GRAPH THEORY

Graph theory originated from the Konigsberg Bridge Problem where two islands linked to each other and the banks of the Pregel River by seven bridges.

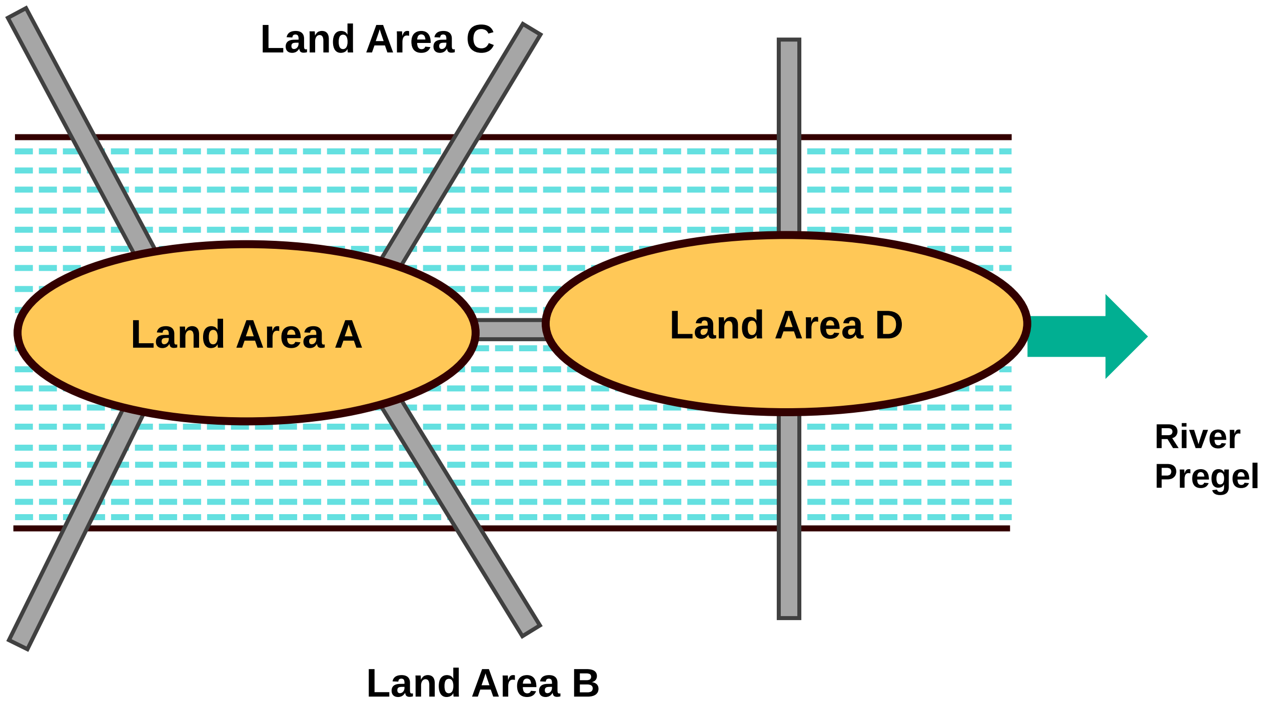


Figure 1: Konigsberg Bridge Problem

The problem was, to begin at any of the four land areas, (Land area A, B, C, D as shown in Figure 1) walk across each bridge exactly once and return to the initial land area. This problem was solved in 1736 by Euler. The Graph theory was first introduced by Euler. Mathematician L. Euler solved the Konigsberg Bridge problem. His theorem is known as the first theorem in Graph theory.

Graph theory has lot of applications in Making a timetable, Register allocation, Google maps, Facebook/LinkedIn, WWW, GPS etc.

Graph

A graph G consists of a finite nonempty set $V(G)$ of n elements called vertices (or points) together with a prescribed set $E(G)$ of m unordered pairs of different elements of $V(G)$(called as edges). We use the notations V and E for $V(G)$ and $E(G)$, respectively. Each pair $e = \{u, v\}$ of elements in $E(G)$ is an edge (or line) of G, and e is said to join u and v. Here, vertex u and edge e are said to be incident with each other, as v and e are. We say that u and v are adjacent vertices. A graph with n vertices and m edges is referred to as an (n, m) graph.

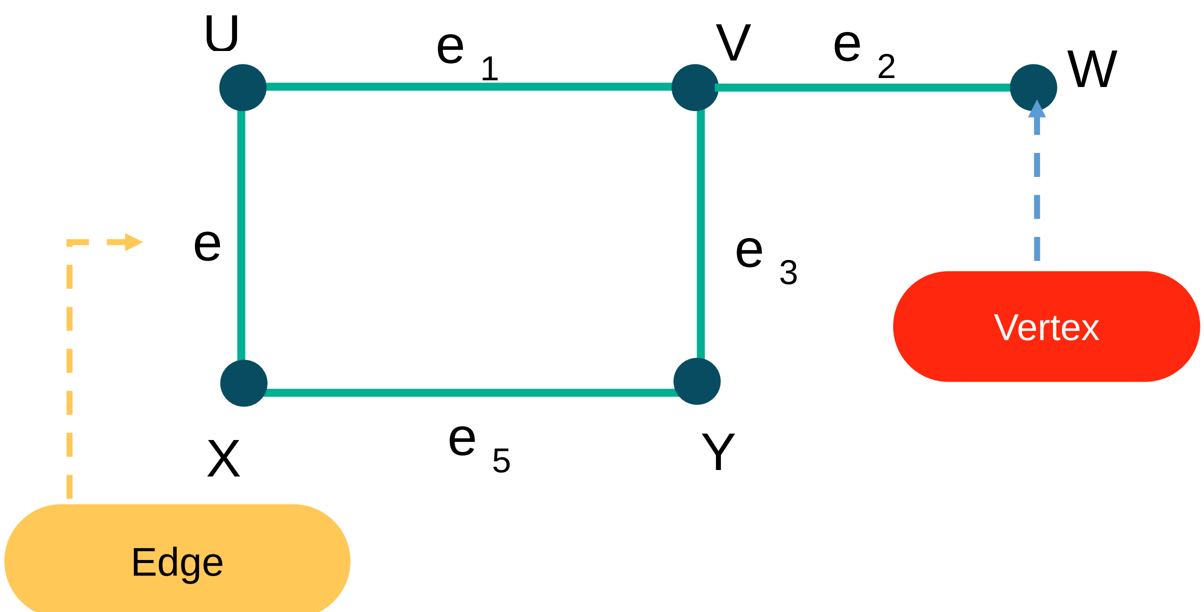


Figure 2: A Graph with 5 Vertices and 5 Edges

Each vertex is represented by a small dot, and each edge is represented by a line segment joining the two vertices with which the edge is incident. In figure 2, $\mathrm{u, v, w, x, y}$ are vertices and e_1, e_2, e_3, e_4, e_5 are edges. In figure 2, vertex u and vertex v are adjacent.

If two distinct edges e and f are incident with a common vertex, then they are said to be adjacent edges. In figure 2, edges e_1 and e_3 adjacent but e_3 and e_4 are not adjacent edges.

An edge with identical ends is called a loop and two edges with the same end vertices are called parallel (multiple) edges. In Figure 3, edge e_3 is a loop and edges e_2 and e_5 are parallel. A graph is simple if it contains neither loops nor parallel edges. In a multigraph, no loops are allowed but parallel edges are permitted. If both loops and parallel edges are permitted, then we have a pseudo graph.

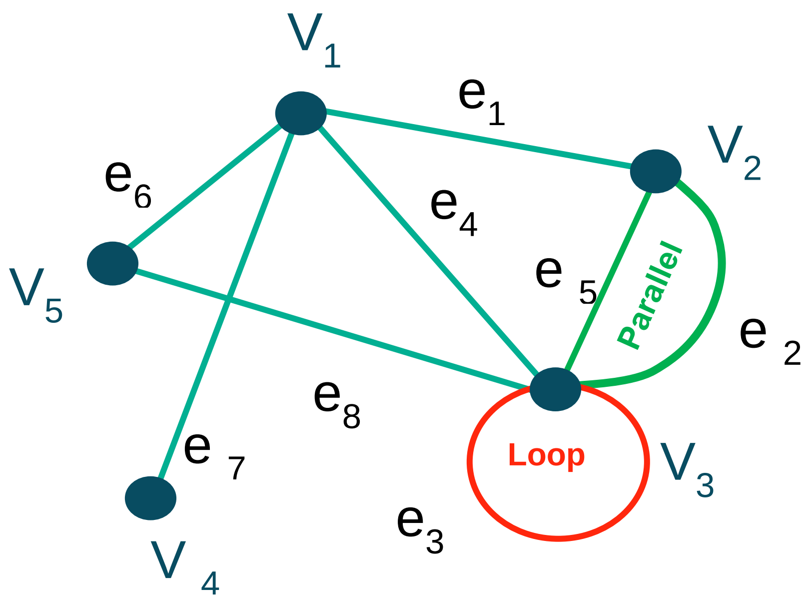


Figure 3: Pseudo Graph

The number of edges incident with v is the **degree** of a vertex v in a graph G. The degree of v is represented by $\deg(v)$.

