CHAPTER 1
TIME VALUE OF MONEY

- Interest: The Cost of Money
- Economic Equivalence
- Interest Formulas – Single Cash Flows
- Equal-Payment Series
- Dealing with Gradient Series
- Composite Cash Flows.

DECISION DILEMMA – TAKE A LUMP SUM OR ANNUAL INSTALLMENTS

- A suburban Chicago couple won the Power-ball.
- They had to choose between a single lump sum $104 million, or $198 million paid out over 25 years (or $7.92 million per year).
- The winning couple opted for the lump sum.
- Did they make the right choice? What basis do we make such an economic comparison?

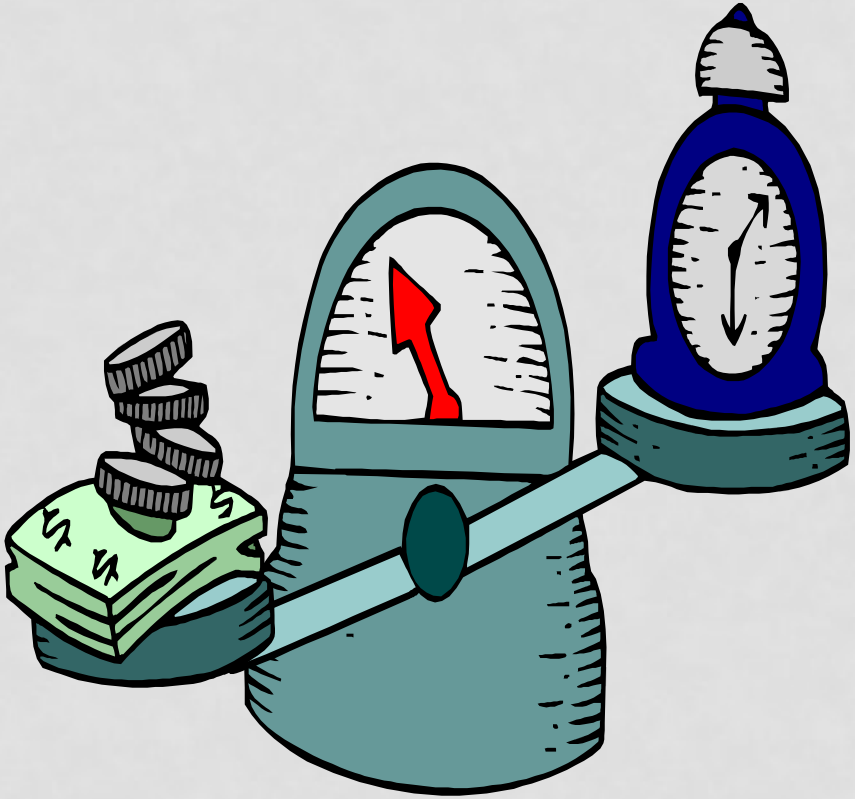
	Option A (Lump Sum)	Option B (Installment Plan)
0	$104 M	
1		$7.92 M
2		$7.92 M
3		$7.92 M
⋮		⋮
25		$7.92 M

INTRODUCTION

- **What is time value of money?**
- **Why it is Important in Economics Analysis?**

