A Discussion of Arnold's Limit Problem and its Geometric Argument

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Abstract

Upon re-examining Arnold's established lemma for explaining his famous limit problem, we have determined that while the lemma itself is correct, there is a defect in the original geometric proof. In this paper, we prove the correctness of the lemma using methods of power series, and construct a counterexample to illustrate the defect in Arnold's geometric proof.

Keywords: limit, counterexample, classical mathematical analysis

MSC Classification: 26A06, 26A03

1 Introduction

In [2], Arnold first formulated his famous limit problem. That is,

$$\lim_{x\to 0} \frac{\tan\sin x - \sin\tan x}{\arctan\arcsin x - \arcsin\arctan x} =?$$

Subsequently, the problem reappeared in [3], known as one of the "100 mathematical problems for physicists".

In general form, this problem can be formulated as

$$\lim_{x\to 0} \frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)} =? \tag{1}$$

with

$$f,\, g \text{ analytic}, \quad f(0) = g(0) = 0, \quad f'(0) = g'(0) = 1. \tag{2}$$

Also in [2], Arnold presented his famous geometric solution. The core of his method is

Lemma 1. *If the graphs of analytic functions f and g do not coincide and they are both tangent to the line $y = x$ at the origin (Fig.1), then $|AB|/|BC|$ and $|BC|/|ED|$ converges to 1 as A is sufficiently close to the origin.*

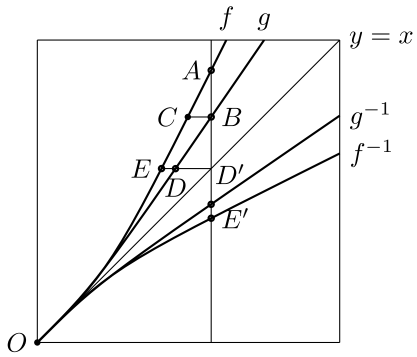

Fig. 1 f and f^{-1}, g and g^{-1} are symmetric about $y = x$ respectively

Arnold's own proof of Lemma 1 is Fig.1. Intuitively, it seems that when A close to O, AC as a cut line transitions to a tangent line, in turn, ACB tends to an isosceles right triangle and $BCED$ tends to a parallelogram. Hence limit (1) is equivalent to

$$\lim \frac{|AB|}{|D'E'|} = \lim \frac{|AB|}{|ED|} = \lim \left(\frac{|AB|}{|BC|} \cdot \frac{|BC|}{|ED|} \right) = 1.$$

This problem has been appropriately generalized in [4], where the limit is determined to be $f'(0)^6$ by means of Faa di Bruno's formula under the weaker condition

$$\begin{cases} f,\, g \text{ odd and analytic in a neighborhood of } 0, \\ f'(0) = g'(0) \neq 0, \\ f^{(3)}(0) = g^{(3)}(0) \neq 0, \\ 0 \neq f^{(5)}(0) \neq g^{(5)}(0) \neq 0, \end{cases} \tag{2'}$$

and some of Arnold's own comments on the problem, with the geometric argument, can be found in [5, 6]. However, an analysis argument of Lemma 1 itself has not yet appeared in the relevant literature. As we attempt to accomplish this, we verified the correctness of Lemma 1, and at the same time found a small defect in the geometric argument above.

Therefore, this paper is organized as follows.

In Section 2, we solved problem (1) – (2) by means of power series, which shows the correctness of Lemma 1 itself. Then we trying to describe "ACB tends to an isosceles right triangle" and "$BCED$ tends to a parallelogram" in analysis, and in completing the latter we found that defect. Furthermore, a counterexample is constructed in the last section to confirm the nature of this defect.

2 An Analysis Argument of Problem (1) – (2)

Since f and g are analytic, they can be represented as power series respectively. We denote by

$$f(x) = \sum_{k\geq 1} a_k x^k \quad \text{and} \quad f^{-1}(x) = \sum_{l\geq 1} b_l x^l. \tag{3}$$

The summations starts with k, $l = 1$, since by the restriction $f(0) = 0$, the coefficient of x^0 must be 0; moreover, due to the existence of inverse there must be a_1, $b_1 \neq 0$.

The connection between two power series in (3) is inscribed by the following Theorem.

Theorem 1. *For coefficients in (3) we have*

$$b_n = \begin{cases} 1/a_1, & n = 1, \\ -a_n/a_1^{n+1} + R_n, & n > 1, \end{cases}$$

where R_n are some terms determined only by a_k for $k < n$.

