GRAPH THEORY

Graph theory originated from the Konigsberg Bridge Problem where two islands linked to each other and the banks of the Pregel River by seven bridges.

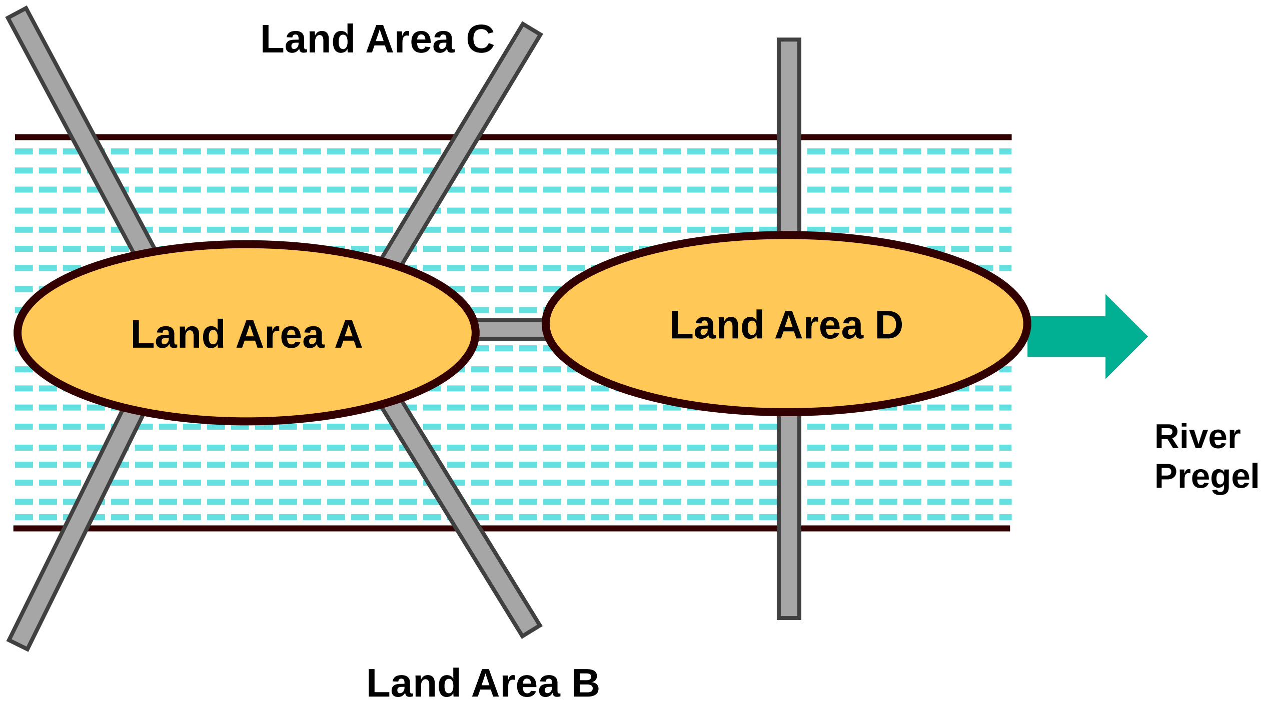


Figure 1: Konigsberg Bridge Problem

The problem was, to begin at any of the four land areas, (Land area A, B, C, D as shown in Figure 1) walk across each bridge exactly once and return to the initial land area. This problem was solved in 1736 by Euler. The Graph theory was first introduced by Euler. Mathematician L. Euler solved the Konigsberg Bridge problem. His theorem is known as the first theorem in Graph theory.

Graph theory has lot of applications in Making a timetable, Register allocation, Google maps, Facebook/LinkedIn, WWW, GPS etc.

Graph

A graph G consists of a finite nonempty set $V(G)$ of n elements called vertices (or points) together with a prescribed set $E(G)$ of m unordered pairs of different elements of $V(G)$(called as edges). We use the notations V and E for $V(G)$ and $E(G)$, respectively. Each pair $e = \{u, v\}$ of elements in $E(G)$ is an edge (or line) of G, and e is said to join u and v. Here, vertex u and edge e are said to be incident with each other, as v and e are. We say that u and v are adjacent vertices. A graph with n vertices and m edges is referred to as an (n, m) graph.