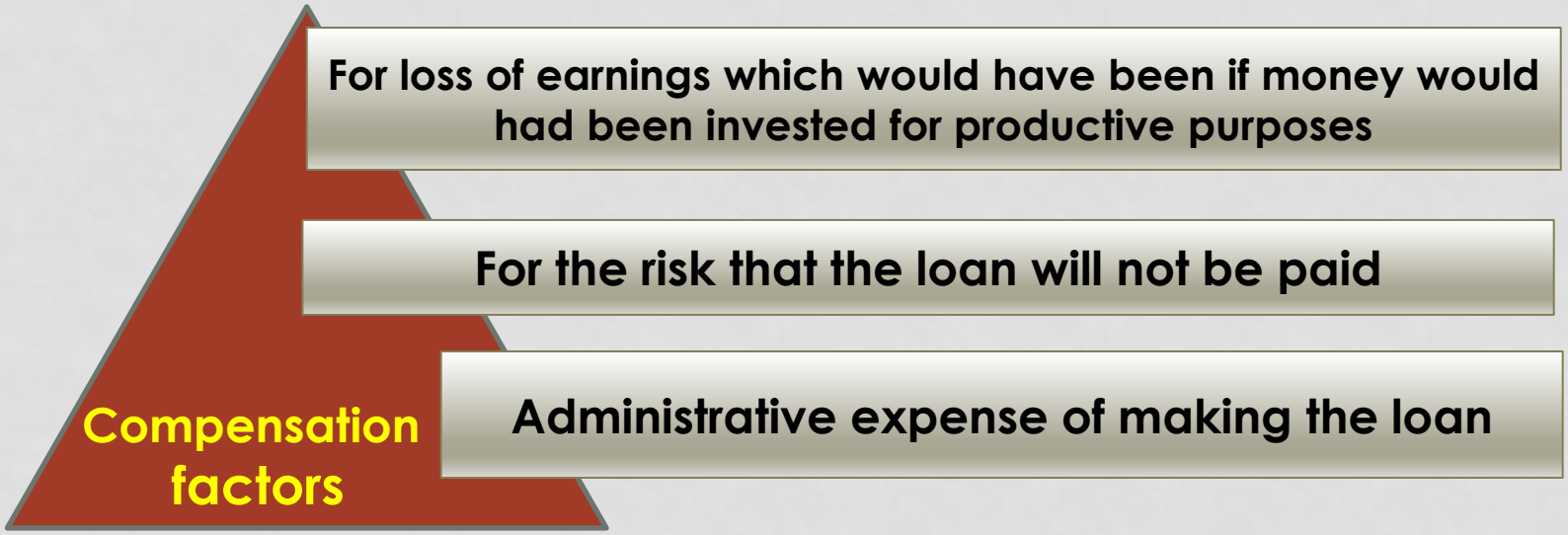
TIME VALUE OF MONEY

- Money has a time value because it can earn more money over time (earning power).
- Money has a time value because its purchasing power changes over time (inflation).
- Time value of money is measured in terms of interest rate.
- Interest is the cost of money—a cost to the borrower and an earning to the lender



Interest

- Rental value of money representing growth of capital per unit period.
- Interest represents earning power of money.
- It is the cost of using the capital.



SIMPLE INTEREST

In this case interest earned is directly proportional to capital involved in the loan.

If, I= interest earned through several time period.

P= Principal amount

i= rate of interest per period

N= number of interest periods (usually years)

Then, **I= P * i *N**

The total amount the borrower is supposed to pay the lender,

F= P + I => P + PiN => P(1+ iN)

COMPOUND INTEREST

- This is the method of charging interest on the interest earned.
- Total amount to pay, varies drastically when compared to simple interest charged.

Year end	Interest Rupees (@15%)	Compound Amount	
0		100	**P**
1	15	115	**$P_1 = P(1+i)$**
2	17.25	132.25	**$P_2 = P_1(1+i) = P(1+i)^2$**
3	19.84	152.09	**$P(1+i)^3$**

$$F = P \times (1 + i)^n$$

P- Principal Amount invested at time 0,

F- Future amount,

i- interest rate compounded annually,

n- period of deposits

- **Alternatively** if we want100 Rs at the end of n_{th} year what should be the amount deposited now?

$$P = \frac{F}{(1+i)^n}$$

- Similarly there are different interest formulas which are very useful for making investment decisions.

TIME VALUE EQUIVALENCE

- From economic point of view in time value conversion the equivalent values of money are determined not on values with equivalent purchasing power.

This can be easily understood with the following example.

- If 1000 is kept in a locker for 2 years or as deposit in bank earning 10% annually.
- Also, earning power could have been used to pay two equal annual installments of Rs.500 each.

(or initial deposit could have been reduced to 868 to pay out the installments.)

TYPES OF INTEREST FACTOR

- **Compound amount factor**
- **Annuity factor**

FOR EQUAL PAYMENT SERIES SINKING FUND

Since the first payment occurs at the end of first period, $F = A(1 + i)^{N-1}$

Similarly each payment treated in the same manner till the last payment N which receives no interest.

Therefore, $F = A(1+i)^{N-1} + A(1+1)^{N-2} + \ldots + A(1+i)^{N-N}$ ----------1

Factoring out A and multiplying through out by (1+i)

$$F(1+i) = A[(1+i)^{N} + (1+i)^{N-1} + (1+i)^{N-2} + \ldots + (1+i)]$$ ----------2

