Summer Bootcamp 2021

Introduction to Computer Science

Lecture 4

The Basics of Combinatorics

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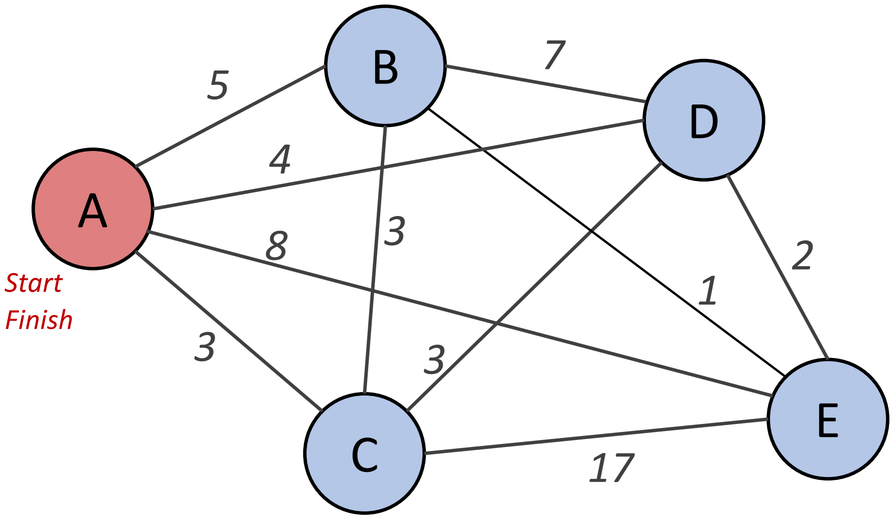
August 05, 2021

INNOPOLIS UNIVERSITY

Introduction: the Traveling Salesman Problem (TSP)

Find the shortest route through all cities, starting and ending at city A,

such that no city is visited twice.



Graph representing the distances (or travel times) between cities

Assumptions:

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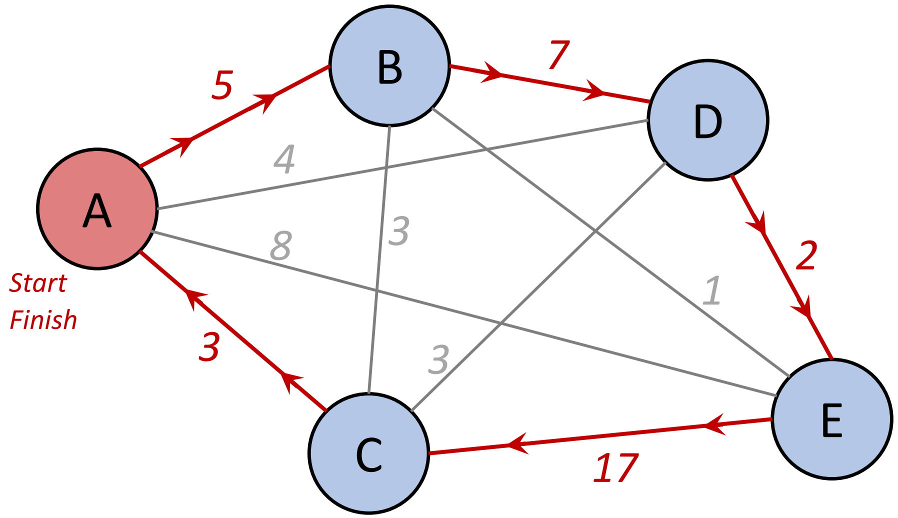

Route length: 5+7+2+17+3 = 34

One possible route (selected in red)

$A \longrightarrow B \longrightarrow$D $\longrightarrow$E $\longrightarrow$C $\longrightarrow$A

has length 5+7+2+17+3 = 34

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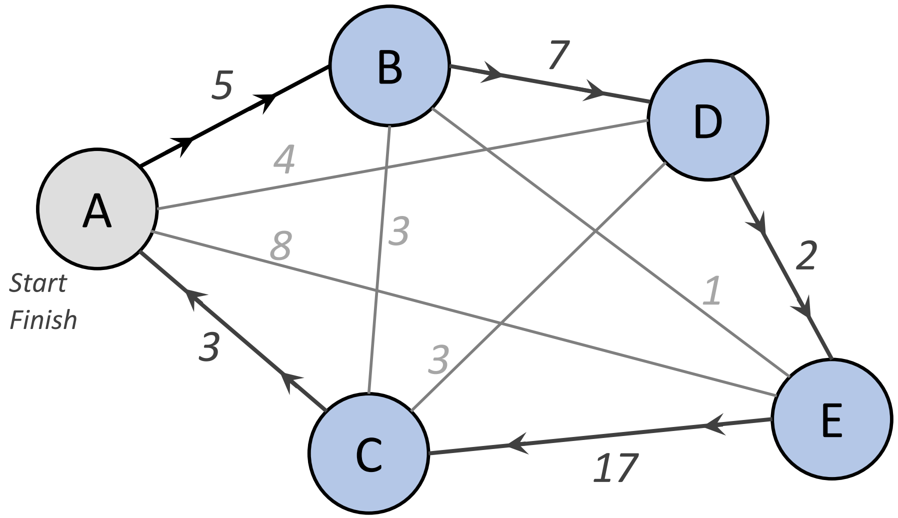
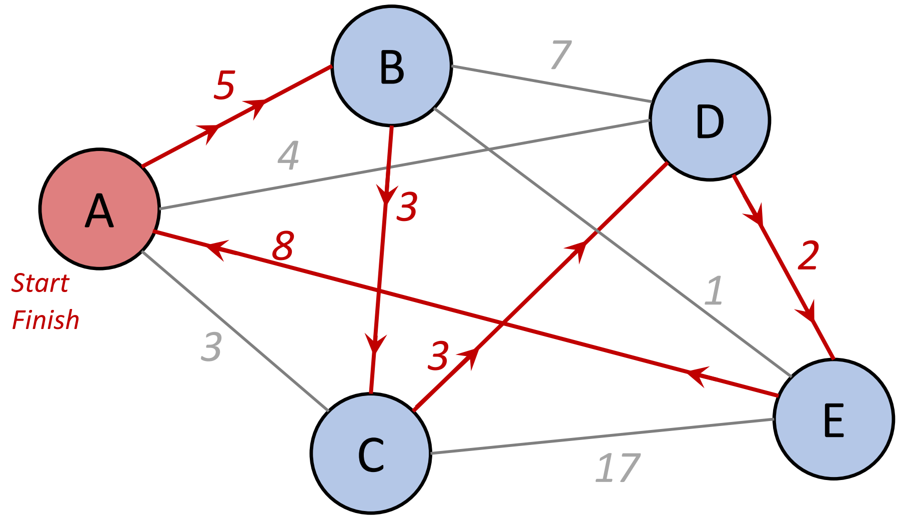
A different shorter route:





Route length: 5+7+2+17+3 = 34

Route length: 5+3+3+2+8 = 21

(But is it the shortest (optimal) one?!)