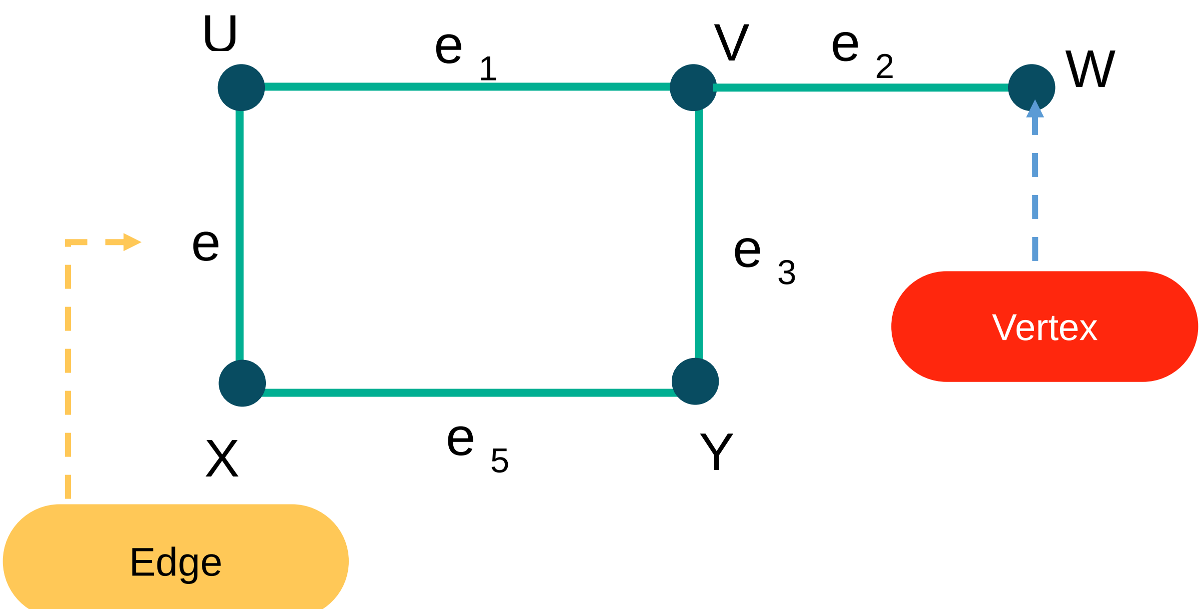


Figure 2: A Graph with 5 Vertices and 5 Edges

Each vertex is represented by a small dot, and each edge is represented by a line segment joining the two vertices with which the edge is incident. In figure 2, $\mathrm{u, v, w, x, y}$ are vertices and e_1, e_2, e_3, e_4, e_5 are edges. In figure 2, vertex u and vertex v are adjacent.

If two distinct edges e and f are incident with a common vertex, then they are said to be adjacent edges. In figure 2, edges e_1 and e_3 adjacent but e_3 and e_4 are not adjacent edges.

An edge with identical ends is called a loop and two edges with the same end vertices are called parallel (multiple) edges. In Figure 3, edge e_3 is a loop and edges e_2 and e_5 are parallel. A graph is simple if it contains neither loops nor parallel edges. In a multigraph, no loops are allowed but parallel edges are permitted. If both loops and parallel edges are permitted, then we have a pseudo graph.

