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Global efficiency of graphs

Bryan Ek[a], Caitlin VerSchneider[b], Darren A. Narayan[a,*]

[a] School of Mathematical Sciences, Rochester Institute of Technology, Rochester, NY 14623, United States
[b] Mathematics Department, Nazareth College, Rochester, NY 14618, United States

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Abstract

The distance $d(i, j)$ between any two vertices i and j in a graph is the number of edges in a shortest path between i and j. If there is no path connecting i and j, then $d(i, j) = \infty$. In 2001, Latora and Marchiori introduced the measure of efficiency between vertices in a graph (Latora and Marchiori, 2001). The efficiency between two vertices i and j is defined to be $\in_{i,j} = \frac{1}{d(i,j)}$ for all $i \neq j$. The *global efficiency* of a graph is the average efficiency over all $i \neq j$. The concept of global efficiency has been applied to optimization of transportation systems and brain connectivity. In this paper we determine the global efficiency for complete multipartite graphs $K_{m,n}$, star and subdivided star graphs, and the Cartesian Products $K_n \times P_n^m$, $K_n \times C_n^m$, $K_m \times K_n$, and $P_m \times P_n$.

Keywords: Distance; Efficiency; Harary index

1. Introduction

The distance $d(i, j)$ between any two vertices i and j in a graph is the number of edges in a shortest path between i and j. If there is no path connecting i and j, then $d(i, j) = \infty$. In 2001, Latora and Marchiori introduced the measure of efficiency between vertices in a graph [1]. The (unweighted) *efficiency* between two vertices i and j is defined to be $\in_{i,j} = \frac{1}{d(i,j)}$ for all $i \neq j$. The *global efficiency* of a graph G (with n vertices) is denoted $E_{glob}(G) = \frac{1}{n(n-1)} \sum_{i \neq j} \in (v_i, v_j)$, which is simply the average of the efficiencies over all pairs of vertices. Global efficiency has emerged in a plethora of real world applications including optimization of transportation systems [1–4], as well as brain connectivity [5,6] and [7]. Recently, Ek, Verschneider, and Narayan applied the concept of global efficiency to the metro system in Atlanta [3].

The concept of reciprocal distance has been studied previously. In 1993, Plavšić, Nikolić, Trinajstić, and Mihalić introduced the *Harary index* of a simple graph [8]. For a simple graph G with vertices $v_1, v_2, \ldots, v_n$ the Harary

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* Corresponding author.
E-mail address: dansma@rit.edu (D.A. Narayan).

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index is denoted $H(G)$ and equals $\sum_{1\leq i<j\leq n}\frac{1}{d(v_i,v_j)}$. We note the close relationship between global efficiency and the Harary index, $E_{glob}(G)=\frac{2}{n(n-1)}H(G)$. There also have been other studies involving the Harary index [9–13].

In this paper, we determine the global efficiency for several families of graphs, including powers of graphs and Cartesian Products. Recall the kth power of a graph G, which is denoted G^k, where $V\left(G^k\right)=V(G)$ and $(u,v)\in E(G^k)$ if and only if the distance between u and v in G is less than or equal to k. We determine the global efficiency for complete multipartite graphs, star and subdivided star graphs, and the Cartesian Products $K_n\times P_n^m$, $K_n\times C_n^m$, $K_m\times K_n$, and $P_m\times P_n$. As a consequence, we determine new results involving the Harary index for these families of graphs.

1.1. Complete multipartite graphs

We will use $K_{t_1,t_2,\ldots,t_r}$ to denote the complete multipartite graph with r parts with t_i vertices where $1\leq i\leq r$. We note that the distance between any pair of vertices in different classes is 1 and the distance between any pair of vertices in the same class is 2.

This leads to our next theorem.

Theorem 1. *Let $G=K_{t_1,t_2,\ldots,t_r}$ where $n=\sum_{i=1}^r t_i$. Then*

$$E_{glob}(G)=\frac{1}{n(n-1)}\sum_{i=1}^{r}t_i\left[\frac{(t_i-1)}{2}+(n-t_i)\right]. \tag{1}$$

Proof. Let v_i be a vertex in a part with t_i vertices. Then the shortest path from v_i to any vertices in the same part is 2 and v_i is adjacent to all vertices in other parts. Hence the efficiencies of all pairs containing v_i is $\sum_{i=1}^r t_i\left[\frac{(t_i-1)}{2}+(n-t_i)\right]$. Averaging over all vertices gives the desired result. ■

1.2. Star graphs and subdivided star graphs

A *star graph* is a complete bipartite graph where one part is a single vertex, and the other part has at least one vertex. We will use $S(k,l)$ to denote the star graph with k spokes each of length l. We next recall the well known operation of an edge subdivision.

Definition 2. An edge subdivision is an operation that is applied to an edge (u,v) where a new vertex w is inserted, and the edge (u,v) is replaced by edges (u,w) and (w,v). A subdivision H of a graph G is a graph that can be obtained by performing a sequence of edge subdivisions.

Definition 3. Let $S_{d,l}$ be the subdivision of the star $K_{1,d}$ where each of the d edges is replaced by a path with l vertices.

The graph $S_{4,3}$ is shown in Fig. 1.

The efficiency matrix for $S_{4,3}$ is given in Fig. 2.

We note that the patches of four identical entries across the top row begin a downward diagonal with an equivalent entry. On each side of the diagonals, there are patches of identical entries. The sum of these diagonals equals $\frac{4(3)}{1}+\frac{4(2)}{2}+\frac{4(1)}{3}$. In the general case we have $\sum_{i=1}^{L}\frac{D(L+1-i)}{i}$, which we divide into two sums. The first sum includes the largest number of patches; the second sum begins with the smallest efficiency patch and includes the patches up to but not including the largest section. In our example the first sum equals $\frac{4(3)}{2}\cdot\frac{1}{2}+\frac{4(3)}{2}\cdot\frac{2}{3}+\frac{4(3)}{2}\cdot\frac{3}{4}$. The second sum equals $\frac{4(3)}{2}\cdot\frac{1}{6}+\frac{4(3)}{2}\cdot\frac{2}{5}$.

Now we proceed with the general case. We will use $S_{D,L}$ to denote a subdivided star graph with D arms of length L. Let $v_{d,l}$ denote the lth vertex along the dth arm. The center vertex is denoted v_0. The distance between non-center vertices is then

$$d(v_{d,l},v_{d',l'})=\begin{cases} l+l' & \text{if } d\neq d'\\ |l-l'| & \text{if } d=d'.\end{cases} \tag{2}$$

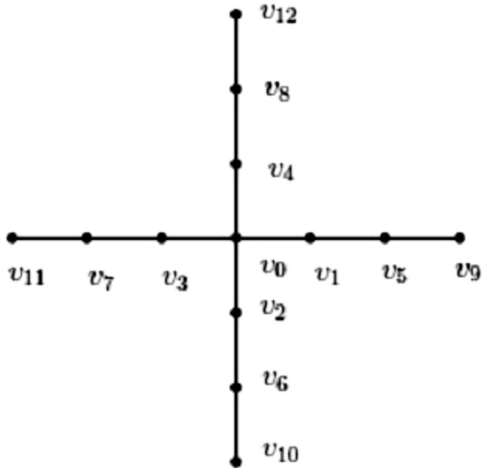


Fig. 1. The star graph $S_{4,3}$.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_0	0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
v_1		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
v_2			0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
v_3				0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
v_4					0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
v_5						0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
v_6							0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	1	$\frac{1}{5}$	$\frac{1}{5}$
v_7								0	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{5}$	1	$\frac{1}{5}$
v_8									0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
v_9										0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
v_{10}											0	$\frac{1}{6}$	$\frac{1}{6}$
v_{11}												0	$\frac{1}{6}$
v_{12}													0

Fig. 2. The efficiency matrix for $S_{4,3}$.

The sum of efficiencies can be broken up into two different sections: efficiencies along the same arm and efficiencies between arms. First, we consider the efficiencies along the same arm. Note each arm is isomorphic to every other arm so the efficiencies along one arm are identical to all other arms. A single arm, including v_0, is isomorphic to P_{L+1} so we have this sum to be

$$D \cdot \sum_{k=1}^{L} \sum_{i=1}^{k} \frac{1}{i} = D \sum_{k=1}^{L} \frac{L+1-k}{k} = D \left[(L+1) \sum_{k=1}^{L} \frac{1}{k} - L \right] = D \left[(L+1) H_L - L \right] \tag{3}$$

where H_L is the Lth harmonic number. Second, we consider the efficiencies along different arms. There are $\binom{D}{2}$ different pairs of arms. Since we are considering unweighted star graphs, each pair is isomorphic. Using Eq. (2), the sum between two arms is:

$$\begin{aligned}
\sum_{j=1}^{L} \sum_{i=1}^{L} \frac{1}{i+j} &= \sum_{k=2}^{L+1} \frac{k-1}{k} + \sum_{k=L+2}^{2L} \frac{2L-(k-1)}{k} \\
&= \sum_{k=1}^{L} \frac{k}{k+1} + \sum_{k=1}^{L-1} \frac{2L-[(k+L+1)-1]}{k+L+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{L} \frac{k}{k+1} + \sum_{k=1}^{L} \frac{2L-[k+L]}{k+L+1} \\
&= \sum_{k=1}^{L} \left[\frac{k}{k+1} + \frac{L-k}{k+L+1} \right] \\
&= \sum_{k=1}^{L} \left[\frac{2L+1}{k+L+1} - \frac{1}{k+1} \right] \\
&= \sum_{k=1}^{L} \frac{L(2k+1)}{(k+L+1)(k+1)}.
\end{aligned}
$$

The last few expressions were given as options. The total from different arms is then

$$
\frac{D(D-1)}{2} \sum_{k=1}^{L} \left[\frac{2L+1}{k+L+1} - \frac{1}{k+1} \right]. \tag{4}
$$

Finally we can combine the two sums: Eqs. (3) and (4), and normalize with $n = DL+1$ to obtain the global efficiency given in Eq. (5).

Theorem 4.

$$
\begin{aligned}
E_{glob}(S_{D,L}) &= \frac{2}{DL(DL+1)} \left(D \sum_{k=1}^{L} \frac{L+1-k}{k} + \frac{D(D-1)}{2} \sum_{k=1}^{L} \left[\frac{2L+1}{k+L+1} - \frac{1}{k+1} \right] \right) \\
&= \frac{2}{L(DL+1)} \sum_{k=1}^{L} \left(\frac{L+1-k}{k} + \frac{D-1}{2} \left[\frac{2L+1}{k+L+1} - \frac{1}{k+1} \right] \right). \qquad (5)
\end{aligned}
$$

The sum of efficiencies for a star graph with D spokes, each of L length is:

$$
\begin{aligned}
\sum_{i,j} \epsilon_{ij} &= 2 \left(\sum_{i=1}^{L} \frac{D(L+1-i)}{i} + \sum_{i=1}^{L} \frac{D(D-1)}{2} \left(\frac{i}{i+1} + \frac{i}{2L+1-i} \right) + \sum_{i=1}^{L-1} \frac{D(D-1)}{2} \cdot \frac{L}{L+1} \right) \\
&= 2D \left(\sum_{i=1}^{L} \left[\frac{(L+1-i)}{i} + \frac{(D-1)}{2} \left(\frac{i}{i+1} + \frac{i}{2L+1-i} \right) \right] - \frac{(D-1)}{2} \cdot \frac{L}{L+1} \right). \qquad (6)
\end{aligned}
$$

The unweighted global efficiency, with $n = D(L) + 1$, is:

$$
\begin{aligned}
&E_{glob}((D, L)) \\
&= \frac{2D}{n(n-1)} \left(\sum_{i=1}^{L} \left[\frac{(L+1-i)}{i} + \frac{(D-1)}{2} \left(\frac{i}{i+1} + \frac{i}{2L+1-i} \right) \right] - \frac{(D-1)}{2} \cdot \frac{L}{L+1} \right) \qquad (7) \\
&= \frac{2}{L(DL+1)} \left(\sum_{i=1}^{L} \left[\frac{(L+1-i)}{i} + \frac{(D-1)}{2} \left(\frac{i}{i+1} + \frac{i}{2L+1-i} \right) \right] - \frac{(D-1)}{2} \cdot \frac{L}{L+1} \right). \qquad (8)
\end{aligned}
$$

We assume that our graph is embedded in the plane. Then we consider the Euclidean distances between vertices where the distance between two adjacent vertices is defined to be 1. Furthermore, we consider all spokes to be linear and evenly spaced around the center vertex, v_0. Creating a weighted efficiency can effectively approximate real life networks such as a subway system [3]. This is found by taking the unweighted global efficiency and dividing by the ideal global efficiency. A partial completion is shown in Fig. 3.

The matrix in Fig. 4 gives the efficiencies of a star graph where each pair of vertices is connected with an edge weighted by the Euclidean distance between them. For example, the Euclidean distance between v_8 and v_{11} is $\sqrt{2^2+3^2} = \sqrt{13}$. Hence $\epsilon_{8,11} = \frac{1}{\sqrt{13}}$.

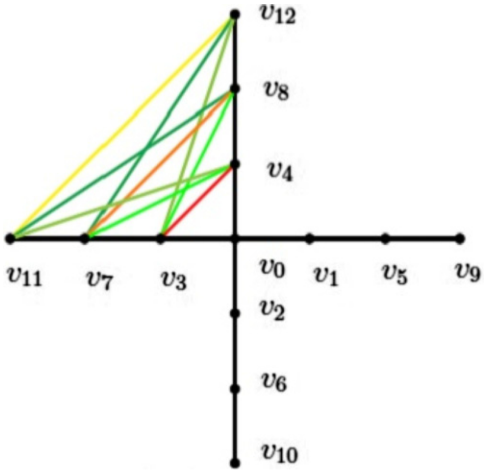


Fig. 3. ‘Ideal’ connections in a star graph.

			$j=1$				$j=2$				$j=3$			
		v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
	v_0	0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
i=1	v_1	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$
	v_2	1	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$
	v_3	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$
	v_4	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$
i=2	v_5	$\frac{1}{2}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	0	$\frac{1}{\sqrt{8}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{8}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$
	v_6	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{8}}$	0	$\frac{1}{\sqrt{8}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$
	v_7	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{8}}$	0	$\frac{1}{\sqrt{8}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$
	v_8	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{8}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{8}}$	0	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1
i=3	v_9	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	0	$\frac{1}{\sqrt{18}}$	$\frac{1}{6}$	$\frac{1}{\sqrt{18}}$
	v_{10}	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{18}}$	0	$\frac{1}{\sqrt{18}}$	$\frac{1}{6}$
	v_{11}	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{6}$	$\frac{1}{\sqrt{18}}$	0	$\frac{1}{\sqrt{18}}$
	v_{12}	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{18}}$	$\frac{1}{6}$	$\frac{1}{\sqrt{18}}$	0

Fig. 4. The Euclidean efficiency matrix of the star graph $S_{4,3}$.

Notice that the diagonal terms equal to 1 are identical to the matrix of the subdivided star graph. As a result, the term for this sum is $\sum_{i=1}^{L} \frac{D(L+1-i)}{i}$. In this case, we are considering the patches to be squares indexed by i and j. The sum of the entries in the block $i = 1$, $j = 2$ equals $8 \cdot \frac{1}{\sqrt{5}} + 4 \cdot \frac{1}{3} = \frac{4}{\sqrt{5}} + \frac{4}{3} + \frac{4}{\sqrt{5}}$. A path from v_1 to v_8 includes an angle of $\frac{\pi}{2}$. Identical angles within a block are shaded alike. The terms can be expressed using the law of cosines,

$$\text{Sum} = \frac{4}{\sqrt{1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos(\frac{\pi}{2})}} + \frac{4}{\sqrt{1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos(\pi)}} + \frac{4}{\sqrt{1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos(\frac{3\pi}{2})}}.$$

The generic terms in a given block can be written as $\frac{D}{\sqrt{i^2+j^2-2ij\cos(\frac{2\pi}{D}\theta)}}$ where θ varies from 1 to $D - 1$. We then sum over all of the blocks and add in the diagonal terms to yield Eq. (9).

The sum of Euclidean (ideal) efficiencies for a star graph with D spokes, each of L length is given by:

$$\sum_{i,j} \epsilon_{ij} = 2 \sum_{i=1}^{L} \frac{D(L+1-i)}{i} + \sum_{i=1}^{L} \sum_{j=1}^{L} \sum_{\theta=1}^{D-1} \frac{D}{\sqrt{i^2 + j^2 - 2ij \cos(\frac{2\pi}{D}\theta)}}. \tag{9}$$

We define the global efficiency ratio ER_{glob} to be the ratio of the unweighted global efficiency to the Euclidean (ideal) efficiency. The global efficiency ratio of the star graph is found by dividing Eq. (6) by Eq. (9).

Theorem 5.

$$ER_{glob}((D,L)) = \frac{\sum_{i=1}^{L}\left[\frac{2(L+1-i)}{i} + (D-1)\left(\frac{i}{i+1} + \frac{i}{2L+1-i}\right)\right] - (D-1)\cdot\frac{L}{L+1}}{\sum_{i=1}^{L}\left[\frac{2(L+1-i)}{i} + \sum_{j=1}^{L}\sum_{\theta=1}^{D-1}\frac{1}{\sqrt{i^2+j^2-2ij\cos(\frac{2\pi}{D}\theta)}}\right]}. \tag{10}$$

We note that as D increases, the global efficiency ratio decreases as a path between two vertices on different spokes must pass through the center vertex. However as L increases the graph bears a closer resemblance to a path. Hence the weighted global efficiency will increase with a limit of 1.

1.3. Cartesian products

We next investigate the efficiency of powers of a path P_n. Su, Xiong, and Gutman obtained the Harary index of P_m^k, from which $E_{glob}(P_m^k)$ can easily be obtained. However, we include a computation of $E_{glob}(P_n^m)$, as it is useful for obtaining the global efficiency for the families $K_n \times P_n^m$ and $K_n \times C_n^m$.

For the global efficiency of path power graphs: $E_{glob}(P_n^m)$, each element of the efficiency matrix is given by

$$\epsilon_{ij} = \frac{1}{\left\lceil \frac{|i-j|}{m} \right\rceil} \tag{11}$$

where i is the row and j is the column of the entry. This value corresponds to the efficiency between vertices i and j. The distance between the vertices in P_n is simply $|i-j|$. In P_n^m, each step can be up to m vertices away. Hence the distance between vertices equals $\left\lceil \frac{|i-j|}{m} \right\rceil$. Taking the inverse gives the formula in Eq. (11). Hence the matrix is:

$$\begin{array}{c|ccccccc}
 & v_1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} & v_n \\
\hline
v_1 & 0 & 1 & 1 & & \frac{1}{\lceil \frac{n-3}{m} \rceil} & \frac{1}{\lceil \frac{n-2}{m} \rceil} & \frac{1}{\lceil \frac{n-1}{m} \rceil} \\
v_2 & 1 & 0 & 1 & \cdots & \frac{1}{\lceil \frac{n-4}{m} \rceil} & \frac{1}{\lceil \frac{n-3}{m} \rceil} & \frac{1}{\lceil \frac{n-2}{m} \rceil} \\
v_3 & 1 & 1 & 0 & & \frac{1}{\lceil \frac{n-5}{m} \rceil} & \frac{1}{\lceil \frac{n-4}{m} \rceil} & \frac{1}{\lceil \frac{n-3}{m} \rceil} \\
\vdots & & \vdots & & \ddots & & \vdots & \\
v_{n-2} & \frac{1}{\lceil \frac{n-3}{m} \rceil} & \frac{1}{\lceil \frac{n-4}{m} \rceil} & \frac{1}{\lceil \frac{n-5}{m} \rceil} & & 0 & 1 & 1 \\
v_{n-1} & \frac{1}{\lceil \frac{n-2}{m} \rceil} & \frac{1}{\lceil \frac{n-3}{m} \rceil} & \frac{1}{\lceil \frac{n-4}{m} \rceil} & \cdots & 1 & 0 & 1 \\
v_n & \frac{1}{\lceil \frac{n-1}{m} \rceil} & \frac{1}{\lceil \frac{n-2}{m} \rceil} & \frac{1}{\lceil \frac{n-3}{m} \rceil} & & 1 & 1 & 0
\end{array}$$

Consider the first vertex of P_n^m. There are $(n-1)$ other vertices to compute the efficiency with. The sum of efficiencies from the first vertex is:

$$\sum_{j=2}^{n} \epsilon_{1,j} = \sum_{j=2}^{n} \frac{1}{\left\lceil \frac{|1-j|}{m} \right\rceil} = \sum_{i=1}^{n-1} \frac{1}{\left\lceil \frac{i}{m} \right\rceil}.$$

For the second vertex we have, $1 \le |2-j| \le n-2$ since $3 \le j \le n$, which yields:

$$\sum_{j=3}^{n} \epsilon_{2,j} = \sum_{j=3}^{n} \frac{1}{\left\lceil \frac{|2-j|}{m} \right\rceil} = \sum_{i=1}^{n-2} \frac{1}{\left\lceil \frac{i}{m} \right\rceil}.$$

Summing the terms for all vertices gives:

$$\sum_{i=1}^{n-1} \frac{1}{\left\lceil \frac{i}{m} \right\rceil} + \sum_{i=1}^{n-2} \frac{1}{\left\lceil \frac{i}{m} \right\rceil} + \cdots + \sum_{i=1}^{2} \frac{1}{\left\lceil \frac{i}{m} \right\rceil} + \sum_{i=1}^{1} \frac{1}{\left\lceil \frac{i}{m} \right\rceil} = \sum_{k=1}^{n-1}\sum_{i=1}^{k} \frac{1}{\left\lceil \frac{i}{m} \right\rceil}.$$

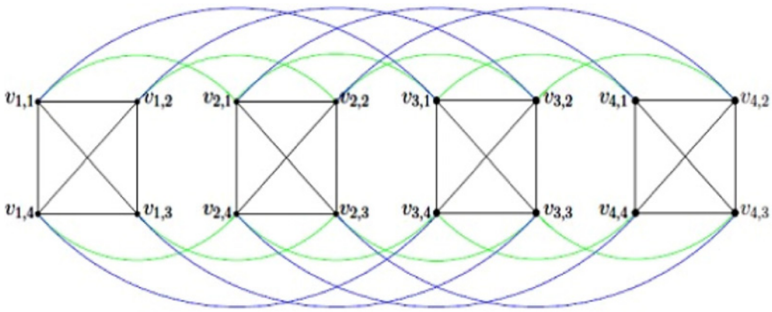
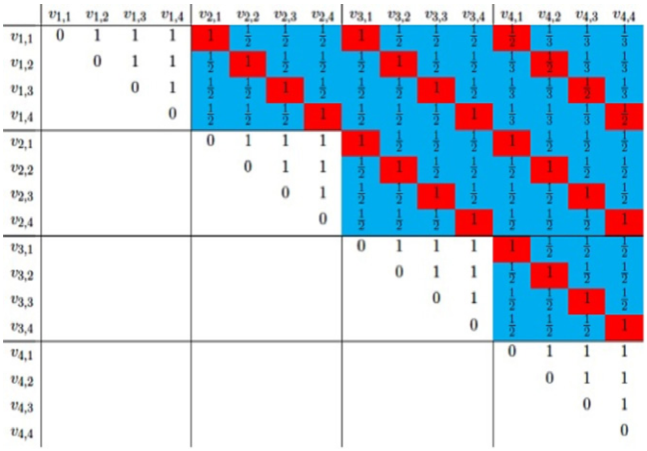


Fig. 5. The Cartesian product of K_4 and P_4^2.



	$v_{1,1}$	$v_{1,2}$	$v_{1,3}$	$v_{1,4}$	$v_{2,1}$	$v_{2,2}$	$v_{2,3}$	$v_{2,4}$	$v_{3,1}$	$v_{3,2}$	$v_{3,3}$	$v_{3,4}$	$v_{4,1}$	$v_{4,2}$	$v_{4,3}$	$v_{4,4}$
$v_{1,1}$	0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$v_{1,2}$		0	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
$v_{1,3}$			0	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
$v_{1,4}$				0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
$v_{2,1}$					0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$v_{2,2}$						0	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
$v_{2,3}$							0	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
$v_{2,4}$								0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$v_{3,1}$									0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$v_{3,2}$										0	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
$v_{3,3}$											0	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
$v_{3,4}$												0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$v_{4,1}$													0	1	1	1
$v_{4,2}$														0	1	1
$v_{4,3}$															0	1
$v_{4,4}$																0

Fig. 6. The efficiency matrix for $K_4 \times P_4^2$.

Finally, we divide this term to get the following result. We note as the matrix is symmetric, we can sum over the upper half of the matrix (ordered pairs) and then multiply our result by 2.

Theorem 6. *Let* $1 \leq m \leq n$. *Then*

$$E_{glob}(P_n^m) = \frac{2}{n(n-1)} \sum_{k=1}^{n-1} \sum_{i=1}^{k} \frac{1}{\lceil \frac{i}{m} \rceil}. \tag{12}$$

Definition 7. The Cartesian product of two graphs G and H is a graph $G \times H$, with the vertex set $V(G) \times V(H)$, where vertices $\{(i_1, i_2), (j_1, j_2)\}$ are adjacent if $\{i_1, j_1\} \in E(G)$ and $i_2 = j_2$, or $\{i_2, j_2\}$ forms an edge in H and $i_1 = j_1$.

In the figure below, we show the graph of the Cartesian product of K_4 and P_4^2 (see Fig. 5).

The efficiency matrix is given in Fig. 6.

We next extend to the general case.

Theorem 8.

$$E_{glob}(K_r \times P_n^m) = \frac{2}{nr(nr-1)} \left[\sum_{k=1}^{n-1} \sum_{i=1}^{k} \left(\frac{1}{\lceil \frac{i}{m} \rceil} + \frac{r(r-1)}{1 + \lceil \frac{i}{m} \rceil} \right) + \frac{nr(r-1)}{2} \right]. \tag{13}$$

Proof. Notice that the matrix is very similar to that of a path power. Each i now corresponds to a block in the kth row from the bottom. Each block has r terms on the main diagonal of a block and these correspond to the pairs of

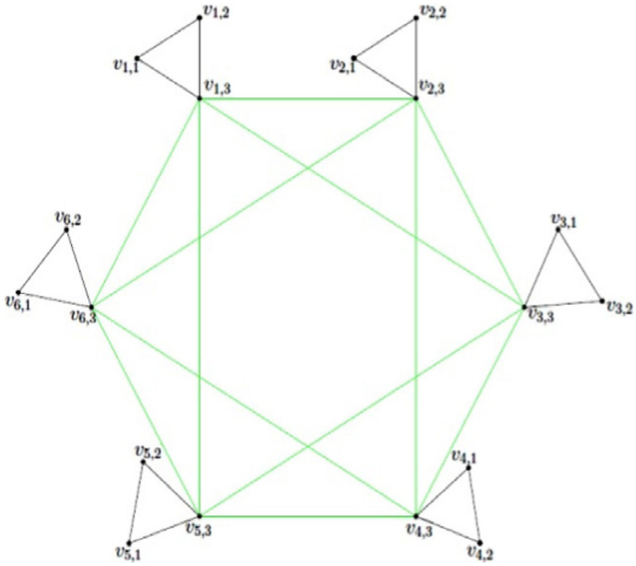


Fig. 7. The Cartesian product of K_3 and C_6^2.

	$v_{1,1}$	$v_{1,2}$	$v_{1,3}$	$v_{2,1}$	$v_{2,2}$	$v_{2,3}$	$v_{3,1}$	$v_{3,2}$	$v_{3,3}$	$v_{4,1}$	$v_{4,2}$	$v_{4,3}$	$v_{5,1}$	$v_{5,2}$	$v_{5,3}$	$v_{6,1}$	$v_{6,2}$	$v_{6,3}$
$v_{1,1}$	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
$v_{1,2}$	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
$v_{1,3}$	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1
$v_{2,1}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{2}$
$v_{2,2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$
$v_{2,3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$v_{3,1}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
$v_{3,2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
$v_{3,3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
$v_{4,1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
$v_{4,2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
$v_{4,3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1
$v_{5,1}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$
$v_{5,2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$
$v_{5,3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$v_{6,1}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1
$v_{6,2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1
$v_{6,3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	0

Fig. 8. The efficiency matrix of $K_3 \times C_6$.

vertices in the r separate path K_r powers. All other terms correspond to the distance between vertices in different path powers and then within the copies of the complete subgraphs. For this, we have r vertices in the initial class to choose from and $r - 1$ vertices in the K_r. Since each class is complete, it will only take one additional step to reach the final vertex, and so the efficiency is only slightly smaller. There are also 'triangles' of 1s next to the main diagonal; these correspond to movements within a single K_r. The number of 1s is then n times the number of edges in K_r which equals $\frac{r(r-1)}{2}$. Averaging the efficiencies over all pairs yields Eq. (13). ■

We next investigate the global efficiency of a Cartesian product of a complete graph and a cycle. The graph of the Cartesian product of K_3 and C_6^2 is shown in Fig. 7.

The efficiency matrix is found in Fig. 8.

For a Cartesian product between a complete graph K_r and a cycle power C_n^m, we must divide the global efficiency into two cases where n is either odd or even.

Theorem 9. *If n is odd then*

$$E_{glob}(K_r \times C_n^m) = \frac{1}{rn-1}\left[2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{\lceil \frac{i}{m}\rceil} + \frac{r-1}{1+\lceil \frac{i}{m}\rceil}\right) + r - 1\right]. \tag{14}$$

If n is even then

$$E_{glob}(K_r \times C_n^m) = \frac{1}{rn-1}\left[2\sum_{i=1}^{n/2}\left(\frac{1}{\lceil \frac{i}{m}\rceil} + \frac{r-1}{1+\lceil \frac{i}{m}\rceil}\right) + r - 1 - \left(\frac{1}{\lceil \frac{n}{2m}\rceil} + \frac{r-1}{\lceil \frac{n}{2m}\rceil + 1}\right)\right]. \tag{15}$$

Proof. Each i corresponds to a single line of each block. Also, each i has one entry that falls on the main diagonal of an $r \times r$ block that corresponds to pairs of vertices within a cycle power. All other terms correspond to pairs of vertices that are in different cycle powers but in different copies of K_r. There are also 1s next to the main diagonal; these correspond to movements within a single complete graph. The number of 1s is then $r-1$: the number of vertices that are available for the final position. Averaging the efficiencies over all pairs of vertices yields Eqs. (14) and (15).

$$\begin{aligned} E_{glob}(K_r \times C_n^m) &= \frac{1}{nr(nr-1)} \cdot nr \cdot \left[2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{\lceil \frac{i}{m}\rceil} + \frac{r-1}{1+\lceil \frac{i}{m}\rceil}\right) + r - 1\right] \\ &= \frac{1}{nr-1}\left[2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{\lceil \frac{i}{m}\rceil} + \frac{r-1}{1+\lceil \frac{i}{m}\rceil}\right) + r - 1\right]. \end{aligned}$$

For the even case, note that the term corresponding to the efficiency of moving across the diameter is counted twice, so it must be subtracted to obtain Eq. (15). ■

Theorem 10.

$$E_{glob}(K_m \times K_n) = \frac{nm+m+n-3}{2(nm-1)}. \tag{16}$$

We obtain the global efficiency for $K_m \times K_n$ using $E_{glob}\left(K_m \times P_n^{n-1}\right)$ and $E_{glob}\left(K_m \times C_n^{\lfloor \frac{n}{2} \rfloor}\right)$.

$$\begin{aligned} E_{glob}(K_m \times K_n) &= E_{glob}(K_m \times P_n^{n-1}) \\ &= \frac{2}{nm(nm-1)}\left[\sum_{k=1}^{n-1}\sum_{i=1}^{k}\left(\frac{m}{\lceil \frac{i}{n-1}\rceil} + \frac{m(m-1)}{\lceil \frac{i}{n-1}\rceil + 1}\right) + \frac{nm(m-1)}{2}\right] \\ &= \frac{2}{n(nm-1)}\left[\sum_{k=1}^{n-1}\sum_{i=1}^{k}\left(\frac{1}{1} + \frac{m-1}{1+1}\right) + \frac{n(m-1)}{2}\right] \\ &= \frac{1}{n(nm-1)}\left[\sum_{k=1}^{n-1}\sum_{i=1}^{k}(m+1) + n(m-1)\right] \\ &= \frac{1}{n(nm-1)}\left[\sum_{k=1}^{n-1}(k(m+1)) + n(m-1)\right] \\ &= \frac{1}{n(nm-1)}\left[\left(\frac{(n-1)(n-1+1)}{2}\right)(m+1) + n(m-1)\right] \\ &= \frac{1}{nm-1}\left[\frac{n-1}{2}(m+1) + (m-1)\right] \end{aligned}$$

$$= \frac{1}{nm-1}\left[\frac{nm}{2} - \frac{m}{2} + \frac{n}{2} - \frac{1}{2} + m - 1\right]$$
$$= \frac{nm+m+n-3}{2(nm-1)}.$$