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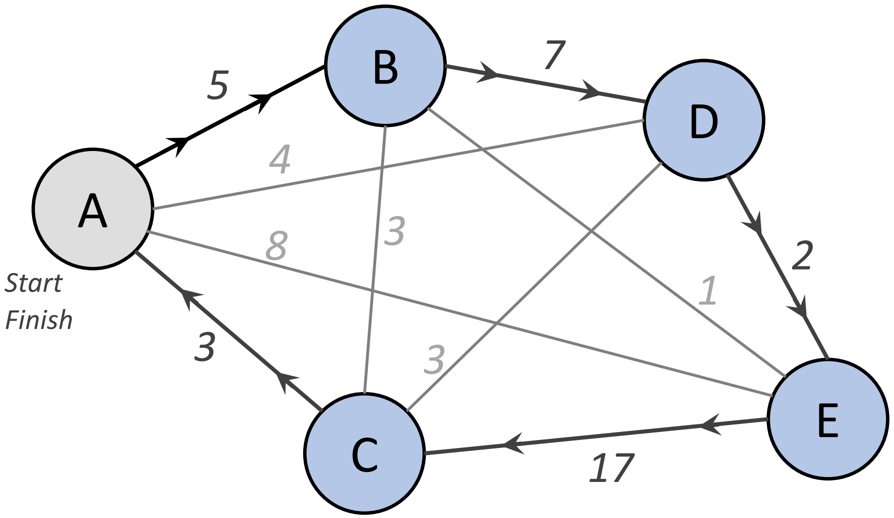

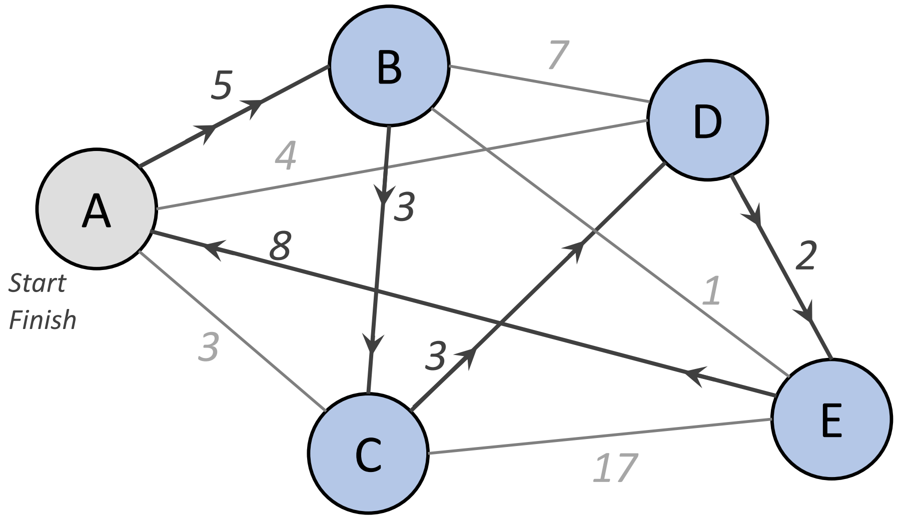

Route length: 5+7+2+17+3 = 34

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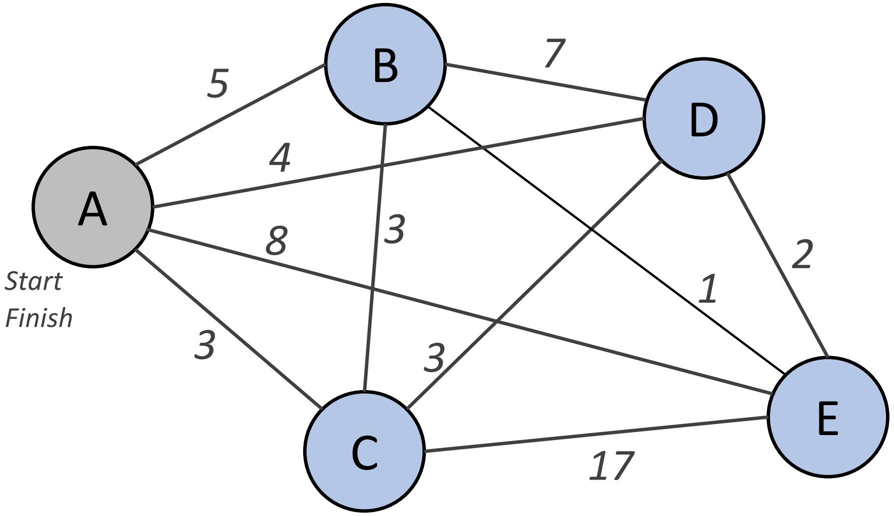
For the case of 4 cities (plus A), 24 routes exist in total

Combinatorics: How many feasible routes are?



Route passes through all cities, starting and ending at A:

A	?	?	?	?	A

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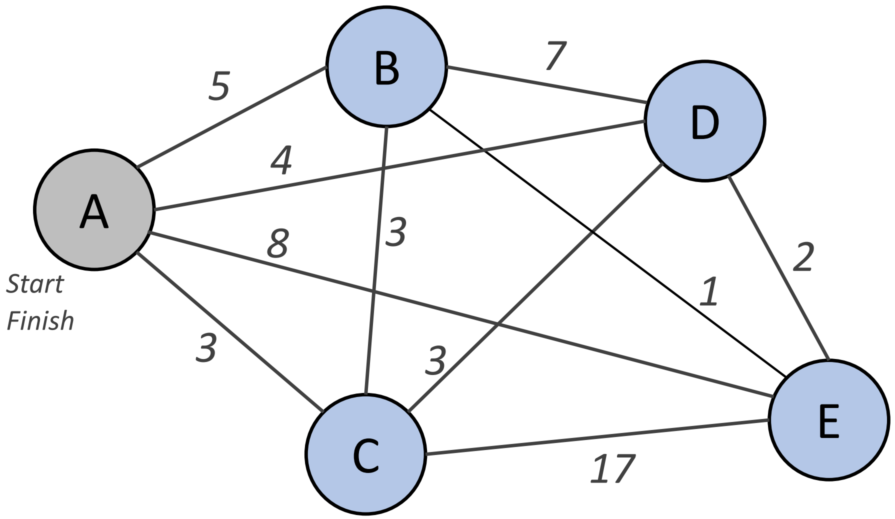

A	?	?	?	?	A

4 possible cases:
B, C, D, or E

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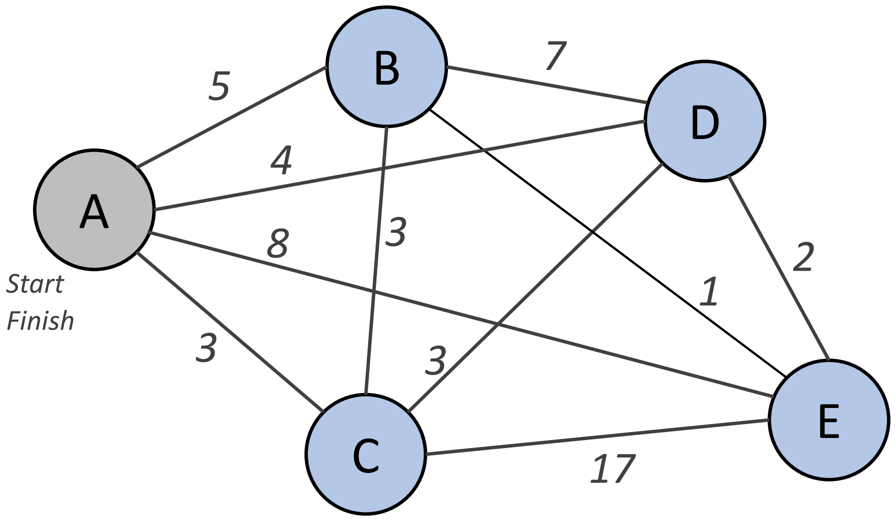

A	?	?	?	?	A

4 cases
(B, C, D, or E)

