Theoretical Computer Science

Tutorial Week 4

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INNOPOLIS UNIVERSITY

Agenda

Finite State Automaton (FSA)

- **Representations of FSA**
  - Formally
  - Graphical Representation
  - State Transition Table
- Operations on FSA
  - Complement
  - Intersection
  - Union
  - Difference
- Examples

FSA (Formal definition)

Definition

A (complete) Finite State Automaton is a tuple $\langle Q, \Sigma, q_0, A, \delta \rangle$, where

Q is a finite set of *states*;
Σ is a finite *input alphabet*;
$q_0 \in Q$ is the *initial* state;
$A \subseteq Q$ is the set of *accepting* states;
$\delta : Q \times \Sigma \to Q$ is a (total) *transition* function.

FSA: formally

Example (by formal definition)

$$M = \langle \{q_0, q_1\}, \{0, 1\}, \\ \{((q_0, 0), q_0), ((q_0, 1), q_1), ((q_1, 0), q_0), ((q_1, 1), q_1)\}, q_0, \{q_1\} \rangle$$

or

Example (by formal definition)

$M = \langle \{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_1\} \rangle$, where
$\delta(q_0, 0) = q_0, \delta(q_0, 1) = q_1, \delta(q_1, 0) = q_0, \delta(q_1, 1) = q_1$

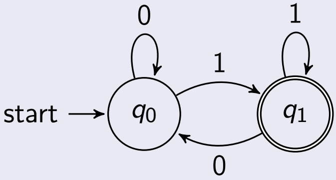
FSA: formally

Example of a FSA (by formal definition)

$$
\begin{array}{ll}
M = \langle & \\
\quad \{q_0, q_1\}, & \text{set of states} \\
\quad \{0, 1\}, & \text{input alphabet} \\
\quad \{((q_0, 0), q_0), ((q_0, 1), q_1), & \\
\qquad ((q_1, 0), q_0), ((q_1, 1), q_1)\}, & \text{total transition function} \\
\quad q_0, & \text{initial state} \\
\quad \{q_1\} & \text{set of final states} \\
\rangle &
\end{array}
$$

FSA: Graphical Representation

State Transition Diagram



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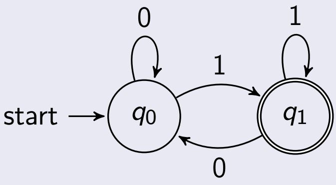
FSA: Table Representation

State Transition Table

	0	1
$\rightarrow q_0$	q_0	q_1
$^*q_1$	q_0	q_1

State Transition Diagram



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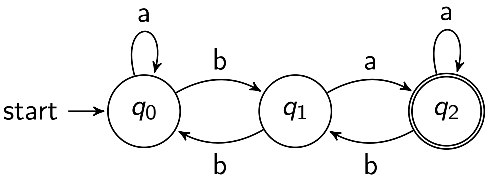
State Transition Table: Example

Given an FSA as a State Transition Diagram, build a State Transition Table

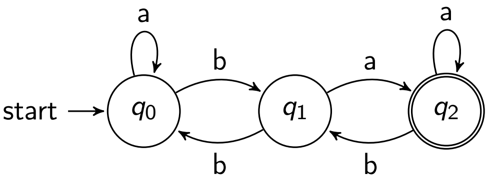


State Transition Table

	a	b
$\rightarrow q_0$		
q_1		
$^* q_2$		

State Transition Table: Example

Given an FSA as a State Transition Diagram, build a State Transition Table



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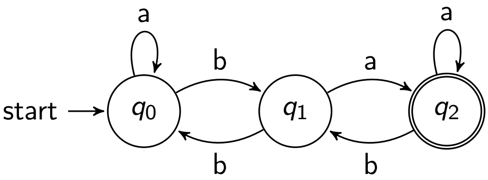

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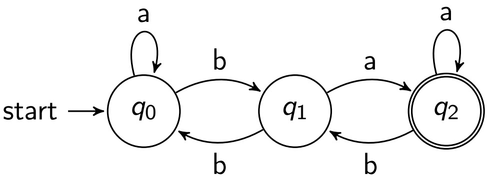

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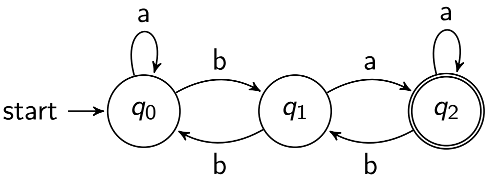

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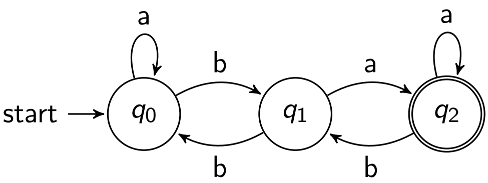

State Transition Table

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q_1	q_2	q_0
$^*q_2$	q_2	

State Transition Table: Example

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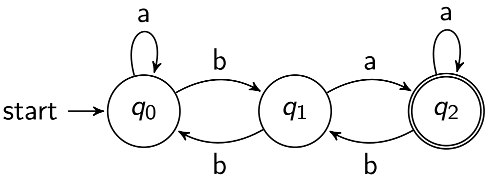
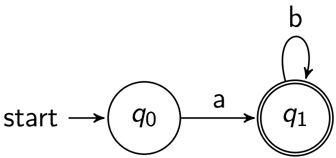


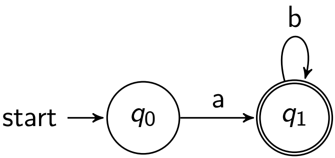
State Transition Table

	a	b
$\rightarrow q_0$	q_0	q_1
q_1	q_2	q_0
$^*q_2$	q_2	q_1

Non-complete FSA

If a FSA is not complete?