Subtracting, 2-1

$$F(1+i-1) = A[(1+i)^{N} - 1]$$

$$A = F[i / (1+i)^{N} - 1]$$

NOTATIONS USED IN INTEREST FORMULAE

P – principal amount

n – number of interest periods

i – interest rate

F – future amount at the end of year n

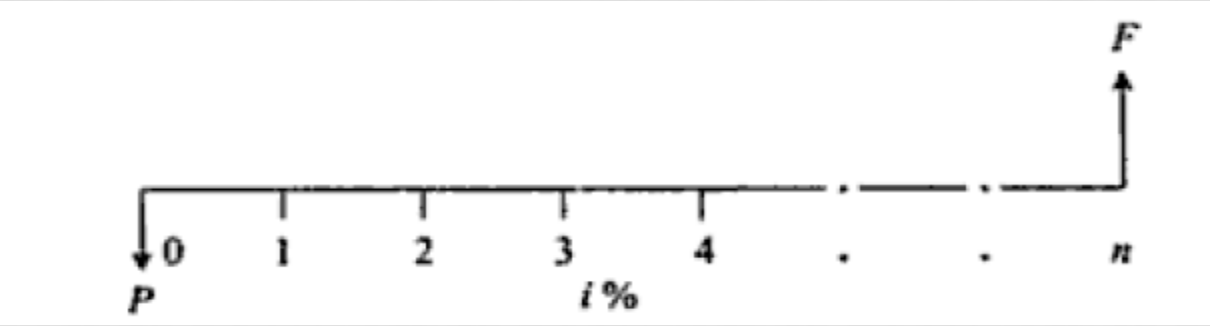
A – equal amount deposited at the end of every interest period

G – uniform amount which will be added/subtracted period after period to/from the amount of deposit A1 at the end of period 1.

TYPES OF COMPOUND INTEREST FORMULAS

1. **Single payment compound amount**

Here the objective is to find the single future sum (F) of initial payment P after n period at interest rate i% compounded every period.



$$F = P(1 + i)^n = P(F/P, i, n)$$

where

$(F/P, i, n)$ is called as single-payment compound amount factor.

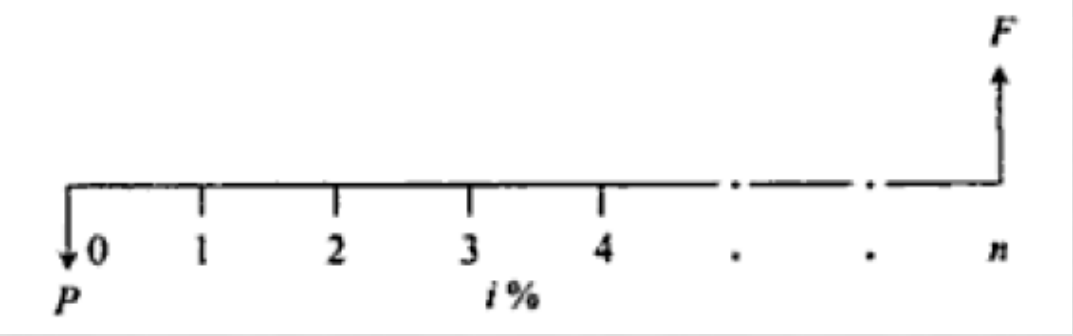
EXAMPLE 1

A person deposits a amount of ₹20,000 into an account bearing an interest rate of 18% compounded annually for 10 years. Find the maturity value after 10 years.

ANSWER: F= Rs. 1,04,680

2. Single Payment Present Worth Amount

Here the objective is to find the present worth amount (P) of a single future sum (F) which will be received after n periods at an interest rate of i% compounded at the end of every interest period.



The formula to obtain the present worth is

$$P = \frac{F}{(1+i)^n} = F(P/F, i, n)$$

where

$(P/F, i, n)$ is termed as *single-payment present worth factor.*

EXAMPLE 2

A person wishes to have a future sum of ₹1,00,000 for his son's education after 10 years from now. What is the single-payment that he should deposit now so that he gets the desired amount after 10 years? The bank gives 15% interest rate compounded annually.

ANSWER: P= Rs. 24,720

NUMERICAL

Suppose you are offered the alternative of receiving either $3,000 at the end of five years or P dollars today. There is no question that the $3,000 will be paid in full (no risk). Because you have no current need for the money, you would deposit the P dollars in an account that pays 8% interest. What value of P would make you indifferent to your choice between P dollars today and the promise of $3,000 at the end of five years?