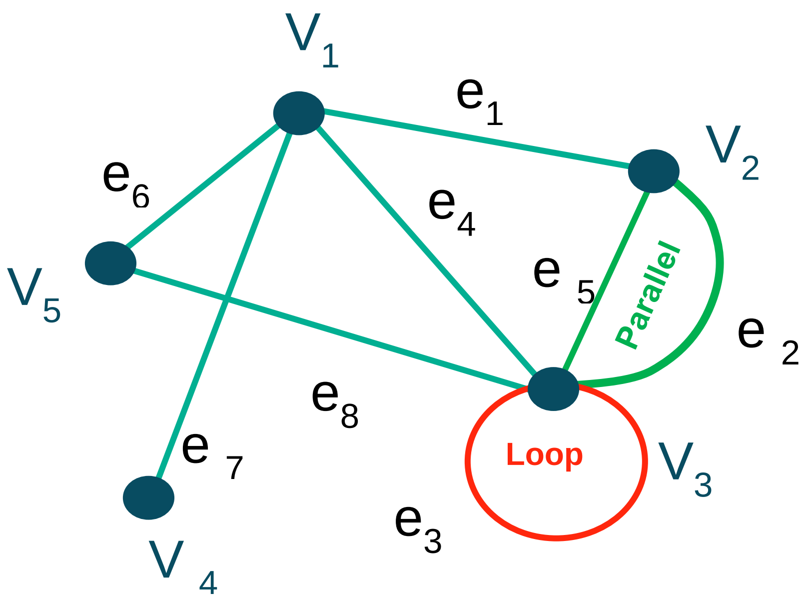


Figure 3: Pseudo Graph

The number of edges incident with v is the **degree** of a vertex v in a graph G. The degree of v is represented by $\deg(v)$.

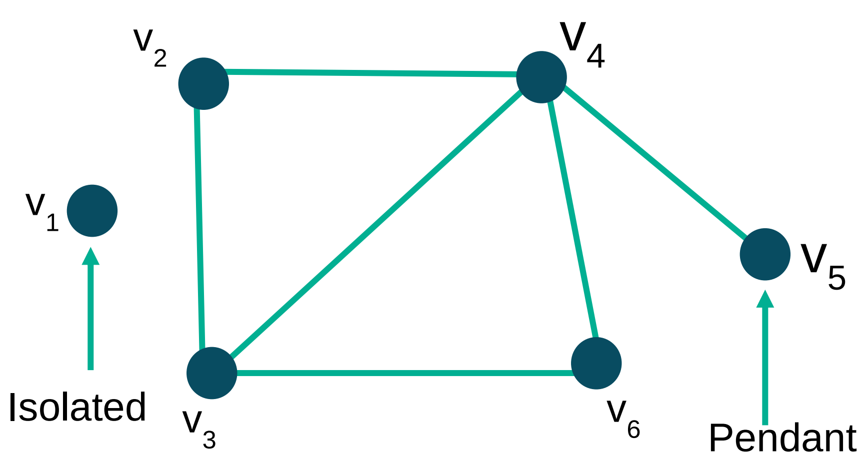


Figure 4: A (5,6) Graph

In Figure 4, $\deg(v_1) = 0$, $\deg(v_2) = 2$, $\deg(v_3) = 3$.

A vertex v of degree 0 is termed an isolated vertex. A vertex v of degree 1 is termed a pendant vertex. The minimum degree among the vertices of G is indicated by $\delta(G)$. The maximum degree among the vertices of G is indicated by $\Delta(G)$.

In Figure 4, v_4 is a vertex with a maximum degree and v_1 is a vertex with a minimum degree. Hence for the above graph $\delta(G) = 0$ and $\Delta(G) = 4$. Vertex v_1 is isolated and vertex v_5 is pendant for the graph as shown in Figure 4.

A graph is finite when both its vertex set, and edge set are finite. The (1, 0) graph is trivial, i.e., a graph with a single vertex and no edge is called trivial. A graph whose edge set is empty, is termed a null graph or a totally disconnected graph.

For a graph G, the number of elements in $V(G)$ is called order of the graph G and denoted by $|V(G)|$ and the number of elements in $E(G)$ is called the size of the graph G denoted by $|E(G)|$.

NOTE: While drawing a graph, it is immaterial whether the edges are drawn straight or curved, long or short. Thus, a diagram of the graph depicts the incidence relation holding between its vertices and edges.

Relation between Number of Edges and Degree

Theorem 1: The sum of the degrees of the vertices of a graph G is twice the number of edges.
Proof: Every edge of G is incident with two vertices. Hence every edge contributes 2 to the sum of the degrees of the vertices. Hence the result follows.

Question: If possible, draw a graph with degree sequence (1,1,2,2,3). If not, justify your answer.

Theorem 2: In any graph G, the number of vertices of odd degrees is even.

Proof: We know $\sum_{v \in V} \deg(v) =$ twice the number of edges. The sum of degrees of the vertices of odd degree (S_O) and even degree (S_E)equals the sum of the degrees of all the vertices of G which is twice the number of edges (an even number).

Thus, the sum of the degrees of the vertices of odd degree (S_O) must be an even number and hence the number of such vertices must be even.