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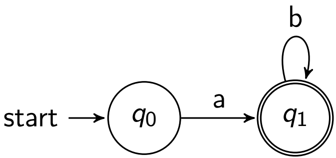

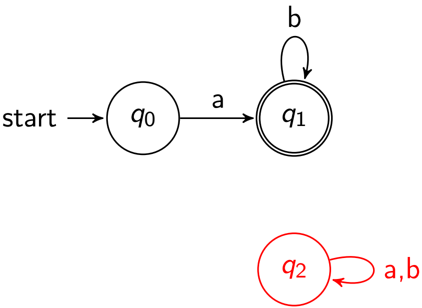

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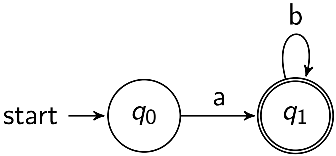

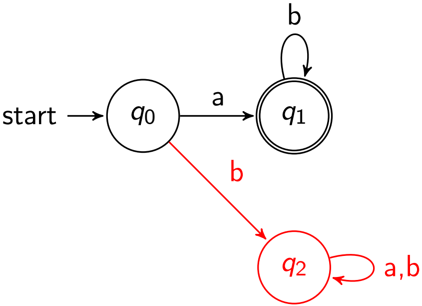

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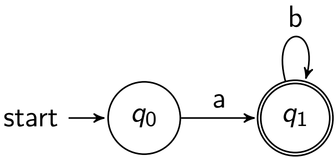

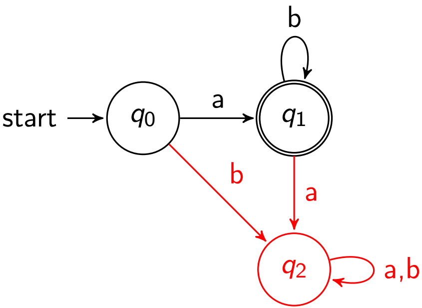

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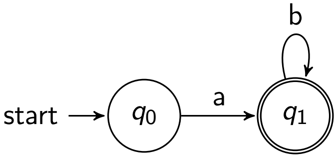

State Transition Table

	a	b
$\rightarrow q_0$	q_1	
$^*q_1$		q_1

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Operations

Problem

If we have an algorithm to accept L, how can we formulate an algorithm to accept L^c?

Problem

Suppose L_1 and L_2 are both languages over the alphabet A.

If we have one algorithm to accept L_1 and another to accept L_2, how can we formulate an algorithm to accept $L_1 \cap L_2$? (similarly, $L_1 \cup L_2$ or $L_1 \backslash L_2$).

Problem

Suppose $M = (Q^1, A, \delta^1, q_0^1, F^1)$ is a finite automaton accepting L.

What is an automaton which accepts L^c?

Problem

Suppose $M^1 = (Q^1, A, \delta^1, q_0^1, F^1)$ and $M^2 = (Q^2, A, \delta^2, q_0^2, F^2)$ are finite automata accepting L_1 and L_2, respectively.

What is an automaton which accepts $L_1 \cap L_2$?
(similarly, $L_1 \cup L_2$, $L_1 \backslash L_2$)?

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Complement

Suppose $M = (Q, A, \delta, q_0, F)$ is a finite automaton accepting L.

The automaton $M^c = (Q, A, \delta, q_0, F^c)$ accepts the language L^c.

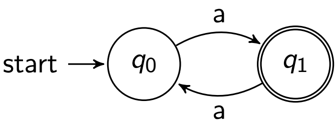
Recall that

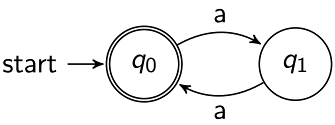
$$F^c = Q \setminus F$$

Complement: Example

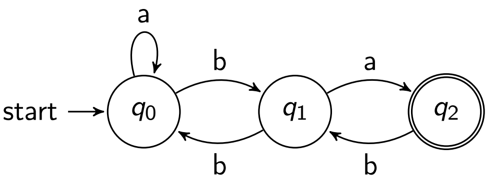
$$M = \langle \{q_0, q_1\}, \{a\}, \{((q_0, a), q_1), ((q_1, a), q_0)\}, q_0, \{q_1\} \rangle$$

$$M^c = \langle \{q_0, q_1\}, \{a\}, \{((q_0, a), q_1), ((q_1, a), q_0)\}, q_0, \{q_0\} \rangle$$



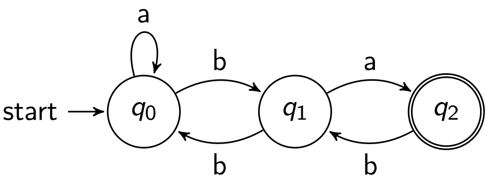


Complement: Example



What would be the complement M^c?

Complement: Example



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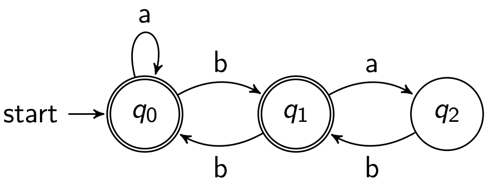
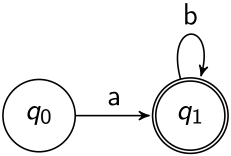


Complement: Example

If a FSA is not complete?