EQUIVALENCE

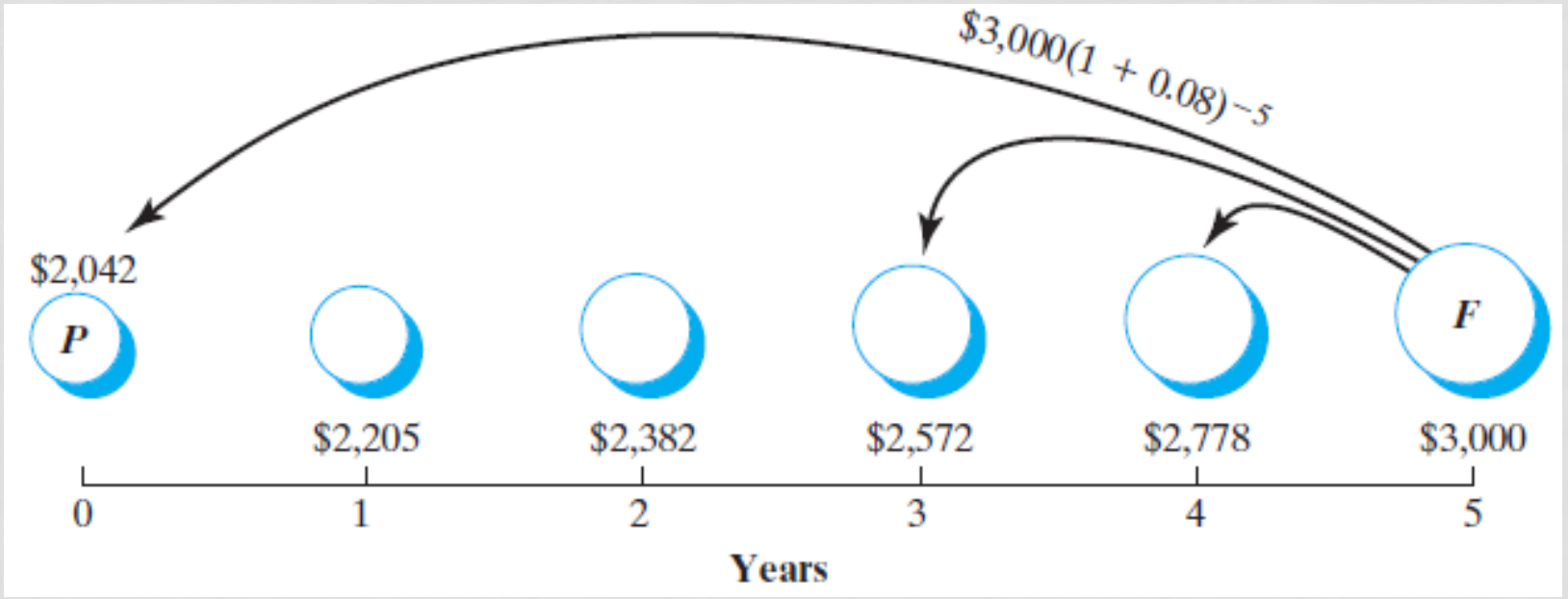


Figure 1: Various dollar amounts that will be economically equivalent to $3,000 in five years, given an interest rate of 8%

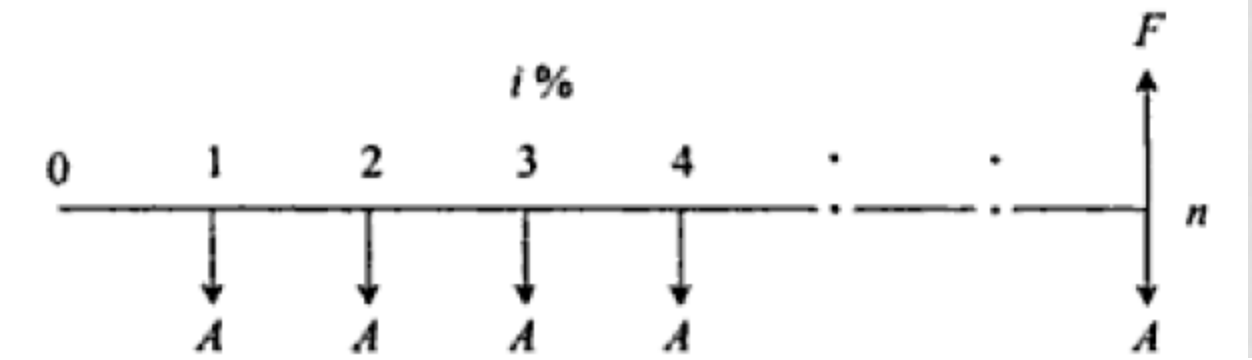
(source: Chan S Park, Contemporary Engineering Economics)

NUMERICAL

A person is planning to invest ₹10,000 now and also an amount of ₹5000 in the 3rd year. He is expected to withdraw an amount of ₹2000 for ever alternate years with first withdrawal starting from year 2. How much money will be left out in the account at the end of tenth year if the interest rate is 10%. (F balance= 20,502.5)

3. Equal Payment Series Compound Amount

Here the objective is to find the future worth of n equal payments which are made at the end of every interest period till the end of n_{th} interest period at an interest rate of i% compounded at the end of each interest period.



