Discrete Mathematics and Logic
Graph Theory
Lecture 5

Andrey Frolov
Professor

Innopolis University

What did we know in the last week?

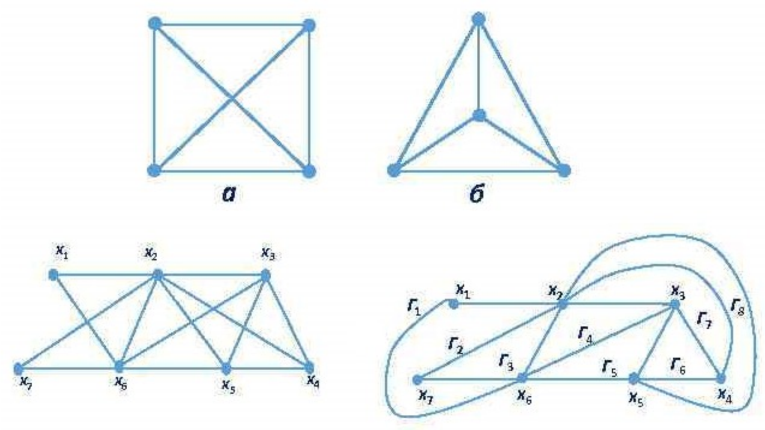
1. Hamilton paths and cycles
2. Hamiltonians
3. Ore's Theorem
4. Dirac's Theorem
5. The traveling salesman problem

Planar graphs

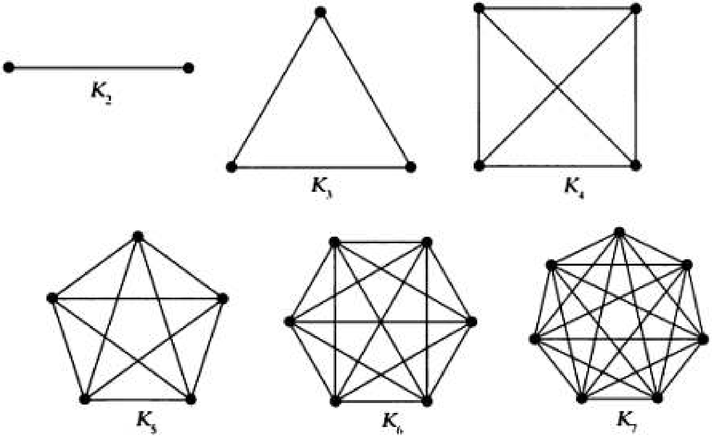
Definition

A graph G is called a **planar graph**, if it has a plane figure $P(G)$, called the **plane embedding** of G, where the lines corresponding to the edges do not intersect each other except at their ends.

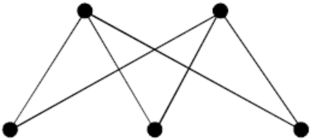
Planar graphs



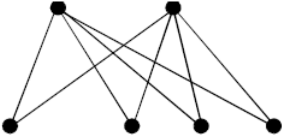
Planar graphs



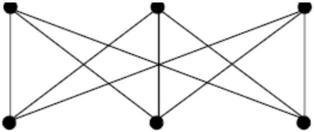
Planar graphs



$K_{2,3}$

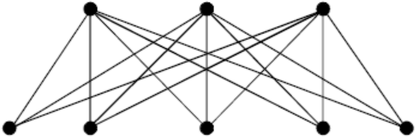


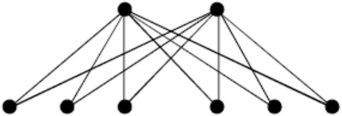
$K_{2,4}$



$K_{3,3}$

Planar graphs

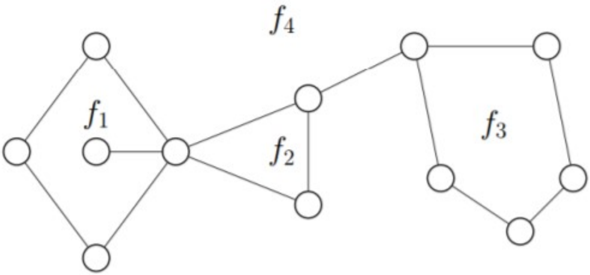




Euler's formula

Definition

A **face** of a plane graph is a region bounded by edges including infinitely large outer region (it is called exterior face).



Euler's formula

Theorem (Euler's formula)

Let G be a connected planar graph, $P(G)$ be any of its plane embeddings. Then

$$v - e + f = 2,$$

where f is the number of faces of $P(G)$, v is the number of vertices, e is the number of edges of G.

Euler's formula

Theorem (Euler's formula)

$$v - e + f = 2,$$

Proof by induction on the number of faces f

1. If $f = 1$ then the graph is a tree. The claim holds.

2. Suppose that the claim is true for all plane embeddings with less than f faces for $f \geq 2$.

3. Let $P(G)$ be a plane embedding of a connected planar graph G such that $P(G)$ has $f \geq 2$ faces.

Euler's formula

Theorem (Euler's formula)

$$v - e + f = 2,$$

Proof by induction on the number of faces f

3. Let $P(G)$ be a plane embedding of a connected planar graph G such that $P(g)$ has $f \geq 2$ faces.

Let e be an edge that lies in some cycle.

