Special Systems of ODE's. Phase Spaces

Theory of System of Linear ODE's with Const. Coeff.

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1. Prologue

2. Autonomous Systems
   - Trajectory
   - Phase portrait
   - Standard Example: $\dot{x} = f(x)$, $x \in \mathbb{R}$ (regarding 2 problems raised)

3. Phase Plane for 2-dimension $\dot{x} = Ax$

   · node

   · saddle point (parabolic)

   · focus

   · centre

Today's Topic

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- Autonomous Systems and Phase Spaces.

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- Phase Plane of Systems of Homogenuous Linear Equations with Constant Coefficients.

Autonomus Systems

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Definition

A system of ODE's is said to be autonomus, if the independent variable t does not appear obviously in it.

$$\dot{x}^i = f^i(x^1, \cdots, x^n), \quad i = 1, \cdots, n \quad \text{in open } \Omega \subset \mathbb{R}^n, \\ f, \partial f / \partial x_i \in C(\Omega). \tag{1}$$

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Theorem

If

$$x^i = \varphi^i(t), \quad i = 1, \cdots, n$$

is a solution of (1), then $\varphi_*^i(t) = \varphi^i(t + c)$ *is also a solution of (1), where* c *– constant.*

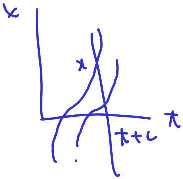
$\varphi(t) - \varphi(t+c)$

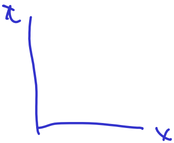
$t, x : \mathbb{R} \times \mathbb{R}^n$

For every solution of (1) $x^i = \varphi^i(t)$, we make the trajectories of points in $\mathbb{R}^n$ correspond to them, i.e.,

Definition

Trajectory refers to $\{(t, x) : x = (x^1, \cdots, x^n) \in \Omega\}$.





Can These Trajectories intersect themselves?

· disjoint trajectories

$\varphi(t+c) = \varphi(t)$, $c = t_1 - t_2$

Suppose for some $t_1 \neq t_2$ we have $\varphi^i(t_1) = \varphi^i(t_2)$. Solution $\varphi^i(t+c) = \varphi^i(t)$ can be extended to $t \in \mathbb{R}$. The c's so-called the period and we denote F the collection of c's.

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Theorem

1) $(c \in F) \Rightarrow (-c \in F)$; $\varphi(t+c_m)$
2) $(c_1, c_2 \in F) \Rightarrow (c_1 + c_2 \in F)$;
3) Suppose a sequence $(c_m) \in F$. *If* $c_m \to c_0$ *as* $m \to \infty$, *then* $c_0 \in F$.

In other word, F is closed in $\mathbb{R}$.

In the context of the above properties of F, we can distinguish two cases.

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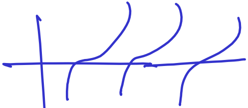
   平衡位置



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2. $F = \{mT : m \in \mathbb{N}, T > 0\}$. Then for all t we have $\varphi^i(t + T) = \varphi^i(t)$, but when $|\tau_1 - \tau_2| < T$, at least for an $I \in \{1, \cdots, n\}$ there holds $\varphi^I(\tau_1) \neq \varphi^I(\tau_2)$ (so-called the periodic trajectory).

(1) disjoint

(2) fixed points

(3) periodic

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In other words, trajectories can only be 3 kinds of morphologies: 1) Fixed point; 2) Periodic trajectory; 3) Self-disjointed trajectory.

Phase Space. Portrait. etc

$x^i = \varphi^i(t)$ $\dot{x} = f(x)$ in open $\Omega \subset \mathbb{R}^n$ vector field

On the other hand, each point $x_0 = (x_0^1, \cdots, x_0^n)$ can be correspond to the vector $f(x_0) = (f^1(x_0^1, \cdots, x_0^n), \cdots, f^n(x_0^1, \cdots, x_0^n))$. Clearly, the velocity vector of the corresponding point $x^i = \varphi^i(t)$ through x_0 coincides with $f(x_0)$.

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Phase portrait refers to a sketch of the phase space (i.e., the space where vector field f is located) together with "directions". f so-called the phase velocity.

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Theorem

$a = (a^1, \cdots, a^n) \in \Omega$ is a fixed point, i.e., $\varphi(t) = a$, iff $f(a) = 0$.

$f^i(a) = \dot{\varphi}^i(t) = \frac{d}{dt} a^i = 0$

Standard Example: $\dot{x} = f(x)$, $x \in \mathbb{R}$ (2 problems raised)

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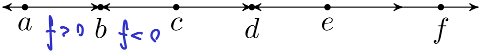
Suppose this ODE has fixed points $N = \{a, b, c, d, e, \cdots\}$, hence $\mathbb{R}$ can be parted to unions of some disjoint intervals $\Sigma = \{(-\infty, a), (a, b), \cdots\}$.



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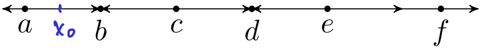

Consider a interval, e.g., (a, b) and fix $x_0 \in (a, b)$ satisfies $f(x_0) > 0$. Suppose

$$\begin{cases} x = \varphi(t), \quad t \in [r_1, r_2], \\ x|_{t=0} = x_0 \quad \text{initial condition} \end{cases}$$

is a solution of this ODE.

