(a) use their series representation to prove that

$$b = \frac{d}{dz}M(a,c;z) = aM(a+1,c+1;z)$$

(b) use an integral representation to prove that

$$M(a,c;z) = e^z M(c-a,c;-z)$$

18.20 The Bessel function $J_v(z)$ can be considered as a special case of the solution M(a,c;z) of the confluent hypergeometric equation, the connection being

$$\lim_{a\to\infty} \frac{M(a,v+1;-z/a)}{\Gamma(v+1)} = z^{-v/2} J_v(2\sqrt{z}).$$

Prove this equality by writing each side in terms of an infinite series and showing that the series are the same.

18.21 Find the differential equation satisfied by the function y(x) defined by

$$y(x) = Ax^{-n}\int_0^x e^{-t}t^{n-1}dt \equiv Ax^{-n}\gamma(n,x),$$

and, by comparing it with the confluent hypergeometric function, express y as a multiple of the solution M(a,c:z) of that equation. Determine the value of A that maes equal to M.

18.22 Show, from its definition, that the Bessel function of the second kind, and of integral order v,can be written as

$$Y_v(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial \mu} - (-1)^v \frac{\partial J_-\mu(z)}{\partial \mu}\right]_{\mu=v}.$$

Using the expicit series expression for $J_\mu(z)$,show that $\partial J_\mu(z)/\partial\mu$ can be written as

$$J_v(z)ln\left(\frac{z}{2}\right) + g(v,z),$$

and deduce that $Y_v(z)$ can be expressed as

$$Y_v(z) = \frac{2}{\pi}J_v(z)ln(\frac{z}{2} + h(v,z),$$

where h(v,z), like g(v,z), is a power series in z.

18.23 Prove two of the properties of the incomplete gamma function $P(a,x^2)$ as follows.

(a) By considering its from for suitable value of α, show the error function can be expressed as particular case of the incomplete gamma function.

(b) The Fresnel integrals, of importance in the study of the diffraction of light, are given by

$$C(x) = \textstyle\int_0^x cos\left(\frac{\pi}{2}t^2\right)dt \qquad S(x) = \int_0^x sin\left(\frac{\pi}{2}t^2\right)dt$$

Show that they can be expressed in terms of the error function by

$$C(x) + iS(x) = Aerf\left[\frac{\sqrt{\pi}}{2}(1-i)x\right],$$

where A is a (complex) constant, which you should determine. Hence express $C(x)+iS(x)$ in terms of the incomplete gamma function.