Proof. Note that

$$f \circ f^{-1}(x) = \sum_{k\geq 1} a_k \left(\sum_{l\geq 1} b_l x^l \right)^k = x := \sum_{n\geq 1} c_n x^n, \tag{4}$$

here the rightmost end is clearly a power series of the form, with all coefficients 0 except $c_1 = 1$. For each $n > 1$, c_n does not absorb b_l that makes $l > n$, otherwise the power of x in such b_l-term will greater than n. Similarly c_n does not contain those a_k that makes $k > n$. However, $\sum_{l\geq 1} b_l x^l$ already contains $b_n x^n$, which means that k here can only be taken as 1, so the b_n-term involved can only be $a_1 b_n x^n$. Similarly, the a_n-term in $a_n \left(\sum_{l\geq 1} b_l x^l\right)^n$ can only be $a_n b_1^n x^n$. Therefore,

$$c_n = 0 = a_1 b_n + a_n b_1^n + K,$$

where K is some terms determined only by a_k and b_l for l, $k < n$; and for $c_1 = 1$ we obviously have $c_1 = a_1 b_1$. Thus we have shown that, b_n can only be determined by a_k for $k \leq n$ and b_l for $l < n$.

Furthermore, by $b_1 = 1/a_1$, b_1 is completely determined by a_1. b_2 is determined by a_1, a_2, b_1, which means by a_1, a_2. Continuing this process, eventually b_n is completely determined by a_k only for $k \leq n$, which finishes our proof. □

Remark 1. *We have indeed verified an even stronger conclusion, which is that, R_n is the same for all f satisfying condition (2). In fact, this proposition can also be seen* *as an application of the Lagrange Inversion Formula (see, e.g., [1]) and we will omit the detailed proof of this.*

Fix

$$g(x) = \sum_{k\geq 1} A_k x^k \quad \text{and} \quad g^{-1}(x) = \sum_{l\geq 1} B_l x^l,$$

then since $f'(0) = g'(0) = 1$, we have $a_1 = A_1 = 1$. Hence

$$b_n = a_n + R_n, \quad B_n = -A_n + R_n.$$

It is clear that $b_n = B_n$ holds only if $a_n = A_n$. Suppose that the first distinct coefficients in the power series of f and g occur in the place of N-th term. In this situation, $b_n - B_N = -(a_N - A_N)$, thus

$$\lim_{x\to 0} \frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)} = \lim_{x\to 0} \frac{(a_N - A_N)x^N + O(x^{N+1})}{(a_N - A_N)x^N + O(x^{N+1})} = 1.$$

3 Motivation to Detect a Defect

Recall Lemma 1 and Fig.1, we will notice that the Lemma itself requires the assumption of analytic f and g, but this condition is not used in Arnold's geometric proof, because we cannot represent such a strong condition of analyticity just in a graph; at least the graph of an analytic function does not look different from the graph of a "smooth" function. So this geometric proof may be tempting to think that Lemma 1 may still holds, after weakening the original assumption to "f, g are 'smooth' functions (e.g. C^∞-functions)". Under the latter weaker condition we try to provide an analysis formulation of limit processes appearing in the geometric proof of Section 1, and the defect appears. To this end, without loss of generality, we suppose the coordinates (see Fig.1)

$$A(x, f(x)), \quad B(x, g(x)), \quad C(f^{-1}\circ g(x), g(x)),$$
$$D(g^{-1}(x), x), \quad D'(x, x), \quad E(f^{-1}(x), x).$$

3.1 Analysis argument of $|AB|/|BC| \to 1$.

We have

$$|AB| = f(x) - g(x), \quad |BC| = x - f^{-1}\circ g(x),$$

which yields

$$\frac{|AB|}{|BC|} = \frac{f(x) - f(f^{-1}\circ g(x))}{x - f^{-1}\circ g(x)}. \tag{5}$$

Applying the Lagrange Mean Value Theorem to (5), we can find a point $x \leq \xi \leq f^{-1}\circ g(x)$ such that $|AB|/|BC| = f'(\xi)$. But $f^{-1}(x)\circ g(x) \to 0$ as $x \to 0$, so we have $f'(\xi) \to f'(0) = 1$ by continuity.

