Discrete Mathematics and Logic
Graph Theory
Lecture 6

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What did we know in the last week?

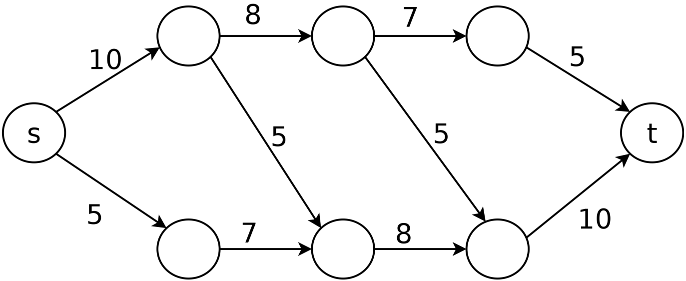
1. Planar graphs
2. Euler's formula $v - e + f = 2$
3. K_5 and $K_{3,3}$ are not planar
4. Kuratowski's Theorem
5. Colouring

Transportation networks

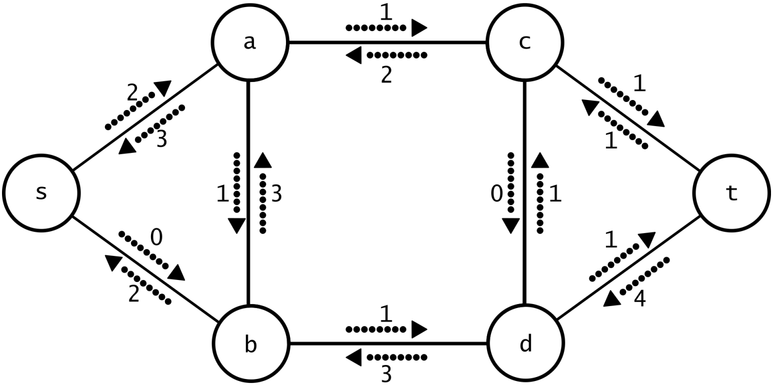
Definition

A $N = (V, E, c, s, t)$ is called a transportation **network**, if
1) (V, E) is a directed graph,
2) $c : E \to \mathbb{R}^+ \cup \{+\infty\}$ is the **capacity** function,
3) s and t are two distinguished vertices,
s is called the **source**, t is called the **sink**.

Transportation networks



Transportation networks



What about undirected graphs?

Flows

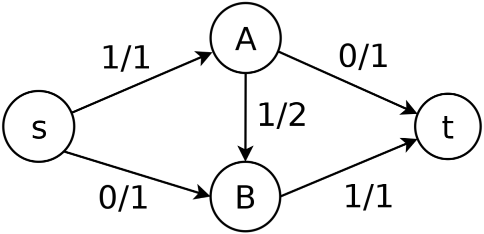
Definition

A **flow** in a network $N = (V, E, c, s, t)$ is a function $f : E \to \mathbb{R}^+$ such that

1) $0 \leq f(e) \leq c(e)$ for all $e \in E$,

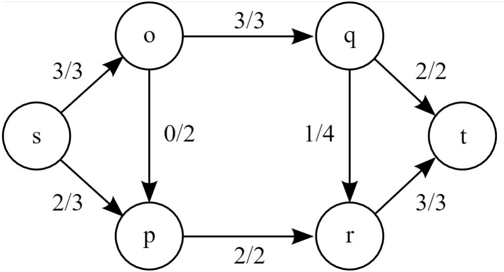
2) $\sum_{w \in V} f(\overrightarrow{v, w}) = \sum_{w' \in V} f(\overrightarrow{w', v})$

Flows



flow / capacity

Flows



flow / capacity

Maximum flows

Problem

For given network, to find a flow with the maximum possible value.

The **value** of a flow f is $\sum_{w \in V} f(\overrightarrow{s, w})$.

The naive wrong algorithm!

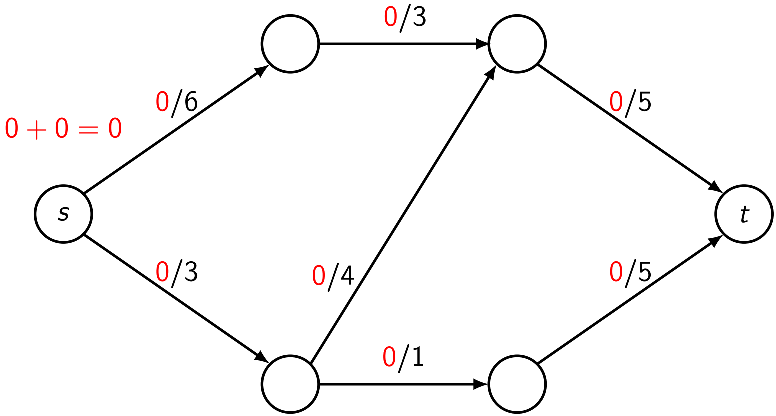
1) $f(u, v) = 0$ for any $u, v \in V$,

2) While there is a path p from s to t such that $c(u, v) - f(u, v) > 0$ for all $(u, v) \in p$:

a) find $c_f(p) = min\{c(u, v) - f(u, v) > 0 \mid (u, v) \in p\}$,

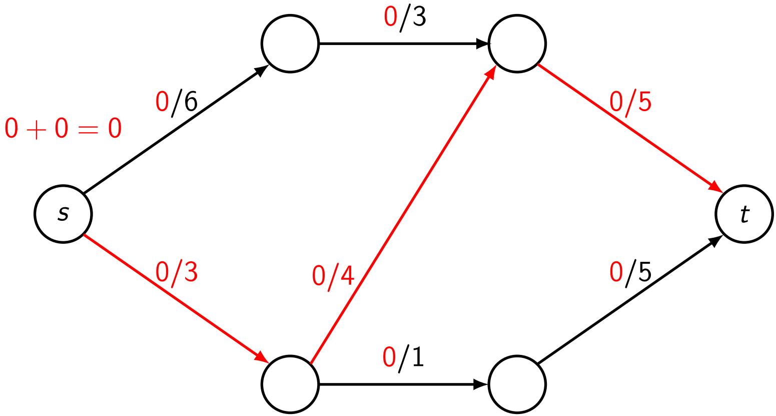
b) for any $(u, v) \in p$, put $f(u, v) := f(u, v) + c_f(p)$.

Step 1



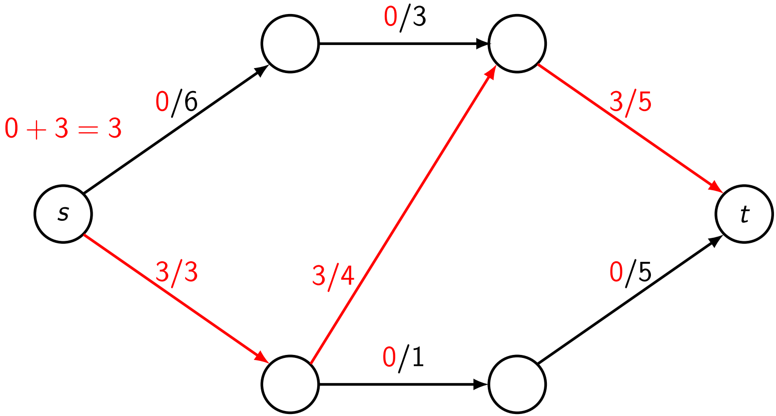
flow / capacity

Step 2



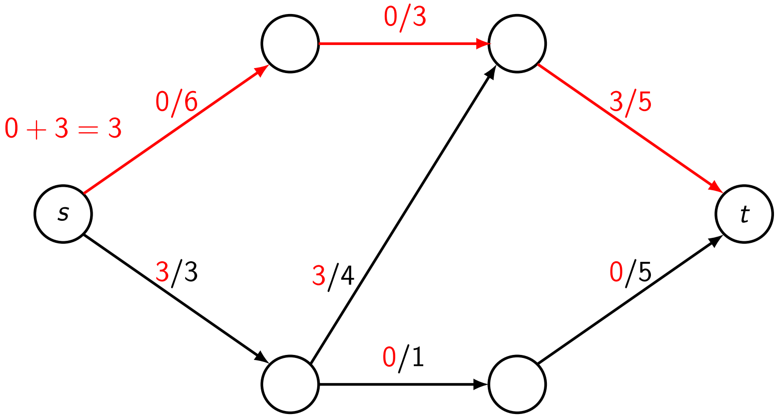
flow / capacity

Step 2



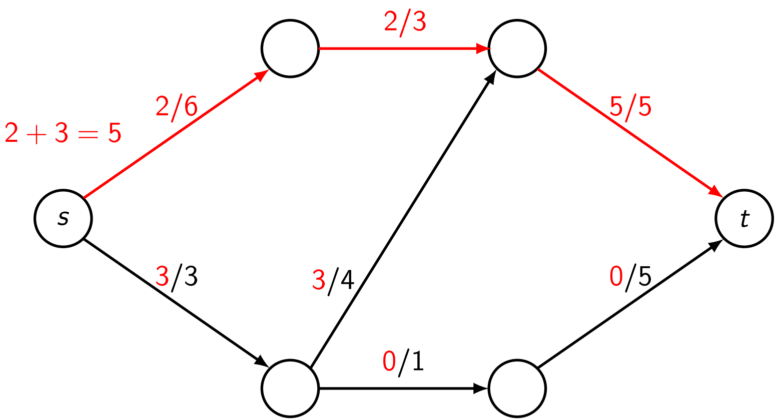
flow / capacity

Step 3



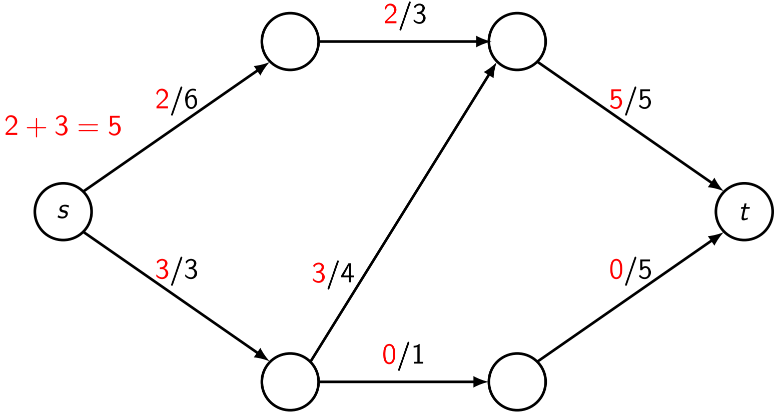
flow / capacity

Step 3



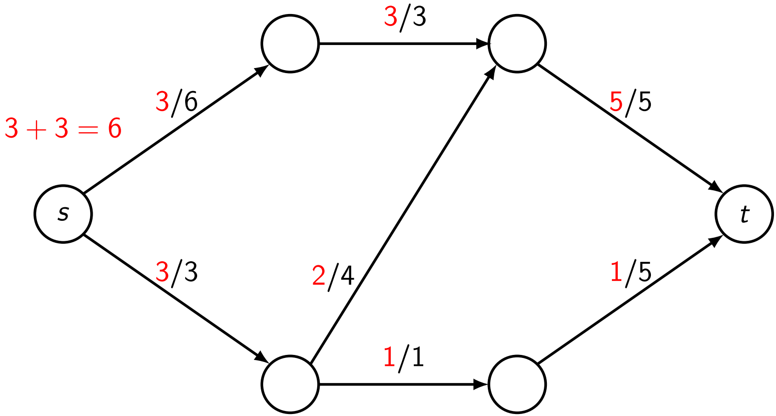
flow / capacity

What is the next step?



5 is not maximum!!

The answer



6 is maximum!!

flow / capacity

Cuts

Definition

For a network (V, E, c, s, t), a **cut** is a set $C \subseteq E$ such that there are sets S and T with

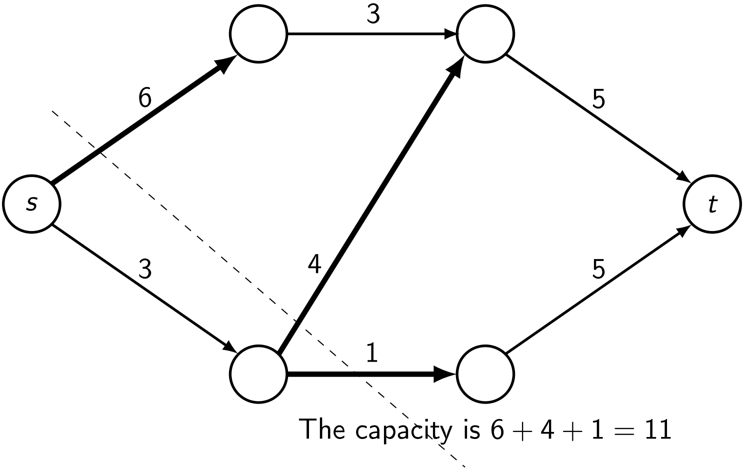
$$C = \{\overrightarrow{(u, v)} \in E \mid u \in S \,\&\, v \in T\},$$