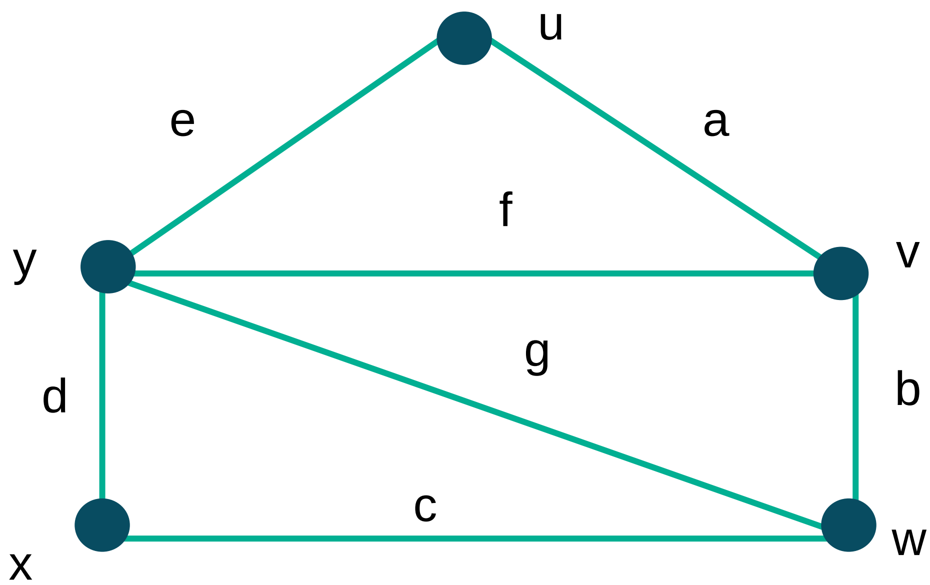


Figure 5: A (5,7) Graph

For the graph in Figure 5,

$$2|E(G)|=\sum_{v \in V} deg(v) = 14.$$

Sum of degrees=2+4+2+3+3=14. Vertices with odd degrees in the graph as shown in figure 5 are vertex v and w.

Question: If possible, draw a graph with degree sequence (0,1,2,3,4). If it is not possible, give the reason.

Perfect Graph: A graph is perfect if no two vertices are of the same degree.

Theorem 3: No graph is perfect or in a (n, m) graph G, there exists at least 2 vertices with same degree.

We note that in a (n, m) graph, $0 \leq \deg(v) \leq n-1$. Suppose there is a vertex with degree 0 in G, then there cannot exist a vertex with degree n-1 and vice versa. Hence n vertices will have n-1 choices. Hence the result follows.

Subgraph

A graph $H = (V_1, E_1)$ is called a subgraph of $G = (V, E)$, if $V_1 \subseteq V$ and $E_1 \subseteq E$.

A subgraph H is called a spanning subgraph of G, if $V_1 = V$.
The graph H is called an induced subgraph of G, if H is the maximal subgraph of G with vertex set V_1. Thus, if H is an induced subgraph of G, two vertices are adjacent in H, if and only if, they are adjacent in G.
For the graph as shown in Figure 6, the graphs in Figure 7, 8 are subgraphs.

