Discrete Mathematics and Logic
Graph Theory
Lecture 4

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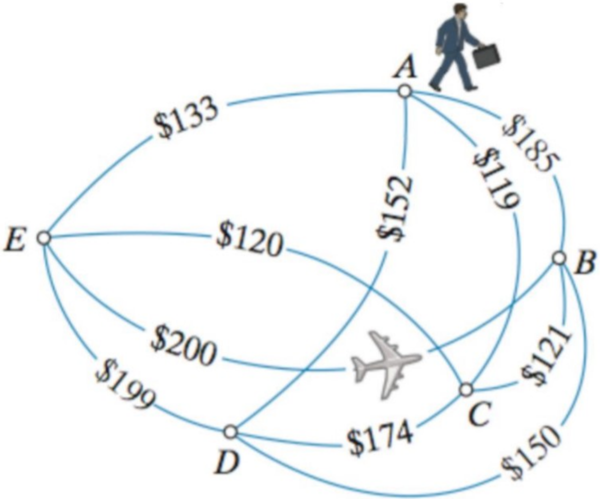
What did we know in the last week?

0. Are you ready to travel?



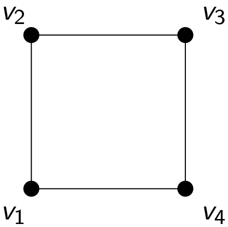
What did we know in the last week?

1. Euler paths and cycles
2. Eulerians
3. The sufficient and necessary condition
4. The Fleury's algorithm

Traversability

Example

$v_1 v_2 v_3 v_1$ – a non-walk
$v_1 v_2 v_3 v_2$ – some walk
$v_1 v_2 v_3 v_2 v_1$ – a closed walk
$v_4 v_3 v_4$ – a closed walk
$v_4 v_3 v_2$ – a path
$v_1 v_2 v_3 v_4$ – a path
$v_1 v_2 v_3 v_4 v_1$ – a cycle



Hamilton paths and cycles

Definitions

A path P is called a **Hamilton path** if P visits every vertex once.

If the P is a cycle, then it is called a **Hamilton cycle**.

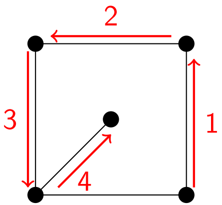
Remark (the terminology problem)

A Hamilton cycle = A hamilton cycle = A Hamilton circuits = ...

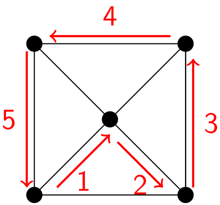
Hamilton paths and cycles

Definition

A path (cycle) P is called a **Hamilton path** (cycle) if P visits every **vertex** once.



a Hamilton path!



a Hamilton cycle!

Hamilton paths and cycles

Relationship

an **Euler** path (cycle)	a **Hamilton** path (cycle)
It visits every edge once	It visits every vertex once

Hamilton paths and cycles

Definitions

A graph is **hamiltonian** if it has a Hamilton cycle.

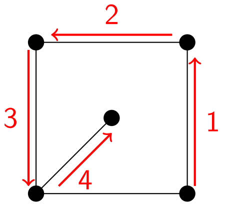
Definitions (only in russian)

A graph is **semi-hamiltonian** if it has a Hamilton a path.

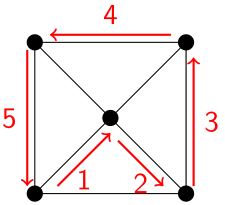
Hamilton paths and cycles

Definition

A path (cycle) P is called a **Hamilton path** (cycle) if P visits every **vertex** once.



a semi-Hamiltonian



a Hamiltonian!

Hamilton paths and cycles

Relationship

an **Eulerian**	a **Hamiltonian**
It contains a cycle which visits every edge once	It contains a cycle which visits every vertex once
sufficient and necessary conditions	
iff every vertex has even degree	Does not exist!

Dirac's theorem

Theorem (Dirac, 1952)

Let G be a simple graph with $n \geq 3$ vertices.
Suppose that for any vertex v

$$deg(v) \geq n/2.$$

Then G is a **hamiltonian**.

Dirac's theorem

Theorem (Dirac, 1952)

$\forall v\, (deg(v) \geq n/2) \Rightarrow G$ is a **hamiltonian**.

Proof

By contradiction, assume that G is not a hamiltonian.

