Trees

A **tree** is a connected graph without any cycles. All the trees with **6 vertices** are given in Figure 1.

Figure 1: Trees on 6 vertices

Trees with n=1,2,3,4 are given in Figure 2.

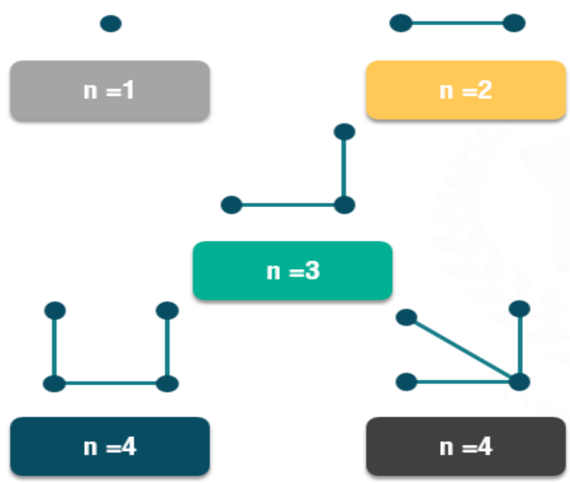


Figure 2: Trees with less than or equal to 4 vertices.

Theorem : A graph G is a tree if and only if between every pair of vertices there exists a unique path.

Proof: Let G be a tree then G is connected. Hence, there exist at least one path between every pair of vertices. Suppose that between two vertices say u and v, there are two distinct paths then union of these two paths will contain a cycle; a contradiction. Thus, if G is a tree, there is at most one path joining any two vertices.
Conversely, suppose that there is a unique path between every pair of vertices in G. Then G is connected. A cycle in the graph implies that there is at least one pair of vertices u and v such that there are two distinct paths between u and v. Which is not possible because of our hypothesis. Hence, G is acyclic and therefore it is a tree.

Theorem : A tree with n vertices has n − 1 edges.
Proof: The proof is by induction on the number of vertices.
If $n = 1$, we get a tree with one vertex and no edge. If $n = 2$, we get a tree with two vertices and one edge. If $n = 3$, we get a tree with three vertices and two edges. Assume that the statement is true with all trees with k vertices $(k < n)$. Let G be a tree with n vertices. Since G is a tree there exist a unique path between every pair of vertices in G. Thus, removal of an edge e from G will disconnect the graph G. Further, $G - e$ consists of exactly two components with number of vertices say r and s with $r + s = n$. Each component is again a tree. By induction, the component with r vertices
has $r - 1$ edges and the component with s vertices has $s - 1$ edges. Thus, the number of edges in G $= r - 1 + s - 1 + 1 = r + s - 1 = n - 1$.

Centers of tree

Theorem : Every tree has a center consisting of either one vertex or two adjacent vertices.
Proof: The result is obvious for the trees K_1 and K_2. We show that any other tree T has the same central vertices as the tree T_1 obtained by removing all end vertices of T. Clearly, the maximum of the distances from a given vertex u of T to any other vertex v of T will occur only when v is an end vertex. Thus, the eccentricity of each vertex in T_1 will be exactly one less than the eccentricity of the same vertex in T. Hence, the vertices of T which possess minimum eccentricity in T are the same vertices having minimum eccentricity in T_1. That is, T and T_1 have the same center. If the process of removing end vertices is repeated, we obtain successive trees having the same center as T. Since T is finite, we eventually obtain a tree which is either K_1 or K_2. In either case all vertices of this ultimate tree constitute the center of T which consists of just a single vertex or of two adjacent vertices.

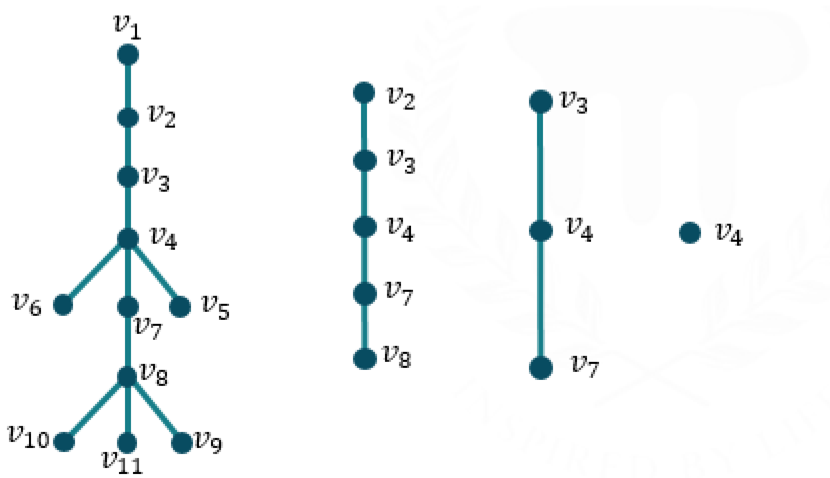


Figure 3: A tree with center v_4

Applying Theorem, we get the center of the tree in Figure 3 is vertex v_4.

A graph is a tree if and only if it is minimally connected i.e., removal of any edge disconnects the graph.
Theorem: Every tree has at least two pendant vertices.

Spanning tree

Definition: A tree T is said to be a **spanning tree** of a connected graph G if T is subgraph of G and T contains all the vertices of G.
In other words, spanning tree in a graph G is a minimal subgraph connecting all the vertices of G.
Every connected graph has a spanning tree.

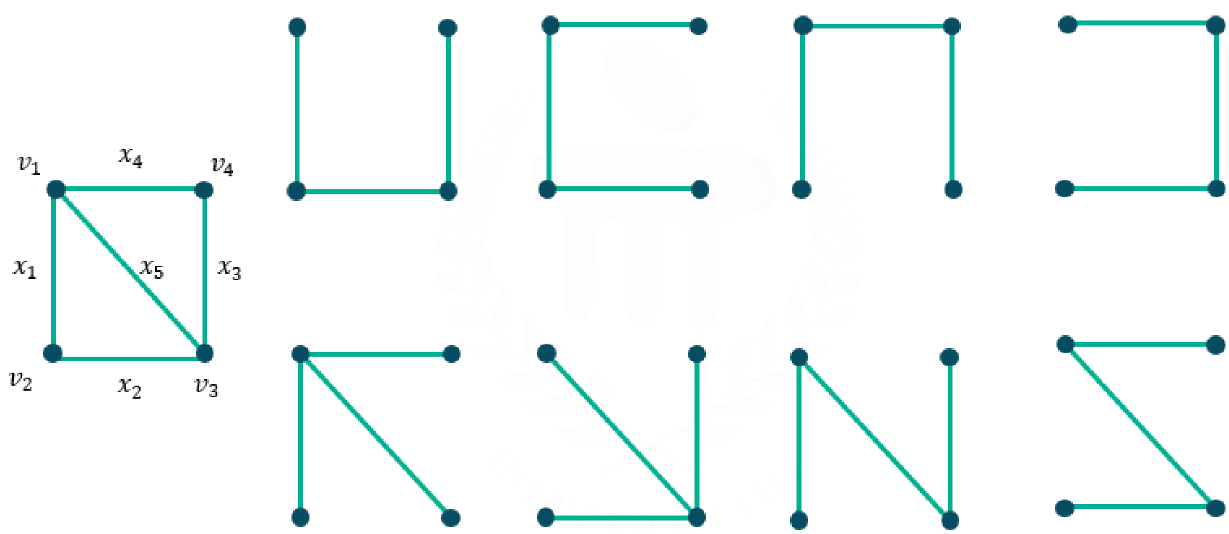


Figure 4: Graph G and its spanning tree

Theorem: Every connected graph has at least one spanning tree.

Proof: If the graph G is a tree, then G itself is a spanning tree.
If G is not a tree, then there exists at least one cycle. Remove one edge from that cycle. Let the resulting graph be G'. Suppose G' is a tree, then G' is a spanning tree. If not, it contains another cycle. Again, removing one edge from G'. Continuing like this we get a graph without any cycles. That graph is a spanning tree for graph G.

Minimal Spanning Tree

If a graph G is a weighted graph, then the weight of a spanning tree T is the sum of the weights of all the edges in T.
In general, different spanning trees of G will have different weights. Among all spanning trees of G the one with the smallest weight is of practical significance
A spanning tree with the smallest weight in a weighted graph is called a **shortest spanning tree** or **shortest-distance spanning tree** or **minimal spanning tree**.

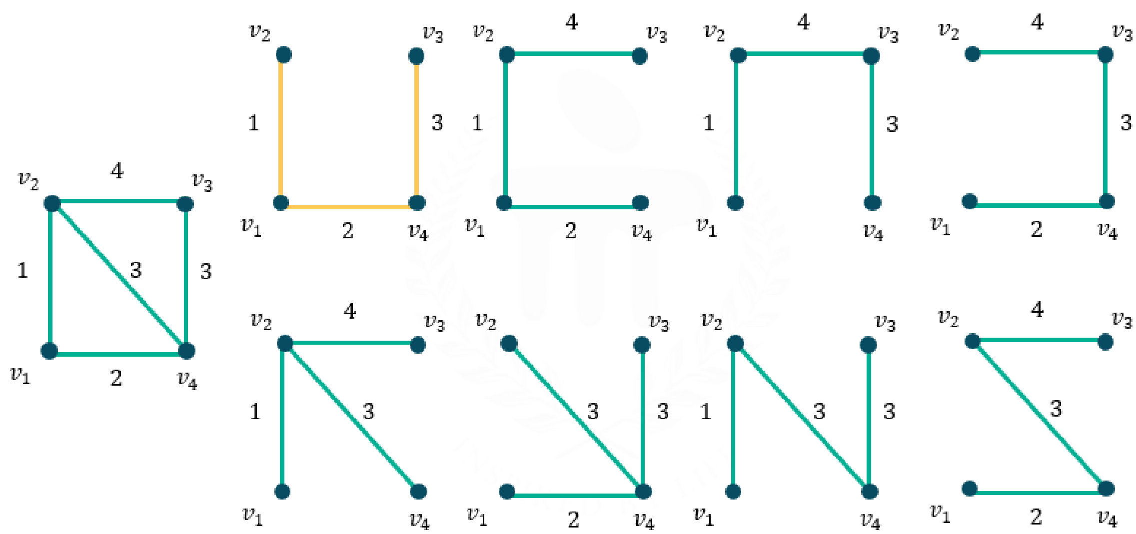


Figure 7: Minimal spanning tree with weight 6

For a given graph, such a spanning tree may not be unique. For example, if G is a weighted graph with n vertices in which every edge has unit weight, then all the spanning trees have a weight of n−1 units.

A practical application of finding the shortest spanning tree is the following: Suppose that we have to connect n cities $v_1, v_2, \dots, v_n$ through a network of roads, given that c_{ij} is the cost of building a direct road between the cities v_i and v_j.

Then the problem of finding the least expensive network that connects all the cities is nothing but the problem of finding a shortest spanning tree.

Graph G and its minimal spanning tree T with weight 20 is given in Figure 8.

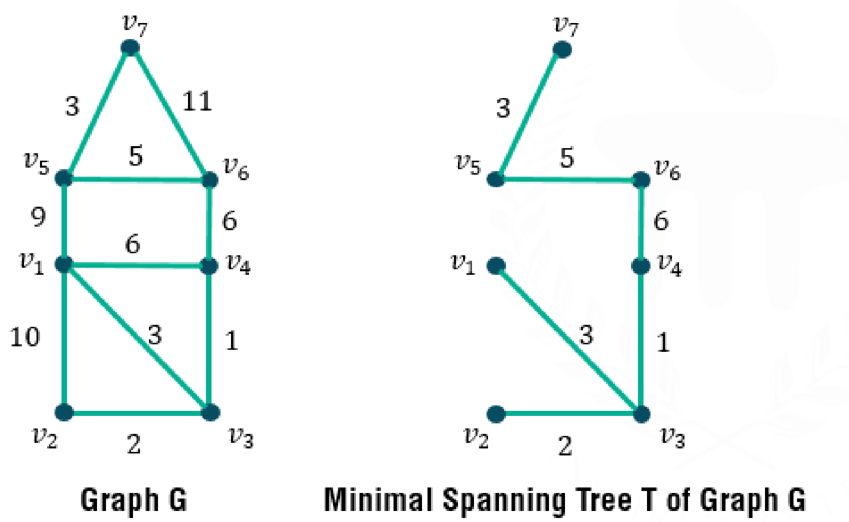
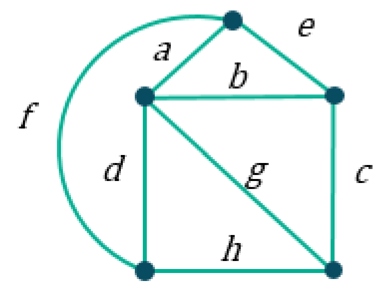


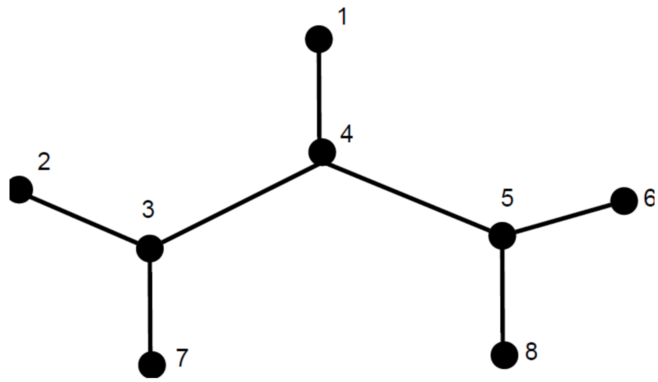
Figure 8: Graph G and its minimal spanning tree T

Questions:

1. For the graph in the figure below find any 5 spanning trees.



2. The Centre of the tree given below is ---.



3. Draw a tree with 6 vertices whose degrees are 5,1,1,1,1,1.
4. The number of edges in any tree with 1000 vertices is ---.
5. Suppose T is a tree with 6 vertices and degrees of its four vertices are 2,2,2,2. Then the remaining vertices will have degree----.
6. If G is a weighed graph on 10 vertices and 15 edges with every edge having weight=3, then the weight of the minimal spanning tree is---
7. Find the number of spanning cycles in $K_{3,3}$.
8. Find the number of spanning cycles in $K_{4,3}$.
9. The number of spanning cycles in $K_4$is ---
10. State true or false: $K_{p,p+1}$ is Eulerian?
11. How many vertices are there in the center of $K_{1,9}$?

Cut Vertex/Bridge: A cut vertex of a graph is one whose removal increases the number of components and bridge is such an edge.
In a tree every edge is a bridge, and every vertex (other than the pendant vertex) is a cut vertex.
In cycle graph C_n, and a complete graph K_n, there is no cut vertex and no bridge.
Note that in a graph there may or may not exist a cut vertex or bridge.

Vertex Connectivity/ Edge connectivity:

Vertex Connectivity $\boldsymbol{k(G)}$, is the minimum number of vertices that need to be removed such that G becomes disconnected or a trivial graph (K_1).
Edge connectivity $\boldsymbol{\lambda(G)}$, is the minimum number of edges that need to be removed such that graph G becomes disconnected or trivial graph (K_1).
A disconnected graph G has $\lambda(G)$=0, and a connected graph G with at least one edge has $\lambda(G) \geq$ 1.

1. $\lambda(C_n) = 2 = k(C_n)$
2. $\lambda(P_n) = 1 = k(P_n)$

Theorem : For any graph G, . $k(G) \leq \lambda(G) \leq \delta(G)$.

Question:
Draw a graph with $k(G) = \lambda(G) = \delta(G) = 1$
Draw a graph with $k(G) = 1, \lambda(G) = 2 \; and \; \delta(G) = 3$
Draw a graph with $k(G) = \lambda(G) = \ \delta(G) = 3$