We note that the ceiling functions were dropped since $1 \le i \le n-1$ implies $0 < \frac{1}{n-1} \le \frac{i}{n-1} \le 1$ which makes the ceiling terms always equal to 1. For the Cartesian product of a complete graph and an odd cycle, we have:

$$\begin{aligned}
E_{glob}(K_m \times K_n) &= E_{glob}\left(K_m \times C_n^{\frac{n-1}{2}}\right)\\
&= \frac{1}{nm-1}\left[2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{\left\lceil\frac{i}{(n-1)/2}\right\rceil} + \frac{m-1}{\left\lceil\frac{i}{(n-1)/2}\right\rceil + 1}\right) + m - 1\right]\\
&= \frac{1}{nm-1}\left[2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{1} + \frac{m-1}{1+1}\right) + m - 1\right]\\
&= \frac{1}{nm-1}\left[2\sum_{i=1}^{\frac{n-1}{2}}\frac{m+1}{2} + m - 1\right]\\
&= \frac{1}{nm-1}\left[\frac{n-1}{2}(m+1) + m - 1\right]\\
&= \frac{nm+m+n-3}{2(nm-1)}.
\end{aligned}$$

For the Cartesian product of a complete graph and an even cycle, we have:

$$\begin{aligned}
E_{glob}(K_m \times K_n) &= E_{glob}\left(K_m \times C_n^{\frac{n}{2}}\right)\\
&= \frac{1}{nm-1}\left[2\sum_{i=1}^{\frac{n}{2}}\left(\frac{1}{\left\lceil\frac{i}{n/2}\right\rceil} + \frac{m-1}{\left\lceil\frac{i}{n/2}\right\rceil + 1}\right) + m - 1 - \left(\frac{1}{\left\lceil\frac{n}{2(n/2)}\right\rceil} + \frac{m-1}{\left\lceil\frac{n}{2(n/2)}\right\rceil + 1}\right)\right]\\
&= \frac{1}{nm-1}\left[2\sum_{i=1}^{\frac{n}{2}}\left(\frac{1}{1} + \frac{m-1}{1+1}\right) + m - 1 - \left(\frac{1}{1} + \frac{m-1}{1+1}\right)\right]\\
&= \frac{1}{nm-1}\left[2\sum_{i=1}^{\frac{n}{2}}\frac{m+1}{2} + m - 1 - \frac{1}{2}(m+1)\right]\\
&= \frac{1}{nm-1}\left[\frac{n}{2}(m+1) + m - 1 - \frac{1}{2}(m+1)\right]\\
&= \frac{1}{nm-1}\left[\frac{n-1}{2}(m+1) + m - 1\right]\\
&= \frac{nm+m+n-3}{2(nm-1)}.
\end{aligned}$$

Thus $E_{glob}(K_m \times K_n)$ is given by Eq. (16).

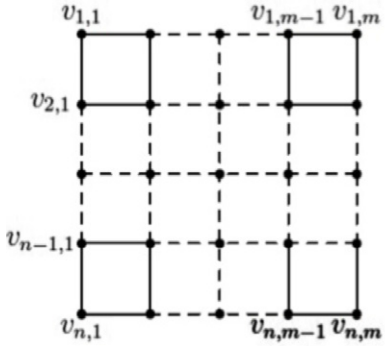


Fig. 9. A grid graph.

Fig. 10. The graph $P_3 \times P_6$.

1.3.1. *Grid graphs: $P_m \times P_n$*

Consider the grid graph $P_m \times P_n$ which is embedded in the plane. The vertex in the upper left corner is labeled $v_{1,1}$ and $v_{i,j}$ is used to label vertex that is obtained by starting at vertex $v_{1,1}$ and traveling $i-1$ positions to the right and then $j-1$ units downward (see Fig. 9).

Now consider the graph $P_3 \times P_6$ (see Fig. 10).

The initial block of 9 vertices from $v_{1,1}$ to $v_{3,3}$ creates the graph $P_3 \times P_3$. Adding sets of 3 additional vertices, $v_{1,4}$ to $v_{3,4}$ up to $v_{1,6}$ to $v_{3,6}$ we obtain the entire graph of $P_3 \times P_6$. This can be seen in Fig. 8. The efficiency matrix below is divided into sections of $P_3 \times P_n$ where $n \leq 6$ (see Fig. 11).

Our first goal is to sum the efficiencies of $P_m \times P_n$. We shall consider the copies of P_m to be 'vertical' and the P_n copies to be 'horizontal'. To sum the efficiencies we begin by considering the n copies of P_m. The sum of efficiencies between a single P_m is simply $\sum_{k=1}^{m-1} \frac{m-k}{k}$. So our total for vertical connections is $n \sum_{k=1}^{m-1} \frac{m-k}{k}$. Similarly, our total for horizontal connections is $m \sum_{k=1}^{n-1} \frac{n-k}{k}$.

Next we determine the remaining efficiencies. Consider two copies of P_m. There are $n-i$ pairs of P_m that are separated by a horizontal distance of $i \leq n-1$. There are $2(m-j)$ pairs of points in separate P_m that are separated by a vertical distance of $j \leq m-1$. Thus the sum of efficiencies of the cross terms is $\sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{2(n-i)(m-j)}{i+j}$. Since the total number of vertices is nm, our global efficiency is:

$$\begin{aligned} E_{glob}(P_m \times P_n) &= \frac{2}{mn(mn-1)} \left[n \sum_{k=1}^{m-1} \frac{m-k}{k} + m \sum_{k=1}^{n-1} \frac{n-k}{k} + \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{2(n-i)(m-j)}{i+j} \right] \\ &= \frac{2}{mn(mn-1)} \left[\sum_{k=1}^{m-1} \frac{nm}{k} - n(m-1) + \sum_{k=1}^{n-1} \frac{mn}{k} - m(n-1) + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{(n-i)(m-j)}{i+j} \right] \\ &= \frac{2}{mn(mn-1)} \left[m + n - 2nm + \sum_{k=1}^{m-1} \frac{nm}{k} + \sum_{k=1}^{n-1} \frac{nm}{k} + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{(n-i)(m-j)}{i+j} \right]. \end{aligned}$$

Using a weight corresponding to the Euclidean distance, we can obtain the global efficiency ratio which compares the efficiency using distances along the lines of the grid versus the ideal Euclidean distance.

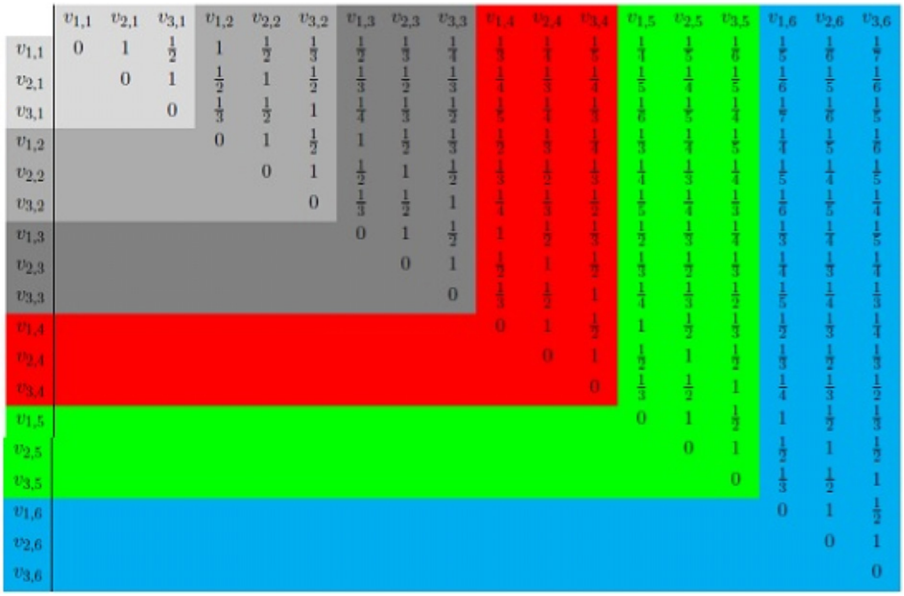


Fig. 11. The efficiency matrix for $P_3 \times P_6$.

Theorem 11. *The global efficiency ratio is given by:*

$$ER_{glob}(P_m \times P_n) = \frac{\frac{2}{mn(mn-1)}\left[m+n-2nm+\sum\limits_{k=1}^{m-1}\frac{nm}{k}+\sum\limits_{k=1}^{n-1}\frac{nm}{k}+2\sum\limits_{i=1}^{n-1}\sum\limits_{j=1}^{m-1}\frac{(n-i)(m-j)}{i+j}\right]}{\frac{2}{mn(mn-1)}\left[m+n-2nm+\sum\limits_{k=1}^{m-1}\frac{nm}{k}+\sum\limits_{k=1}^{n-1}\frac{nm}{k}+2\sum\limits_{i=1}^{n-1}\sum\limits_{j=1}^{m-1}\frac{(n-i)(m-j)}{\sqrt{i^2+j^2}}\right]}. \tag{17}$$

We can use the close relationship between global efficiency and the Harary index, $E_{glob}(G) = \frac{2}{n(n-1)}H(G)$, to obtain new results.

Corollary 12. *Let $H(G)$ be the Harary index of a graph G. Then we have:*

(i) $H\left(K_n \times P_n^m\right) = \binom{n^2}{2}\left[\frac{2}{nr(nr-1)}\left[\sum_{k=1}^{n-1}\sum_{i=1}^{k}\left(\frac{1}{\left\lceil\frac{i}{m}\right\rceil}+\frac{r(r-1)}{1+\left\lceil\frac{i}{m}\right\rceil}\right)+\frac{nr(r-1)}{2}\right]\right]$

(ii) $H\left(K_n \times C_n^m\right) = \binom{n^2}{2}\left[\frac{1}{nr-1}\left[2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{\left\lceil\frac{i}{m}\right\rceil}+\frac{r-1}{1+\left\lceil\frac{i}{m}\right\rceil}\right)+r-1\right]\right]$

(iii) $H\left(K_m \times K_n\right) = \binom{n\cdot m}{2}\left[\frac{nm+m+n-3}{2(nm-1)}\right]$

(iv) $H\left(P_m \times P_n\right) = \binom{n\cdot m}{2}\left[\frac{2}{mn(mn-1)}\left[m+n-2nm+\sum_{k=1}^{m-1}\frac{nm}{k}+\sum_{k=1}^{n-1}\frac{nm}{k}+2\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\frac{(n-i)(m-j)}{i+j}\right]\right]$.

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