Theoretical Computer Science

Tutorial Week 2

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Agenda

- The naive set theory
  - **Basic definition**
  - Operations
  - Properties
  - Power Set and Cardinality
- Formal Languages
  - Alphabets and Strings
  - Languages
  - Operations

The naive set theory

Definition

$$A = \{x \in \mathbf{U} \mid P(x)\}$$

$$A = \{a_1, a_2, \ldots, a_n\}$$

Example

$$\{x \in \mathbb{Z} \mid x < 0\}$$

The naive set theory

The naive set theory	Logic
The empty set $\emptyset$	False F | 0
The universe $\mathbf{U}$	True T | 1

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Complement

$$A^c = \overline{A} = \{x \in \mathbf{U} \mid x \notin A\}$$

Example

If $\mathbf{U} = \{1, 2, 3, 4\}$ and $A = \{1, 3\}$, then

$$\overline{A} = \{2, 4\}$$

Union

$$A \cup B = \{x \in \mathbf{U} \mid x \in A \vee x \in B\}$$

Example

If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then

$$A \cup B = \{1, 2, 3, 4\}$$

The naive set theory

Intersection

$$A \cap B = \{x \in \mathbf{U} \mid x \in A \,\&\, x \in B\}$$

Example

If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then

$$A \cap B = \{2, 3\}$$

The naive set theory

Difference

$$A \setminus B = \{x \in \mathbf{U} \mid x \in A \,\&\, x \notin B\}$$

Example

If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then

$$A \setminus B = \{1\}$$

Definition

$X \times Y = \{(x, y) \mid x \in X \,\&\, y \in Y\}$

Example

If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

The naive set theory

Definition

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_1 \in X_1 \;\&\; \ldots \;\&\; x_n \in X_n\}$$

Example

$$\underbrace{X \times \cdots \times X}_{n \text{ times}} = X^n$$

$\mathbb{R}^3$

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The naive set theory

$$A \cap \emptyset = \emptyset \qquad A \cup \emptyset = A$$

$$A \cap \mathbf{U} = A \qquad A \cup \mathbf{U} = \mathbf{U}$$

$$A \cap \overline{A} = \emptyset \qquad A \cup \overline{A} = \mathbf{U}$$

$$\overline{\overline{A}} = A$$

The naive set theory

$A \cap B = B \cap A \quad A \cup B = B \cup A$	Commutativity
$A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$	Associativity
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributivity
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	Morgan's laws

The naive set theory

Definition

$A \subseteq B$ if, for any x,

$$x \in A \rightarrow x \in B$$

Examples/Properties

For any A,

$$\emptyset \subseteq A$$

$$A \subseteq \mathbf{U}$$

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The naive set theory

Definition

For a set A, the power of A is the set

$$2^A = \mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Examples

1) If $A = \{a\}$ then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$
2) If $A = \{a, b\}$ then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Definition

Intuitively, the cardinality of a set A, denotes by $|A|$, is the number of elements of A.

Examples

1. $|\emptyset| = 0$
2. if $A = \{2\}$ then $|A| = 1$
2. if $A = \{1, 2, 3\}$ then $|A| = 3$
3. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \omega$
4. $|\mathbb{R}| = 2^{\omega}$

The naive set theory

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \times B| = |A| \cdot |B|$$

Properties

$$|A \times B| = |A| \cdot |B|$$

Example

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$.

	a_1	a_2	$\ldots$	a_n
b_1	(a_1, b_1)	(a_2, b_1)	$\ldots$	(a_n, b_1)
b_2	(a_1, b_2)	(a_2, b_2)	$\ldots$	(a_n, b_2)
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
b_m	(a_1, b_m)	(a_2, b_m)	$\ldots$	(a_n, b_m)

Obviously, the table contains $n \times m$ elements.

Properties

$$|A \times B| = |A| \cdot |B|$$

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$$

$$|A^n| = |A|^n$$

The naive set theory

$$|2^A| = 2^{|A|}$$

$A = \{a_1, a_2, \ldots, a_{n-1}, a_{n-2}, a_n\}$

a_1	a_2	$\ldots$	a_{n-1}	a_n	*Subsets*
0	0	$\ldots$	0	0	$\emptyset$
0	0	$\ldots$	0	1	$\{a_n\}$
0	0	$\ldots$	1	0	$\{a_{n-1}\}$
0	0	$\ldots$	1	1	$\{a_{n-1}, a_n\}$
		$\ldots$		$\ldots$	
1	1	$\ldots$	0	0	$\{a_1, a_2, \ldots, a_{n-2}\}$
1	1	$\ldots$	0	1	$\{a_1, a_2, \ldots, a_{n-2}, a_n\}$
1	1	$\ldots$	1	0	$\{a_1, a_2, \ldots, a_{n-2}, a_{n-1}\}$
1	1	$\ldots$	1	1	$A = \{a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n\}$