3 cases:
all cities (B, C, D, or E),
except the one visited before

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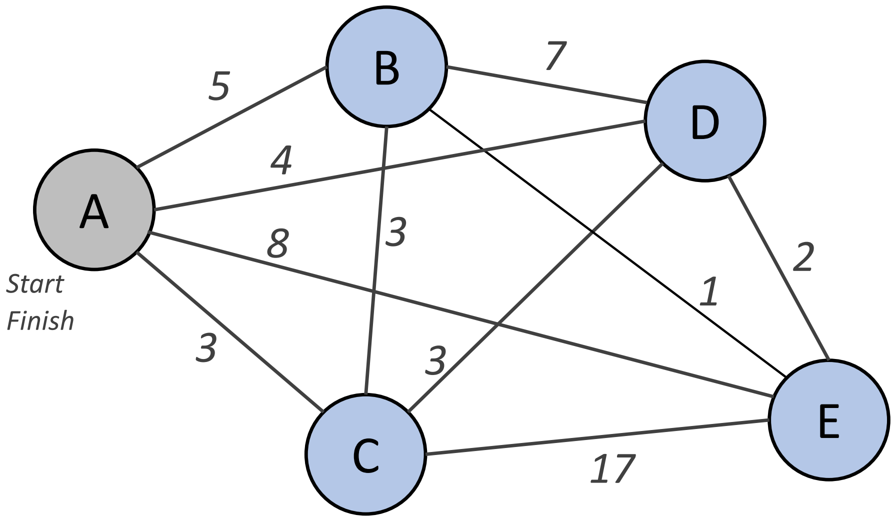

A	?	?	?	?	A

4 cases
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3 cases

2 cases
(2 cities out of 4 have been already visited)

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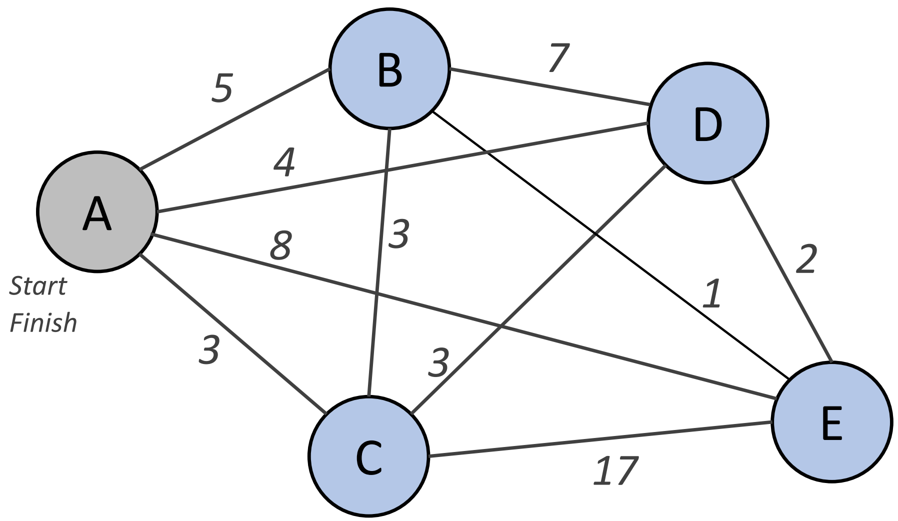

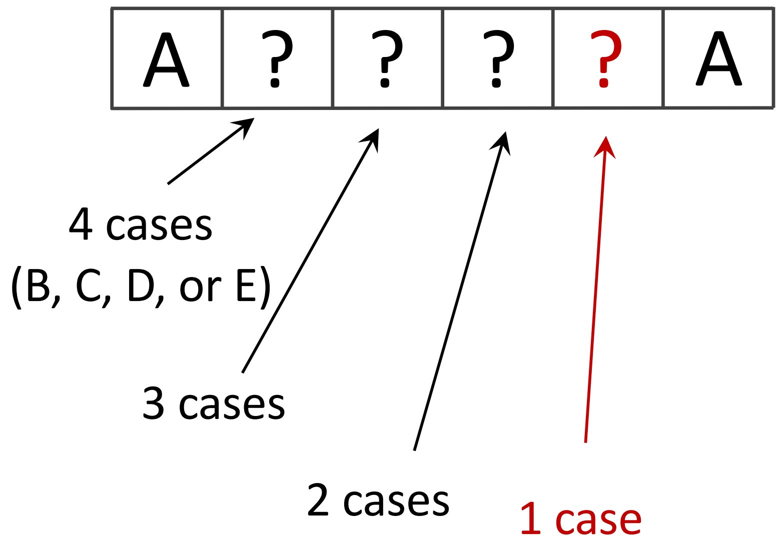

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Combinatorics:
How many feasible routes are there in the graph?

There are 4×3×2 ×1 = 24 feasible routes
to be examined for our graph, which are:

$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow A$
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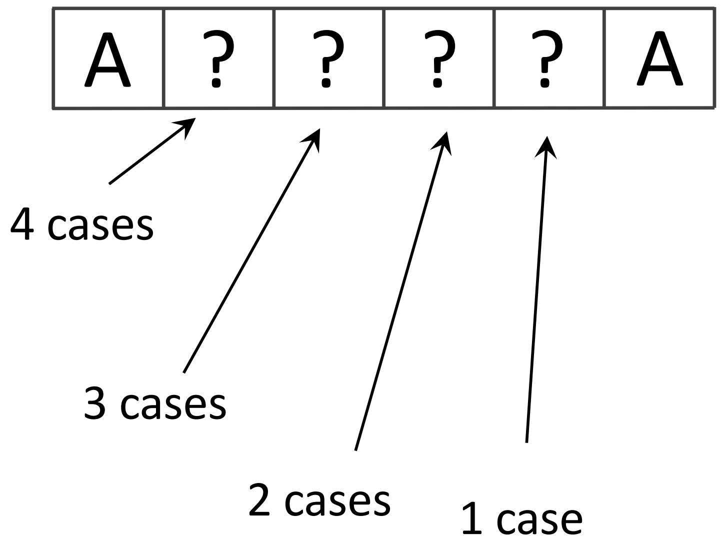

4×3×2 ×1 = 24 cases in total

Combinatorics studies the following questions:

How many possible routes are?

How many feasible combinations or permutations of objects exist?

Etc.

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Various subfields of combinatorics are identified, including:

- Enumerative combinatorics;
- Graph theory;
- Probabilistic combinatorics;
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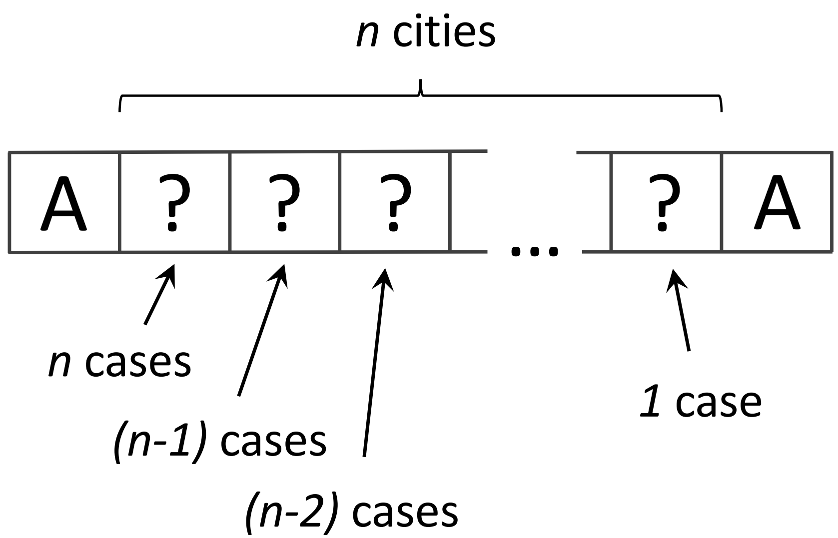
Combinatorics is the foundation for computer science and probability theory (of a discrete variable) in particular

Combinatorics:
How many feasible routes are there in the graph?

General case for *n* cities:



The number of feasible routes to be examined equals to:

$$\boxed{n! = n\times(n-1)\times(n-2)\times\cdots\times3\times2\times1}$$

("n factorial")

Factorial: $n! \stackrel{\text{def}}{=} 1\times 2\times 3\times\cdots\times(n-2)\times(n-1)\times n$

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800
11	39916800
12	479001600
13	6227020800
14	87178291200
15	1307674368000
16	20922789888000
17	355687428096000
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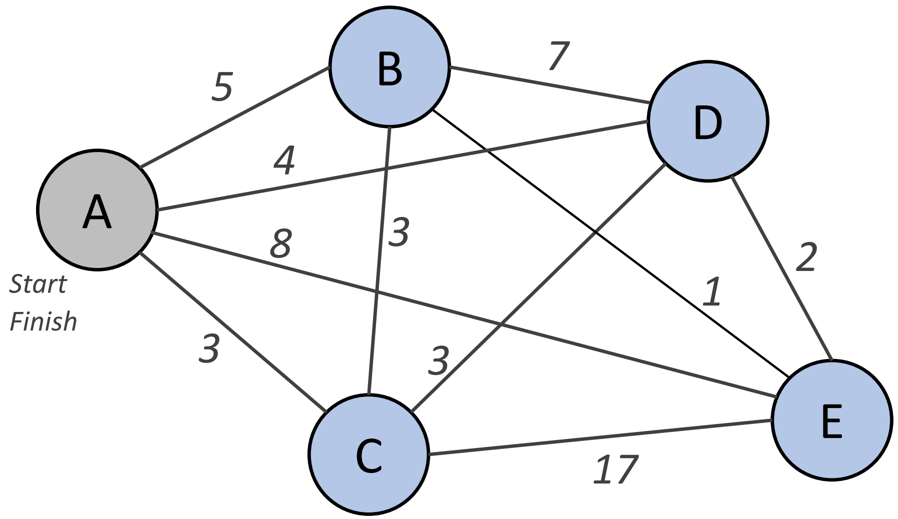
Examples: Brute-force enumeration (or search) The Nearest Neighbour Heuristic

Generalization of the Traveling Salesman Problem (TSP)

Find the shortest route through all cities, starting and ending at city A



Graph representing the distances (or travel times) between cities

Possible generalizations:

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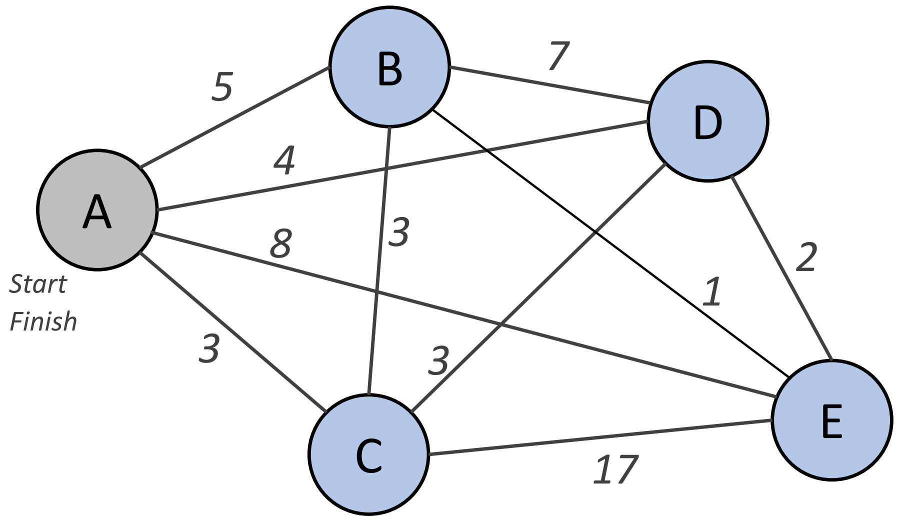

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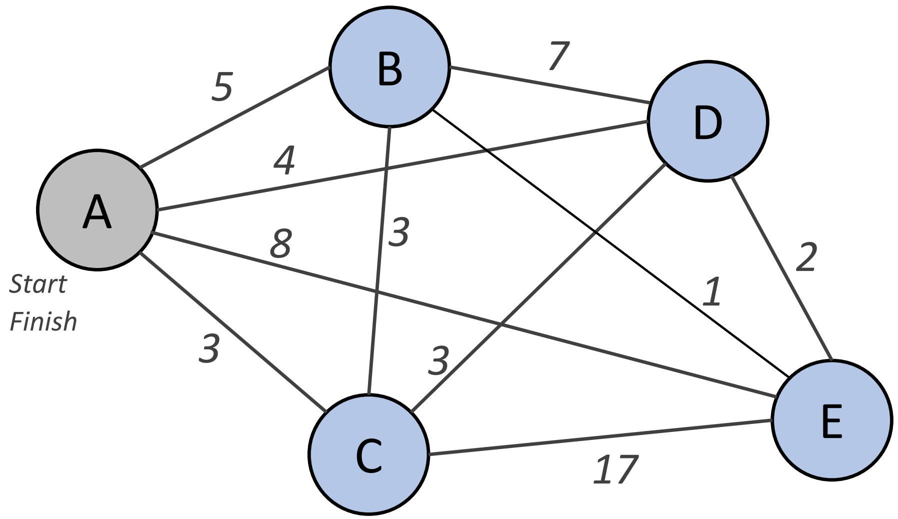

Possible generalizations:

- Cities can be visited multiple times;
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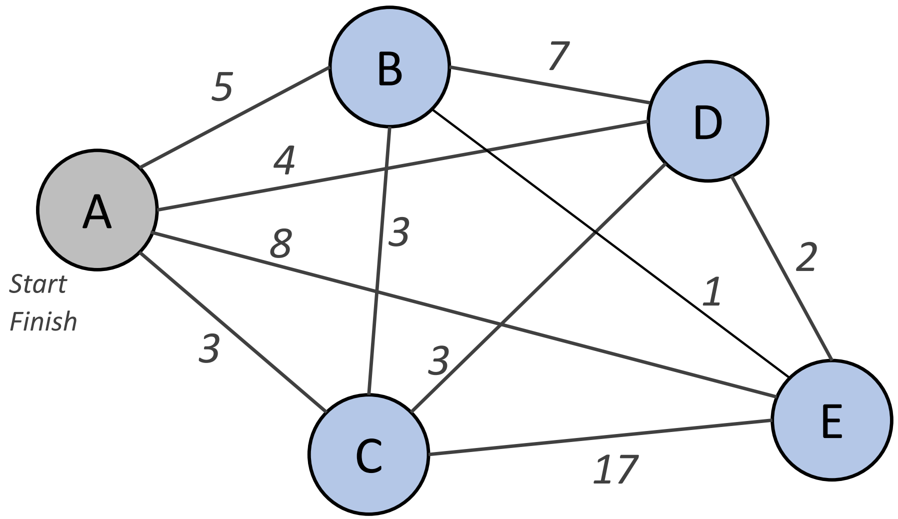
- Not all cities must be visited;
- Etc.

