Discrete Mathematics and Logic
Graph Theory
Preparation for Final Exam

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Today plan

1. The final exam rules
2. The final exam structure
3. Main topics of the course
4. Questions and discussions (instead of Tutorial)

Announcement

The final exam will be held on Dec 15 in rooms 105 and 108 between 10:00 and 12:00.

The exam will take an hour and a half.

Main rules

The exam is written.

No books, no cell phones, no laptops!

Just your pen and your mind alone!

Syllabus

Type	Points
Attendance	20
Mid Term (Logic)	40
Final Exam (Disc Math)	40

The structure

Block A. Quizzes (10 points)

Block B. Exercisers (15 points)

Block C. Theoretical tasks (15 points)

Total. 40 points

The structure of Block A

Block A. 5 quizzes (10 points)

Example

(2 points) What is a tree?

a. A tree is a perennial plant with an elongated stem.

b. A tree is a graph without loops.

c. A tree is a graph without cycles.

If you are not ready for the exam, then Block A can be difficult!

The structure of Block B

Block B. 3 exercisers (15 points)

Example (see below)

(5 points) Let $G = (V, E)$ be a graph such that $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (1, 3), \ldots\}$. Find all spanning trees. What of them are pairwise non-isomorphic?

The structure of Block C

Block C. 3 theoretical tasks (15 points)

Example

(5 points) You need to prove that . . .

Topics

1. Basic definitions and handshaking lemma
2. Trees and spanning trees
3. Euler and Hamilton paths
4. Planar graphs
5. Maximum flow problem

Main Definition

Definition (undirected simple)

A pair $G = (V, E)$ is called a **graph**, if

$$E \subseteq \{\{u, v\} \mid u, v \in V \,\&\, u \neq v\}$$

The elements of V are **vertices**, the elements of E are **edges**.

Definitions

The degree $d_G(v)$ of a vertex v is the number of its neighbours:

$$d_G(v) = |N_G(v)|$$

Handshaking Lemma

Lemma (Handshaking lemma)

For each graph $G = (V_G, E_G)$,

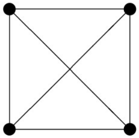
$$\sum_{v \in V_G} d_G(v) = 2 \cdot |E_G|.$$

In particular, the sum of all degrees is even.

Handshaking Lemma

Task 1

For some simple graph, let D be a set of its degrees. Draw such graph.
(a) $D = \{3, 3, 3, 3\}$, (b) $D = \{1, 2, 1, 4, 1\}$.



(b) By handshaking lemma, the sum of all degree is even. But $1 + 2 + 1 + 4 + 1 = 9$ is odd. Therefore, does not exit such graph.

Trees

Definition

A connected graph having no cycle is called a **tree**.

Theorem (the characteristic property for trees)

Let G be a connected graph with v vertices and e edges.

$$G \text{ is a tree iff } v = e + 1.$$

Spanning trees

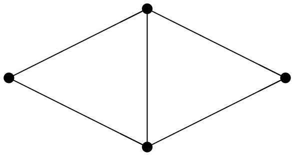
Definition

Let $G = (V, E)$ be a connected graph. A graph $G' = (V, E') \subseteq G$ is called a **spanning tree**, if G' is a tree.

Spanning trees

Task 2

For a given graph, draw all spanning trees and determine which of them are not pairwise isomorphic?

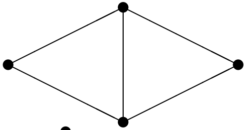


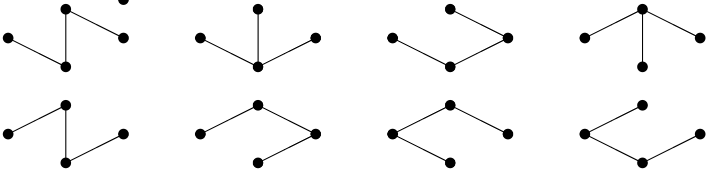
For a given graph G, $v_G = 4, e_G = 5$.
For its spanning tree, $v_T = v_G = 4$ and $e_T = v_T - 1 = 4 - 1 = 3$
We need to delete $e_G - e_T = 5 - 3 = 2$ edges.

Spanning trees

Task 2

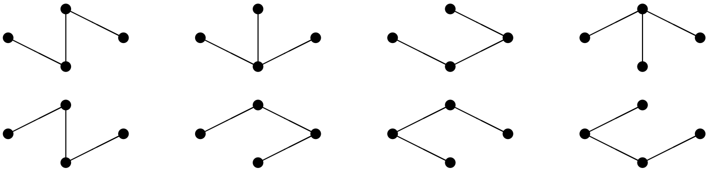
For a given graph, draw all spanning trees and determine which of them are not pairwise isomorphic?





Spanning trees

8 spanning trees

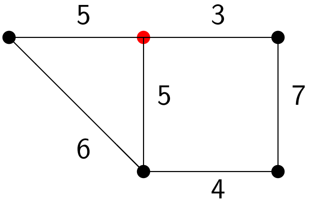


2 non-isomorphic

The Prim's algorithm

Task 3

For a given graph, find a spanning tree with minimum total weight.



Euler and Hamilton paths

Definition

A path or cycle is called **Euler**, if it contains every edge and only once.

Definition (only in Russian)

A graph having an Euler path is called a **semi-Eulerian**.

Definition

A graph having an Euler cycle is called an **Eulerian**.

Euler and Hamilton paths

Definitions

A path or cycle is called **Hamilton** if it visits every vertex and only once.

Definitions (only in Russian)

A graph having a Hamilton path is called a **semi-Hamiltonian**.

Definitions

A graph having a Hamilton cycle is called a **Hamiltonian**.

Euler and Hamilton paths

Task 3

Construct a graph which is an Eulerian but not a Hamiltonian.

Euler and Hamilton paths

Theorem (the sufficient and necessary condition for Eulerians)

A non-trivial connected graph is an Eulerian iff every vertex has even degree.

Theorem (Dirac, 1952)

Let G be a simple graph with $n \geq 3$ vertices. Suppose that, for any vertex v, $deg(v) \geq n/2$. Then G is a **Hamiltonian**.

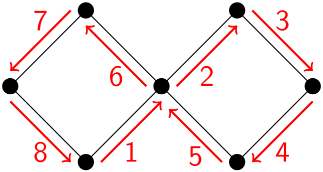
Theorem (Ore, 1960)

Let G be a simple graph with $n \geq 3$ vertices. Suppose that, for any non-adjacent vertices v, v', $deg(v) + deg(v') \geq n$. Then G is a **Hamiltonian**.

Euler and Hamilton paths

Task 3

Construct a graph which is an Eulerian but not a Hamiltonian.



Planar graphs

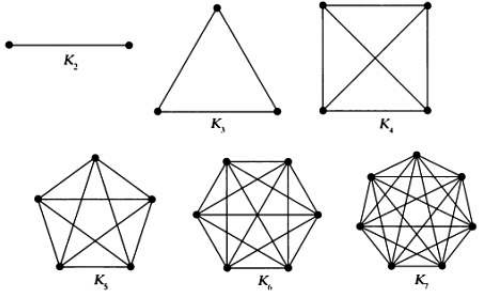
Definition

A graph G is called a **planar graph**, if it has a plane figure $P(G)$, called the **plane embedding** of G, where the lines corresponding to the edges do not intersect each other except at their ends.

Planar graphs

Task 4

Find all n such that K_n is planar.



Planar graphs

Task 4

Find all n such that K_n is planar.

Answer

For $n = 1, 2, 3, 4$, the graph K_n is planar.

For $n \geq 5$, it is not.

Euler's formula

Task 5

Prove that if G is a planar graph and $v \geq 3$ then $e \leq 3v - 6$, where v is number of all its vertices, e is number of all its edges.

Euler's formula

Theorem (Euler's formula)

Let G be a connected planar graph, $P(G)$ be any of its plane embeddings. Then

$$v - e + f = 2,$$

where f is the number of faces of $P(G)$, v is the number of vertices, e is the number of edges of G.

Euler's formula

Task 5

Prove that if G is a planar graph and $v \geq 3$ then $e \leq 3v - 6$, where v is number of all its vertices, e is number of all its edges.

Proof

Since, (1) each face contains at least three edges on its boundary, and (2) each edge lies on at most two faces,
we have $3f \leq 2e$.

Since $f = e - v + 2$, we have that $3(e - v + 2) \leq 2e$.

Therefore, $e \leq 3v - 6$.

Remark

Kuratowski's Theorem

A graph is planar iff it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Remark

You need to know Kuratowski's Theorem, but no proofs uses it!

Colourings

Definition

A k-colouring of a graph G is a mapping $\alpha : V_G \to \{1, \ldots, k\}$.

The colouring α is **proper**, if adjacent vertices have different colours, i.e., $\alpha(v) \neq \alpha(v')$ for any $vv' \in E_G$.

Definition

A graph G is k-colourable, if there is a proper k-colouring for G.

