Discrete Mathematics and Logic
Graph Theory
Lecture 1

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Books

Mathematics for Computer Science
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Books

GRAPH THEORY WITH APPLICATIONS

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Problem with graph terminology

GRAPH THEORY

by *Frank Harary*

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”In order to avoid quibbling at conferences on graph-theory, it has been found convenient to adopt the procedure that each man state in advance the graph theoretic language he would use. Even the very word "graph" has not been sacrosanct.“

Frank Harary

Main Definition

Definition (undirected simple)

A pair $G = (V, E)$ is called a **graph**, if

$$E \subseteq \{\{u, v\} \mid u, v \in V \,\&\, u \neq v\}$$

Terminology

The elements of V	vertices	nodes	points
The elements of E	edges	arcs	lines or links

An edge $\{x, y\}$ is written as xy, $x - y$, or (x, y).

Adjacency and Incidence

Definitions

1) **Two vertices** x, y of G are **adjacent** or **neighbours**, if xy is an edge of G.

2) A **vertex** v is **incident** with an **edge** e, if $v \in e$, that is $e = (v, \cdot)$ or $(\cdot, v)$.

3) Two vertices incident with an edge are its **endvertices** or **ends**.

4) **Two edges** v, w of G are **adjacent** or **neighbours**, if one of their ends is the same.

Isomorphisms

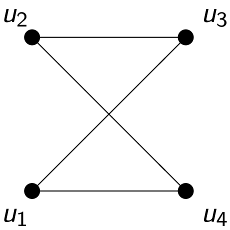
Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' **isomorphic**, and write $G \cong G'$, if there is a bijection $\varphi : V \to V'$ (φ is called an **isomorphism**) such that for all $x, y \in V$

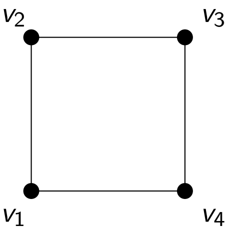
$$(x, y) \in E \Leftrightarrow (\varphi(x), \varphi(y)) \in E'.$$

Isomorphisms

Example



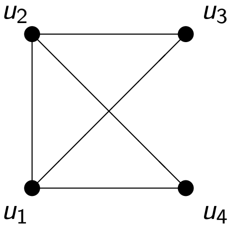
$G_1 \cong G_2$



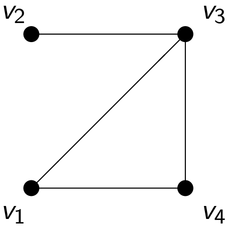
$\phi : u_1 \rightarrow v_1, u_2 \rightarrow v_3, u_3 \rightarrow v_2, u_4 \rightarrow v_4$

Isomorphisms

Example



$G_1 \ncong G_2$



Degrees

Definitions

1) The set of neighbours of a vertex v in $G = (V_G, E_G)$ is denoted by $N_G(v)$ or shortly $N(v)$.

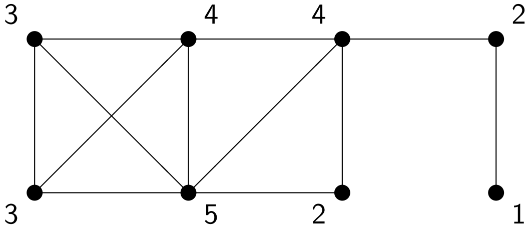
$$N_G(v) = \{u \in V_G \mid (v, u) \in E_G\}$$

2) The degree (or valency) $d_G(v) = d(v)$ of a vertex v is the number of its neighbours:

$$d_G(v) = |N_G(v)|$$

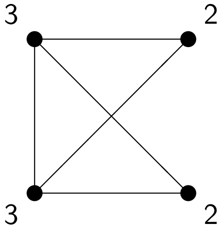
Degrees

Example

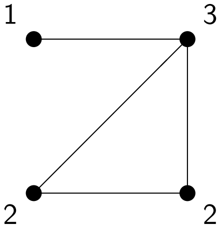


Degrees and Isomorphisms

Example



$G_1 \ncong G_2$



Degrees and Isomorphisms

Question

Suppose that two graphs have the same collections of vertices. Then is it true that these two graphs are isomorphic?

Handshaking Lemma

Lemma (Handshaking lemma)

For each graph $G = (V_G, E_G)$,

$$\sum_{v \in V_G} d_G(v) = 2 \cdot |E_G|.$$

If several people shake hands, then the number of hands shaken is even.

Handshaking Lemma

Lemma (Handshaking lemma)

For each graph $G = (V_G, E_G)$,

$$\sum_{v \in V_G} d_G(v) = 2 \cdot |E_G|.$$

Proof

Note that every edge $e \in E_G$ has two ends.

Handshaking Lemma

Question

Is there a graph with 3 vertices whose degrees are $(2, 3, 4)$?

Handshaking Lemma

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Is there a graph with 3 vertices whose degrees are $(2, 3, 4)$?

Answer

No!

Handshaking Lemma

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Is there a graph with 3 vertices whose degrees are $(2, 3, 4)$?

Answer

No!

Why?

Handshaking Lemma

Question

Is there a graph with 3 vertices whose degrees are $(2, 3, 4)$?

Answer

No!

Why?

By handshaking lemma, the sum of degrees must be even, but $2 + 3 + 4 = 9$ is odd.

Walks and Paths

Definitions

The sequence $u_1 u_2 \ldots u_k$ is called a **walk** or a **way** from u_1 to u_k, if any vertices u_i and u_{i+1} are neighbours.

Definition

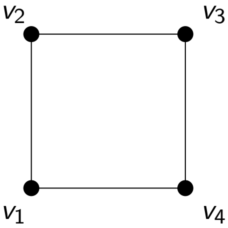
Let the sequence $u_1, u_2, \ldots, u_k$ be a walk. We say that

- it is **closed**, if $u_1 = u_k$.
- it is a **path**, if $(u_i, u_{i+1}) \neq (u_j, u_{j+1})$ for all $i \neq j$.
- it is a **cycle**, if it is a closed path.

Walks and Paths

Example

$v_1 v_2 v_3 v_1$ – a non-walk
$v_1 v_2 v_3 v_2$ – some walk
$v_1 v_2 v_3 v_2 v_1$ – a closed walk
$v_4 v_3 v_4$ – a closed walk
$v_4 v_3 v_2$ – a path
$v_1 v_2 v_3 v_4$ – a path
$v_1 v_2 v_3 v_4 v_1$ – a cycle



Connectivity

Definition

A non-empty graph G is called **connected** if G contains a path (walk) from v to w for any its vertices v, w.

Otherwise, G is called **disconnected**.

Definition

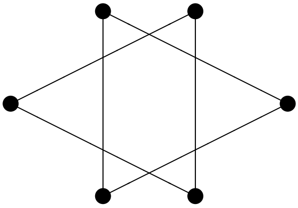
Let $G = (V, E)$ be a graph. A maximal connected subgraph of G is called a **component** of G.

$G' = (V', E')$ is subgraph of G, if $V' \subseteq V$ and $E' \subseteq E$.

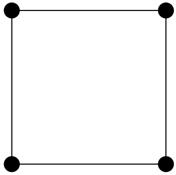
Connectivity

Example

Disconnected



Connected



Connectivity

Definition

$G' = (V', E')$ is called a **subgraph** of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

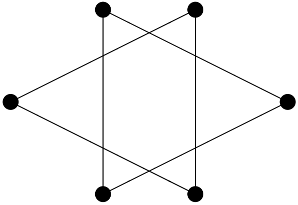
Definition

Let $G = (V, E)$ be a graph. A maximal connected subgraph of G is called a **component** of G.

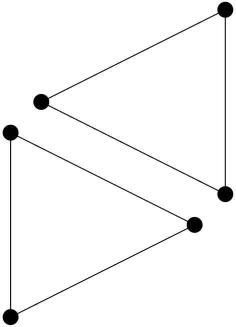
Connectivity

Example

Disconnected



2 components



Connectivity

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We set

$$G \cup G' \leftrightharpoons (V \cup V', E \cup E')$$

$$G \cap G' \leftrightharpoons (V \cap V', E \cap E')$$

Proposition

Any graph is a disjoint union of all its connected components.

Proof

It is obvious. :)

What we knew today?

1. My favorite question is ”Why?“.
2. Books
3. The basic terminology of Graph Theory.
4. Handshaking lemma.
5. Connectivity.

Thank you for your attention!