Summary of the Traveling Salesman Problem (TSP)

Find the shortest route through all cities, starting and ending at city A



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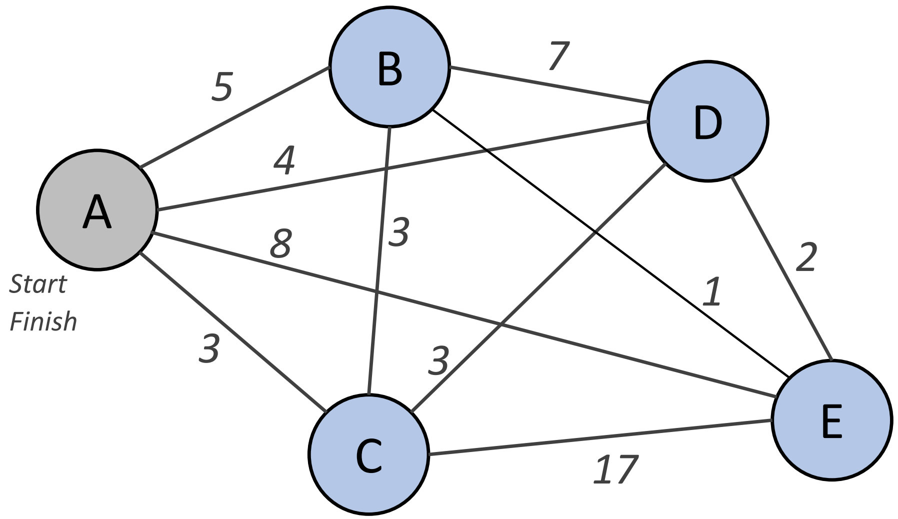

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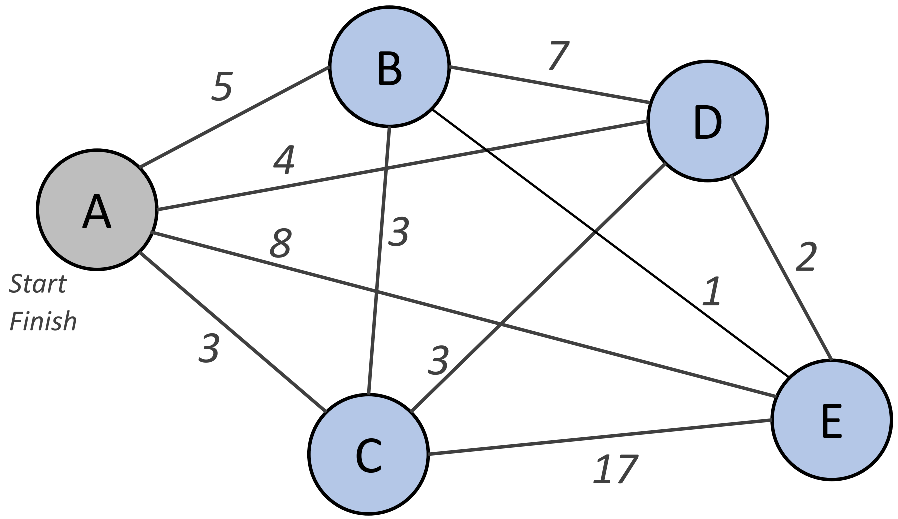

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The problem is fundamental for Computer Science, e.g. an optimal routing of data packets in a computer network.

Combinatorics: Key Terms

Permutation of objects

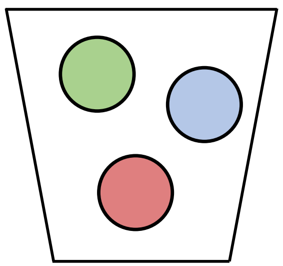
Combination of objects *with* repetition

Combination of objects *without* repetition

Permutation – an ordered set of objects (the order matters!)

Example:

There are 3 balls of different colours in a basket



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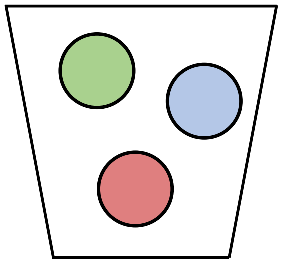

We remove all balls from the basket, one by one

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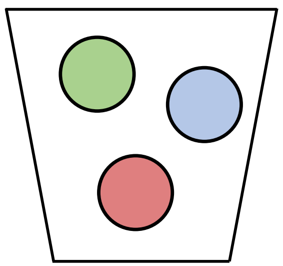

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How many different outcomes are feasible?

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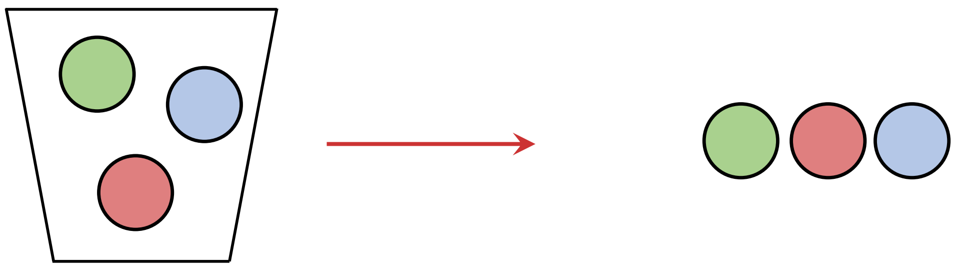

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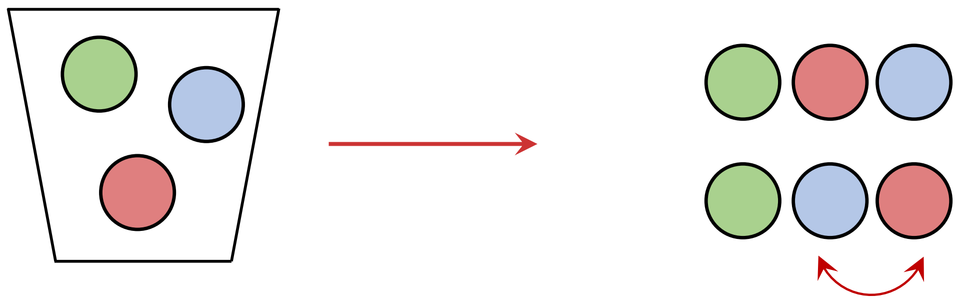

We remove all balls from the basket, one by one

Or instead we take out a blue ball before red one

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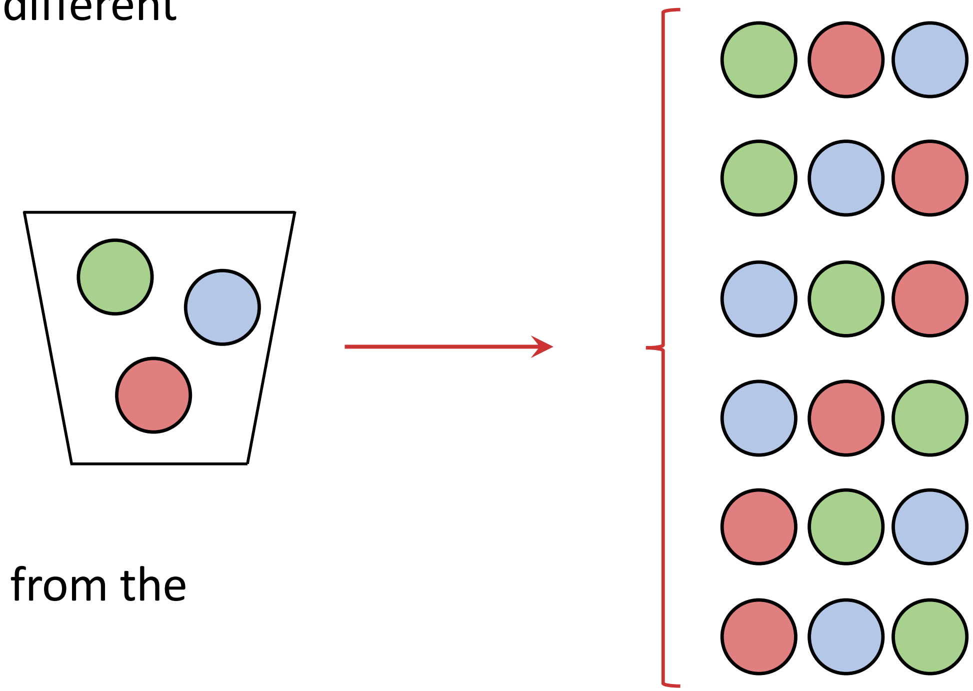

6 feasible permutations

We remove all balls from the basket, one by one

How many different outcomes are feasible?

