Theoretical Computer Science

Tutorial Week 7

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Recap - Operations

Operations on DPDA

- Union
- Intersection
- Difference
- Complement

Operations on NPDA

- Union
- Intersection
- Difference
- Complement

Operations on FSA

Suppose L_1 and L_2 are both regular languages. Then

- $L_1 \cup L_2$ is also regular,
- $L_1 \cap L_2$ is also regular,
- $L_1 \setminus L_2$ is also regular,
- L_1^c is also regular.

Operations on FSA

Definition

We say that the class of languages $\mathcal{C}$ is **closed under an operation** O, if, for any languages $L_1, \ldots, L_n \in \mathcal{C}$, we have $O(L_1, \ldots, L_n) \in \mathcal{C}$.

Examples

$C(L) = L^c$
$\bigcup(L_1, L_2) = L_1 \cup L_2$

Operations on FSA

Let **FSA** be the class of regular (recognized by a FSA) languages. If $L_1, L_2 \in$ **FSA**, then

- $L_1 \cup L_2 \in$ **FSA**,
- $L_1 \cap L_2 \in$ **FSA**,
- $L_1 \setminus L_2 \in$ **FSA**,
- $L_1^c \in$ **FSA**.

Fact

FSA is closed under $\cup, \cap, \setminus, {}^c$

Operations on FSA

Definition

- **FSA** is the class of recognized by a FSA languages (regular),
- **DPDA** is the class of recognized by a DPDA languages,
- **NPDA** is the class of recognized by a NPDA languages (context-free).

Operations on FSA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA				
NPDA				

yes means the class of languages is closed under an operation
no means the class of languages is not closed under an operation

Recap - Operations

Operations on DPDA

- **Union**
- Intersection
- Difference
- Complement

Operations on NPDA

- Union
- Intersection
- Difference
- Complement

Operations on DPDA

Example

Suppose L_1 and L_2 are the following languages over the alphabet $\Sigma = \{a, b\}$:

$$L_1 = \{a^n b^n | n \geq 1\}$$

$$L_2 = \{a^n b^{2n} | n \geq 1\}$$

$$L_1 \cup L_2 = \{a^n b^n | n \geq 1\} \cup \{a^n b^{2n} | n \geq 1\}$$

This language cannot be recognized by any DPDA,

Operations on DPDA

Example

Suppose L_1 and L_2 are the following languages over the alphabet $\Sigma = \{a, b\}$:

$$L_1 = \{a^n b^n | n \geq 1\}$$

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$$L_1 \cup L_2 = \{a^n b^n | n \geq 1\} \cup \{a^n b^{2n} | n \geq 1\}$$

This language cannot be recognized by any DPDA,

thus **DPDA** is not closed under union.

Operations on DPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no			
NPDA				

yes means the class of languages is closed under an operation
no means the class of languages is not closed under an operation

Recap - Operations

Operations on DPDA

- Union
- **Intersection**
- Difference
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Operations on NPDA

- Union
- Intersection
- Difference
- Complement

Operations on DPDA

Example

Suppose L_1 and L_2 are languages over the alphabet $\Sigma = \{a, b, c\}$:

$$L_1 = \{a^n b^n c^m | n, m \geq 0\}$$

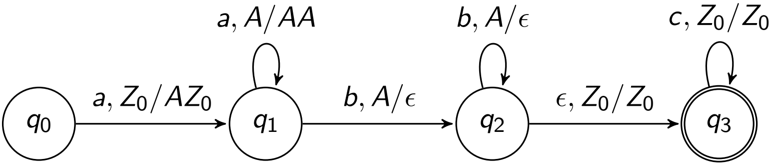
$$L_2 = \{a^m b^n c^n | n, m \geq 0\}$$

$$L_1 \cap L_2 = \{a^n b^n c^n | n \geq 0\}$$

PDA. Examples

Example 2

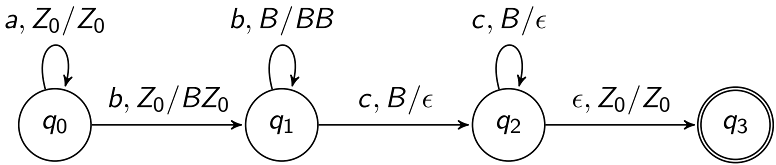
$L_1 = \{a^n b^n c^m \mid n, m \geq 0\} \in \textbf{DPDA}$.



PDA. Examples

Example 2

$L_2 = \{a^m b^n c^n \mid n, m \geq 0\} \in \textbf{DPDA}$.



Operations on DPDA

Example 2

Suppose L_1 and L_2 are languages over the alphabet $\Sigma = \{a, b, c\}$:

$$L_1 = \{a^n b^n c^m | n, m \geq 0\} \in \textbf{DPDA}$$

$$L_2 = \{a^m b^n c^n | n, m \geq 0\} \in \textbf{DPDA}$$

$$L_1 \cap L_2 = \{a^n b^n c^n | n \geq 0\} \in \textbf{DPDA}???$$

Pumping lemma for PDA

Bar-Hillel lemma

If $L \subseteq \Sigma^*$ is a recognized by a NPDA language then there exists $m \geq 1$ such that any $w \in L$ with $|w| \geq m$ can be represented as $w = x_1 x_2 x_3 x_4 x_5$ such that

- $|x_2 x_4| > 0$,
- $|x_2 x_3 x_4| \leq m$,
- $x_1 x_2^i x_3 x_4^i x_5 \in L$ for any $i \geq 1$.

Example

$L = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ is not recognized by any DPDA (moreover, is not recognized by any NPDA).

Operations on DPDA

Example 2

Suppose L_1 and L_2 are languages over the alphabet $\Sigma = \{a, b, c\}$:

$$L_1 = \{a^n b^n c^m | n, m \geq 0\} \in \textbf{DPDA}$$

$$L_2 = \{a^m b^n c^n | n, m \geq 0\} \in \textbf{DPDA}$$

$$L_1 \cap L_2 = \{a^n b^n c^n | n \geq 0\} \notin \textbf{DPDA}!$$

This language cannot be recognized by any DPDA,

Operations on DPDA

Example 2

Suppose L_1 and L_2 are languages over the alphabet $\Sigma = \{a, b, c\}$:

$$L_1 = \{a^n b^n c^m | n, m \geq 0\} \in \textbf{DPDA}$$

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This language cannot be recognized by any DPDA,

thus **DPDA** is not closed under union.

Operations on DPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no		
NPDA				

yes means the class of languages is closed under an operation
no means the class of languages is not closed under an operation

Recap - Operations

Operations on DPDA

- Union
- Intersection
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- **Complement**

Operations on NPDA

- Union
- Intersection
- Difference
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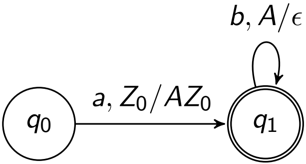
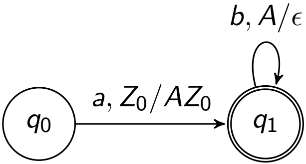
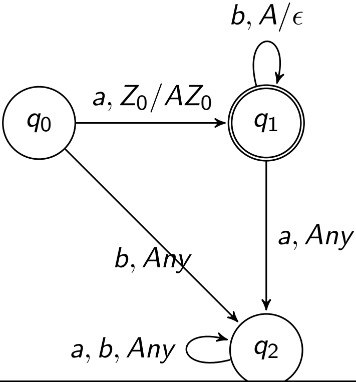
Operations on DPDA

The idea



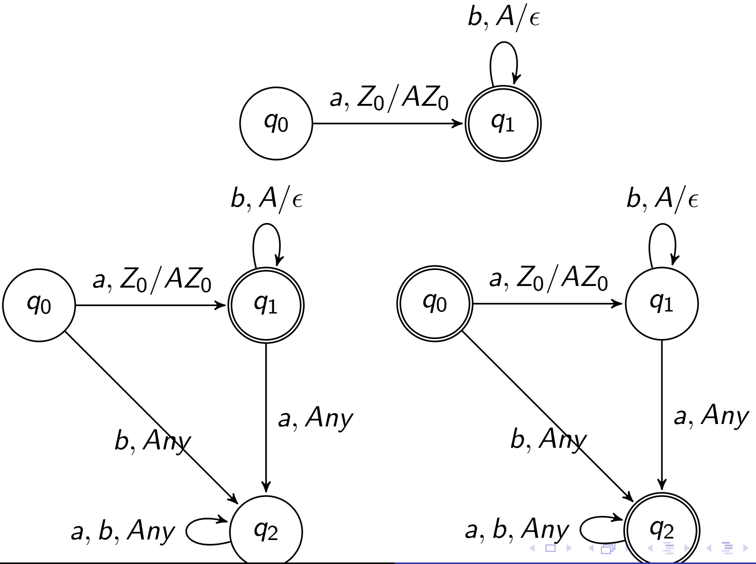
The idea





Operations on DPDA

The idea



Operations on DPDA

$$L \in \textbf{DPDA} \Longrightarrow L^c \in \textbf{DPDA}$$

Operations on DPDA

$$L \in \textbf{DPDA} \implies L^c \in \textbf{DPDA}$$

Fact

DPDA is closed under ^{c}

Operations on DPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no		yes
NPDA				

yes means the class of languages is closed under an operation
no means the class of languages is not closed under an operation

Recap - Operations

Operations on DPDA

- Union
- Intersection
- **Difference**
- Complement

Operations on NPDA

- Union
- Intersection
- Difference
- Complement

Operations on DPDA

Suppose that **DPDA** is closed under $\setminus$. Then, since

$$L_1 \cap L_2 = L_1 \setminus L_2^c,$$

DPDA must be closed under $\cap$, but it is not!!!

Contradiction!

Therefore, **DPDA** is not closed under $\setminus$

Operations on DPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no	no	yes
NPDA				

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Recap - Operations

Operations on DPDA

- Union
- Intersection
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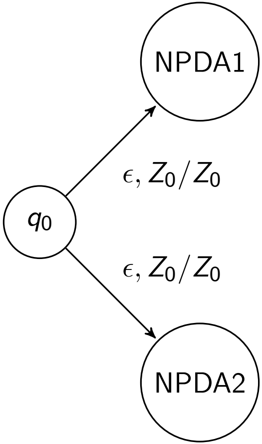
Operations on NPDA

- **Union**
- Intersection
- Difference
- Complement

Operations on NPDA

Let NPDA1 and NPDA2 be two NPDAs.

- If NPDA1 recognizes a language L_1
- and NPDA2 recognizes a language L_2,
- then the following NPDA recognizes the language $L_1 \cup L_2$.



Operations on NPDA

$$L_1, L_2 \in \textbf{NPDA} \Longrightarrow L_1 \cup L_2 \in \textbf{NPDA}$$

Operations on NPDA

$$L_1, L_2 \in \textbf{NPDA} \implies L_1 \cup L_2 \in \textbf{NPDA}$$

Fact

NPDA is closed under $\cup$

Operations on NPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no	no	yes
NPDA	yes			

yes means the class of languages is closed under an operation
no means the class of languages is not closed under an operation

Recap - Operations

Operations on DPDA

- Union
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Operations on NPDA

- Union
- **Intersection**
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Pumping lemma for PDA

Bar-Hillel lemma

If $L \subseteq \Sigma^*$ is a recognized by a NPDA language then there exists $m \geq 1$ such that any $w \in L$ with $|w| \geq m$ can be represented as $w = x_1 x_2 x_3 x_4 x_5$ such that

- $|x_2 x_4| > 0$,
- $|x_2 x_3 x_4| \leq m$,
- $x_1 x_2^i x_3 x_4^i x_5 \in L$ for any $i \geq 1$.

Example

$L = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ is not recognized by any NPDA.

Operations on NPDA

Example

Suppose L_1 and L_2 are languages over the alphabet $\Sigma = \{a, b, c\}$:

$$L_1 = \{a^n b^n c^m | n, m \geq 0\} \in \mathbf{DPDA} \Rightarrow L_1 \in \mathbf{NPDA}$$

$$L_2 = \{a^m b^n c^n | n, m \geq 0\} \in \mathbf{DPDA} \Rightarrow L_2 \in \mathbf{NPDA}$$

$$L_1 \cap L_2 = \{a^n b^n c^n | n \geq 0\} \notin \mathbf{NPDA}!$$

This language cannot be recognized by any NPDA,

Operations on NPDA

Example

Suppose L_1 and L_2 are languages over the alphabet $\Sigma = \{a, b, c\}$:

$$L_1 = \{a^n b^n c^m | n, m \geq 0\} \in \textbf{DPDA} \Rightarrow L_1 \in \textbf{NPDA}$$

$$L_2 = \{a^m b^n c^n | n, m \geq 0\} \in \textbf{DPDA} \Rightarrow L_2 \in \textbf{NPDA}$$

$$L_1 \cap L_2 = \{a^n b^n c^n | n \geq 0\} \notin \textbf{NPDA}!$$

This language cannot be recognized by any NPDA,

thus **NPDA** is not closed under union.

Operations on NPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no	no	yes
NPDA	yes	no		

yes means the class of languages is closed under an operation
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Recap - Operations

Operations on DPDA

- Union
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Operations on NPDA

- Union
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- Difference
- **Complement**

Operations on NPDA

Suppose that **NPDA** is closed under ^{c}. Then, since

$$L_1 \cap L_2 = (L_1^c \cup L_2^c)^c,$$

NPDA must be closed under $\cap$, but it is not!!!

Contradiction!

Therefore, **NPDA** is not closed under ^{c}

Operations on NPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no	no	yes
NPDA	yes	no		no

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Recap - Operations

Operations on DPDA

- Union
- Intersection
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Operations on NPDA

- Union
- Intersection
- **Difference**
- Complement

Operations on NPDA

Suppose that **NPDA** is closed under $\setminus$. Then, since

$$L^c = \Sigma^* \setminus L,$$

NPDA must be closed under ^{c}, but it is not!!!

Contradiction!

Therefore, **NPDA** is not closed under $\setminus$

Operations on NPDA

	$\cup$	$\cap$	$\setminus$	^{c}
FSA	yes	yes	yes	yes
DPDA	no	no	no	yes
NPDA	yes	no	no	no

yes means the class of languages is closed under an operation
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Thank you for your attention!