Analytical Geometry and Linear Algebra. Lecture 2.

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End of Lecture #1

Review

- Points and Vectors
- Vector Addition. Scalar Vector Multiplication
- Properties of Vector Arithmetic
- Vector spaces, Subspaces
- Span, Linear Independence
- Basis and Coordinates

Quiz in class

Go to `http://b.socrative.com`

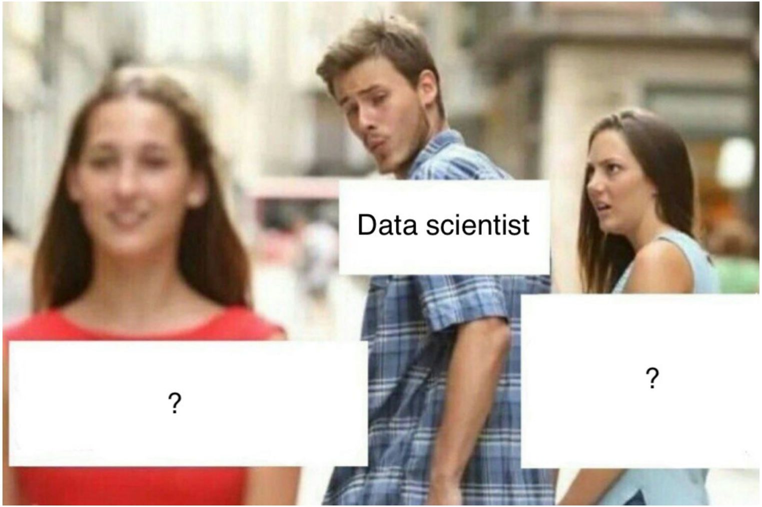
Type Room: **LINAL**

Answer 3 questions.
(you can use anonymous login)

Lecture 2. Outline

- Part 1. The Dot Product and its properties
  - Norm of a vector
  - Cauchy-Schwarz inequality
  - Triangle Inequality
- Part 2. Vector Cross Product
- Part 3. Matrices and Operations (2x2, 3x3).

How would you label the blanks?

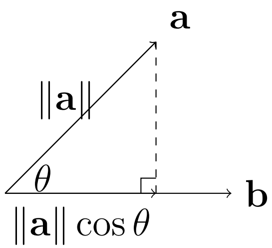


Part 1. Dot Product

Geometric view (in $\mathbb{R}^2$ and $\mathbb{R}^3$)

Scalar/dot product

$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



Scalar projection

Scalar projection of vector $\mathbf{a}$ on vector $\mathbf{b}$ is **a scalar**:
$a_b = \|\mathbf{a}\| \cos\theta$

Find the scalar projections $\mathbf{a_b}$ and $\mathbf{b_a}$.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Orthogonal projection

Orthogonal projection of vector $\mathbf{a}$ on vector $\mathbf{b}$ is **a vector**:
$\mathbf{a_b} = \hat{\mathbf{b}}||\mathbf{a}|| \cos\theta$

$\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$

Dot Product. Algebraic view

Definition

Let V be a vector space over $\mathbb{R}$.
By a dot product on V we mean a real valued function $\mathbf{u} \cdot \mathbf{v}$ on $V \times V \to \mathbb{R}$ which satisfies the following axioms:

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- $\mathbf{u} \cdot (\mathbf{w} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \quad , \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

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- $(\lambda \mathbf{u}) \cdot \mathbf{v} = \lambda(\mathbf{u} \cdot \mathbf{v}) \quad , \quad \forall \mathbf{u}, \mathbf{v} \in V, \lambda \in \mathbb{R}$

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- $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u} \quad , \quad \forall \mathbf{u},\mathbf{v}\in V$
- $\mathbf{u}\cdot(\mathbf{w}+\mathbf{v})=\mathbf{u}\cdot\mathbf{w}+\mathbf{u}\cdot\mathbf{v} \quad , \quad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in V$
- $(\lambda\mathbf{u})\cdot\mathbf{v}=\lambda(\mathbf{u}\cdot\mathbf{v}) \quad , \quad \forall \mathbf{u},\mathbf{v}\in V,\lambda\in\mathbb{R}$
- $\mathbf{u}\cdot\mathbf{u}\geq 0 \quad , \quad \forall \mathbf{u}\in V$
- $\mathbf{u}\cdot\mathbf{u}=0\Leftrightarrow\mathbf{u}=\mathbf{0}$

Notation

$$\mathbf{u}\cdot\mathbf{v}=(\mathbf{u},\mathbf{v})=\mathbf{u}^\top\mathbf{v}=\langle\mathbf{u},\mathbf{v}\rangle$$

Dot Product. Calculation

Dot product in $\mathbb{R}^n$

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

If $\mathbf{u}, \mathbf{v}$ are column vectors, then

$$\mathbf{u}^\top \mathbf{v} = u_1 v_1 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i = \mathbf{u} \cdot \mathbf{v}$$

Question. Find the angle between $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$$

Hint

$$\| \mathbf{u} \| \equiv \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Norm (or a length) of a vector

A norm on any vector space is defined as follows:

Definition

We say $\| \mathbf{u} \|$ is a norm on a vector space V if $\forall \mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$,

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b) $\| \mathbf{u} \| \geq 0$
c) $\| \mathbf{u} \| = 0 \Leftrightarrow \mathbf{u} = 0$
d) $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$

Norm (or a length) of a vector

A norm on any vector space is defined as follows:

Definition

We say $\parallel \mathbf{u} \parallel$ is a norm on a vector space V if $\forall \mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$,

a) $\parallel \alpha \mathbf{u} \parallel = |\alpha| \parallel \mathbf{u} \parallel$

b) $\parallel \mathbf{u} \parallel \geq 0$

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d) $\parallel \mathbf{u} + \mathbf{v} \parallel \leq \parallel \mathbf{u} \parallel + \parallel \mathbf{v} \parallel$

Cauchy-Schwarz inequality

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

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For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Proof

Consider the expression $\|\mathbf{x} - \lambda\mathbf{y}\|^2$. We must have

$$\begin{aligned} \|\mathbf{x} - \lambda\mathbf{y}\|^2 &\geq 0 \\ (\mathbf{x} - \lambda\mathbf{y}) \cdot (\mathbf{x} - \lambda\mathbf{y}) &\geq 0 \\ \lambda^2\|\mathbf{y}\|^2 - \lambda(2\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2 &\geq 0. \end{aligned}$$

Cauchy-Schwarz inequality

Consider the expression $\|\mathbf{x} - \lambda \mathbf{y}\|^2$. We must have $\lambda^2\|\mathbf{y}\|^2 - \lambda(2\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{x}\|^2 \geq 0$.

Viewing this as a quadratic in λ, we see that the quadratic is non-negative. Thus, it cannot have 2 different real roots. The discriminant $\Delta = b^2 - 4ac \leq 0$. So

$$
\begin{aligned}
4(\mathbf{x}\cdot\mathbf{y})^2 &\leq 4\|\mathbf{y}\|^2\|\mathbf{x}\|^2 \\
(\mathbf{x}\cdot\mathbf{y})^2 &\leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \\
|\mathbf{x}\cdot\mathbf{y}| &\leq \|\mathbf{x}\|\|\mathbf{y}\|.
\end{aligned}
$$

Problem solving

Maximize $x + 2y + 3z$
given the constrain $x^2 + y^2 + z^2 = 1$

Write some code

Here we open Google Colab...

... to check Cauchy-Schwarz inequality

https://colab.research.google.com/drive/1QKCs22fjRaLks5oSA2QjssqXYgBHMn1A?usp=sharing

Triangle inequality

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$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

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$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) \\
&= \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

Orthogonality

Definition

Let V be vector space with a dot product.
Vectors $\mathbf{u}, \mathbf{v} \in V$ are said to be **orthogonal** if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Examples

Here we open the Geogebra :)

Homework

Show that the difference between a vector $\mathbf{a}$ and its orthogonal projection on a vector b is orthogonal to the vector b.

If

$$\mathbf{p} = \mathbf{a} - \mathbf{a_b}$$

then

$$\mathbf{p} \cdot \mathbf{b} = 0$$

Break

5 min. break

Part 2. Vector Cross Product

Vector Cross product

Apart from the scalar product, we can also define the *vector product*. However, this is defined only for $\mathbb{R}^3$ space, but not spaces in general.

Vector cross product

Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Define the *vector cross product*

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{n}} \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta,$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Vector cross product

Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Define the *vector cross product*

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{n}} ||\mathbf{a}|| \, ||\mathbf{b}|| \sin \theta,$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Since there are two (opposite) unit vectors that are perpendicular to both of them, we pick $\hat{\mathbf{n}}$ to be the one that is perpendicular to $\mathbf{a}, \mathbf{b}$ in a *right-handed* sense.

Vector cross product

Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Define the *vector cross product*

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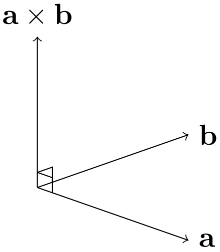
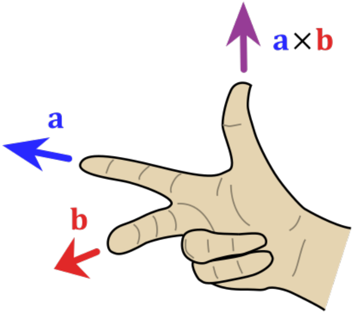
Vector cross product defined only for 3-dimensional vectors!!!

Geometric view



Right hand rule



Properties of cross-product

The vector product satisfies the following properties:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
2. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
3. $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \lambda \mathbf{b}$ for some $\lambda \in$ (or $\mathbf{b} = \mathbf{0}$).
4. $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda (\mathbf{a} \times \mathbf{b})$.
5. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

Coordinate form of cross product

There is a way of calculating vector products:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \times (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}} \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}\end{aligned}$$

There are more convenient ways to calculate vector cross products.

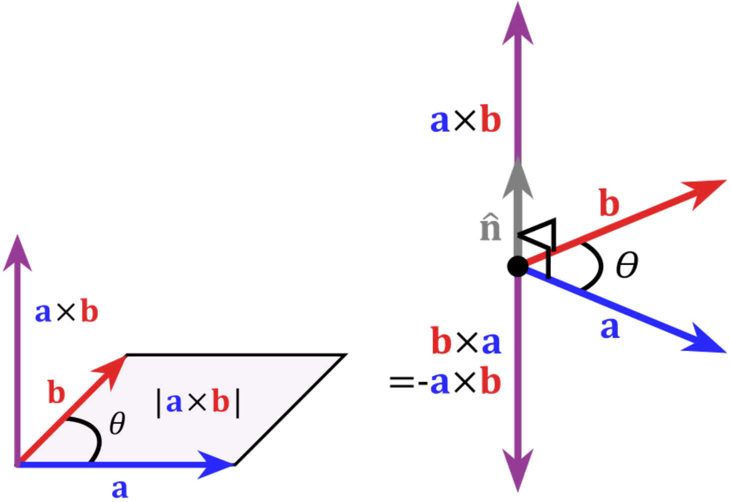
Examples

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2c - 3b \\ 3a - 1c \\ 1b - 2a \end{bmatrix}$$

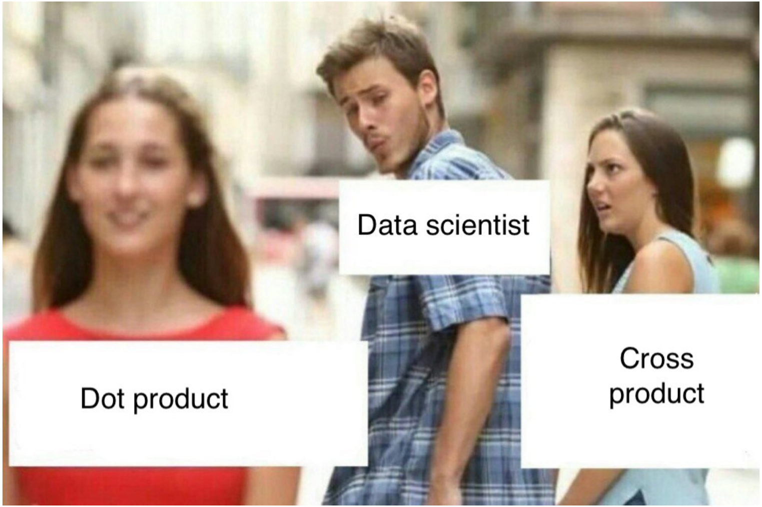
Simplify expression

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$$

Interpretation



Why would you assign such labels?