Permutation – an ordered set of objects (the order matters!)

P_n – the number of feasible permutations of n distinct objects

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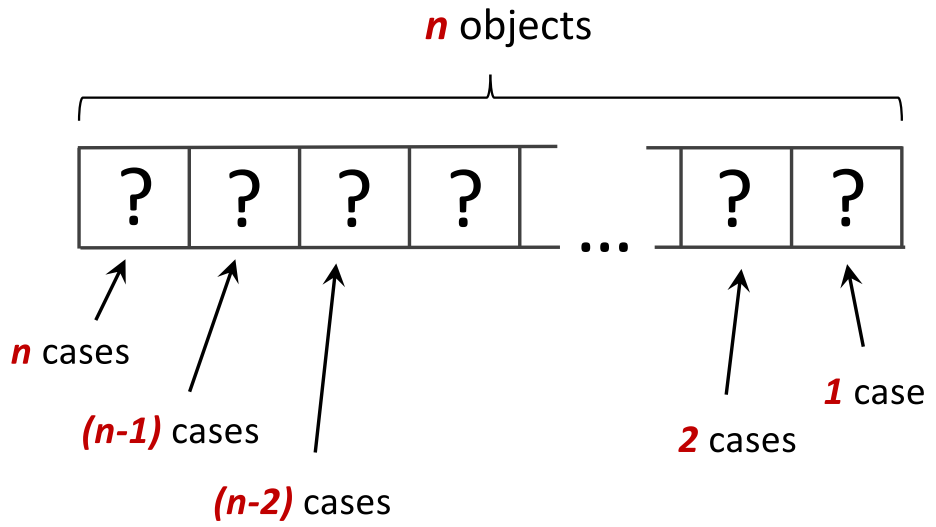
General case: There are $n!$ feasible permutations of *n* distinct objects:



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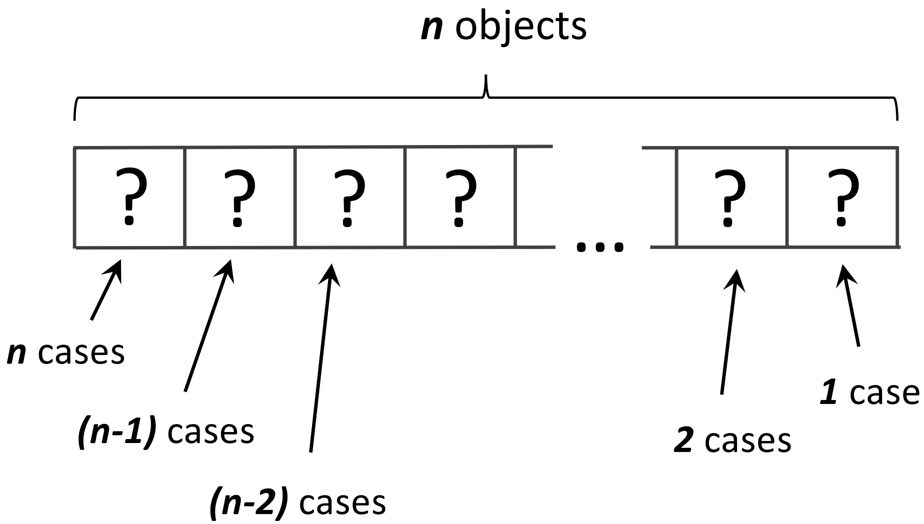

The number of permutations $P(n)$:

$$P_n = n\times(n-1)\times(n-2)\times\cdots\times3\times2\times1 = n!$$

Partial Permutation

Example:

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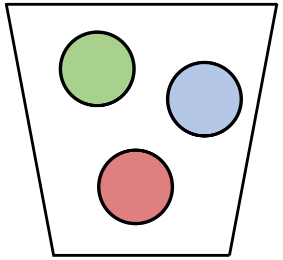

We now take out only 2 balls from the basket, one by one

Partial Permutation

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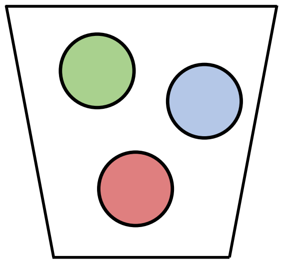

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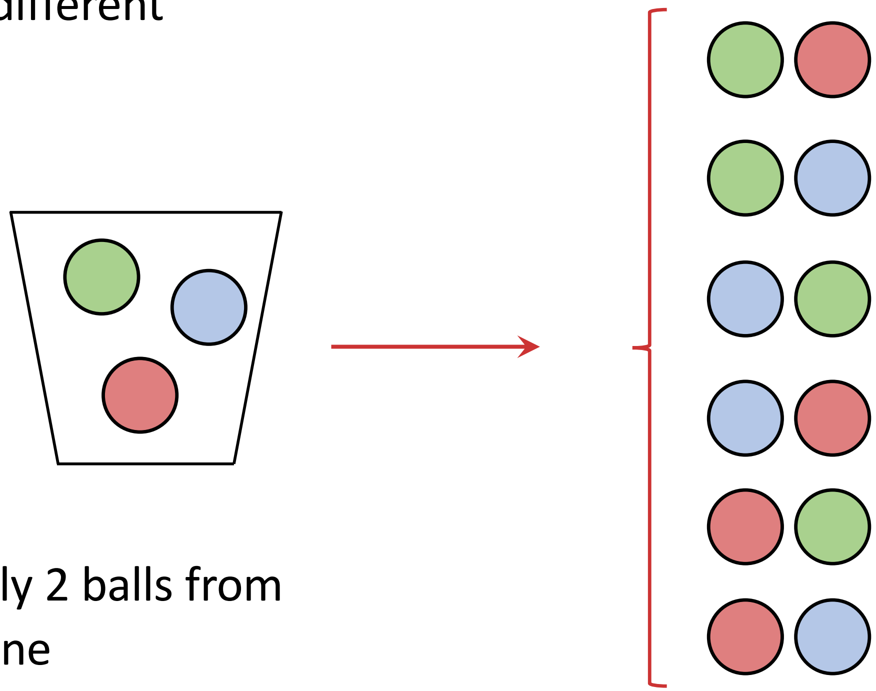


6 feasible partial permutations as well

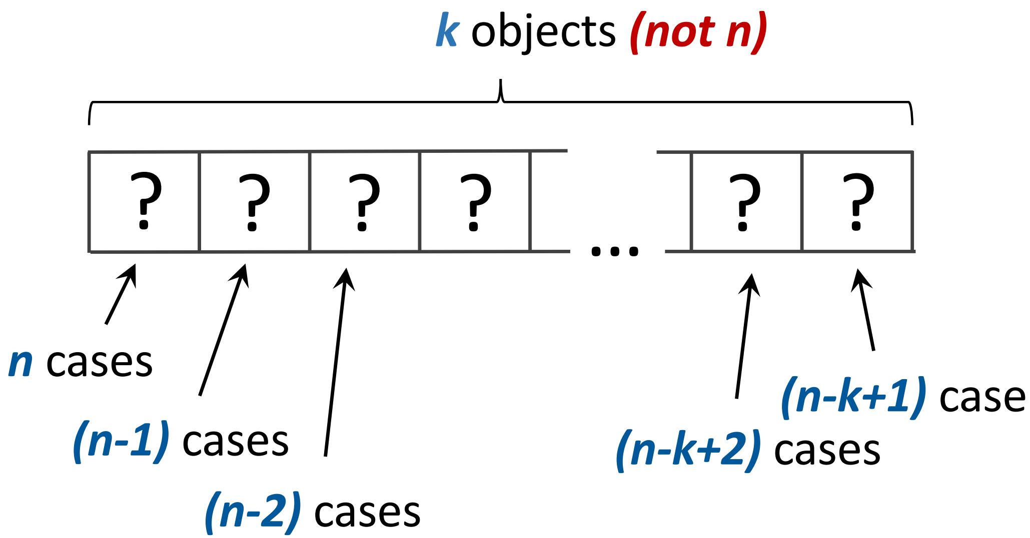
(coincidence)

We now take out only 2 balls from the basket, one by one

How many different outcomes are feasible?