3.2 Analysis argument of $|ED|/|BC| \to 1$.

The defect in Arnold's geometric proof will be cited here. In this case we should note that, $BCED$ is never a parallelogram, but tends to be as A approaches O. So as shown in Fig.2,

$$|ED| = |EF| + |FD| = |DD'|\frac{|EF|}{|DD'|} + |FD| = |FD'| + |DD'|\left(\frac{|EF|}{|DD'|} - 1\right).$$

Note that $|BC| \sim |FD'|$ as $A \to O$, we have

$$\frac{|ED|}{|BC|} \sim 1 + \frac{|DD'|}{|FD'|}\left(\frac{|EF|}{|DD'|} - 1\right). \tag{6}$$

In the case of Fig.2, we have $|EF|/|DD'| - 1 \to 0$ and $|DD'|/|FD'|$ bounded (since $|DD'| < |FD'|$), so there is indeed $|ED|/|BC| \to 1$. But if we consider case of Fig.3, i.e., the quasi-parallelogram is very narrow and $|BC|$, $|DE|$ are small enough, it is easy to verify that (6) still holds, but $|DD'| > |FD'|$, which might lead to a divergent $|DD'|/|FD'|$. This situation will always occur if f and g are moving closer together than they are to $y = x$. This suggests to us that there exist such C^∞-functions, which make Lemma 1 incorrect, but Arnold's geometric proof still holds.

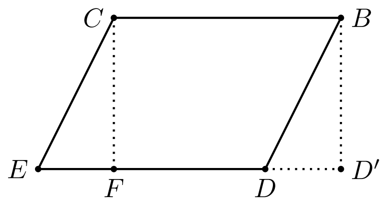

Fig. 2 Convergent case: F is the projection of C onto ED, which is to the left of D.

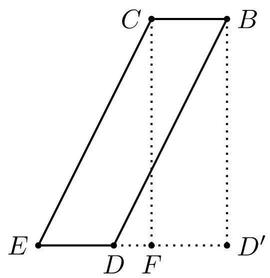

Fig. 3 Divergent case: F is the projection of C onto ED, which lies between D and D'.

4 Construction of the Counterexample

We start with

Definition 1. *We define $\theta(x) : \mathbb{R} \to \mathbb{R}$ by*

$$\theta(x) = \begin{cases} e^{-1/|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The following property of $\theta(x)$ is well known.

Proposition 1. *$\theta(x) \in C^\infty$ but not analytic.*

The proof of this proposition can be found in many textbooks, so we will omit it here.

The whole problem is symmetric for f, g and their inverses. Define $p = f^{-1}$ and $q = g^{-1}$, then

$$\frac{|BC|}{|ED|} = \frac{x - p \circ g(x)}{q(x) - p(x)}.$$

In order for p, q to also satisfy the requirements stated at the end of Section 3, we may set

$$p(x) = q(x) + \theta(x), \quad q(x) = x + x^2, \tag{7}$$

we choose the x^2 term in (7) so that it serves the purpose of controlling graphs of p, q away from $y = x$. Denote $t = g(x)$, hence $x = q(t)$, and

$$\frac{|BC|}{|ED|} = \frac{q(t) - p(t)}{\theta(x)} = \frac{\theta(t)}{\theta(q(t))} = \frac{\theta(t)}{\theta(t + t^2)}. \tag{8}$$

As an example, let's take the right-hand limit of (8) as t tends to 0. Since $t \to 0^+$ as $x \to 0^+$, we have

$$\lim_{x \to 0^+} \frac{|BC|}{|ED|} = \lim_{t \to 0^+} \frac{\theta(t)}{\theta(t + t^2)} = \lim_{t \to 0^+} e^{-1/(t+1)} = e^{-1} \neq 1, \tag{9}$$

which finishes our construction.

5 Conclusion Remarks

As a conclusion, we first note that the drawing of Fig.1 does not affect the generality of our discussion. If we place f and g on both sides of the line $y = x$, it can be verified that the discussion similar to that in Section 3 still holds, with the only change being that ED is now above BC.

As shown in Section 4, just using the intuitive perspective of geometry, we will not be able to explain why Arnold's proof fails at this point, since adding a $\theta(x)$ near 0 to the graph makes no difference. Therefore, this geometric proof, although not said to be wrong, does at least have its imperfections.

If we look into the nature of the problem, we will find that the problem lies in the process of "$BCED$ tends to a parallelogram". If f, g are analytic functions, then they approach each other at a rate close to the rate at which they approach $y = x$, and $BCED$ is indeed a parallelogram in the limit; whereas if f, g are constructed as we have done in Section 4, $BCED$ only becomes more and more elongated and tends to "diverge". But the subtle differences in the properties of the functions, and the tendency of the graphs to change, cannot be captured in a static figure. Therefore, the discussion in this paper makes us realize once again that, although intuitive graphical representations can help us quickly identify the essence of the problem, rigorous proofs are still indispensable.

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