EXISTENCE OF A CLASS OF DIFFERENTIAL FORMS ON A SUBSET OF $\mathbb{R}^n$

WEIHONG GAO AND KEISING HON

ABSTRACT. This paper concerns the existence of a solution to the problem $du = f$, where f is a fixed $(p + 1)$-form. Following similar guidelines for the so-called $\bar{\partial}$-problem, we convert our problem into a L^2-estimation on Hilbert spaces and prove that the problem always has a solution u on a special class of subsets of $\mathbb{R}^n$ that have similar properties to the pseudo-convex domain on $\mathbb{C}^n$. In order to state the above result accurately, we also establish some properties of the subsets mentioned above.

1. INTRODUCTION

This paper concerns the problem

$$du = f \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1}$$

where f is a fixed $(p + 1)$-form. In general, the solvability of this problem is closely related to the topological properties of Ω.

A slightly more specific formulation than problem (1) is the famous $\bar{\partial}$-problem defined on a subset of $\mathbb{C}^n$, the study of which starting with the widespread interest in the several complex analysis in 1960s and continuing to the present, has yileded ample results. These works are mainly based on the L^2-estimates developed by Morrey, Kohn and Hormander (see [3, 7, 11]), which reduces the $\bar{\partial}$-problem to an abstract problem on Hilbert spaces, and then to an inequality. Taking the properties of the pseudo-convex domain on $\mathbb{C}^n$ into account, people claimed that the $\bar{\partial}$-problem always has a solution in the pseudo-convex domain (e.g., see [12]).

The L^2-method is still being developed today (e.g., see [5, 16]), in particular it has been used to extend the $\bar{\partial}$-problem to infinite dimensional complex spaces. This generalization was originally proposed by Raboin (see [13]), and a much more complete result was made by Wang, Yu and Zhang (see [8, 18]).

A more general formulation than problem (1) is the existence of a class of closed forms on manifold M that cannot be expressed as differentials. The best known result in this respect is Poincare's Lemma (see [2], also [1]), i.e., if M is a vector space, then every closed $(p + 1)$-form is an exterior differential of some p-form (see, e.g. [17]).

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Thus, in general, the problem (1) considered in this paper is more general than the $\bar{\partial}$-problem, and more specific than that of differential forms on arbitrary manifolds. In this paper, we shall extend the L^2-method, which was originally used on a subset of $\mathbb{C}^n$, to $\Omega \subset \mathbb{R}^n$, and use it to determine the conditions for the existence of a solution to the problem (1). Thus, our work is distinguished from the $\bar{\partial}$-problem not only in terms of the space under consideration, but also in terms of the methodology employed and from closed form theory in general.

Therefore this paper is organized as follows.

In Section 2 we introduce the terminologies and notations we will be using soon, in particular the most important subspaces used in the L^2-method; we will also have a quick review of some properties of the operations on differential forms, as well as some approximation theorems. Section 3 establishes an abstract formulation of problem (1) and transforms it into an L^2-estimation inequality, and finds the conditions under which this inequality holds. Section 4 focuses on the topological properties of Ω, specifically, defining a notion parallel to pseudo-convex domain in $\mathbb{C}^n$ for a subset of $\mathbb{R}^n$, and finally in section the main result is stated, which is that the problem (1) always has a solution on the pseudo-convex domain in our definition of $\mathbb{R}^n$.

2. Preliminaries

We denote $\mathbb{R}^n$ the real n-dimensional space, $\Omega \subset \mathbb{R}^n$ a domain. K is a relatively compact subset with respect to Ω, denoted by $K \Subset \Omega$. We denote $C^\omega(\Omega)$ the space of all continuous functions with continuous derivatives of up to ω-th order defined on Ω, and in particular, $C(\Omega)$ represents the space of all continuous functions, $C_0^\omega(\Omega)$ represents the spce of all $C^\omega(\Omega)$-functions with compact supports. The support of function φ we denote by $\operatorname{supp}(\varphi)$. Moreover, the usual distance between $x, y \in \mathbb{R}^n$ we denote as $\rho(x, y)$, and $B(\xi; r) := \{x \in \mathbb{R}^n : \rho(\xi, x) < r\}$ denotes a open ball with center at ξ and radius at r.

Let $I = (i_1, \cdots, i_p)$ a multi-index, $|I| = i_1 + \cdots + i_p$ and satisfies $1 \leq i_1 < \cdots < i_p \leq n$. A differential form f is so-called of type (p) if for some coefficients f_I,

$$f = \sum_{|I|=p} f_I dx^I,$$

where $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

Now we import the following space notations.

Definition 2.1. We denote by

(1) $L^2(\Omega) := \left\{f : \int_\Omega |f|^2 \, dm < \infty\right\}$;
(2) $L^2(\Omega, \varphi) := \left\{f : \int_\Omega |f|^2 e^{-\varphi} \, dm < \infty\right\}$;
(3) $L^2_{(p)}(\Omega, \varphi) := \{f : f \text{ is of forms of type } (p) \text{ with coefficients in } L^2(\Omega, \varphi)\}$;
(4) $L^2(\Omega, \text{loc}) := \left\{f : \int_K |f|^2 \, dm < \infty \text{ for all } K\right\}$;

(5) $D_{(p)}(\Omega) := \{f : f$ is of forms of type (p) satisfies $f_I \in C_0^\infty(\Omega)\}$
where $\varphi \in C(\Omega)$, dm the normal Lebesgue measure on $\mathbb{R}^n$.

It is clear that $L^2(\Omega, \varphi)$ is a subspace of $L^2(\Omega, \mathrm{loc})$. Endowed with the norm

$$\|f\|_\varphi^2 := \int_\Omega |f|^2 e^{-\varphi}\, dm$$

and natural (pointwise) ordering, the space $L^2(\Omega, \varphi)$ is a Hilbert lattice.

The following statements is well known (e.g., see [9, 12]), so we will omit their proofs.

