Discrete Mathematics and Logic
Graph Theory
Lecture 3

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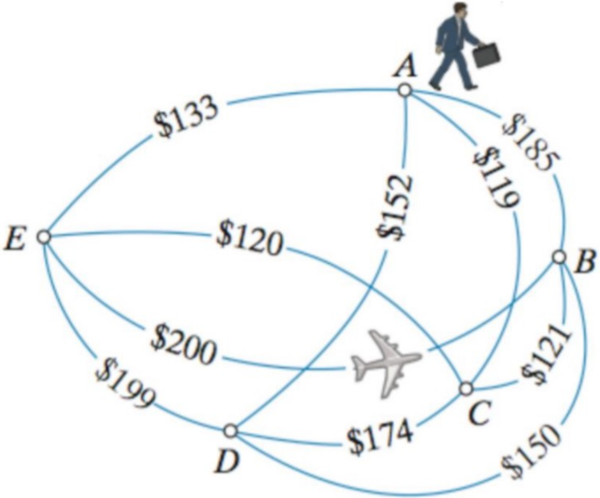
What did we know in the last week?

1. Trees and forests
2. The characteristic property for trees
3. Spanning trees
4. Weighted graphs
5. Minimal spanning trees
6. The Prim's algorithm
7. Never forget that my favorite question is ”Why?“!

Traversability

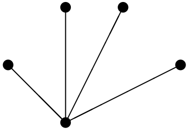


Traversability

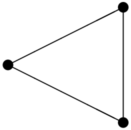
Definition

A connected graph having no cycle is called a **tree**.

a tree



not a tree



What about cycles?

Traversability

Definitions

The sequence $u_1 u_2 \ldots u_k$ is called a **walk** (or a **way**) from u_1 to u_k, if any vertices u_i and u_{i+1} are neighbours.

Definition

Let the sequence $u_1, u_2, \ldots, u_k$ be a walk. We say that

it is **closed**, if $u_1 = u_k$.

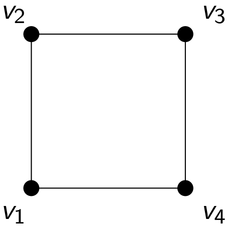
it is a **path**, if $(u_i, u_{i+1}) \neq (u_j, u_{j+1})$ for all $i \neq j$.

it is a **cycle**, if it is a closed path.

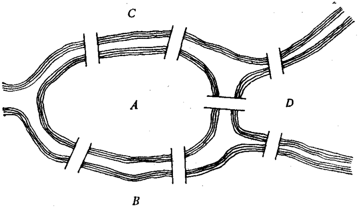
Traversability

Example

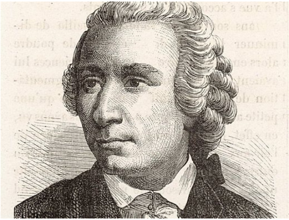
$v_1 v_2 v_3 v_1$ – a non-walk
$v_1 v_2 v_3 v_2$ – some walk
$v_1 v_2 v_3 v_2 v_1$ – a closed walk
$v_4 v_3 v_4$ – a closed walk
$v_4 v_3 v_2$ – a path
$v_1 v_2 v_3 v_4$ – a path
$v_1 v_2 v_3 v_4 v_1$ – a cycle



Traversability



It is impossible to cross each
of the seven bridges of Königsberg
once and only once during a walk (1736).



Leonhard Euler
(1707 - 1783)

Euler paths

Definition

A path is called **Euler**, if it contains every edge and only once.

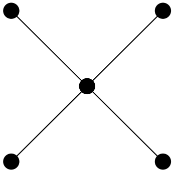
Remark (the terminology problem)

An Euler path = An Euler trail.

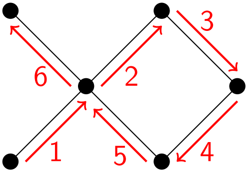
Euler paths

Definition

A path is called **Euler**, if it contains every edge and only once.



No!



Yes!

Euler paths

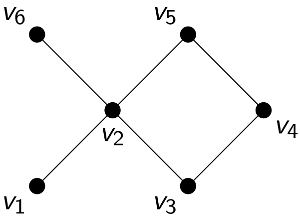
Definition

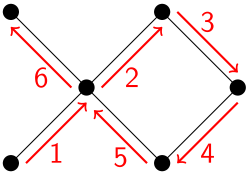
A path is called **Euler**, if it contains every edge and only once.

Definition

A graph having an Euler path is called **semi-Eulerian** (only in Russian), has no name in English.

Euler paths





The graph has a Euler path =

= it is a semi-Eulerian

The path $v_1, v_2, v_5, v_4, v_3, v_2, v_6$ is an Euler path

Euler cycles

Definition

A cycle is called **Euler**, if it contains every edge and only once.

So, a closed Euler path is an Euler cycle.

