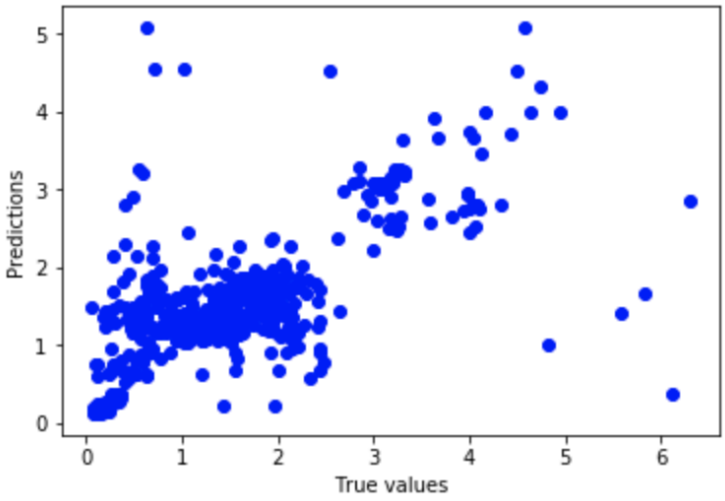
Part 1

(a)

With my implementation, I have gotten the following RMSE-value:

$$RMSE = 0.8243064553494787$$

And the following scatter plot:

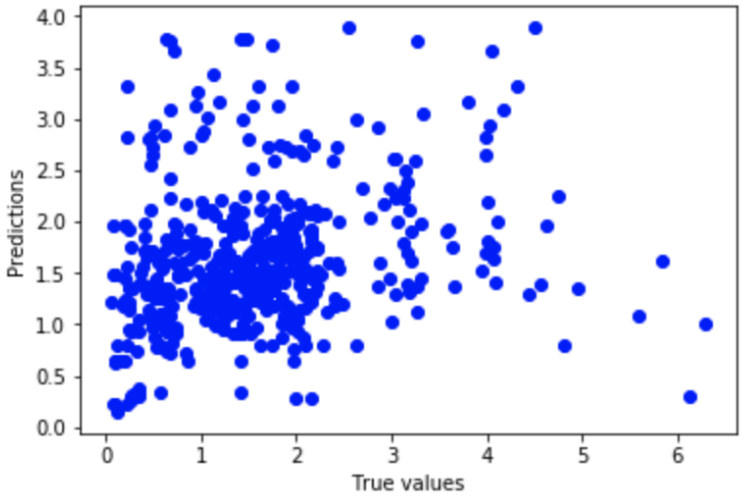


(b)

With this new distance formula, I have optained the following RMSE-value:

$$RMSE = 1.0997971796682453$$

And the following scatter plot:



The matrix M, makes it possible to tune the impact that the different features have on the distance between the two data point, thus also the impact on the prediction. For instance, the given example matrix

$$\begin{bmatrix} 0.00001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.00001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.00001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.00001 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00001 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.00001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.00001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00001 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

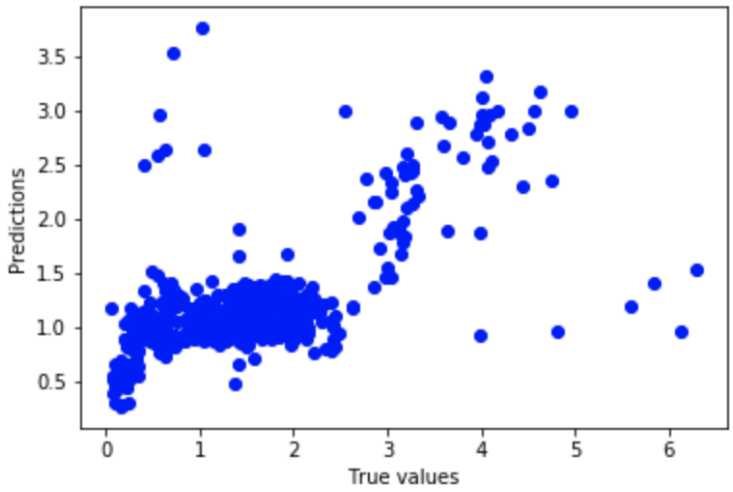
makes the two last features have a bigger impact on the distance and prediction, than the first eight features. This could imply, that the two last feature are more "important", than the first eight features.