$$S \cup T = V, S \cap T = \emptyset$$

$$s \in S, t \in T$$

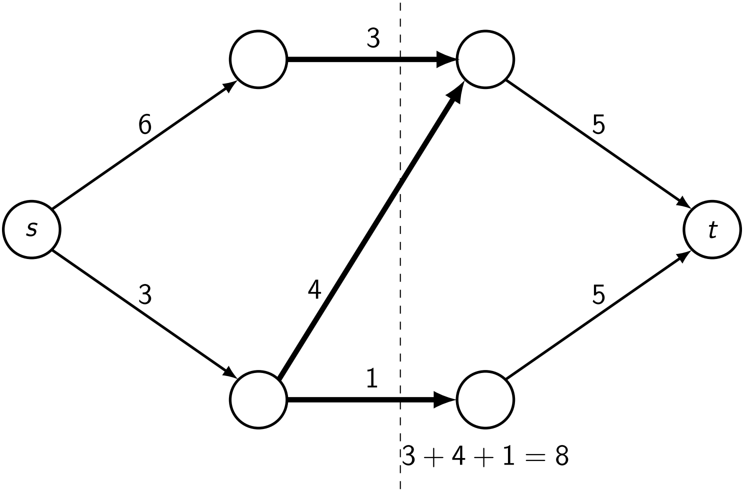
The capacity of a cut C is the sum of $c(e)$ for $e \in C$.

Cuts

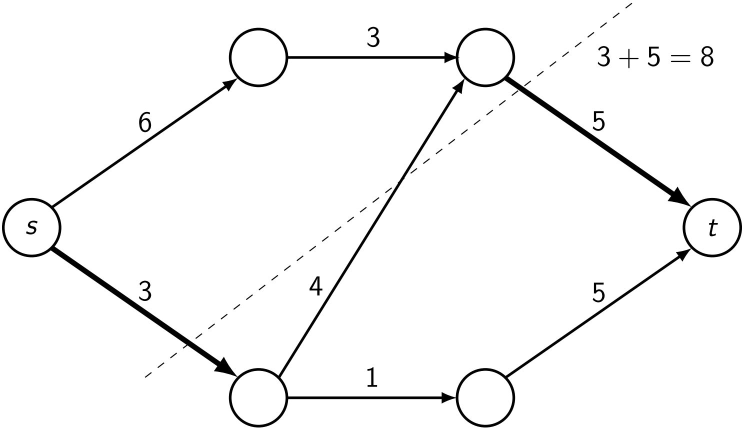


The capacity is $6 + 4 + 1 = 11$

Cuts



Cuts

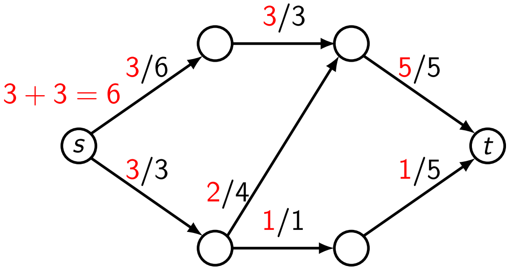


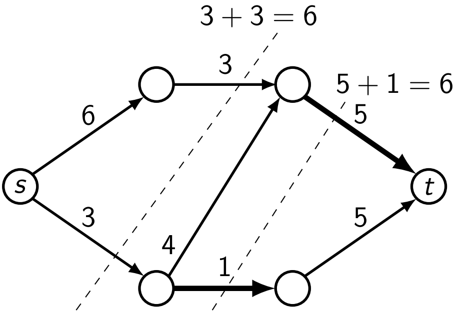
The Ford-Fulkerson theorem

Max-flow min-cut theorem

The maximum value of a flow equals to the minimum capacity over all cuts.

