PROGRAMMING ASSIGNMENT #6
CS 2223 D-TERM 2020
BACKTRACKING
AND
THE n-QUEENS PROBLEM

ONE HUNDRED POINTS
DUE: MONDAY, MAY 11, 2020 2PM

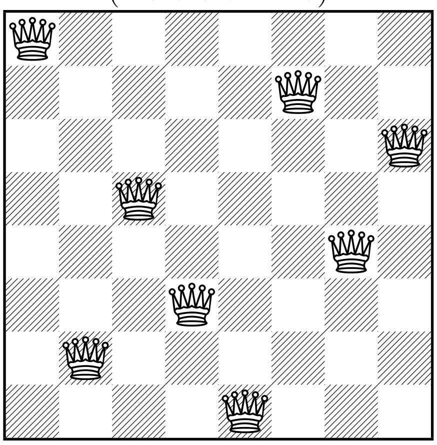
We crown the term with the n-Queens Problem.

The challenge is to place n Queens on an $n \times n$ board (rectangular array?), so that no two attack each other, i.e. no two Queens may be on the same rank (row), file (column), or diagonal (????).

1. (20 Points) isLegalPosition(board,n)

   Write a method isLegalPosition(board,n) that takes a (possibly partial) position and n as arguments and returns TRUE if and only if no two Queens attack each other.

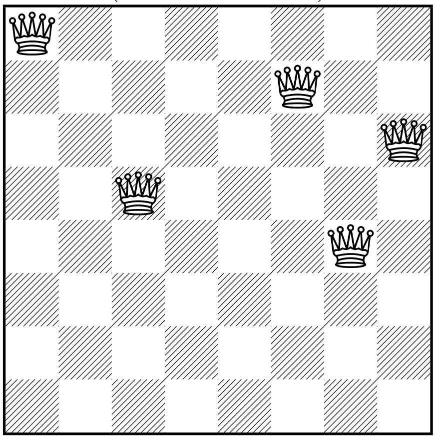
Here is a solution to the 8-Queens Problem:
(1 6 8 3 7 4 2 5)



Thus, isLegalPosition((1 6 8 3 7 4 2 5),8) should return TRUE.

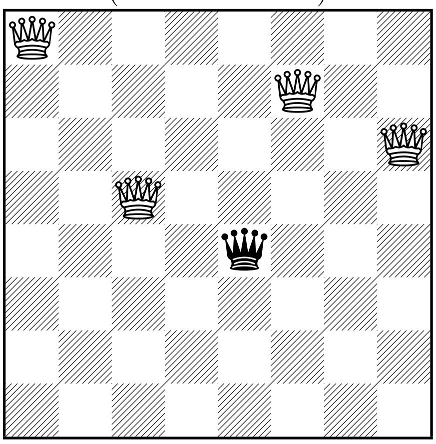
Because we are implementing a backtracking algorithm, we will restrict ourselves to positions which fill from the top of the board. We will insist then that the first $k \leq n$ positions be filled, i.e. non-zero, but the remaining $n - k$ positions may be zeroes. So the partial solution:

(1 6 8 3 7 0 0 0)



should also have isLegalPosition((1 6 8 3 7 0 0 0),8) return TRUE, while

(1 6 8 3 5 0 0 0)



should cause isLegalPosition((1 6 8 3 5 0 0 0),8) to return FALSE.

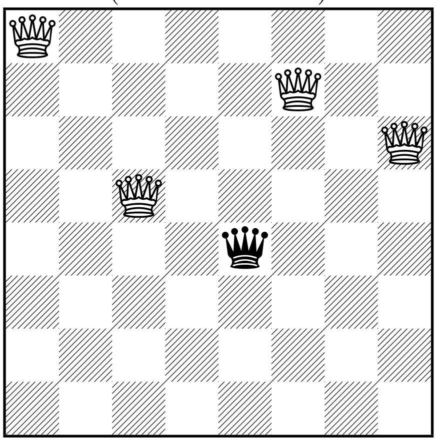
Why?

Do you see an elegant way to check that?

2. (20 Points) NextLegalPosition(board,n)

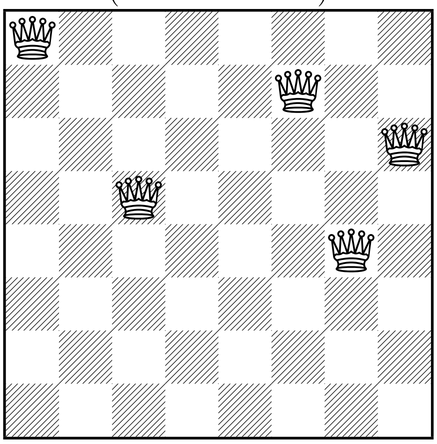
   From any (possibly partial) position, we need to be able to find the *next* legal position. There are, perhaps, three cases here. First, the next legal position from an illegal partial position; second, the next legal position from a *legal* partial position, and third, the next legal position after a full-fledged solution. We will fill our board from the top down and from left to right, so the next legal position after (illegal) partial position:

   (1 6 8 3 5 0 0 0)



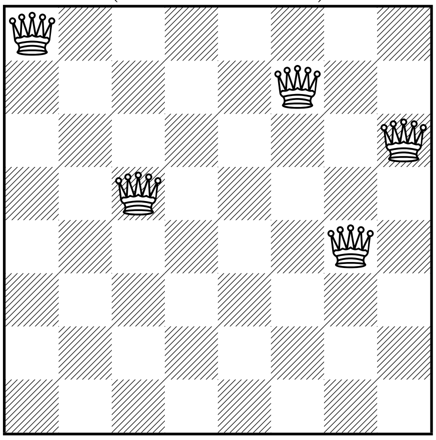
   is

   (1 6 8 3 7 0 0 0)



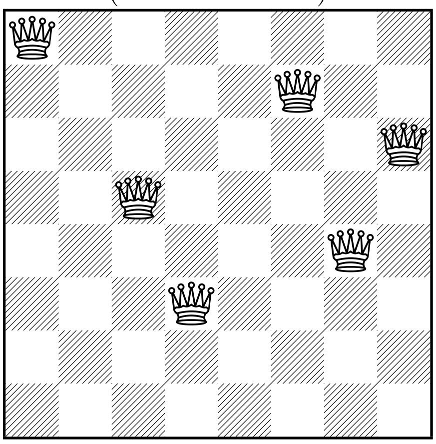
And the next legal position after (legal!) partial position:

(1 6 8 3 7 0 0 0)



is

(1 6 8 3 7 4 0 0)

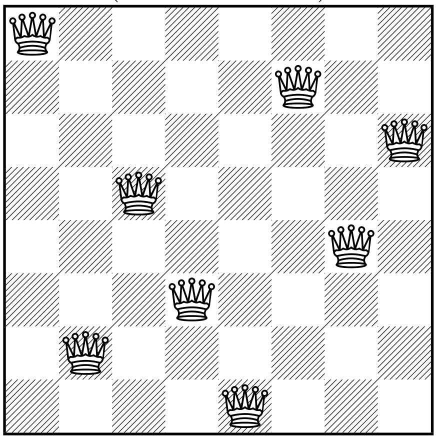


Will the next legal position *from* a legal position *always* add a Queen to the next rank?

Why? / Why not?

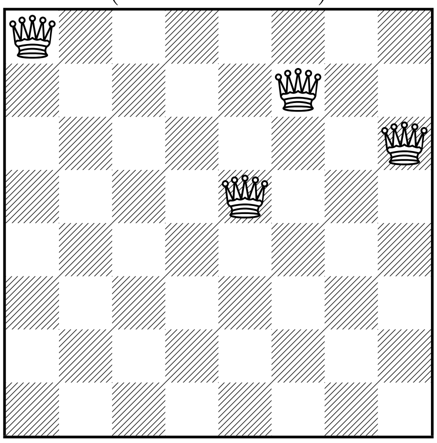
Lastly, the next legal position after our solution:

(1 6 8 3 7 4 2 5)



is

(1 6 8 5 0 0 0 0)



(Understanding this is understanding the backtracking–and then the forwarding–we are doing. This is the crux of the method.)

Write a method NEXTLEGALPOSITION(BOARD,n) that takes a (possibly partial) position and n as arguments and returns a board/position/array that represents the next legal position, or $(0_1, 0_2, \ldots, 0_n)$ if no legal position succeeds "board".

Hint: It may be useful to write another method SUCCESSOR(BOARD, n) that returns the next position to "board", whether legal or not.

3. (30 Points) Find the "first" solution to the n-Queens Problem for $n = 4...100^{\dagger}$.

   With isLegalPosition(board,n) and NextLegalPosition(board,n) in your hip pocket, write a program which solves the n-Queens problem for all values between 4 and 100, inclusive.

   Your output should give a single solution to each instance of the problem, and it should be the *first* solution lexicographically.

   We saw that the 4-Queens problem has solutions (2, 4, 1, 3) and (3, 1, 4, 2) as its distinct solutions. Your output should be the first of these.

   Is our solution to the 8-Queens problem the first one?

4. (30 Points) Find all solutions to the n-Queens Problem for a particular n.

   With isLegalPosition(board,n) and NextLegalPosition(board,n) in your hip pocket, write a program/method which finds (counts) all solutions to the n-Queens problem for each instance of the problem with $4 \leq n \leq 20^{\ddagger}$. Your output should be:

   There are 2 solutions to the 4-Queens Problem.
   There are 10 solutions to the 5-Queens Problem. ...
   ...
   ...

   (What constitutes a "solution"?)

For parts 2-4, you can get some gains in efficiency by modifying isLegalPosition(board,n). We will *build* positions from *legal* positions. This means that only the last Queen, the last non-zero entry, can cause a position to be illegal. Do you see?

Why are you going to want increased efficiency?

You may find the Wikipedia entry on the "8-Queens puzzle" to be helpful... maybe even interesting.

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†OK, you will NOT be able to go this high – we're searching and pruning an n^n tree so we can do only so much. See how high you can go!

‡This is probably out of reach, too; here are more then 2 million solutions to $n = 15$.