differential equ. - o.d.e: 常微分方程. 庞特里亚金, ver 8, ch4
- p.d.e:

· Rudin, Principle of MA

Mathematical Induction

With standard examples

· Zorich. MA

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November 11, 2023

$$\boxed{1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}}$$

1. Induction: a General intro

2. Standard examples
   - Bernoulli's inequality
   - Binomial expansion
   - A classical problem in elementary combinatorics
   - The inclusion-exclusion principle
   - Cauchy's method to prove the mean-value inequality

Peano's axiom

Each nonempty set $S \subset \mathbb{N}$ must have a minimal element.

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Theorem

To verify that the proposition $M(n)$ holds for every $n \in \mathbb{N}$, it is sufficient to verify the following two terms:

- *$M(n_0)$ holds, where n_0 is a known number.*
- *Assume that $M(n)$ holds, and verify that $M(n+1)$ holds under this condition.*

$\phi \quad \{\phi\} \quad \{\phi, \{\phi\}\} \quad \{\phi, \{\phi, \{\phi\}\}\}$

$0 \quad 1 \quad 2 \quad 3 \quad \ldots$

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Proof.

Let

$$S = \{s : M(s) \text{ doen't hold}, \ s \in \mathbb{N}\}$$

is nonempty. By Peano's axiom, S has a minimal element s_0. Then $M(s_0)$ does not hold, while $M(s)$ holds for any $s < s_0$. This contradicts our ability to show that $M(s_0)$ holds under the assumptions. $\square$

Example (Bernoulli's inequality)

$$\boxed{(1+x_1)(1+x_2)\cdots(1+x_n) \geqslant 1+x_1+x_2+\cdots+x_n,}$$

where $x_1, \cdots, x_n$ are numbers with the same sign and > -1.

- $n=1$ $\quad 1+x_1 = 1+x_1$ ✓

- $n=m \Rightarrow n=m+1$

$(1+x_1)\cdots(1+x_m)(1+x_{m+1}) \geq (1+x_1+\cdots+x_m)(1+x_{m+1})$

$= \boxed{1+x_1+\cdots+x_m+x_{m+1}} + \underbrace{x_1x_{m+1}+\cdots+x_mx_{m+1}}$

$\geq 1+x_1+\cdots+x_{m+1}$ #

$\boxed{\binom{n}{k} = C_n^k}$

$x_1 = \cdots x_n = x$

$\boxed{(1+x)^n \geq 1+nx}$

$C_n^0 1^n x^0 + C_n^1 1^{n-1} x^1 + (\text{sth} \geq 0)$

$1 + nx$

$\geq 1+nx$ #

Example (Binomial expansion)

Define $a^{[n]} = a(a-h)\cdots(a-(n-1)h)$ and $a^{[0]} = 1$. Show that

$$(a+b)^{[n]} = \sum_{m=0}^{n} C_n^m a^{[n-m]} b^{[m]}.$$

Set $h=0$

$a^{[n]} = a^n$

$\cdot\ n=1.\quad (a+b)^{[1]} = a+b = \sum_{m=0}^{1} C_1^m a^{[1-m]} b^{[m]} = a+b$

$n=k \Rightarrow n=k+1$

$(a+b)^{[k+1]} = (a+b)^{[k]} \cdot (a+b-kh)$

$= (a+b-kh) \sum_{m=0}^{k} C_k^m a^{[k-m]} b^{[m]}$

$= (a+b-kh)\left(C_k^0 a^{[k]} b^{[0]} + C_k^1 a^{[k-1]} b^{[1]} + \cdots \right.$

$$+ \cdots C_k^k a^{[0]} b^{[k]} \Big) \qquad = a - kh + h + b - h$$

$$= \big((a-kh)+b\big) C_k^0 a^{[k]} b^{[0]} + \big(a-(k-1)h + (b-h)\big) C_k^1 a^{[k-1]} b^{[1]}$$

$$+ \cdots + (a+b-kh) C_k^k a^{[0]} b^{[k]}$$

$$= C_k^0 a^{[k+1]} b^{[0]} + C_k^0 a^{[k]} b^{[1]} + C_k^1 a^{[k]} b^{[1]} + C_k^1 a^{[k-1]} b^{[2]} + \cdots$$

$$+ C_k^k a^{[1]} b^{[k]} + C_k^k a^{[0]} b^{[k+1]} \qquad \boxed{C_k^0 + C_k^1 = C_{k+1}^1}$$

$$= C_k^0 a^{[k+1]} b^{[0]} + C_{k+1}^1 a^{[k]} b^{[1]} + \cdots C_{k+1}^2 \cdots$$

