Homework 2

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1.

(i) Show that given finitely many countable sets $A_1, A_2, \cdots, A_n$, the set $A_1 \times A_2 \times \cdots \times A_n$ is also countable.

$$|A_k| = |\mathbb{N}| \Rightarrow \exists f_k : \mathbb{N} \to A_k \text{ is bijective}$$
$$f(x_1, x_2, \cdots, x_n) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n)) : x_1, x_2, \cdots, x_n \in \mathbb{N}$$
$$f : \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \to A_1 \times A_2 \times \cdots \times A_n \text{ is bijective}$$

$$h_2(x_1, x_2) = 2^{n-1}(2m-1) : x_1, x_2 \in \mathbb{N}$$
$$h_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ is bijective}$$
$$h_3(x_1, x_2, x_3) = h_2(h_2(x_1, x_2), x_3) : x_1, x_2, x_3 \in \mathbb{N} h_2(x_1, x_2)$$
$$h_3 : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ is bijective}$$
$$h_k(x_1, x_2, \cdots, x_k) = h_{k-1}(h_{k-1}(x_1, x_2, \cdots, x_{k-1}), x_k) : x_1, x_2, \cdots, x_k \in \mathbb{N}$$
$$h_k : \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \to \mathbb{N} \text{ is bijective}$$
$$\text{Let } g_n = h_n^{-1} \Rightarrow g_n : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \text{ is bijective}$$

$$f \circ g : \mathbb{N} \to A_1 \times A_2 \times \cdots \times A_n \text{ is bijective}$$
$$\therefore$$
$$\boxed{|A_1 \times A_2 \times \cdots \times A_n| = |\mathbb{N}|}$$

(ii) Is it true that given countably many countable sets $A_1, A_2, \cdots$, the set $A_1 \times A_2 \times \cdots$ is also countable? Justify your answer.

Assume bijection, all output tuples can be represented as:

f_{nm} as in section (i) is bijective between some A_k and $\mathbb{N}$

$$\begin{matrix} (f_{11}(x_{11}) & f_{12}(x_{12}) & \cdots & f_{1m}(x_{1m}) & \cdots) \\ (f_{21}(x_{21}) & f_{22}(x_{22}) & \cdots & f_{2m}(x_{2m}) & \cdots) \\ (f_{n1}(x_{n1}) & f_{n2}(x_{n2}) & \cdots & f_{nm}(x_{nm}) & \cdots) \end{matrix}$$

Consider the tuple: $R = (f_{11}(x_{11}+1), f_{22}(x_{22}+1), \cdots, f_{nn}(x_{nn}+1), \cdots)$

R will differ from every row by at least one value along the diagonal because f_{nn} is bijective, so $a \neq b \Rightarrow f_{nn}(a) \neq f_{nn}(b)$, and $x_{nn} \neq x_{nn} + 1$. However, $R \in A_1 \times A_2 \times \cdots$ which means it should be indexable in the bijection, a contradiction.

$$\therefore$$

$$\boxed{|A_1 \times A_2 \times \cdots \times A_n| \neq |\mathbb{N}|}$$

2.

(i) Let $\mathcal{F}$ be the collection of all functions $f : \{0, 1\} \to \mathbb{N}$. Is $\mathcal{F}$ countable? Justify your answer.

$$f \in \mathcal{F} \Rightarrow f(x) = \begin{cases} n & \text{if } x = 0 : n \in \mathbb{N} \\ m & \text{if } x = 1 : m \in \mathbb{N} \end{cases}$$

$$g = (n, m) \mapsto (x \mapsto \begin{cases} n & \text{if } x = 0 \\ m & \text{if } x = 1 \end{cases} : n, m \in \mathbb{N})$$

$$g : \mathbb{N} \times \mathbb{N} \to \mathcal{F} \text{ is bijective}$$

$$|\mathcal{F}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

$$\therefore$$

$$\boxed{|\mathcal{F}| = |\mathbb{N}|}$$

(ii) Let $\mathcal{F}$ be the collection of all functions $f : \mathbb{N} \to \{0, 1\}$. Is $\mathcal{F}$ countable? Justify your answer.

$$F \subset \mathcal{F}, F \in \{f(x) = \text{round}(a^x) : n \in \mathbb{N}, a \in (0, 1)\}$$

$$g = a \mapsto (x \mapsto \text{round}(a^x)) : n \in \mathbb{N}, a \in (0, 1)$$

$$g : (0, 1) \to F \text{ is bijective}$$

$$|\mathcal{F}| \geq |F| = |(0, 1)| = |\mathbb{R}| > |\mathbb{N}|$$

$|(0, 1)| = |\mathbb{R}|$ shown in **3**(ii)

$$\therefore$$

$$\boxed{|\mathcal{F}| \neq |\mathbb{N}|}$$

(iii) Let $\mathcal{F}$ be the collection of all functions $f : \mathbb{N} \to \mathbb{N}$. Is $\mathcal{F}$ countable? What about $f : \mathbb{R} \to \mathbb{R}$? Justify your answers.

$$F \subset \mathcal{F}, F = \{f(n) = \lceil rn \rceil : n \in \mathbb{N}, r \in \mathbb{R}\}$$

$$g = r \mapsto (n \mapsto \lceil rn \rceil) : n \in \mathbb{N}, r \in \mathbb{R}$$

$$g : \mathbb{R} \to F \text{ is bijective} \Rightarrow |F| = |\mathbb{R}|$$

$$|\mathcal{F}| \geq |F| = |\mathbb{R}| > |\mathbb{N}|$$

$$\therefore$$

$$\boxed{|\mathcal{F}| \neq |\mathbb{N}| : \mathcal{F} \text{ set of all } f : \mathbb{N} \to \mathbb{N}}$$

$$F \subset \mathcal{F}, F = \{f(x) = rx : x \in \mathbb{R}, r \in \mathbb{R}\}$$
$$g = r \mapsto (x \mapsto rx) : x, r \in \mathbb{R}$$
$$g : \mathbb{R} \to F \text{ is bijective} \Rightarrow |F| = |\mathbb{R}|$$
$$|\mathcal{F}| \geq |F| = |\mathbb{R}| > |\mathbb{N}|$$

$$\therefore$$

$$\boxed{|\mathcal{F}| \neq |\mathbb{N}| : \mathcal{F} \text{ set of all } f : \mathbb{R} \to \mathbb{R}}$$

3.

(i) Let $(0, 1)$ be the open interval from 0 to 1. Show that $|(0, 1)| = |\mathbb{R}^+|$ by finding an explicit bijection.

$$\boxed{f(x) = \frac{1}{x+1}}$$

(ii) Show that $|(0, 1)| = |\mathbb{R}|$ by finding an explicit bijection.

$$\boxed{f(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}}$$

(iii) Show that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}^+|$.

Consider $r \in \mathbb{R}^+$ can be represented using floating point notation. The function f

4. A real number x is said to be *algebraic* (over $\mathbb{Q}$) if it satisfies some polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, where each $a_i \in \mathbb{Q}, a_n \neq 0$. If x is not algebraic, it is called *transcendental*.

(i) Show that the set of all polynomials over $\mathbb{Q}$ is countable.

$$P = \{a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n : a_0, a_1, a_2, \cdots, a_n \in \mathbb{Q}\}$$
$$f = (c_0, c_1, c_2, \cdots, c_n) \mapsto ((x) \mapsto c_0 + c_1 x + c_2 x^2 \cdots + c_n x^n)$$
$$f : \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q} \to P \text{ is bijective}$$

set of all polynomials over $\mathbb{Q}$

$$|\mathbb{Q}| = |\mathbb{N}| \Rightarrow \exists q : \mathbb{N} \to \mathbb{Q} \text{ is bijective}$$
$$g(x_0, x_1, \cdots, x_n) = (q(x_0), q(x_1), \cdots, q(x_n))$$
$$g : \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \to \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q} \text{ is bijective}$$

$$|\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}| = |\mathbb{N}| \Rightarrow \exists h : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \text{ is bijective}$$
$$f \circ g \circ h : \mathbb{N} \to P \text{ is bijective}$$
$$\therefore$$
$$\boxed{|P| = |\mathbb{N}|}$$

proven in work of **1**(i)

(ii) Prove that there are countably many algebraic numbers.

$$x$$

(iii) Prove that there are uncountably many transcendental numbers.

Let T denote the set of transcendental numbers:

$$S \subset T, S = \{r\pi : r \in \mathbb{R}\}$$
$$s(r) = r\pi : r \in \mathbb{R}$$
$$s : \mathbb{R} \to S \text{ is bijective}$$
$$|T| \geq |S| = |\mathbb{R}| > |\mathbb{N}|$$
$$\boxed{|T| \neq |\mathbb{N}|}$$