Analytical Geometry and Linear Algebra. Lecture 4.

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End of Lecture #3

Review. Lecture 3

- Part 1 (recap). Matrices. Transpose, Addition, Scalar multiplication
- Part 2. Matrix multiplication
- Part 3. Determinants. Scalar Triple Product

Quiz in class

Go to `http://b.socrative.com`

Type Room: **LINAL**

Answer questions.

Lecture 4. Outline

- Part 1. Change of basis and coordinates
- Part 2. Matrix rank
- Part 3. Matrix inverse

A couple words about linear maps...

Linear map

Definition

Given two vector spaces E and F, a linear map between E and F is a function $f : E \to F$ satisfying the following two conditions:

- $f(a+b) = f(a) + f(b), \forall a, b \in E$
- $f(\lambda a) = \lambda f(a), \forall a \in E, \lambda \in \mathbb{R}$

Example

Let E be the vector space $R[X]^4$ of polynomials of degree at most 4, let F be the vector space $R[X]^3$ of polynomials of degree at most 3, and let the linear map be the **derivative map** $d : E \to F$:

- $d(P+Q) = dP + dQ$
- $d(\lambda P) = \lambda dP, \lambda \in \mathbb{R}$

Example

We choose $(1, x, x^2, x^3, x^4)$ as a basis of E
and $(1, x, x^2, x^3)$ as a basis of F.

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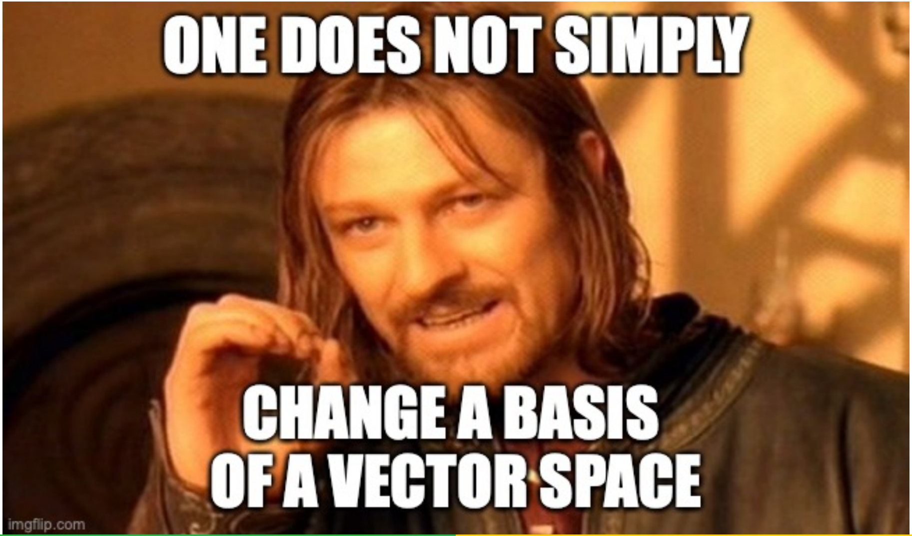
and $(1, x, x^2, x^3)$ as a basis of F.
Then the 4×5 matrix D associated with map d is obtained by expressing the derivative dx^i of each basis vector x^i for $i = 0, 1, 2, 3, 4$ over the basis $(1, x, x^2, x^3)$.

Example

Matrix D =

Change of basis and coordinates


ONE DOES NOT SIMPLY
CHANGE A BASIS
OF A VECTOR SPACE
imgflip.com

Here we are going to derive the formula.

Break, 5 min.

Matrix rank

Consider the following matrices ($a \neq b \neq 0$)

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} a & 0 & a \\ 0 & b & b \end{bmatrix}$$

$$D = \begin{bmatrix} a & 0 & a & -2a \\ 0 & b & b & -2b \end{bmatrix}\ E = \begin{bmatrix} a & 0 & a & -2a & 3a \\ 0 & b & b & -2b & 2b \end{bmatrix}$$

What can you say about columns-vectors inside each matrix?
Which matrices contain basis for $\mathbb{R}^2$?
Which matrices contain 'redundant' information about space spanned by column-vectors?

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For ANY $m \times n$ matrix the column rank equals to row rank.

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For ANY $m \times n$ matrix the column rank equals to row rank.

So, there is only one matrix rank. $rank(A) = rank(A^\top)$

Examples. Calculate rank of a matrix and its transpose

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} a & 0 & a \\ 0 & b & b \end{bmatrix}, rank(C) = rank(C^\top) =?$$

$$D = \begin{bmatrix} a & 0 & a & -2a \\ 0 & b & b & -2b \end{bmatrix},$$

More examples

For the following $m \times m$ matrices, which value of λ would give each matrix rank $m - 1$?

$$A = \begin{bmatrix} 1 & 3 \\ 1 & \lambda \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & \lambda \\ 1 & 0 & 3 \end{bmatrix}$$

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  so $rank(A) \leq \min(m, n)$
  If matrix has a maximum possible rank, it is called a **full rank matrix**
- $rank(A + B) \leq rank(A) + rank(B)$
- $rank(AB) \leq \min(rank(A), rank(B))$
- $rank(A) = rank(AA^\top) = rank(A^\top A) = rank(A^\top)$

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What about $rank(\lambda A)$?
$\lambda \in \mathbb{R}$

Break, 5 min.

Matrix inverse

Simple view

Matrix B is called inverse of a square matrix A if

$$AB = BA = I$$

Notation

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Example of inverse matrix

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$$\begin{bmatrix} 4*2+1*(-2) & 4*(-1)+1*(4) \\ 2*2+2*(-2) & 2*(-1)+2*4 \end{bmatrix} = \tfrac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = I$$

What if matrix A is **nonsquare**?

Left and Right inverse

Left inverse

Consider an $m \times n$ matrix A and $n \times m$ matrix B.
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Right inverse

Consider an $m \times n$ matrix A and $n \times m$ matrix C.
If $AC = I$, then we say C is the **right inverse** of A.

Let A be a square matrix. Show that its left and right inverses are the same.
Hint: use associative property of matrix multiplication.

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Are all matrices invertible?

Provide a simple counter-example of noninvertible 3×3 matrix.

Important property

If A and B are invertible and AB is invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Prove it, using pen and paper.

Hint: multiply $(B^{-1}A^{-1})$ by (AB).

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Step 0: Find determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = **ad-bc.** If $det(A) = 0$, then A^{-1} **does not exist**.

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Step 1: Swap **main** diagonal elements:

$$\begin{bmatrix} \mathbf{d} & b \\ c & \mathbf{a} \end{bmatrix},$$

Step 2: Multiply off-diagonal elements by -1:

$$\begin{bmatrix} d & \mathbf{-b} \\ \mathbf{-c} & a \end{bmatrix}$$

Step 3: Divide by $det(A)$. So, $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & \mathbf{-b} \\ \mathbf{-c} & a \end{bmatrix}$

Exercise

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; A^{-1} = $\frac{1}{det(A)}\begin{bmatrix} d & \textbf{-b} \\ \textbf{-c} & a \end{bmatrix}$

Check with pen and paper

$$A^{-1}A = \ldots$$

Example

Find the inverse and confirm that $AA^{-1} = A^{-1}A = I$

A= $\begin{bmatrix} 3 & 1 \\ 0 & b \end{bmatrix}$

Important case: Orthogonal matrix

$$A^{-1} = A^\top$$

For orthogonal matrix:

$$AA^\top = A^\top A = I$$

Example

Rotation matrix is an example of an orthogonal matrix.

Rotation matrix

$$R = \begin{bmatrix} \cos(\theta) & \text{-}\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}; \; R^\top = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \text{-}\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Assignment:

- Find R^{-1}
- Show that $det(R) = 1$ (this is true for any rotation matrix)
- If $det(A) = -1$ what kind of transformation you can think of?

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- $A^\top$ is invertible
- The rows of matrix A form a basis for $\mathbb{R}^n$

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- A is invertible
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- The columns of matrix A form a basis for $\mathbb{R}^n$
- The rank of the matrix A is n
- $A^\top$ is invertible
- The rows of matrix A form a basis for $\mathbb{R}^n$
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution ($\mathbf{x} = A^{-1}\mathbf{b}$)
- $A\mathbf{x} = \mathbf{0}$ has only a *trivial* solution ($\mathbf{x} = \mathbf{0}$, zero vector)

Break. 5 min.

End of Lecture #4

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila