Determinant

A spanning subgraph H of G is an elementary spanning subgraph is each component of H is either K_2 or a cycle.

Theorem: Let G be a labelled graph on n vertices with adjacency matrix $A(G)$. Then,

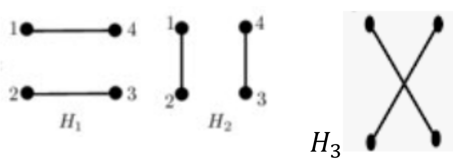
$$\det\big(A(G)\big) = \sum(-1)^{n-c_1(H)-c(H)}\, 2^{c(H)}$$

Where the summation is over all spanning subgraph H of G and $c(H), c_1(H)$ are the number of components in H which are cycles and $K_2's$ respectively.

Example: K_4

Observe that K_4 has 3 elementary spanning subgraphs

K4



H_3

For H_1, H_2, H_3 $c(H_1) = c(H_2) = c(H_3) = 2, c(H_1) = c(H_2) = c(H_3) = 0$

For H_4, H_5, H_6 $c(H_4) = c(H_5) = c(H_6) = 0, c(H_4) = c(H_5) = c(H_6) = 1$

$$\det(k_4) = 3(-1)^{4-2-0}2^0 + 3(-1)^{4-0-1}2^1 = -3$$

Determinant of P_n: When n is odd, there are no elementary spanning subgraphs possible. Hence $\det\big(A(P_n)\big) = 0$

When n is even: there is one elementary spanning subgraphs possible which is union of $\frac{n}{2}$ - $K_2's$

Hence, $\det\big(A(P_n)\big) = (-1)^{n-\frac{n}{2}}$, which is 1 or -1 depending upon n is a multiple of 4 or not.

$$\det(A(P_n)) = \begin{cases} 0; & n \text{ is odd} \\ 1; & n \text{ is a multiple of } 4 \\ -1; & \text{otherwise} \end{cases}$$

Determinant of C_n: When n is odd, there is only one elementary spanning subgraph possible which is C_n itself. Hence $\det(A(C_n)) = (-1)^{n-1}2^1 = 2$

When n is even: there are 3 elementary spanning subgraphs possible.

Two are union of $\frac{n}{2}$ - $K_2's$

And one is C_n itself.

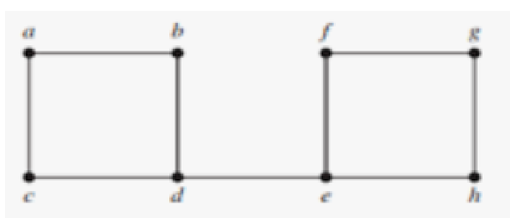
Hence, $\det(A(C_n)) = 2(-1)^{n-\frac{n}{2}} + (-1)^{n-1}2^1 = 2(-1)^{n-\frac{n}{2}} - 2$, which is 0 or -4 depending upon n is a multiple of 4 or not.

$$\det(A(C_n)) = \begin{cases} 2; & n \text{ is odd} \\ 0; & n \text{ is a multiple of } 4 \\ -4; & \text{otherwise} \end{cases}$$

Obtain $\det\left(A(K_{p,q})\right),\ p \neq q.$

As there are no elementary spanning subgraphs possible, $\det\left(A(K_{p,q})\right) = 0.$

Exercise: 1. Obtain determinant of the following graph by computing all its elementary spanning subgraphs.



2. Obtain determinant of $A(K_{2,2})$

Eigenvalues : Eigenvalues of a graph are the eigenvalues of its adjacency matrix.

The roots of the characteristic equation $|A(G) - \lambda I| = 0$ are the eigenvalues.

The vector $X \neq 0$, satisfying $AX = \lambda X$ is the corresponding eigenvector.

Theorem: If G is regular graph with regularity r, then r is an eigenvalue of A(G) with eigenvector $X = [1, 1, \dots, 1]^T$.

Proof: Let $X = [1, 1, \dots, 1]^T$.

As $A(G)X = rX,$ implies r is a n eigenvalue of A(G) with eigenvector $X = [1, 1, \dots, 1]^T$.

Theorem: The eigenvalues of $A(K_n)$ are $(n-1)$ with multiplicity 1 and (-1) With multiplicity n-1.

Proof:

$$|A(K_n) - \lambda I| = \begin{bmatrix} -\lambda & 1 & 1 & \dots & 1 & 1 \\ 1 & -\lambda & 1 & \dots & \dots & 1 \\ 1 & 1 & . & . & . & . \\ . & . & . & . & . & . \\ 1 & . & . & . & -\lambda & 1 \\ 1 & 1 & .. & .. & 1 & \lambda \end{bmatrix}$$

By performing $R_i : R_i - R_{i+1}, 1 \leq i \leq n-1$ we get

$$|A(K_n) - \lambda I| = \begin{vmatrix} -\lambda - 1 & \lambda + 1 & 0 & \dots & 0 & 0 \\ 0 & -\lambda - 1 & \lambda + 1 & \dots & \dots & 0 \\ 0 & 0 & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & \lambda + 1 & 0 \\ 1 & 1 & .. & .. & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$(\lambda + 1)^{n-1} \begin{vmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & \dots & 0 \\ 0 & 0 & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 1 & 0 \\ 1 & 1 & .. & .. & \lambda - 1 & \lambda + 1 \end{vmatrix}$$

From the above its clear that (-1) is a n eigenvalue with multiplicity (n-1).

As trace(A(G))=0=sum of eigenvalues of A(G)

$\Rightarrow n - 1$ is the eigenvalue of A(G).

As G is regular with regularity n-1, we can conclude the result.