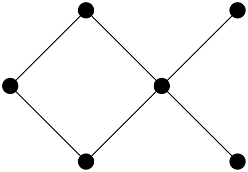
Remark (the terminology problem)

An Euler cycle = An Euler tour.

Euler cycles

Definition

A cycle is called **Euler**, if it contains every edge and only once.



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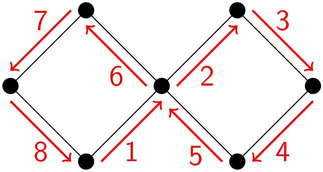


Yes!

Euler cycles

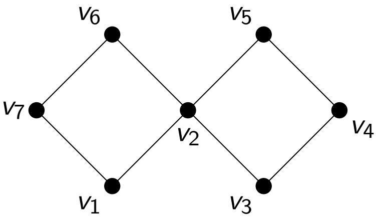
Definition

A cycle is called **Euler**, if it contains every edge and only once.

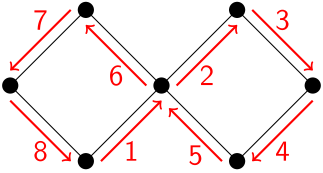
Definition

A graph having an Euler cycle is called an **Eulerian**.

Euler cycles



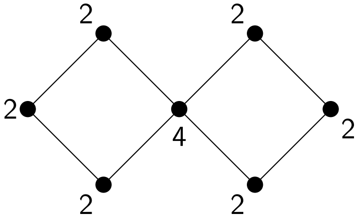
The graph is an Eulerian



The cycle $v_1, v_2, v_5, v_4, v_3, v_2, v_6, v_7, v_1$ is an Euler cycle

Theorem (the sufficient and necessary condition)

A non-trivial connected graph is Eulerian iff every vertex has even degree.

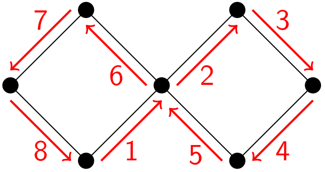


Theorem (the sufficient and necessary condition)

A non-trivial connected graph is Eulerian iff every vertex has even degree.

Proof (⇒)

If a vertex appearing k times in a Euler tour, then it must have degree $2k$.



The Euler cycle $v_1, v_2, v_5, v_4, v_3, v_2, v_6, v_7, v_1$

Theorem (the sufficient and necessary condition)

A non-trivial connected graph is Eulerian iff every vertex has even degree.

Proof ($\Leftarrow$)

Suppose that G is a non-Eulerian connected graph with at least one edge and all vertices have even degree.

Since each vertex of G has degree at least 2, the G must contain a cycle.

Theorem (the sufficient and necessary condition)

A non-trivial connected graph is Eulerian iff every vertex has even degree.

Proof ($\Leftarrow$)

Since the G contains a cycle, we can find a cycle C of maximum possible length.

Obviously, C is itself Eulerian. So, each vertex of C has even degree.

Theorem (the sufficient and necessary condition)

A non-trivial connected graph is Eulerian iff every vertex has even degree.

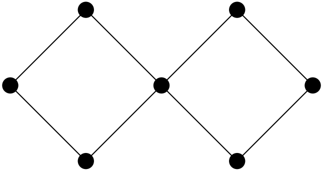
Proof ($\Leftarrow$)

Therefore, each vertex of $G - E(C)$ has also even degree and is not trivial (since G is not an Eulerian).

Thus, C can be extended. It contradicts with the choice of C.

Theorem (the sufficient and necessary condition)

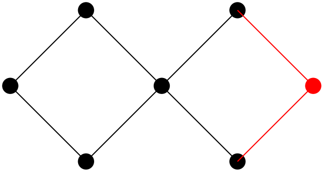
A non-trivial connected graph is Eulerian iff every vertex has even degree.



Euler paths and cycles

Corollary

A connected graph has an Euler path iff it either has no, or has exactly two vertices with odd degrees.



The Fleury's algorithm

1. Let v_0 be an arbitrary vertex, and $W_0 = \emptyset$.

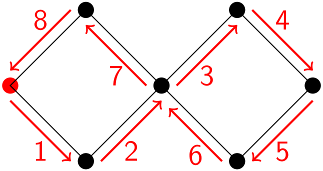
2. Repeat the following procedure for $i = 1, 2, \dots$ as long as possible:

Suppose that $W_i = \{e_1, e_2, \dots, e_i\}$ has been constructed, where $e_j = (v_{j-1}, v_j)$. Choose a new edge e_{i+1} such that

a) $e_{i+1} \notin W_i$ and e_{i+1} has an end v_i,

b) $G - \{e_1, e_2, \dots, e_i, e_{i+1}\}$ does not contain two non-trivial connected components, unless there is no alternative.

The Fleury's algorithm



What we knew today?

1. Euler paths and cycles
2. Eulerians
3. The sufficient and necessary condition
4. The Fleury's algorithm

Thank you for your attention!

Don't forget that my favorite question is "Why?"!