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STEADY GROUNDWATER FLOW IN LEAKY MULTIPLE-AQUIFER SYSTEMS

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ABSTRACT

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The hydraulic properties of stratified formations usually vary from layer to layer. Within the area of influence of a pumped well, drain, or section of a river, the subsoil can often be schematized into a limited number of aquifers each separated by less pervious layers.

A general method has been developed for the analytical solution of steady flow problems in leaky multiple-aquifer systems comprising any number of aquifers. This method has been used to investigate potential distributions in multilayered aquifers of infinite extent, being pumped or recharged at a constant rate using completely penetrating wells and drains in one or more of the aquifers.

The same approach is applicable to groundwater flow problems with more complicated boundary conditions, such as the influence of a wide river partially penetrating the leaky top-layer.

The well flow solution has been applied to evaluate hydraulic properties of several aquifers and aquitards from pumping-tests in a six-layer leaky aquifer system.

In a general sense the method presented may be regarded as a combined analytical—numerical technique.

1. INTRODUCTION

Many aquifers in nature lose or gain water through adjacent confining beds of relatively low permeability. However, such leaky aquifers are often only part of multiple-aquifer systems. When water is withdrawn or recharged in one particular aquifer the head distribution in the entire system will be influenced.

One of the main assumptions in the theory of steady groundwater flow in leaky aquifer systems is that the under- and/or overlying aquifers or phreatic surface supply or discharge the entire leakage, while the hydraulic head in these layers remains unaltered.

With respect to the number of layers within a considered system, it is

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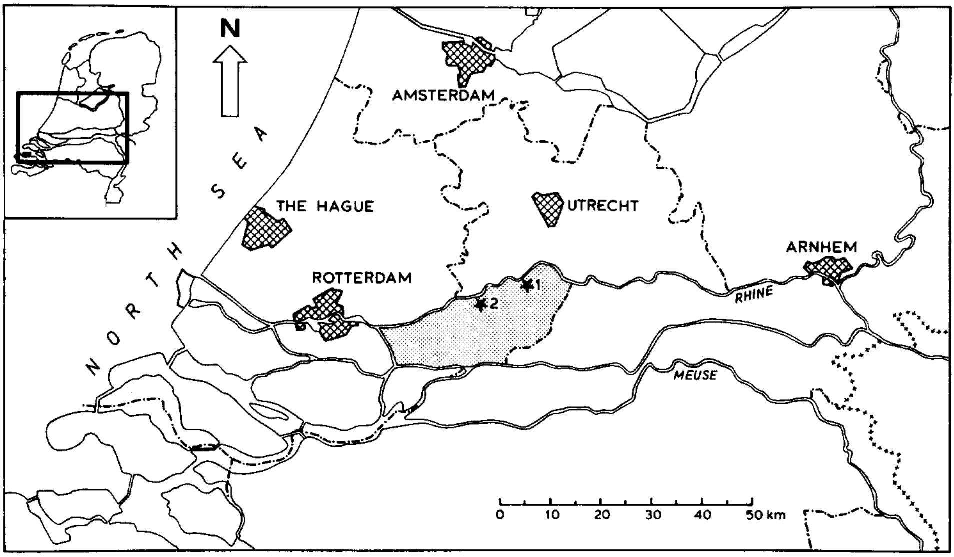

Fig. 1. Map of the central part of The Netherlands showing the location of the polder areas "Alblasserwaard" and "Vijfheerenlanden", and the pumping-test sites: (*1*) "Lexmond" and (*2*) "Langerak".

evident that the single-aquifer type with an impervious base leads to the simplest solutions. Solutions for radial flow (Kooper, 1914; DeGlee, 1930) and parallel flow (Mazure, 1932) are well known and often used.

Pumping-test data show that the assumption of zero drawdown in unpumped aquifers is often violated. Since this can lead to severe errors in the derived hydraulic properties, Huisman and Kemperman (1951) have extended the analytical well flow solution to a two-layer leaky aquifer system bounded by an impervious base. A further development to a two-layer leaky-base solution has been reported by Polubarinova-Kochina (1962).

In 1967 a series of publications on transient flow in two-aquifer systems was initiated by Hantush. Presented solutions refer either to a bounded system with impermeable upper and lower horizon or to a two-layer aquifer with the upper layer unconfined but not recharged (Hantush, 1967; Neuman and Witherspoon, 1969a, b, 1972). As these systems are, in a hydraulic sense, separated from their surroundings, no steady-state flow can be achieved and solutions for large values of time will deviate increasingly from steady-state drawdowns in leaky multiple-aquifer systems. Saleem and Jacob (1973, 1974) extended these two-aquifer solutions to unconfined systems replenished by precipitation.

The steady- and unsteady-state solutions for well flow in a leaky two-aquifer system were proposed by Bruggeman (1972). Similar solutions for a larger number of aquifers may be derived in the same way (G.A. Bruggeman, pers. commun., 1983) and Bruggeman's formula can also be used to analyse pumping-test data. As the effect of storage in the confining beds is neglected, however, the use of early drawdown data to calculate transmissivity may lead to values which are too large.

In 1978 the Dutch Water Supply Company "De Alblasserwaard en de Vijfheerenlanden" began geohydrological investigations to determine the feasibility of developing a new well field in the polder area located between 30 and 50 km east of Rotterdam (Fig. 1). The study included a test-hole drilling program and two aquifer tests. Up to the investigated depth of 250 m the unconsolidated deltaic (marine and fluvial) deposits showed an alternation of 10—60 m thick, permeable sand and gravel layers and thinner deposits of clay, loam and fine-grained silt, in accordance with the data available from surrounding areas. By borehole-logging all important permeable layers were recognized and in each test-hole several (2—8) piezometers were installed. Using borehole data and the observed hydraulic heads of the different layers which were influenced by 20,000-m^3-day^{-1} groundwater extraction at Lexmond, it was shown that at least four aquifers should be distinguished within the study area.

Drawdowns at the "Langerak" test-site showed that even a larger number of aquifers should be taken into account locally.

The above-mentioned pumping tests were performed in 1979 and 1980, but since no appropriate analysis method was known from the literature, a solution using groundwater flow in multilayered aquifer systems was explored. The resulting theoretical approach is presented in this paper.

2. SOLUTION METHOD

Consider a system of n leaky aquifers as shown schematically in Fig. 2. Since semi-pervious layers (aquitards) occur at the top and bottom as well as between all the aquifers, their number will be $(n + 1)$. Each layer is

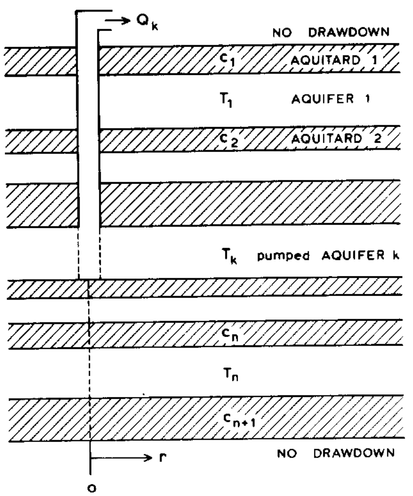

Fig. 2. Schematic diagram of a well in a leaky multiple-aquifer system.

homogeneous, of uniform thickness and horizontal. It is further assumed that the ratio of thickness and vertical conductivity of the aquifers, D/K_v, is sufficiently small, compared with the vertical hydraulic resistance of aquitards, D'/K_v', to allow the horizontal component of flow to be practically independent of vertical position within each aquifer. This means that the flow direction in aquifers need not be horizontal (as, for example, in unpumped aquifers in the vicinity of a well), while heads along any vertical line may be assumed constant within each aquifer. A similar argument relating to horizontal and vertical flow components in aquitards yields the condition $K_H'D' \ll K_H D$. In the case of isotropic layers both conditions can be written in the combined form:

$$K'/K \ll D/D' \ll K/K'$$

where K, D and K', D' are respectively the hydraulic conductivity and thickness of any aquifer and aquitard. Since conductivity contrasts frequently exceed three orders of magnitude, while the thickness ratio is often less than 10, the conditions are generally satisfied. When conductivity contrasts are less well pronounced, approximate solutions may be obtained by subdividing the layer(s) of intermediate conductivity into a number of separate aquifers and aquitards, leaving the total transmissivity $\Sigma K_i D_i$ and total vertical resistance $\Sigma(D_i'/K_i')$ unaltered.

As already mentioned, the upper and lower limits of a complete system are chosen in such a way that the piezometric heads of over- and underlying groundwater bodies will not be influenced by the considered groundwater flow. As this groundwater flow is restricted to that resulting from conditions within the system itself, all heads outside the system may be set to zero (principle of superposition).

The problem to be solved can thus be stated as the determination of the steady-state head distribution in all aquifers of the leaky system, depending on the given (fixed) boundary conditions. Apart from these conditions, all problems of this type can be written in a general form as n differential equations for the unknown functions h_i:

$$\mathscr{L}h_i = (h_i - h_{i-1})/T_i c_i + (h_i - h_{i+1})/T_i c_{i+1}, \qquad i = 1, 2, \ldots, n \tag{1}$$

where $\mathscr{L}$ = a linear differential (Laplacian) operator, depending on the type of flow; h_i = piezometric head in ith aquifer [L]; T_i = transmissivity of ith aquifer = $K_i D_i$ [$\mathrm{L}^2\ \mathrm{T}^{-1}$]; c_i = vertical hydraulic resistance of ith semi-pervious layer = D_i'/K_i' [T]; K_i, K_i' = hydraulic conductivity of ith aquifer and ith semi-pervious layer, respectively [$\mathrm{L\,T}^{-1}$]; and D_i, D_i' = thickness of ith aquifer and ith semi-pervious layer, respectively [L].

The operator notation $\mathscr{L}$ is adopted for convenience only. When two-dimensional flow is considered it is defined by:

$$\mathscr{L}h = \partial^2 h/\partial x^2 + \partial^2 h/\partial y^2$$

If the problem is only one dimensional this reduces to:

$$\mathscr{L}h = \mathrm{d}^2h/\mathrm{d}x^2$$

in the case of parallel flow, or to:

$$\mathscr{L}h = \mathrm{d}^2h/\mathrm{d}r^2 + \mathrm{d}h/r\mathrm{d}r$$

in the case of axial symmetry.

By introducing the substitutions $a_i = 1/T_i c_i$ and $b_i = 1/T_i c_{i+1}$, eq. 1 can be replaced by:

$$\mathscr{L}h_i = -a_i h_{i-1} + (a_i + b_i)h_i - b_i h_{i+1} \qquad i = 1, 2, \ldots, n \tag{2}$$

Let the vector **h** be defined by:

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_n \end{bmatrix}$$

then the system of eqs. 2 can be rewritten in the form:

$$\mathscr{L}\mathbf{h} = \mathbf{Ah} \tag{3}$$

in which A is the non-symmetric tridiagonal $n \times n$ matrix:

$$\mathrm{A} = \begin{bmatrix} a_1 + b_1 & -b_1 & 0 & \ldots & 0 & 0 \\ -a_2 & a_2 + b_2 & -b_2 & \ldots & 0 & 0 \\ 0 & -a_3 & a_3 + b_3 & \ldots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \ldots & -a_n & a_n + b_n \end{bmatrix} \tag{4}$$

Obviously, use is made here of the assumptions $h_0 = 0$ and $h_{n+1} = 0$.

The solution for eq. 3 is similar to that for the one-aquifer system when A is reduced to diagonal form. This is achieved by decomposing the A matrix into its eigenvalues and eigenvectors. If w is an eigenvalue of A and **v** is an eigenvector of A, associated with w, then by definition (Bodewig, 1959):

$$\mathrm{A}\mathbf{v} = w\mathbf{v}$$

In general, there will be n eigenvalues (w_i) and n associated eigenvectors ($\mathbf{v}_i$). Hence AV = VW, and:

$$\mathrm{A} = \mathrm{VWV}^{-1} \tag{5}$$

where W is the $n \times n$ diagonal matrix with eigenvalues $w_1, w_2, \ldots, w_n$ along the main diagonal; V is a $n \times n$ matrix containing the corresponding eigenvectors of A in its columns; and $\mathbf{V}^{-1}$ is the $n \times n$ inverse matrix of V.

By substituting eq. 5 into eq. 3, premultiplying both sides by $\mathbf{V}^{-1}$ and defining $\mathbf{t} = \mathbf{V}^{-1}\,\mathbf{h}$, we obtain:

$$\mathscr{L}\mathbf{t} = \mathbf{W}\mathbf{t} \tag{6}$$

Since any further solution to the multilayer-aquifer problem is dependent on the type of flow and boundary conditions involved, three applications are presented to illustrate the utility of eq. 6.

2.1. Parallel flow to canals or drains completely penetrating one or more aquifers

Consider a system of n leaky aquifers, drained (or recharged) in one or more aquifers at the same vertical position $x = 0$ (Fig. 3). All the above-mentioned assumptions apply and additional conditions are:

(1) Canals or drains of infinitesimal width which completely penetrate one or more aquifers.

(2) Either discharging rates or resulting heads in the drains are given for each aquifer separately.

(3) The aquifer system is of infinite extent.

Eq. 6 for parallel flow provides the governing matrix differential equation:

$$\mathrm{d}^2\mathbf{t}/\mathrm{d}x^2 = \mathbf{W}\mathbf{t} \tag{7}$$

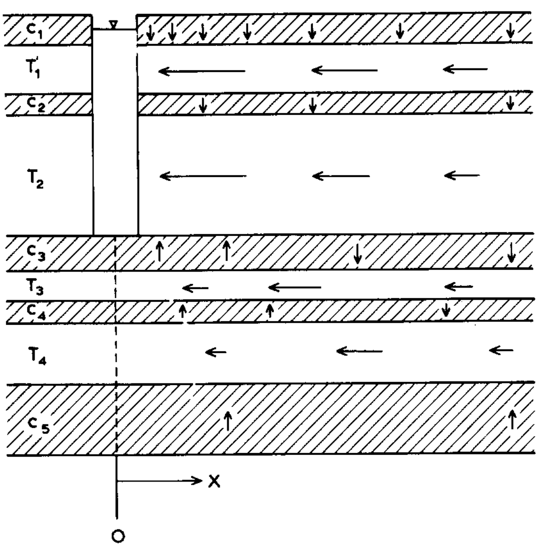

Fig. 3. Schematic diagram of parallel flow to a drain completely penetrating two aquifers of a leaky four-aquifer system.

The boundary conditions are:

$\mathbf{h} = \mathbf{0}$	(null vector)	at	$x = \infty$
$d\mathbf{h}/dx = \mathbf{f}$	(vector defined below)	at	$x = 0$

From the first condition and the definition $\mathbf{t} = \mathbf{V}^{-1}\,\mathbf{h}$ it follows that each linear combination of $h(\infty)$ will be zero. Thus:

$$\mathbf{t}(\infty) = \mathbf{0} \tag{8}$$

The second boundary condition applies to problems with given discharge rates. Fixed heads will be treated later. When $q(0)$ is the discharge per unit width from a drained aquifer (both sides), then using Darcy's law:

$$dh(0)/dx = q(0)/2T$$

Since in undrained aquifers $q(0) = 0$, this condition can be generalised as:

$$\mathbf{f} = \begin{bmatrix} q_1(0)/2T_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n(0)/2T_n \end{bmatrix} \tag{9}$$

The solution of eq. 7, when satisfying the boundary condition $\mathbf{t}(\infty) = \mathbf{0}$, can be written as a vector function:

$$\mathbf{t}(x) = \mathbf{Da}$$

where $\mathbf{D}$ is a $n \times n$ diagonal matrix with elements $\exp(-x\sqrt{w_i})$ along the main diagonal and $\mathbf{a}$ is a vector of constants to be determined from the second boundary condition. Since $\mathbf{h} = \mathbf{Vt}$, we obtain:

$$\mathbf{h} = \mathbf{VDa} \tag{10}$$

Differentiating this system of equations with respect to x yields:

$$\mathbf{h}' = -\mathbf{VD}'\mathbf{a} \tag{11}$$

where, for simplicity of notation, $d\mathbf{h}/dx$ is denoted by $\mathbf{h}'$ and $\mathbf{D}'$ is a diagonal matrix with non-zero elements $\sqrt{w_i}\,\exp(-x\sqrt{w_i})$. The boundary condition $\mathbf{h}'(0) = \mathbf{f}$ can now be used in eq. 11, giving:

$$\mathbf{f} = -\mathbf{VD}'(0)\,\mathbf{a} \qquad \text{and} \qquad \mathbf{a} = -\mathbf{D}'(0)^{-1}\,\mathbf{V}^{-1}\,\mathbf{f}$$

Substitution of this result in eq. 10 yields the solution:

$$\mathbf{h} = -\mathbf{VD}\,\mathbf{D}'(0)^{-1}\,\mathbf{V}^{-1}\,\mathbf{f}$$

or

$$\mathbf{h} = -\mathbf{VEV}^{-1}\,\mathbf{f} \tag{12}$$

where E is a $n \times n$ diagonal matrix with elements $(1/\sqrt{w_i}) \exp(-x\sqrt{w_i})$ along the main diagonal and V, V^{-1} and f are as defined in eqs. 5 and 9.

Setting $L_i = 1/\sqrt{w_i}$, the elements of E appear as $L_i \exp(-x/L_i)$ and resemblance with the one-aquifer solution shows L_i to be the n leakage factors relating to a n-layered leaky system.

Instead of a matrix presentation the problem solution can also be written as a summation:

$$h_j = \sum_{k=1}^{n} (-q_k(0)/2T_k) \sum_{i=1}^{n} v_{ji} z_{ki} L_i \exp(-x/L_i) \tag{13}$$

to express the piezometric head in aquifer j as a function of horizontal distance x, when aquifer k is discharged at a rate $q_k(0)$. Here v_{ji} is the jth element of the ith eigenvector of A and z_{ki} is the kth element of the ith column of matrix Z, the transposed inverse of V; $Z^T = V^{-1}$.

If one aquifer at a time is discharged at an equal rate $q(0)$, then, from the principle of reciprocity (Bruggeman, 1972) $h_{jk} = h_{kj}$, where the second index indicates the discharged aquifer. Thus, for every x:

$$(q(0)/2T_k) \sum_{i=1}^{n} v_{ji} z_{ki} L_i \exp(-x/L_i) = (q(0)/2T_j) \sum_{i=1}^{n} v_{ki} z_{ji} L_i \exp(-x/L_i) \tag{14}$$

The proof of this equality is nontrivial. Only a closer look at the construction of matrix A can reveal the explanation. From definition of the tridiagonal matrix A (see eq. 4) it can easily be seen that A can be considered as a product of two matrices: a symmetric matrix containing all the information relating to aquitard hydraulic resistances, premultiplied by a diagonal matrix containing the inverse transmissivity values. Based upon this property it can be shown that when all eigenvectors $\mathbf{v}_i$ are normalized according to:

$$\mathbf{v}_i^* = \mathbf{v}_i \Bigg/ \left(\sum_j v_{ji}^2 T_j \right)^{1/2} \tag{15}$$

eq. 13 may be written as:

$$h_j = \sum_{k=1}^{n} (-q_k(0)/2) \sum_{i=1}^{n} v_{ji}^* v_{ki}^* L_i \exp(-x/L_i) \tag{16}$$

thus demonstrating equality (14). This manner of normalizing proves very useful when evaluating h, as the need to compute inverse matrices is avoided.

When constant heads in the drains are given instead of discharging rates, h(0) can be calculated from eq. 12:

$$\mathbf{f} = -\mathbf{V}\mathbf{E}^{-1}(0)\,\mathbf{V}^{-1}\mathbf{h}(0)$$

f_i being zero in each undrained aquifer. If only one aquifer is drained or recharged at a level $h_k(0)$, the resulting heads are:

$$h_{jk} = h_k(0)\left[\sum_i v_{ji} z_{ki} L_i \exp\left(-x/L_i\right)\right] \Big/ \left[\sum_i v_{ki} z_{ki} L_i\right]$$

2.2. Radial flow to wells completely penetrating one or more aquifers

Since the multi-aquifer well flow problem can be solved in essentially the same way as that for parallel flow, the present derivation will be restricted to some deviating elements. As is usual in well-flow formulae, drawdowns rather than heads are used and the well is situated at the origin of coordinates. The radius of the well, r_w, is not necessarily very small, but groundwater flow occurring within this radius in the unscreened and uncased aquifers is neglected. It is further assumed that the aquifer system is of infinite extent.

The matrix differential equation for radial flow can, according to eq. 6, be expressed as:

$$d^2\mathbf{t}/dr^2 + d\mathbf{t}/r dr = \mathbf{W}\mathbf{t} \tag{17}$$

where $\mathbf{t} = \mathbf{V}^{-1}\mathbf{s}$ and $\mathbf{s}$ is the drawdown vector: $\mathbf{s} = -\mathbf{h}$. The boundary conditions are:

$$\mathbf{s} = \mathbf{0} \qquad \text{at} \qquad r = \infty$$

$$r d\mathbf{s}/dr = -\mathbf{g} \qquad \text{(vector defined below)} \qquad \text{at} \qquad r = r_w$$

When Q_i is the discharge rate from aquifer i, then:

$$r_w \, ds(r_w)/dr = -Q_i/2\pi T_i$$

so that:

$$\mathbf{g} = \begin{bmatrix} Q_1/2\pi T_1 \\ \cdot \\ \cdot \\ \cdot \\ Q_n/2\pi T_n \end{bmatrix} \tag{18}$$

The system of linear differential equations (17) is similar to the one-aquifer well flow case of modified Bessel type. The general solution can be written as:

$$\mathbf{t}(r) = \mathbf{I}\mathbf{a} + \mathbf{K}\mathbf{b} \tag{19}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors, and $\mathbf{I}$ and $\mathbf{K}$ are $n \times n$ diagonal matrices with $I_0(r\sqrt{w_i})$ and $K_0(r\sqrt{w_i})$ as non-zero elements, respectively. Since the first boundary condition requires that $\mathbf{a} = \mathbf{0}$, eq. 19 reduces to:

$$\mathbf{t} = \mathbf{K}\mathbf{b} \qquad \text{and} \qquad \mathbf{s} = \mathbf{V}\mathbf{K}\mathbf{b} \tag{20a, b}$$

By differentiating eq. 20b and noting that $xK_1(x) \approx 1$ for small values of x, we obtain:

$$-r\,s' \approx \mathrm{Vb} \qquad \text{for} \qquad r\sqrt{w_i} \ll 1 \tag{21}$$

which together with the inner boundary condition yields the complete solution:

$$\mathrm{s} = \mathrm{VKV}^{-1}\mathrm{g} \tag{22}$$

Returning to a notation using summations, eq. 22 can be rewritten as:

$$s_j = \sum_{k=1}^{n} (Q_k/2\pi T_k) \sum_{i=1}^{n} v_{ji} z_{ki} K_0(r/L_i) \tag{23}$$

recalling that $\mathrm{Z}^{\mathrm{T}} = \mathrm{V}^{-1}$ and $1/L_i = \sqrt{w_i}$.

When normalizing the columns of V according to eq. 15, we obtain:

$$s_j = \sum_{k=1}^{n} (Q_k/2\pi) \sum_{i=1}^{n} v_{ji}^* v_{ki}^* K_0(r/L_i) \tag{24}$$

which shows that the well flow formula is also in agreement with the principle of reciprocity.

The similarity between the resulting parallel and radial flow equations is obvious and eq. 12 can in fact, be derived from eq. 22 using the equality:

$$\int K_0\,[(a^2 + y^2)^{1/2}/L]\;\mathrm{d}y \;=\; \pi L \exp(-a/L)$$

2.3. Parallel flow to canals or rivers partially penetrating the leaky top-layer

As a third example of the applicability of eq. 6 a problem with more complicated boundary conditions is chosen. Fig. 4 schematically shows a typical Dutch seepage situation in which a wide river, partially penetrating a leaky top-layer, recharges the surrounding polder area. It is assumed that the groundwater flow is parallel, symmetrical with respect to $x = 0$ and that the polder area is of infinite extent. Solutions for one and two aquifer systems have recently been published by Heij and Kester (1980).

The problem can be described by the following two sets of n differential equations and their boundary conditions:

$$\begin{array}{llll}
\mathrm{h}'' = \mathrm{Ah} + \mathrm{c} & & \text{for} & 0 \leqslant x \leqslant L \\
\mathrm{h}'' = \mathrm{Bh} & & \text{for} & L \leqslant x < \infty \\
\mathrm{h}' = 0 & & \text{at} & x = 0 \\
\mathrm{h} = \mathrm{h}(L) & \text{and} \quad \mathrm{h}' = \mathrm{h}'(L) & \text{at} & x = L \\
\mathrm{h} = 0 & & \text{at} & x = \infty
\end{array}$$

where c = a vector with its first component equal to $h_{\mathrm{r}}/T_1 c_{\mathrm{r}}$ and all other components zero; h_{r} = river level [L]; c_{r} = hydraulic resistance of semi-pervious riverbed [T]; $\mathrm{A} = n \times n$ tridiagonal matrix containing hydraulic

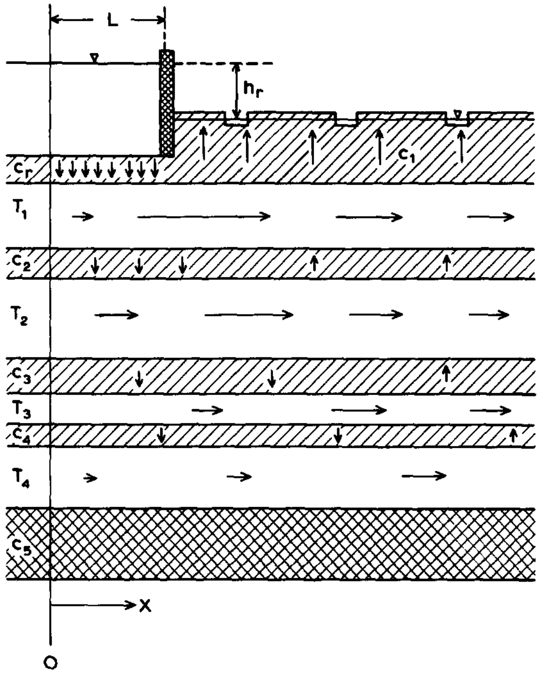

Fig. 4. Schematic diagram of a river partially penetrating the leaky top-layer of a multiple-aquifer system and recharging the surrounding polder area.

properties of all layers in the river area ($x \leqslant L$); and **B** = similar to A, but relating to the region $x \geqslant L$. All other notations are as previously defined.

Solution of the problem is obtained by first solving the differential equations for each region separately while satisfying the boundary conditions at $x = 0$ and $x = \infty$. In a second stage these results are coupled by applying the common boundary conditions. When all the eigenvectors of matrices **A** and **B** constitute the columns of matrices **V** and **Q**, respectively, while the corresponding eigenvalues are elements of the diagonal matrices **W** and **U**, we can decompose **A** and **B** and write:

$$\mathbf{A} = \mathbf{VWV}^{-1} \quad \text{and} \quad \mathbf{B} = \mathbf{QUQ}^{-1} \tag{25}$$

(a) *Inner region* ($0 \leqslant x \leqslant L$). Using eq. 25, the differential equation becomes:

$$(\mathbf{V}^{-1}\mathbf{h})'' = \mathbf{W}(\mathbf{V}^{-1}\mathbf{h}) + \mathbf{V}^{-1}\mathbf{c}$$

A solution satisfying the inner boundary condition can thus be calculated, yielding:

$$\mathbf{h} = \mathbf{VMa} - \mathbf{VW}^{-1}\mathbf{V}^{-1}\mathbf{c} \tag{26}$$

where **a** is a vector to be determined from the conditions at $x = L$ and **M** is a diagonal matrix with $\cosh(x\sqrt{w_i})$ as elements. Differentiating eq. 26 and using the still unknown $\mathbf{h}(L)$ gives:

$$\mathbf{h}'(L) = \mathbf{V}\mathbf{M}'(L)\,\mathbf{M}^{-1}(L)\mathbf{V}^{-1}\,[\mathbf{h}(L) + \mathbf{V}\mathbf{W}^{-1}\mathbf{V}^{-1}\mathbf{c}] \tag{27}$$

(b) *Outer region* ($L \leqslant x < \infty$). From eq. 25, the differential equation can be written as:

$$(\mathbf{Q}^{-1}\mathbf{h})'' = \mathbf{U}(\mathbf{Q}^{-1}\mathbf{h})$$

With the boundary condition at $x = \infty$ this yields:

$$\mathbf{h} = \mathbf{Q}\mathbf{N}\mathbf{b} \tag{28}$$

where $\mathbf{b}$ is a vector to be determined from the condition at $x = L$ and N is a diagonal matrix with $\exp(-x\sqrt{u_i})$ as elements. Continuing in the same way as above and eliminating $\mathbf{b}$, it will follow that:

$$\mathbf{h}'(L) = \mathbf{Q}\mathbf{N}'(L)\mathbf{N}^{-1}(L)\,\mathbf{Q}^{-1}\mathbf{h}(L) \tag{29}$$

Piezometric heads in all aquifers at the boundary $x = L$ can now be obtained by equating eqs. 27 and 29. Thus:

$$\mathbf{h}(L) = (\mathbf{Q}\mathbf{N}'\mathbf{N}^{-1}\mathbf{Q}^{-1} - \mathbf{V}\mathbf{M}'\mathbf{M}^{-1}\mathbf{V}^{-1})^{-1}\ \mathbf{V}\mathbf{M}'\mathbf{M}^{-1}\mathbf{W}^{-1}\mathbf{V}^{-1}\mathbf{c}$$

or

$$\mathbf{h}(L) = (\mathbf{Q}\mathbf{N}^{*}\mathbf{Q}^{-1} - \mathbf{V}\mathbf{M}^{*}\mathbf{V}^{-1})^{-1}\ \mathbf{V}\mathbf{P}^{*}\mathbf{V}^{-1}\mathbf{c} \tag{30}$$

where $\mathbf{N}^*$, $\mathbf{M}^*$ and $\mathbf{P}^*$ are diagonal matrices with the elements $-\sqrt{u_i}$, $\sqrt{w_i} \times \tanh(L\sqrt{w_i})$ and $[\tanh(L\sqrt{w_i})]/\sqrt{w_i}$, respectively. Once $\mathbf{h}(L)$ has been calculated all other heads can be obtained from:

$$\mathbf{h} = \begin{cases} \mathbf{V}\mathbf{M}\mathbf{M}^{-1}(L)\mathbf{V}^{-1}\,[\mathbf{h}(L) + \mathbf{V}\mathbf{W}^{-1}\mathbf{V}^{-1}\mathbf{c}] - \mathbf{V}\mathbf{W}^{-1}\mathbf{V}^{-1}\mathbf{c}, & \text{for } 0 \leqslant x \leqslant L \\[2ex] \mathbf{Q}\mathbf{N}\mathbf{N}^{-1}(L)\,\mathbf{Q}^{-1}\mathbf{h}(L), & \text{for } L \leqslant x < \infty \end{cases} \tag{31}$$

At first glance this result appears rather complex, but once the eigenvalues and eigenvectors are evaluated the calculation is straightforward and can easily be carried out on a microcomputer.

In comparison with the simple solutions for leaky multiple-aquifer parallel and radial flow (eqs. 12 and 22), the latter example is less generally applicable. However, it illustrates quite well how the presented solution method can be used to analytically solve a variety of steady-state problems.

3. ANALYSIS OF FLOW IN MULTILAYERED AQUIFERS

The presented approach to multiple-aquifer systems originated from the need to analyse multilayer pumping-test data. Although the method used to determine hydraulic properties is not dealt with in this paper, final results are included as an example of the applicability of the derived well flow formulae. First, however, some results of calculated head distributions in multilayered aquifers are presented for different types of flow.

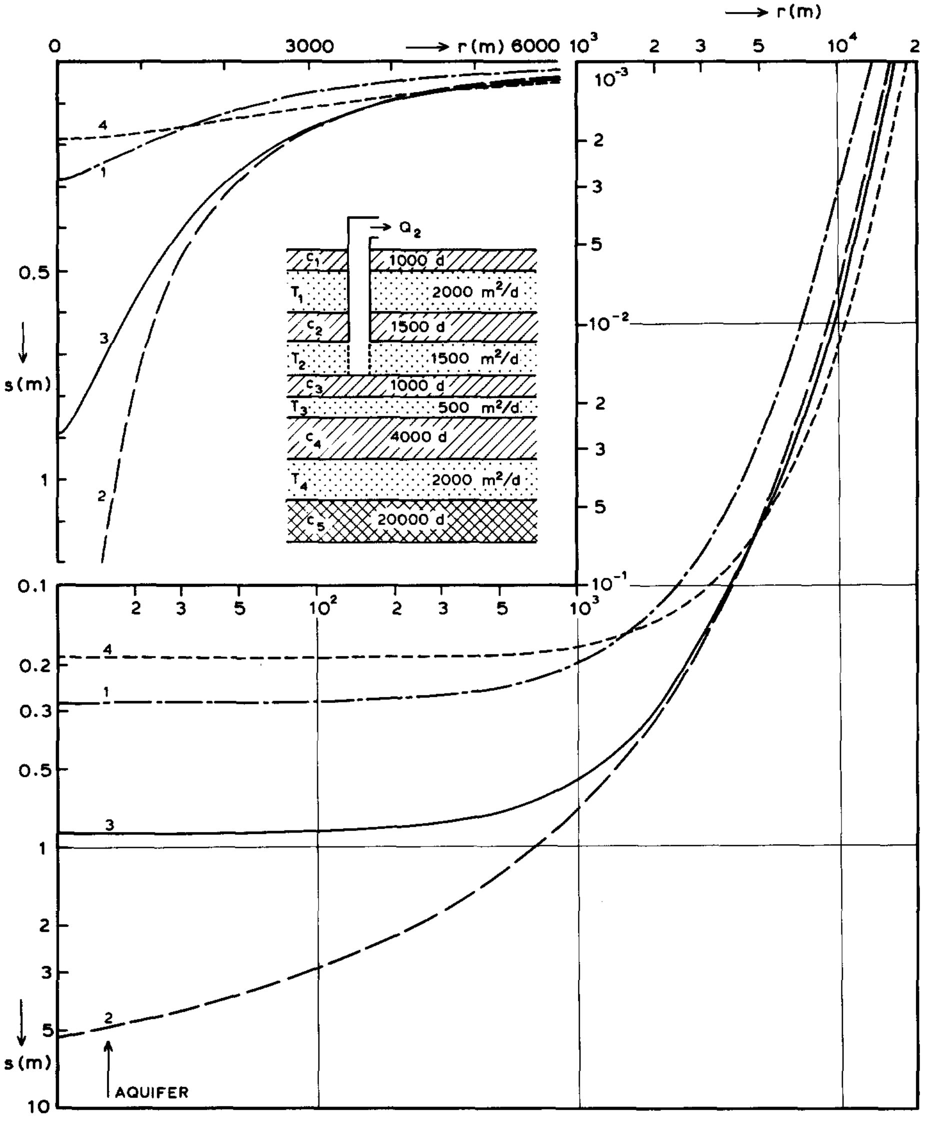

Fig. 5. Linear and double logarithmic distance—drawdown variations around a well discharging 10,000 m^3 day^{-1} from the second aquifer of a leaky four-aquifer system.

In the polder area around Lexmond (Fig. 1) four different semi-confined aquifers may be distinguished within the sand and clay deposits of Pleistocene—Holocene age, reaching a depth of 250 m. Based on the pumping-test results, transmissivities are assumed to average 2000, 1500, 500 and 2000 m^2 day^{-1} for the top and deeper aquifers, respectively. Estimates of hydraulic resistance for the covering and intermediate semi-pervious layers are 1000, 1500, 1000 and 4000 days. The base is probably nearly impervious

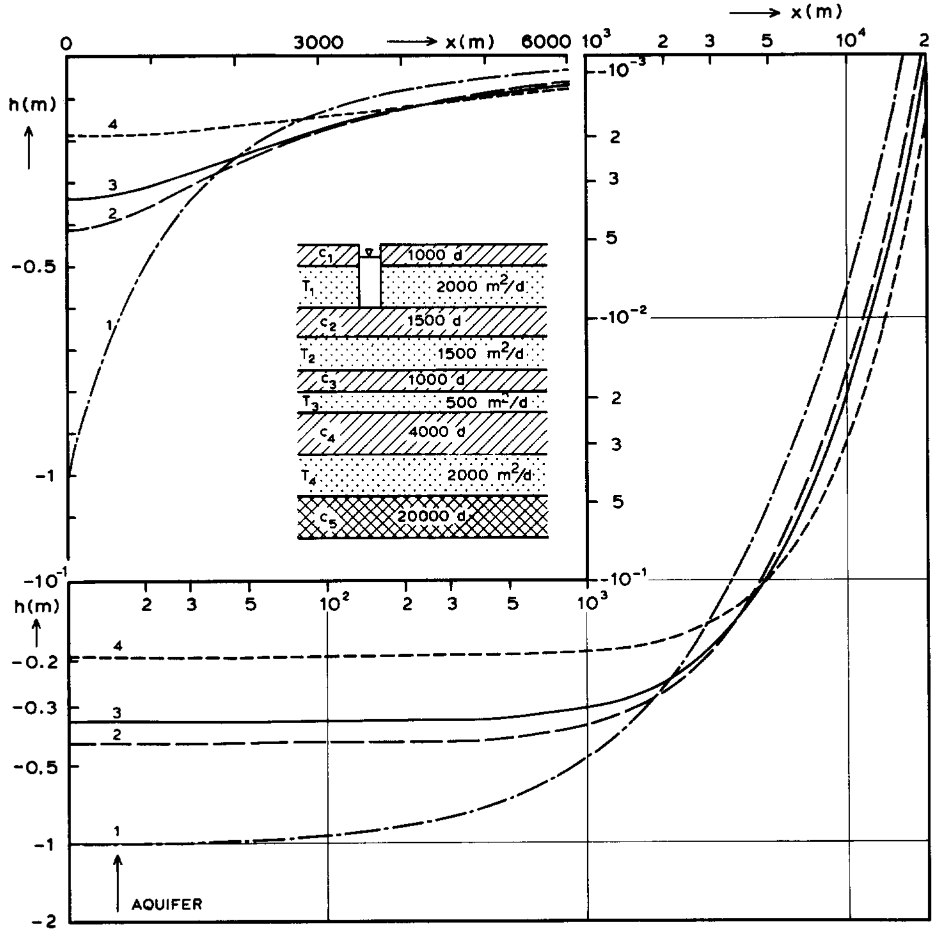

Fig. 6. Linear and double logarithmic distance—drawdown variations perpendicular to a drain or gallery in the top aquifer of a leaky four-aquifer system. Discharge capacity amounts to 3.15 m^2 day^{-1} per metre lowering of the water level.

and is characterised by a value of 20,000 days, all other large values leading to essentially the same results.

To predict well-field drawdown the influence of a single well screened in the second aquifer and discharging 10,000 m^3 day^{-1} is computed. Results are presented in Fig. 5 and may be compared with the head distribution resulting from a fully penetrating gallery in the upper aquifer, as given in Fig. 6. In these examples the effects of induced infiltration by the nearby Rhine River are not taken into account. All values of matrix V^*, comprising the normalized eigenvectors, together with the associated eigenvalues and leakage factors, are shown in Table I.

Referring to Figs. 5 and 6, one finds that at an increasing distance from the well or gallery, drawdown in the different aquifers decreases in such a

TABLE I

Eigenvectors, eigenvalues and leakage factors of the four-aquifer system shown in Figs. 5 and 6

V^*	v_1^*	v_2^*	v_3^*	v_4^*
	−0.00181	−0.01893	0.01103	0.00408
	0.01279	0.01085	0.01730	0.00928
	−0.03855	0.01403	0.01342	0.01170
	0.00159	−0.00201	−0.01044	0.01961
w (10^{-6} m^{-2})	3.1840	1.0244	0.3106	0.0754
L (m)	560.4	988.0	1,794.3	3,641.8

way that the largest values, occurring first in the pumped aquifer, gradually descend to deeper levels. This feature is easily explained, since a deeper aquifer, drained in the vicinity of the well (or gallery), can at some distance only be recharged by its upper boundary, assuming an essentially impervious base. From drawdown formula (24) it is clear that at an increasing distance the term with the largest L_i approaches zero most slowly. This implies that whichever aquifer is pumped, independent of the flow type (radial or parallel), drawdown in all aquifers will, at an increasing distance, approach a ratio determined by the eigenvector associated with the largest leakage factor. Table I shows that for the set of hydraulic properties concerned, any recharge or discharge will cause a long-distance drawdown ratio of 0.00408: 0.00928:0.01170:0.01961.

Similar reasoning may be used to calculate the ratio of residual drawdown during pseudo-steady-state flow. Although this ratio is of practical importance when analysing pumping-test data, transient flow is beyond the scope of this paper and will not be discussed here.

From the boundary conditions of the unpumped aquifers it is obvious that ds/dr approaches zero in the vicinity of a well. Therefore, a constant drawdown ratio in the unpumped aquifers must be present and it can be shown that if the kth aquifer is pumped:

$$s_j/s_m = \sum_i v_{ji}^* v_{ki}^* \ln(L_i) \Big/ \sum_i v_{mi}^* v_{ki}^* \ln(L_i), \quad j,\ m \neq k$$

In the case of parallel flow, $\ln(L_i)$ should be replaced by L_i.

A further example considers parallel groundwater flow resulting from a recharging river, such as the Rhine in the polder area of western Holland (Fig. 1). Assuming the same set of hydraulic properties as in the well-flow example, a river level of +1 m and an effective river-bed width and resistance of 500 m and 100 days, respectively, one can compute the resulting head distribution with eqs. 30 and 31. The rate of stream depletion (both sides) is given by:

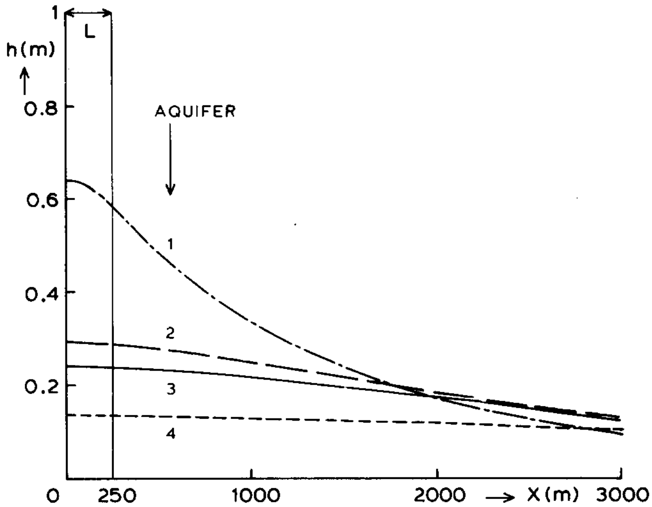

Fig. 7. Distance—drawdown variations perpendicular to the river with semi-pervious bed indicated in Fig. 4. Hydraulic properties of the leaky four-aquifer system are as shown in Fig. 6. Infiltration capacity amounts to 1.90 m^2 day^{-1} per metre rise in river level.

$$q_{\mathrm{TOTAL}} = -\sum_{i=1}^{n} 2T_i h_i'(L)$$

where $h_i'(L)$ can be obtained from eq. 29. Calculation results are presented in Fig. 7 and the recharge rate was found to be 1.90 m^3 m^{-1} day^{-1}.

Some results of a multilayer pumping-test analysis may now be presented to give an example of the practical use for the derived multiple-aquifer well-flow formula. A brief description of the hydrogeology for the area concerned has been given in the Introduction. In a 200-m deep borehole drilled at the "Langerak" test-site (Fig. 1), six individual aquifers could be distinguished and at least one piezometer was installed in each. Two pumping-wells were drilled at 41.2- and 30.5-m distance from the borehole and were screened in the second and third aquifers, respectively. Additional piezometers were installed in the upper three aquifers.

The pumping test lasted 19 days, continuously discharging the second aquifer at a rate of 1950 m^3 day^{-1}. During the last 8 days the third aquifer was pumped at a rate of 1750 m^3 day^{-1}. From the analysis of time-drawdown curves for all piezometers, including control piezometers outside the zone of influence, steady-state drawdowns were separately determined for both aquifer tests. Table II summarizes these results.

It is clear that the observed drawdowns provide insufficient information to determine transmissivity values for all six aquifers and resistances for all seven aquitards. A separation must therefore be made between hydraulic characteristics whose values need to be assumed (e.g., from other studies, estimations based on thickness and lithology) and those which may be determined. In the example presented seven parameters are assumed known,

TABLE II
Corrected measured drawdowns for the "Langerak" pumping-test

Aquifer	Piezometer depth (m)	Drawdown (cm)					
		2nd aquifer $Q = 1950\,\mathrm{m^3\,day^{-1}}$			3rd aquifer $Q = 1750\,\mathrm{m^3\,day^{-1}}$		
		$r \approx 0.2$ m	$r = 10.7$ m	$r = 41.2$ m	$r \approx 0.2$ m	$r = 10.7$ m	$r = 30.5$ m
1	18–20	2	2	3	1	2	3
2	80–82	(377)	168	128	14	16	16
3	110–112	14	14	14	(754)	316	237
4	127–129	—	—	12	—	—	51
5	142–144	—	—	4	—	—	11
6	167–169	—	—	2	—	—	6

Drawdowns near pumping-screens (*377*) and (*754*) are not used for further analysis.

TABLE III
Difference between calculated and measured drawdowns for the "Langerak" pumping-test

Aquifer	Piezometer depth (m)	Drawdown (cm)					
		2nd aquifer $Q = 1950\,\mathrm{m^3\,day^{-1}}$			3rd aquifer $Q = 1750\,\mathrm{m^3\,day^{-1}}$		
		$r \approx 0.2$ m	$r = 10.7$ m	$r = 41.2$ m	$r \approx 0.2$ m	$r = 10.7$ m	$r = 30.5$ m
1	18–20	+ 1.6	+ 1.6	+ 0.6	+ 0.8	− 0.2	− 1.2
2	80–82	—	+ 1.3	− 1.9	− 0.2	− 2.2	− 2.3
3	110–112	+ 1.3	+ 1.3	+ 1.3	—	− 0.1	+ 0.1
4	127–129	—	—	− 0.2	—	—	+ 0.1
5	142–144	—	—	+ 0.3	—	—	− 0.5
6	167–169	—	—	+ 1.0	—	—	+ 0.2

Reduced sum of squares is 29.6 $\mathrm{cm^2}$.

viz. $T_1 = 2100$, $T_5 = 300$ and $T_6 = 1200\,\mathrm{m^2\,day^{-1}}$, $c_1 = 900$, $c_2 = 7000$, $c_6 = 1200$ and $c_7 = 10{,}000$ days. Regression analysis allows derivation of the following values: $T_2 = 965$ (2%), $T_3 = 368$ (3%), $T_4 = 585$ (21%) $\mathrm{m^2\,day^{-1}}$ and $c_3 = 3986$ (18%), $c_4 = 478$ (20%), $c_5 = 5356$ (36%) days; percentages indicate a measure of the result uncertainty. To illustrate the close approximation of measured and calculated drawdowns, differences are given in Table III.

4. CONCLUDING REMARKS

Numerous analytical solutions to steady-state single-aquifer groundwater flow problems are available, and as each problem has its leaky multiple-aquifer counterpart it is not feasible to explore all possible solutions. For this reason, particular attention has been paid here to the method of solution rather than to the solutions themselves. Many steady-state solutions may be derived, however, even concerning impervious layers and phreatic conditions.

One of the general assumptions made was a semi-pervious top and bottom, but if either were regarded as impermeable a_1 or b_n in eq. 4 will be zero.

If the system is confined ($a_1 = b_n = 0$), a special case is obtained, since the smallest eigenvalue will be zero and the largest leakage factor will consequently be infinite. In this event a solution for steady-state flow can still be derived as the elements of the diagonal matrices have a limit, when L_i approaches infinity. Since all eigenvector components associated with $L_i = \infty$ are equal, it follows from eq. 15 that $(v_i^*)^2$ can be replaced by $(\Sigma_j T_j)^{-1}$, reducing the term in question to the solution of the allied confined one-aquifer problem. When, for example, two wells are present, discharging aquifer k at distance r and recharging aquifer m at distance t, the drawdown in aquifer j will be given by the appropriate modification of eq. 24, yielding:

$$s_j = (Q/2\pi) \sum_{i=1}^{n-1} v_{ji}^* \quad [v_{ki}^* K_0(r/L_i) - v_{mi}^* K_0(t/L_i)] + (Q/2\pi \Sigma T_i) \ln (t/r)$$

Solutions for leaky systems with an unconfined top aquifer can be obtained if differences in saturated thickness remain comparatively small. In this case a_1 will again be zero in eq. 4, but if the unconfined aquifer is recharged from above by infiltrating soil water the differential equation should be extended with an extra term such as c in the example of the infiltrating, partially-penetrating river. Solutions of this type include the two-layer problem solved by Motz (1978).

Problems involving non-homogeneities of relatively simple form can in some cases be solved by superposition. One example is the effect of circular natural or artificial hydraulic connections between two aquifers, which can be treated as a sink and a source, discharging and recharging at the same rate and compensating the local difference in head between both aquifers.

In the discussion relating horizontal and vertical flow directions to conductivity contrasts and thickness ratios of multi-aquifer systems, it was suggested that layers of intermediate conductivity should be subdivided into a number of aquifers and aquitards to find approximate solutions of the head distribution. However, it should be noted that the multi-aquifer solution, which is purely analytical in a richly contrasting aquifer and aquitard system, tends to a semi-numerical solution when horizontal and vertical flow components are separated in the model by introducing a number of sublayers. Instead of treating the elements in a numerical model the x—z (or x—r) -plane is now modelled by a number of layers with finite or infinite horizontal extent. Discretization is in the z-direction only. Just as numerical solutions yield values for only a predetermined, finite number of points, multi-aquifer solutions are one-dimensional analytical solutions for only a predetermined finite number of horizontal lines.

The technique of introducing sublayers proved to be useful when vertical flow components for the deepest aquifer of a multilayer system had to be calculated in order to predict the rate of upconing brackish groundwater induced by a planned well field.

F. Székely (pers. commun., 1983) indicated that Hungarian hydrologists have been studying multilayer well-flow problems for several years, with results published in Hungarian and Russian (Halász, 1975; Székely, 1978; Halász and Székely, 1979). By means of integral transforms and polynomial techniques a steady-state well-flow formula has been developed (Halász and Székely, 1979, formula (12)):

$$s_j(r) = \sum_{i=1}^{n} A_{ji} K_0(ra_i)$$

This formula may be compared with well-flow equation (24) if only one aquifer is discharged. A_{ji} may be equated with $Qv_{ji}^* \, v_{ki}^*/2\pi$, while a_i^2 is identical to the eigenvalues w_i. The essential difference is that this matrix A depends not only on the hydraulic properties and the discharge rate, but also on the flow conditions (the pumped aquifer in question), while matrix $\mathbf{V}^*$ is determined by the aquifer system alone.

Other multilayer solutions presented by Halász and Székely concern pseudo-steady state and transient well-flow and future investigations will show in which way separately developed solutions will mutually contribute.

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