Let $G' \supseteq G$ be a maximal non-hamiltonian with the same vertices.

It means that $G' + vv'$ is a hamiltonian for any non-adjacent vertices v, v'.

Note that G' is not a complete (see Lab exercises).

Dirac's theorem

Proof

Choose some non-adjacent vertices v, v'.
Since $G' + vv'$ is a hamiltonian, G' contains a Hamilton path from v to v':

$$v_1 \to \cdots \to v_{i-1} \to v_i \to \cdots \to v_n$$

where $v_1 = v, v_n = v'$.

Dirac's theorem

Proof

1) Suppose that there exists $i(1 < i < n)$ such that $vv_i \in G'$ and $v_{i-1}v' \in G'$.

Hence, $v_1 \dots v_{i-1}v_nv_{n-1} \dots v_iv_1$ is a Hamilton cycle.

$$\overbrace{v_1 \to \cdots \to v_{i-1} \to v_i}\! v_1 \to \cdots \to \underbrace{v_{i-1} \to v_i \to \cdots \to v_n}$$

This is a contradiction.

Dirac's theorem

Proof

2) Suppose that there does not exist i $(1 < i < n)$ such that

$$vv_i \in G' \text{ and } v_{i-1}v' \in G'. \tag{1}$$

Let $D_1(v) = \{v_i \mid vv_i \in E'\}$ and $D_2(v') = \{v_i \mid v_{i-1}v' \in E'\}$.

$|D_1(v) \cup D_2(v')| < n$ (since $v_1 v_n \notin E'$)
and $D_1(v) \cap D_2(v') = \emptyset$ (by (1) above).

Therefore, $d(v) + d(v') < n$.

This is a contradiction with $d(u) \geq n/2$ for any u.

Ore's theorem

Theorem (Ore, 1960)

Let G be a simple graph with $n \geq 3$ vertices.
Suppose that for any non-adjacent vertices v, v'

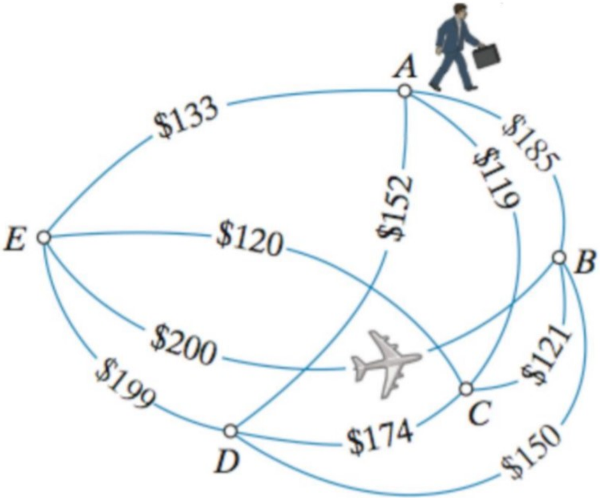
$$deg(v) + deg(v') \geq n.$$

Then G is a **hamiltonian**.

Proof

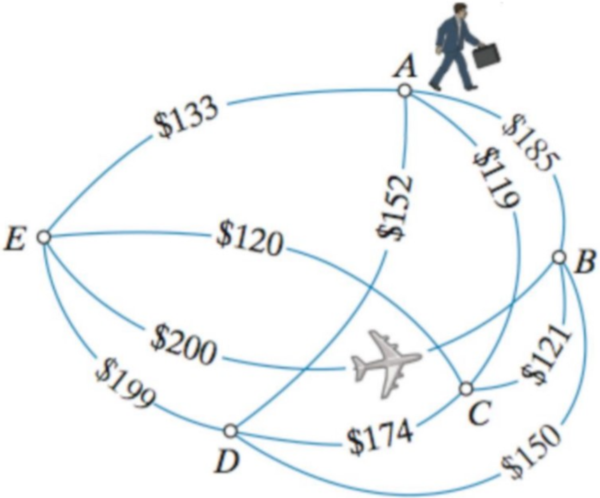
The same proof (Bondy and Chvatal, 1974).

A traveling salesman



A traveling salesman wants to tour five cities A, B, C, D, E and starts at point A. He needs to choose the best Hamilton path!

A traveling salesman



There is no algorithm in polynomial time!

What we knew today?

1. Hamilton paths and cycles
2. Hamiltonians
3. Ore's Theorem
4. Dirac's Theorem
5. The traveling salesman problem

Thank you for your attention!