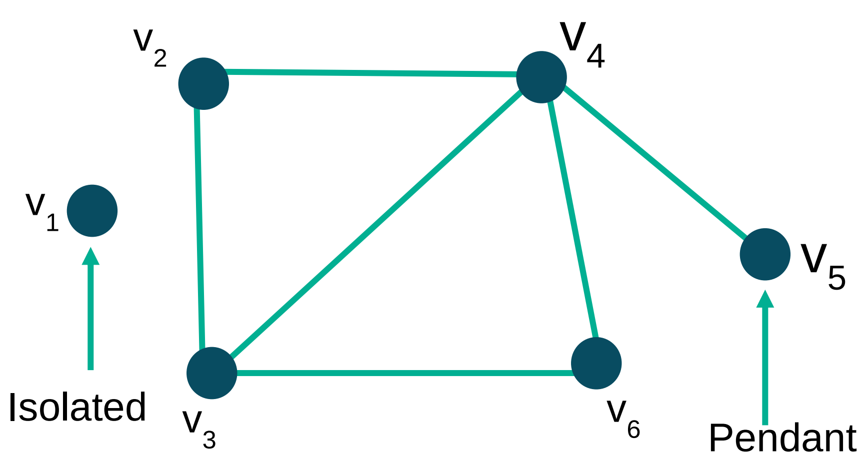


Figure 4: A (5,6) Graph

In Figure 4, $\deg(v_1) = 0,\ \deg(v_2) = 2, \deg(v_3) = 3$.

A vertex v of degree 0 is termed an isolated vertex. A vertex v of degree 1 is termed a pendant vertex. The minimum degree among the vertices of G is indicated by $\delta(G)$. The maximum degree among the vertices of G is indicated by $\Delta(G)$.

In Figure 4, v_4 is a vertex with a maximum degree and v_1 is a vertex with a minimum degree. Hence for the above graph $\delta(G) = 0$ and $\Delta(G) = 4$. Vertex v_1 is isolated and vertex v_5 is pendant for the graph as shown in Figure 4.

A graph is finite when both its vertex set, and edge set are finite. The (1, 0) graph is trivial, i.e., a graph with a single vertex and no edge is called trivial. A graph whose edge set is empty, is termed a null graph or a totally disconnected graph.

For a graph G, the number of elements in $V(G)$ is called order of the graph G and denoted by $|V(G)|$ and the number of elements in $E(G)$ is called the size of the graph G denoted by $|E(G)|$.

NOTE: While drawing a graph, it is immaterial whether the edges are drawn straight or curved, long or short. Thus, a diagram of the graph depicts the incidence relation holding between its vertices and edges.

Relation between Number of Edges and Degree

Theorem 1: The sum of the degrees of the vertices of a graph G is twice the number of edges.
Proof: Every edge of G is incident with two vertices. Hence every edge contributes 2 to the sum of the degrees of the vertices. Hence the result follows.

Question: If possible, draw a graph with degree sequence (1,1,2,2,3). If not, justify your answer.

Theorem 2: In any graph G, the number of vertices of odd degrees is even.

Proof: We know $\sum_{v \in V} \deg(v) =$ twice the number of edges. The sum of degrees of the vertices of odd degree (S_O) and even degree (S_E)equals the sum of the degrees of all the vertices of G which is twice the number of edges (an even number).

Thus, the sum of the degrees of the vertices of odd degree (S_O) must be an even number and hence the number of such vertices must be even.

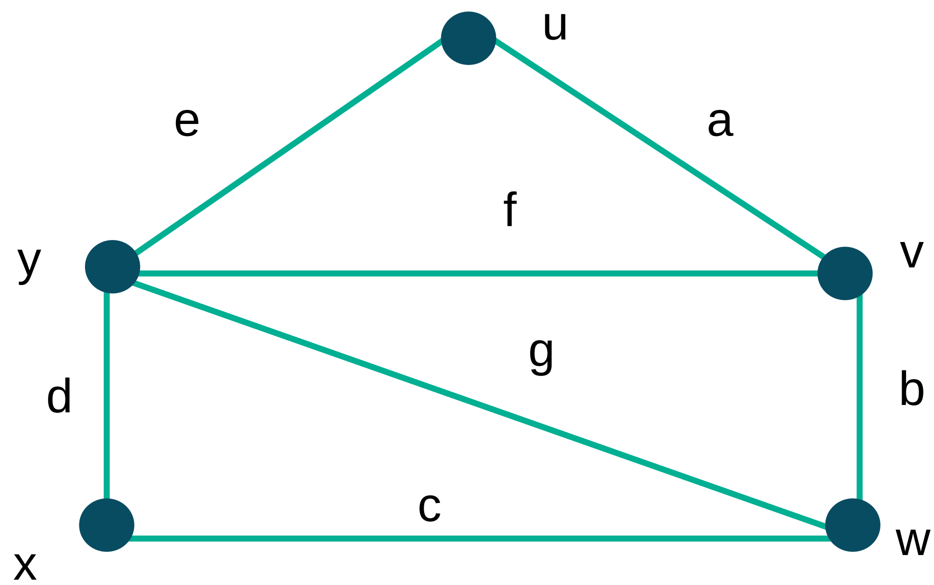


Figure 5: A (5,7) Graph

For the graph in Figure 5,

$$2|E(G)|=\sum_{v \in V} deg(v) = 14.$$

Sum of degrees=2+4+2+3+3=14. Vertices with odd degrees in the graph as shown in figure 5 are vertex v and w.

Question: If possible, draw a graph with degree sequence (0,1,2,3,4). If it is not possible, give the reason.

Perfect Graph: A graph is perfect if no two vertices are of the same degree.

Theorem 3: No graph is perfect or in a (n, m) graph G, there exists at least 2 vertices with same degree.

We note that in a (n, m) graph, $0 \leq \deg(v) \leq n-1$. Suppose there is a vertex with degree 0 in G, then there cannot exist a vertex with degree n-1 and vice versa. Hence n vertices will have n-1 choices. Hence the result follows.

Subgraph

A graph $H = (V_1, E_1)$ is called a subgraph of $G = (V, E)$, if $V_1 \subseteq V$ and $E_1 \subseteq E$.

A subgraph H is called a spanning subgraph of G, if $V_1 = V$.

The graph H is called an induced subgraph of G, if H is the maximal subgraph of G with vertex set V_1. Thus, if H is an induced subgraph of G, two vertices are adjacent in H, if and only if, they are adjacent in G.

For the graph as shown in Figure 6, the graphs in Figure 7, 8 are subgraphs.

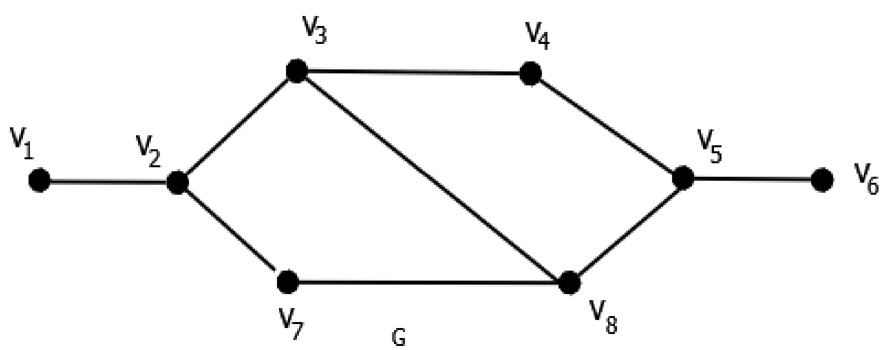


Figure 6: A Graph G

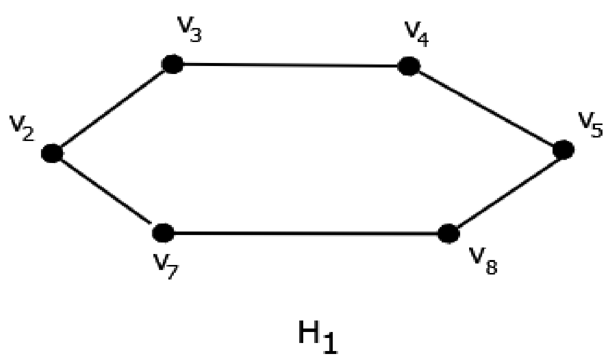


Figure 7: Graph H_1 is a Subgraph of G

The graph in Figure 8 is a spanning subgraph of G.

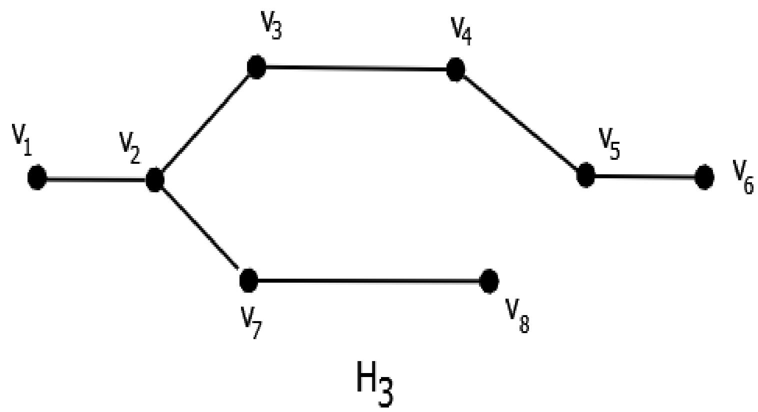


Figure 8: H_3 Spanning Subgraph of G

Figure 9 shows a graph G, induced subgraph H_1 induced by the set $V_1 = \{A, G, J, D, C\}$ and a spanning subgraph H_2.

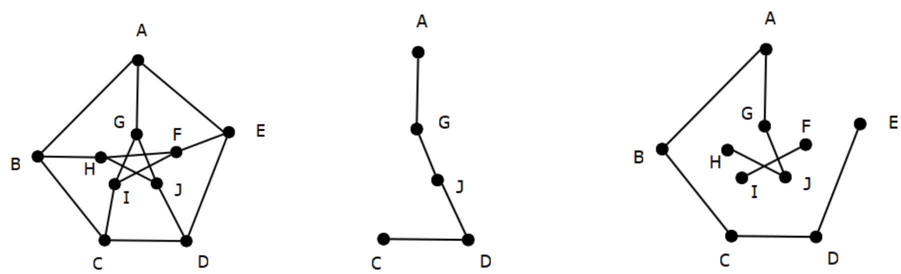


Figure 9: Graphs G, H_1, H_2

Removal of Vertices and Removal of Edges

The removal of a vertex v from a graph *G* results in that subgraph $G - v$ of *G* consisting of all vertices of *G* except v and all edges not incident with v (Figure 10).

The removal of an edge *e* from *G* yields the spanning subgraph *G* – *e* containing all edges of *G* except *e* (Figure 10)

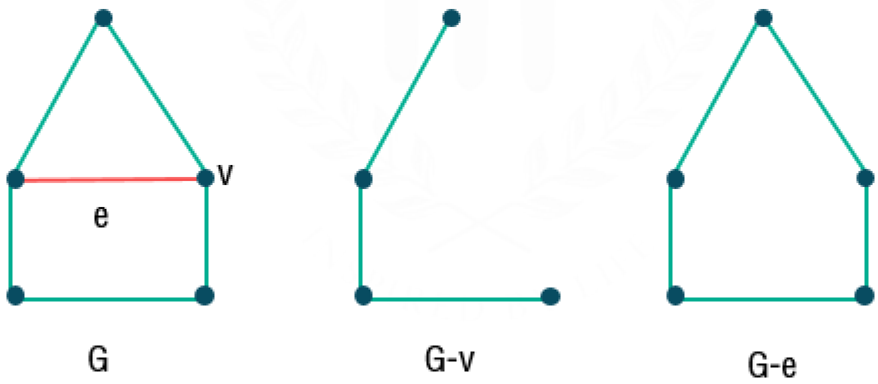


Figure 10: Graph G, G-v, G-e

Similarly, removal of vertices and removal of edges can be defined. Figure 11 shows Graph G along with $G - \{b, g, e\}$ and $G - \{v_7, v_4\}$.

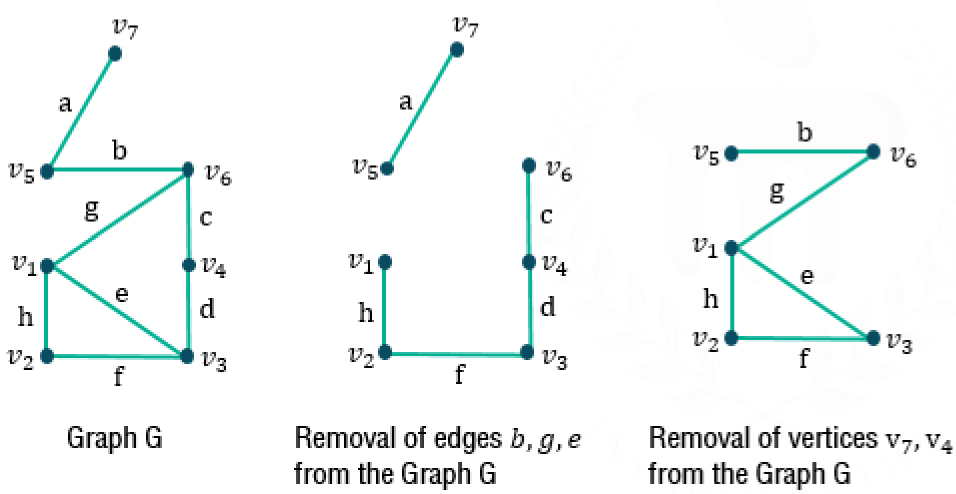


Figure 11: Graph G, $G = \{b, g, e\}, G - \{v_4, v_7\}$

Types of Graphs

Regular Graph

A graph G in which all the vertices have the same degree is called a regular graph of degree r.

Figure 12 represents a regular graph of degree 2.

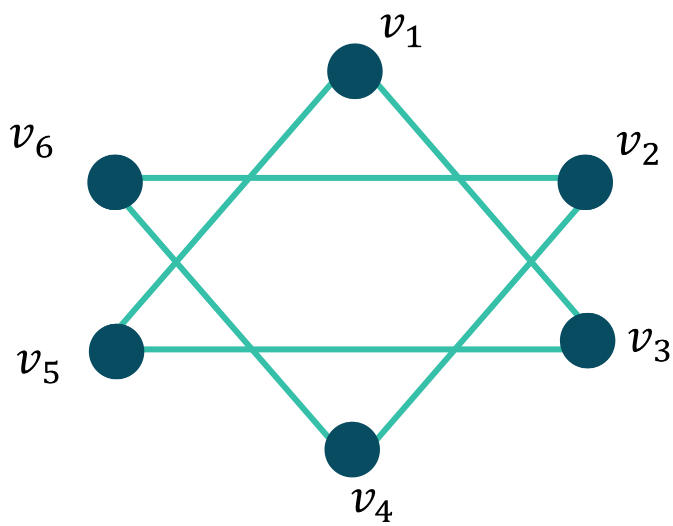


Figure 12: A Regular Graph of Degree 2

A regular graph of degree 3 is also known as a cubic graph.

Question: Construct a cubic graph (if possible) (i) on 5 vertices (ii) 6 vertices. If that is not possible then give a reason.

Complete Graph

A graph in which any two distinct vertices are adjacent is called a **complet**e graph. The complete graph with n vertices is denoted by K_n. Figure 13 represents a Complete graph on 6 vertices.

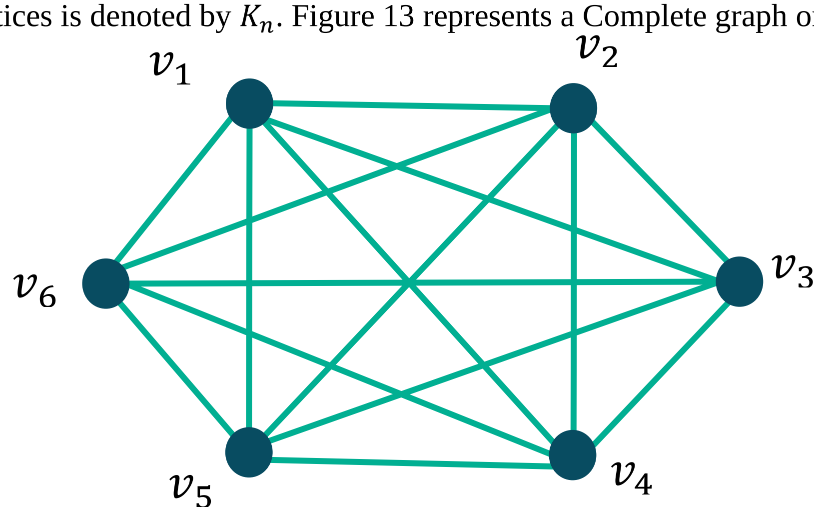


Figure 13: A Complete Graph on 6 Vertices Denoted by K_6

K_3 is also known as a triangle.

Question: Draw a regular graph on regularity 4 and number of vertices is 6.

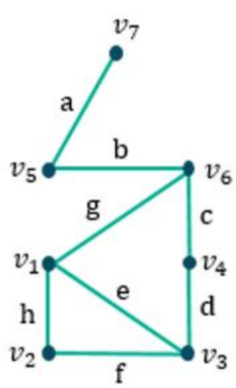
Question: Draw a complete graph on 7 vertices.

A **labelled graph** is a graph in which every vertex and every edge is labelled. We consider a graph always as a labelled graph.

Clique: A clique of graph G is a maximal complete subgraph.

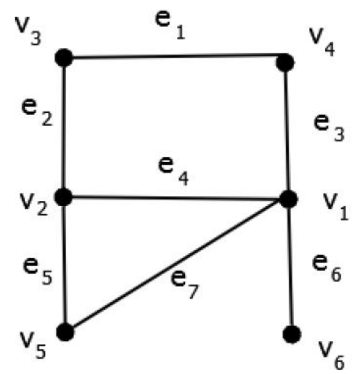
The clique of K_p is K_p itself.

For the graph below the clique of G is K_3. (Induced subgraph with $S = \{v_1, v_2, v_3\}$ forms a complete subgraph)



Graph G

Question 1: Draw a labelled graph *G* having (4, 3, 2, 2, 2, 1) as degree sequence.



Question 2: Draw graph *G* having (2, 2, 2, 2, 2, 2) as degree sequence. Is there more than one graph on a given degree sequence?

Question 3: Draw graph *G* having (4, 4, 3, 3, 2) as degree sequence. How many edges can this graph contain?

Weighted Graph

A weighted graph is a graph consisting of weight, or a number associated with each edge.

Application: Suppose that we have to connect n cities $v_1, v_2, \ldots, v_n$ through a network of roads, given that c_{ij} is the cost of building a direct road between the cities v_i and v_j. The problem is finding the least expensive network that connects all the cities. A weighted graph G is as shown in Figure 14.

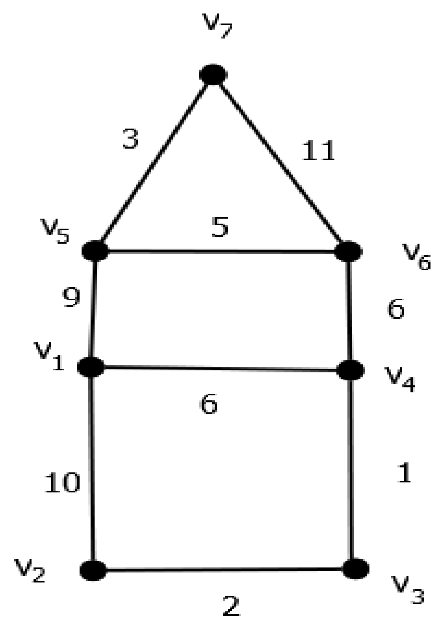


Figure 14: A Weighted Graph

Directed Graph

A directed graph (or digraph) is a triplet containing a vertex set $V(G)$, an edge set $E(G)$ and function assigning each edge, an ordered pair of vertices (Figure 15).

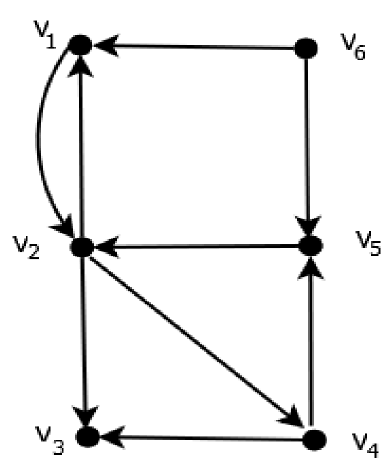


Figure 15: Directed Graph G

For the graph in Figure 15, there is a directed edge from vertex v_2 to v_4. But to travel from $v_4$to v_2, we have to travel via v_5.

Application: Given a list of cities and the distances between each pair of them.

We need to find the shortest probable route to visit each city exactly once and return to the original city. For the graph in Figure 16, to travel from B to E instead of a direct road (distance 10), if we travel from C, we will reach E with minimum distance (5+3).

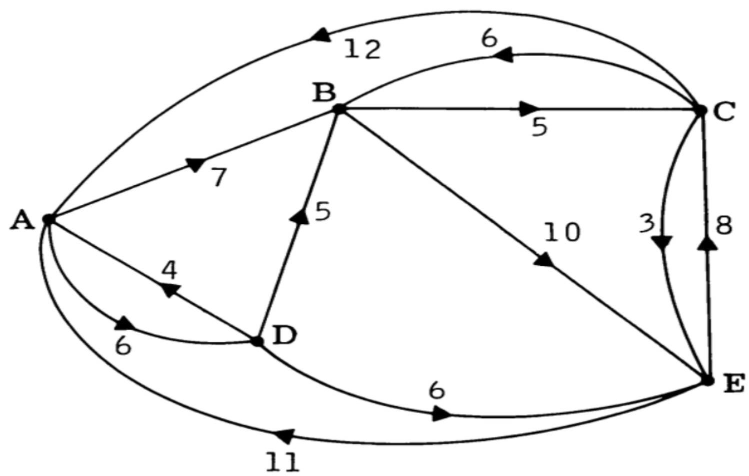


Figure 16: Weighed Directed Graph

Complement of a graph

Let $G = (V, E)$ be a Graph. The complement $\bar{G}$ of G is defined to be the graph which has V as its set of vertices and two vertices are adjacent in $\bar{G}$ only when they are not adjacent in G

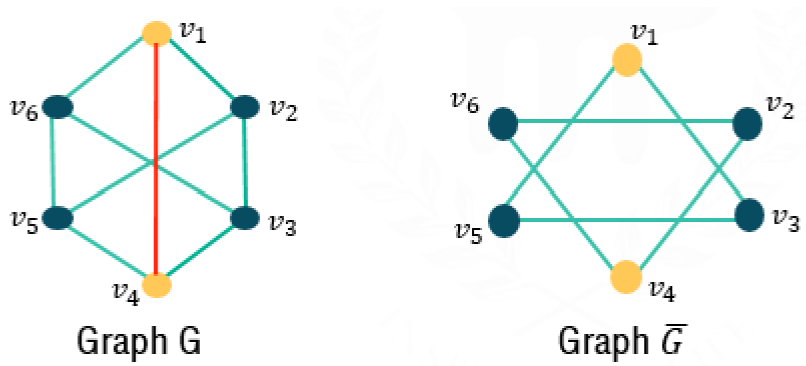
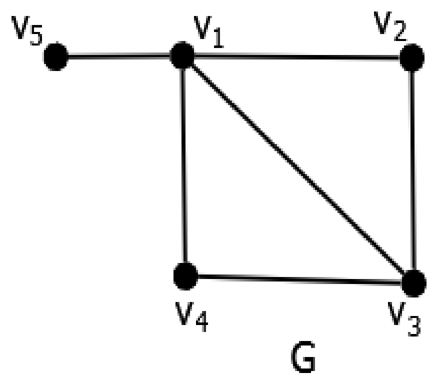


Figure 19: Graph G and its complement $\bar{G}$

In graph G (Figure 19) there is an edge from v_1 to v_4, but in $\bar{G}$, the edge between v_1 to v_4 is not there. $\bar{G}$ is disconnected graph with 2 components.

Question: Find the complement of the graph G.



Theorem 4: For any Graph G with six vertices, G or $\bar{G}$ contains a triangle.

Proof: Let G be a graph with six vertices. Let v be any vertex in G. Since v is adjacent to other five vertices either in G or in $\bar{G}$. We assume that v is adjacent with $v1$, $v2$, $v3$ in G. If any 2 of these vertices say $v1$, $v2$ are adjacent then $v1$, $v2$, v forms a triangle in G. If no two of them are adjacent in G then $v1$, $v2$, $v3$ are the vertices of a triangle in $\bar{G}$.

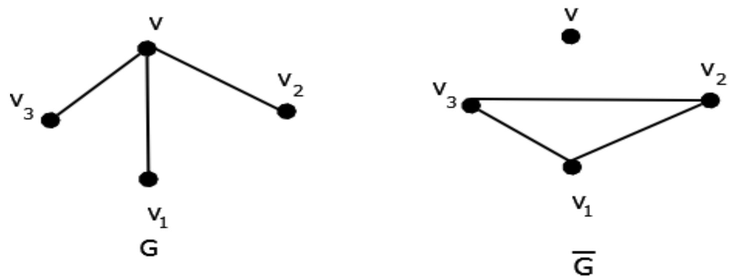


Figure 20:

The above theorem can be stated as follows.

At any party of 6 people, there are 3 mutual friends or there are 3 people who do not know each other. In other words, there is no induced subgraph which is either a K_3 or a $\overline{K_3}$.

Question: In any group of 7 people make connections in such a way that there are no 3 people who know each other and there are no 4 people who do not know each other. (Draw a graph with 7 vertices that contains no induced subgraph which is either K_3 or 4 isolated vertices).

Isomorphism: Two graphs G and H are isomorphic (written $G \cong H$ or sometimes $G = H$) if there exists a one-to-one correspondence between their vertices and between their edges such that structure is preserved.

Example:

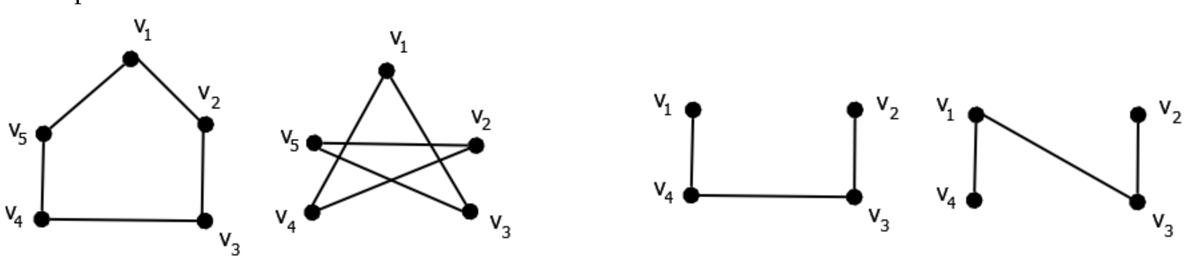
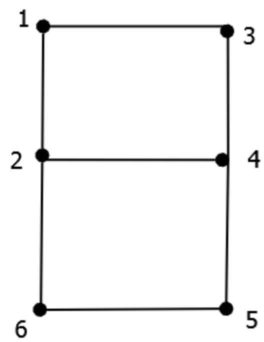
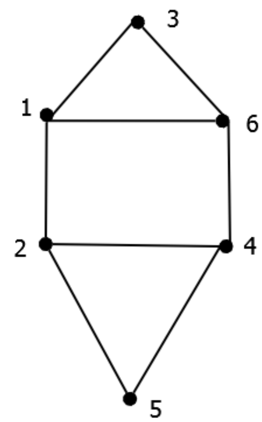
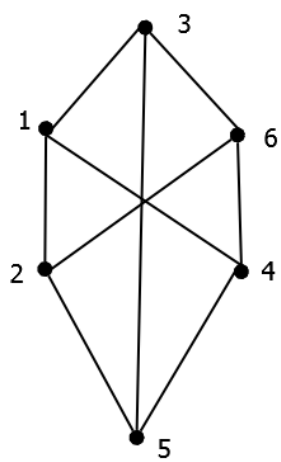


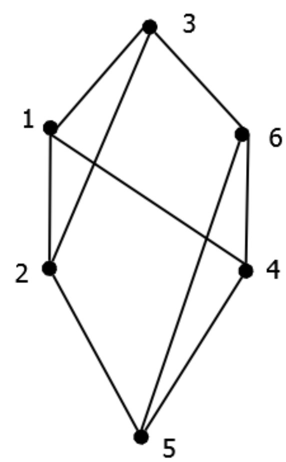
Figure: Isomorphism graphs

Question: Check whether the following set of graphs are isomorphic? If no give reason.









Self-complementary Graphs: A graph *G* is said to be **self-complementary** if *G* is isomorphic to its complement.

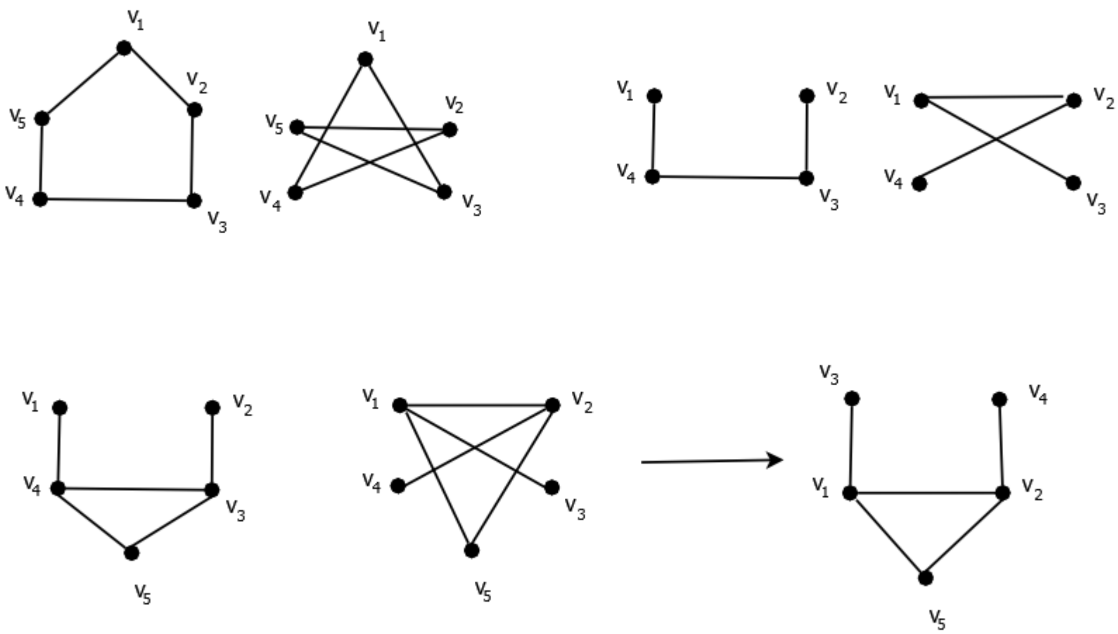


Figure 28: Self complementary graphs

Theorem 5: Let G be a self-complementary graph. Show that the number of vertices in G is of the form 4n or 4n + 1.

Proof: Let G be a (p, q) graph. Number of edges in $Kp = \frac{p(p-1)}{2} = {}^{p}C_2$

Since G is self-complementary, number of edges in G = number of edges in G = q.

Number of edges in Kp = number of edges in G + number of edges in G.

Number of edges in G = $\frac{p(p-1)}{2} - q$

$$q = \frac{p(p-1)}{2} - q,$$
$$4q = p(p-1)$$

Therefore, $q = \frac{p(p-1)}{4}$

$\Rightarrow \frac{4}{p}\ or \frac{4}{p-1}$

$p = 4n\ or\ p - 1 = 4n$

$p = 4n\ or\ p = 4n + 1$

Walk, trail, path, cycle

A walk of a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_n$ beginning and ending with vertices such that every edge e_i is incident with v_{i-1} and v_i. The walk joins v_0 and v_n, and it is called a $v_0 - v_n$ walk. v_0 is called the initial vertex, and v_n is called the terminal vertex of the walk.

In a simple graph, the walk is denoted by $v_0, v_1, \cdots, v_n$. The number of edges in the walk is called the length of this walk. A single vertex is taken as a walk of length 0.

A walk should begin and end with a vertex only. It cannot start or end at an edge.

A walk is called a **trail** if all its edges are distinct and is called a **path** if all its vertices are distinct.

A $v_0 - v_n$ walk is called **closed** if $v_0 = v_n$. A closed path $v_0, v_1, \cdots, v_n = v_0$ in which $n \geq 3$ is called a **cycle** of length n.

In Figure 15, u, v, w, y, v, w, x is a walk of length 6 (the vertices v, w and edge b is repeated), v, w, x, y, v, u is a trail (vertex v repeats but none of the edges repeat) and y, u, v, y, w, x, y is not a cycle (as vertex y repeats).

Note that y, v, w, x is a path of length 3, x, w, v, w, x is a closed walk, and y, v, w, x, y is a cycle of length 4.

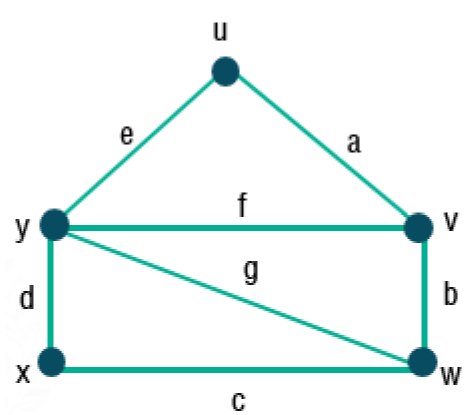


Figure 15: A (5,8) graph

Connected and disconnected graphs

Two vertices u and v of a graph G are said to be connected if there exists a $u - v$ path in G. A graph G is said to be connected if every pair of its vertices are connected. A graph that is not connected is said to be disconnected.

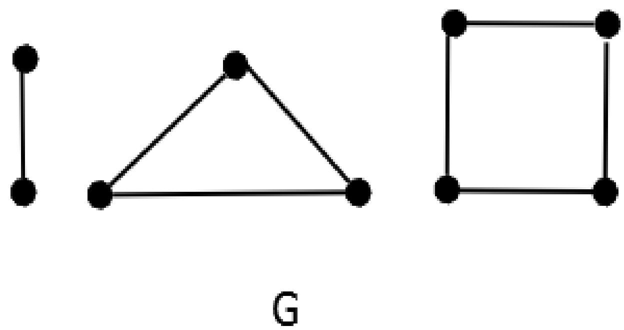


Figure 16: Disconnected graph G

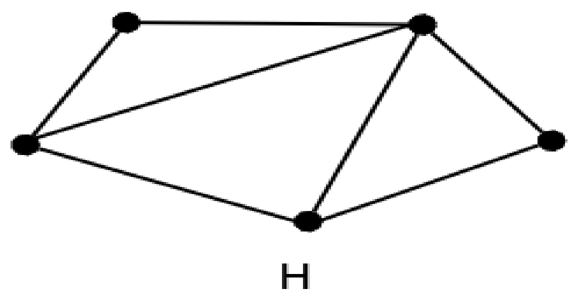


Figure 17: Connected graph H

Graph G in Figure 16 is a disconnected graph with 3 components, whereas H is a connected graph (Figure 17). In G there is no path from any vertex in a triangle to a vertex in the square (Cycle of length 4).
A maximal connected subgraph of G is called component of G. Thus, a disconnected graph has at least 2 components. Graph G in Figure 16 is a disconnected graph with 3 components. A graph G is connected if and only if it has exactly one component.
A connected regular graph with regularity two is called a cycle. A cycle on n vertices is denoted by C_n.

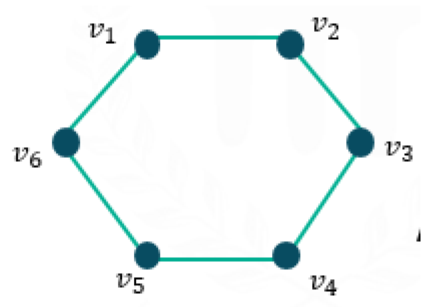


Figure 18: A cycle graph on 6 vertices.

The cycle of length 3, i.e., C_3, is also known as a triangle.

A **Path** graph denoted by P_n on n vertices is a graph with vertices $\{v_1, v_2, .., v_n\}$ and edges are (v_i, v_{i+1}), $1 \leq i \leq n-1$.
P_n has n-1 edges.

Question: Draw a cycle graph on 9 vertices.

Question: Draw the complement of cycle graph C_9.

Question: The clique in the graph $C_n, n \geq 4$ is ----.

Theorem 6: If a graph has exactly 2 vertices of odd degree, then there must be a path joining the 2 vertices.

Proof: Let u and v be any two vertices.

Suppose the graph is connected, then there exists a path joining u and v. Suppose G is disconnected, then there exist at least 2 components. Now if both the vertices u and v are in one of the components, then as component is connected, there exists a path between u and v.

We note that u and v cannot be in 2 different components as in G there are exactly 2 vertices with odd degrees. If u is in one component, then in that component we get only one vertex of odd degree which is not possible.

Hence u and v be in one of the components and there exists a path between them.

Theorem 7: For any graph G, show that either G or $\overline{\mathbf{G}}$ is connected.

Proof: If *G* itself is connected, there is nothing to prove.

Suppose that graph *G* is disconnected and has two components *C*1 and *C*2. Let *u* and *v* be any two vertices, we have the following cases.

If *u* and *v* are in different components and are not adjacent in *G*. Then *u* and *v* are adjacent in *G*. We have, uv path, hence *G* is connected.

If *u* and *v* belong to the same component but they are not adjacent in G. Hence, they are adjacent in *G*. Hence, we have *uv* path.

Suppose that *u* and *v* are adjacent in *G* (Obviously, they belong to the same component). Then we can find *w* in another component (which does not contain *u* and *v*). We have a *uv* path via w in *G*. That is, *u* *w* and *v* *w*.

Theorem 8: Show that if G has p vertices and minimum degree $\delta(G) \geq \frac{p-1}{2}$, then G is connected.
Proof: Suppose that graph *G* is disconnected. Let us assume that *G* has two (or more) components, say *C*1 and *C*2. Suppose that a component *C*1 has a vertex of minimum degree (*p* − 1)/2. Then, *C*1 must contain at least [(*p*−1)/2+1] vertices. Similarly, suppose that a component *C*2 has a vertex of minimum degree (*p*−1)/2. Then, *C*2 must contain at least [(*p*−1)/2+1] vertices. Now, the total number of vertices in *G* is equal to [(*p* − 1)/2 + 1 + (*p* −1)/2+1] = *p* −1+2 =*p* +1 which is a contradiction to the fact that *G* has *p* vertices. Hence, *G* is connected.

Distance

Let G be a connected graph and let u, v be two vertices in G.

The shortest path between u and v in G is a $(u, v) -$ path with the minimum number of edges in it. The distance between u and v in G is denoted by $d(u, v)$ is the length of a shortest path between them.

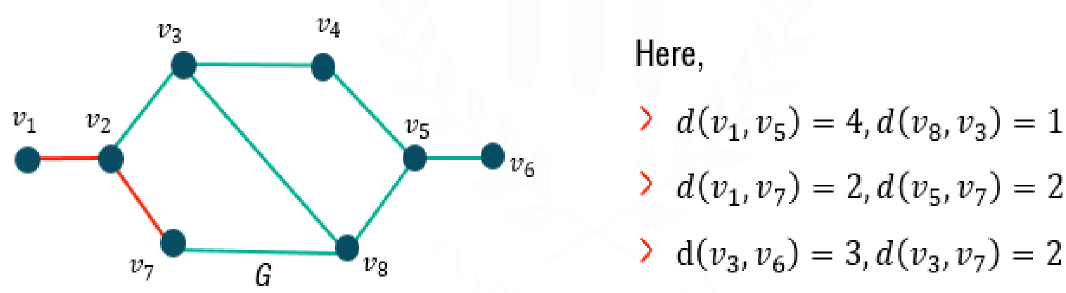


Figure 21: A graph G with diameter 5

To calculate $d(v_1, v_7)$, (Figure 21) we take a path from $v_1 - v_2 - v_7$ not $v_1 - v_2 - v_3 - v_8 - v_7$ as this is not the shortest path.

Eccentricity of a vertex v in a connected graph G, denoted by $e(v)$ is defined as:

$$e(v) = \max_{u \in V} d(u, v)$$

For the graph in Figure 21, $d(v_3, v_1) = 2$, $d(v_3, v_2) = 1$, $d(v_3, v_4) = 1, d(v_3, v_5) = 2$, $d(v_3, v_6) = 3$, $d(v_3, v_7) = 2$, $d(v_3, v_8) = 1$.

Hence, $\boldsymbol{e(v_3) = \max_{u \in V} d(u, v_3) = 3}$
Similarly, $e(v_1) = 5$.

The minimum and maximum of the eccentricities of vertices of G are **radius** and **diameter** of the graph G. A vertex v in G with minimum eccentricity is called a **central vertex** and set of all central vertices in G is called the centre of G.

For the graph in Figure 21, $e(v_1) = 5, e(v_2) = 4$, $e(v_3) = 3$, $e(v_4) = 3$, $e(v_5) = 4$, $e(v_6) = 5$, $e(v_7) = 3$, $e(v_8) = 3$.

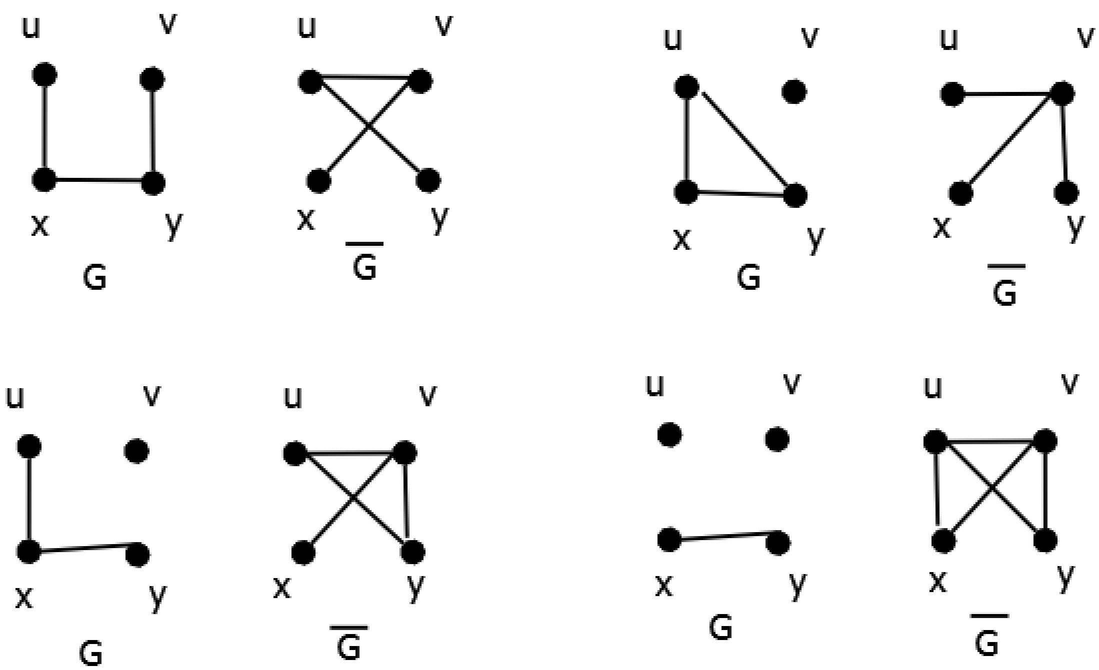
So, Centre of G=$\{v_3, v_4, v_7, v_8\}$, Radius(G)=3, Diameter(G)=5.

A graph G is said to be self-centered if every vertex of G has the same eccentricity. In such a graph, the radius is equal to the diameter.
The cycle graph C_n is a self-centered graph and is the complete graph K_n.
Diameter and radius of K_n is 1
Diameter and radius of C_n is $\left\lfloor \frac{n}{2} \right\rfloor$.
In graph, it is not essential that the relation $diamete(G)$ = 2. $radius(G)$ gets satisfied.
Example: In complete graph K_n, $diameter(G) = radius(G)$.
The **Girth** of a graph G is the length of the smallest cycle (if any) in G and is denoted by $g(G)$.

The **circumference** of graph G is the length of the longest cycle (if any) in G and is denoted by $c(G)$.

Theorem 8: If $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$

Proof: Let *x* and *y* be any two non-adjacent vertices in *G*. Since $diam(G) \geq 3$, there exist vertices *u* and *v* at distance 3 in *G*.



Since *u* and *v* have no common neighbors in *G*, both *x* and *y* are each adjacent to *u* or *v* in *G*. It follows that $diam(x, y) \leq 3$ in *G* and hence $diam(\overline{G}) \leq 3$.

Theorem 9: Every nontrivial self-complementary graph has diameter 2 or 3.

Proof: Let G be a self-complementary graph. Clearly, G cannot have diameter 1.
Since $G \not\cong K_n$ which is not self-complementary graph. Hence, self-complementary graphs have diameter at least 2.
Suppose that diam(G) ≥ 3. By the above theorem, diam($\overline{G}$)≤ 3. Hence, the diameter of every self-complementary graph is either 2 or 3.

Question: The complete graph K_p has————————edges.
Question: The cycle graph C_n has ———————— edges.
Question: The complete graph K_p has diameter = ————————-
Question : Draw a regular graph on 6 vertices with regularity 1.

Eulerian Graph

Eulerian Graph: A graph G is said to be Eulerian if it contains a closed trail that traverses each edge of the graph G. Eulerian trail should start from one vertex and travel through all other remaining edges exactly once and reach back to the starting vertex.

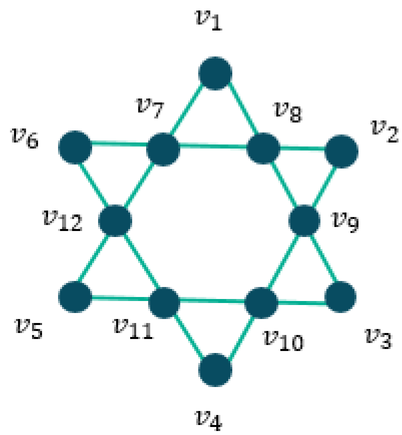


Figure 21: Eulerian Graph G

For the graph in Figure 21, $v_1, v_8, v_2, v_9, v_3, v_{10}, v_4, v_{11}, v_5, v_{12}, v_6, v_7, v_{12}, v_{11}, v_{10}, v_9, v_8, v_7, v_1$ is an Eulerian Trail.

First Theorem in Graph Theory: A nonempty connected graph is Eulerian if and only if all of its vertices are of even degree.

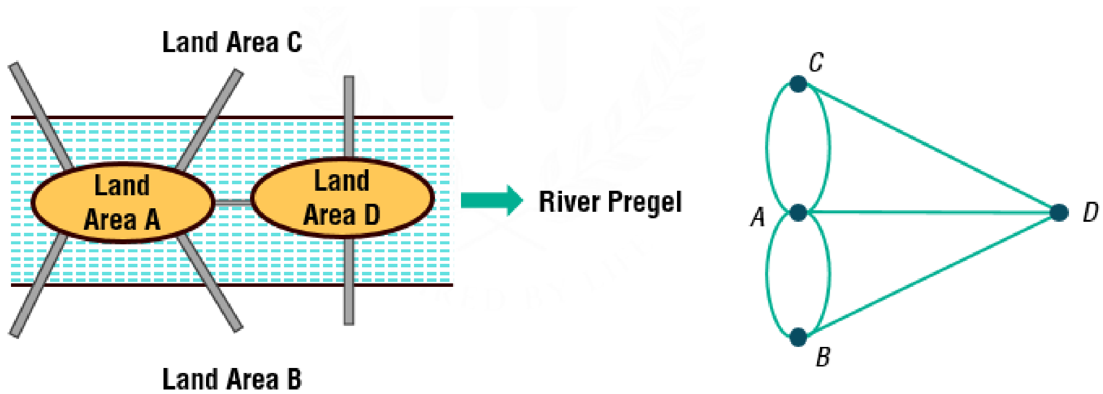


Figure 22 : Graph representing Konigsberg Bridge Problem

This theorem concludes that the bridge problem has no solution.

The Konigsberg Bridge problem has no solution. Figure 22 represents the Konigsberg Bridge.

Theorem 10: A non-empty connected graph is Eulerian if and only if all of its vertices of even degree.

Proof: Suppose that G is connected and Eulerian. Since G has an Eulerian trail which passes through each edge exactly once, goes through all vertices.
Hence, all of its vertices are of even degree.
Conversely, Let G be a connected graph such that every vertex of G is of even degree. Since, G is connected, no vertex can be of degree zero. Thus, every vertex of degree ≥ 2, so G contains a cycle. Let C be a cycle in a graph G. Remove edges of the cycle C from the graph G. The resulting graph (say G1) may not be connected, but every vertex of the resulting graph is of even degree.

Suppose G consists only of this cycle C, then G is obviously Eulerian. Otherwise, there is another cycle C1 with a vertex v in common with C. The walk beginning at v and consisting of the cycles C and C1 in succession is a closed trial containing the edges of these two cycles. By continuing this process, we can construct a closed trial containing all edges of G, hence G is Eulerian.

Questions:

1. The number of edges in a complete graph on 7 vertices is ____________.
2. State True or False: There exists a regular graph of degree 3 with 5 vertices. Justify your answer.
3. Let G be a simple graph with 5 vertices and degrees of these 5 vertices be 2,3,3,4,4. Then number of edges in G = ____________.
4. State True or False: There exists a simple graph with degrees of its 5 vertices be 1,2,3,4,4.
5. The number of edges in a regular graph of degree 3 with 6 vertices is ____________.
6. Draw a complete graph on 5 vertices.
7. The number of edges in the complement of a cycle graph on 7 vertices is ---.
8. Draw a cycle graph on 8 vertices.
9. The diameter of a cycle graph on 9 vertices is ---.
10. The number of edges in a cycle graph on 50 vertices is ---
11. Girth of P_{20} is
12. Circumference of C_{50} is

Hamiltonian Graph:

Around the World Game: A game invented by William Hamilton in 1859, uses a regular solid **dodecahedron**, whose 20 vertices are labelled with the names of famous cities.

The player is challenged to travel "around the world" by finding a closed cycle along the edges which passes through each vertex exactly once.

The object of the game is to find a spanning cycle in the graph of the dodecahedron. The points of the graph are marked 1,2, • • •, 20 (rather than cities' name) so that the existence of a spanning cycle is evident.

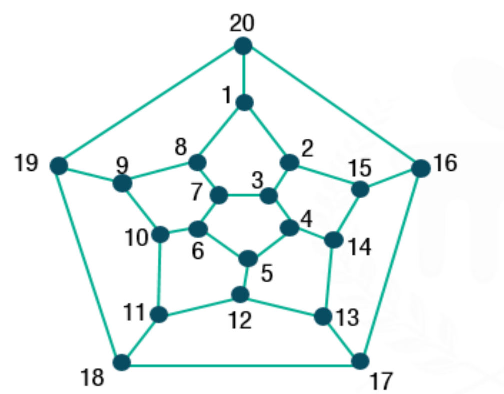


Figure 23: Graph of the Dodecahedron

Cities are numbered as 1,2,..,20 (figure 23) to find a closed cycle that passes through each city exactly once.

Hamiltonian Graph: A graph which contains a closed path that traverses each vertex of the graph is a Hamiltonian graph. In a Hamiltonian graph it is not required that the closed path should cover all the edges.

The edges which are not numbered (red coloured) in the figure 24 are not covered in the closed path that traverses through all the vertices.

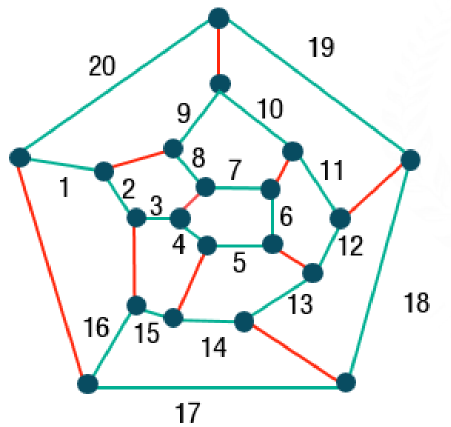


Figure 24 : Hamiltonian graph G

The complete graph and cycle graph are examples for Hamiltonian graph.

Example of Non-Hamiltonian Graph:

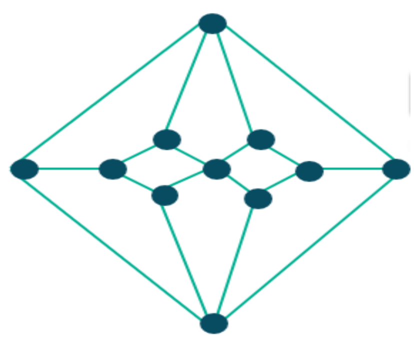


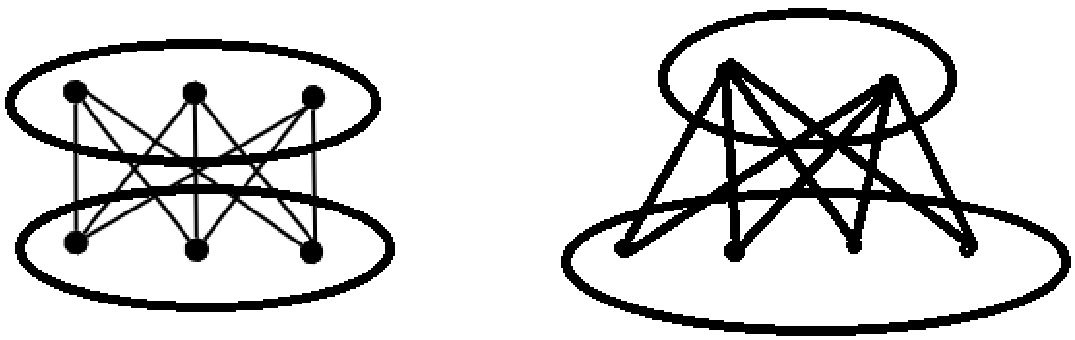
Figure 25: A non-Hamiltonian graph

There is no closed path which traverses all the vertices of the graph shown in Figure 25.

Question: Is the cycle graph on 11 vertices Hamiltonian and Eulerian?

Bipartite Graph:
A bipartite graph is one whose vertex set can be partitioned into 2 subsets X and Y so that each edge has one end vertex in X and one end vertex in Y . Such a partition (X, Y) is called a bipartition of the graph G.

A complete bipartite graph is a bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y ; if |X| = m and |Y | = n, such a graph is denoted by $K_{m,n}$.
The graphs (a) and (b) below are complete bipartite $K_{3,3}$ and $K_{2,4}$ respectively.



Complete bipartite graph $K_{3,3}$ on 6 vertices and $K_{2,4}$ on 6 vertices.
Number of edges in $K_{m,n}$ is ———.

Theorem 11: A graph is bipartite if and only if all its cycles are even.
Proof: Let G be a connected bipartite graph. Then its vertex set V can be partitioned into two sets V1 and V2 such that every edge of G joins a vertex of V1 with a vertex of V2. Thus, every cycle $v_1, v_2, \ldots, v_n, v_1$ in G necessarily has its oddly subscripted vertices in V1(say), i.e, $v_1, v_3, \ldots, \in V_1$ and other vertices $v_2, v_4, \ldots \in V_2$. In a cycle $v_1, v_2, \ldots, v_n, v_1$: v_n, v_1 is an edge in G.
Since, $v_1 \in V_1$ we must have $v_n \in V_2$. This implies n is even. Hence, the length of the cycle is even.
Conversely, suppose that G is a connected graph with no odd cycles. Let $u \in V(G)$ be any vertex.
Let $V_1 = \{v \in V(G) | d(u,v) = even\}, V_2 = \{v \in V(G) | d(u,v) = odd\}$.
Then, $V = V_1 \cup V_2, V_1 \cap V_2 = \phi$
We must prove that no two vertices in V_1 and V_2 are adjacent. Suppose that $x, w \in V_1$ be adjacent. Then, $d(u,w) = 2k$ and $d(u,x) = 2l$. Thus, the path u − w − x − u forms a cycle of length 2k + 2l + 1, odd a contradiction. Therefore, x and w cannot be adjacent. No two vertices in V_1 are adjacent. Similarly, we can prove that no two vertices in V_2 are adjacent. Hence, the graph is bipartite.

Question: Check whether C_8, C_7 are bipartite graphs?