Partial Permutation

P_n^k – the number of feasible permutations of n distinct objects



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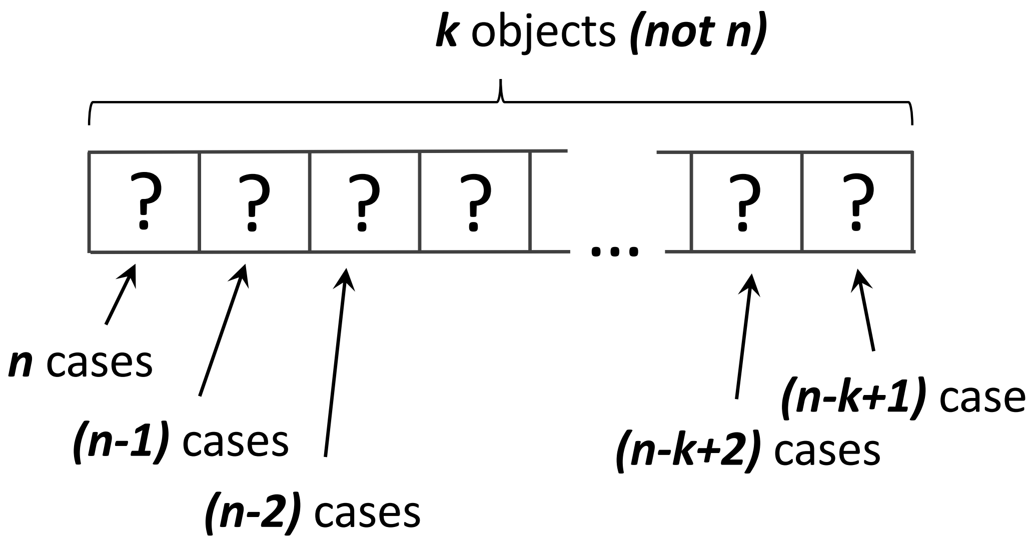

The number of permutations $P(n, k)$:

$$P_n^k = n \times (n-1) \times (n-2) \times \cdots \times (n-k+2) \times (n-k+1)$$

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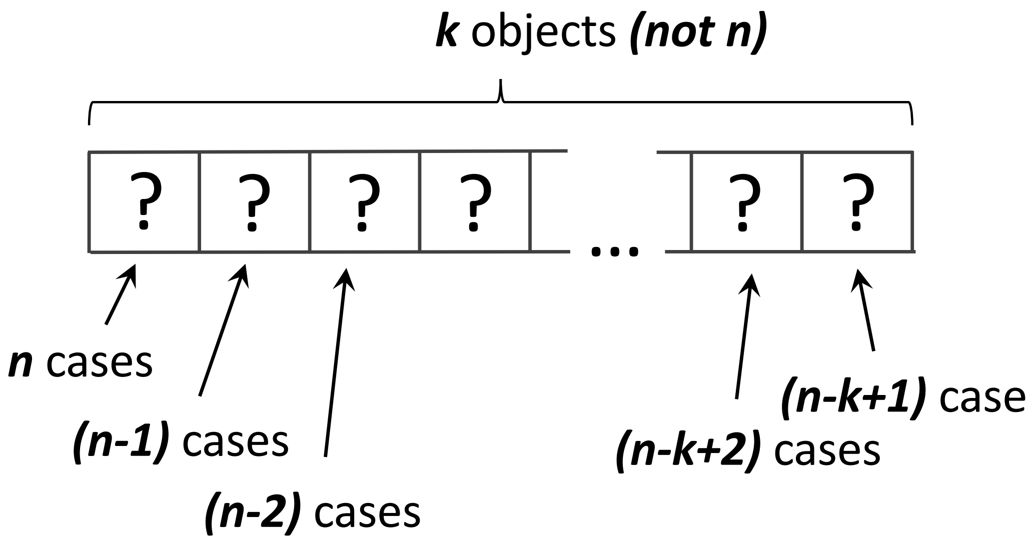
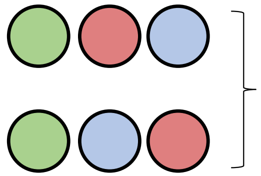


The number of permutations $P(n, k)$:

$$P_n^k = n \times (n-1) \times (n-2) \times \cdots \times (n-k+2) \times (n-k+1) = {n!}/{(n-k)!}$$

Permutations vs. Combinations

Permutation – the order matters:



Two different permutations

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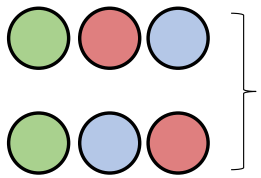
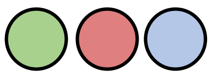
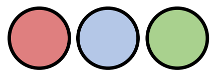
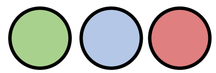


Two different permutations

Combination – the order does not matter

Combination  = *Combination*  = *Combination* 

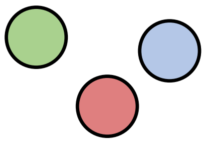
Combination of Objects (the order does not matter!)

Combination – a selection of *k* objects out of *n* types, with $k \leq n$, either with or without repetition

Combination of Objects (the order does not matter!)

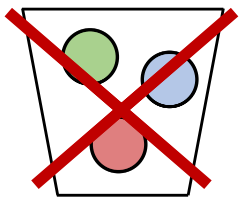
Combination – a selection of k objects out of n types, with $k \leq n$, either with or without repetition

Example: Possible combinations of 2 balls out of 3 types of balls:



Types of balls available (3 in total)

An infinite number of balls is available for each type

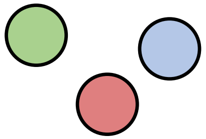


No basket with a limited number of objects

Combination of Objects (the order does not matter!)

Combination – a selection of *k* objects out of *n* types, with $k \leq n$, either with or without repetition

Example: Possible combinations of 2 balls out of 3 types of balls:

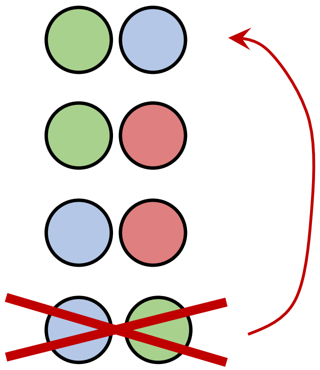


Types of balls available (3 in total)

An infinite number of balls is available for each type

without repetition of types:

3 cases

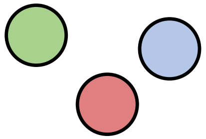


This combination is already present
(recall that the order does not matter)

Combination of Objects (the order does not matter!)