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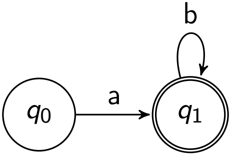

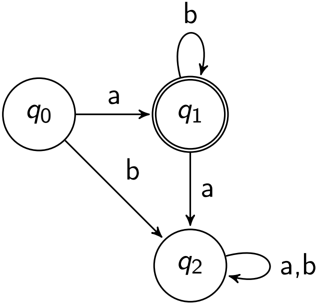

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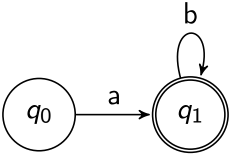

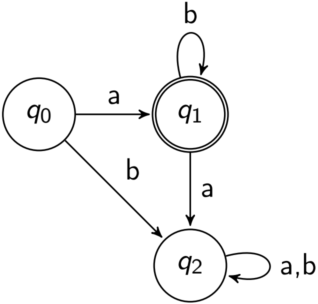

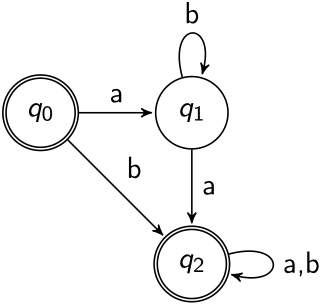

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Intersection

Suppose $M^1 = (Q^1, A, \delta^1, q_0^1, F^1)$ and $M^2 = (Q^2, A, \delta^2, q_0^2, F^2)$ are finite automata accepting L_1 and L_2, respectively.

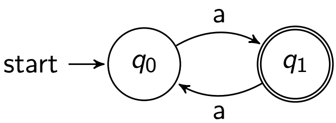
$$\begin{aligned}
&Q = Q^1 \times Q^2 \\
&A \\
&q_0 = (q_0^1, q_0^2) \\
&\delta((q,p),a) = (\delta^1(q,a), \delta^2(p,a)) \\
&F = \{(q,p) \in Q^1 \times Q^2 \mid q \in F^1 \ \& \ p \in F^2\}
\end{aligned}$$

The automaton $M = (Q, A, \delta, q_0, F)$ accepts the language $L_1 \cap L_2$.

$$M = M_1 \cap M_2$$

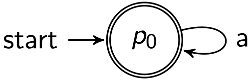
Intersection: Example 1

$$M^1 = \langle \{q_0, q_1\}, \{a\},$$
$$\{((q_0, a), q_1), ((q_1, a), q_0)\},$$
$$q_0, \{q_1\}\rangle$$



Intersection: Example 1

$$M^2 = \langle \{p_0\}, \{a\},\\ \{((p_0, a), p_0)\},\\ p_0, \{p_0\}\rangle$$



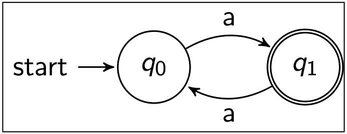
Intersection: Example 1

$$\begin{aligned} M^1 = \langle & \{q_0, q_1\}, \{a\}, \\ & \{((q_0, a), q_1), ((q_1, a), q_0)\}, \\ & q_0, \{q_1\}\rangle \end{aligned}$$

$$\begin{aligned} M^2 = \langle & \{p_0\}, \{a\}, \\ & \{((p_0, a), p_0)\}, \\ & p_0, \{p_0\}\rangle \end{aligned}$$

$$\begin{aligned} (M^1 \cap M^2) = \langle & \{(q_0, p_0), (q_1, p_0)\}, \{a\}, \\ & \Big\{\Big(((q_0, p_0), a), (q_1, p_0)\Big), \Big(((q_1, p_0), a), (q_0, p_0)\Big)\Big\}, \\ & (q_0, p_0), \{(q_1, p_0)\}\rangle \end{aligned}$$

Intersection: Example 1



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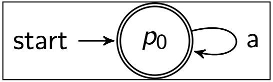

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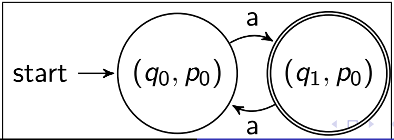
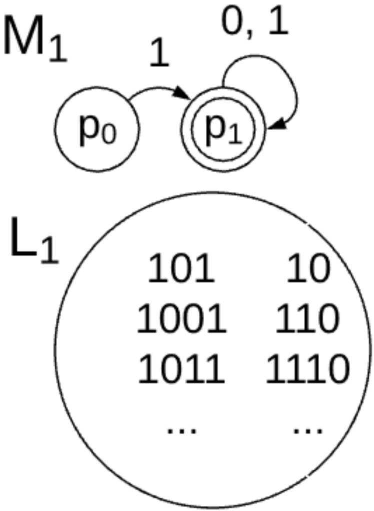


Intersection: Example 2



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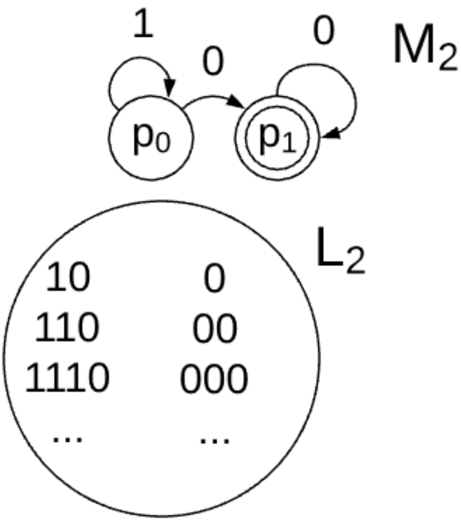

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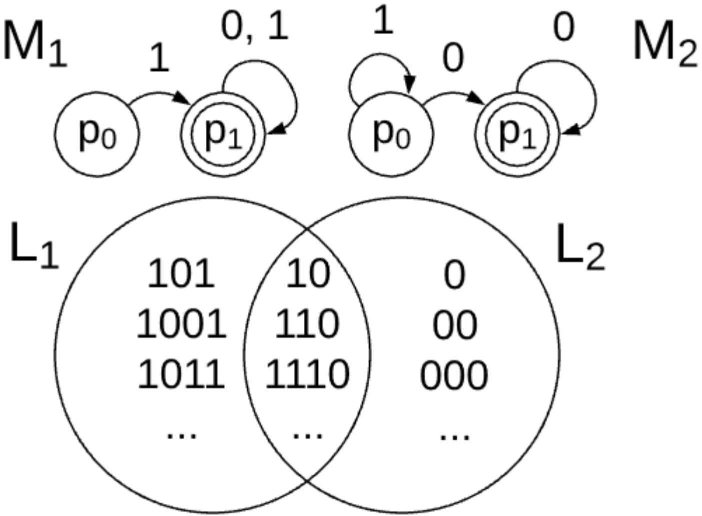

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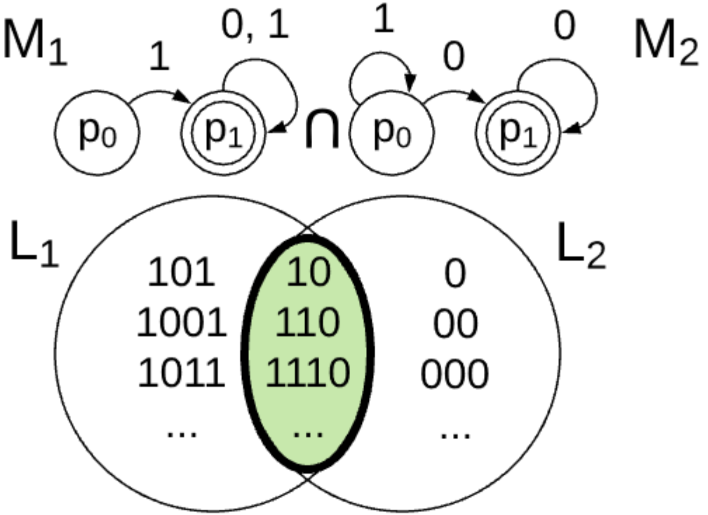

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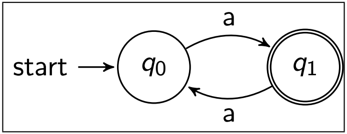
\end{aligned}$$

The automaton $M = (Q, A, \delta, q_0, F)$ accepts the language $L_1 \cup L_2$.

$$M = M_1 \cup M_2$$

Union: Example 1



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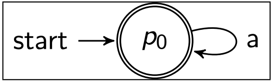

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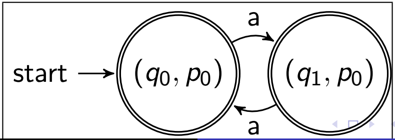

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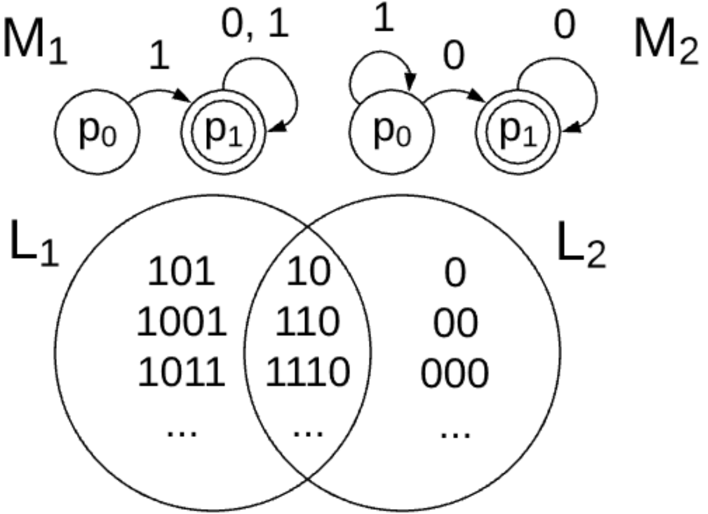

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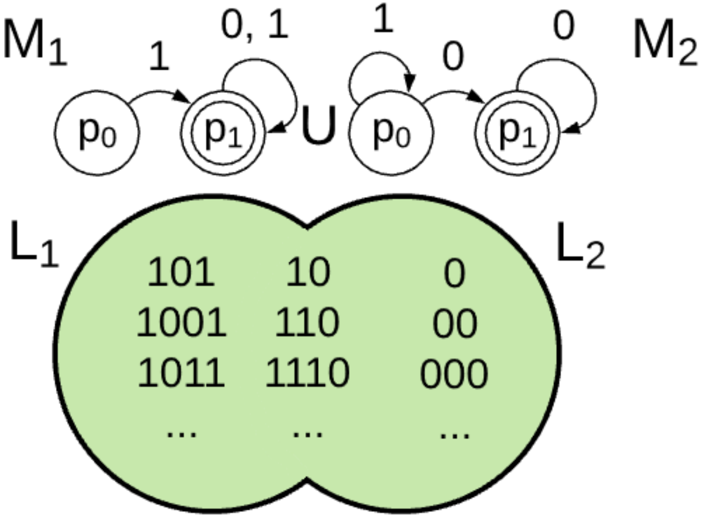

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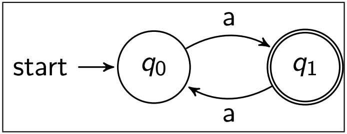
&F = \{(q,p) \in Q^1 \times Q^2 \mid q \in F^1 \;\&\; p \notin F^2\}
\end{aligned}$$

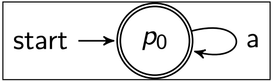
The automaton $M = (Q, A, \delta, q_0, F)$ accepts the language $L_1 \setminus L_2$.

$$M = M_1 \setminus M_2$$

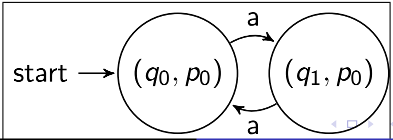
Difference (Example 1 $L_1 \setminus L_2$)



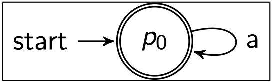
$$\setminus$$



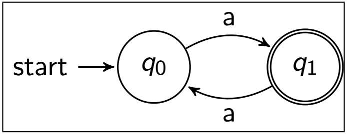
$$=$$



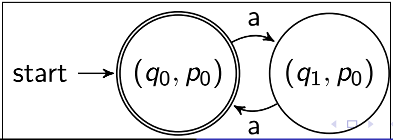
Difference (Example 2 $L_2 \setminus L_1$)



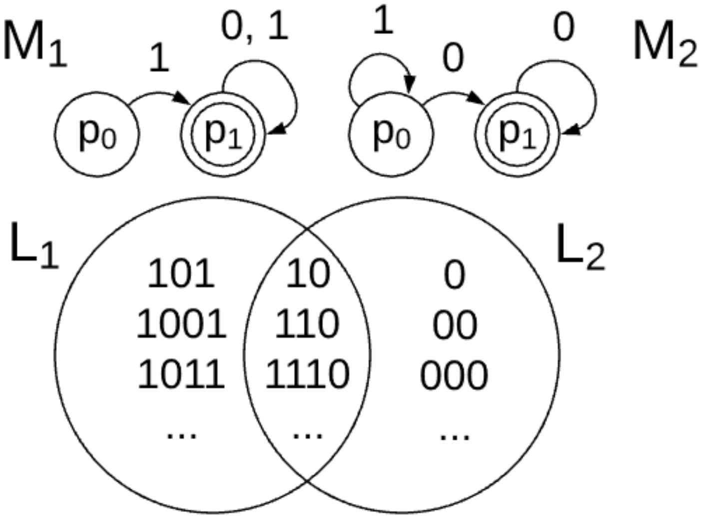
$$\setminus$$



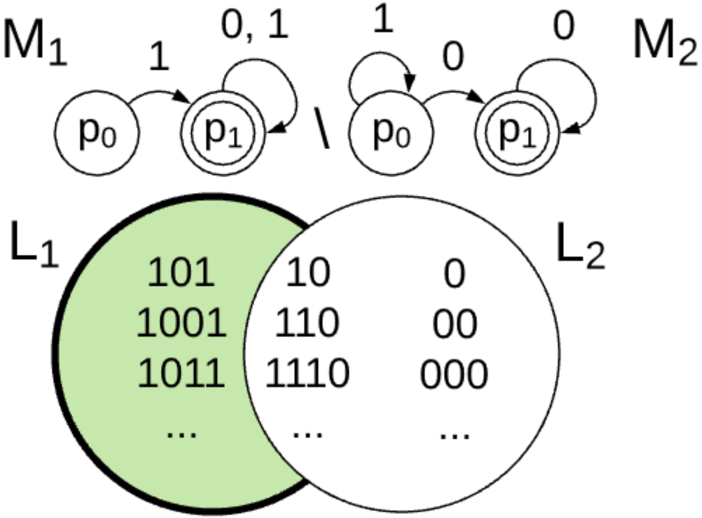
$$=$$



Difference (Example 3)



Difference (Example 3)

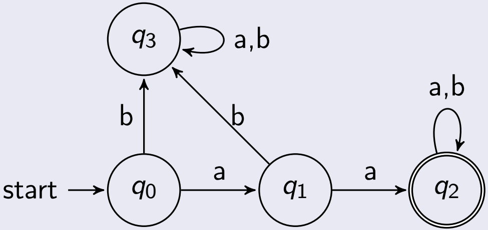


Finite State Automaton (FSA)

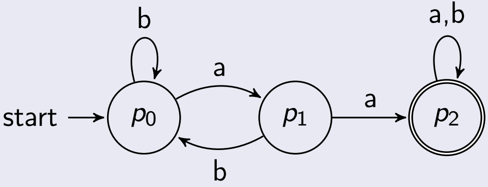
- Representations of FSA
  - Formally
  - Graphical Representation
  - State Transition Table
- Operations on FSA
  - Complement
  - Intersection
  - Union
  - Difference
- **Examples**

Examples: Complement

M_1

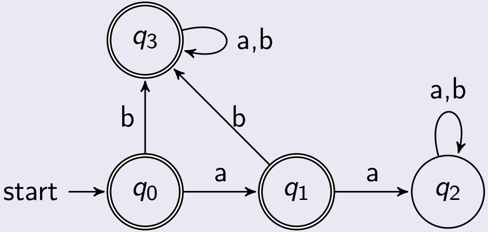


M_2

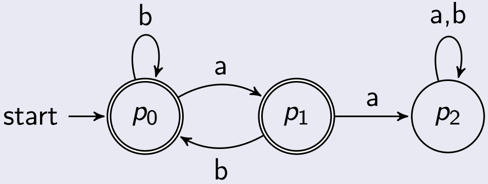


Examples: Complement

M_1^c

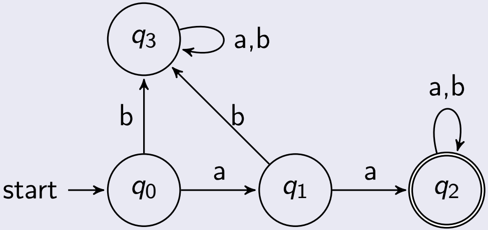


M_2^c

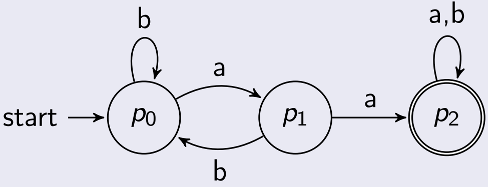


Examples: Intersection

M_1

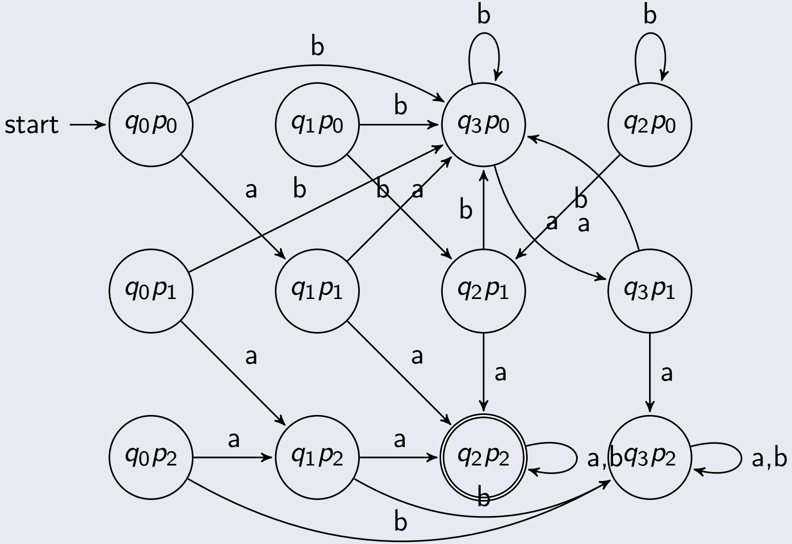


M_2



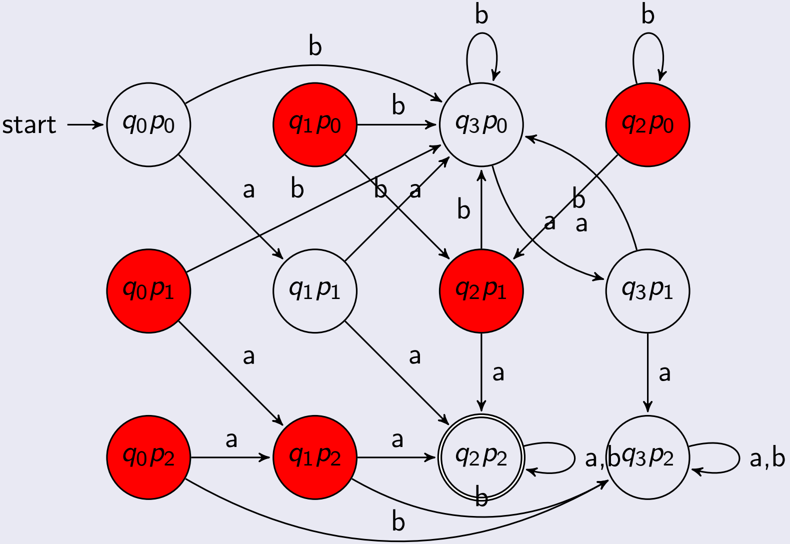
Examples: Intersection

$M_1 \cap M_2$



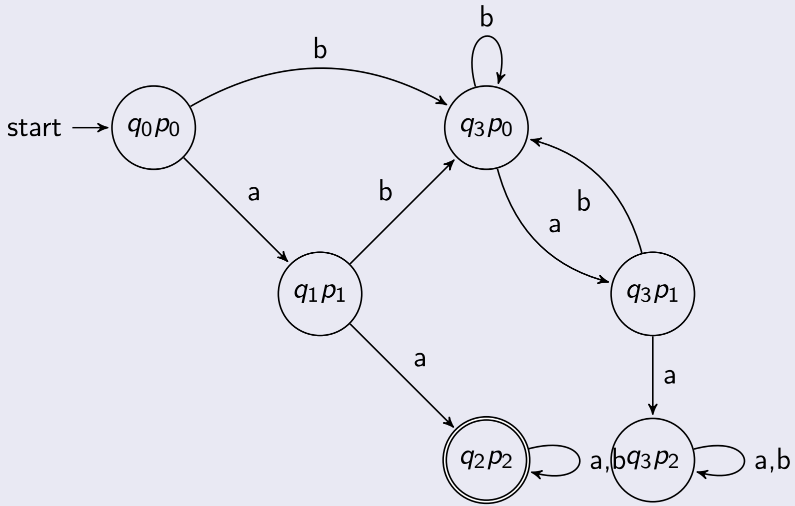
Examples: Intersection

$M_1 \cap M_2$



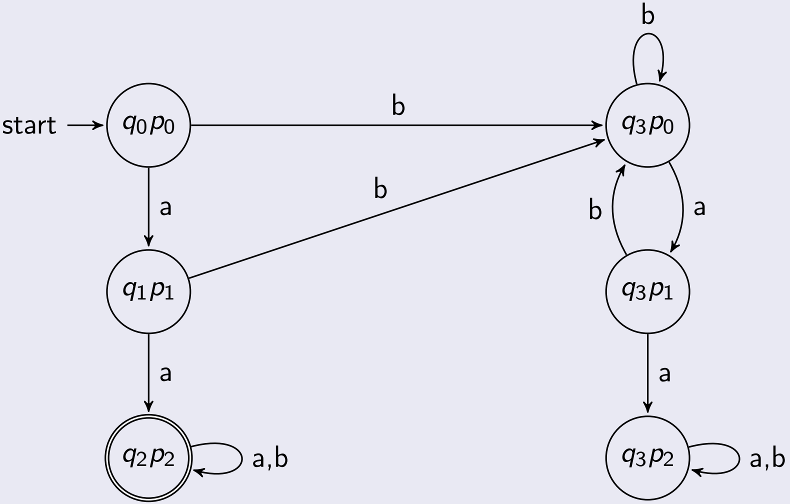
Examples: Intersection

$M_1 \cap M_2$



Examples: Intersection

$M_1 \cap M_2$



Examples: Intersection

Table representation of M_1

δ	a	b
$\rightarrow q_0$	q_1	q_3
q_1	q_2	q_3
$^*q_2$	q_2	q_2
q_3	q_3	q_3

Table representation of M_2

δ	a	b
$\rightarrow p_0$	p_1	p_0
p_1	p_2	p_0
$^*p_2$	p_2	p_2

Examples: Intersection

Table representation of $M_1 \cap M_2$

δ	a	b
$\rightarrow (q_0 p_0)$	$(q_1 p_1)$	$(q_3 p_0)$
$(q_1 p_0)$	$(q_2 p_1)$	$(q_3 p_0)$
$(q_2 p_0)$	$(q_2 p_1)$	$(q_2 p_0)$
$(q_3 p_0)$	$(q_3 p_1)$	$(q_3 p_0)$
$(q_0 p_1)$	$(q_1 p_2)$	$(q_3 p_0)$
$(q_1 p_1)$	$(q_2 p_2)$	$(q_3 p_0)$
$(q_2 p_1)$	$(q_2 p_2)$	$(q_2 p_0)$
$(q_3 p_1)$	$(q_3 p_2)$	$(q_3 p_0)$
$(q_0 p_2)$	$(q_1 p_2)$	$(q_3 p_2)$
$(q_1 p_2)$	$(q_2 p_2)$	$(q_3 p_2)$
$^*(q_2 p_2)$	$(q_2 p_2)$	$(q_2 p_2)$
$(q_3 p_2)$	$(q_3 p_2)$	$(q_3 p_2)$

Let us remove unreachable states

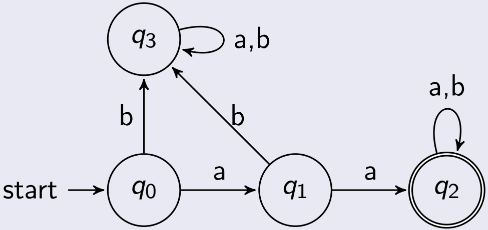
Examples: Intersection

Table representation of $M_1 \cap M_2$

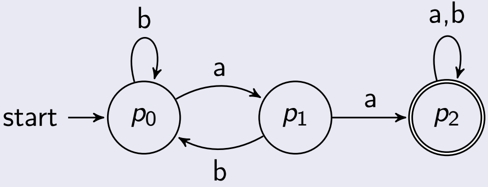
δ	a	b
$\rightarrow (q_0 p_0)$	$(q_1 p_1)$	$(q_3 p_0)$
$(q_3 p_0)$	$(q_3 p_1)$	$(q_3 p_0)$
$(q_1 p_1)$	$(q_2 p_2)$	$(q_3 p_0)$
$(q_3 p_1)$	$(q_3 p_2)$	$(q_3 p_0)$
$^*(q_2 p_2)$	$(q_2 p_2)$	$(q_2 p_2)$
$(q_3 p_2)$	$(q_3 p_2)$	$(q_3 p_2)$

Examples: Union

M_1

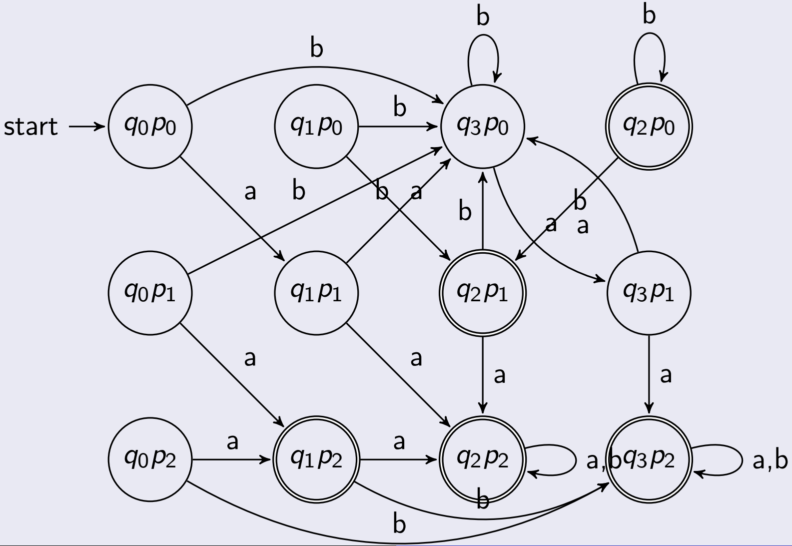


M_2



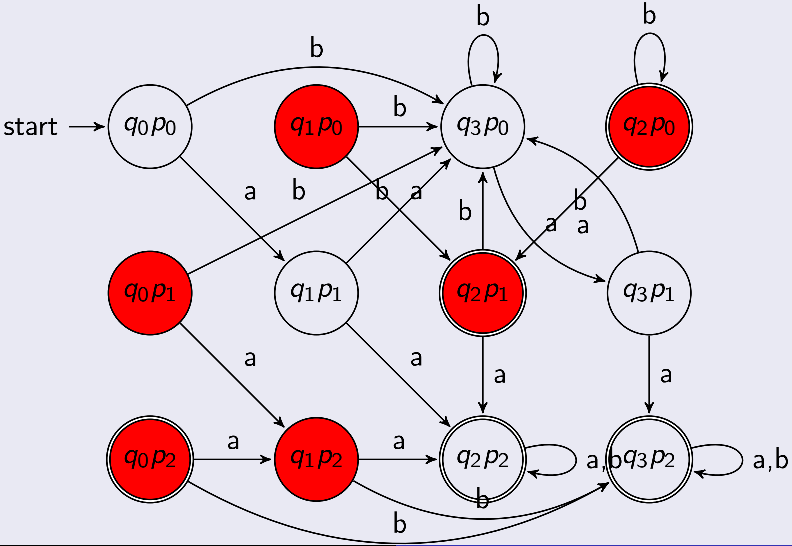
Examples: Union

$M_1 \cup M_2$



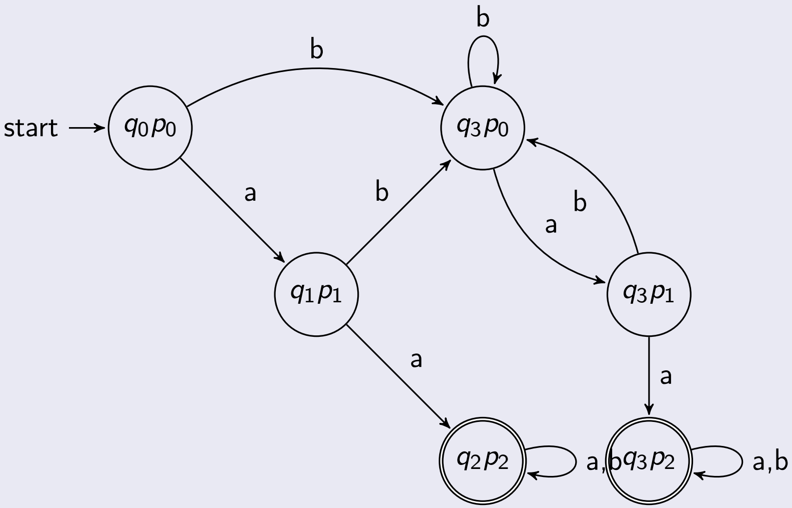
Examples: Union

$M_1 \cup M_2$



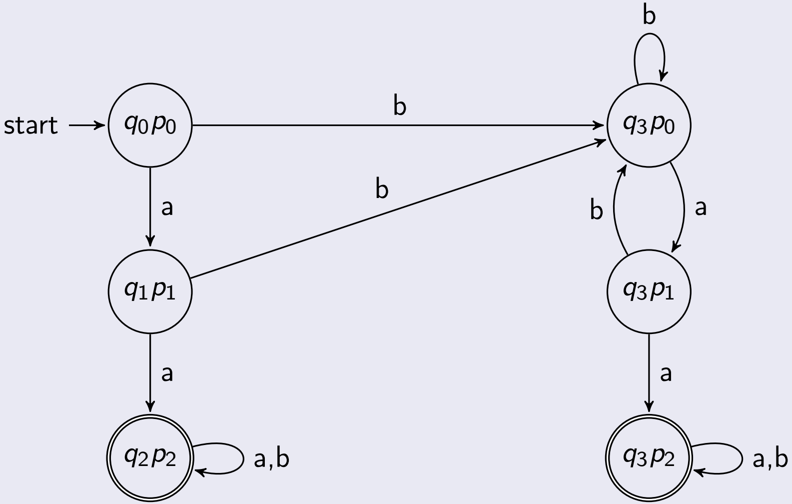
Examples: Union

$M_1 \cup M_2$



Examples: Union

$M_1 \cup M_2$



Examples: Union

Table representation of $M_1 \cup M_2$

δ	a	b
$\rightarrow (q_0 p_0)$	$(q_1 p_1)$	$(q_3 p_0)$
$(q_3 p_0)$	$(q_3 p_1)$	$(q_3 p_0)$
$(q_1 p_1)$	$(q_2 p_2)$	$(q_3 p_0)$
$(q_3 p_1)$	$(q_3 p_2)$	$(q_3 p_0)$
$^*(q_2 p_2)$	$(q_2 p_2)$	$(q_2 p_2)$
$^*(q_3 p_2)$	$(q_3 p_2)$	$(q_3 p_2)$

Examples: Difference

The accepting state of M_1 is q_2. The accepting state of M_2 is p_2

Table representation of $M_1 \setminus M_2$

δ	a	b
$\rightarrow (q_0 p_0)$	$(q_1 p_1)$	$(q_3 p_0)$
$(q_3 p_0)$	$(q_3 p_1)$	$(q_3 p_0)$
$(q_1 p_1)$	$(q_2 p_2)$	$(q_3 p_0)$
$(q_3 p_1)$	$(q_3 p_2)$	$(q_3 p_0)$
$(q_2 p_2)$	$(q_2 p_2)$	$(q_2 p_2)$
$(q_3 p_2)$	$(q_3 p_2)$	$(q_3 p_2)$

Examples: Difference

The accepting state of M_1 is q_2. The accepting state of M_2 is p_2

Table representation of $M_2 \setminus M_1$

δ	a	b
$\rightarrow (q_0 p_0)$	$(q_1 p_1)$	$(q_3 p_0)$
$(q_3 p_0)$	$(q_3 p_1)$	$(q_3 p_0)$
$(q_1 p_1)$	$(q_2 p_2)$	$(q_3 p_0)$
$(q_3 p_1)$	$(q_3 p_2)$	$(q_3 p_0)$
$(q_2 p_2)$	$(q_2 p_2)$	$(q_2 p_2)$
${}^*(q_3 p_2)$	$(q_3 p_2)$	$(q_3 p_2)$

Thank you for your attention!