

Summer Bootcamp 2021
Introduction to Computer Science
Lecture 3 (Part II)

Iterative and Recursive Algorithms

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Slides prepared with the help of Mikhail Martovitskiy

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Factorial Function – a widely used function in the theory of algorithms

$$n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$$

Factorial Function: a possible implementation

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Implementation is based on
for-loop iterations

Factorial Function: another possible implementation

$$n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$$

This implementation instead
explores the recursive property:

$$n! = n \times (n - 1)!$$

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Base case of the recursion

Factorial Function: possible Implementations

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Factorial Function: Iterative and Recursive Implementations

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Function repeats until some condition fails
(employs loop constructions)

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Function calls itself, until some condition is met,
called as a base case

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Factorial Function: Comparison of Iterative and Recursive Implementations

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Which approach computes faster?

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Which approach computes faster?

We need to perform runtime analysis!

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Which approach computes faster?

We need to perform runtime analysis!

Key idea: each operation takes time (or has its cost)

Runtime Estimation for Iterative Implementation

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Operation
Integer variable declaration

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Operation	<i>cost</i>
Integer variable declaration	c_1

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```


Operation	<i>cost</i>	<i>Invocation times</i>
Integer variable declaration	c_1	2

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Function repeats until some condition fails
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    unsigned long res = 1;  
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Operation	<i>cost</i>	<i>Invocation times</i>
Integer variable declaration	c_1	2
"if" condition in "for" loop	c_2	n

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Operation	<i>cost</i>	<i>Invocation times</i>
Integer variable declaration	c_1	2
"if" condition in "for" loop	c_2	n
"for" loop decrement	c_3	$n - 1$

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Integer variable declaration	c_1	2
"if" condition in "for" loop	c_2	n
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Multiplication	c_3	$n - 1$

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```

A total estimated execution time:

The number of times this function is invoked

$$t^{\text{iter}} = 1 \times (2 * c_1 + n * c_2 + (2 * n - 2) * c_3)$$
$$= 2 * c_1 + n * c_2 + (2 * n - 2) * c_3$$

Parameters of given equations are not expensive

Operation	<i>cost</i>	<i>Invocation times</i>
Integer variable declaration	c_1	2
“if” condition in “for” loop	c_2	n
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Multiplication	c_3	$n - 1$
Function calls number	1	

Recursive and Iterative implementations: Example of a Factorial Computation

Recursive

Function calls itself, until some condition is met, called as a base case

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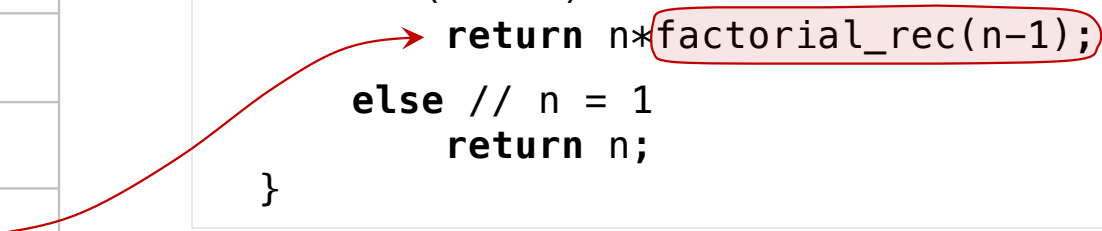
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Function call overhead	c_4	1

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Total time of recursive version will be:

$$t^{\text{recurs}} = n \times (c_2 + c_3 + c_4) = n * c_2 + n * c_3 + n * c_4$$

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Function calls number (due to recursion)	n	

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Cost of function call is greater than cost of other operation due to implementation details (the usage of execution stack)

$$c_4 \gg c_1, c_2, c_3$$

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$$t^{\text{recurs}} = n * c_2 + n * c_3 + n * c_4$$

$c_4 > c_1, c_2, c_3$ in average case

$$t^{\text{recurs}} > t^{\text{iter}}$$

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Works faster (for larger n)

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Slower, due to function call overheads
(overhead depend on a programming language and many other aspects)

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Consumes more memory

Any recursive implementation can be transformed into an iterative implementation
(possibly, with a use of an additional data structure, such as stack)

Types of Recursive Functions

```
unsigned long factorial_ntrec(unsigned n) {  
    if (n == 1) return n;  
    return n*factorial_ntrec(n-1);  
}
```

Recursive call is not the last function operation

Types of Recursive Functions

Non-Tail Recursion

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Tail Recursion

Recursive call is the last function operation

```
unsigned long factorial_trec(unsigned n,  
                             unsigned long a) {  
    if (n == 1) return a;  
    return factorial_trec(n-1, a*n);  
}  
  
unsigned long factorial(unsigned n) {  
    return factorial_trec(n, 1);  
}
```

Faster, due to an optimized usage of a function call stack

Comparison of Iterative and Recursive Implementation

Properties	Function implementations	
	Iterative	Recursive
Run-time complexity in an average case	Proportional to $n * \log(n)$	
Speed	Faster	Slower
Memory	Lower	Higher
Complexity	High	Low

Recursion depth problem: The number of recursive calls can be limited by the Operating System