$\ddot{x} = F(x, \dot{x}, t)$

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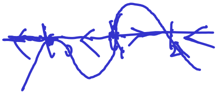

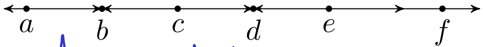

(a, b) a.b finite $t \to \infty$ interval

Consider a interval, e.g., (a, b) and fix $x_0 \in (a, b)$ satisfies $f(x_0) > 0$. Suppose

$$\int dt = \int \frac{dx}{f(x)} \qquad \frac{dx}{dt} = f(x)$$

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is a solution of this ODE. which $f(t) \in (a, b)$. $t \in (0, \infty)$?

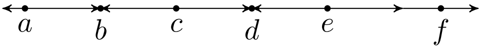
$\dot{x}\,(x-1)(2-x)$ $F[1,2]$

Theorem

In this situation, when $t \in (r_1, r_2)$, we have

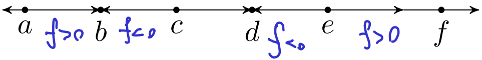
$$\lim_{t\to\infty} \varphi(t) = \infty$$

$$\boxed{\varphi(t) \in (a, b),} \quad \lim_{t \to r_1} \varphi(t) = a, \quad \lim_{t \to r_2} \varphi(t) = b.$$



Upon distinguishing different morphologies of $a, b, c, d, e, \cdots$ we raise 3 cases:

portrait 相曲线



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1. Stable (b, d), i.e., portraits approach it from both sides;
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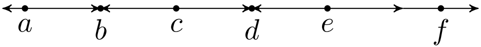

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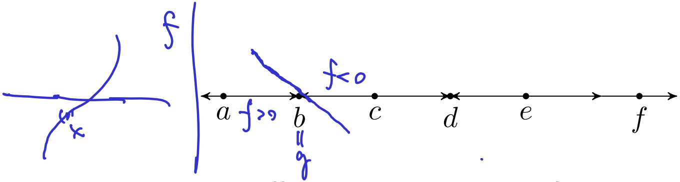

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Finally, for fixed point g suppose $\dot{f}(g) \neq 0$, then $f(x)$ has the same symbol as $\dot{f}(g)(x-g)$. Hence g is stable (unstable, resp.) if $\dot{f}(g) < 0$ (> 0, resp.).

<0 in some neighborhood

$x<g$:

$x>g$:

Case of fixed-point (0,0)

Jacobian

$(0,0)$ (c^1, c^2)

see Pontryagin[1], also see [3].

$$\dot{x} = Ax$$

$$\dot{x} = J \begin{pmatrix} x^1 - c^1 \\ x^2 - c^2 \end{pmatrix}$$

$$\dot{x} = Af$$

$$\begin{cases} f^1(x^1, x^2) & > \\ f^2(x^1, x^2) & < \end{cases} 0$$

1. L.S.Pontryagin, *Ordinary Differential Equations.* Pergamon Press, Paris, 1962.

2. V.I.Arnold, *Mathematical Methods of Classical Mechanics.* (GTM) Springer, New York, 1998.

3. W.Walter, *Ordinary Differential Equations.* (GTM) Springer, New York, 1998.

4. V. I. Arnold, O.D.E.

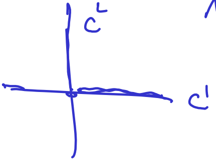
$$\dot{x} = Ax \iff \dot{x} = J\begin{pmatrix} x^1 - b^1 \\ x^2 - b^2 \end{pmatrix}$$

eigenvalue of A

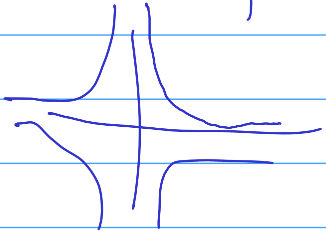
- real
  - $\lambda_2 = 0$
    - $\lambda_1 = 0$: f.p. $\forall$
    - $\lambda_1 \neq 0$: ...
  - $\lambda_1 \cdot \lambda_2 \neq 0$. have the same sign
    - $\lambda_2 < \lambda_1 < 0$: stable node
    - $\lambda_2 > \lambda_1 > 0$: un... node
  - unique λ
    - $\lambda < 0$ stable improper
    - $\lambda > 0$ un...
  - $\lambda_1 \cdot \lambda_2 \neq 0$. $\lambda_1 < 0 < \lambda_2$: saddle point (hyperbolic point)
- complex, $\lambda_{1,2} = \mu \pm i\nu$
  - $\mu \neq 0$
    - $\mu < 0$: stable focus
    - $\mu > 0$: unstable focus
  - $\mu = 0$: centre

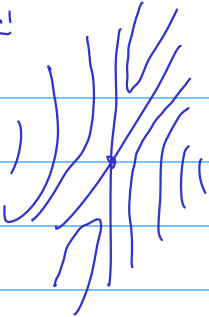
$\xi^1 = c^1 e^{\lambda_1 t}$. $\xi^2 = c^2 e^{\lambda_2 t}$

$\xi^1 = -c^1 e^{\lambda_1 t}$



$\lambda_2 < \lambda_1 < 0$





h_1, h_2

$$\begin{cases} h = \frac{1}{2}(h_1 - i h_2) \\ \bar{h} = \frac{1}{2}(h_1 + i h_2) \end{cases}$$

$$\dot{x} = A f(x)$$

$$= A \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$$

$$A \sim \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = (3-\lambda)(4-\lambda) - 2 = 0$$