*$0 =$ "$a_i \notin A$" & $1 =$ "$a_i \in A$"

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Formal Languages

Definition

Alphabet is a finite set of symbols

Examples

$$\{0, 1\}$$

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\{a, b, c, d, \ldots, x, y, z\}$$

Formal Languages

Definition

String over an alphabet Σ is a finite sequence of symbols in Σ

Examples

For $\Sigma = \{0, 1\}$,

$$010011$$

$$11100011$$

Formal Languages

Definition

String over an alphabet Σ is a finite sequence of symbols in Σ

Examples

For $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$$123456$$

$$666$$

$$2022$$

Formal Languages

Definition

String over an alphabet Σ is a finite sequence of symbols in Σ

Examples

For $\Sigma = \{a, b, c, d, \ldots, x, y, z\}$,

$$peace$$

$$war$$

$$dfklgnkjrbgjrbg$$

Formal Languages

Definition

Length of a string s is the number of symbols of s and denotes as $|s|$

Examples

$$|peace| = 5$$

$$|war| = 3$$

$$|dfklgnkjrbgjrbg| = 15$$

Definition

ϵ is the **null** string (empty string) over any alphabet.

Property

$$|\epsilon| = 0$$

Formal Languages

Definition

For two strings x and y, the concatenation $x \cdot y$ is the operation of joining "end-to-end".

Examples

For $x = 123$ and $y = 987$,

$$x \cdot y = 123987$$

Definition

For two strings x and y, the concatenation $x \cdot y$ is the operation of joining "end-to-end".

Examples

For $x = \mathit{back}$ and $y = \mathit{end}$,

$$x \cdot y = \mathit{backend}$$

Property

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Examples

For $x = ab$, $y = cd$ and $z = ef$,

$$(x \cdot y) \cdot z = (abcd) \cdot ef = abcdef$$

$$x \cdot (y \cdot z) = ab \cdot (cdef) = abcdef$$

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Definition

The set of all strings over Σ is denoted by Σ^*

Examples

For $\Sigma = \{0, 1\}$,

$$\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$$

Formal Languages

Definition

A language L is a set of strings over an alphabet Σ.

Equivalent definition

$$L \subseteq \Sigma^*$$

Formal Languages

Alphabet

For $\Sigma = \{0, 1\}$,

$$\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$$

Languages

$$L_1 = \{00000000, 00000001, \ldots, 11111110, 11111111\} =$$

$$= \{x \in \{0, 1\}^* \mid |x| = 8\}$$

$$L_2 = \{0, 00, 01, 000, 001, 010, \ldots\} = \{0 \cdot x \mid x \in \Sigma^*\}$$

Formal Languages

Alphabet

For $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$$\Sigma^* = \mathbb{N} \cup \{\epsilon\}$$

Languages

$$L_1 = \{0, 2, 4, 6, 8, 10, \ldots\} = \{x \in \Sigma^* \mid x \text{ is even } \}$$

$$L_2 = \{2, 3, 5, 7, 13, \ldots\} = \{x \in \Sigma^* \mid x \text{ is prime } \}$$

Formal Languages

Alphabet

For $\Sigma = \{a, b, c, d, \ldots, x, y, z\}$

Languages

English, Italian, French,...

Formal Languages

Alphabet

For $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, =\}$

Arithmetic

$$\{0 + 0 = 0, 0 - 0 = 0, \ldots, 12 + 32 = 44, \ldots, 52 - 39 = 13, \ldots\}$$

Formal Languages

Alphabet

For $\Sigma = \{A, B, C, \ldots, {}^c, \cup, \cap, \setminus, (,)\}$

Formulas of the naive set theory

$$A^c, A \cup B, \ldots, (A \cup B^c) \cap (A^c \cup B), \ldots$$

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Operations

$$\overline{L} = L^c$$

$$L_1 \cup L_2$$

$$L_1 \cap L_2$$

$$L_1 \setminus L_2$$

Concatenation

$$L_1 \cdot L_2 = \{x \cdot y \mid x \in L_1 \,\&\, y \in L_2\}$$

Example

If $L_1 = \{1, 2, 3\}$ and $L_2 = \{a, b\}$, then

$$L_1 \cdot L_2 = \{1a, 1b, 2a, 2b, 3a, 3b\}$$

Concatenation

$$L_1 \cdot L_2 = \{x \cdot y \mid x \in L_1 \,\&\, y \in L_2\}$$

Example

If $L_1 = \{1, 12\}$ and $L_2 = \{\epsilon, 2\}$, then

$$L_1 \cdot L_2 = \{1, 12, 122\}$$

Thank you for your attention!