The formula to get F is

$$F = A\frac{(1+i)^n - 1}{i} = A(F/A, i, n)$$

where

$(F/A, i, n)$ is termed as *equal-payment series compound amount factor.*

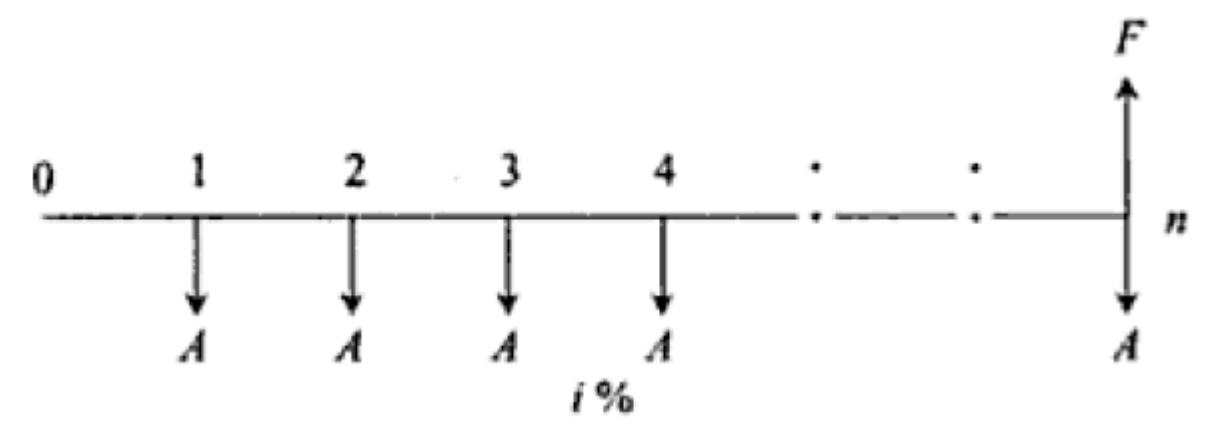
EXAMPLE 3

A person who is now 35 years old is planning for his retired life. He plans to invest any equal sum of ₹10,000 at the end of every year for the next 25 years starting from the end of the next year. The bank gives 20% interest rate, compounded annually. Find the maturity value of his account when he is 60 years old.

ANSWER: F= Rs. 47,19,810

4. Equal Payment Series Sinking Fund

Here the objective is to find the equal amount (A) that should be deposited at the end of every interest period for n period to realize a future sum (F) at the end of n_{th} period at an interest rate of i%.



The formula to get F is

$$A = F\frac{i}{(1+i)^n - 1} = F(A/F, i, n)$$

where

$(A/F, i, n)$ is called as *equal-payment series sinking fund factor.*

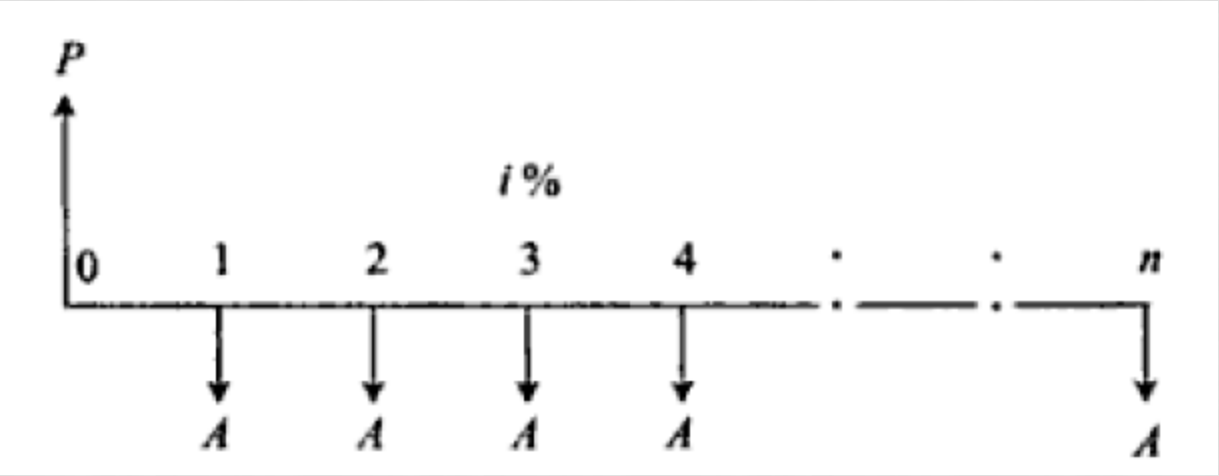
EXAMPLE 4

A company has to replace a present facility after 15 years at an outlay of ₹5,00,000. It plans to deposit an equal amount at the end of every year for the next 15 years at an interest rate of 18% compounded annually. Find the equivalent amount that must be deposited at the end of every year for the next 15 years.

ANSWER: A= Rs. 8200 per year

5. EQUAL PAYMENT SERIES PRESENT WORTH

Objective is to find present the worth of an equal payment made at end of every interest period for n periods.



The formula to compute P is

$$P = A\frac{(1+i)^n - 1}{i(1+i)^n} = A(P/A, i, n)$$

where

$(P/A, i, n)$ is called *equal-payment series present worth factor.*

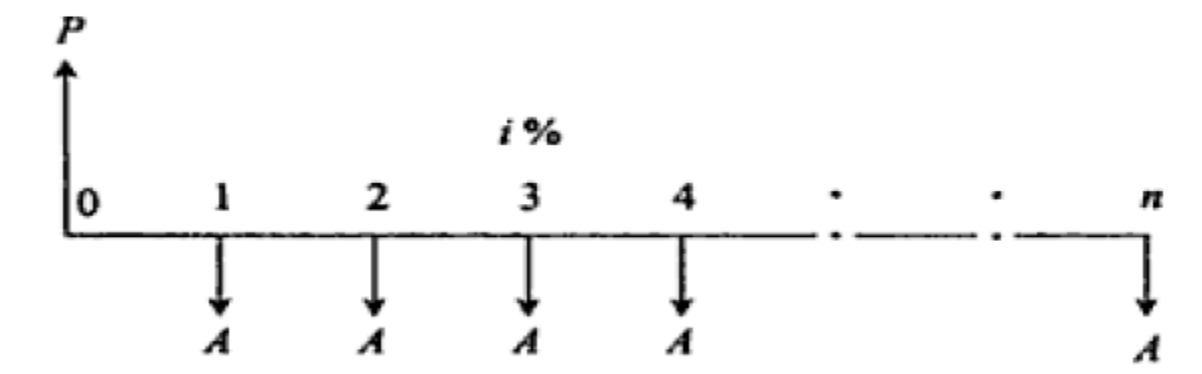
EXAMPLE 5

A company wants to set up a reserve which will help the company to have an annual equivalent amount of ₹10,00,000 for the next 20 years towards its employee welfare measures. The reserve is assumed to grow at the rate of 15% annually. Find the single-payment that must be made now as the reserve amount.

ANSWER: P= Rs. 62,59,300

6. EQUAL PAYMENT SERIES CAPITAL RECOVERY AMOUNT

Objective of this mode of investment is to find the annual equivalent amount (A) which is to be recovered at the end of every interest period for n interest periods for a loan (P) which is sanctioned now at an interest rate i% compounded every period.



The formula to compute P is as follows:

$$A = P\frac{i(1+i)^n}{(1+i)^n - 1} = P(A/P, i, n)$$

where,

$(A/P, i, n)$ is called *equal-payment series capital recovery factor.*

EXAMPLE 6

A bank gives a loan to a company to purchase an equipment worth ₹10,00,000 at an interest rate of 18% compounded annually. This amount should be repaid in 15 yearly equal installments. Find the installment amount that the company has to pay to the bank.

ANSWER: A= Rs. 1,96,400 Per Year

EXAMPLES

NUMERICAL 1

- How much money will be accumulated in 25 years if Rs. 800 is deposited at the end of 2nd year from now, Rs. 2400 six years from now and Rs. 3300 eight years from now all at an interest rate of 18% per year. (146739.2)
- Also, find its equivalent annual worth (A) for this time period of 25 years.

NUMERICAL 2

- A person plans to have a retirement policy which will give him return when he reaches an age of 50. For this person whose age is 35 years now has to make annual premium payment of Rs. 19760 till he reaches an age of 49. if the interest rate is 8% compounded annually, what is the lumpsum he is getting an maturity for this policy

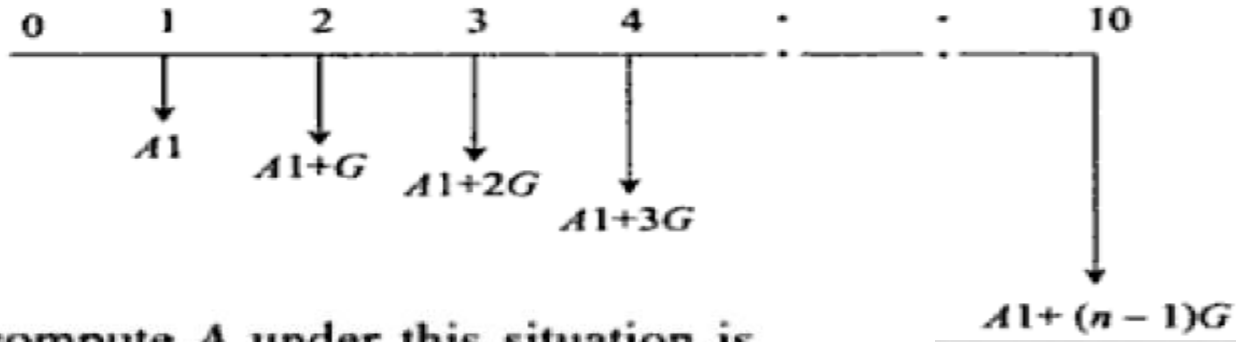
ANSWER: F= Rs. 5,16,765.77

NUMERICAL 3

- A boy is now 11 years old. On his fifth birthday he received a gift of $5,000 from his grandparents, which was invested in a 10 year fixed deposit bearing an interest rate of 6% per year compounded annually. His parents plan to have $6,000 available each year for the boy's nineteenth to twenty second birthdays to help finance his college education. To assist the financing, the fixed deposit will be reinvested when it matures. If required how much equal amount should the parents deposit each year, beginning from his next birthday, so that one year after the last deposit they can start making payments to their son. All future investments will earn 6.5% per year compounded annually.