Part 2

With my implementation, I got the following RMSE-value:

$$RMSE = 0.8925765347348245$$

And the following scatter plot:



Exercise 5

Part 1

We have been given the following expression

$$p(t_n|\boldsymbol{x}_n;\boldsymbol{w}) = \frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n} exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}$$

And we wish to find the joint conditional density:

$$p(\boldsymbol{t}|\boldsymbol{X};\boldsymbol{w})$$

Which extends to

$$p(t_1, ..., t_N|\boldsymbol{x}_1, ..., \boldsymbol{x}_N;\boldsymbol{w})$$

Following the formula for joint probability distribution

$$p(A, B) = p(A) \cdot p(B)$$

We get the result

$$p(\boldsymbol{t}|\boldsymbol{X};\boldsymbol{w}) = \prod_{n=1}^{N} p(t_n|\boldsymbol{x}_n;\boldsymbol{w}) = \prod_{n=1}^{N} \frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n} exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}$$

Part 2

Since, it says in the exercise desciption, how we only should include the essential steps for deriving the expression, some of the descriptions of actions in the following exercise have been left out.
We have been given the expression for $p(w|\boldsymbol{t}, \boldsymbol{X}, \mu_0, \sigma_0^2)$, which we extend

$$p(w|\boldsymbol{t}, \boldsymbol{X}, \mu_0, \sigma_0^2) = \frac{\prod_{n=1}^{N}\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n} exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right)\frac{1}{(2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}}\exp(-\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0))}{p(\boldsymbol{t}|\boldsymbol{X})}$$

To find $\log p(w|\boldsymbol{t}, \boldsymbol{X}, \mu_0, \sigma_0^2)$, we start of by taking the log on both sides and reducing a little:

$$\log p(w|\boldsymbol{t}, \boldsymbol{X}, \mu_0, \sigma_0^2) = \log\left(\frac{\prod_{n=1}^{N}\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right)\frac{1}{(2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}}\exp(-\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0))}{p(\boldsymbol{t}|\boldsymbol{X})}\right)$$

$$= \log\left(\prod_{n=1}^{N}\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right)\right) + \log\left(\frac{1}{(2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}}\right) + \log\left(\exp(-\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0))\right) - \log\left(p(\boldsymbol{t}|\boldsymbol{X})\right)$$

For the sake of readability, I choose to split the proof into three sections:

First Section

In this section, I will prove

$$\log\left(\prod_{n=1}^{N}\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right)\right) = \sum_{n=1}^{N}\left(t_n\log f(\boldsymbol{x}_n;\boldsymbol{w}) - f(\boldsymbol{x}_n;\boldsymbol{w}) - \log t_n!\right)$$

This is done by the following

$$\log\left(\prod_{n=1}^{N}\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right)\right) = \sum_{n=1}^{N}\log\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right) =$$
$$\sum_{n=1}^{N}\left(\log(f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))) - \log t_n!\right) =$$
$$\sum_{n=1}^{N}\left(\log f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n} + \log\exp(-f(\boldsymbol{x}_n;\boldsymbol{w})) - \log t_n!\right) =$$
$$\sum_{n=1}^{N}\left(t_n\log f(\boldsymbol{x}_n;\boldsymbol{w}) - f(\boldsymbol{x}_n;\boldsymbol{w}) - \log t_n!\right)$$

Second Section

In this section, I will prove

$$\log\left(\exp(-\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0))\right) = -\frac{1}{2\sigma_0^2}(\boldsymbol{w}-\mu_0)^T(\boldsymbol{w}-\mu_0)$$

This is done by

$$\log\left(\exp(-\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0))\right) = -\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0) =$$
$$-\frac{1}{2}(\boldsymbol{w}-\mu_0)^T(\frac{1}{\sigma_0^2}\boldsymbol{I})(\boldsymbol{w}-\mu_0) = -\frac{1}{2}\frac{1}{\sigma_0^2}(\boldsymbol{w}-\mu_0)^T\boldsymbol{I}(\boldsymbol{w}-\mu_0) =$$
$$-\frac{1}{2\sigma_0^2}(\boldsymbol{w}-\mu_0)^T(\boldsymbol{w}-\mu_0)$$

Third section

In this section, I will prove

$$\log\left(\frac{1}{(2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}}\right) = -\log\left(\sqrt{2\pi\sigma_0^2}^{D+1}\right)$$

This is done by

$$\log\left(\frac{1}{(2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}}\right) = \log(1) - \log((2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}) =$$
$$-\log((2\pi)^{\frac{D+1}{2}}\left(\sigma_0^{2\cdot(D+1)}\right)^{\frac{1}{2}}) = -\log((2\pi)^{\frac{D+1}{2}}\left(\sigma_0^2\right)^{\frac{D+1}{2}}) =$$
$$-\log((2\pi)^{\left(\frac{1}{2}\right)^D}\left(\sigma_0^2\right)^{\left(\frac{1}{2}\right)^D}) = -\log(2\pi\sigma_0^2)^{\left(\frac{1}{2}\right)^D} = -\log\sqrt{(2\pi\sigma_0^2)}^D$$

Substituting the results of these three sections into $\log p(w|\boldsymbol{t}, \boldsymbol{X}, \mu_0, \sigma_0^2)$ gives the following

$$\log p(w|\boldsymbol{t}, \boldsymbol{X}, \mu_0, \sigma_0^2) = \log\left(\prod_{n=1}^N\left(\frac{f(\boldsymbol{x}_n;\boldsymbol{w})^{t_n}\exp(-f(\boldsymbol{x}_n;\boldsymbol{w}))}{t_n!}\right)\right) +$$
$$\log\left(\frac{1}{(2\pi)^{\frac{D+1}{2}}\det(\sigma_0^2\boldsymbol{I})^{\frac{1}{2}}}\right) + \log\left(\exp(-\tfrac{1}{2}(\boldsymbol{w}-\mu_0)^T(\sigma_0^2\boldsymbol{I})^{-1}(\boldsymbol{w}-\mu_0))\right) -$$
$$\log\left(p(\boldsymbol{t}|\boldsymbol{X})\right) = \sum_{n=1}^N\left(t_n\log f(\boldsymbol{x}_n;\boldsymbol{w}) - f(\boldsymbol{x}_n;\boldsymbol{w}) - \log t_n!\right) - \log\sqrt{(2\pi\sigma_0^2)}^D -$$
$$\frac{1}{2\sigma_0^2}(\boldsymbol{w}-\mu_0)^T(\boldsymbol{w}-\mu_0) - \log\left(p(\boldsymbol{t}|\boldsymbol{X})\right)$$

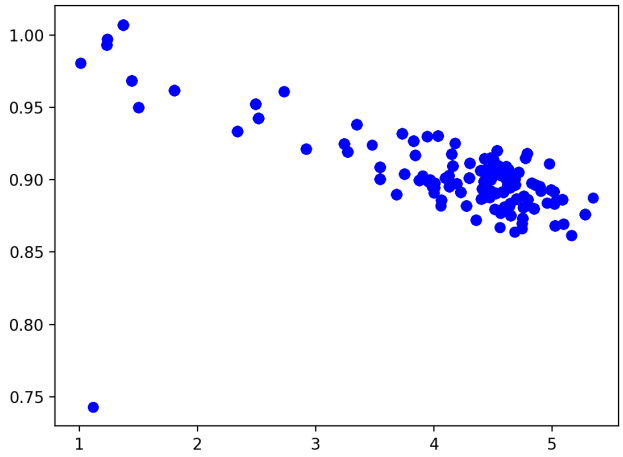
Rearranging, the equation a little, and we get to the result:

$$\sum_{n=1}^N\left(t_n\log f(\boldsymbol{x}_n;\boldsymbol{w}) - f(\boldsymbol{x}_n;\boldsymbol{w}) - \log t_n!\right) - \log\sqrt{(2\pi\sigma_0^2)}^D - \frac{1}{2\sigma_0^2}(\boldsymbol{w}-$$
$$\mu_0)^T(\boldsymbol{w}-\mu_0) - \log\left(p(\boldsymbol{t}|\boldsymbol{X})\right)$$
$$= \sum_{n=1}^N\left(t_n\log f(\boldsymbol{x}_n;\boldsymbol{w}) - f(\boldsymbol{x}_n;\boldsymbol{w}) - \log t_n!\right) - \frac{1}{2\sigma_0^2}(\boldsymbol{w}-\mu_0)^T(\boldsymbol{w}-\mu_0) -$$
$$\log\sqrt{(2\pi\sigma_0^2)}^D - \log\left(p(\boldsymbol{t}|\boldsymbol{X})\right), \qquad Q.E.D$$

Part 3

See `code/sunspot.py` for the source code of my implementation

Part 4



Looking at the 2D plot, I would say the samples are correlated, as most of them, are very close to each other and they are "on the same path".

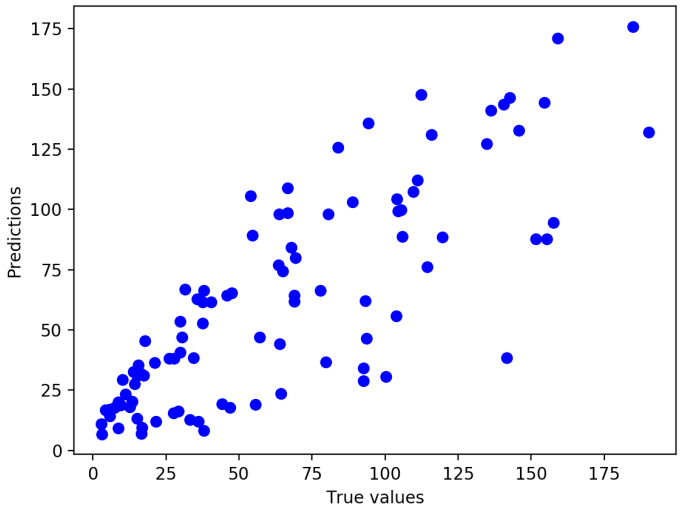
Part 5

When I ran my implementation of the Metropolis-Hasting algorithm on the three different model selections over 5 different sets of samples, I got the following RMSE values:

Iteration	Selection 1	Selection 2	Selection 3
1	29.05787	44.4726	19.8786
2	29.0298	44.9460	19.1823
3	29.0466	44.1827	20.3028
4	29.0616	45.1014	18.5558
5	29.0325	44.7816	20.1317
Average	29.0457	44.6969	19.61020

Looking at these RMSE values, the model with selection 3 clearly gives the best prediction results, as it is the model with the lowest RMSE value. Next is the model with selection 1, and worst is the model with selection 2.

For model selection 1, I have made the following scatter plot of the test set and the predictions:



It is quite difficult to specify how many samples are need for my implementation. This is due to the fact, that the more samples used, the lower RMSE the implementation gets, where the RMSE decreases less, the greater the amount of samples used. By trying out different values of the amount of samples for the different model selections, an amount of 3000 accepted samples gives some good predictions and not to long sample generation period.

By looking at the generated samples, as well as trying out different values, I have fixed the burn-in time and skip steps to respectively 30 and 5. The burn-in discards the first 30 samples of the accepted samples, as those might not be representative. On the other hand, by only accepting every 5th sample, the samples will be more independent. This number could be increased further for more independence between the samples, however, I have not chosen to do so, since this will also result in worse predictions. More independence could be obtained further by increasing the variance in the proposal, so each generated sampled would "walk" further away from the last generated sample, however, I have chosen a small numer (at 0.1), since this number resultet in better predictions.