Part 3. Matrices

Definition and examples

Matrix A is a rectangular table of numbers with m rows and n columns.

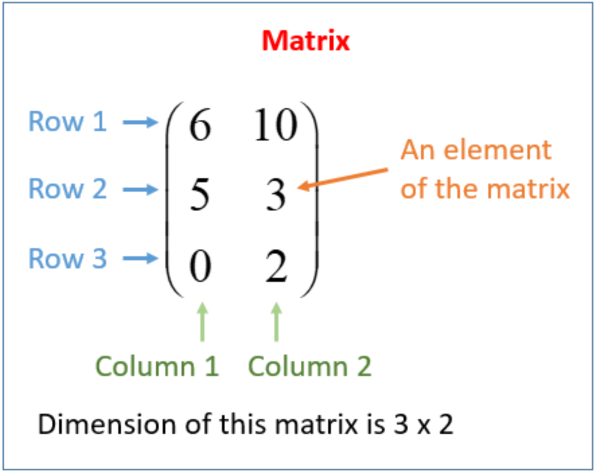
Example of a 3×3 matrix

A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Example of a 2×3 matrix

A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Sizes


Matrix
Row 1
Row 2
Row 3
6 10
5 3
0 2
An element of the matrix
Column 1
Column 2
Dimension of this matrix is 3 x 2

Quick check. What are the sizes?

$$A = \begin{bmatrix} 3 & 4 & 9 \\ 12 & 11 & 35 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -6 & 13 \\ 32 & -7 & -23 \\ -9 & 9 & 15 \\ 8 & 25 & 7 \end{bmatrix}$$

Different kinds of matrices

1. Square and rectangular matrix
2. Symmetric matrix
3. Triangle matrix
4. Diagonal matrix
5. Identity matrix
6. Zero matrix

Examples

(1) Square matrix : (#rows = #columns)

Main diagonal

$$\begin{bmatrix} 3 & -1 & -3 \\ 2 & 4 & 0 \\ -1 & 5 & 6 \end{bmatrix}$$

(8) Row matrix (n – vector):

$\begin{bmatrix} 1 & 7 & -3 \end{bmatrix}$ is a 3 - vector

(9) Column matrix (n – vector)

$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is a 3 - vector

(2) Upper triangular matrix :

$$\begin{bmatrix} 3 & -1 & -3 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

(3) Lower triangular matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & 7 & -6 \end{bmatrix}$$

(5) Diagonal matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

(6) Identity matrix :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) Zero matrix :

$$O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Source: `https://medium.com/@nithishraghav/linear-algebra-for-aspiring-data-scientists-part-i-37a9b63c031f`

Operations. Transpose a matrix

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\forall A, A^{\top\top} = A$$

What does it mean if $A^\top = A$

Operations. Addition, multiplication by a scalar

Element-wise addition:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 1+a & 4+d \\ 2+b & 5+e \\ 3+c & 6+f \end{bmatrix}$$

Properties. A, B, C are matrices

1. $A + B = B + A$ (commutative)
2. $A + (B + C) = (A + B) + C$ (associative)
3. $B = \lambda A, \lambda \in \mathbb{R}$ (element-wise multiplication)

$$B = \lambda A, \quad \forall 1 \le i \le m; 1 \le j \le n : b_{ij} = \lambda a_{ij}$$

Addition defined only for matrices of the same size!

Examples

Trace of a matrix

A is $(m \times m)$ matrix

$$Tr(A) = \sum_{i=1}^{m} a_{ii}$$

Trace can be applied to square matrices only!

Examples

Homework. Linearity of the trace

Write a program to demonstrate Linearity of the trace

$$Tr(A + B) = Tr(A) + Tr(B)?$$

$$\lambda \in \mathbb{R}, \quad Tr(\lambda A) = \lambda Tr(A)?$$

End of Lecture #2

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila