Theoretical Computer Science

Tutorial Week 5

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Agenda

Regular languages

- **Myhill-Nerode criteria**
  - Positive Examples
  - Negative Examples
- Pumping Lemma

Regular Languages

Definition

A languages is called regular, if it recognized by a FSA.

Problem

Which languages are regular?

Myhill-Nerode criteria

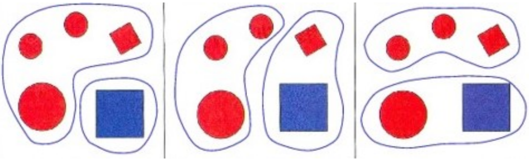
For a language L over an alphabet A,

$$s_1 \equiv_L s_2 \Leftrightarrow (\forall t \in A^*)\,(s_1 t \in L \leftrightarrow s_2 t \in L)$$

$\equiv_L$ is an equivalence relation

Myhill-Nerode criteria

What are equivalence relations in general?



Myhill-Nerode criteria

For a language L over an alphabet A,

$$s_1 \not\equiv_L s_2 \Leftrightarrow (\exists t \in A^*)\,[(s_1 t \notin L \,\&\, s_2 t \in L) \vee (s_1 t \in L \,\&\, s_2 t \notin L)]$$

t is called a **distinguishing extension**.

Myhill-Nerode criteria

Myhill-Nerode theorem

A language L is regular iff $\equiv_L$ has a finite number of equivalent classes.

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Regular languages

- Myhill-Nerode criteria
  - **Positive Examples**
  - Negative Examples
- Pumping Lemma

Myhill-Nerode method. Examples

$$s_1 \not\equiv_L s_2 \Leftrightarrow (\exists t \in A^*)\,[(s_1 t \notin L \,\&\, s_2 t \in L) \vee (s_1 t \in L \,\&\, s_2 t \notin L)]$$

Example 1: $L_1 = \{0x \mid x \in \Sigma^*\}$, where $\Sigma = \{0, 1\}$

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$$1 \not\equiv_{L_1} 0, \text{ since } 1 \notin L_1, 0 \in L_1$$

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Example 1: $L_1 = \{0x \mid x \in \Sigma^*\}$

$[\epsilon] = \{\epsilon\}$, $[0] = \{0x \mid x \in \Sigma^*\} = L_1$, $[1] = \{1x \mid x \in \Sigma^*\}$

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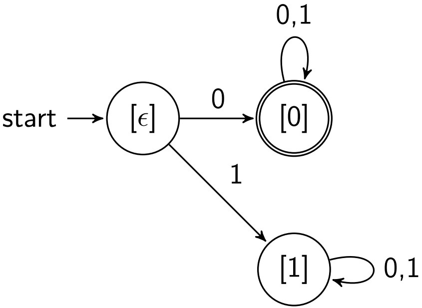

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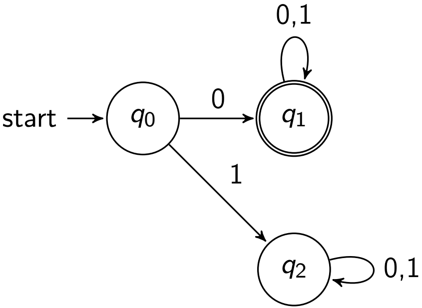

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Example 2: $L_2 = \{x00 \mid x \in \Sigma^*\}$

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Myhill-Nerode method. Examples

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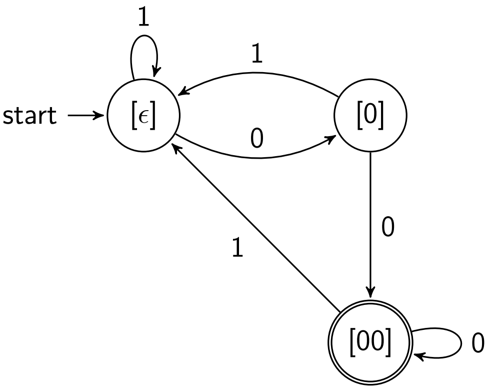

Myhill-Nerode method. Examples

Example 3: $L_3 = \{x \in \Sigma^* \mid x$ is a binary representation of an integer divisible by 5 and it begins with $1\}$, where $\Sigma = \{0, 1\}$

1. $[\epsilon]$
2. $[0] = \{0x \mid x \in \Sigma^*\}$
3. $[1] = \{x \mid$ the remainder after dividing x by 5 is $1\}$
4. $[10] = \{x \mid$ the remainder after dividing x by 5 is $2\}$
5. $[11] = \{x \mid$ the remainder after dividing x by 5 is $3\}$
6. $[100] = \{x \mid$ the remainder after dividing x by 5 is $4\}$
7. $[101] = \{x \mid x$ is divisible by $5\}$

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- Myhill-Nerode criteria
  - Positive Examples
  - **Negative Examples**
- Pumping Lemma

Myhill-Nerode method. Examples

Negative Example 1

$L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Proof

For $m \neq k$,

$$a^m \not\equiv_{L_1} a^k,$$

since $a^m b^k \notin L_1, a^k b^k \in L_1$ (a distinguishing ext. is b^k).
Therefore, there are infinity many equivalence classes!
So, L_1 is not regular.

Myhill-Nerode method. Examples

Negative Example 2

$L_2 = \{a^n b a^n \mid n \in \mathbb{N}\}$ is not regular.

Proof

For $m \neq k$,

$$a^m b \not\equiv_{L_2} a^k b,$$

since $a^m b a^k \notin L_2, a^k b a^k \in L_2$ (a distinguishing ext. is a^k).
Therefore, there are infinity many equivalence classes!
So, L_2 is not regular.

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- Myhill-Nerode criteria
  - Positive Examples
  - Negative Examples
- **Pumping Lemma**

Pumping lemma

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If $L \subseteq \Sigma^*$ is a regular language then there exists $m \geq 1$ such that any $w \in L$ with $|w| \geq m$ can be represented as $w = xyz$ such that

- $y \neq \epsilon$,
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How Pumping lemma is useful?

- **Can we use this theorem to prove that a set is regular?**
  No, because it gives only a necessary condition for a language to be regular (and not a sufficient condition).

- **We can use it to prove that a language is not regular. How?**

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Corollary

If for any $m \geq 1$ there is $w \in L$ such that $|w| \geq m$ and for any representation $w = xyz$ with $y \neq \epsilon$

$$xy^iz \notin L \text{ for some } i \geq 1.$$

Then L is not a regular language.

Pumping lemma. Examples

Example 1

$L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Proof

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1) $\underbrace{a^{n-p_1-p_2}}_{x}\underbrace{(a^{p_1})}_{y}\underbrace{a^{p_2}b^n}_{z}$ and $p_1 \neq 0$

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Let $w = xyz \in L_2$ and $y \neq \epsilon$

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Negative Example 1

$L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Proof

For each m, let $w = a^k b^k$, where $k \geq m$. If $w = xyz$ and $|xy| \leq m$, then $w = \underbrace{a^{n-p_1-p_2}}_{x} \underbrace{(a^{p_1})}_{y} \underbrace{a^{p_2} b^n}_{z}$ and hence $xy^2z \notin L$

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Thank you for your attention!