Game Theory
Programming Assignment – Gale-Shapley Algorithm

Programming Problem: Write a Matlab function that simulates the Gale-Shapley algorithm. The input/output syntax should be of the form:

```
[Match,NumStages] = GaleShapleyAlgorithm(P1,P2)
```

where

- `P1` and `P2` are the preference structures for Group 1 and Group 2 respectively. Both matrices are of dimension $m \times n$ (i.e., m number of rows and n number of columns), where n is the size of Group 1 and m is the size of Group 2. With regards to Question #2 given in Lecture #5, Group 1 is the drivers ($n = 5$), Group 2 is the riders ($m = 3$), $P1$ is the matrix representing the Drivers' preferences, and $P2$ is the matrix representing the Riders' preferences where

$$P1 = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 1 & 1 & 2 \\ 3 & 2 & 3 & 2 & 1 \end{bmatrix} \qquad P2 = \begin{bmatrix} 2 & 1 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \\ 3 & 1 & 4 & 2 & 5 \end{bmatrix}$$

- `Match` is a binary matrix of dimension $m \times n$ which indicates the matches after running the Gale-Shapley algorithm. For example, consider the case:

$$\text{Match} \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

  This situation corresponds to the scenario where the first individual in Group 1 is unmatched, second individual in Group 1 is matched with the first individual in Group 2, third individual in Group 1 is matched with the second individual in Group 2, fourth individual in Group 1 is unmatched, and fifth individual in Group 1 is matched with the third individual in Group 2. Note that there can be at most a single entry of 1 in any row or column of the match matrix.

- `NumStages` is an integer returning the number of stages that it took the Gale-Shapley algorithm to return a proposal.

1. Verify your answers for Questions 1 and 2 from Lecture Notes #5. Include a screen shot of the results of your Matlab code being executed on the appropriate matrices.

2. Three examples of a medical school admission problem are posted on Canvas. Here, P1 indicates the hospital preferences (row i describes hospital i's preferences over the students) and P2 indicates the student preferences (column j describes student j's preferences over the hospitals). Solve each of the three matching problems using the Gale-Shapley algorithm when (a) applicants apply to medical schools and (b) when medical schools make bids to applicants. Then, for each example, answer the following questions:

(a) Is there a unique stable matching?

(b) How many stages did the Gale-Shapley algorithm take when applicants made bids to medical schools? What about when medical schools made bids to applicants?

(c) Discuss the optimality of the solution for both groups.

3. Based on your answers to the previous question, do you think the Gale-Shapley algorithm should be run with the applicants proposing or with the hospitals proposing? Does it make a difference? Explain your reasoning.

   Note: This question was heavily debated and researched within the context of the National Resident Matching Program (NRMP) in the US. After coming up with your own answer, you can read this article to learn more about how the NRMP addressed this issue.

4. Now, consider the case where each hospital has a quota pertaining to the number of applicants they can accept. For example, if a hospital has a quota of 3, then the hospital can accept up to 3 applicants. Make a modification to the your algorithm where the new input/output syntax should be of the form:

   ```
   [Match,NumStages] = GaleShapleyAlgorithmQuota(P1,P2,quota)
   ```

   where

   - `quota` is a vector of appropriate length with the quota for each hospital.

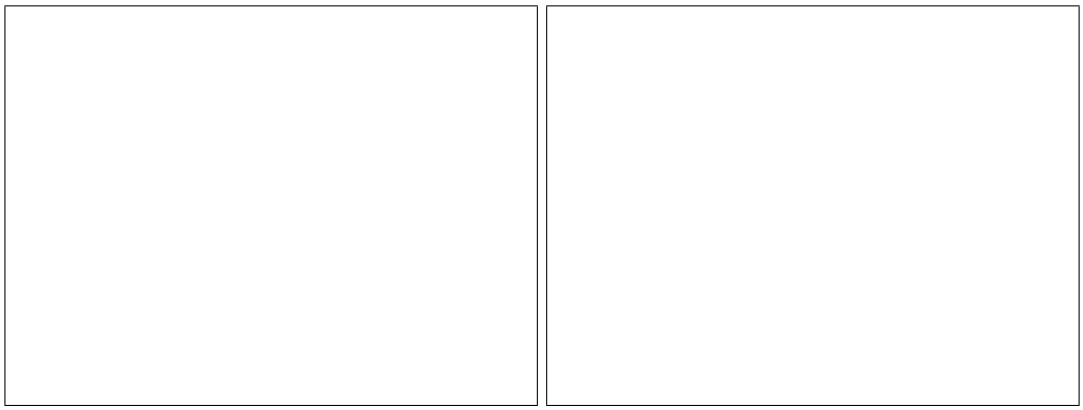
   Complete the exercise from (b) where you now consider the quotas of the hospitals. Make sure to answer parts i-iii for both examples.

5. Did you observe multiple stable matchings in either example? If so, discuss the difference in the number of unoccupied residency positions in each matching. If not, construct a simple example of a residency admissions problem with quotas that has multiple stable matchings. Does a hospital fill more of its quota when hospitals make bids in the Gale-Shapley algorithm? Can you think of any real-world implications that your observations might suggest?

   Note: This question points to an interesting feature of the Gale-Shapley algorithm. After coming up with your own answer, you can read this article to learn more about this issue.

Write-up Template

1. Verify your answers for Questions 1 and 2 from Lecture Note #5. Include a screen shot of the results of your Matlab code being executed on the appropriate matrices.



2. Solve each of the three matching problems using the Gale-Shapley algorithm when (a) applicants apply to medical schools and (b) when medical schools make bids to applicants. Then, for each example, answer the following questions:

   (a) Is there a unique stable matching?

   (b) How many stages did the Gale-Shapley algorithm take when applicants made bids to medical schools? What about when medical schools made bids to applicants?

   (c) Discuss the optimality of the solution for both groups.

3. Based on your answers to the previous question, do you think the Gale-Shapley algorithm should be run with the applicants proposing or with the hospitals proposing? Does it make a difference? Explain your reasoning.

4. Solve each of the three matching problems **with quotas** using the Gale-Shapley algorithm when (a) applicants apply to medical schools and (b) when medical schools make bids to applicants. Then, for each example, answer the following questions:

   (a) Is there a unique stable matching?

   (b) How many stages did the Gale-Shapley algorithm take when applicants made bids to medical schools? What about when medical schools made bids to applicants?

   (c) Discuss the optimality of the solution for both groups.

5. Did you observe multiple stable matchings in either example? If so, discuss the difference in the number of unoccupied residency positions in each matching. If not, construct a simple example of a residency admissions problem with quotas that has multiple stable matchings. Does a hospital fill more of its quota when hospitals make bids in the Gale-Shapley algorithm? Can you think of any real-world implications that your observations might suggest?