Functional Analysis

Chapter 1. Seminorms

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Fall 2023

Why we need functional analysis?

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- $(\mathbb{R}, |\cdot|) \longrightarrow (X, \|\cdot\|)$
- generalized function (applications in pdes, etc)

Preliminaries

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Definition

A real-valued function $p(x)$ defined on a linear space X is called a seminorm iff
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(2) $p(\alpha x) = |\alpha| p(x)$.

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(2) $p(\alpha x) = |\alpha| p(x)$.

(3) $p(0) = 0$;
(4) $p(x_1 - x_2) \geqslant |p(x_1) - p(x_2)|$. In particular, $p(x) \geqslant 0$.

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is called the Minkowski functional of M.

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$p_M(x)$ is a seminorm.

Bourbaki: Define a topology on X

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Let's consider a family of seminorms

$$\{p_\gamma(x) : \gamma \in \Gamma\}$$

satisfies

Axiom of separation

$$\forall x_0 \neq 0 \exists \gamma_0 \in \Gamma (p_{\gamma_0}(x) \neq 0).$$

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Hausdorff's axiom

$$\forall x_1 \neq x_2 \exists \text{ disjoint open sets } G_1,\ G_2(x_1 \in G_1,\ x_2 \in G_2).$$

Bourbaki: Define a topology on X

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2. Consider $G \subset X$, which contains a neighborhood of each of its points, then totality $\{G\}$ is a open set system.

Bourbaki: Define a topology on X

Theorem

linear space $X \xleftrightarrow[\textit{topologized}]{\textit{Minkowski functional}}$ *locally convex, linear topological space* X

Notations and Nouns

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- The support of $f : \Omega \to \mathbb{C}$, denoted by $\operatorname{supp}(f)$, is the smallest closed set containing the set

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where $\Omega \subset \mathbb{R}^n$.
- $C^k(\Omega)$ and $C_0^k(\Omega)$.
- Differential operator
$$D^s = \frac{\partial^{s_1+\cdots+s_n}}{\partial^{s_1} x_1 \cdots \partial^{s_n} x_n}, \quad |s| = s_1 + \cdots + s_n.$$

Application: a theorem of approximation

Theorem

Any $f \in C_0^0(\mathbb{R}^n)$ *can be approximated by functions of* $C_0^\infty(\mathbb{R}^n)$ *uniformly on* $\mathbb{R}^n$*.*

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$$p_{K,m}(f) = \sup_{|s| \leqslant m, x \in K} |D^s f(x)|, \quad f \in C^k(\Omega).$$

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$$\mathfrak{C}^\infty = \mathfrak{C}.$$

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$$p_{K,m}(f) = \sup_{|s| \leqslant m, x \in K} |D^s f(x)|.$$

Let $\mathfrak{D}$ denote the set of all $f \in C_0^\infty(\Omega)$ with compact supports.

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$\mathfrak{D}$ is the "strict inductive limit" of $\mathfrak{D}_K$, i.e., each convex subset of $\mathfrak{D}$ to be a 0-neighborhood iff its intersection with each $\mathfrak{D}_K$ constructs a 0-neighborhood under the topology of $\mathfrak{D}_K$.

Normed linear space

Definition

A locally convex space is called a normed linear space iff its topology is defined by just one seminorm.

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Theorem

(1) $(x_n \to x) \Rightarrow (\|x_n\| \to \|x\|)$;
(2) $(x_n \to x,\ \alpha_n \to \alpha) \Rightarrow (\alpha_n x_n \to \alpha x)$;
(3) $(x_n \to x,\ y_n \to y) \Rightarrow (x_n + y_n \to x + y)$.

Standard examples: S – topological space

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Example

$C(S)$ with $\|x\| = \sup_{s \in S} |x(s)|$.

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$L^p(S, \mathfrak{B}, m)$ with $\|x\| = \left(\int_S |x(s)|^p m(ds)\right)^{1/p}$.

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Theorem

$$\lim_{p\to\infty} \left(\int_S |x(s)|^p m(ds)\right)^{1/p} = \operatorname{ess.\,sup}_{s\in S} |x(s)|, \quad x(s) \in L^\infty(S).$$

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(l^p) with $\|x\| = \left(\sum_n |x_n|^p\right)^{1/p}$.

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(l^p) with $\|x\| = \left(\sum_n |x_n|^p\right)^{1/p}$.

Example

$(l^\infty) = (m)$ with $\|x\| = \sup_n |x_n|$.

Standard examples: space of measures

Example

$A(S, \mathfrak{B})$ with $\|\varphi\| = V(\varphi; S) = \sup_{\sup|x(s)|\leqslant 1} \left|\int_S x(s)\varphi(ds)\right|$.

Quasinorms

Definition

A quasinorm is similar to a norm, just replace

$$\|\alpha x\| = |\alpha|\|x\|$$

by

$$\|-x\| = \|x\|, \quad \lim_{\alpha_n \to 0} \|\alpha_n x\| = 0, \quad \lim_{\|x_n\| \to 0} \|\alpha x_n\| = 0.$$

Quasinorms

Theorem

In a quasinormed linear space X we also have
(1) $(x_n \to x) \Rightarrow (\|x_n\| \to \|x\|)$;
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In other words,

$$\left(\lim_{n\to\infty} \|x_n\| = 0\right) \Rightarrow \left(\lim_{n\to\infty} \|\alpha x_n\| = 0\right)$$

uniformly in α on any bounded set of α.

Examples

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Example

$\mathfrak{C}^k(\Omega)$.

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Example

$M(S, \mathfrak{B}, m)$ with $\|x\| = \int_S |x(s)|(1 + |x(s)|)^{-1} m(ds)$.

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Example

$\mathfrak{D}_K(\Omega)$.

Definition

A linear space X is called a pre-Hilbert space iff its norm satisfies

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

We define, for X over $\mathbb{R}$, the inner product

$$(x, y) = 4^{-1}(\|x+y\|^2 - \|x-y\|^2)$$

and

$$(x, y)_{\mathbb{C}} = (x, y) + i(x, iy)$$

for X over $\mathbb{C}$.

Simple examples

Example

$L^2(S, \mathfrak{B}, m)$.

Example

(l^2).

Spaces $\hat{H}^k(\Omega)$ and $\hat{H}_0^k(\Omega)$

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Let $\Omega \subset \mathbb{R}^n$ is open and $k \in \mathbb{N}$. Then $f \in C^k(\Omega)$ for which

$$\|f\|_k = \left(\sum_{|j| \leqslant k} \int_\Omega |D^j f(x)|^2 \, dx \right)^{1/2} < \infty$$

constitutes a pre-Hilbert space $\hat{H}^k(\Omega)$.

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constitutes a pre-Hilbert space $\hat{H}^k(\Omega)$.
Replace C^k by C_0^k we get space $\hat{H}_0^k(\Omega)$.

Space $A^2(G)$

Let G be a bounded open domain of the complex z-plane. Let $A^2(G)$ be the set of all holomorphic functions $f(z)$ defined on G such that

$$\|f\| = \left(\int_G |f(z)|^2 \, dxdy \right)^{1/2} < \infty, \quad z = x + iy.$$

Hardy-Lebesgue class $H\text{-}L^2$

That is the set of all $f(z)$ which are holomorphic in the unit disk $\{z : |z| \leqslant 1\}$ of the complex z-plane and such that

$$\sup_{r\in(0,1)} \left(\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta\right) < \infty.$$

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Theorem

The Hardy-Lebesgue class is in 1-1 correspondence with the pre-Hilbert space (l^2) as

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

B-spaces and F-spaces

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Definition

A quasinormed (or normed) linear space X is called an F-space (or B-space) if it is complete, i.e., if every Cauchy sequence $\{x_n\} \subset X$ converges strongly to a point $x_\infty \in X$:

$$\lim_{n \to \infty} \|x_n - x_\infty\| = 0.$$

A complete pre-Hilbert space is called a Hilbert space.

Standard examples

Example

$\mathfrak{C}(\Omega)$ is an F-space.

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Example

$M(S, \mathfrak{B}, m)$ with $m(B) < \infty$ is an F-space.

Next TODO:

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- Sobolev spaces $W^{k,p}$ (B-space)

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- Generalized functions