The Ford-Fulkerson theorem





The Ford-Fulkerson algorithm

1) $f(u,v) = 0$ for any $u, v \in V$,

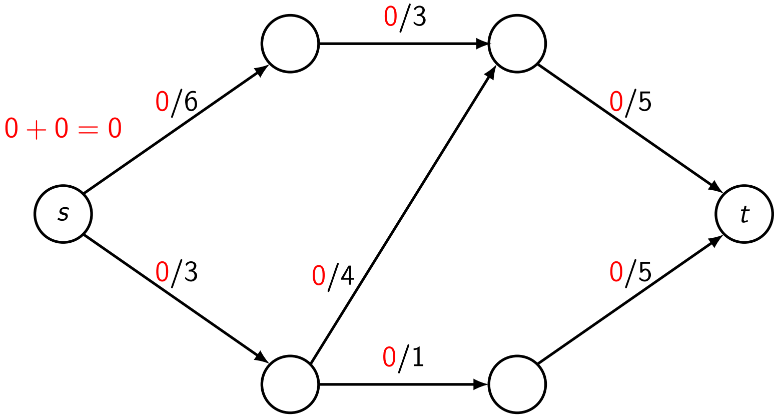
2) While there is a path p from s to t ignoring the direction of the edges such that $c(u,v) - f(u,v) > 0$ for all $(u,v) \in p$:

a) find $c_f(p) = min\{c(u,v) - f(u,v) > 0 \mid (u,v) \in p\}$,

b) put $f(u,v) := f(u,v) + c_f(p)$, if $\overrightarrow{(u,v)} \in p$,

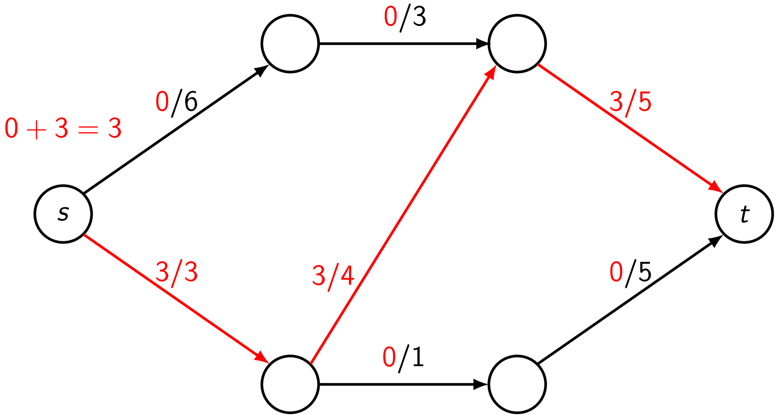
c) put $f(u,v) := f(u,v) - c_f(p)$, if $\overrightarrow{(v,u)} \in p$,

Step 1



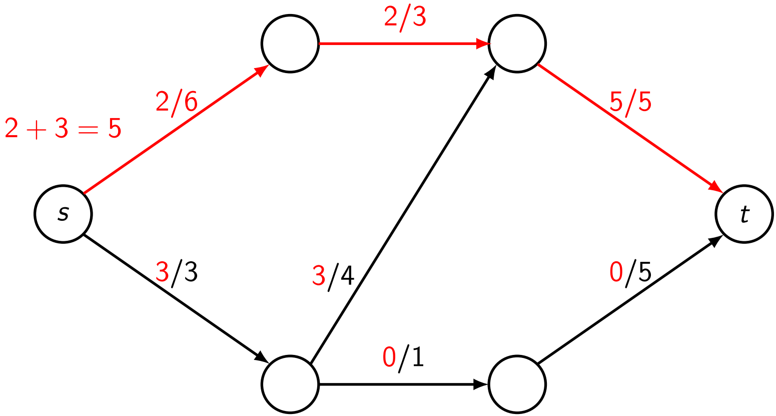
flow / capacity

Step 2



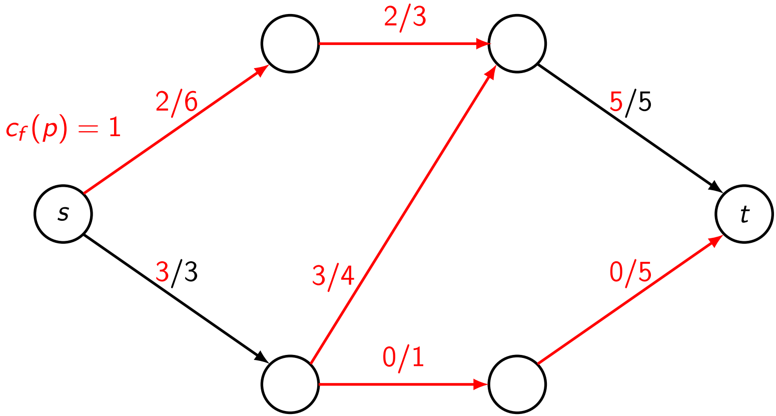
flow / capacity

Step 3



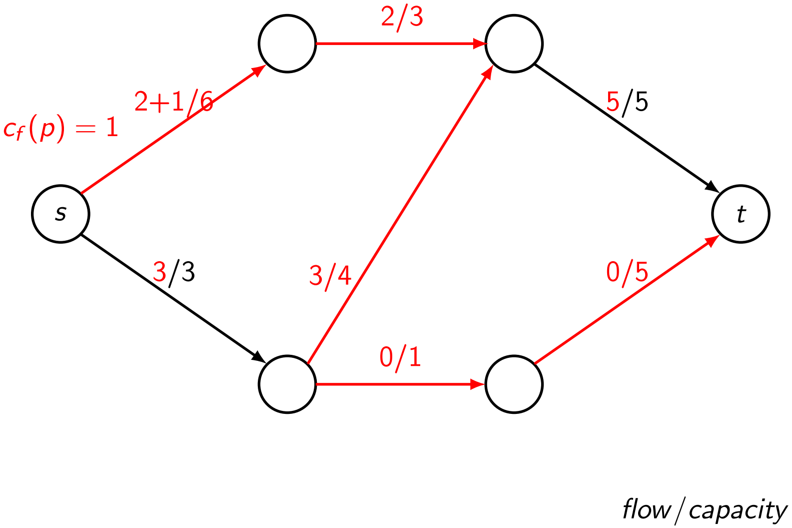
flow / capacity

Step 4-a

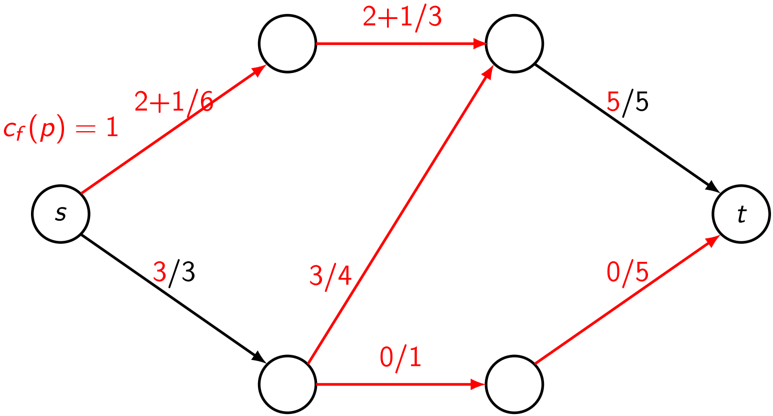


flow / capacity

Step 4-b

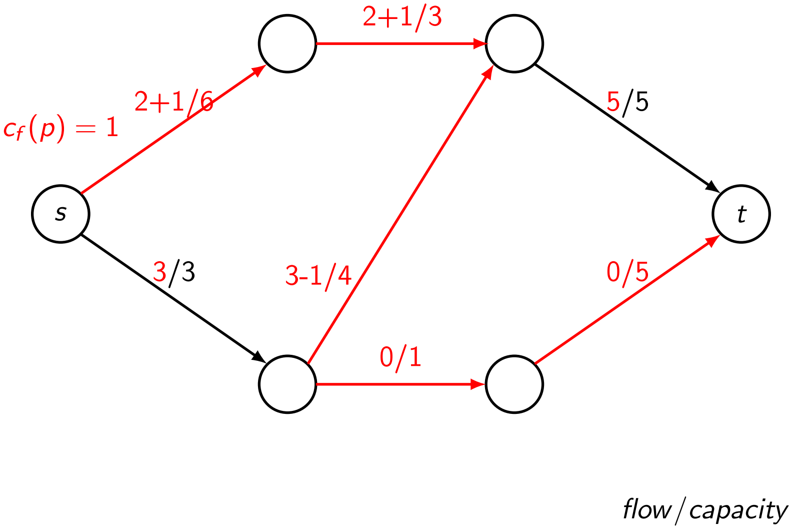


Step 4-c

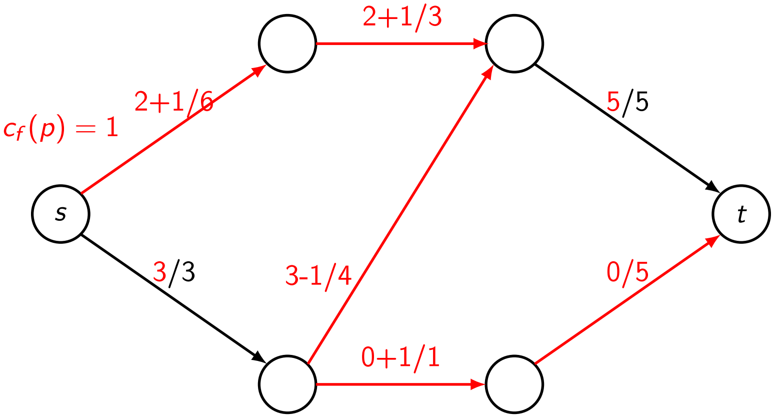


flow / capacity

