Riemann-Integrals

(Part I)

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Definition

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Suppose $f\colon [a,b] \to \mathbb{R}$. By a partition Δ of $[a,b]$ we mean a finite set of points $x_0, x_1, \cdots, x_n$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Fix the Riemann Summation

$$R(f, \Delta, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

where $\xi_i \in [x_{i-1}, x_i]$ arbitrarily. Consider $\lim_{\max |x_i - x_{i-1}| \to 0} R(f, \Delta, \xi)$. If it exists, we write it as

$$\int_a^b f(x)\,dx,$$

this is the Riemann-inetgral of f over $[a,b]$.

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- Consider (Darboux's upper and lower integral)

$$U(f, \Delta, \xi) = R\left(f, \Delta, \sup_{\xi \in [x_{i-1}, x_i]} f(\xi_i)\right), \quad L(f, \Delta, \xi) = R\left(f, \Delta, \inf_{\xi \in [x_{i-1}, x_i]} f(\xi_i)\right).$$

Theorem (Darboux)

f is Riemann-integrable iff

$$\lim_{\max |x_i - x_{i-1}| \to 0} U(f, \Delta, \xi) = \lim_{\max |x_i - x_{i-1}| \to 0} L(f, \Delta, \xi).$$

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- Lebesgue's test:

Theorem (Lebesgue)

f is Riemann-integrable on $[a, b]$ iff f continuous on $[a, b]$ a.e.

Example

Suppose f is nonnagetive and Riemann-integrable on $[0, 2]$. Calculate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n^{2021} f\left(\frac{2k-1}{n}\right)}{(n^2+k)^{1011}}.$$

Example

Using definition, calculate

$$\int_0^1 x^2\,dx.$$

Newton-Leibniz's Fundamental Theorem

Theorem (Newton-Leibniz)

If $f \in \mathfrak{R}[a, b]$ *and fix*

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F \in \mathfrak{C}[a, b]$*. Furthermore, if* f *continuous at* x_0*, then* F *is differentiable at* x_0*, and*

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Corollary

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'(x) = f(x)$*. We write*

$$\int f(x)\,dx = F(x) + C.$$

Theorem

Suppose F, G are differentiable functions on $[a, b]$, where $F' = f$, $G' = g$ are $\in \mathfrak{R}$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Theorem

Suppose F, G are differentiable functions on $[a, b]$, where $F' = f$, $G' = g$ are $\in \mathfrak{R}$. Then

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Theorem

Suppose $\varphi : [\alpha, \beta] \to [a, b]$ is differentiable, $f : [a, b] \to \mathbb{R}$ is Riemann-integrable. Then

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_\alpha^\beta f \circ \varphi(u)\,d\varphi(u).$$

Example

$$\int_0^x \exp(t^2)\,dt \sim (2x)^{-1}\exp(x^2), \quad x \to \infty.$$

Example

Suppose $f(x) \in \mathfrak{C}[0, \infty)$ and $f(x) \to A$ as $x \to \infty$. Calculate

$$\lim_{n\to\infty} \int_0^1 f(nx)\,dx.$$

Example

Calculate

$$J = \int_0^x e^t \cos t dt.$$

Next todo: Methods of finding F

Example

$$\int \frac{dx}{x\sqrt{x^2-1}}.$$

Example

$$\int \frac{x \arctan x}{(1+x^2)^2}\,dx.$$

Example

$$\int \frac{\sin x dx}{2\sin x + 3\cos x}.$$

Example

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}}.$$

Rational functions

Let's calculate

$$\int \frac{p(x)}{q(x)}\,dx,$$

where

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$
$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0.$$

Wlog, suppose $a_n = b_m = 1$ and $n < m$.

Lemma

The following constant transformation exists:

$$\frac{p(x)}{q(x)} = \left(\frac{a_{1,1}}{x-\alpha_1} + \cdots + \frac{a_{1,r_1}}{(x-\alpha_1)^{r_1}}\right) + \cdots +$$
$$+ \left(\frac{a_{k,1}}{x-\alpha_k} + \cdots + \frac{a_{1,r_k}}{(x-\alpha_k)^{r_k}}\right) +$$
$$+ \left(\frac{b_{1,1}x + c_{1,1}}{x^2+\beta_1 x+\gamma_1} + \cdots + \frac{b_{1,s_1}x + c_{1,s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right) + \cdots +$$
$$+ \left(\frac{b_{l,1}x + c_{l,1}}{x^2+\beta_l x+\gamma_l} + \cdots + \frac{b_{l,s_l}x + c_{l,s_l}}{(x^2+\beta_l x+\gamma_l)^{s_l}}\right).$$

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R.H.S consists 4 kinds of terms:

$$(1)\frac{a}{x+b}, \quad (2)\frac{a}{(x+b)^n}, \quad (3)\frac{ax+b}{x^2+px+q}, \quad (4)\frac{ax+b}{(x^2+px+q)^n},$$

notice that

$$x^2+px+q = (x+\alpha)^2+\beta$$

and using the substitution

$$u = (x+\alpha)/\sqrt{\beta}.$$

Example

$$\int \frac{x^7 - 2x^6 + 4x^5 - 5x^4 + 4x^3 - 5x^2 - x}{(x-1)^2(x^2+1)^2}\,dx.$$

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- If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then set $t = \sin x$;
- If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, then set $t = \tan x$.

Example

$$\int \frac{\cos 2x}{\sin^4 x + \cos^4 x} dx.$$

More examples

Example

$$\int \frac{1}{x}\sqrt{\frac{x+2}{x-2}}\,dx.$$

Example

$$\int \sqrt{1 + \sin x}\, dx.$$

Example

$$\int \arctan \sqrt{\frac{a-x}{a+x}}\, dx, \qquad a > 0.$$