Agenda

Modular Arithmetic Intro

Count pairs whose sum mod m is 0

GCD Intro

Properties of GCD

Delete one

$x \% 5 =$

Modulo (%)

$A \% M$ = Remainder when A is divided by M

Range of $A \% M \rightarrow [0, M-1]$

Why do we need Mod?

Limit the range of the data

$$-\infty \ldots \infty \quad \% M = \quad 0 \ldots M-1$$

Int $\rightarrow 2 \times 10^9$

Long $\rightarrow 4 \times 10^{18}$

ans $\% \; 10^9 + 7$

$[0 \rightarrow 10^9 + 6]$

(Int)

Properties of Mod (on arithmetic operators)

① $(a+b) \% m = (a \% m + b \% m) \% m$

Eg. $a = 9$, $b = 8$ $m = 5$

$$(9+8)\%5 = 17\%5 = 2 \qquad \Big| \qquad (9\%5 + 8\%5)\%5 = (4+3)\%5 = 7\%5 = 2$$

(2) $(a \times b)\%m = (a\%m \times b\%m)\%m$

Eg. $a = 9,\ b = 8 \quad m = 5$

$$(9\times8)\%5 \Rightarrow 72\%5 \Rightarrow 2 \qquad \Big| \qquad (9\%5 \times 8\%5)\%5 \Rightarrow (4 \times 3)\%5 \Rightarrow 12\%5 = 2$$

(3) $(a\%m)\%m = a\%m$

$$72\%5 = 2\%5 = 2\%5 = 2\ldots$$

(4) $(a+m)\%m = (a\%m + \overset{0}{m\%m})\%m$

$$= (a\%m)\%m$$

$$\boxed{(a+m)\%m = a\%m}$$

$$(18+5)\%5 = 23\%5 = 3 \qquad 18\%5 = 3$$

eg. $(-4)\%6 = (-4+6)\%6 = 2\%6 = 2$

$(-8)\%6 = (-8+6)\%6 = -2\%6 = (-2+6)\%6$

$= 4 \% 6 = 4$

⑤ $(a-b) \% m = (a \% m - b \% m + m) \% m$

For eg. $a = 17 \quad b = 8 \quad m = 5$

$\Rightarrow (17 - 8) \% 5$	$(17 \% 5 - 8 \% 5) \% 5$
$\Rightarrow 9 \% 5 = 4$	$(2 - 3) \% 5 = (-1) \% 5$
	$= 4$

$(-1) \% 5 = (-1 + 5) \% 5 = 4 \% 5 = 4$

⑥ $(a^b) \% m = (a \times a \times a \ldots \text{b times}) \% m$

$= ((a \% m) \times (a \% m) \times (a \% m) \ldots$

$\text{b times}) \% m$

$= ((a \% m)^b) \% m$

Quiz : $(37^{103} - 1) \% 12$

$\Rightarrow ((37^{103}) \% 12 - 1 \% 12 + 12) \% 12$

$\Rightarrow (1 - 1 + 12) \% 12 = 0$

① $(a-b) \% m = (a \% m - b \% m + m) \% m$

② $a^b \% m = ((a \% m)^b) \% m$

$37^{103} \% 12 = ((37 \% 12)^{103}) \% 12$

$= (1^{103}) \% 12 = 1$

If $A \% M = 0$

A is completely divisible by M

M is a factor of A

Prob 1 : Given N array elements, find count of pairs (i, j) such that (ar[i] + ar[j]) % m = 0

i != j and pair (i, j) is same as pair (j, i)

2 idx (i, j) → pair sum divisible by m

A = < 4, 3, 6, 3, 8, 12 > ans = 3

m = 6

(6 + 12) % 6 = 0

(3 + 3) % 6 = 0

(4 + 8) % 6 = 0

BF : Go to all unique pairs, calculate their sum. If their sum % m == 0 cnt ++

```
int ans = 0
for (i = 0 ; i < n ; i++) {
    for (j = i+1 ; j < n ; j++) {
        if (a[i] + a[j] % m == 0)
            ans++
    }
}
return ans
```

TC: $O(N^2)$

SC: $O(1)$

Optimised Approach

$$(a+b)\,\%\,m = (\overbrace{a\,\%\,m}^{r_1} + \overbrace{b\,\%\,m}^{r_2})\,\%\,m$$

$(a+b)\,\%\,m \rightarrow 0$ $\qquad$ $(a\%m + b\%m)\%m \rightarrow 0$

Look for a pair whose sum is divisible by m

sum of remainders should be divisible by m

$(r_1 + r_2)\,\%\,m = 0$

$r_1, r_2 \rightarrow [0 \quad m-1]$

$1 \rightarrow M-1$

$2 \rightarrow M-2$

$3 \rightarrow M-3$

$\vdots$

$\boxed{x \rightarrow M-x}$

$(r_1 + r_2)\,\%\,6 = 0$

$r_1, r_2 \rightarrow [0 \quad 5]$

$1 \rightarrow 5$

$2 \rightarrow 4$

$3 \rightarrow 3$

$0 \rightarrow 0$

Observation : $r_1 + r_2 = M$

	0	1	2	3	4	5
A =	4,	3,	6,	3,	8,	12
$r_1 = A\%M$	[4	3	1	3	3	2
r_2	1	2	4	2	2	3

M = 5

ans = ~~0~~ ~~1~~ 4

rem	0	1	2	3	4
cnt	0	~~0~~ 1	~~0~~ 1	~~0~~ ~~1~~ ~~2~~ 3	~~0~~ 1

	0	1	2	3	4	5	
A =	10,	3,	6,	9,	8,	12	9
A % M =	4	3	0	3	2	0	3
	2	3	0	3	4	0	3

M = 6

ans = ~~0~~ ~~1~~ ~~2~~ 3

rem	0	1	2	3	4	5
cnt	~~0~~ ~~1~~ 2	0	~~0~~ 1	~~0~~ ~~1~~ 2	~~0~~ 1	0

ans = 3 → +2 → 5

idx → rem   freq [rem]

```
int  countPairs (int A[ ], int N, int M) {

    int  freq [M] = {0}     // idx → 0 to M-1
    int  ans = 0

    for (int i = 0 ; i < N ; i++) {

        int r1 = A[i] % m
        int r2 = m - r1
        if (r1 == 0)
                      r2 = 0

        ans += freq[r2]
        freq [r1] ++
```

}

return ans

}

TC : O(N)

SC : O(M)

Break till 10:42

GCD → Greatest Common Divisor

HCF → Highest Common Factor

If x is a factor of A

$\Rightarrow A \% x = 0$

GCD(A, B) = Greatest factor that divides both A and B

$GCD(A, B) = x$

① $A \% x = 0$

② $B \% x = 0$

③ x is largest no. which divides both

$GCD(15, 25) = 5$

15	25
1	1
3	5
5	25
15	

$GCD(12, 30) = 6$

12	30
1	1
2	2
3	3
4	5
6	6
12	10
	15
	30

$GCD(a, b) = GCD(a, -b) = GCD(-a, b)$
$= GCD(-a, -b)$

$GCD(0, 4) = 4$

0	4
1	1
2	2
3	4
⋮	
∞	

$\frac{0}{0} \rightarrow$ undefined

$0 \% 1 = 0$

$0 \% 2 = 0$ ...

$GCD(0, a) = a$

0: 1, 2, 3, ..., ∞

a: 1, ..., a

$GCD(0, 0) = \infty$

0: 1, 2, 3, ..., ∞

0: 1, 2, 3, ..., ∞

$GCD(4, 7) = 1$ (Co-primes : pair of nos. whose GCD is 1)

4: 1, 2, 4

7: 1, 7

$GCD(-15, 25) = 5$

-15: ± 1, ± 3, ± 5

25: 1, 5, 25

$GCD(-15, 25) = GCD(15, 25)$

Properties of GCD

1. $GCD(A, B) = GCD(B, A)$
2. $GCD(0, A) = A$
3. $GCD(A, B, C) = GCD(GCD(A, B), C)$

$= GCD(GCD(A, C), B)$

$= GCD(GCD(B, C), A)$

$$GCD(2,3,4) = GCD(GCD(2,3),4) = 1$$

$$= GCD(GCD(3,4),2) = 1$$

④ $GCD(1, A) = 1$

$$1 \quad 1$$
$$\vdots$$
$$A$$

⑤ Given $A \geq B > 0$

$$\boxed{GCD(A,B) = GCD(A-B, B)}$$

$$GCD(A,B) = GCD(A-B, B)$$
$$= GCD(\underbrace{A-B-B}_{A-2B}, B)$$
$$= GCD(\underbrace{A-B-B-B}_{A-3B}, B)$$
$$= GCD(\underbrace{A-3B-B}_{A-4B}, B)$$

⑥ $\boxed{GCD(A,B) = GCD(A \% B, B)}$

eg. $GCD(17,5) = GCD(\overset{12}{17-5}, 5)$
$$= GCD(\overset{7}{17-5-5}, 5)$$
$$= GCD(\overset{2}{17-5-5-5}, 5)$$
$$= GCD(17 \% 5, 5)$$

e.g. GCD (24, 16) = GCD (8, 16)

↓

GCD (8, 16)

GCD (A, B) = GCD (B, A % B)

GCD (24, 16) = GCD (16, 8)

= GCD (8, 0) = 8

GCD (16, 24) = GCD (24, 16)

```
int gcd (int a, int b) {
    if (b==0) return a
    return gcd (b, a%b)
}
```

TC : $O(\log_2(\min(a,b)))$

SC : $O(\log_2(\min(a,b)))$

Prob: Calculate GCD of entire array

arr[3] = < 6, 12, 15 >

ans → 0   6   6   3

ans = 3

```
int ans = 0
for (i = 0 ; i < N ; i++) {
    ans = gcd (ans, a[i])
}
return ans
```

$GCD(x,y) \xrightarrow{TC} O(\log_2 \min(a,b))$

$GCD(ans, a[i]) \rightarrow O(\log_2 \max ele)$

TC : $O(N * \log_2 \max ele)$

$$LCM(a,b) = \frac{a * b}{GCD(a,b)}$$

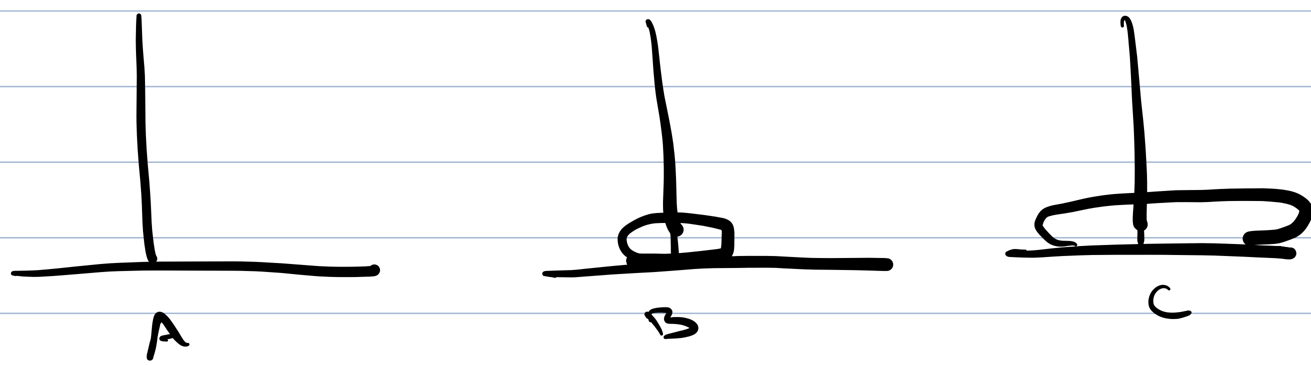
[10, 3, 6, 9, 8, 12]   M = 6

r1	4	3	0	3	2	0
r2	2	3	0	3	4	0

ans = ~~0~~ ~~1~~ ~~2~~ 3

rem	0	1	2	3	4	5
freq	~~0~~ ~~1~~ 2	0	~~0~~ 1	~~0~~ ~~1~~ 2	~~0~~ 1	0

N = 2



src   helper   dest

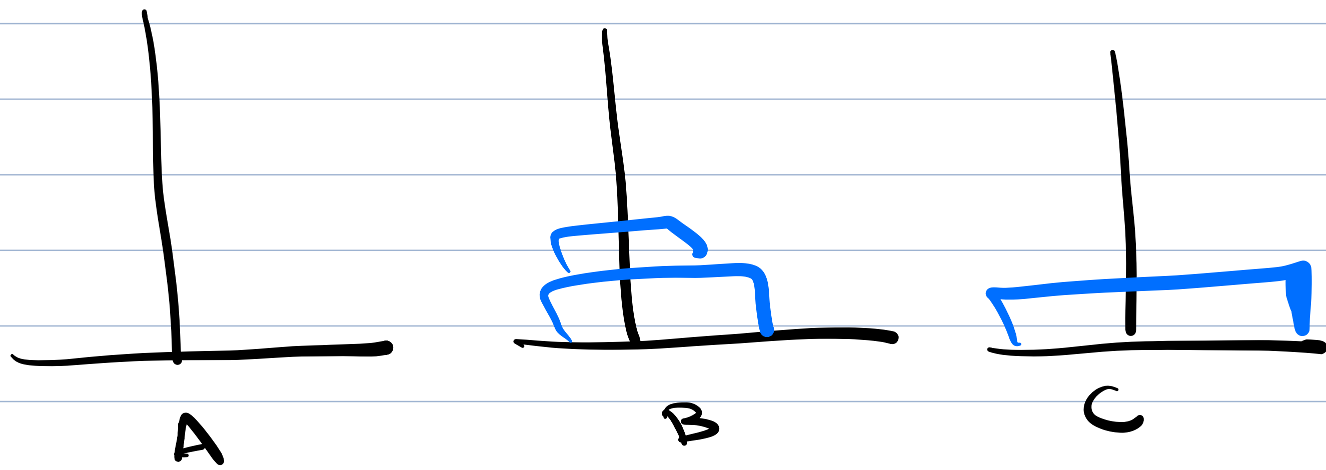
1. Move S disc   Src → Helper
2. Move L disc   Src → Dest
3. Move S disc   Helper → Dest

1. Move smaller N-1 discs src → helper

2. Move 1 disc Src → Dest

3. Move smaller N-1 discs helper → dest

N = 3



N	Src	Help	Dest
3	A	B	C
	↓		
2	A	C	B

1. 1 A → C
2.

③

<

2 B A C

// Given n discs, it will move src → dest using helper

```
void toh (int N, char src, char help, char dest){
    if (N = =0) return

    toh (N -1, src, dest, help)
    print( src → dest)
    toh (N-1, help, src, dest)
}
```

toh (N, A, B, C)