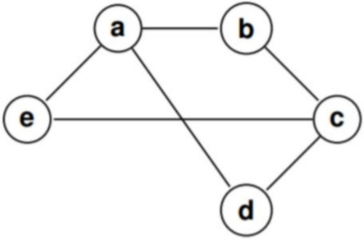
The chromatic number is

$$\chi(G) = min\{k \mid G \text{ is } k\text{-colourable}\}$$

Colourings

Task 6

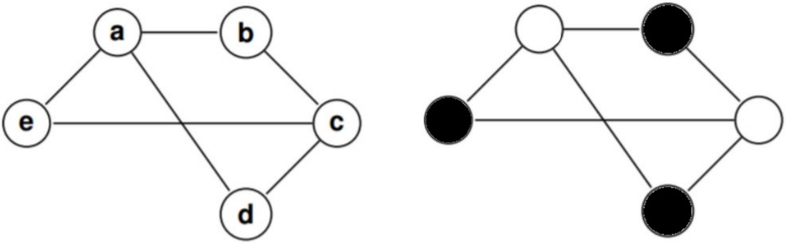
Find the chromatic number of the following graphs.



Colourings

Task 6

Answer: the picture below shows that it is 2-colourable. And, obviously, 1 colour is not enough.



Transportation networks

Definition

A $N = (V, E, c, s, t)$ is called a transportation **network**, if
1) (V, E) is a directed graph,
2) $c : E \to \mathbb{R}^+ \cup \{+\infty\}$ is the **capacity** function,
3) s and t are two distinguished vertices,
s is called the **source**, t is called the **sink**.

Flows

Definition

A **flow** in a network $N = (V, E, c, s, t)$ is a function $f : E \to \mathbb{R}^+$ such that

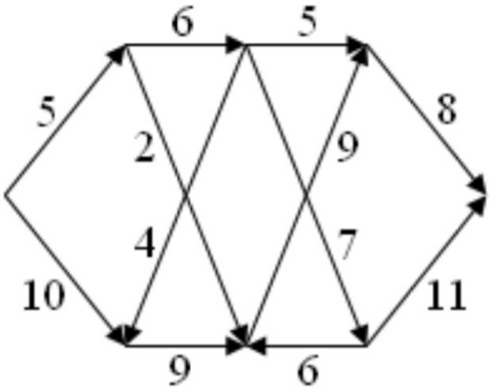
1) $0 \le f(e) \le c(e)$ for all $e \in E$,

2) $\sum_{w \in V} f(\overrightarrow{v, w}) = \sum_{w' \in V} f(\overrightarrow{w', v})$ for all $v \in V \setminus \{s, t\}$.

Flows

Task 7

Find a flow with the maximum possible value.



Cuts

Definition

For a network (V, E, c, s, t), a **cut** is a set $C \subseteq E$ such that there are sets S and T with

$$C = \{\overrightarrow{(u, v)} \in E \mid u \in S \,\&\, v \in T\},$$

$$S \cup T = V, S \cap T = \emptyset$$

$$s \in S, t \in T$$

The capacity of a cut C is the sum of $c(e)$ for $e \in C$.

The Ford-Fulkerson theorem

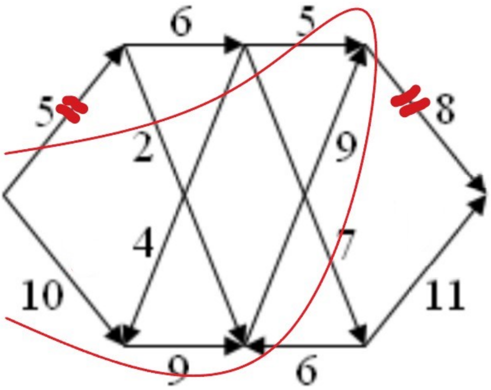
Max-flow min-cut theorem

The maximum value of a flow equals to the minimum capacity over all cuts.

Flows

Task 7

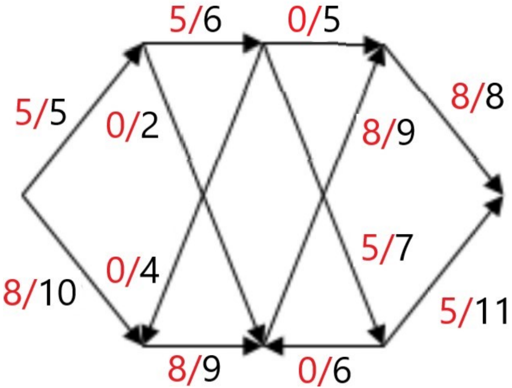
Answer: the minimum cut is $5 + 8 = 13$.



Flows

Task 7

Answer: the maximum flow is $5 + 8 = 13$.



Good luck!