7. UNIFORM GRADIENT SERIES ANNUAL EQUIVALENT AMOUNT

The objective of this mode of investment is to find the annual equivalent amount of a series with an amount $A1$ at the end of the first year and with an equal increment (G) at the end of each of the following $n - 1$ years with an interest rate i compounded annually.



The formula to compute A under this situation is

$$A = A1 + G\,\frac{(1+i)^n - in - 1}{i(1+i)^n - i}$$

$$= A1 + G\,(A/G, i, n)$$

where

$(A/G, i, n)$ is called *uniform gradient series factor.*

$F = G(1+i)^{n-2} + 2G(1+1)^{n-3} + 3g(1+i)^{n-4} + \ldots + (n-2)G(1+i) + (n-1)G$-----**1**

Multiply B.S. of eq 1 by (1+i)

$F(1+i) = G[(1+i)^{n-1} + (1+1)^{n-2} + \ldots + (n-2)(1+i)^2 + (n-1)(1+i)]$--------**2**

Eq. 2 -1

$Fi = G(1+i)^{n-1} + G(1+1)^{n-2} + \ldots\ldots\ldots\ldots\ldots - (n-1)G$

$Fi = G[(1+i)^{n-1} + (1+1)^{n-2} + \ldots + (1+i)^2 + (1+i) + 1] - nG$

$Fi = G(F/A,i,n) - nG$

Multiply BS by (A/F,i,n)

$Fi(A/F,i,n) = G(F/A,i,n)(A/F,i,n) - nG(A/F,i,n)$

$Ai = G - nG(A/F,i,n)$

EXAMPLE 7

A person is planning for his retired life. He has 10 more years of service. He would like to deposit 20% of his salary, which is ₹4,000, at the end of the first year, and thereafter he wishes to deposit the amount with an annual increase of ₹500 for the next 9 years with an interest rate of 15%. Find the total amount at the end of the 10^{th} year of the above series.

A= ₹5,691.60 / year; F= ₹1,15,562.25

EXAMPLE CONTINUED…

NUMERICAL 4

- A couple would like to determine what amount they must deposit in a savings account bearing 12% interest rate so that they will get Rs.5000/- at the end of 10^{th} year and will get an increase of Rs.1000/- each year for the next 10 years. Draw the cash flow diagram.
- Determine the present amount.
- If the interest rate is compounded quarterly what is the present amount?

NUMERICAL 5

Reconstruct the Cash flow diagram whose Future worth equation is as follows (n=15, i= 12%)

F=100(F/P,12%,14)+250(F/P,12%,12)+350(F/P,12%,10)+
450(F/P,12%,9)+750(F/P,12%,7)+1000(F/P,12%,5)+
1300(F/P,12%,4)+1600(F/P,12%,2)+1700

NUMERICAL 6

You will plan to retire 33 years from now, you expect that you will live 27 years after retiring. You want to have enough money upon reaching retirement age to withdraw 180,000 from the account at the beginning of each year. You expect to live , and yet still have 2,500,000 left in the account at the time of your expected death (60 years from now). You plan to accumulate the retirement fund by making equal annual deposits at the end of each year for the next 33 years. You expect that you will be able to earn 12% per year on your deposits. However, you only expect to earn 6% per year on your investment after you retire, since you will choose to place the money in less risky investments. What equal annual deposits you should make each year to achieve your retirement goal? (P at 33 for requirement is 30,39,469; and A to invest is 8875.2)

NOMINAL INTEREST RATES

- Interest rate are normally calculated on an annual basis.
- However interest may be compounded several times in an year quarterly, half yearly or monthly etc. this is nominal interest rate.

Example- 1000 earning an interest rate of 8% compounded quarterly.

PRACTICE EXAMPLES

Put these interest rates to standard nominal rate representation (yearly).

1. **2% per month**
2. **4% per semi-annual**
3. **5% per quarter**

Determine the interest on compounding period for these nominal rates.

1. **6% compounded semi-annually**
2. **7% per semi-annual**
3. **9% compounded quarterly**

EFFECTIVE INTEREST RATES

- It is the ratio of interest charge for 1year to the principal,
- This term actually eliminates the confusion over actual interest earned.

Consider the previous example of, 1year loan of Rs.1000 at a nominal interest rate of 8% compounded quarterly.

If compounded biannually at 8%, effective interest becomes?

