Essentials of Analytical Geometry and Linear Algebra. Lecture 9.

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Lecture 9. Outline

- Part 1. Quadratic curves
- Part 2. Ellipse
- Part 3. Hyperbola
- Part 4. Parabola

Curves

Part 1. Quadratic curves

In general any quadratic curve is a set of points satisfying the equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Without proof...

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Any five (5) points on a plane uniquely define a quadratic curve.

Goals

- Understand similarities
- Understand differences
- Solve some basic problems

Part 2. Ellipse

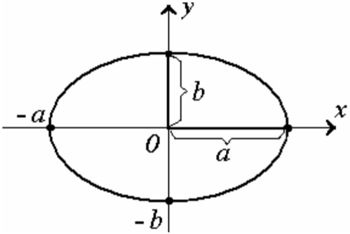
Definition

Ellipse. Canonical equation of an ellipse

An ellipse is a plane curve, which is represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in some Cartesian coordinate system.



Parametric form of the equation of an ellipse

Given

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

Parametric form of the equation of an ellipse

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$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

and eliminating the parameter t, we get

$$\begin{cases} \frac{x^2}{a^2} = \cos^2 t \\ \frac{y^2}{b^2} = \sin^2 t \end{cases}$$

This gives you a nice way to plot a point M(x,y) on an ellipse.

Question

Is this an equation of an ellipse?

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

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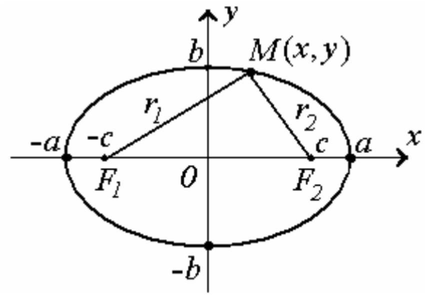
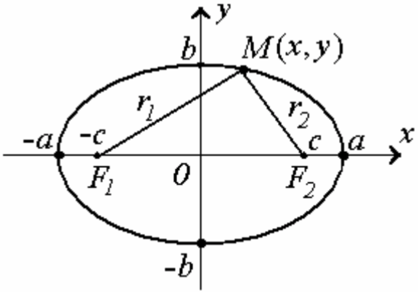
Center of the ellipse is at $M(x_0, y_0)$.

Ellipse: foci, eccentricity and focal distances

Foci and eccentricity



Foci and eccentricity

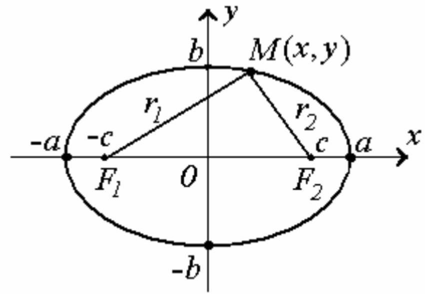


Foci (aka focuses)

Given an ellipse with major axis $2a$. Foci (plural from *focus*) are points $F_1(-c, 0)$ and $F_2(c, 0)$ that satisfy:

$$c^2 = a^2 - b^2$$

Foci and eccentricity



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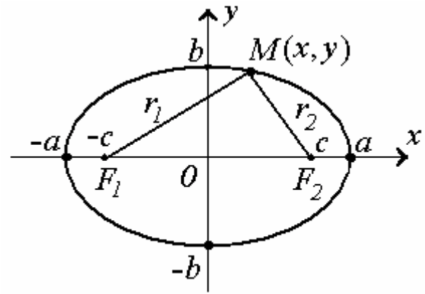
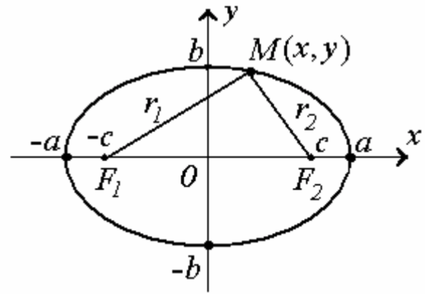


Eccentricity

Given an ellipse with major axis $2a$ and foci $F_1(-c, 0)$, $F_2(c, 0)$, the eccentricity of ellipse is denoted as ε: $\varepsilon = \frac{c}{a}$

Foci and eccentricity



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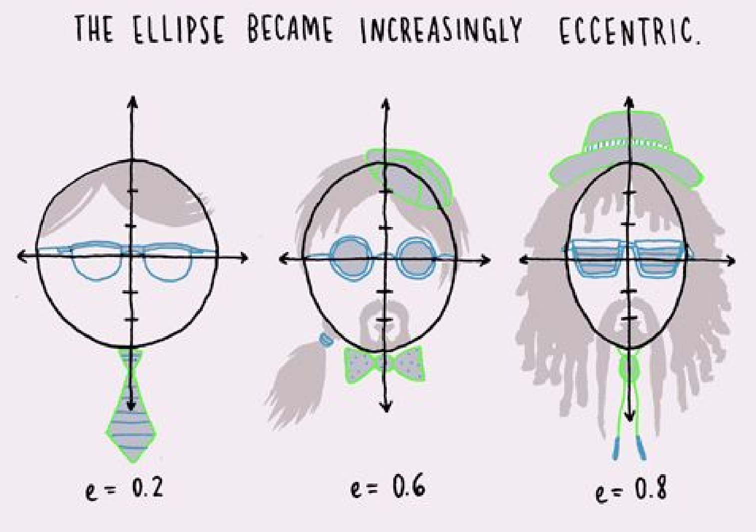
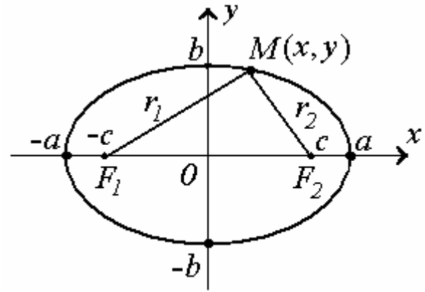
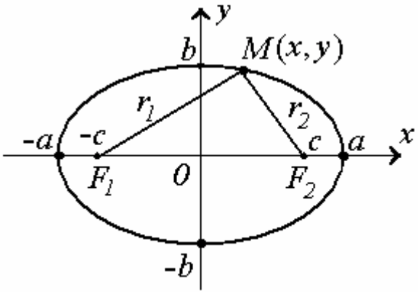
What is the range for eccentricity?

"Eccentric Ellipse"



Focal distances





Focal distances

Distance from a point $M(x, y)$ on an ellipse to each of foci.

$r_1 = a + x\varepsilon$

$r_2 = a - x\varepsilon$

Proof

We need to show that $r_1 = a + x\varepsilon$.

$r_1 = \sqrt{(x+c)^2 + y^2}$

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$y^2 = (a^2 - x^2)\frac{b^2}{a^2} = a^2 - a^2\varepsilon^2 - x^2 + x^2\varepsilon^2$

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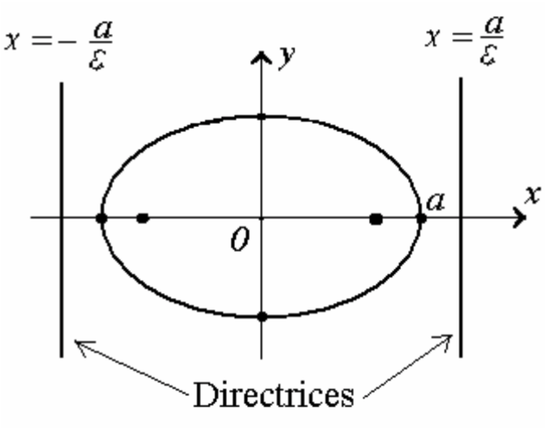
$r_1^2 = (x+c)^2 + y^2 = x^2 + 2xc + c^2 + a^2 - a^2\varepsilon^2 - x^2 + x^2\varepsilon^2 = (a + x\varepsilon)^2$

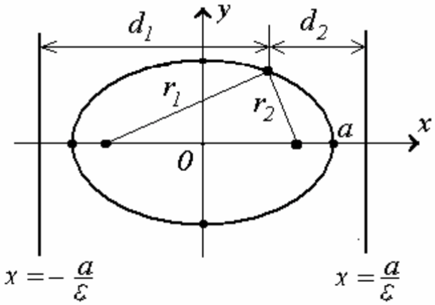
Note, that $r_2 = a - x\varepsilon$ and hence $r_1 + r_2 = 2a$

Directrices



What are the equations of the directrices?

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon$$



Why $\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon$?

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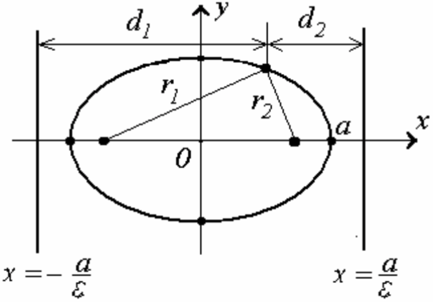
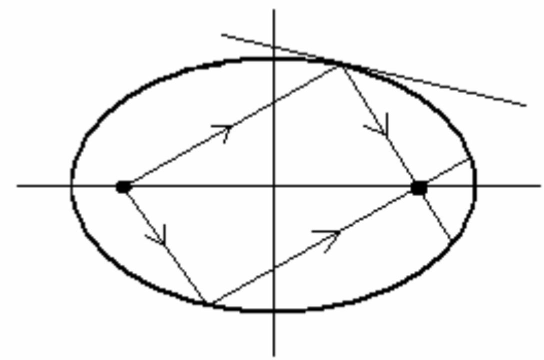


Why $\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon$?
$d_1 = \frac{a}{\varepsilon} + x = \frac{1}{\varepsilon} r_1$; $d_2 = \frac{a}{\varepsilon} - x = \frac{1}{\varepsilon} r_2$

Tangent lines



Example

Check whether this equation is an equation of ellipse?

$$2x^2 + 4x + 3y^2 - 12 = 1$$

Break, 5 min.

Interesting question to study:

Propose a formula for the length (perimeter) of an ellipse.

or

Write a program to calculate it.

Part 3. Hyperbola

Hyperbola

Definition. Canonical equation

A hyperbola is a plane curve, which can be represented in some Cartesian coordinate system by one of the below equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad or \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

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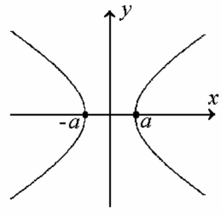

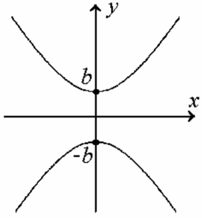

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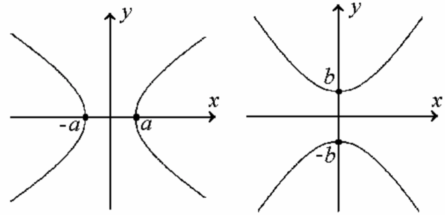

If $a = b$, then it is a **equilateral hyperbola.**

Question

Is this an equation of a hyperbola?

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = \pm 1$$

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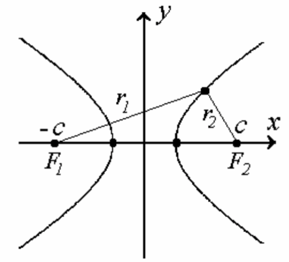
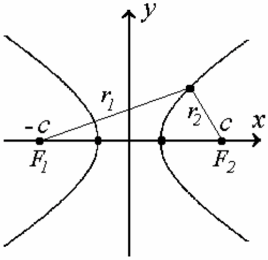
Center of the hyperbola is at $M(x_0, y_0)$.

Hyperbola: foci, eccentricity and focal distances

Foci and eccentricity



Foci and eccentricity

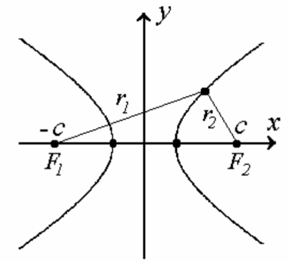


Foci (aka focuses)

Given a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Foci are points $F_1(-c, 0)$ and $F_2(c, 0)$ that satisfy:

$$c^2 = a^2 + b^2$$

Foci and eccentricity



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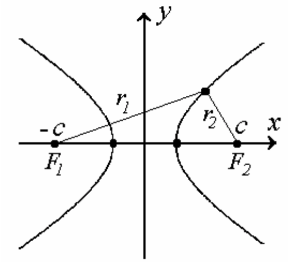


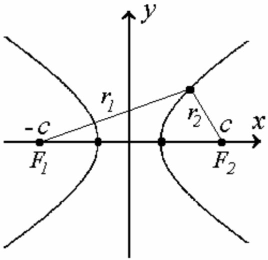
Eccentricity

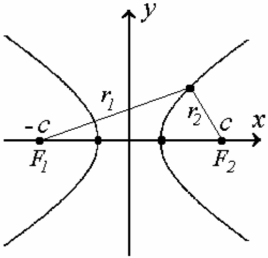
Given $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Eccentricity of the hyperbola ε :

$$\varepsilon = \frac{c}{a}$$

Note, $\varepsilon > 1$

Focal distances





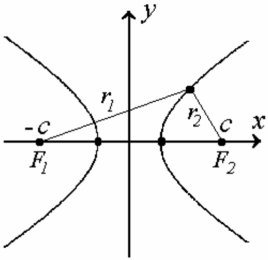
Focal distances

Distance from a point $M(x, y)$ on a hyperbola to each of foci.

$r_1 = \pm(x\varepsilon + a)$

$r_2 = \pm(x\varepsilon - a)$

Focal distances



Focal distances

Distance from a point $M(x, y)$ on a hyperbola to each of foci.

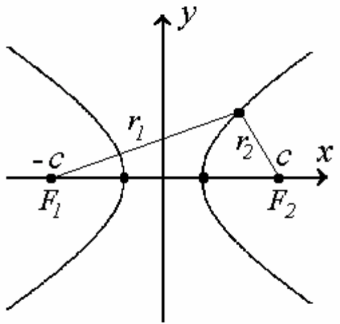
$r_1 = \pm(x\varepsilon + a)$

$r_2 = \pm(x\varepsilon - a)$

For any point of hyperbola:

$$r_1 - r_2 = \pm 2a$$

Focal distances



$$r_1 = \pm(x\varepsilon + a)$$
$$r_2 = \pm(x\varepsilon - a)$$

$r_1 - r_2 = \pm 2a$ (or, just $|r_1 - r_2| = 2a$)

Why do we use $\pm$ here?

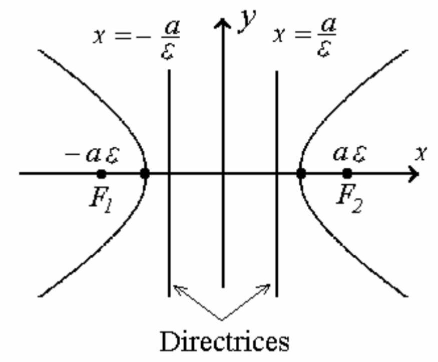
Hint: we have 2 branches (for one x is positive for another one $x < 0$)

Assignment

Define foci, eccentricity and focal distances if a hyperbola has the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Directrices



Directrices (plural from *directrix*) are two vertical lines $x = \pm\frac{a}{\varepsilon}$

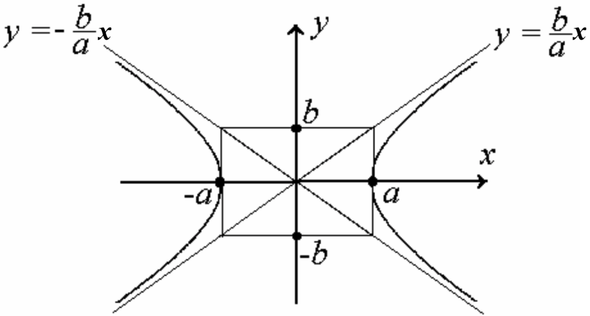
$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon$$

Directrix is not DirectX

Directrix a fixed line used in describing a curve or surface.

Thus, Ellipse, Hyperbola (and also a parabola) can be defined using a **directrix** and a **point.**

Assymptotes



Given a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Two straight lines $y = \pm\frac{b}{a}x$ are the asymptotes of the hyperbola.

Use limits to prove it (as $x \to \pm\infty$).

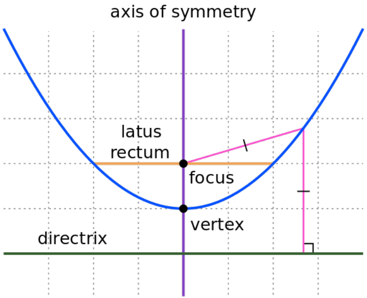
Example

Reduce the equation

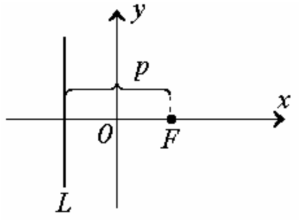
$$x^2 - 6x + y^2 + 8y = 0$$

to the canonical form

Part 4. Parabola



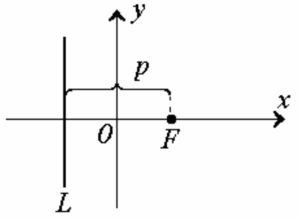
Parabola. Definitions



Parabola

- A parabola is the locus of points, which are equidistant from a given point **F** and line **L**.

Parabola. Definitions



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- The point **F** is called the **focus**.

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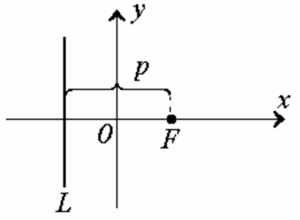
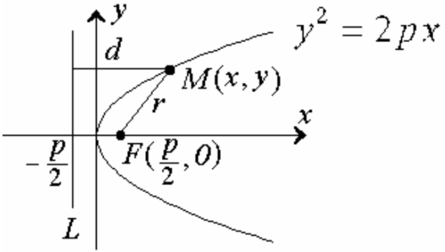


Parabola

- A parabola is the locus of points, which are equidistant from a given point **F** and line **L**.
- The point **F** is called the **focus**.
- The line **L** is called the **directrix** of the parabola.

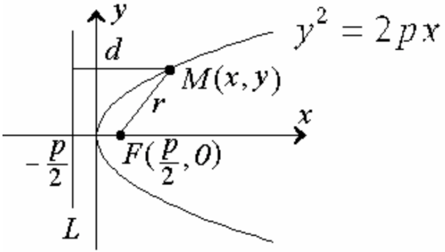
Sometimes, p is denoted as $2a$.

Parabola. Canonical equation



$d = r,$

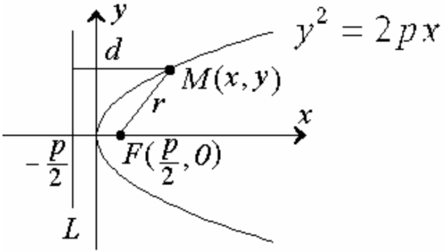
Parabola. Canonical equation



$d = r,$

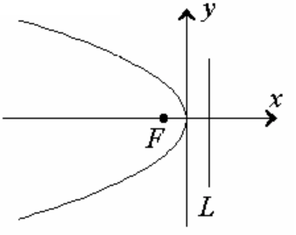
$x + \frac{p}{2} = \sqrt{(x - \frac{p}{2})^2 + y^2}$

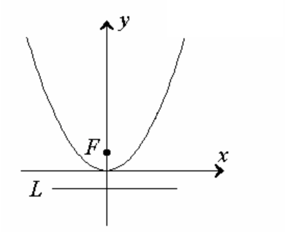
Parabola. Canonical equation

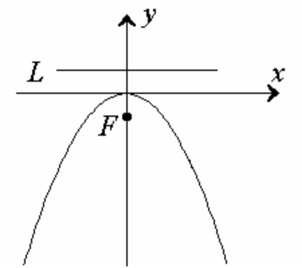


$d = r,$
$x + \frac{p}{2} = \sqrt{(x - \frac{p}{2})^2 + y^2}$
$y^2 = 2px$

Parabola. Other cases







Write the canonical equations.

Parabola. Eccentricity

Recall how we defined eccentricity for ellipse and hyperbola.

Parabola. Eccentricity

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What is the eccentricity of a parabola?

Parabola. Eccentricity

Recall how we defined eccentricity for ellipse and hyperbola.
What is the eccentricity of a parabola?

$$\varepsilon = \frac{r}{d} = 1$$

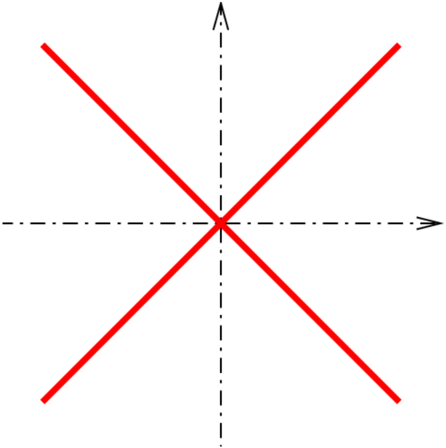
Reduce the equation

$$x^2 + 4x - 3y = -5$$

to the canonical form.

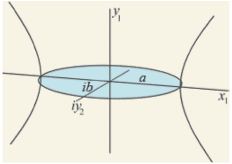
Interesting case 1

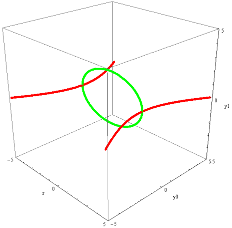
- $x^2 - y^2 = 0$ (a pair of intersecting lines)



Interesting case 2

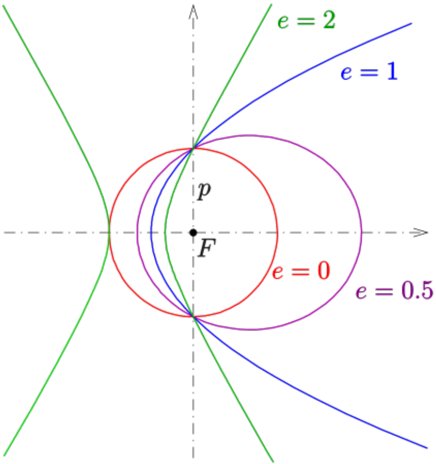
- $2x^2 + 3y^2 = -1$ (an imaginary ellipse)





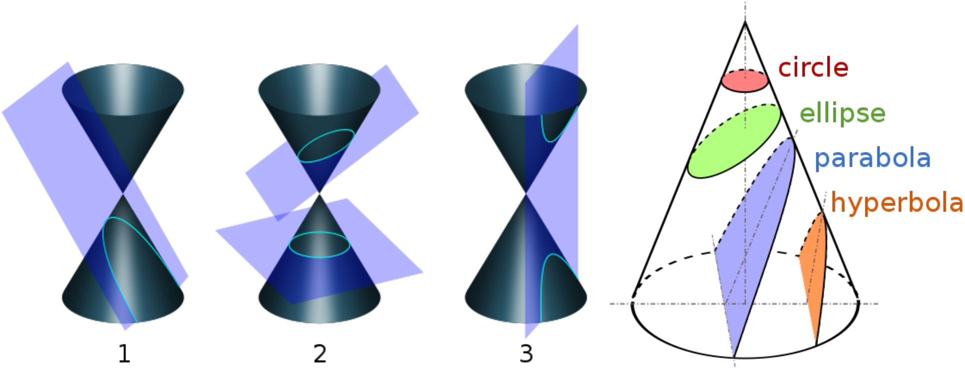
Summary

Summary.

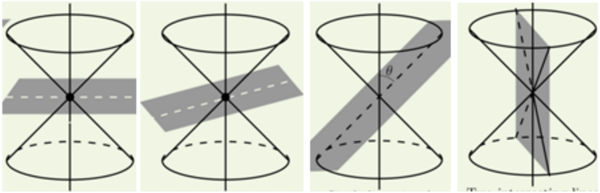


Quadratic curves as sections of a circle cone

Conics!



Degenerate conics



Relation to Quadratic forms and Matrices

Conic sections are the sets of points whose coordinates satisfy a second-degree polynomial equation (A, B, C, D, E, F are numbers):

$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

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$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

In matrix form (it is the **same** equation):

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F = 0.$$

Relation to Quadratic forms and Matrices

Conic sections are the sets of points whose coordinates satisfy a second-degree polynomial equation (A, B, C, D, E, F are numbers):

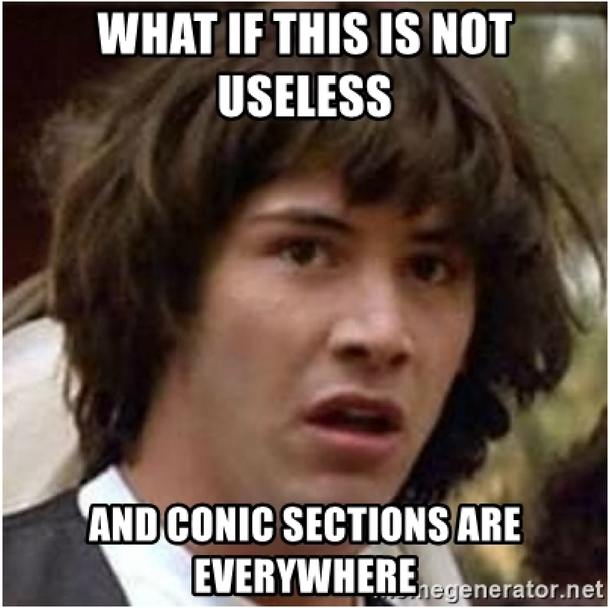
$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

In matrix form (it is the **same** equation):

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F = 0.$$

This following expression is called the **quadratic form**: $Ax^2 + Bxy + Cy^2$.

Matrix of the **quadratic form** : $\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$


WHAT IF THIS IS NOT USELESS
AND CONIC SECTIONS ARE EVERYWHERE

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila