Computer Architecture. Week 3

Number Systems

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Content of the class

- Number System
- The MIPS Number System
- Numbers and Numerals
- Numeric Systems
- Base Conversion
- Finite Precision Numbers
- Positive and Negative Numbers
- Binary and Floating Point Arithmetic Operations
- Standard IEEE 754
- Fractional Representation

Number System

- A set of values used to represent different quantities is known as Number System

Number System for Human

- As humans, we generally count and perform arithmetic using decimal having 10 digits from 0 to 9.
- Historically, it seems that the main reason we usedecimal (i.e., base 10) is that humans have ten fingers
- Numbers may be represented in any base.
  - For example, 123 base 10 = 1111011 base 2.

Number System for Computers

- Numbers are kept in computer hardware as a series of high and low electronic signals
- Computers perform all of their operations using the binary (base 2) number system.
- All program code and data are stored and manipulated in binary form.
- Calculations are performed using binary arithmetic.
- Each digit in a binary number is known as a bit (for binary digit) and can have only one of two values, 0 or 1.

MIPS Architecture: Historical Aspects

- MIPS - Microprocessor without Interlocked Pipeline Stages
- MIPS was developed at Stanford University by John Hennessey and the team
- MIPS Computer Systems was founded 1984
- MIPS Computer Systems spun out as MIPS Technologies in 1998
- MIPS Technologies (as part of Wave Computing company) went bankrupt in 2020
- RISC-V architecture is the replacement of MIPS (and strongly inspired by MIPS design)

CISC vs. RISC Architectures

- RISC (Reduced Instruction Set Computer) principles (MIPS, ARM, PowerPC, SPARC, HP-PA, Alpha):
  - Simplicity favors regularity
  - Make the common case fast
  - Smaller is faster
  - Good design demands, good compromises
- CISC (Complex Instruction Set Computer) - an alternative for RISC (Intel's x86, Motorola 68000, DEC VAX)
- VLIW (Very Long Instruction Word) architecture - something between RISC and CISC, made of multiple RISC-like execution units

The MIPS Number System

- The drawing below shows the numbering of bits within a MIPS word and the placement of the number

31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

(32 bits wide)

Example:

$$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_2 = 0_{10}$$
$$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0001_2 = 1_{10}$$
$$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_2 = 2_{10}$$

......

$$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_2 = 4\ 294\ 967\ 293_{10}$$
$$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1110_2 = 4\ 294\ 967\ 294_{10}$$
$$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111_2 = 4\ 294\ 967\ 295_{10}$$

Numbers and Numerals

- **Number**: abstract entity
- **Numeral**: string of symbols that represent a number in a given system
- The same number can be represented by different numerals in different numeric systems
- Example I
  - 234 in decimal system
  - CCXXXIV using roman system
- Example II
  $(10)_{10} = 10$
  $(10)_2 = (2)_{10} = 2$

Numeric Systems – 1/5

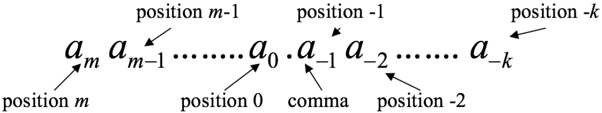
- To define a Numeric System we need:
  - A set of symbols that we will call digits (E.g. 1,2,3, A,C, ...)
  - Some rules to build up numbers
- We can define Positional Numeric System or Non Positional Numeric Systems
- Non Positional Numeric System: the value of digits in the number is position independent
- Example: (Roman Numeric System)
  the symbol V means 5 always, but IV $\neq$ VI . . . .

Numeric Systems – 2/5

- Positional Numeric System: digit value depends on its position within the number (weight)
  - Each digit represents the coefficient of a power of the base
  - Exponent is given by the position of the digit within the number

$$\text{base} = b$$

$$\text{used symbols} = 0 \le a_i \le b - 1$$



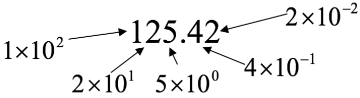
$$N = \sum_{i=-k}^{m} a_i b^i$$

Numeric Systems – 3/5

- Example (The Decimal System)

$$\text{base} = 10$$

$$\text{used symbols} = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$



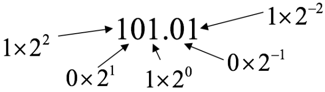
$$125.42 = 1 \times 10^2 + 2 \times 10^1 + 5 \times 10^0 + 4 \times 10^{-1} + 2 \times 10^{-2}$$

Numeric Systems – 4/5

- Example (The Binary System)

$$\text{base} = 2$$

$$\text{used symbols} = 0, 1$$



$$101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$$

Numeric Systems – 5/5

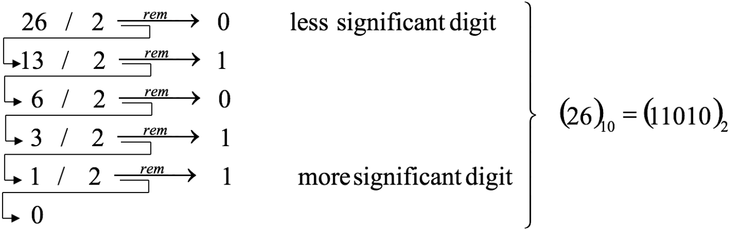
- Other Systems Used

- Octal System
  Base = 8
  Symbols used = 0,1,2,3,4,5,6,7

- Hexadecimal System:
  Base = 16
  Symbols used = 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

Base Conversion (decimal to binary)

Decimal number X is converted into binary format by a repeated division of X by 2; the remainder of integer division at each step is the digit of the binary number:



Note that less significant bit goes to the right (the same as for decimal representation)

Base Conversion (binary to decimal)

- Generalizing the point, in any number base, the value of ith digit d is d * Base^i
- For example, 11010_2 represents:

$$
\begin{aligned}
&((1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0))_{10} \\
&= ((1 \times 16) + (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1))_{10} \\
&= (16 + 8 + 0 + 2 + 0)_{10} \\
&= 26_{10}
\end{aligned}
$$

Finite Precision Numbers

- Definition: numbers with a finite number of digits.
- Some properties are lost:
  - Operators closure
  - Distributive and associative properties
  - Holes in the representation of real numbers

Finite Precision Numbers – Example 1

- Let's use integer numbers with 2 decimal digits
  - Interval represented: [0, 99]
  - Lost closure with respect to operator +
    76+30 = ???
    (106 is out of the interval)
  - Lost associativity: 25+(90-30) $\neq$ (25+90)-30

Finite Precision Numbers – Example 2

- Let's use consider rational numbers with two decimal digit after the point
  - Interval represented: [0, 0.99]
  - We cannot represent any additional number between 0.05 and 0.06
  - Again lost closure with respect to operator +: $0.90+0.30 = ???$ (1.20 is out of the interval)
  - Again lost associativity: $0.25+ (0.90\text{-}0.30) \neq (0.25+ 0.90)\text{-}0.30$

Represented Intervals

- Let us assume integer numbers
- By representing positive integers in binary notation, n digits (bits), cover the interval $[0, 2^n\text{-}1]$
- If the maximum number re-presentable using n bit is

  $$X = 2^n - 1$$

  then to represent number X, the necessary number of bits is

  $$n = \lceil log_2(X) \rceil + 1$$

- Example: To represent numbers in interval [0, 7], we need 3 bits:

0	000	4	100
1	001	5	101
2	010	6	110
3	011	7	111

Signed Number Representation

- In mathematics, positive numbers (including zero) are represented as unsigned numbers. That is we do not put the +ve sign in front of them to show that they are positive numbers.
- However, when dealing with negative numbers we do use a -ve sign in front of the number to show that the number is negative.
- However, in digital circuits there is no provision made to put a plus or even a minus sign to a number, since digital systems operate with binary numbers that are represented in terms of 0's and 1's.

Sign and Magnitude Representation

- Sign and Magnitude is one of the method used to represent signed numbers in binary format
  - The first bit is used for the sign 0 mean + ; 1 mean –
  - n-1 bits are used for the magnitude
  - Represented interval: [-2^{n-1} +1, 2^{n-1} -1]

- Example: Using n=4 interval [-7, 7] is completely represented

  5 0101

  -5 1101

Issues with Sign and Magnitude

Example: Using n=4 interval [-7, 7] is completely represented

Pattern	Value Represented Sign Magnitude
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	-0
1001	-1
1010	-2
1011	-3
1100	-4
1101	-5
1110	-6
1111	-7

- Having two patterns to represent 0 is wasteful.
- The signed magnitude representation has the advantage that it is easy to read the value from the pattern.
- But does it have the binary arithmetic property?
- For instance, what is the result of pattern(-1) + pattern(1)?