Theorem 2.2. *Let $\mu \in C_0^\infty(\mathbb{R}^n)$ with $\int \mu\, dm = 1$ and $\mu_\varepsilon(x) = \varepsilon^{-n}\mu(x/\varepsilon)$, $x \in \mathbb{R}^n$. If $g \in L^2(\mathbb{R}^n)$ it follows that*

$$g_\varepsilon(x) := g * \mu_\varepsilon(x) := \int g(y)\mu_\varepsilon(x-y)\, dy = \int g(x - \varepsilon y)\mu(y)\, dy$$

is a $(C^\infty \cap L^2)(\mathbb{R}^n)$-function such that $\|g_\varepsilon - g\|_{L^2} \to 0$ as $\varepsilon \to 0$. Furthermore, $\mathrm{supp}(g_\varepsilon) \subset \{x : \rho(x, \mathrm{supp}(g)) \leq \varepsilon\}$.

Proposition 2.3. *For all $x \in \Omega$ and $\rho(x, \partial\Omega) > \varepsilon$, if the weak derivative of a-th order of f, denoted by $D^a f$ exists, then*

$$D^a(f * \mu_\varepsilon)(x) = ((D^a f) * \mu_\varepsilon)(x),$$

where μ_ε defined in Theorem 2.2.

Applying the above Theorem 2.2 on Ω and its regular exhaustion Ω_k, we get

Theorem 2.4. *There exists such $\{\eta_k\} \subset C_0^\infty(\Omega)$ satisfies $0 \leq \eta_k(x) \leq 1$ for all $x \in \Omega$, and furthermore, $\eta_k(x) = 1$ on K for sufficiently large j.*

Proof. Taking g_k the indicator of Ω_k, $\delta_k = \rho(\Omega_k, \partial\Omega)$ and $\eta_k(x) = (g_k * \mu_{\delta_k/2})(x)$. □

We list two lemmas on the properties of operations on differential forms and C^2-smooth functions. The proof of the former appears in [2], while the latter can be verified by direct calculation.

Lemma 2.5. *Suppose $f = \sum_{|I|=p} f_I dx^I$, then*

$$|df|^2 = \sum_I \sum_{j=1}^n \left|\frac{\partial f_I}{\partial x_j}\right|^2 - \sum_I \sum_{j,k=1}^n \frac{\partial f_{jI}}{\partial x_k}\frac{\partial f_{kI}}{\partial x_j}.$$

Lemma 2.6. *For $C^2(\Omega)$-function $\varphi : \Omega \to \mathbb{R}$, define*

$$\delta_j w = e^\varphi \frac{\partial}{\partial x_j}(w e^{-\varphi}) = \frac{\partial w}{\partial x_j} - w\frac{\partial \varphi}{\partial x_j},$$

then

(1) *for u, $v \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u \frac{\partial v}{\partial x_j} e^{-\varphi}\, dm = -\int_\Omega (\delta_j u) v e^{-\varphi}\, dm,$$

(2) *for* $\{w_j\}_{j=1}^n \subset C_0^\infty(\Omega)$ *we have*

$$\sum_{j,k=1}^n \int_\Omega \left((\delta_j w_j)(\delta_k w_k) - \frac{\partial w_j}{\partial x_k}\frac{\partial w_k}{\partial x_j} \right) e^{-\varphi}\, dm = \sum_{j,k=1}^n \int_\Omega w_j w_k \frac{\partial^2 \varphi}{\partial x_j x_k} e^{-\varphi}\, dm.$$

Finally, for a linear operator T we denote $D(T)$ the domain of T and $N(T)$ the null space of T.

3. An Abstract Problem Associated with Problem (1)

Note that the differential operator d defines a linear, closed, and densely defined operator

$$T : L^2_{(p)}(\Omega, \varphi_1) \to L^2_{(p+1)}(\Omega, \varphi_2), \quad \varphi_1, \varphi_2 \in C(\Omega), \tag{2}$$

then problem (1) can be reduced to the following form: Suppose $H_1 = L^2_{(p)}(\Omega, \varphi_1)$, $H_2 = L^2_{(p+1)}(\Omega, \varphi_2)$, $H_3 = L^2_{(p+2)}(\Omega, \varphi_3)$ endowed with the norm $\|\cdot\|_{\varphi_1}$, $\|\cdot\|_{\varphi_2}$, $\|\cdot\|_{\varphi_3}$ resp., T, S are linear operators defined by d in the sense of (2) and satisfies

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3.$$

We claim that $TS = 0$ since $dd = 0$. The theorem listed below investigates whether the problem $Tu = f$ has a solution in H_1 under the condition $Sf = 0$.

Theorem 3.1. *Suppose* $D(T^*) \cap D(S)$ *is dense in* H_2. *Problem*

$$Tu = f, \quad u \in H_1, \quad Sf = 0$$

has a solution for $f \in N(S)$, *if there exists a constant* C *such that*

$$\|g\|^2_{\varphi_2} \le C(\|T^* g\|^2_{\varphi_1} + \|Sg\|^2_{\varphi_3}), \quad \forall g \in D(T^*) \cap D(S). \tag{3}$$

Furthermore, the solution u *has the following properties:*

(1) $\|u\| \le \sqrt{C}\|f\|$;
(2) *There exists* $v \in D(T^*)$ *such that* $u = T^* v$;
(3) $u \in N(T)^\perp$.

We will omit its proof since that can be found easily in [12], also in [11, 15] etc.

Therefore, all that remains to be done is to find the conditions under which the inequality (3) holds. To this end, we first formulate some lemmas. The first one gives the adjoint operator T^* .

Lemma 3.2. *Suppose* $f = \sum_{|I|=p} f_I dx^I \in D(T^*)$, *then*

$$T^* f = -\sum_{|I|=p}\sum_{j=1}^n \left(\frac{\partial f_{jI}}{\partial x_j} - f_{jI}\frac{\partial \varphi_2}{\partial x_j} \right) e^{\varphi_1 - \varphi_2}\, dx^I.$$

Proof. Fix $g = \sum_{|J|=p} g_J dx^J \in D_{(p)}(\Omega)$, then

$$Tf = \sum_{|J|=p} \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx^I.$$

Direct calculation shows that

$$(T^*g, f)_{\varphi_1} = \sum_{|I|=p} \int_\Omega (T^*g)_I f_I e^{-\varphi_1}\, dm$$

and

$$\begin{aligned}(g, Tf)_{\varphi_2} &= \sum_{|I|=p} \sum_{j=1}^{n} \int_\Omega g_{jI} \frac{\partial f_I}{\partial x_j} e^{-\varphi_2}\, dm \\ &= -\sum_{|I|=p} \sum_{j=1}^{n} \int_\Omega \frac{\partial}{\partial x_j} \left(g_{jI} e^{-\varphi_2}\right) f_I\, dm,\end{aligned}$$

here we use the integration by parts since f has a compact support. Note that $(f, Tg)_{\varphi_2} = (T^*f, g)_{\varphi_1}$, which yields

$$(T^*g)_I = -\sum_{j=1}^{n} \left(\left(\frac{\partial g_{jI}}{\partial x_j} - g_{jI} \frac{\partial \varphi_2}{\partial x_j} \right) e^{\varphi_1 - \varphi_2} \right),$$

hence we finishes the proof. $\square$

Lemma 3.3. *Suppose $\eta \in C_0^\infty(\Omega)$ and satisfies*

$$e^{-\varphi_{j+1}} \sum_{s=1}^{n} \left| \frac{\partial \eta}{\partial x_s} \right|^2 \le e^{-\varphi_j}, \quad j = 1,\, 2,$$

then

(1) *for a fixed $(p+1)$-form f we have*

$$|d(\eta f) - \eta df|^2 e^{-\varphi_{j+1}} \le |f|^2 e^{-\varphi_j},$$

(2) *if $f \in D(T^*)$, then $\eta f \in D(T^*)$.*

Proof. (1) Note that

$$d\eta = \sum_{l=1}^{n} \frac{\partial \eta}{\partial z_l} dz_l, \quad d(\eta f) = d\eta \wedge f + \eta df,$$

we have

$$\begin{aligned}|d(\eta f) - \eta df|^2 = |d\eta \wedge f|^2 = &\left| \sum_I \sum_{l=1}^{n} \frac{\partial \eta}{\partial z_l} f_I dx_l \wedge dx^I \right|^2 \le \\ &\le \sum_I \sum_{l=1}^{n} \left| \frac{\partial \eta}{\partial x_l} \right|^2 |f_I|^2,\end{aligned}$$

and then by the condition of this Lemma,

$$|d(\eta f) - \eta df|^2 e^{-\varphi_{j+1}} = \sum_I |f_I|^2 \sum_{l=1}^n \left| \frac{\partial \eta}{\partial x_l} \right|^2 e e^{-\varphi_{j+1}} < |f|^2 e^{-\varphi_j}.$$

(2) We have to show that for all $u \in D(T)$, there exists $v \in L^2_{(p,q)}(\Omega, \varphi_1)$ such that

$$(Tu, \eta f)_{\varphi_2} = (u, v)_{\varphi_1},$$

i.e., $(Tu, \eta f)_{\varphi_2}$ is a linear bounded functional of u. To complete this just need to note $(Tu, \eta f)_{\varphi_2} = (u, \eta T^* f)_{\varphi_1} + (\eta Tu - T(\eta u), f)_{\varphi_2}$ and taking the Cauchy-Schwartz Inequality into account. □

Lemma 3.4. *Suppose $\varphi_2 \in C^1(\Omega)$ and*

$$e^{-\varphi_{j+1}} \sum_{s=1}^n \left| \frac{\partial \eta_k}{\partial x_s} \right|^2 \le e^{-\varphi_j}, \quad j = 1, 2, \quad k = 1, 2, \cdots, \tag{4}$$

where η_k has been defined in Theorem 2.4, then there exists $h_k \in D_{(p+1)}(\Omega)$ such that for all $f \in D(T^*) \cap D(S)$,

$$\|h_k - f\|_{\varphi_2} \to 0, \quad \|T^* h_k - T^* f\|_{\varphi_1} \to 0, \quad \|Sh_k - Sf\|_{\varphi_3} \to 0 \tag{5}$$

as $k \to \infty$.

Proof. We begin by proving the weaker conclusion that there exists a $g_k \in D(T^*) \cap D(S)$ with compact supports such that

$$\|g_k - f\|_{\varphi_2} \to 0, \quad \|T^* g_k - T^* f\|_{\varphi_1} \to 0, \quad \|Sg_k - Sf\|_{\varphi_3} \to 0. \tag{6}$$

Indeed, $g_k = \eta_k f$ already satisfies (6). First note that g_k has compact support since η_k has. By Lemma 3.3,

$$|d(\eta_k f) - \eta_k df|^2 e^{-\varphi_3} < |f|^2 e^{-\varphi_2},$$

the right-hand end of above equation is integrable since $f \in L^2_{(p+1)}(\Omega, \varphi_2)$. By Lebesgue's Dominated Convergence Theorem we have

$$\begin{aligned} \lim_{k\to\infty} \|S(\eta_k f) - \eta_k Sf\|^2_{\varphi_3} &= \int_\Omega \lim_{k\to\infty} |S(\eta_k f) - \eta_k Sf|^2 e^{-\varphi_3}\, dm = \\ &= \lim_{l\to\infty} \int_{\Omega_l} \lim_{k\to\infty} |S(\eta_k f) - \eta_k Sf|^2 e^{-\varphi_3}\, dm = 0, \end{aligned} \tag{7}$$

where $\{\Omega_l\}$ is a regular exhaustion of Ω. (7) holds since $\eta_k = 1$ for all sufficiently large l. By definition $Sf \in L^2_{(p+2)}(\Omega, \varphi_3)$ and note that $|\eta_k Sf - Sf| \le 2|Sf|$, then also based on Dominated Convergence Theorem,

$$\begin{aligned} \lim_{k\to\infty} \|\eta_k Sf - Sf\|_{\varphi_3} &= \int_\Omega \lim_{k\to\infty} |\eta_k Sf - Sf|^2 e^{-\varphi_3}\, dm = \\ &= \lim_{l\to\infty} \int_{\Omega_l} \lim_{k\to\infty} |\eta_k Sf - Sf|^2 e^{-\varphi_3}\, dm = 0. \end{aligned} \tag{8}$$

Synthesizing the result of (7) and (8), we have

$$\|Sg_k - Sf\|_{\varphi_3} \le \|S(\eta_k f) - \eta_k Sf\|_{\varphi_3} + \|\eta_k Sf - Sf\|_{\varphi_3} \to 0.$$

Moreover, by Lemma 3.3, $\eta_k f \in D(T^*)$, thus

$$\begin{aligned} |(T^*(\eta_k f) - \eta_k T^* f, v)_{\varphi_1}| &= |(T^*(\eta_k f), v)_{\varphi_1} - (\eta_k T^* f, v)_{\varphi_1}| = \\ &= |(\eta_k f, Tv)_{\varphi_1} - (T^* f, \eta_k v)_{\varphi_1}| = \\ &= |(f, \eta_k Tv)_{\varphi_1} - (f, T(\eta_k v))_{\varphi_1}| = |(f, \eta_k Tv - T(\eta_k v))_{\varphi_2}|, \end{aligned}$$

and

$$\begin{aligned} \left|\int_\Omega (T^*(\eta_k f) - \eta_k T^* f) v e^{-\varphi_1}\, dm\right| &\le \int_\Omega |f||\eta_k Tv - T(\eta_k v)| e^{-\varphi_2}\, dm \le \\ &\le \int_\Omega |f||v| e^{-\varphi_1/2 + \varphi_2/2} \cdot e^{-\varphi_2}\, dm = \\ &= \int_\Omega |f||v| e^{-\varphi_1/2 - \varphi_2/2}\, dm. \end{aligned}$$

Fix $v = T^*(\eta_k f) - \eta_k T^* f$ and note that support of v is contained by $\Omega \setminus \Omega_k$, then

$$\begin{aligned} &\int_\Omega |T^*(\eta_k f) - \eta_k T^* f|^2 e^{-\varphi_1}\, dm \le \\ &\le \int_\Omega |f||T^*(\eta_k f) - \eta_k T^* f| e^{-\varphi_1/2 - \varphi_2/2}\, dm \le \\ &\le \int_{\Omega\setminus\Omega_k} |f| e^{-\varphi_2/2} |T^*(\eta_k f) - \eta_k T^* f| e^{-\varphi_1/2}\, dm \le \\ &\le \left(\int_{\Omega\setminus\Omega_k} |f|^2 e^{-\varphi_2}\, dm\right)^{1/2} \left(\int_{\Omega\setminus\Omega_k} |T^*(\eta_k f) - \eta_k (T^* f)|^2 e^{-\varphi_1}\, dm\right)^{1/2}, \end{aligned}$$

hence

$$\|T^*(\eta_k f) - \eta_k T^* f\|_{\varphi_1} \le \left(\int_{\Omega\setminus\Omega_k} |f|^2 e^{-\varphi_2}\, dm\right)^{1/2} \to 0$$

as $k \to \infty$, which implies that

$$\|T^* g_k - T^* f\|_{\varphi_1} \to 0.$$

The remaining $\|g_k - f\|_{\varphi_2} \to 0$ in (6) can also be verified directly by the Dominated Convergence Theorem.

To complete our proof, we have to construct h_k. Fix $f = \sum_I f_I dx^I \in D(T^*) \cap D(S)$ with compact support in Ω, i.e., f_I has the same support $K \Subset \Omega$. Similar to Theorem 2.2, denote

$$(f * \mu_\varepsilon)(x) = \sum_I (f_I * \mu_\varepsilon)(x) dx^I,$$

that is,

$$f_\varepsilon(x) = \sum_I (f_I)_\varepsilon dx^I.$$

We can choose ε sufficiently small such that $\operatorname{supp}((f_I)_\varepsilon) \Subset \Omega$, thus $(f_I)_\varepsilon \in C_0^\infty(\Omega)$ and $\|(f_I)_\varepsilon - f_I\|_{L^2} \to 0$, therefore $\|f_\varepsilon - f\|_{\varphi_2} \to 0$ as $\varepsilon \to 0$. By Proposition 2.3, $Sf_\varepsilon = (Sf)_\varepsilon$, thus

$$\|Sf_\varepsilon - Sf\|_{\varphi_3} = \|(Sf)_\varepsilon - Sf\|_{\varphi_3} = 0.$$

We define

$$\theta f = -\sum_I \sum_{j=1}^n \frac{\partial f_{jI}}{\partial x_j}\, dx^I$$

and

$$\Theta f = \sum_I \sum_{j=1}^n f_{jI} \frac{\partial \varphi_2}{\partial x_j}\, dx^I,$$

then by Lemma 3.2,

$$e^{\varphi_2 - \varphi_1} T^* f = \theta f + \Theta f.$$

Taking Proposition 2.3 into consideration, we have

$$\begin{aligned}(\theta + \Theta) f_\varepsilon &= \theta f_\varepsilon + \Theta f_\varepsilon = (\theta f)_\varepsilon + \Theta f_\varepsilon) = \\ &= ((\theta+\Theta)f)_\varepsilon - (\Theta f)_\varepsilon + \Theta f_\varepsilon \to (\theta+\Theta)f - \Theta f + \Theta f = (\theta+\Theta)f\end{aligned}$$

in L^2, which means that $\|T^* f_\varepsilon - T^* f\|_{\varphi_1} \to 0$ as $\varepsilon \to 0$.

Now we have shown that for all $f \in D(T^*) \cap D(S)$, there exists a sequence of $\{g_k\} \subset D(T^*) \cap D(S)$ such that (6) holds, and for each g_k, there exists a sequence of $\{g_{kl}\} \subset D_{(p+1)}(\Omega)$ such that (5) holds. We choose one of g_{kl}'s as h_k such that $\|g_k - h_k\|_{\varphi_2} < k^{-1}$ and makes all the limits in (5) hold, and to this end we just have to fix k sufficiently large such that

$$\|f - g_k\|_{\varphi_2} < \varepsilon/2, \quad \|g_k - h_k\|_{\varphi_2} < k^{-1} < \varepsilon/2,$$

and in so doing, $h_k = f_{k^{-1}}$. $\square$

The above Lemma, as an application of Theorem 2.2 and 2.4, gives $D_{(p+1)}(\Omega)$-asymptotic estimate of $\|f\|_{\varphi_2}$, $\|T^* f\|_{\varphi_1}$ and $\|Sf\|_{\varphi_3}$. Thus, we just need to consider the conditions under which the inequality (3) holds for $f \in D_{(p+1)}(\Omega)$. To this end, first construct φ_j's that satisfies the condition (4).

Lemma 3.5. *For any $\varphi \in C^2(\Omega)$, define*

$$\varphi_1 = \varphi - 2\psi, \quad \varphi_2 = \varphi - \psi, \quad \varphi_3 = \varphi,$$

where $\psi(x) = \ln \sup_k \sum_{l=1}^n \left| \frac{\partial \eta_k(x)}{\partial x_l} \right|^2$, then φ_1, φ_2 and φ_3 satisfy (4).

The Lemma above can be directly verified, so we will omit its proof. We will now state the most important result of this section.

Main Theorem 3.6. *Maintain the notations in Lemma 3.5. Inequality (3) holds for all $f \in D_{(p+1)}(\Omega)$ (hence for all $f \in D(T^*) \cap D(S)$ by Lemma 3.4) if*

$$\sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \xi_j \xi_k \geq 2(|d\psi|^2 + e^\psi) \sum_{j=1}^n |\xi_j|^2. \tag{9}$$

Proof. Recall Lemma 3.2, 3.5 and 2.6, we write

$$e^{\psi}T^*f = X + Y,$$

where

$$X = (-1)^{p-1}\sum_I\sum_{j=1}^n \delta_j f_{jI}dx_I, \quad Y = (-1)^{p-1}\sum_I\sum_{j=1}^n f_{jI}\frac{\partial\psi}{\partial x_j}dx_I,$$

which yields that

$$\|e^{-\psi}X\|^2_{\varphi_1} \le (\|T^*f\|_{\varphi_1} + \|e^{-\psi}Y\|^2_{\varphi_1} \le 2\|T^*f\|^2_{\varphi_1} + 2\|e^{-\psi}Y\|^2_{\varphi_1}. \tag{10}$$

Substitute the expressions for X and Y into the above equation, then substitute the result into (10), we get

$$\int_\Omega \sum_I\sum_{j,k=1}^n (\delta_j f_{jI})(\delta_k f_{kI})e^{-\varphi}\,dm \le 2\|T^*f\|^2_{\varphi_1} + 2\int_\Omega |f|^2|d\psi|^2e^{-\varphi}\,dm.$$

Integrate the $|df|^2$ expression appearing in Lemma 2.5 along Ω and add the result to both ends of the above equation, we obtain

$$\int_\Omega \sum_I\sum_{j,k=1}^n \left((\delta_j f_{jI})(\delta_k f_{kI}) - \frac{\partial f_{jI}}{\partial x_k}\frac{\partial f_{kI}}{\partial x_j}\right)e^{-\varphi}\,dm + \int_\Omega\sum_I\sum_{j=1}^n\left|\frac{\partial f_I}{\partial x_j}\right|^2 e^{-\varphi}\,dm =$$
$$= \sum_I\int_\Omega\sum_{j,k=1}^n f_{jI}f_{kI}\frac{\partial^2\varphi}{\partial x_j x_k}e^{-\varphi}\,dm + \sum_I\sum_{j=1}^n\int_\Omega\left|\frac{\partial f_I}{\partial x_j}\right|^2 e^{-\varphi}\,dm =$$
$$\le 2\|T^*f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3} + 2\int_\Omega |f|^2||d\psi|^2e^{-\varphi}\,dm,$$

at the equality we used Lemma 2.6, and we took $w_j = f_{kI}$. Moreover, in this situation we have

$$\sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial x_j x_k}f_{jI}f_{kI} \ge 2(|d\psi|^2 + e^{\psi})\sum_{j=1}^n |f_{jI}|^2,$$

by (9), hence

$$2\|T^*f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3} + 2\int_\Omega |f|^2|d\psi|^2e^{-\varphi}\,dm \ge$$
$$\ge \sum_I\int_\Omega 2(|d\psi|^2 + e^{\psi})\sum_{j=1}^n |f_{jI}|^2e^{-\varphi}\,dm + \sum_I\sum_{j=1}^n\int_\Omega\left|\frac{\partial f_{jI}}{\partial x_j}\right|^2 e^{-\varphi}\,dm =$$
$$= 2\int_\Omega |f|^2|d\psi|^2e^{-\varphi}\,dm + 2\|f\|^2_{\varphi_2} + \sum_I\sum_{j=1}^n\int_\Omega\left|\frac{\partial f_{jI}}{\partial x_j}\right|^2 e^{-\varphi}\,dm \ge$$
$$\ge 2\int_\Omega |f|^2|d\psi|^2e^{-\varphi}\,dm + 2\|f\|^2_{\varphi_2}.$$

□

4. Pseudo-convex Domains on $\mathbb{R}^n$

We establish in this section some of the concepts with respect to the domain $\Omega \subset \mathbb{R}^n$, which are mostly parallel to that in the case of $\mathbb{C}^n$, especially pseudo-convex domain in $\mathbb{R}^n$. We will also establish relationships between these concepts. Although these properties usually also hold for that in $\mathbb{C}^n$, for the sake of completeness, we write out the proofs.

We start with

Definition 4.1. Suppose $U := B(0;1)$ the open unit ball, $\{\varphi_i\}_{i\geq 1} : \bar{U} \to \Omega$ is a sequence of C^∞-functions on $\bar{D}$. Ω is said to be

(1) with the D-property, iff

$$\bigcup_{i\geq 1} \varphi_i(\partial U) \Subset \Omega \Rightarrow \bigcup_{i=1}^{n} \varphi_i(U) \Subset \Omega;$$

(2) Levi pseudo-convex, iff Ω has the C^2-boundary and the Levi-form

$$L_x(\varphi, y) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi(x)}{\partial x_j \partial x_k} y_j y_k \geq 0$$

holds for all locally defined function φ near all $x \in \partial\Omega$ and $y \in T\mathbb{R}^n_x(\partial\Omega)$;

(3) pseudo-convex, iff there exists convex $C(\Omega)$-exhaustion functions.

Proposition 4.2. *Suppose G has the D-property, $\rho(x) := \rho(x, \partial U)$ denotes the distance between $x \in \Omega$ and ∂U, then $x \to -\ln \rho(x)$ is a convex map on Ω.*

Proof. By contraposition, suppose that $-\ln \rho(x)$ is not convex, and without loss of generality, loses its convexity at the origin, i.e., there exists $r_\pm = (\pm r, 0, \cdots, 0)$ such that

$$-\ln \rho(0) > -2^{-1} \ln d(r_\pm).$$

We define $U_r = \{\lambda \in \mathbb{R} : |\lambda| < r\}$, then there exists a linear function $h \in C(\bar{D}_r) \cap C^3(D_r)$ such that $h(\pm r) = -\ln \rho(r_\pm)$, thus

$$h(0) = 2^{-1}(h(-r) + h(r)) = 2^{-1}(-\ln \rho(r_-) - \ln \rho(r_+)) < -\ln \rho(0).$$

Furthermore, we denote by $\delta := -\ln \rho(0) - h(0) > 0$, $g = h + \delta$, then $g(\pm r) = h(\pm r) + \varepsilon = -\ln \rho(r_\pm) + \delta$ and $g(0) = h(0) + \delta = -\ln \rho(0)$. Fix $\xi \in \partial U$ such that $\rho(0) = \rho(0, \xi) = |\xi|$ and $e_1 := (1, 0, \cdots, 0)$. The $C(\bar{U}_r)$-function

$$\varphi_k(\lambda) = \lambda e_1 + (1 - k^{-1}) e^{-g(\lambda)} \frac{\xi}{|\xi|}$$

is analytic on U_r and

$$|\varphi_k(\lambda) - \lambda e_1| < |e^{-g(\lambda)}| = e^{-\delta} \rho(\lambda e_1),$$

which implies that $\varphi_k(\pm r) \in B(\pm re_1; e^{-\delta}\rho(\pm re_1))$, hence

$$\varphi_k(\pm r) \in \Omega, \quad \bigcup_{k\geq 1} \varphi_n(\partial U_r) \Subset \Omega.$$

Finally we note that, under the condition $\lambda = 0$, we have

$$\lim_{k\to\infty} \varphi_k(0) = e^{-g(0)}\frac{\xi}{|\xi|} = \rho(0)\frac{\xi}{|\xi|} = \xi \in \partial\Omega,$$

but Ω is open, hence $\overline{\bigcup_{n\geq 1} \varphi_k(U)}$ is not compact with respect to Ω. This contradiction finishes our proof. □

Theorem 4.3. *Ω is a pseudo-convex domain iff it has the D-property.*

Proof. Suppose Ω is a pseudo-convex domain and φ the exhaustion function. Define

$$F_i : \bar{U} \to \Omega, F_i \in C^1(U) \cap C(\bar{U}), \quad \bigcup_{i\geq 1} F_i(\partial U) \Subset \Omega.$$

Since $V = \bigcup_{i\geq 1} \overline{F_i(\partial U)}$ is compact,

$$\forall x \in V \exists M \in \Omega(\varphi(x) \leq M)$$

by continuity. Furthermore, note that $\varphi \circ F_i$ is convex on U (see [14], it is also similar to, see, e.g. [4, 6]), hence

$$\sup \varphi \circ F_i(U) = \max \varphi \circ F_i(\partial U) \leq \max \varphi(V) \leq M,$$

which means

$$\bigcup_{i\geq 1} F_i(U) \subset \{x \in \Omega : \varphi(x) \leq M\}.$$

By the definition of exhaustion, the right-end of the above equation is compact with respect to Ω, so its left-end is even more so.

Conversely, suppose that Ω has ths D-property, then by Proposition 4.2, the $-\ln\rho(x)$ is convex on Ω. If Ω is bounded, $-\ln\rho(x)$ is indeed a exhaustion, so next we consider thd case that Ω is unbounded.

Fix

$$\varphi(x) = |x|^2 - \ln\rho(x),$$

it is clear that φ is convex and $\{x \in \Omega : |x|^2 - \ln\rho(x) \leq M\}$ a bounded set, which implies that Ω is a pseudo-convex domain. □

Theorem 4.4. *A bounded domain with C^2-boundary Ω is pseudo-convex iff Ω is Levi pseudo-convex.*

Proof. Suppose Ω is a pseudo-convex domain in $\mathbb{R}^n$. By Theorem 4.3, Ω has the D-property, and by Proposition 4.2, $-\ln\rho$ is convex on Ω. Since Ω is bounded with C^2-boundary, there exists an neighborhood U of $\partial\Omega$ such that $\rho(x) \in C^2(U)$ (see [15]), thus the Levi-form of x, denoted by L_x, satisfies

$$L_x(-\ln\rho, y) \geq 0, \quad x \in U, \quad y \in \mathbb{R}^n.$$

Fix the defined function of Ω by

$$r(x)=\begin{cases}-\rho(x), & x\in\Omega,\\ 0, & x\in\partial\Omega,\\ \rho(x,\Omega), & x\in\mathbb{R}^n\setminus\Omega.\end{cases}$$

Direct calculation shows that

$$\begin{aligned}\frac{\partial^2(-\ln\rho(x))}{\partial x_j\partial x_k}&=-(\rho(x))^{-1}\frac{\partial^2\rho(x)}{\partial x_k\partial x_k}+(\rho^2(x))^{-1}\frac{\partial\rho(x)}{\partial x_j}\frac{\partial\rho(x)}{\partial x_k}=\\ &=-(r(x))^{-1}\frac{\partial^2 r(x)}{\partial x_k\partial x_k}+(r^2(x))^{-1}\frac{\partial r(x)}{\partial x_j}\frac{\partial r(x)}{\partial x_k},\end{aligned}$$

which yields

$$L_x(-\log\rho,y)=-(\rho(x))^{-1}L_x(r,y)+(r^2(x))^{-1}\left|\sum_{j=1}^n\frac{\partial r(x)}{\partial x_j}y_j\right|^2.$$

For fixed x we choose $y\in\mathbb{R}^n$ such that $\sum_{j=1}^n\frac{\partial r(x)}{\partial x_j}y_j=0$, then

$$L_x(r,y)\geq 0,\quad \frac{\partial r(x)}{\partial x_j}y_j=0.$$

Letting $x\to x_0\in\partial\Omega$, it is clear that Ω is Levi pseudo-convex.

