Mini-project 1
N-Body Problem: Gravity Assists

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1 Introduction

The N-body problem encompasses the modelling of multiple bodies, generally more than two, and their motion under various forces. In the gravitational N-body problem, each body is affected by the gravitational force exerted by every other body. This makes the equations of motion extremely complex to do by hand for more than two bodies. But with coding ad computation, simulating many bodies and forces becomes much easier. In this project, I have attempted to simulate a satellite using a gravity assist to exit the solar system.

2 VPython

This project was done using Vpython to plot the paths of the planets and stars. VPython uses Python programming language with the addition of 3 dimensional graphics module called Visual. This module also enables us to work with position, velocity and acceleration as 3 dimensional vectors, and more easily calculate relative positions, magnitudes, etc.

3 Forward integration using Leapfrog

Using the initial conditions of a body, it is possible to calculate its position and velocity after a very small time period.

$$x(\Delta t) = x(0) + v(0)\Delta t$$
$$v(\Delta t) = v(0) + a(0)\Delta t$$

Where the acceleration is calculated from Newton's Law of gravitation:

$$F = \frac{-G \times M_1 \times M_2}{r^2}$$
$$a = \frac{F}{M_1}$$

In vector form: $a = \dfrac{-G \times M_2 \times \hat{r}}{r^2}$

In the n-body problem, the force due to gravity from multiple bodies are superimposed to get the overall acceleration for one body.

However, this method has significant error. This method uses $v(0)$ to calculate the position $x(\Delta t)$, but between $x(0)$ and $x(\Delta t)$, the velocity is not constant - it is changing according to the acceleration, which itself is changing. The Leapfrog method can help reduce the error. Using this method, we calculate $x(\Delta t)$ using the velocity in the middle of the interval between $t = 0$ and $t = \Delta t$, which is closer to the mean of the velocity in the interval. To calculate the value of $x(2\Delta t)$, we use the value of $v(\frac{3}{2}\Delta t)$, in the middle of the interval. Therefore, we use the equations:

$$x(\Delta t) = x(0) + v(\frac{\Delta t}{2})\Delta t$$
$$v(\frac{\Delta t}{2}) = v(0) + a(0)\frac{\Delta t}{2}$$

The general form of the equation is:

$$x(t + \Delta t) = x(t) + v(t + \frac{\Delta t}{2})\Delta t$$
$$v(t + \frac{\Delta t}{2}) = v(t - \frac{\Delta t}{2}) + a(t)\frac{\Delta t}{2}$$

This method gives much less error, and produces more accurate orbits for the planets and satellite.

4 Gravity Assist - Theoretical Background

In order to observe and collect data on interstellar space, we need to send satellites outside the solar system. Ordinarily, we cannot launch a satellite or spacecraft with sufficient initial velocity to exit the solar system. Due to how massive the sun is, any satellite we can feasibly launch would simply begin to decelerate and fall into an elliptical orbit around the sun.

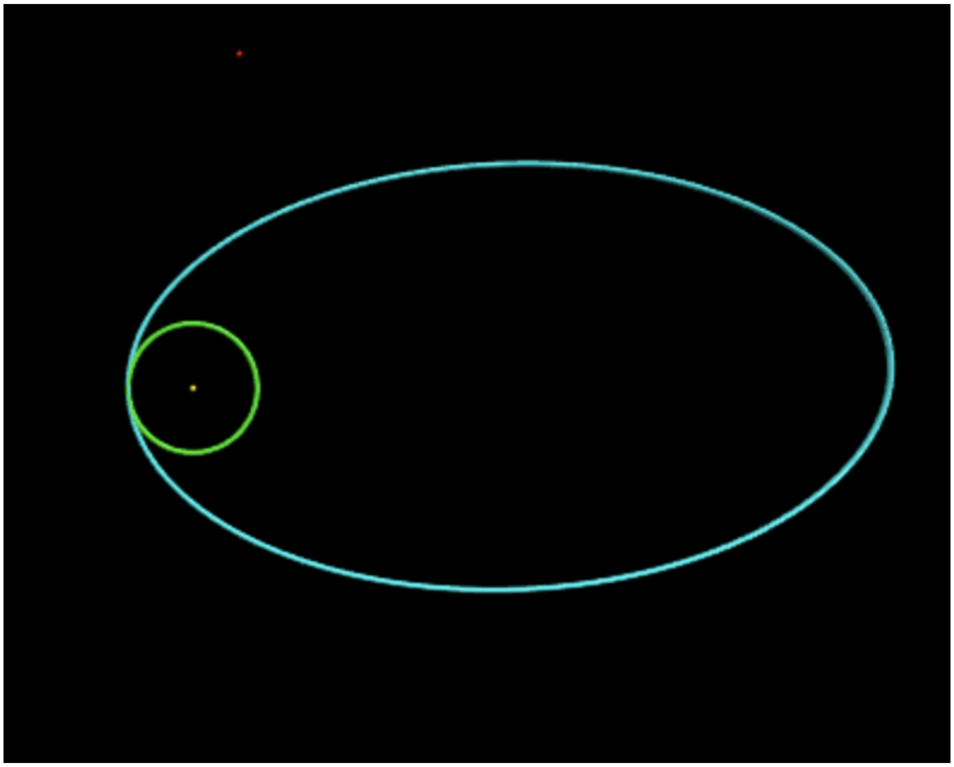

Figure 1: Elliptical orbit of a satellite without gravity assist.

One option would be to accelerate the space craft away from the sun. But this would require large amounts of fuel, which would add to the mass of the spacecraft, which would in turn require more fuel, and so on. Instead of adding mass to the satellite, an alternate possibility is to use a gravity assist from a planet to accelerate the craft, giving it a boost enough to escape the solar system.

In a gravity assist or fly-by, the satellite's launch and trajectory is planned such that it can come close to a planet with minimal use of fuel. As the satellite approaches, it begins to accelerate in the direction of the planet. The satellite's speed and momentum keeps it going past the planet itself, and it begins to decelerate , as the force is now acting in the opposite direction of its movement.

With a stationary planet, the amount of kinetic energy gained by the satellite as it falls towards the planet is equal to the amount of potential energy lost. As it moves past the planet, it begins lose kinetic energy and gain potential energy, ultimately leaving with the same amount of kinetic energy. Due to the law of conservation of energy, the satellite cannot gain speed in this situation.

However, if the planet is moving, such as in an elliptical orbit around the Sun, what occurs is essentially a perfectly elastic collision between the planet and satellite. The sum total kinetic energy of the two bodies remains the same, but the 'collision' between the two allows the satellite (being much less massive than the planet) to accelerate. This acceleration can be a change in direction of the satellite's motion, or an increase in velocity, or a combination of both. In this case, the magnitude of velocity of the satellite remains the same with respect to the planet, but from the sun's reference frame, the velocity has increased. Since the velocity (and thus the kinetic energy) of the satellite has increased, the kinetic energy of the planet must have decreased accordingly. The velocity of the planet does decrease, but due to the much greater mass of the planet compared to the satellite, the decrease in velocity is very small and not significant.

We can calculate the final velocity of the satellite using the initial momenta of the planet and satellite:

$$\begin{aligned}
&\text{Mass of the satellite} = m_s \\
&\text{Mass of the planet} = m_p \\
&m_s u_s + m_p u_p = m_s v_s + m_p v_p \\
&m_s(u_s - v_s) = m_p(v_p - u_p) \rightarrow (1)
\end{aligned}$$

The kinetic energy of both bodies is conserved as well.

$$\begin{aligned}
&\frac{m_s u_s^2}{2} + \frac{m_p u_p^2}{2} = \frac{m_s v_s^2}{2} + \frac{m_p v_p^2}{2} \\
&m_s(u_s^2 - v_s^2) = m_p(v_p^2 - u_p^2) \rightarrow (2)
\end{aligned}$$

Dividing equation (2) by (1):

$$
\begin{aligned}
\frac{m_s(u_s^2 - v_s^2)}{m_s(u_s - v_s)} &= \frac{m_p(v_p^2 - u_p^2)}{m_p(v_p - u_p)} \\
u_s + v_s &= u_p + v_p \\
v_p &= u_s - u_p + v_s
\end{aligned}
$$

Substituting this in the equation for momentum:

$$
\begin{aligned}
m_s u_s + m_p u_p &= m_s v_s + m_p(u_s - u_p + v_s) \\
m_s u_s + 2m_p u_p - m_p u_s &= (m_s + m_p)v_s \\
v_s &= \frac{m_s - m_p}{m_s + m_p} u_s + \frac{2m_p}{m_s + m_p} u_p
\end{aligned}
$$

Similarly, substituting $(v_s = u_p - u_s + v_p)$ in the momentum equation gives:

$$
v_p = \frac{2m_s}{m_s + m_p} u_s \frac{m_p - m_s}{m_s + m_p} u_p
$$

Since the mass of the planet is much greater than the mass of the satellite, we consider $(m_s + m_p \approx m_p)$ and $(\frac{m_s}{m_p} \approx 0)$. This gives:

$$
\begin{aligned}
v_s &\approx -u_s + 2u_p \\
v_p &\approx u_p
\end{aligned}
$$

As we can see from the equations, the velocity of the planet is not changed significantly, but the satellite's velocity increases considerably.

Gravity assists can also be used to help satellites study the interior planets of the solar system, by using the gravitational pull of a planet to slow down and resist the force exerted by the Sun. As the satellite would be falling towards the sun, it would accelerate considerably, and move quickly past any inner planet. In order for a satellite to orbit one of the planets rather than fall past it, the satellite would need to be slowed down by a gravity assist.

5 Simulating the Solar system

In order to simulate the satellites trajectory, we first require a simulation of the orbits of planets.

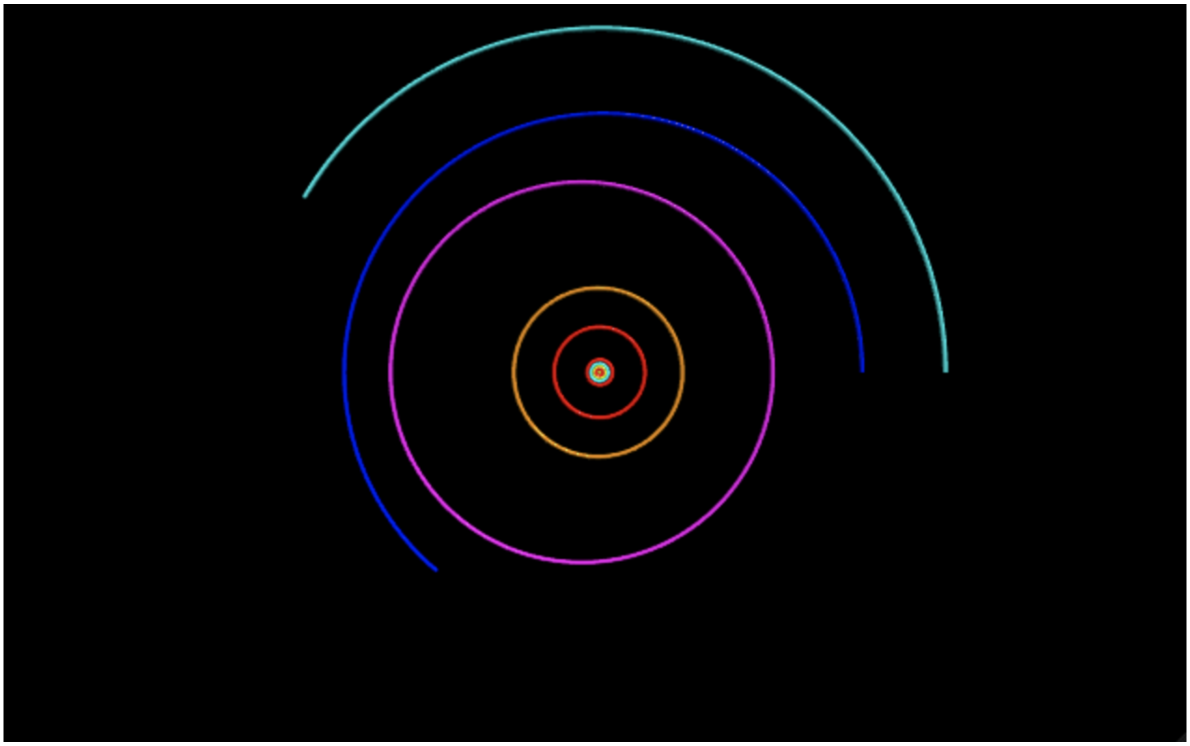

Figure 2: The orbits of the 8 planets of the solar system, and Pluto. Due to the very large time period of Neptune and Pluto, it was not possible to capture the complete orbit. The planets visible in this plot are Jupiter (red), Saturn (orange), Uranus (pink), Neptune (blue), Pluto (cyan).

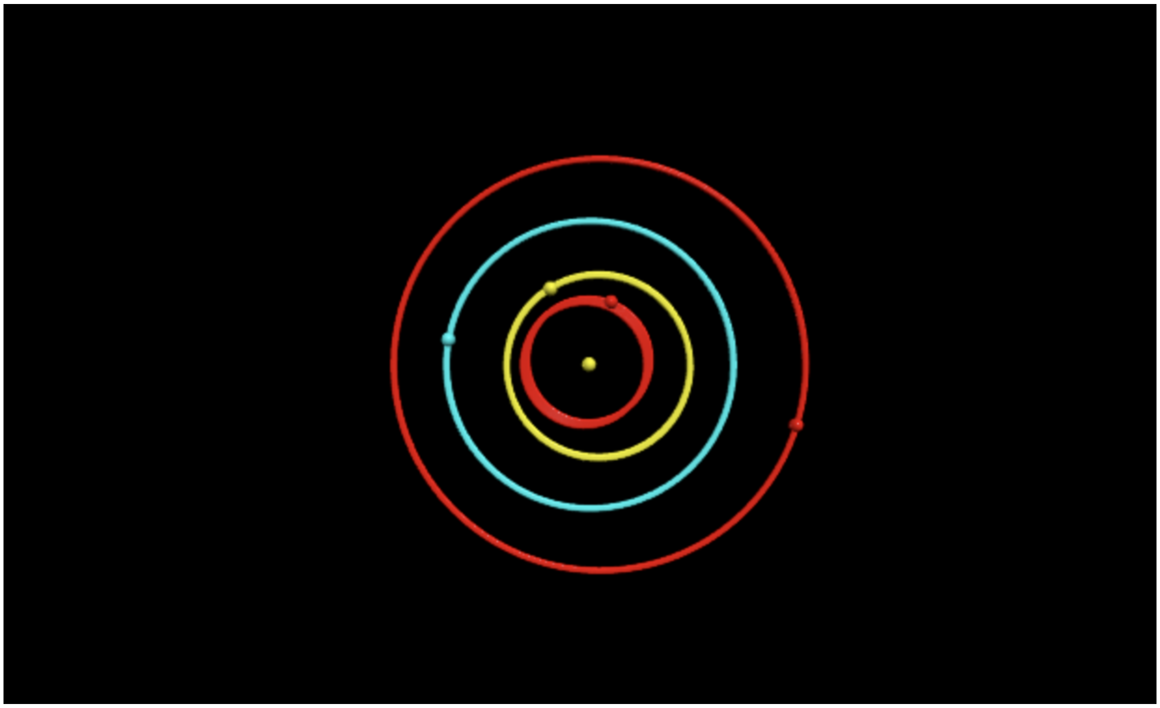

Figure 3: Close-up view of the inner four planets. Mars (red), Earth (cyan), Venus (Yellow), Mercury (red).

Planet	Mass of planet (kg)	Distance from the Sun (in AU)	Initial velocity (m/s)
Mercury	3.28 e23	0.397 AU	49080
Venus	4.87 e24	0.7 AU	33846
Earth	6 e24	1 AU	29785
Mars	6.39 e23	1.5 AU	23753
Jupiter	1.9 e27	5.2 AU	13059
Saturn	5.7 e26	9.5 AU	9757
Uranus	8.7 e25	19.8 AU	7020
Neptune	1 e26	30 AU	5415
Pluto	1.3 e22	39.5 AU	4744

Table 1: Table of initial values for the planets plotted. Distance from the sun for each planet is given in units of Astronomical units, or the distance between the Earth and the Sun. 1 AU = 149.6×10^9. The initial velocity for each planet was calculated using the radius and time period: Velocity $= \frac{2\pi \times \text{radius}}{\text{Time period}}$

Due to the size of the time step of the leapfrog calculation, there is some precession visible in the orbit of Mercury. To plot this kind of orbit by any analytical method would be extremely difficult due to the number of forces and bodies interacting, but the computational method makes it relatively easy to observe the orbits.

6 Simulating the gravity assist for the satellite

In my simulation, I have used the Voyager 2 satellite launched in 1977 as a rough model. Although the positions of the planets and the acceleration from the gravity assist are not precisely the same, I have modelled the satellite on a trajectory that uses Jupiter, Saturn and Uranus to boost its speed, similar to the Voyager 2.

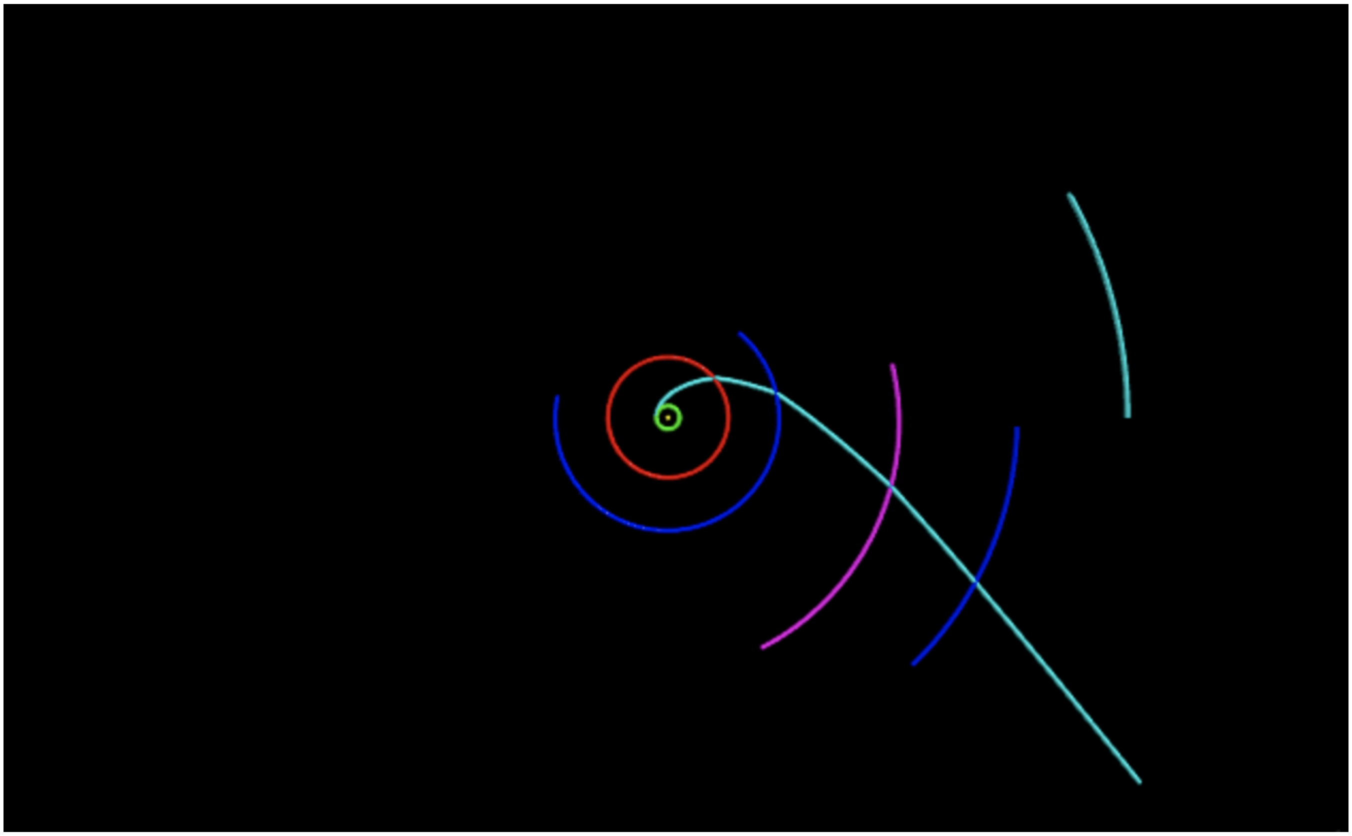

Figure 4: Trajectory of a satellite launched from Earth. Satellite (cyan) receives a gravity assist from Jupiter (red), Saturn (Blue) and Uranus (pink).

In this simulation, the satellite was launched with an initial velocity of 40500 m/s. The mass and initial velocities used for the planets were the same as those in the previous solar system code. To save computational power, the inner planets apart from Earth were not depicted in this set.

To ensure that the satellite passes in close proximity to the required planets, this velocity was kept constant, and the angle was adjusted until it followed the desired trajectory. The satellite passed within 4.5×10^{10}m of the center of Jupiter and accelerated, and the magnitude of velocity increasing by about 5000 m/s. It also accelerates (although the change in magnitude is lesser) when it passes Saturn and Uranus. These three gravity assists give the satellite sufficient velocity such that it can escape the gravitational pull of the sun, and escape the solar system.

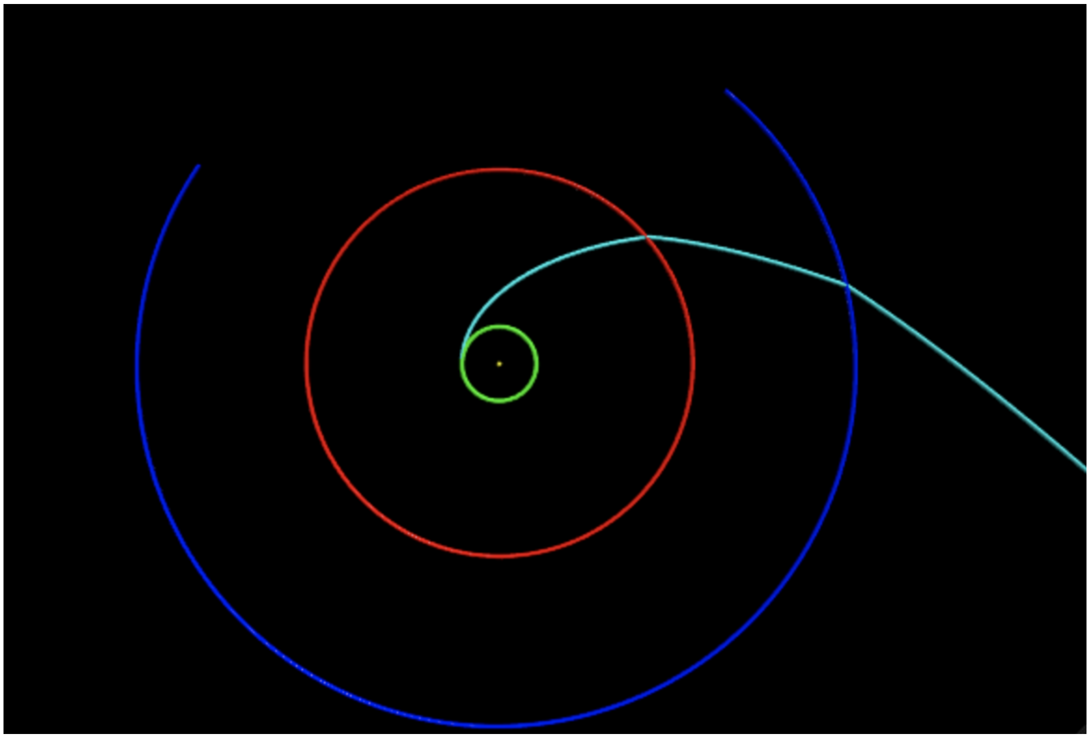

Figure 5: Close-up of the trajectory of the satellite. The change in the direction of motion as it passes Jupiter and Saturn are more clear.

The satellite in Figure 1 had the same initial velocity and direction as that in Figure 4 and 5. In Figure 1, after the satellite is launched, the gravitational pull of the sun causes the velocity to decrease. Without any gravity assists, eventually the satellite begins to fall back towards the Sun, and settles into an elliptical orbit. In the second simulation, after the satellite is launched, it also begins to slow down due to the gravitational pull of the sun. However, the trajectory of the satellite is planned such that before it starts moving back towards the Sun, it encounters Jupiter at a close range. The gravitational force of Jupiter exerted on the satellite causes it to accelerate in the direction of the planet. The satellite passes just behind Jupiter, which causes it to veer in the direction of Jupiter's motion. The satellite leaves Jupiter with a greater velocity that is sufficient to carry it near Saturn, where a similar gravity assist manoeuvre occurs.

Thus, the successive gravity assist manoeuvres are sufficient to overcome the gravitational pull of the Sun and allow the satellite to fly out of the Solar system.

7 Gravity assist with the Sun

It is also possible to use the sun itself in a gravity assist manoeuvre. Since the Sun is moving at the same velocity as the solar system (i.e. it is stationary within the reference frame of the entire solar system), it is not possible to use a gravity assist manoeuvre around the Sun to accelerate relative to other planets. However, the sun's gravitational force could potentially be used to accelerate a spacecraft to an interstellar trajectory.

In the images below, I simulated 3 variations of a gravity assist manoeuvre using the Sun. In all three cases, the satellite was launched at an angle $45°$ below the Earth-Sun line, with a magnitude of 7071 m/s. In the first image, the program only included the satellite, the Earth and the Sun. In the second image, the program included the satellite, Venus, Earth, and the Sun. In the third image, the program included the satellite, Mercury, Venus, Earth and the Sun.

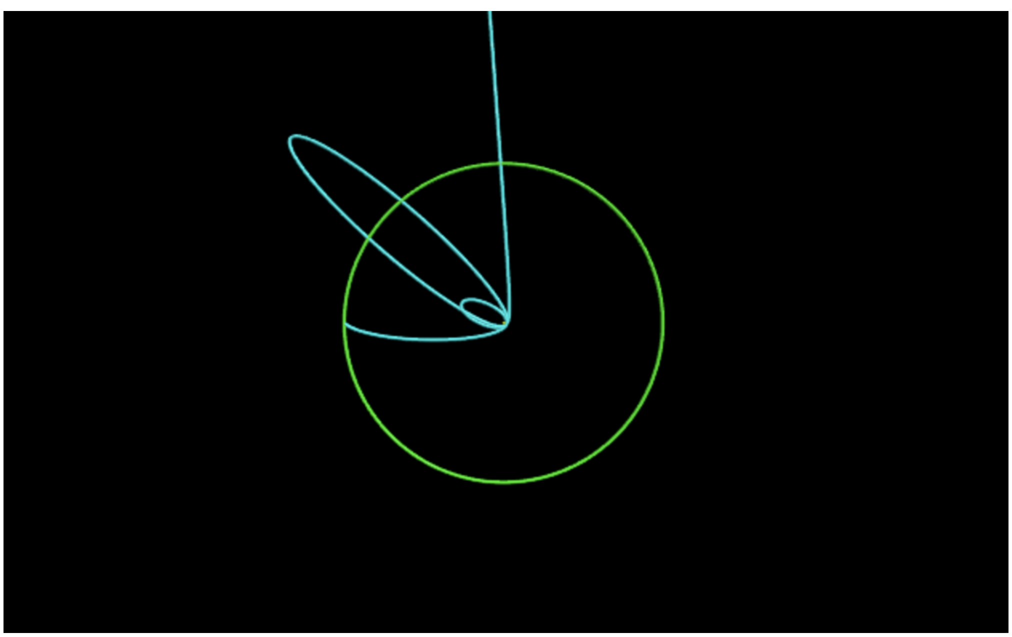

Figure 6: A slingshot with only the forces of Earth and Sun acting on the satellite.

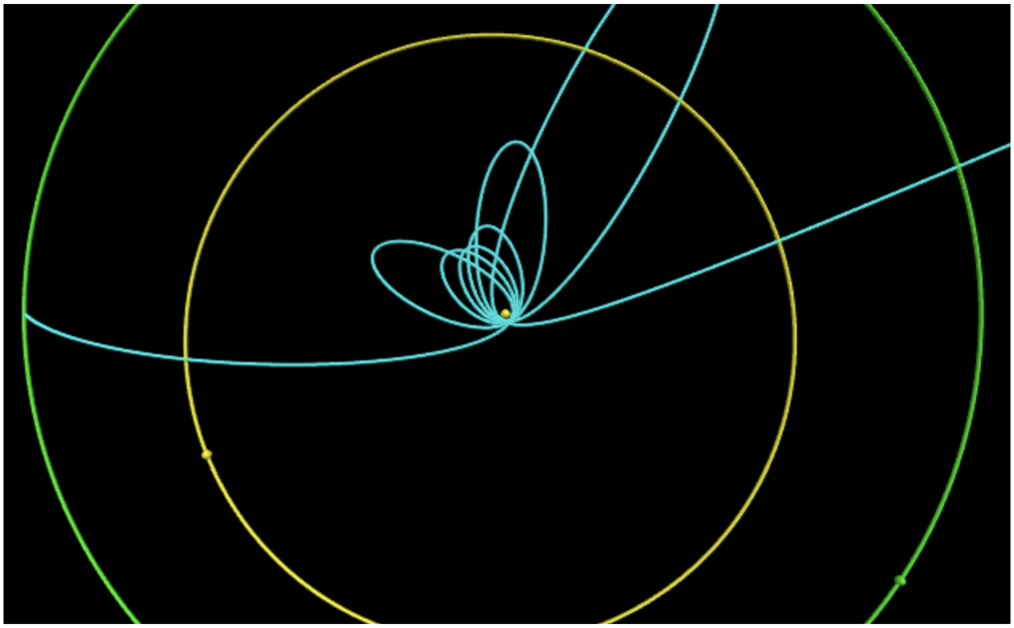

Figure 7: A slingshot with the forces of the Sun, Venus and Earth acting on the satellite.

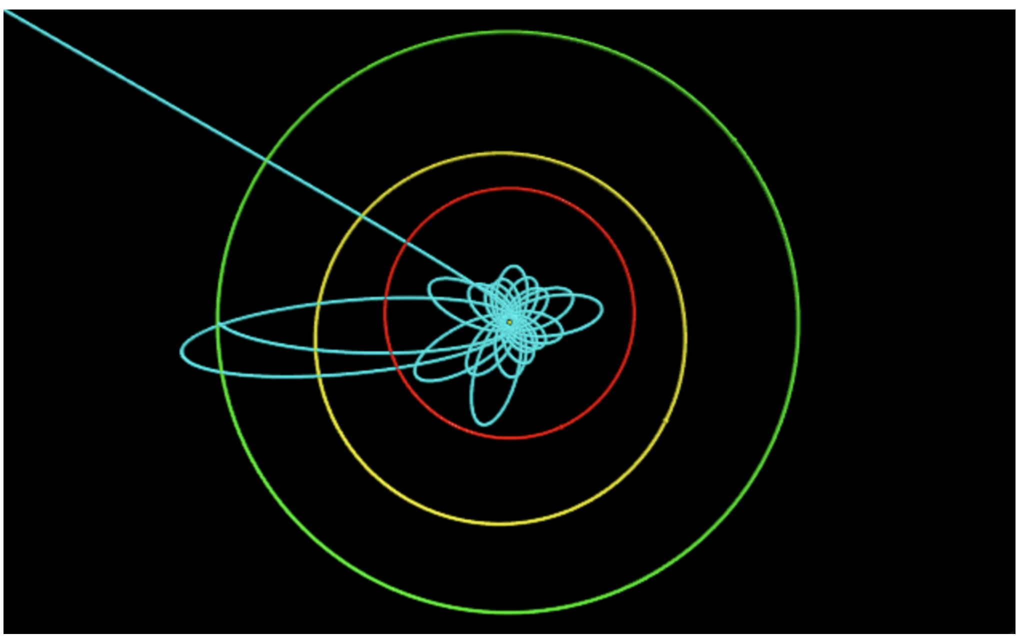

Figure 8: A slingshot with the forces of the Sun, Mercury, Venus and Earth acting on the satellite.

As we can see from the images, as more gravitational forces as superimposed on the satellite, the elliptical orbits it follows becomes more complex. Each ellipse is affected by the final velocity of the previous ellipse, and the positions of the planets in their orbits at the time. In all three cases, the satellite moves around the sun until it gains enough velocity and is thrown beyond the inner solar system.

8 Sources of Error

The main source of error for these simulations is the time step Δt used in the leapfrog calculation. In order to generate orbits and satellite trajectories in a reasonable amount of time, the time step used was quite large - $60 \times 60 \times 20$ or 20 hours. This would have caused the error in the calculated velocity and position to accumulate. Thus, the accuracy of the final plots may be low.

9 Discussion

9.1 Oberth Effect

The Oberth effect is a powered fly-by in which a spacecraft uses a gravity assist along with fueled acceleration to achieve an even greater velocity than an ordinary fly-by. In this manoeuvre, the spacecraft is pulled by the gravitational pull of a body and falls towards it. At the deepest point in the gravitational 'well', the engines are used to further accelerate it in it's fall. This lowest point is called the orbital periapsis, where the orbital velocity is greatest and the powered fly-by is the most efficient.

This manoeuvre was used by the Indian space probe Mangalyaan, launched in 2013. Mangalyaan used seven powered fly-bys around Earth to increase velocity, before going to orbit Mars.

9.2 Frame-Dragging

The simulations done above are concerned only with Newtonian mechanics. However, according Einstein's General Relativity theory, an additional factor called frame-dragging (also known as the Lense-Thirring effect) affects gravity assists, particularly around very massive bodies such as black holes. According to general relativity, when a body rotates, it drags space-time with it in the direction of the spin. This prediction was confirmed by Gravity Probe B, a satellite mission launched in 2004, that successfully detected the frame-dragging effect caused by the rotation of the Earth. Frame-dragging cause an region of space-time around the object to move, carrying an other bodies in the region as well. Theoretically, it may be possible to gain energy from travelling through this area, similar to a gravity assist, by falling towards it.

10 References

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- `https://en.wikipedia.org/wiki/MarsOrbiterMission`
- `https://en.wikipedia.org/wiki/Frame-dragging`