① General eq-n of 2nd order curve

$\Rightarrow$ canonical eq-n.

$$a_{11}x^2 + \underbrace{a_{12}xy} + a_{22}y^2 + a_1x + a_2y + a_0 = 0$$

general eq-n.

Step 1: eliminate $xy$ term by rotation at an angle $\alpha$.

$(x', y')$
$$\begin{cases} x = x'\cos\alpha - y'\sin\alpha \\ y = x'\sin\alpha + y'\cos\alpha \end{cases}$$

$$a_{11}(x'\cos\alpha - y'\sin\alpha)^2 + a_{12}(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha) +$$
$$+ a_{22}(x'\sin\alpha + y'\cos\alpha)^2 + \ldots = 0$$

$$x'^2(a_{11}\cos^2\alpha + a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha) +$$
$$+ y'^2(a_{11}\sin^2\alpha - a_{12}\cos\alpha\sin\alpha + a_{22}\cos^2\alpha) +$$
$$+ x'y\underbrace{(-2a_{11}\cos\alpha\sin\alpha + a_{12}\cos^2\alpha - a_{12}\sin^2\alpha + 2a_{22}\cos\alpha\cdot\sin\alpha)}_{=0}$$
$$+ \ldots = 0$$

$$a_{12}(\cos^2\alpha - \sin^2\alpha) - 2(a_{11}-a_{22})(\cos\alpha\cdot\sin\alpha)=0$$

$\sin 2\alpha$

$$\underline{a_{12}}\cos 2\alpha - (a_{11}-a_{22})\sin 2\alpha = 0 \qquad a_{12}\neq 0$$

$$\cos 2\alpha = \frac{a_{11}-a_{22}}{a_{12}}\cdot \sin 2\alpha$$

$$\boxed{ctg\, 2\alpha} = \frac{a_{11}-a_{22}}{a_{12}} \qquad \alpha = \frac{1}{2}\operatorname{arcctg}\left(\frac{a_{11}-a_{22}}{a_{12}}\right)$$

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$$a'_{11}x'^2 + \underline{a'_{12}}x'y' + a'_{22}y'^2 + a'_1x' + a'_2y' + a'_0 = 0$$

$$\begin{cases} a'_{11} = a_{11}\cos^2\alpha + a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha \\ a'_{12} = \underline{\dots} = (a_{22}-a_{11})\sin 2\alpha + a_{12}\cos 2\alpha \quad (=0) \\ a'_{22} = a_{11}\sin^2\alpha - a_{12}\cos\alpha\sin\alpha + a_{22}\cos^2\alpha \\ a'_1 = a_1\cos\alpha + a_2\sin\alpha \\ a'_2 = -a_1\sin\alpha + a_2\cos\alpha \\ a'_0 = a_0 \end{cases}$$

$$a_{11}' x'^2 + a_{22}' y'^2 + a_1 x' + a_2 y' + a_0' = 0$$

Step 2: eliminate linear terms

$$\begin{cases} x' = x'' + a \\ y' = y'' + b \end{cases}$$

$$a_{11}' (x'' + a)^2 + a_{22}' (y'' + b)^2 + a_1 (x'' + a) + a_2 (y'' + b) + a_0' = 0$$

$$a_{11}' x''^2 + a_{22}' y''^2 + \underline{(a_1' + 2a_{11}' a)} x'' + \underline{(a_2' + 2a_{22}' b)} y'' +$$

$$+ (a_0' + a_{11}' a^2 + a_{22}' b^2) + \ldots = 0$$

$$a_1' + 2a_{11}' a = 0 \qquad a_{11}' \neq 0$$

$$a_2' + 2a_{22}' b = 0 \qquad a_{22}' \neq 0$$

$$a = -\frac{a_1'}{2a_{11}'}$$

$$b = -\frac{a_2'}{2a_{22}'}$$

$$a_{11}'' x''^2 + a_{22}'' y''^2 + a_0'' = 0$$

// easy.

Canonical form.

## Discriminant of a 2nd ORDER curve

$$\underline{a_{11}} x^2 + \underline{a_{12}} xy + \underline{a_{22}} y^2 + a_1 x + a_2 y + a_0 = 0$$

quadratic coef.

$$\underline{D} = a_{12}^2 - 4 a_{11} a_{22}$$

$$D' = a_{12}'^2 - 4 a_{11}' a_{22}' = \dots\dots\dots = a_{12}^2 - 4 a_{11} a_{22} = D$$

after Rotation

Discriminant is invariant

- under Rotation
- under Shift

(1) Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$D = 0 - 4 \frac{1}{a^2} \frac{1}{b^2} = - \frac{4}{a^2 b^2} < 0$$

(2) Parabola : $D = 0$

(3) Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ : $D = 0 - 4 \frac{1}{a^2} \left(-\frac{1}{b^2}\right) = \frac{4}{a^2 b^2} > 0$

# Degenerate Cases

$$y = \pm kx$$

$$y^2 = k^2 x^2$$

$$y^2 - k^2 x^2 = 0$$

$$\frac{y^2}{k^2} - \frac{x^2}{1} = 0$$ 2 lines. (crossing each other)

pair of lines

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$ $x = y = 0$

a point