Combination – a selection of k objects out of n types, with $k \leq n$, either with or without repetition

Example: Possible combinations of 2 balls out of 3 types of balls:



Types of balls available (3 in total)

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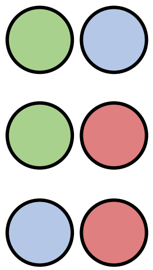
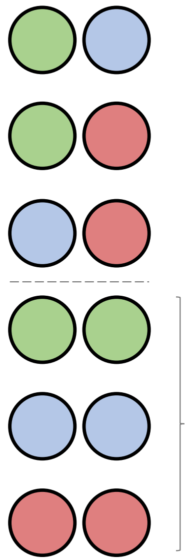
without repetition of types:

3 cases



with repetitions of types:

6 cases



repetitions

Combination of Objects Without Repetitions

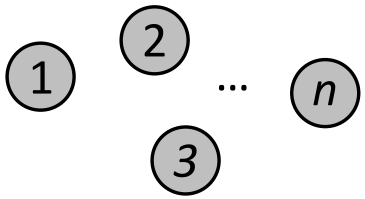
$C_n^k,\ C(n,k), \binom{n}{k}$ – the number of feasible *k*-combinations of *n* objects (without repetitions)

Combination of Objects Without Repetitions

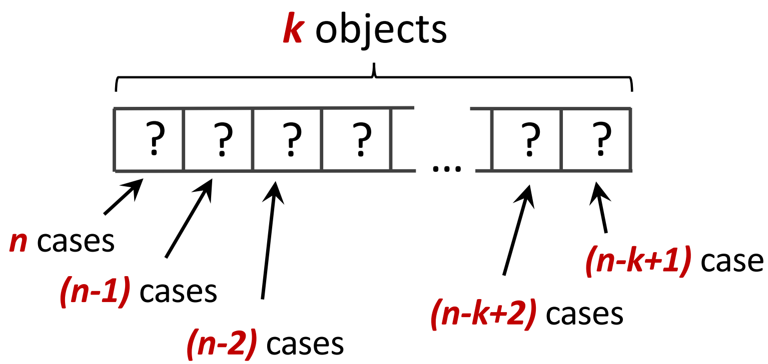
$C_n^k,\ C(n,k),\ \binom{n}{k}$ – the number of feasible *k*-combinations of *n* objects (without repetitions)

Types of balls available

(*n* in total):



An infinite number of balls is available for each type



Number of combinations without repetitions:

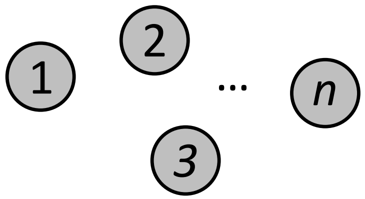
$$C_n^k = \frac{n!}{(n-k)! * k!}$$

Combination of Objects Without Repetitions

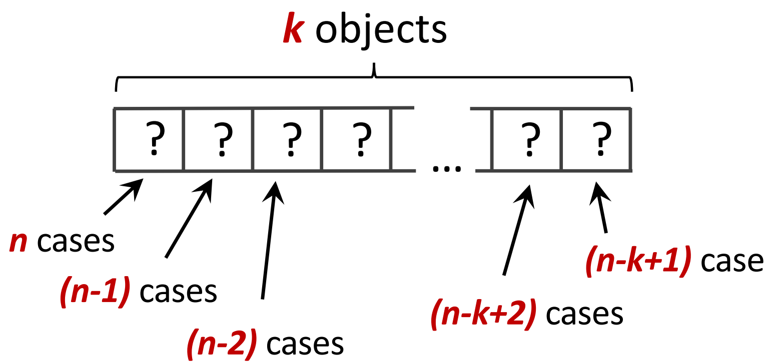
C_n^k, $C(n,k)$, $\binom{n}{k}$ – the number of feasible *k*-combinations of *n* objects (without repetitions)

Types of balls available

(*n* in total):



An infinite number of balls is available for each type



Number of combinations without repetitions:

$$C_n^k = \frac{n!}{(n-k)! * k!}$$

Number of *k*-permutations out of *n* objects (see prev. slides)

To exclude combinations of the same objects, but of different orders (their number in total is *k*!)

Combination of Objects With Repetitions

Number of feasible combinations of k objects out of n types, with repetitions, is computed by:

$$C_{n+k-1}^{k} = \frac{(n+k-1)!}{(n-1)! * k!}$$

(The computation is outside the scope of this presentation)

Binomial Theorem

The number of feasible *k*-combinations of *n* objects

$$C_n^k = \frac{n!}{(n-k)! * k!}$$

is also known as the ***binomial coefficient***

$$(x+y)^n = C_n^0 x^0 y^n + C_n^1 x^1 y^{n-1} + C_n^2 x^2 y^{n-2} + \cdots + C_n^{n-1} x^{n-1} y^1 + C_n^n x^n y^0$$

Binomial coefficients

Binomial Theorem

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$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

Pascal's Triangle – an efficient way to compute binomial coefficients $C_n^k = \frac{n!}{(n-k)! * k!}$

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						1						
					1		1					
				1				1				
			1						1			
		1								1		
	1										1	
1												1

Pascal’s Triangle – an efficient way to compute binomial coefficients $C_n^k = \frac{n!}{(n-k)!*k!}$

						1						
					1		1					
				1		+ **2**		1				
			1						1			
		1								1		
	1										1	
1												1

Pascal’s Triangle – an efficient way to compute binomial coefficients $C_n^k = \frac{n!}{(n-k)!*k!}$

						1						
					1		1					
				1		2		1				
			1		**3**				1			
		1								1		
	1										1	
1												1

Pascal's Triangle – an efficient way to compute binomial coefficients $C_n^k = \frac{n!}{(n-k)! * k!}$

						1						
					1		1					
				1		2		1				
			1		**3**		**3**		1			
		1								1		
	1										1	
1												1

Pascal's Triangle – an efficient way to compute binomial coefficients $C_n^k = \frac{n!}{(n-k)!*k!}$

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		**10**		10		5		1	
1		6		15		20		15		6		1

Pascal's Triangle – an efficient way to compute binomial coefficients $C_n^k = \frac{n!}{(n-k)!*k!}$

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal’s Triangle – an efficient way to compute binomial coefficients

Interpretation:

$$C_n^k = \frac{n!}{(n-k)! * k!}$$

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal's Triangle – an efficient way to compute binomial coefficients

Interpretation:

$$C_{n}^{k} = \frac{n!}{(n-k)! * k!}$$

Row number

0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1

Pascal’s Triangle – an efficient way to compute binomial coefficients

Interpretation:

Position in that row

$$C_{n}^{k} = \frac{n!}{(n-k)! * k!}$$

Row number

0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1

Pascal’s Triangle – an efficient way to compute binomial coefficients

Interpretation:

Position in that row

C_6^4

Row number

0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		$\boxed{15}$		6		1
	0		*1*		*2*		*3*		*4*				

Note: Rows and columns are numbered from 0, not 1

Summary: Basic Formulas of Combinatorics

The number of

Permutations of *n* objects	$P(n) = n!$
k-Permutations of *n* objects	$P(n) = {}^{n!}/_{(n-k)!}$
k-combinations of *n* objects, without repetition	$C(n,k) = \dfrac{n!}{(n-k)!\,k!}$
k-combination of *n* objects, with repetition	$C(n+k-1,k) = \dfrac{(n+k-1)!}{(n-1)!\,k!}$

Combinatorics is the foundation for probability theory that we study next time

Terminology

Combinatorics

Traveling Salesman Problem (TSP)

An optimal route

Brute-force search

Exhaustive enumeration

General case

Particular case

Factorial

Feasible scenarios

An optimal solution

Problem complexity

Computational problem

Computational complexity of an algorithm

Exact solution

Approximate solution

Permutation

Combination

Repetition

Distinct objects

Binomial theorem

Binomial coefficient

Pascal’s triangle