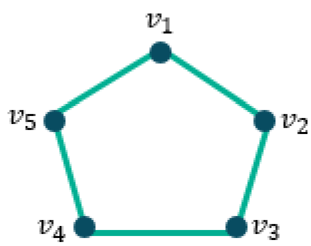
1. Adjacency Matrix

The adjacency matrix $A(G)$ of a simple (n, m) graph G is a $n \times n$ matrix whose rows and columns are indexed by $V(G)$ such that:

$$(A(G))_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ and vertex } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

We often denote $A(G)$ simply by A.

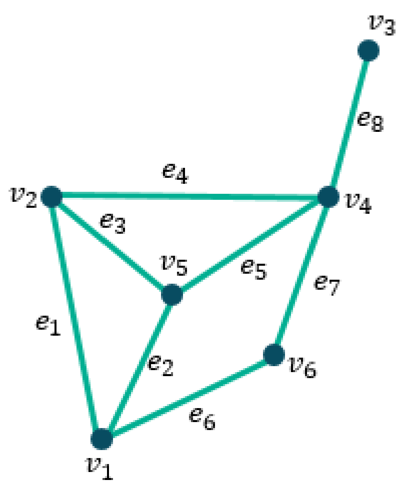
A graph G along with its adjacency matrix $A(G)$ is given in figure 1 and 2.



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Figure 1: Graph G and its Adjacency Matrix

For a simple graph, the adjacency matrix $A(G)$ is a 0-1 symmetric matrix with diagonal entry equal to zero.



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2: Graph G and its Adjacency matrix

The degree of a vertex equals the number of 1’s in the corresponding row or column of $A(G)$.

1.1 Properties of Adjacency Matrix

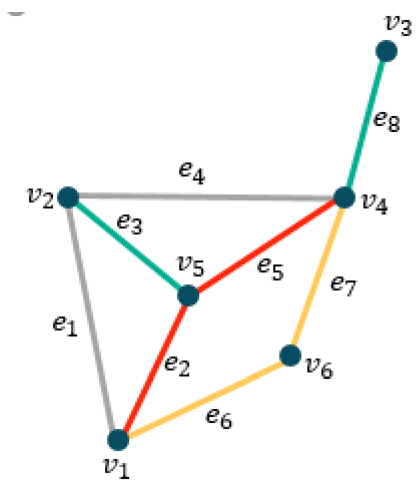
Theorem 1.1: Let G be a labelled graph with adjacency matrix A. Then the $i,\ j$ entry of A^n is the number of walks of length n from v_i to v_j.

Let G be a labelled graph with adjacency matrix A.

The $i,\ j$ entry of A^2 is the number of walks of length 2 from v_i to v_j. The $i,\ i$ entry of A^2 is the degree of v_i.

For the graph G as shown in Figure 3, we note the following.

- The 1,1 entry is 3, i.e., $\deg(v_1) = 3$.
- The 3,3 entry is 1, i.e., $\deg(v_3) = 1$.
- The 1,4 entry of A^2 is 3. The number of paths of length 2 from v_1 to v_4 are $\{e_2, e_5\}, \{e_6, e_7\}, \{e_1, e_4\}$.
- The 3,5 entry of $A^2$is 1. The number of paths of length 2 from v_3 to v_5 is $\{e_8, e_5\}$.



$$A^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Figure 3: Graph G and the matrix A^2

The $i,\ i^{th}$ entry of A^3 is twice the number of triangles containing v_i.

$$A^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}$$

- The 1,5 entry of $A^3$is 7.
- The 7 different edge sequences of three edges between v_1 and v_5 are: $\{e_1, e_1, e_2\}$, $\{e_2, e_2, e_2\}$, $\{e_6, e_6, e_2\}, \{e_2, e_3, e_3\}, \{e_6, e_7, e_5\}, \{e_2, e_5, e_5\}$, $\{e_1, e_4, e_5\}$.

- The 5,5 entry of A^3 is 4. Hence, there are 2 triangles containing the vertex v_5.

Theorem 1.2: The sum of the 2×2 minors of A equals −|E(G)|.

Proof: For i≠ j the principal submatrix of A formed by the rows and columns i, j is the zero matrix if $i \nsim j$ and otherwise it equals $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The determinant of this matrix is −1. Thus the absolute value of the sum of all 2×2 principal minors of A is the number of edges in G in magnitude.

Theorem 1.3: The sum of the 3×3 principal minors of A equals twice the number of triangles in G.

Proof: Consider three distinct vertices $i, j, k \in V(G)$. The pricipal submatrix formed by the rows and columns i, j, k will be non-singular if and only if they are adjacent to each other. In which case the submatrix formed these rows and columns is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, whose determinant is 2.

Suppose i ≁ j then the submatrix formed by the rows and columns i, j, k is $\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix}$, where a, b are 1 or 0 depending on i ~ k, j ~ k or not, whose determinant is zero contradicting to non singularity. Hence, two times the number of cycles of length 3 equals the sum of all 3×3 principal submatrices of A(G).

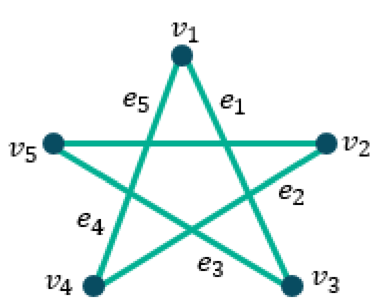
2. Incidence Matrix

For a graph G with $V(G) = \{v_1, v_2, . . . , v_n\}$ and $E(G) = \{e_1, e_2, . . . , e_m\}$ the (vertex- edge) incidence matrix of G, which is denoted by $B(G)$ is the $n \times m$ matrix defined as follows:

The $(i, j)^{th}$– entry of $B(G)$ is 0 if vertex v_i and edge e_j are not incident, and otherwise $(i, j)^{th}$– entry of B(G) is 1.

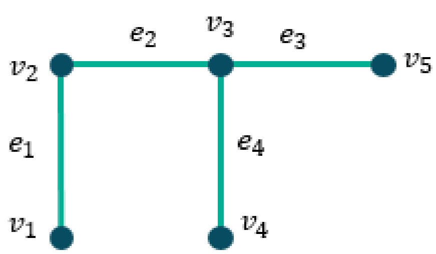
This is often referred to as the (0, 1)− incidence matrix.

The graph G and its incidence matrix $B(G)$ is shown in Figure 4,5.



$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Figure 4: Graph G and its $B(G)$



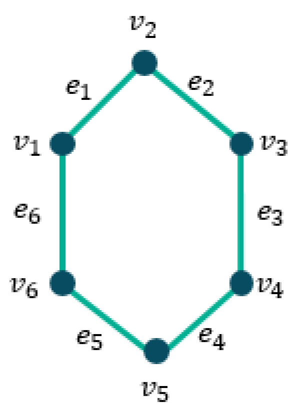
$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Figure 5: Graph G and its $B(G)$

2.1 Properties of Incidence Matrix:

Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Theorem 2.1: Let C_n be the cycle on the vertices $\{v_1, v_2. . . , v_n\}$, $n \geq 3$, and let B(G) be its incidence matrix." Then det $B(G)$ equals 0 if n is even and 2 if n is odd.



$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 6: Graph G and $B(G)$

Det (B(G))=0 for the graph G as shown in the Figure 6.

3. Distance Matrix

Let G be a connected graph with $V(G) = \{v_1, v_2 . . . , v_n\}$.

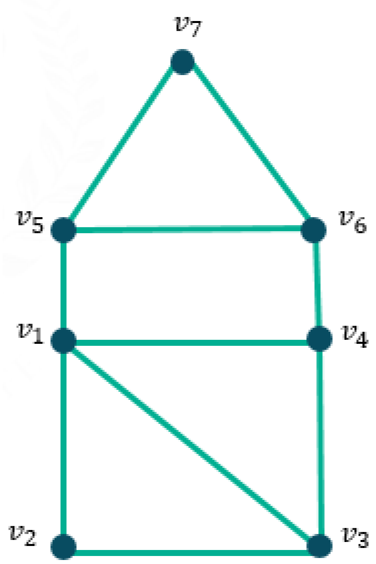
The distance matrix $D(G)$ of G is an $n \times n$ matrix with its rows and columns indexed by $V(G)$.

For $i \neq j$, the (i, j) —entry d_{ij} of G is equal to $d(v_i, v_j)$.

Also, $d_{ii} = 0,\ i = 1, \ldots, n$. Here $d(v_i, v_j)$ is the distance between the vertices v_i and v_j of G (is the length of a shortest path from v_i to v_j).

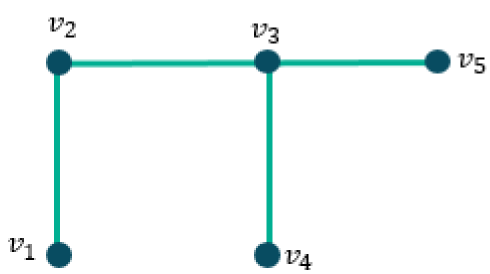
$D(G)$ can be denoted simply by D.

It is a symmetric matrix with zeros on the diagonal.



$$D(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 0 & 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 0 & 2 & 1 & 2 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 3 & 2 & 1 & 1 & 0 & 1 \\ 2 & 3 & 3 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Figure 7: Graph G and its distance matrix $D(G)$

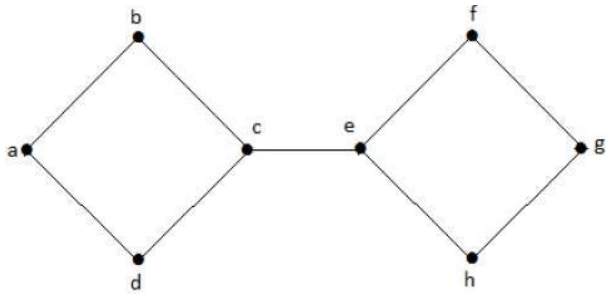


$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 & 3 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Figure 8: Graph G and its distance matrix

Questions:

1. For the graph in the figure below find (i) Adjacency Matrix (ii) Incidence Matrix and (iii) Distance Matrix.



2. For the graph in the figure below find (i) Adjacency Matrix (ii) Incidence Matrix and (iii) Distance Matrix.