Reg. No. | | | | | | | | | |

MANIPAL INSTITUTE OF TECHNOLOGY
MANIPAL
(A constituent unit of MAHE, Manipal)

FIFTH SEMESTER B.TECH (DATA SCIENCE)
MID SEMESTER EXAMINATIONS, SEPTEMBER 2024

MATHEMATICAL FOUNDATION FOR DATA SCIENCE [MAT-3135]

REVISED CREDIT SYSTEM

Date: 27.09.2024 Time: 2.15PM-3.45PM

SCHEME

Q.NO	**Questions**	***Marks***
11	(i) Prove that for non-zero a, $$a \mid bc \Leftrightarrow \frac{a}{(a,b)} \Big\vert c.$$ (ii) Find all the incongruent solutions mod 35, if any, of the linear congruence $$15\,x \equiv 25 \ (\text{mod } 35).$$ *Soln:* (i) $15x \equiv 25 \ (\text{mod } 35)$ ½ $\Rightarrow 3x \equiv 5 \ (\text{mod } 7)$ ½ $\Rightarrow 5(3x) \equiv 5 \times 5 \ (\text{mod } 7)$ ½ $\Rightarrow x \equiv 4 \ (\text{mod } 7)$ ½ $\Rightarrow x = 4, 11, 18, 25, 32$ ½	***4***

	(11) $a \mid bc \iff \frac{a}{(a,b)} \mid c.$ Let $(a,b)=d. \Rightarrow a=dr$ & $b=ds$ (where $(r,s)=1$) $\Rightarrow d=\frac{a}{r}=\frac{b}{s}$ Let $a \mid bc. \Rightarrow bc=ak$ $\Rightarrow (ds)c=ak$ (1M) $\Rightarrow \left(\frac{a}{d}\right)k=cs$ $\Rightarrow \frac{a}{d} \mid cs$ $\Rightarrow \frac{a}{d} \mid c$ $\left(\because \left(\frac{a}{d},s\right)=(r,s)=1\right)$ $\Rightarrow \frac{a}{(a,b)} \mid c$ Conversely, let $\frac{a}{d} \mid c$ $\Rightarrow \left(\frac{a}{d}\right)k=c$ $\Rightarrow ak=cd$ $\Rightarrow ak=c\frac{b}{s}$ $\Rightarrow aks=cb$ (1M) $\Rightarrow a(ks)=bc$ $\Rightarrow a \mid bc$	
12	Let (a,b)denote the greatest common divisor of a and b. Find integers x and y, if exists, such that $$4001x+2689y=(4001,2689).$$ **Soln:** $gcd=1 \longrightarrow$ (1) $x=-1117 \longrightarrow$ (1) $y=1662 \longrightarrow$ (1)	***3***

13	Find the last two digits in the ordinary decimal representation of 3^{400}. **Soln:** 13 $3^{\phi(100)} \equiv 1 \pmod{100} \rightarrow 1$ $\Rightarrow 3^{40} \equiv 1 \pmod{100} \rightarrow 1$ $\Rightarrow (3^{40})^{10} \equiv 1 \pmod{100}$ $\Rightarrow 3^{400} \equiv 1 \pmod{100}$ $\rightarrow 1$ Ans: $\underline{\underline{01}}$	***3***
14	Find the smallest positive integer x that gives the remainders 1,2,3 when divided by 3,4,5 respectively. Soln: 14 $\left.\begin{array}{l} x \equiv 1 \pmod{3} \\ x \equiv 2 \pmod{4} \\ x \equiv 3 \pmod{5} \end{array}\right\}$ $\frac{1}{2}$ $M = 3 \times 4 \times 5 = 60 \rightarrow \frac{1}{2}$ $\left.\begin{array}{ll} M_1 = 20, & N_1 = 2 \\ M_2 = 15, & N_2 = 3 \\ M_3 = 12, & N_3 = 3 \end{array}\right\} \rightarrow 1\frac{1}{2}$ $\therefore x = (1 \times 20 \times 2) + (2 \times 15 \times 3)$ $+ (3 \times 12 \times 3)$ $\frac{1}{2}$ $= 238 \equiv \underline{\underline{58}} \pmod{60}$	***3***
15	Let G be a connected graph with at least 3 vertices. Then, show that G is bipartite if and only if it has no odd cycles. Proof: Let G be a connected bipartite graph. Then its vertex set V can be partitioned into two sets V1 and V2 such that every edge of G joins a vertex of V1 with a vertex of V2. Thus, every cycle $v_1, v_2, \ldots, v_n, v_1$ in G necessarily has its oddly subscripted	***3***

	vertices in V1(say), i.e, $v_1, v_3, \ldots, \in V_1$ and other vertices $v_2, v_4, \ldots \in V_2$. In a cycle $v_1, v_2, \ldots, v_n, v_1$: v_n, v_1 is an edge in G. Since, $v_1 \in V_1$ we must have $v_n \in V_2$. This implies n is even. Hence, the length of the cycle is even. (1.5M) Conversely, suppose that G is a connected graph with no odd cycles. Let $u \in V(G)$ be any vertex. Let $V_1 = \{v \in V(G) \| d(u,v) = even\}, V_2 = \{v \in V(G) \| d(u,v) = odd\}$. Then, $V = V_1 \cup V_2, V_1 \cap V_2 = \phi$ We must prove that no two vertices in V_1 and V_2 are adjacent. Suppose that $x, w \in V_1$ be adjacent. Then, $d(u,w) = 2k$ and $d(u,x) = 2l$. Thus, the path u – w – x – u forms a cycle of length 2k + 2l + 1, odd a contradiction. Therefore, x and w cannot be adjacent. No two vertices in V_1 are adjacent. Similarly, we can prove that no two vertices in V_2 are adjacent. Hence, the graph is bipartite. (1.5M)	
16	Prove that tree T with n vertices has n-1 edges. **Proof**: The proof is by induction on the number of vertices. If $n = 1$, we get a tree with one vertex and no edge. If $n = 2$, we get a tree with two vertices and one edge. If $n = 3$, we get a tree with three vertices and two edges. Assume that the statement is true with all trees with k vertices ($k < n$). (1M) Let G be a tree with n vertices. Since G is a tree there exist a unique path between every pair of vertices in G. Thus, removal of an edge e from G will disconnect the graph G. Further, $G - e$ consists of exactly two components with number of vertices say r and s with $r + s = n$. (1M) Each component is again a tree. By induction, the component with r vertices has $r - 1$ edges and the component with s vertices has $s - 1$ edges. Thus, the number of edges in G $= r - 1 + s - 1 + 1 = r + s - 1 = n - 1$. (1M)	***3***
17	Draw two regular graphs G and H on 8 vertices such that (i) G contains a cut vertex and a bridge (ii) H contains neither a cut vertex nor a bridge Soln: (i) Path on 8 vertices (1M) (ii) Cycle on 8 vertices (1M)	***2***
18	Prove that if a graph G has n vertices and minimum degree $\delta(G) \geq \frac{n-1}{2}$, then G is connected. Proof: Suppose that graph *G* is disconnected. Let us assume that *G* has two (or more) components, say *C*1 and *C*2. Suppose that a component *C*1 has a vertex of minimum degree (*p* − 1)/2. Then, *C*1 must contain at least [(*p*−1)/2+1] vertices. Similarly, suppose that a component *C*2 has a vertex of minimum degree (*p*−1)/2. Then, *C*2 must contain at least [(*p*−1)/2+1] vertices. Now, the total number of vertices in *G* is equal to [(*p* − 1)/2 + 1 + (*p* −1)/2+1] = *p* −1+2 =*p* +1 which is a contradiction to the fact that *G* has *p* vertices. Hence, *G* is connected. (2M)	***2***
19	Prove that in a non-trivial self-complementary graph G, the number of vertices n is of the form 4k or $4k + 1, k \geq 1$. Proof: Let G be a (n, m) graph. Number of edges in $Kn = \frac{n(n-1)}{2} = {}^nC_2$ Since G is self-complementary, number of edges in G = number of edges in G = m. Number of edges in Kn = number of edges in G + number of edges in G. Number of edges in G = $\frac{n(n-1)}{2} - m$	***2***

	$$m = \frac{n(n-1)}{2} - m,$$ $$4m = n(n-1)$$ Therefore, m $= \frac{n(n-1)}{4}$ $\Rightarrow \frac{4}{n} \ \ or \frac{4}{n-1}$ $p = 4k \, or \, n - 1 = 4k$ $p = 4k \, or \, p = 4k + 1$	