Conversely, If Ω is Levi pseudo-convex, we have to prove that $-\ln\rho(x)$ is convex near $\partial\Omega$. Indeed, If that is not the case, with out loss of generality, we suppose that $-\ln\rho$ loses its convexity at the origin $x_0=0$, therefore there exists $a\in\mathbb{R}^n$ such that

$$-\ln\rho(0)>-\ln\rho(-a)>-\ln\rho(a),$$

and clearly, $-\ln\rho(\lambda a)$ is not convex at $\lambda=0$, thus we have

$$\left.\frac{\partial^2\ln\rho(\lambda a+x_0)}{\partial\lambda^2}\right|_{\lambda=0}>0.$$

Fix $\xi\in\partial\Omega$ such that $\rho(x_0,\xi)=\rho(x_0)=|\xi|$ and expand $\varphi(\lambda)=\ln\rho(x_0+\lambda a)$ at $\lambda=0$ into a power series as

$$\varphi(\lambda)=\varphi(0)+A\lambda+B\lambda^2+O(\lambda^3),$$

where

$$A=\sum_{j=1}^n\frac{\partial(\ln\rho(x_0))}{\partial x_j}a_j,\quad B=2^{-1}\sum_{j,k=1}^n\frac{\partial^2(\ln\rho(x_0))}{\partial x_j\partial x_k}a_ja_k.$$

Thus,

$$\rho(\lambda a)=\exp(\varphi(0)+A\lambda+B\lambda^2+O(\lambda^3))=\rho(0)\exp(A\lambda)(B\lambda^2+1+O(\lambda^3)).$$

We define $\chi(\lambda)=x_0+a\lambda+\xi e^{A\lambda}$. Note that

$$|\xi|e^{A\lambda}=\rho(\chi(\lambda,x_0+\lambda a)\geq\rho(x_0+\lambda a)-\rho(\chi(\lambda)),$$

we have

$$\rho(\chi(\lambda)) \geq \rho(\lambda a) - \rho(0)e^{A\lambda} = \rho(0)e^{A\lambda}(B\lambda^2 + O(\lambda^3)),$$

hence $\rho(\chi(\lambda)) > 0$ for $B > 0$ and sufficiently small λ. Meanwhile $\rho(\chi(0)) = \rho(\xi) = 0$, so $\chi(\lambda)$ gets a local minimum at $\lambda = 0$, thus

$$\left.\frac{\partial \rho(\chi(\lambda))}{\partial \lambda}\right|_{\lambda=0} = 0.$$

In this situation, we have

$$\begin{aligned}\rho(\chi(\lambda)) &= \rho(\chi(0)) + \frac{\partial}{\partial \lambda}\rho(\chi(0))\lambda + \frac{\partial^2}{\partial \lambda^2}\rho(\chi(0))\lambda^2 + O(\lambda^3) = \\ &= \frac{\partial^2}{\partial \lambda^2}\rho(\chi(0))\lambda^2 + O(\lambda^3).\end{aligned}$$

It is clear that $\frac{\partial^2}{\partial \lambda^2}\rho(\chi(0)) > 0$ since $\rho(\chi(\lambda)) > 0$ near 0.

Recall the defined function $r = -\rho$, we have

$$0 = -\left.\frac{\partial \rho(\chi(\lambda))}{\partial \lambda}\right|_{\lambda=0} = \left.\frac{\partial r(\lambda)}{\partial \lambda}\right|_{\lambda=0} = \sum_{j=1}^{n} \frac{\partial r(\xi)}{\partial x_j}\frac{\partial x_j(0)}{\partial \lambda}$$

and

$$0 > -\left.\frac{\partial^2 \rho(\chi(\lambda))}{\partial \lambda^2}\right|_{\lambda=0} = \left.\frac{\partial^2 r(\lambda)}{\partial \lambda^2}\right|_{\lambda=0} = \sum_{j,k=1}^{n} \frac{\partial^2 r(\xi)}{\partial x_j x_k}\frac{\partial x_j(0)}{\partial \lambda}\frac{\partial x_k(0)}{\partial \lambda},$$

which implies that Hesse matrix of $r(x)$ is not semi-positive definite at ξ. This contradiction finishes our proof. $\square$

Theorem 4.5. *Suppose Ω is a pseudo-convex domain on $\mathbb{R}^n$, then there exists convex C^∞-exhaustion function on Ω.*

Proof. Define

$$\mu(x) = \begin{cases} c\exp((x^2-1)^{-1}), & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$$

where $c = \int_{|x|<1} \exp((x^2-1)^{-1})\, dx$, and

$$G_k := \{x \in \Omega : \mu(x) \leq k\}.$$

This μ satisfies the condition mentioned in Theorem 2.2. We determine μ_ε as it in Theorem 2.2 and denote $u_\varepsilon = \mu_\varepsilon * u$ for continuous exhaustions u. Direct calculation shows that

$$\begin{aligned}u_\varepsilon &= \varepsilon^{-n}\int_{\mathbb{R}^n} u(x-y)\mu(y/\varepsilon)\, dy = \int_{|y|<1} u(x-\varepsilon y)\mu(y)\, dy = \\ &= \int_{|y|<1} u(x+\varepsilon y)\mu(y)\, dy,\end{aligned}$$

thus (see [10])

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial \varepsilon} &= \int_{|y|<1} \frac{\partial u(x+\varepsilon y)}{\partial \varepsilon} y\, dy = \int_{B(x;\varepsilon)} \frac{\partial u(y)}{\partial \varepsilon} \frac{y-x}{\varepsilon}\, dy = \\ &= \int_{B(x;\varepsilon)} \frac{\partial u}{\partial n}\, dm = \int_0^\varepsilon \int_{\partial B(x;\varepsilon)} \frac{\partial u}{\partial n}\, dr dS = \\ &= \int_0^\varepsilon \int_{B(x;\varepsilon)} \Delta u\, dx dr \geq 0,\end{aligned}$$

where n, dr, dS the unit exteroir normal vector, radius and surface element of $B(x;\varepsilon)$, Δ the normal Laplacian. Hence u_ε is increasing with respect to ε, and $u_\varepsilon \to u$ as $\varepsilon \to 0$.

Now we construct the exhaustion function ψ required by the theorem from the sequence $\{u_\varepsilon\}$ and $\varphi \in C^\infty(\mathbb{R})$ satisfying

$$\begin{cases} \varphi(x) = 0, & x \leq 0, \\ \varphi'(x),\, \varphi''(x) > 0, & x > 0. \end{cases}$$

We define

$$\psi_k := \varphi(u_0(x)+1) + \sum_{j=1}^{k} a_j \varphi(u_j(x)+1-j), \quad a_j > 0,$$

and use induction to determine $a_1, \cdots, a_k$ such that $\psi_k(x) \geq u(x)$. It is clear that $\varphi(u_0(x)+1)$ is convex and $\varphi(u_0(x)+1) \geq u(x)$ for all $x \in G_0$. Now, suppose that there exists $a_1, \cdots, a_{k-1}$ such that $\psi_{k-1}(x) \geq u(x)$ for all $x \in G_{k-1}$.

Note that $G_k = G_{k-1} \cup (G_k \setminus G_{k-1})$ and $\psi_k(x) = \psi_{k-1}(x) + a_k\varphi(u_k+1-k) \geq u(x)$ holds for $x \in G_{k-1}$, we need only consider the case of $x \in G_k \setminus G_{k-1}$. In this situation we have $k-1 < u(x) \leq u_k(x)$, so we can fix a_k sufficiently large such that

$$a_k \varphi(u_k + 1 - k) \geq k \geq u,$$

hence ψ_k convex and $\psi_k \geq u$ holds in G_k.

Let $K \Subset \Omega$ and $\Omega_k \to \Omega$ an exhaustion of Ω, then $K \subset \Omega_s$ for some s. Since u_ε converges uniformly on K (see also [10]), for $k \geq s+2$ and all $x \in K$ we have $|u_k - u| < 1$, which yields

$$u_k(x) + 1 - k < u(x) + 2 - k < m + 2 - k \leq 0,$$

hence $\varphi(u_k(x)+1-k) = 0$, therefore $\psi_k(x) = \psi_{s+1}(x)$ for all $x \in K$ and $k \geq s+2$. Overall, sequence ψ_k converges uniformly on K, we denote by $\psi = \lim_{k\to\infty} \psi_k$. Obviously we have

$$\{\psi \leq k\} \subset \{u \leq k\} \Subset \Omega$$

and

$$\psi = \sup_{k \geq s+2} \{\psi_k\},$$

hence we finished the proof. $\square$

5. Main Results

In this section we will prove that, (3) will always hold as long as Ω is a pseudo-convex domain.

Main Theorem 5.1. *If $f \in L^2_{(p,q+1)}(\Omega, \mathrm{loc})$ satisfies $df = 0$, then problem (1) has a solution $u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$.*

Proof. By Theorem 4.5, there exists convex C^∞-exhaustion function η on Ω, thus for all $t \in \mathbb{R}$ we have

$$G_t := \{x \in \Omega : \eta(x) \leq t\} \Subset \Omega$$

and for all $x \in \Omega$, $\xi \in \mathbb{R}^n$,

$$L_x(\eta, \xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \eta(x)}{\partial x_j \partial x_k} \xi_j \xi_k > 0.$$

For all x, we denote by $0 < m(x) = \min_{x \in \partial U} L_x(\eta, \xi)$, then

$$L_x(\eta, \xi) \geq m(x) \sum_{j=1}^{n} |\xi_j|^2.$$

Take ψ in Lemma 3.5 and define

$$h(t) := 2 \sup_{G_t} \frac{|d\psi|^2 + e^\psi}{m(x)},$$

this h is continuous and monotonically increasing since G_t is compact. Fix $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$g' \geq \max\{h, 0\}, \quad g'' \geq 0$$

and let $\varphi = g \circ \eta$, then for $w \in \mathbb{R}$ we get

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} w_j w_k = g''(\eta(x)) \left| \sum_{j=1}^{n} \frac{\partial \eta}{\partial x_j} w_j \right|^2 + g'(\eta(x)) L(x, w) \geq$$

$$\geq g'(\eta(x)) L(x, w) \geq g'(\eta(x)) m(x) \sum_{j=1}^{n} |w_j|^2.$$

For fixed z put $t = \eta(z)$, thus

$$g'(t) \geq h(t) \geq 2 \frac{|d\psi|^2 + e^{\psi(x)}}{m(x)},$$

which yields

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi(x)}{\partial x_j \partial x_k} w_j w_k \geq 2(|d\psi|^2 + e^\psi) \sum_{j=1}^{n} |w_j|^2,$$

this claims that φ satisfies (9).

Meanwhile, note that $f \in L^2_{(p+1)}(\Omega, \mathrm{loc})$, thus

$$b_j := \int_{G_{k+1}\setminus G_k} |f^2|\, dm < \infty$$

for all j. Choosing δ_j such that $\sum_{j\geq 1} b_j\delta_j < \infty$. Now, if the g constructed above is required to additionally satisfy

$$g(t) \geq \ln \delta_j^{-1} + \sup\{\psi(x) : x \in G_{j+1} \setminus G_j\}, \quad j < t \leq j+1,$$

in this situation, $j < \eta(x) \leq j+1$, therefore

$$\varphi(x) = g(\eta(x)) \geq \ln \delta_j^{-1} + \psi(x),$$

or equivalently

$$e^{\psi-\varphi} \leq \delta_j.$$

Hence

$$\|f\|^2_{\varphi_2} = \int_\Omega |f|^2 e^{\psi-\varphi}\, dm = \sum_{j=1}^{n} \int_{G_{j+1}\setminus G_j} |f|^2 e^{\psi-\varphi}\, dm \leq$$

$$\leq \sum_{j=1}^{\infty} b_j\delta_j < \infty.$$

This implies that $f \in L^2_{(p+1)}(\Omega, \varphi_2)$, hence (1) exists a solution

$$u \in L^2_{(p)}(\Omega, \varphi_1) \subset L^2_{(p)}(\Omega, \mathrm{loc}).$$

$\square$

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Tsuiying Honors College, Lanzhou University, No. 222 South Tianshui Road, Lanzhou 730000, Gansu Province, P.R.China

Email address: gaowh21@lzu.edu.cn

School of Mathematics and Statistics, Lanzhou University, No. 222 South Tianshui Road, Lanzhou 730000, Gansu Province, P.R.China

Email address: hanqch21@lzu.edu.cn