Parareal computation of SDEs with time-scale separation

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In a nutshell

Goal Simulate slow-fast SDEs over long time, quickly

Model Slow-fast system of SDEs, and a macroscopic model taken from the "fast" limit

Method Parallel-in-time algorithm that iteratively improves the macroscopic result

Result Reduction in wall clock time

Bonus Lower variance than full microscopic model

Microscopic model

- Slow-fast system of coupled SDEs

$$\mathrm{d}X = (-X_t^3 + X_t + Y_t^2)\,\mathrm{d}t + \sqrt{\frac{2}{\beta}}\,\mathrm{d}B_t^{(x)}$$
$$\mathrm{d}Y = \frac{1}{\epsilon}(X_t - Y_t)\,\mathrm{d}t + \sqrt{\frac{2}{\beta\epsilon}}\,\mathrm{d}B_t^{(y)}.$$

- Modeled as an ensemble $\mathcal{X}_t$ of particles with positions (X_t^p, Y_t^p) and weight W^p
- Time integrator: a Lie-Trotter splitting, updating X_t first, then Y_t
- Validation: *deterministic* solution given by the Fokker-Planck equation, akin to the macroscopic model

Macroscopic model

- Only slow variable, *assume* the fast Y_t is equilibrated and use only the expected value of the term Y_t^2

$$\mathrm{d}Z = (-Z_t^3 + Z_t + Z_t^2 + \frac{1}{\beta})\,\mathrm{d}t + \sqrt{\frac{2}{\beta}}\,\mathrm{d}B^{(x)},$$

or in potential form

$$\mathrm{d}Z = -\partial_z V_{\text{eff}}(Z_t)\,\mathrm{d}t + \sqrt{\frac{2}{\beta}}\,\mathrm{d}B^{(x)}.$$

- The associated Fokker-Planck equation reads

$$\partial_t \rho(z) = \partial_z\left(\rho(z)\partial_z V_{\text{eff}}\right) + \frac{1}{\beta}\partial_{zz}\rho(z).$$

- The macroscopic state is represented by integral quantities over a regular grid

Coupling

Restriction ($\mathcal{R}$, from micro to macro) sum the weights of all particles in each bin

Matching ($\mathcal{M}$, from macro to micro) reweight particles from a known microstate $(\bar{X}^p, \bar{Y}^p, \bar{W}^p)$

Resampling ($\mathcal{M}^*$, optional) retrieve an ensemble with all particles equal in weight

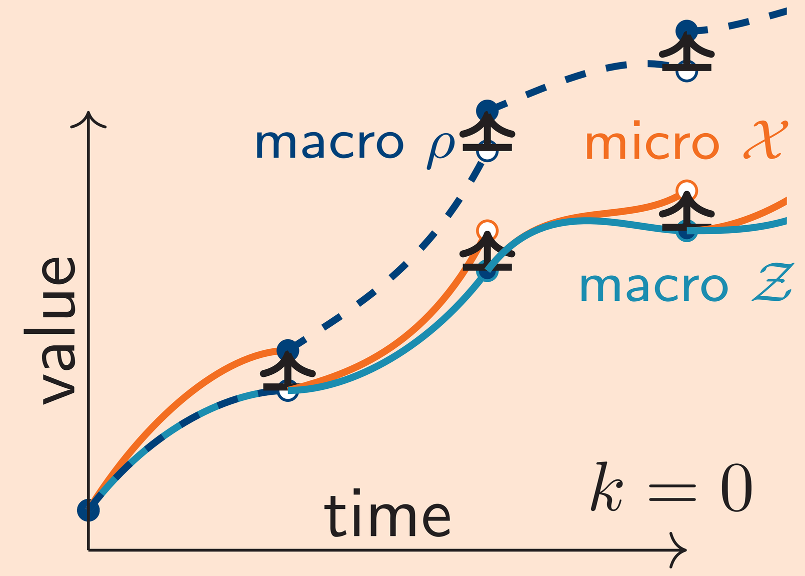
The parareal algorithm

Iteratively improves the macroscopic propagator by computing the discrepancies between the macroscopic and the microscopic models *in parallel*

Parallel use of the microscopic propagator gives a reduction in *wall clock time* if there are fewer iterations needed than time steps