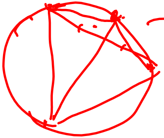
$$+ C_{k+1}^k a^{[1]} b^{[k]} + C_{k+1}^{k+1} a^{[0]} b^{[k+1]} \qquad \#$$

Example (*American Mathematics Monthly*, 1977, V.84)

Given any n points on the circumference of a circle, determine the number of regions $R_n = 1 + C_n^2 + C_n^4$ in the circle divided by C_n^2 strings. Assume that any 3 strings do not intersect in the circle. 证

$\boxed{n-1 \Rightarrow n}$

$$R_n = R_{n-1} + C_{n-1}^3 + n - 1 = 1 + C_n^2 + C_n^4$$



Example (The inclusion-exclusion principle) 容斥原理

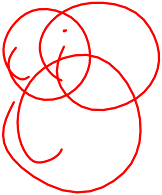
For finite sets $A_1, \cdots, A_n$ show that

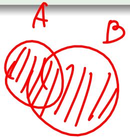
$A_1 \cup A_2 \cdots \cup A_n$ $\qquad A = \{a_1 \ldots a_n\}$ $\qquad |A| = n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \underline{\sum_{i=1}^{n} |A_i|} - \underline{\sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j|} + \cdots$$

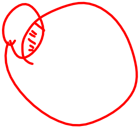
$$+ (-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |A_{i_1} \cap \cdots \cap A_{i_k}| + \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$

$|A|$ = # of elements in A





$A \cup B$



$A \cap B$

Example (The mean-value inequality, proved by Cauchy's method)

There're n numbers say A, B, C, D, $\cdots$. We have

$ab \leq \left(\frac{a+b}{2}\right)^2$

$$ABCD\cdots \leqslant \left(\frac{A+B+C+D+\cdots}{n}\right)^n.$$

- Step 1: for $n=2$, we obviously have

$$AB = \left(\frac{A+B}{2}\right)^2 - \left(\frac{A-B}{2}\right)^2 \leqslant \left(\frac{A+B}{2}\right)^2.$$

- Step 2: for $n = 4, 8, \cdots, 2^m$:

$$ABCD \leqslant \left(\frac{A+B}{2}\right)^2 \left(\frac{C+D}{2}\right)^2 \leqslant \left(\frac{A+B+C+D}{4}\right)^4,$$

$$ABCDEFGH \leqslant \left(\frac{A+B+C+D}{4}\right)^4 \left(\frac{E+F+G+H}{4}\right)^4 \leqslant \left(\frac{A+B+C+D+E+F+G+H}{8}\right)^8,$$

$$\vdots$$

$$ABCD\cdots \leqslant \left(\frac{A+B+C+D+\cdots}{2^m}\right)^{2^m}. \qquad (*)$$

- Step 3: Finally, suppose $n \notin P = \{2, 4, \cdots, 2^m\}$. Fix a term in P large than n, say 2^m, and suppose

$$K = \frac{A + B + C + D + \cdots}{n}.$$

Let the last $2^m - n$ factor in the $L.H.S$ of equation (*) be equal to K, so that

$$ABCD \cdots K^{2^m - n} \leqslant \left(\frac{A + B + C + D + \cdots + (2^m - n)K}{2^m} \right)^{2^m},$$

i.e.,

$$ABCD \cdots K^{2^m - n} \leqslant K^{2^m},$$

hence

$$ABCD \cdots \leqslant K^n = \left(\frac{A + B + C + D + \cdots}{n} \right)^n.$$