Analytical Geometry and Linear Algebra. Lecture 1.

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Outline

- Part 1. About the course
- Part 2. Introduction. Vector spaces. Linear independence. Basis
- Part 3. Dot product

Main questions for today's lecture

- What is this course about?

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- How to use this course in your projects?

Main questions for today's lecture

What is this course about?

Topics of the course

1. Vector spaces, matrices and transformations in 2D and 3D
2. Lines and planes
3. Conics or quadric curves
4. Quadratic surfaces
5. Polar and spherical coordinates

Goals of this course

What you will learn in this course?

1. to use vectors and matrices to solve applied problems
2. to change basis in a vector space
3. to calculate determinants
4. to recognise different transformations, such as rotation, reflection, shear, etc.
5. to work with lines and planes in 2D and 3D
6. to operate with quadric curves, such as ellipse, hyperbola and parabola
7. many more + some examples in Python :)

Main questions for today's lecture

How to get a high grade in this course?

Grading in the course

1. Labs 5%
2. Test 1 15%
3. Midterm 35%
4. Test 2 15%
5. Final Exam 30%

In total, 100 %

How to get the highest grade?

1. Attend classes (either online or offline)
    1. Labs
    2. Tutorials
    3. Lectures
2. Solve assignments (also at home) on your own and in groups
3. Read books (check the list in moodle)
4. Come to office hours (either online or offline)

Repeat :)

What is the exact process you can follow?

1. **Friday**
   1. attend lecture
   2. attend tutorial
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4. **Tuesday - Thursday**
   1. apply your knowledge by some programming (yay!)
   2. do not forget about other courses

Team of the course and Materials

- Vladimir Ivanov (PhD), Principal Instructor, Lectures
- Mohammedreza Bahrami (PhD), Tutorials
- Anastasia Puzankova, Labs
- Oleg Bulichev, Labs
- Eugene Marchuk, Labs

Resources: Books, Assignments, Useful links, etc.

Please, check Moodle!

Main questions for today's lecture

Applications of Linear Algebra and Analytical Geometry

Applications of AGLA in Computer Science and Engineering

Areas:

1. Computer Graphics and Computer Games
2. Machine Learning, Data Analysis
3. Natural Language Processing
4. Robotics
5. Computer Vision

6. and many, many other areas...
7. maybe, even in the backend...

Applications

Computer Graphics and Computer Games

- 2D/3D graphics
- Projective geometry, Homogeneous coordinates
- Collision detection in games. Calculation of trajectories

Machine Learning, Data Analysis

- Linear Regression
- Eigendecomposition
- Singular Value Decomposition
- Covariance matrix
- Linear Layers, Attention Mechanisms in Neural Networks

Agenda: Week 1

Vectors. Linear Independence

- Points and Vectors
- Vector Addition. Scalar Vector Multiplication
- Properties of Vector Arithmetic
- Vector spaces, Subspaces
- Span, Linear Independence
- Vector Bases and Vector Coordinates in Basis

Notation

- We denote points by capital italic letters, e.g., $A, B, ..., Q, ...$
- We denote numbers by Greek letters, e.g., $\alpha, \beta, ..., \lambda, \theta, ...$ and sometimes by Latin letters, $a, b, ..., v, u, x, ...$
- We denote vectors by **bold** letters, e.g., $\mathbf{a}, \mathbf{b}, ..., \mathbf{v}, \mathbf{u}, \mathbf{x}, ...,$
- and also we denote vectors by a letter with an arrow, e.g. $\vec{a}, \vec{b}, \vec{u}$
- and sometimes we denote vectors by end-points, e.g. $\overline{AB}, \overline{BC}, \overline{OA}$
- $\mathbb{R}$ is the set of real numbers
- $\mathbb{C}$ is the set of complex numbers

Introduction

Points and Vectors (**informally**). Direction

Vector. Geometrical point of view. Vectors as 'arrows' in plane or in 3D space

Let A and B be two points.
A directed line segment from A to B is denoted by: $\overline{AB}$
This directed line segment constitutes a vector.

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Let A and B be two points.
A directed line segment from A to B is denoted by: $\overline{AB}$
This directed line segment constitutes a vector.

Thus, each vector can be associated with a notion of *direction*. In this case, we can think of a vector as an “arrow” in space.

Points and Vectors (**informally**). Magnitude

Length (or Magnitude) of a Vector

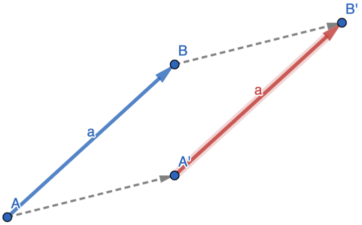
Also, often (**but not always!**) vector has a *length* (or a magnitude). The length of a vector is denoted by $\|\mathbf{v}\|$.

Unit vector

A *unit vector*, $\mathbf{u}$ is a vector with unit length (so $\|\mathbf{u}\|$=1). We can derive a unit vector as $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$.

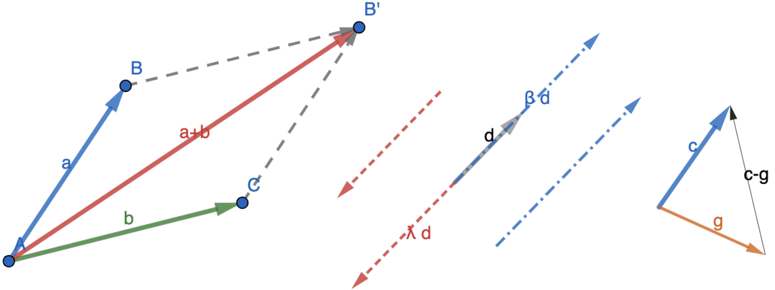
The length of a vector is closely related to the **dot product**, an operation which will be discussed in the next lecture. Therefore, $\mathbf{v}/\|\mathbf{v}\|$ is called a normalized vector.

If you move the line segment to another line segment with the same direction and length, they constitute **the same vector**.



Examples: Points and Vectors (informally)

Note that vector $\lambda \mathbf{d}$ is either parallel ($\lambda > 0$) to or anti-parallel ($\lambda < 0$) to $\mathbf{d}$.



$$\lambda, \beta \in \mathbb{R}$$

In this figure: $\lambda > 0$?

What if $\lambda = 0$?

Vector spaces

Vector space definition

Vector space

A *vector space* V over $\mathbb{R}$ (or $\mathbb{C}$) is a collection of vectors, together with two operations:

- $\mathbf{a} + \mathbf{b}$, addition of two vectors and
- $\lambda \mathbf{a}$, multiplication by a scalar ($\lambda \in \mathbb{R}$)

A scalar is a number from $\mathbb{R}$ or $\mathbb{C}$, respectively.

Addition and multiplication SHOULD satisfy following axioms

Vector addition axioms

Vector addition $\mathbf{a} + \mathbf{b}$ is defined $\forall \mathbf{a}, \mathbf{b} \in V$

Vector addition has to satisfy the following axioms:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity)
3. There is a vector $\mathbf{0}$ (zero vector) such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$. (identity)
4. For each vector $\mathbf{a}$, there exists a vector $(-\mathbf{a})$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (inverse)

Scalar multiplication axioms

$\lambda\mathbf{a}$ is defined $\forall\lambda \in \mathbb{R}, \forall\mathbf{a} \in V$

Scalar multiplication has to satisfy the following axioms:

1. $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.
2. $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
3. $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$.
4. $1\mathbf{a} = \mathbf{a}$.

The scalar is called a *scalar*, because it **scales** a vector :)

Homework Assignment

Prove

The zero vector is unique.

Prove

The inverse vector $(-\mathbf{a})$ is unique for any vector $\mathbf{a}$.

Vectors as lists of numbers

Column vectors. Examples

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ — we will use **this notation**! We represent vectors as **columns**!

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Row vectors. Examples

$\begin{bmatrix}3 & 4\end{bmatrix}$, $\begin{bmatrix}3 & 4 & 5\end{bmatrix}$, $\begin{bmatrix}x & y & z\end{bmatrix}$ Even though vectors can be represented as rows.

Vectors as lists of numbers

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$$\begin{bmatrix}3 & 4\end{bmatrix} \neq \begin{bmatrix}3\\4\end{bmatrix}$$

Transposition

Transposition

$$\begin{bmatrix} 3 & 4 \end{bmatrix}^\top = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}^\top = \begin{bmatrix} 3 & 4 \end{bmatrix} \tag{2}$$

This operation transforms a row vector a to column vector and back

For any vector

$$(\mathbf{v}^\top)^\top = \mathbf{v}$$

Examples

Example

Vector space V consisting of all functions $f(x)$ that are continuous on $\mathbb{R}$

$$V = \{f(x), \texttt{such that} f(x) \texttt{is continuous on } \mathbb{R}\}$$

Linear combination and linear independence

Linear combination

Vector $\mathbf{w} \in V$ is a linear combination of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n} \in V$ with coefficients $c_k \in \mathbb{R}; (k = 1..n)$
such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \sum_{k=1}^{n} c_k\mathbf{v_k}$$

Span

Span

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\} \subset V$.

$$span(S) \equiv \left\{ \mathbf{w} \in V : \mathbf{w} = \sum_{k=1}^{n} c_k \mathbf{v_k}, \quad \forall c_k \in \mathbb{R} \right\}$$

Basically, $W = span(S)$ is the set of all (possible) linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$.
Note that W is a subspace of V.

Subspace

Definition

W is a subspace of V if

a)	$W \subset V$	(subset)
b)	$\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$	(closure under addition)
c)	$\mathbf{u} \in W, \lambda \in \mathbb{R} \Rightarrow \lambda\mathbf{u} \in W$	(closure under scalar multiplication)

Examples

Linear independence in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Linearly independent vectors in $\mathbb{R}^2$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *linearly independent*
if for $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1\mathbf{a} + \alpha_2\mathbf{b} = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = 0$.

Linear independence in $\mathbb{R}^2$ and in $\mathbb{R}^3$

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Linearly independent vectors in $\mathbb{R}^3$

Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are *linearly independent* if for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1\mathbf{a} + \alpha_2\mathbf{b} + \alpha_3\mathbf{c} = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

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Try to give a definition for Linearly independent vectors in $\mathbb{R}^n$

Basis of a vector space

Basis

A **set** of vectors is a *basis* of a vector space if it spans a vector space and this set is **linearly independent**.

Basis in $\mathbb{R}^2$ and $\mathbb{R}^3$

Basis in $\mathbb{R}^2$

A **set** of vectors is a *basis* of $\mathbb{R}^2$ if it spans $\mathbb{R}^2$ and this set is **linearly independent**.

Standard basis in $\mathbb{R}^2$

$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\} = \{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$. They are the standard basis in $\mathbb{R}^2$.

Standard basis in $\mathbb{R}^3$

$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\} = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$. They are the standard (canonical) basis in $\mathbb{R}^3$.

Examples

Representation of a Vector in Vector Space

Theorem

Let V be a vector space over $\mathbb{R}^n$ and let $\{\mathbf{e_1}, ..., \mathbf{e_n}\}$ be a basis.
Then each vector $\mathbf{u}$ can be identified with its coordinates $\{u_1, ..., u_n\}$ in the basis.

$$\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{e_k}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$$

Homework Assignment

Let P_3, be a set of all polynomials of degree 3 or less.

Show that P_3 is a vector space over $\mathbb{R}$.
Hint: check axioms of vector space.

What could be a basis of P_3?
Give examples of two bases in P_3.
Express the polynomial $x^3 - 2x^2 + 3$ in the basis.

End of Lecture 1.

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila
- http://brilliant.com