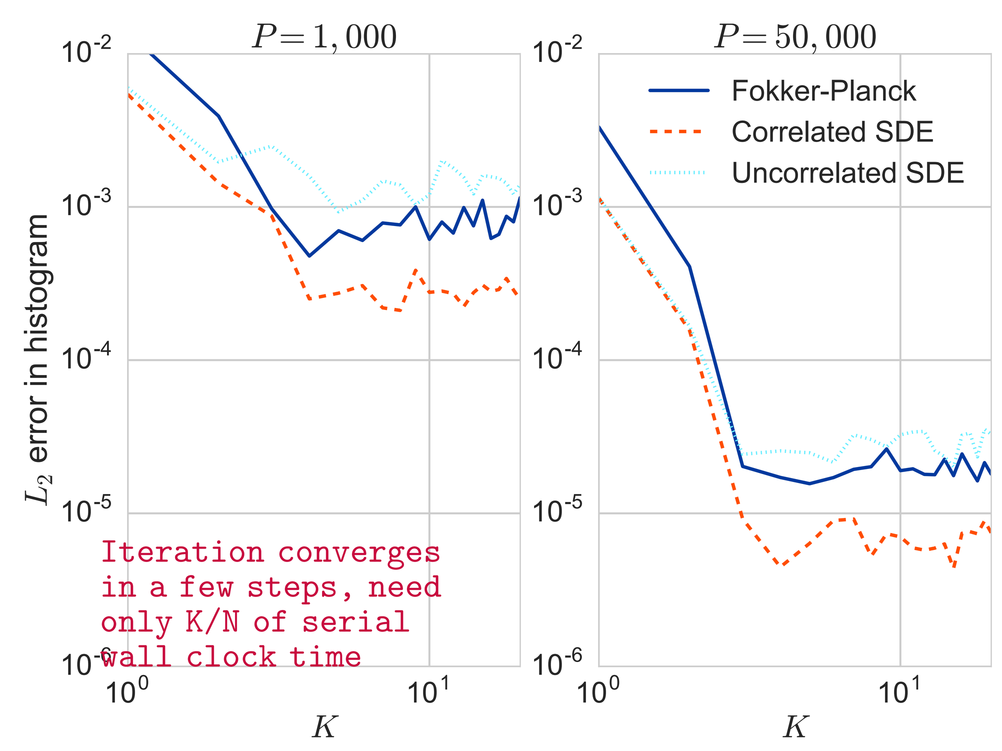
Variance of the stochastic microscopic propagator dominates the error (see figures →)

Reduction in variance by using a particle propagator with correlated noise for the macroscopic model in computing the discrepancies between the models

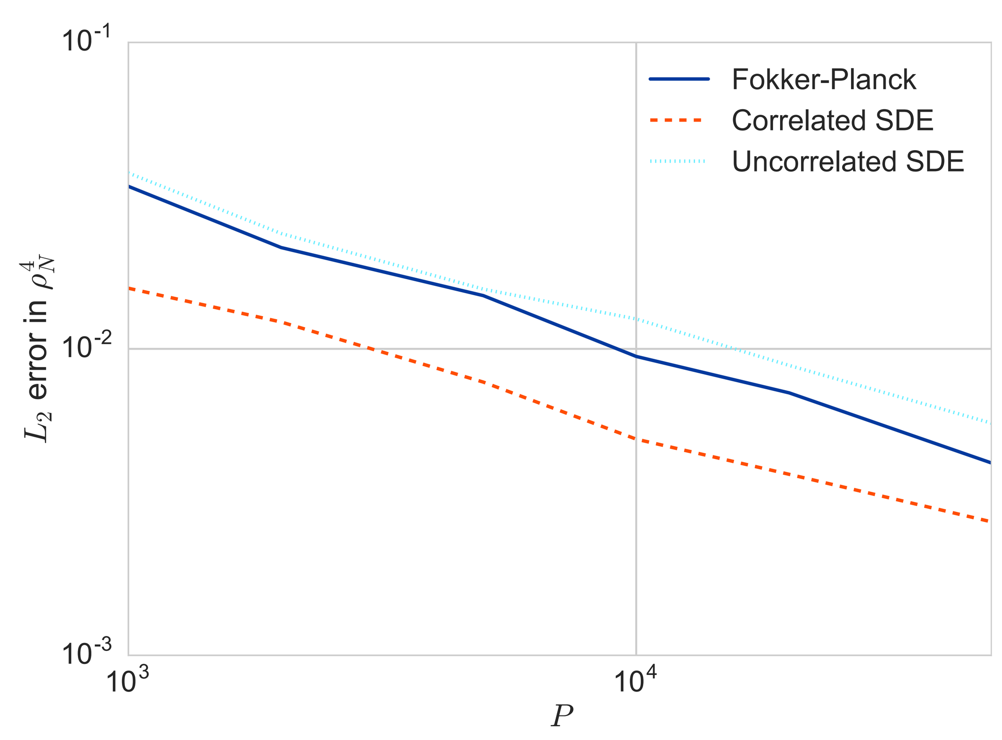


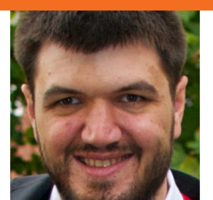
Parameters used below: $\beta = 5, \epsilon = 0.1, K = N = 20, \Delta t = 0.025, \delta t = 1.0 \times 10^{-4}$

Convergence in iterations



Convergence in number of particles





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