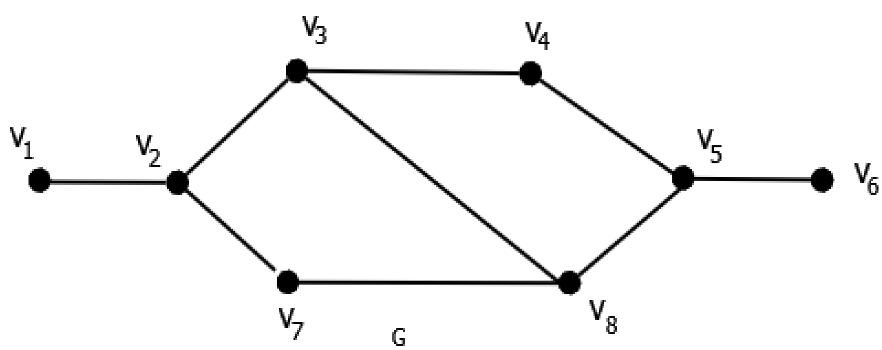


Figure 6: A Graph G

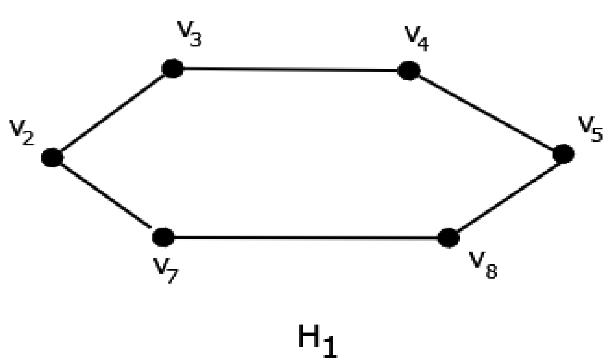


Figure 7: Graph H_1 is a Subgraph of G

The graph in Figure 8 is a spanning subgraph of G.

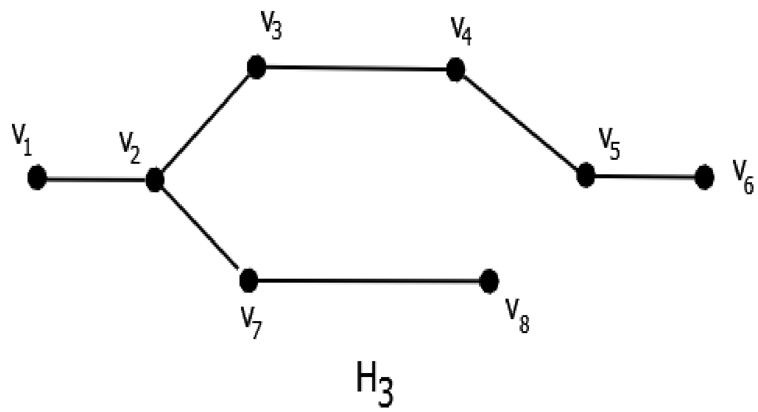


Figure 8: H_3 Spanning Subgraph of G

Figure 9 shows a graph G, induced subgraph H_1 induced by the set $V_1 = \{A, G, J, D, C\}$ and a spanning subgraph H_2.

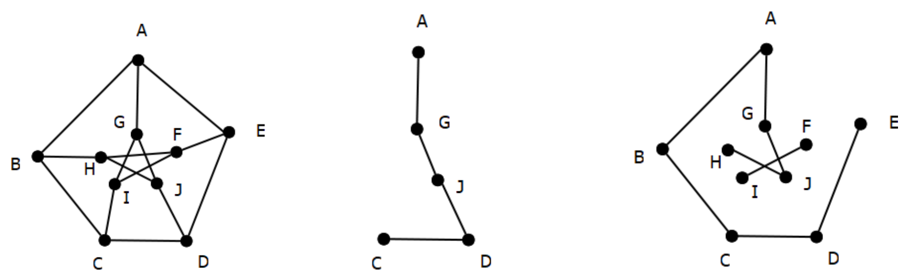


Figure 9: Graphs G, H_1, H_2

Removal of Vertices and Removal of Edges

The removal of a vertex v from a graph *G* results in that subgraph $G - v$ of *G* consisting of all vertices of *G* except v and all edges not incident with v (Figure 10).

The removal of an edge *e* from *G* yields the spanning subgraph *G* – *e* containing all edges of *G* except *e* (Figure 10)

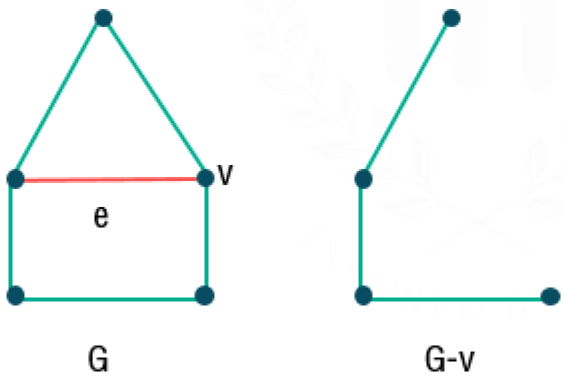


Figure 10: Graph G, G-v, G-e

Similarly, removal of vertices and removal of edges can be defined. Figure 11 shows Graph G along with $G - \{b, g, e\}$ and $G - \{v_7, v_4\}$.

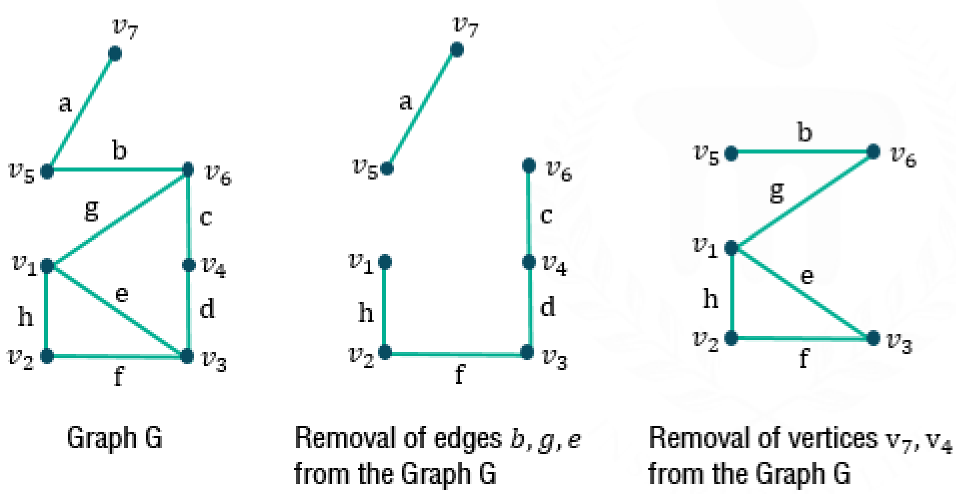


Figure 11: Graph G, $G = \{b, g, e\}, G - \{v_4, v_7\}$

Types of Graphs

Regular Graph

A graph G in which all the vertices have the same degree is called a regular graph of degree r.

Figure 12 represents a regular graph of degree 2.

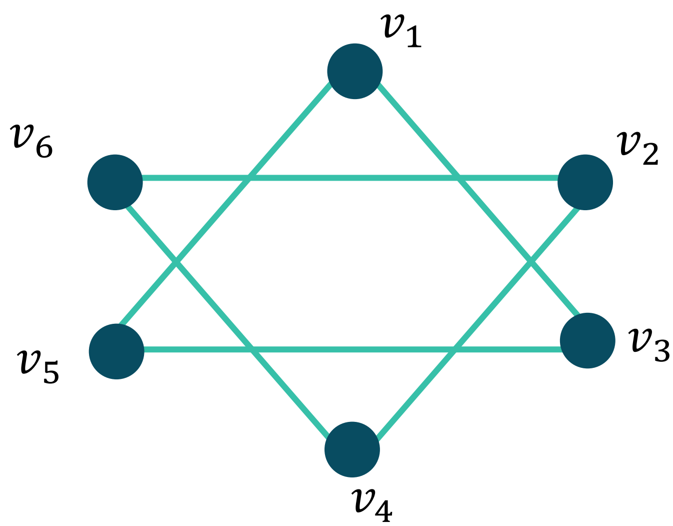


Figure 12: A Regular Graph of Degree 2

A regular graph of degree 3 is also known as a cubic graph.

Question: Construct a cubic graph (if possible) (i) on 5 vertices (ii) 6 vertices. If that is not possible then give a reason.

Complete Graph

A graph in which any two distinct vertices are adjacent is called a **complet**e graph. The complete graph with n vertices is denoted by K_n. Figure 13 represents a Complete graph on 6 vertices.

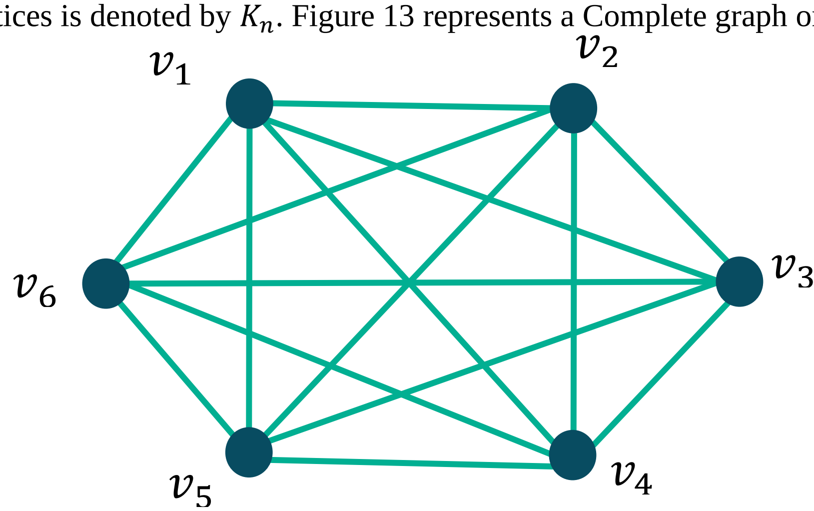


Figure 13: A Complete Graph on 6 Vertices Denoted by K_6

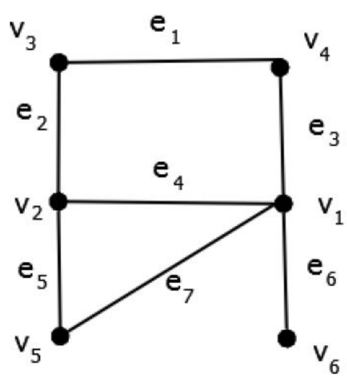
K_3 is also known as a triangle.

Question: Draw a regular graph on regularity 4 and number of vertices is 6.

Question: Draw a complete graph on 7 vertices.

A **labelled graph** is a graph in which every vertex and every edge is labelled. We consider a graph always as a labelled graph.

Question 1: Draw a labelled graph G having (4, 3, 2, 2, 2, 1) as degree sequence.



Question 2: Draw graph G having (2, 2, 2, 2, 2, 2) as degree sequence. Is there more than one graph on a given degree sequence?

Question 3: Draw graph G having (4, 4, 3, 3, 2) as degree sequence. How many edges can this graph contain?

Weighted Graph

A weighted graph is a graph consisting of weight, or a number associated with each edge.

Application: Suppose that we have to connect n cities $v_1, v_2, \ldots, v_n$ through a network of roads, given that c_{ij} is the cost of building a direct road between the cities v_i and v_j. The problem is finding the least expensive network that connects all the cities. A weighted graph G is as shown in Figure 14.

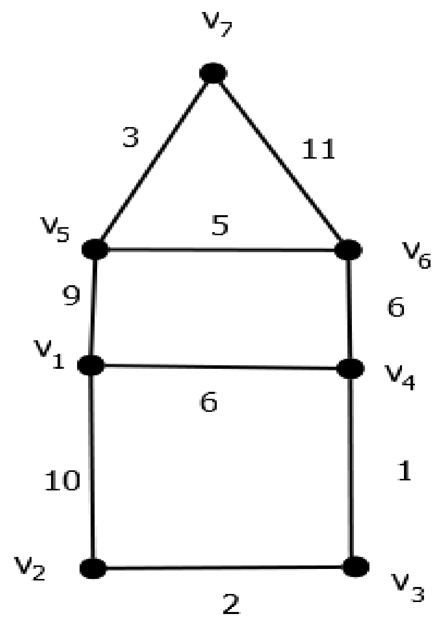


Figure 14: A Weighted Graph

Directed Graph

A directed graph (or digraph) is a triplet containing a vertex set $V(G)$, an edge set $E(G)$ and function assigning each edge, an ordered pair of vertices (Figure 15).

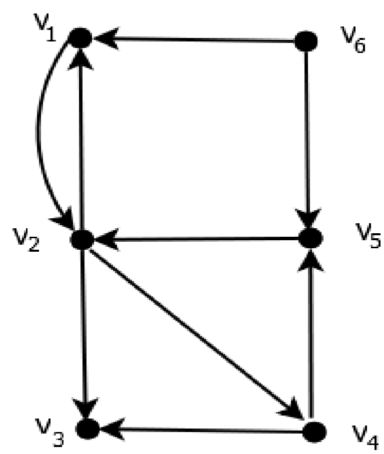


Figure 15: Directed Graph G

For the graph in Figure 15, there is a directed edge from vertex v_2 to v_4. But to travel from $v_4$to v_2, we have to travel via v_5.

