Block A

You need to choose only one answer.
Each question is evaluated with 2 points (+2 extra points).

A0. What is the favorite question of prof. Frolov? (2 extra points)

a. What?
b. Where?
c. When?
d. Why?

A1. What are the elements of a graph?

a. Points and loops
b. Vertices and edges
c. Lines and circles
d. Trees

A2. What is the degree of a vertex?

a. The number of all edges
b. The number of all vertices
c. The number of edges incident with the vertex
d. The number of vertices incident with the vertex

A3. What is true in a tree?

a. Any two vertices are incident
b. Any two vertices are adjacent
c. Any two vertices are connected by a unique path
d. Any two vertices are connected by a unique circle

A4. What are an Eulerian?

a. It is a graph with a cycle which traverses each vertex once
b. It is a graph with a cycle which traverses each edge once
c. It is a graph with a cycle which traverses each vertex twice
d. It is a graph with a cycle which traverses each edge twice

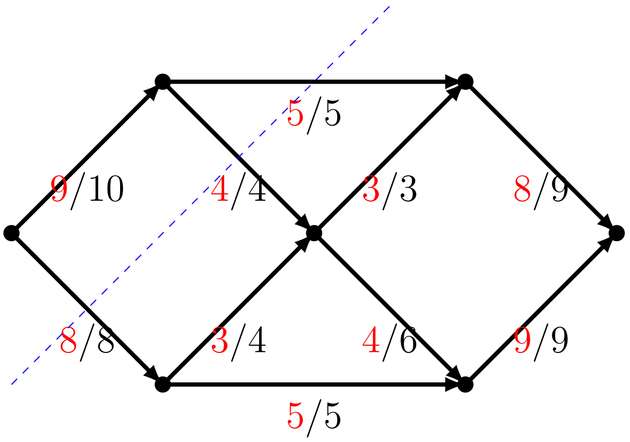
A5. What is true in any network?

a. The value of maximum flow equals the capacity of a maximum cut
b. The value of maximum flow equals the capacity of a minimum cut
c. The value of minimum flow equals the capacity of a minimum cut
d. The value of minimum flow equals the capacity of a maximum cut

Block B

Each question is evaluated with 5 points.

B1. Find a flow with the maximum possible value. Why is it maximum?



Answer: the flow is $8 + 9 = 17$. It is maximum, since it coincides with the cut (the blue line): $8 + 4 + 5 = 17$.

Remarks:

- There are different answers (flows and cuts). The answer above is just one of them.
- Only correct flow = 3 points (red numbers).
- With correct cut = 5 points (black numbers).

B2. Come up with an example of an undirected weighted graph with at least 5 vertices and 9 edges and demonstrate how Dijkstra's algorithm works on this graph. Note that the weights in the graph should not be repeated!

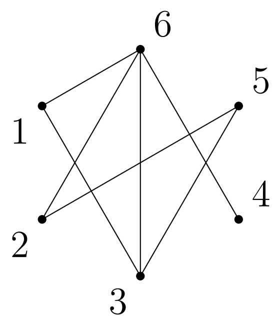
Remarks:

- A correct example with a correct answer = 2 points.
- With a correct demonstration (step by step, for example, in the table) = 5 points.