Effective interest rate is $(1 + r/m)^n - 1$

Where r- nominal interest rate

m- number of compounding periods per year

CONSIDER THE FOLLOWING EXAMPLES

Given information	Standard representation (Nominal rate, r)	Effective interest rate (on yearly)
0.5% per month		
4.5% per semi-annual		
2% per quarter		
16% compounded quarterly		

NUMERICAL 7

An amount of 1200 per year is to be paid into an account each for the next five years. Using a nominal interest rate of 12 % determine the total amount the account will have at the end of 5^{th} year.

Deposit made at the end of each year with interest compounded monthly.

(ans. Effective interest = 12.68%; F= Rs. 7727)

NUMERICAL 8

Visteon, a spin-off company of Ford Motor Company, supplies major automobile components to auto manufacturers worldwide and is Ford's largest supplier. An engineer is on a Visteon committee to evaluate bids for new-generation coordinate-measuring machinery to be directly linked to the automated manufacturing of high-precision components. Three vendor bids include the interest rates. Visteon will make payments on a semi-annual basis only. The engineer is confused about the effective interest rates. What they are annually and over the payment period (PP) of 6-months.

Bid 1: 9% per year, compounded quarterly

Bid 2: 3% per quarter, compounded quarterly

Bid 3: 8.8% per year, compounded monthly

Determine the effective rate for each bid on the basis of semi-annual payments, and construct cash flow diagrams for each bid rate.

What are the effective annual rates? These are to be a part of the final bid selection.

Which bid has the lowest effective annual rate?

ANSWER

	Semiannual Rates			Annual Rates		
Bid	Nominal per 6 Months, r	CP per PP, m	Equation [4.8], Effective i	Nominal per Year, r	CP per Year, m	Equation [4.8], Effective i
#1	4.5%	2	4.55%	9%	4	9.31%
#2	6.0%	2	6.09%	12%	4	12.55%
#3	4.4%	6	4.48%	8.8%	12	9.16%

Source: Chan S Park, Contemporary Engineering Economics

THREE CASES ON EFFECTIVE INTEREST

When the Interest Compounding is Less than Payment Period ($i_{cp} < PP$)

$$i_{eff} = (1 + r/m)^n - 1$$

Example:

If you deposit an amount of 3000 every quarter into a investment account that pays an interest rate of:

A)15% compounded monthly

B) 8% per semi-annual compounded monthly.

What is the total accumulated amount at the end of 5 years?

THREE CASES ON EFFECTIVE INTEREST

When the Interest Compounding is Equal to Payment Period (i_{cp} = PP)

$$i_{eff} = r/m$$

Example:

If you deposit an amount of 3000 every quarter into a investment account that pays an interest rate of:

A)15% compounded quarterly

B) 8% per semi-annual compounded quarterly

C) 5% per quarter compounded quarterly.

What is the total accumulated amount at the end of 5 years?

THREE CASES ON EFFECTIVE INTEREST

When the Interest Compounding is Greater than the Payment Period ($i_{cp} > PP$)

$$i_{eff} = (1 + r/m)^{1/n} - 1$$ [here *n= 1/n*]

Example:

If you deposit an amount of 3000 every quarter into a investment account that pays an interest rate of:

A)15% compounded yearly

B)18% compounded quarterly.

What is the total accumulated amount at the end of 5 years?

NUMERICAL 3

A company is planning to invest Rs. 6000 once in 6 months, the investment is made at the end of every 6th month, for next 5 years. The company is planning to utilize this amount accumulated at the end of 5th year for buying an asset. Identify the amount accumulated at the end of 5th year under following cases:

a) If interest is 12% compounded semi-annually. (Ans- 79084.7)
b) If interest is 12% compounded annually. (Ans-) (78456.22)
c) If interest is 12% compounded quarterly. (Ans- 79419.83)

NUMERICAL 9

An entrepreneur intending to start a new business knows that the first few years are the most difficult.

To lessen the chance of failure, a loan plan for start up capital is proposed in which interest paid during the first two years will be at 3% and at 6% for the next two years of the 6 years loan.

How large a loan can be justified for proposed repayments at the end of years 2,4,and 6 respectively Rs. 20,000 , Rs. 30,000 and Rs.50,000?

NUMERICAL 10

A series of monthly cash flows is deposited into account that earns 12% nominal interest compounded monthly. Each monthly deposit is equal to $2100. The first monthly deposit occurred on June 1, 1998 and the last monthly deposit will be on January 1, 2005. The account (the series of monthly deposits, 12% nominal interest, and monthly compounding) also has equivalent quarterly withdrawals from it. The first quarterly withdrawal is equal to $5000 and occurred on October 1, 1998. The last $5000 withdrawal will occur on January 1, 2005. How much remains in the account after the last withdrawal?