Application: Given a list of cities and the distances between each pair of them.

We need to find the shortest probable route to visit each city exactly once and return to the original city. For the graph in Figure 16, to travel from B to E instead of a direct road (distance 10), if we travel from C, we will reach E with minimum distance (5+3).

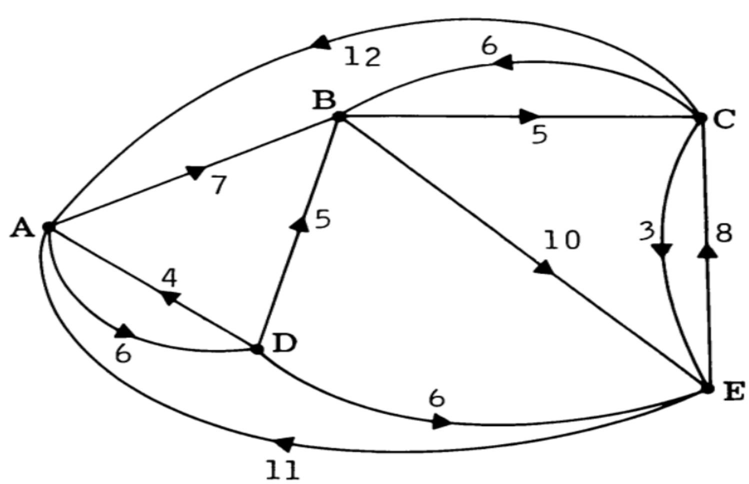


Figure 16: Weighed Directed Graph

Complement of a graph

Let $G = (V, E)$ be a Graph. The complement $\bar{G}$ of G is defined to be the graph which has V as its set of vertices and two vertices are adjacent in $\bar{G}$ only when they are not adjacent in G

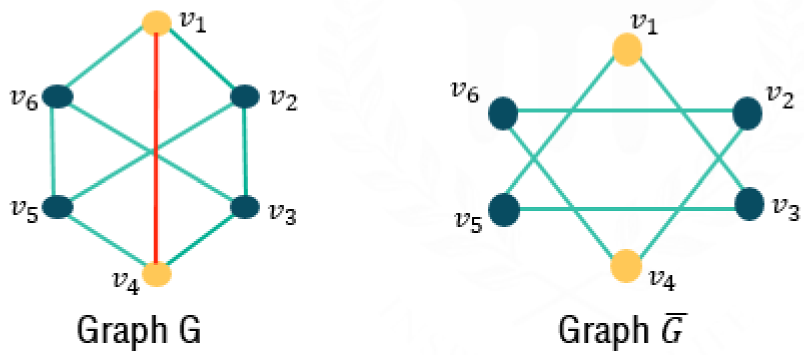
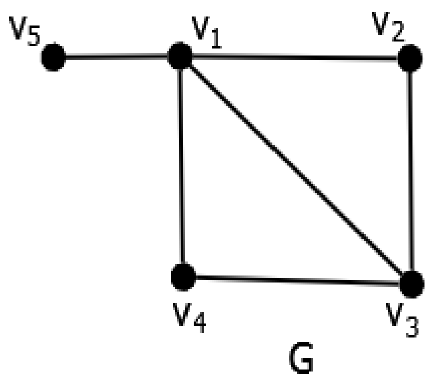


Figure 19: Graph G and its complement $\bar{G}$

In graph G (Figure 19) there is an edge from v_1 to v_4, but in $\bar{G}$, the edge between v_1 to v_4 is not there. $\bar{G}$ is disconnected graph with 2 components.

Question: Find the complement of the graph G.



Question: The complete graph K_p has——————edges.

Question : Draw a regular graph on 6 vertices with regularity 1.

1. The number of edges in a complete graph on 7 vertices is ____________.
2. State True or False: There exists a regular graph of degree 3 with 5 vertices. Justify your answer.
3. Let G be a simple graph with 5 vertices and degrees of these 5 vertices be 2,3,3,4,4. Then number of edges in G = ____________.
4. State True or False: There exists a simple graph with degrees of its 5 vertices be 1,2,3,4,4.
5. The number of edges in a regular graph of degree 3 with 6 vertices is ____________.
6. Draw a complete graph on 5 vertices