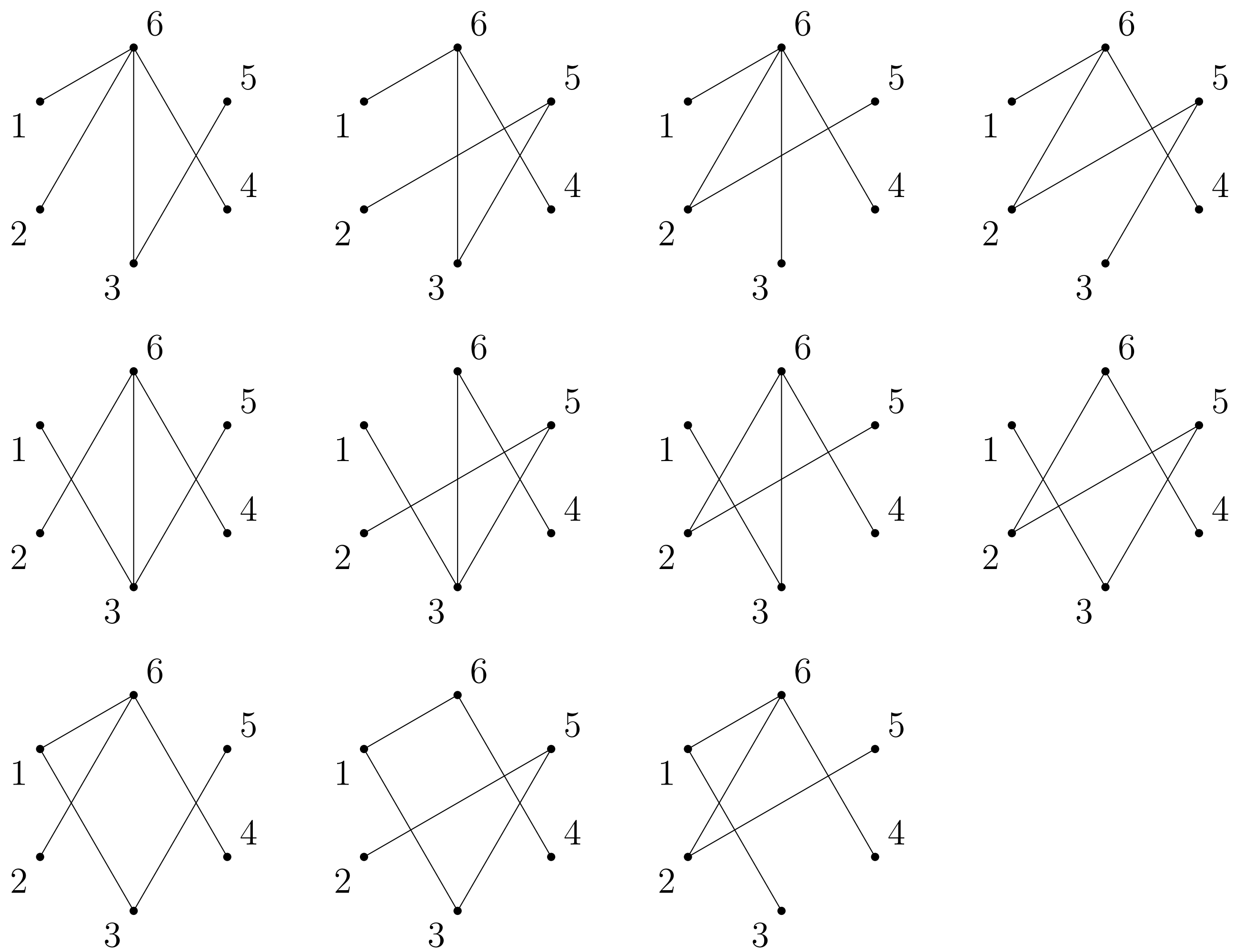
B3. Let $G = (V, E)$ be a graph, where $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 6\}\}$.

Find all spanning trees. What of them are pairwise non-isomorphic?

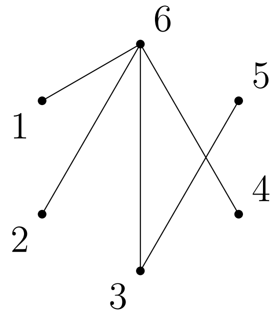
The graph G:

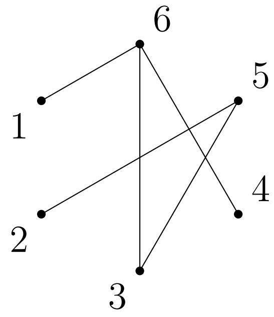


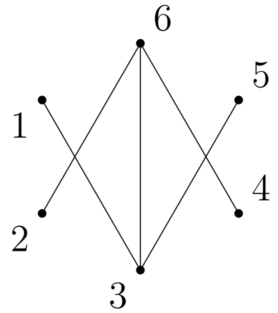
All spanning trees of G (11 graphs):

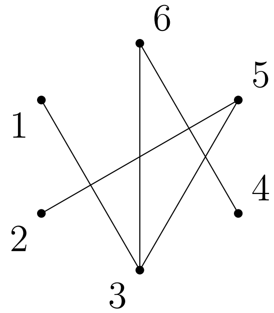


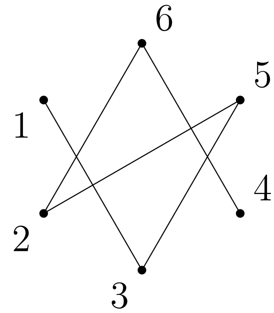
The pairwise non-isomorphic spanning trees of G (5 graphs):











Remarks:

- The exercise contains two parts: (a) find all spanning trees (3 points), and (b) what of them are pairwise non-isomorphic? (2 points), 5 points in total.

- 11 spanning trees = 3 points. If you found not all graphs, there is a penalty: 9–10 spanning trees = 1 penalty point ($3-1=2$ points), 7–8 spanning trees = 2 penalty points ($3-2=1$ points), less than 7 spanning trees = 0 points in total.

- 5 pairwise non-isomorphic spanning trees = 2 points. If you found not all graphs, there is a penalty: 4 graphs = 1 penalty point ($2-1=1$ points), less than 4 graphs = 0 points in total.

- The same penalty idea for repetitions. It means, for example, if you "found" 12 spanning trees or 6 pairwise non-isomorphic spanning trees, then there is 1 point penalty.

Block C

Each question is evaluated with 5 points.

C1. Give and prove Handshaking Lemma.

Lemma 1 (Handshaking lemma)**.** *For each graph* $G = (V_G, E_G)$*,*

$$\sum_{v \in V_G} d_G(v) = 2 \cdot |E_G|.$$

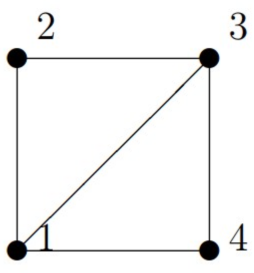
Proof. Every edge $e \in E_G$ gives "+1" to degrees of two vertices.
Therefore, $\sum\limits_{v \in V_G} d_G(v) = 2 \cdot |E_G|$. □

Remarks:

- Only statement of the lemma = 2 points.
- With a correct proof = 5 points.
- "The sum of degrees is even" is the corollary of handshaking lemma, so, it is not enough! A correct proof of this = 3 points.

C2. Give with explanations an example of a 4-colourable graph such that it is Hamiltonian, but is not Eulerian.

Example:



The graph is not Eulerian, since it contains vertices with odd degree (vertices 1 and 3). The graph is Hamiltonian, since it contains Hamilton cycle: $1 \to 2 \to 3 \to 4 \to 1$. The graph is 4-colourable, since the following mapping α is 4-colouring: $\alpha(1)$ = red, $\alpha(2)$ = blue, $\alpha(3)$ = green, $\alpha(4)$ = brown. In other way, since the graph is planar, it is 4-colourable (see the theorem about 4 colours).

Remarks:

- "A graph contains K_4" does not imply "the graph is 4-colourable". If a graph contains K_4 as a subgraph, then the graph is k-colourable for $k \geq 4$.

- To show that a graph is 4-colourable, you need to give 4-colouring mapping from the set of vertices to 4 colours or numbers $\{1, 2, 3, 4\}$. Note that the mapping must not to be a surjection in general.

- In another way, you could draw a plane graph and use the theorem about 4 colours: any planar graph is 4-colourable.

- Only graph and no explanations = 0 points.

- A correct example without explanations about 4-colourable = 4 points.

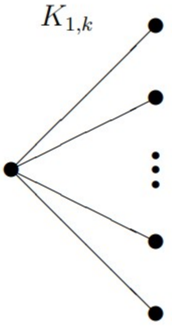
C3. Find all n, m such that $K_{n,m}$ is planar. Explain your answer. (You can not use Kuratowskis Theorem.)

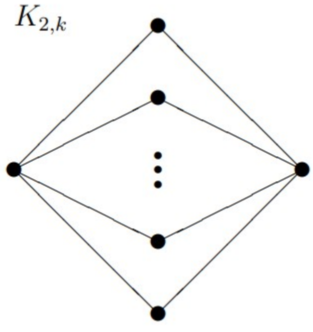
For any $k \geq 1$, $K_{1,k}$, $K_{k,1}$, $K_{2,k}$ and $K_{k,2}$ are planar (see pictures below).

For others, firstly, we prove that $K_{3,3}$ is not planar. Suppose (for a contradiction) that $K_{3,3}$ is planar. Hence, the number of faces f for $K_{3,3}$ exists. By Euler's formula, $f = 2 - v + e = 2 - 6 + 9 = 5$, since $v = 6$ and $e = 9$.

Since any cycle of $K_{3,3}$ is even, each face contains at least four edges on its boundary. Each edge lies on at most two faces. It follows from the last two facts that $4f \leq 2e$. This is a contradiction, since $f = 5$, $e = 9$ and $4 \cdot 5 \not\leq 2 \cdot 9$. Therefore, $K_{3,3}$ is not planar.

Finally, if $n \geq 3$ and $m \geq 3$ then $K_{n,m}$ contains a subgraph isomorphic to $K_{3,3}$. Hence, such $K_{n,m}$ is not planar.





Remarks:

- Just the answer = 0 points.
- Proof the planarity of $K_{1,k}$, $K_{k,1}$, $K_{2,k}$ and $K_{k,2}$ (using pictures or explanations in words) = 2 points (without explanations = 0 points).
- With proof the planarity of $K_{3,3}$ (without the general case) = 4 points.
- Let's consider the following reasoning (shortly):
  Let $K_{n,m}$ be planar. Then $v = n + m$, $e = n \cdot m$. From $4f \leq 2e$ and $v - e + f = 2$ it follows that $nm \leq 2(n + m) - 4$. This holds only for $n \leq 2$ or $m \leq 2$. IMPORTANT. It means that if $n \geq 3$ and $m \geq 3$ then $K_{n,m}$ is not planar. And only this! We needed to prove additionally that $K_{n,m}$ is planar if $n \leq 2$ or $m \leq 2$. If you missed this proof then the task is graduated at 3 points.