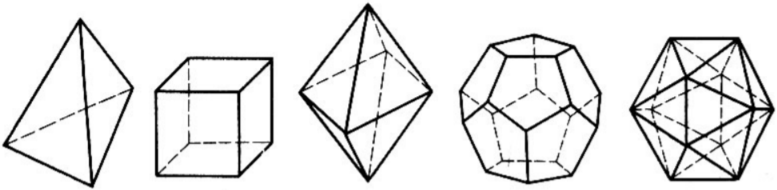
$G - e$ is planar and $P(G - e)$ has $f - 1$ faces, since the two faces of $P(G)$ that are separated by e are merged into one face of $P(G - e)$.

By induction hypothesis, $v_{G-e} - e_{G-e} + (f - 1) = 2$, and hence $v_G - (e_G - 1) + (f - 1) = 2$, and the claim follows.

Euler's formula

Euler's formula for a convex polyhedra

$$v - e + f = 2,$$



Corollaries

Corollary 1

If G is a planar graph and $v_G \geq 3$ then $e_G \leq 3v_G - 6$.

Proof

Each face contains at least three edges on its boundary.

Each edge lies on at most two faces.

Hence, $3f \leq 2e_G$.

So, $3(e_G - v_G + 2) \leq 2e_G$. Therefore, $e_G \leq 3v_G - 6$.

Corollaries

Corollary 1

If G is a planar graph and $v_G \geq 3$ then $e_G \leq 3v_G - 6$.

Corollary 2

K_5 is not planar.

Proof

$v = 5$, $e = 10$

If K_5 is planar then $10 \leq 3 \cdot 5 - 6 = 9$.

This is a contradiction.

Corollaries

Corollary 1

If G is a planar graph and $v_G \geq 3$ then $e_G \leq 3v_G - 6$.

The corollary is not a sufficient for planarity!

For $K_{3,3}$, $v = 6$, $e = 9$

$9 \leq 3 \cdot 6 - 6 = 12$

Corollaries

Corollary 3

$K_{3,3}$ is not planar.

Proof

$v = 6$, $e = 9$. If $K_{3,3}$ is planar then $f = 9 - 6 + 2 = 5$.

Since $K_{3,3}$ is bigraph, each face contains at least **four** edges on its boundary.

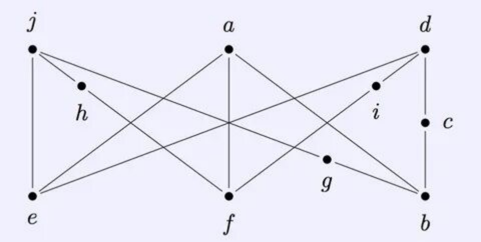
Hence, $4f \leq 2e$. I.e., $4 \cdot 5 \leq 2 \cdot 9$.

This is a contradiction.

Kuratowski's Theorem

Definition

An edge $e = uv$ is **subdivided**, when it is replaced by a path uxv by introducing a new vertex x. A **subdivision** H of a graph G is obtained from G by a sequence of subdivisions.



Kuratowski's Theorem

Lemma

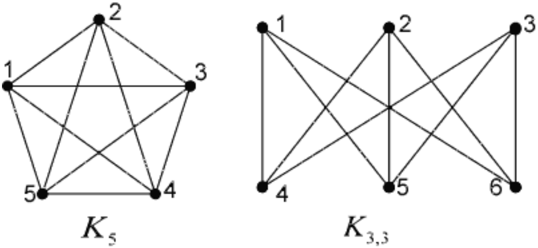
A graph is planar iff its subdivisions are planar.

It is obvious.

Kuratowski's Theorem

Kuratowski's Theorem

A graph is planar iff it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.



Colourings

Definition

A k-colouring of a graph G is a mapping $\alpha : V_G \to \{1, \ldots, k\}$.

The colouring α is **proper**, if adjacent vertices have different colours, i.e., $\alpha(v) \neq \alpha(v')$ for any $vv' \in E_G$.

Definition

A graph G is k-colourable, if there is a proper k-colouring for G.

The chromatic number is

$$\chi(G) = min\{k \mid G \text{ is } k\text{-colourable}\}$$

Colourings

Exercises

0. $\chi(G) = 0$ iff $V_G = \emptyset$,

1. $\chi(G) = 1$ iff $V_G = O_n$ ($|V_G| = n$, $E_G = \emptyset$),

2. $\chi(G) = 2$ iff G is a non-trivial bigraph.

3. $\chi(K_n) = n$

Colourings

Theorem (about 4 colours)

Each planar graph is 4-colourable.

What we knew today?

1. Planar graphs
2. Euler's formula $v - e + f = 2$
3. K_5 and $K_{3,3}$ are not planar
4. Kuratowski's Theorem
5. Colouring

Thank you for your attention!