Step 4-d

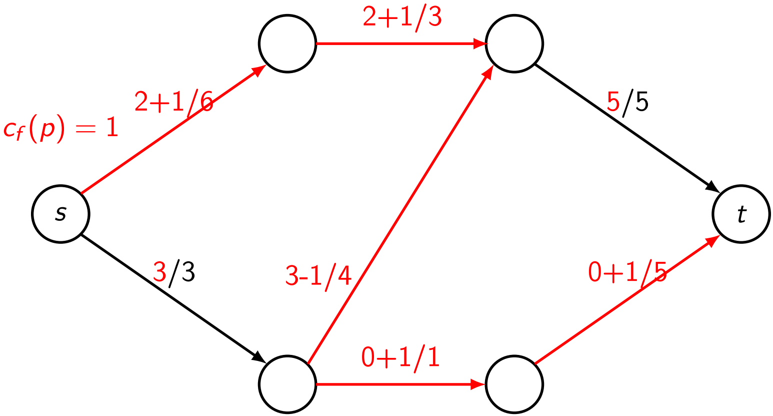


Step 4-e



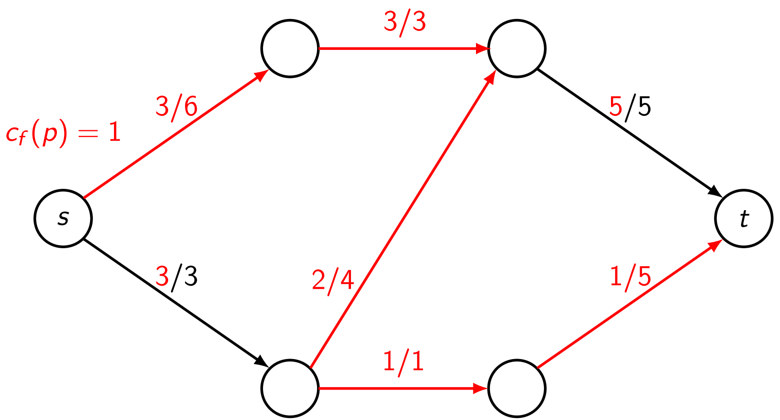
flow / capacity

Step 4-f



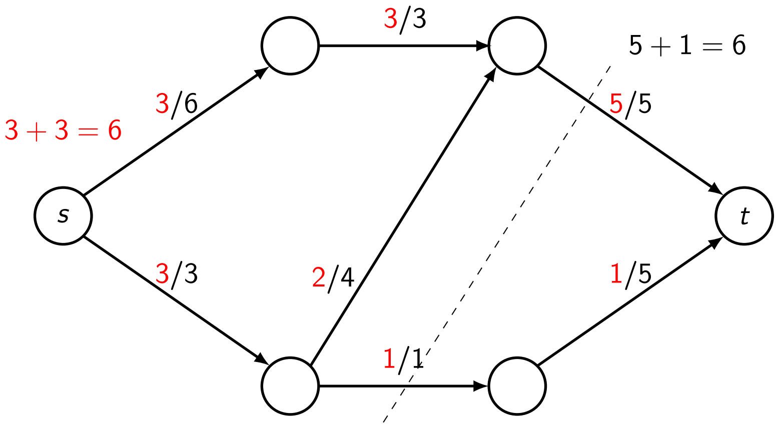
flow / capacity

Step 4-g



flow / capacity

Step 4-g



flow / capacity

What we knew today?

1. Transportation network
2. Flows
3. Cuts
4. Max-flow min-cut theorem
5. The Ford-Fulkerson algorithm

Thank you for your attention!