One's Complement Representation

- One's complement number representation is used for signed numbers in binary format
  - The leftmost bit defines the sign of the number (0 for +; 1 for -)
  - The integer is changed to binary (the sign is ignored).
  - 0s are added to the left of the number to make a total of N bits
  - If the sign is positive, no more action is needed. If the sign is negative, every bit is inverted
  - Represented interval: [$-2^{n-1}+1$, $2^{n-1}-1$]

Example: using n=4 interval [-7, 7] is completely represented

5   0101

-5   1010

Issues with One's Complement

Pattern	Value Represented	
	Sign Magnitude	1's complement
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-0	-7
1001	-1	-6
1010	-2	-5
1011	-3	-4
1100	-4	-3
1101	-5	-2
1110	-6	-1
1111	-7	-0

- A negative number is represented by flipping all the bits of a positive number.
- We still have 2 patterns for 0.
- It is still easy to read a value from a given pattern.
- How about the arithmetic property?
- Suggestion: try the following
  -1 + 1 = ??
  -0 + 1 = ??
  0 + 1 = ??

Two's Complement Representation

- It is most common and widely used representation today.
  - The leftmost bit defines the sign of the number (0 for +; 1 for -)
  - The integer is changed to binary, (the sign is ignored).
  - 0s are added to the left of the number to make a total of N bits
  - If the sign is positive, no more action is needed. If the sign is negative, every bit is complemented and 1 is added.
  - Represented interval: [-2^{n-1} , 2^{n-1} -1]

Example
using n=4 interval [-8, 7] is completely represented
5 0101
-5 1011

Two's Complement Representation

By using 3 bits, we are able to represent these numbers:

$$
\begin{aligned}
000 &= +0 \\
001 &= +1 \\
010 &= +2 \\
011 &= +3 \\
100 &= -4 \\
101 &= -3 \\
110 &= -2 \\
111 &= -1
\end{aligned}
$$

Two's Complement Detailed Example

- Procedure to convert negative number "-28" into two's complement 8-bit binary representation:
  - Omit sign "-", and represent number "28" in an 8-bit binary format: 00011100
  - Invert the bits ("0" becomes "1", and vice versa): 11100011
  - Add 1 to the inverted representation above: 11100100

Thus, $(-28)_{10}$ is $(11100100)_2$ in 8-bit two's complement notation

Signed Number Representation (Summary)

Pattern	Value Represented		
	Sign Magnitude	1's complement	2's complement
0000	0	0	0
0001	1	1	1
0010	2	2	2
0011	3	3	3
0100	4	4	4
0101	5	5	5
0110	6	6	6
0111	7	7	7
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

Note: 0 has exactly one representation.
It holds the arithmetic property, but the reading of a negative pattern is not trivial.

Signed Number Representation (Summary)

Pattern	Value Represented		
	Sign Magnitude	1's complement	2's complement
0000	0	0	0
0001	1	1	1
0010	2	2	2
0011	3	3	3
0100	4	4	4
0101	5	5	5
0110	6	6	6
0111	7	7	7
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

Note: 0 has exactly one representation.
It holds the arithmetic property, but the reading of a negative pattern is not trivial.

Excess Notation

- Excess 8 notation indicates that the value for zero is the bit pattern for 8, that is 1000
- The bit patterns are 4 bits long
- Positive numbers are above it in order and negative numbers are below it in order.

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An Excess Eight Conversion Table

Bit pattern	Value represented
1111	7
1110	6
1101	5
1100	4
1011	3
1010	2
1001	1
1000	0
0111	−1
0110	−2
0101	−3
0100	−4
0011	−5
0010	−6
0001	−7
0000	−8

Excess Notation (Continue)

- That is the zero point for Excess 128 notation is 128; the zero point for excess 64 notation is 64; and so forth.
- For example, let's say we want to determine the pattern for 15 in Excess 128 notation.
- The decimal number would be 128 + 15, or 143. Therefore, the bit pattern would be 10001111

Excess Notation (Continue)

- **Example:** Represent –25 in Excess 127 using an 8-bit allocation.
- **Solution:**
  - $127 + (-25) = 102$
  - 102 in binary 1100110
  - Add 0's to the left to make it 8 bit 01100110

Note: again we do not "lose" any more a number: 0 has exactly one representation

stored number	=	actual number	+	excess notation

actual number	=	stored number	–	excess notation

Binary Representation of Signed Numbers: Summary

- Multiple binary notations for signed numbers are available:
  - Sign-and-magnitude (the leftmost bit represents sign)
  - One's complement
  - Two's complement
  - Excess notation , such as base-(-2)
- Each system has advantages (e.g. the simplicity of resulted logic circuits) and disadvantages (e.g. several binary representations for the same decimal number like "0")
- There is no notation strongly dominating all others; all notations are currently in use

Binary Arithmetic: Sum

- Sum in binary notation is performed bit by bit carrying the rest to next digit

$$\begin{array}{ll} 0+0=0 & \\ 0+1=1 & \\ 1+0=1 & \\ 1+1=0 & \quad \text{carry } 1 \end{array}$$

- Example

$$\begin{array}{cc} 3+ & 0011+ \\ 2 & 0010 \\ \hline 5 & 0101 \end{array}$$

Binary Arithmetic in Two's Complement Notation

- Summation and subtraction are managed similar to non-signed binary numbers, by throwing away carry value:

$$\begin{array}{r} 65+ \\ \text{-}2 \\ \hline 63 \end{array} \qquad (1) \begin{array}{r} 0100\ 0001+ \\ 1111\ 1110 \\ \hline 0011\ 1111 \end{array}$$

- If two terms have different sign the result remains correct!
- If two terms have the same sign but the result has a different one, then we have an overflow/underflow error:
- Example of the sum 100 + 100 using 8-bit two's complement representation:

$$\begin{array}{r} 0110\ 0100+ \\ 0110\ 0100 \\ \hline 1100\ 1000 \end{array} \qquad \text{Overflow error!}$$

What will be the output of the following program?

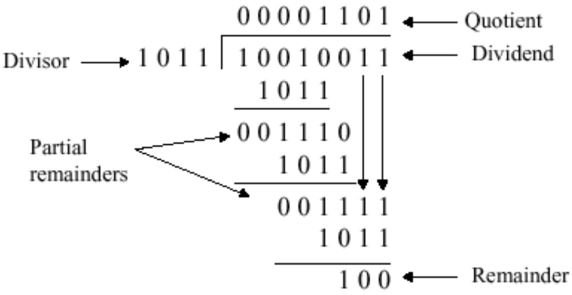
```
#include <stdio.h>
#include <limits.h>

int main()
{
    int count;
    count =INT_MAX;     //2147483647
    count = count +1;
    printf("%d", count);
}
```

Output: -2147483648

Binary Arithmetic: Multiplication

$$
\begin{array}{cccccccccll}
 & & & & 1 & 1 & 0 & 1 & & (13)_{10} & \text{Multiplicand M} \\
 & & & \times & 1 & 0 & 1 & 1 & & (11)_{10} & \text{Multiplier Q} \\
\cline{4-8}
 & & & & 1 & 1 & 0 & 1 & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] & & \\
 & & & 1 & 1 & 0 & 1 & & & \text{Partial products} & \\
 & & 0 & 0 & 0 & 0 & & & & & \\
 & 1 & 1 & 0 & 1 & & & & & & \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & & (143)_{10} & \text{Product P}
\end{array}
$$

Binary Arithmetic: Division



Floating-Point Number Representation

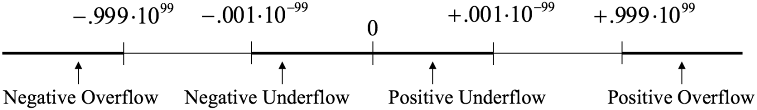
- Representation of floating point number is not unique.
- For example: The number 55.66 can be represented as
  5.566×10^1
  0.5566×10^2
  0.05566×10^3 and so on..
- It is important to note that floating-point numbers suffer from loss of precision when represented with a fixed number of bits (e.g., 32-bit or 64-bit).
- Reason: This is because there are infinite number of real numbers (even within a small range of says 0.0 to 0.1).

Floating-Point Number Representation

- On the other hand, a $n - bit$ binary pattern can represent a finite 2^n distinct numbers.
- Hence, not all the real numbers can be represented. The nearest approximation will be used instead, resulted in loss of accuracy.
- Modern computers adopt IEEE 754 standard for representing floating-point numbers.
- IEEE Standard 754 floating point is the most common representation today for real numbers on computers, including Intel-based PC's, Macs, and most Unix platforms.
- There are two representation schemes:
  - 32-bit single-precision
  - 64-bit double-precision.

Floating Point Notation

- Example: Using Base 10
- Using numerals with 5 digits of the kind ± .XXX ±EE
- Mantissa = ± .XXX three signed digits $0.001 \leq m < 1$
- Exponent = ±EE two signed digits $-99 \leq e \leq 99$
- Represented numbers are:



Represented interval is

$$-.999 \cdot 10^{99} \leq N \leq -.001 \cdot 10^{-99}; \quad .001 \cdot 10^{-99} \leq N \leq .999 \cdot 10^{99}$$

Standard IEEE 754 (1985)

- Standard definition (i.e., architecture independent)
- Single precision (uses 32 bits to represent sign, exponent and mantissa

+/- (1 bit)	Exponent (8 bits)	Mantissa (23 bits)

- Double precision (uses 64 bits)

+/- (1 bit)	Exponent (11 bits)	Mantissa (52 bits)

- Some configurations of the exponent are reserved (i.e. not standardized)

Unum (Number Format)

- The unum (universal number) format is a format similar to floating point, proposed by John Gustafson.
- It is an alternative to the now ubiquitous IEEE 754 format.
- Unum implementations have been explored in Julia.
  - Julia is a high-level, high-performance dynamic programming language for numerical computing.

Ternary Number System

- The ternary (or trinary) numeral system also called base-3
- Analogous to a bit (**bi**nary digi**t**), a trit is a ternary digit (**tr**inary dig**it**, "trinary" is the synonym for "ternary")
- Variations of ternary systems:
  - Standard (unbalanced) system: uses values 0, 1, 2
  - Balanced system: uses values -1, 0, 1
  - Unknown state logic: False, ?, True (analogous to Fuzzi Logic)
- One trit is equivalent to $log_2 3$ (about 1.58496) bits of information
- An advantage is lower number of trits to represent large values
- Ternary systems attract increasing interest in research; see "Is There Any Advantage of Ternary Logic as Compared with Binary?" by Cassee and Strutt, IEEE Transactions on Computers

Ternary Number System

- The first modern electronic ternary computer **Setun** was built in 1958 in the Soviet Union at the Moscow State University by Nikolay Brusentsov.
- Why not famous as binary system?
  - It is much harder to build components that use more than two states.
  - If you use more than two states you need to be compatible to binary, because the rest of the world uses it.
- Still, increasing memory requirements of modern applications return research interest to ternary systems

Saturation Arithmetic – 1/2

- It is a version of arithmetic in which all operations such as addition and multiplication are limited to a fixed range between a minimum and maximum value.
- If the result of an operation is greater than the maximum, it is clamped to the maximum.
- If it is below the minimum, it is clamped to the minimum
  - For example, if the valid range of values is from -100 to 100

$$\begin{aligned} 60 + 30 &= 90 \\ 60 + 43 &= 100 \\ (60 + 43) - (75 + 75) &= 0 \\ 99 \times 99 &= 100 \\ 30 \times (5 - 1) &= 100 \end{aligned}$$

Saturation Arithmetic – 2/2

- Saturation arithmetic enables efficient algorithms for many problems, particularly in digital signal processing.
- For example:
  - Adjusting the volume level of a sound signal can result in overflow, and saturation causes significantly less distortion to the sound than wrap-around.

Summary

- The MIPS Number System
- Numeric Systems
- Base Conversion
- Finite Precision Numbers
- Positive and Negative Numbers
- Binary and Floating Point Arithmetic Operations
- Standard IEEE 754 (Floating point)
- Ternary Number System
- Unum (Universal Number)
- Saturation Arithmetic

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