

Homework 1

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February 5, 2025

1. Let \mathbb{F} be an ordered field and $a, b, \epsilon \in \mathbb{F}$.

(i) Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

\mathbb{F} is ordered $\Leftrightarrow P \exists \mathbb{F}$ where P is the positive set and $\epsilon \in P$

order axiom

$$a < b + \epsilon$$

$$a - b < \epsilon$$

< additivity

$$a - b < \epsilon \Rightarrow a - b \neq \epsilon$$

$$a - b \neq \epsilon \Leftrightarrow a - b \notin P$$

$$a - b \notin P \Rightarrow a - b = 0 \vee -(a - b) \in P$$

trichotomy

$$b - a = 0 \vee b - a \in P$$

$$\therefore$$

$$\boxed{a \leq b}$$

(ii) Use (i) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$.¹

$$|a - b| < \epsilon$$

$$-\epsilon < a - b < \epsilon$$

FT abs-value

$$-\epsilon < a - b \wedge a - b < \epsilon$$

$$b < a + \epsilon \wedge a < b + \epsilon$$

$$b \leq a \wedge a \leq b$$

$$\therefore$$

$$\boxed{a = b}$$

¹Proof using trichotomy in notes

2. Let $A \subseteq \mathbb{R}$. Define $-A = \{-a : a \in A\}$. Suppose that A is non-empty and bounded below. Show that $\inf(A) = -\sup(-A)$.

$$\begin{aligned}
 x \in A &\Rightarrow x \geq \inf(A) \\
 -x &\leq -\inf(A) \\
 x' \in -A &\Rightarrow x' = -x : x \in A \Rightarrow x' \leq -\inf(A) \\
 -\inf(A) &\text{ is an upper bound of } -A \\
 \text{Given } \epsilon > 0, &\exists x \in A : x < \inf(A) + \epsilon \\
 -x &> -\inf(A) - \epsilon \\
 \exists k \in -A : &k = -x, k > -\inf(A) - \epsilon \\
 \therefore & \\
 \sup(-A) &= -\inf(A)
 \end{aligned}$$

inf analytic
definition
lower bound

satisfies both
requirements
for $\sup(-A)$

$$\boxed{\inf(A) = -\sup(-A)}$$

3. Let $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Prove that $\sup(A) = 1, \inf(A) = \frac{1}{2}$.

$$A = \{f(n) : n \in \mathbb{N}, f(x) = \frac{x}{x+1}\}$$

$$\begin{aligned}
 \frac{n}{n+1} &\geq \frac{1}{2} \\
 2n &\geq n+1 \\
 n \geq 1 &\Rightarrow \frac{1}{2} \text{ is a lower bound of } A \\
 f(1) &= \frac{1}{2} \\
 \frac{1}{2} - f(1) &= 0 \\
 \epsilon + \frac{1}{2} - f(1) &= \epsilon \\
 (\frac{1}{2} + \epsilon) - f(1) &> 0 \\
 f(1) &< \frac{1}{2} + \epsilon \\
 \text{Given } \epsilon > 0, &\exists x \in A : x < \frac{1}{2} + \epsilon
 \end{aligned}$$

$n \geq 1$ is valid
by definition

where $\epsilon > 0$

when $x = f(1)$

\therefore

$$\boxed{\inf(A) = \frac{1}{2}}$$

$$\frac{n}{n+1} \leq 1$$

$$n \leq n+1$$

$$0 \leq 1 \Rightarrow 1 \text{ is an upper bound of } A$$

$$\frac{n}{n+1} < 1 - \epsilon$$

where $\epsilon > 0$

$$\epsilon < 1 - \frac{n}{n+1}$$

$$0 < \epsilon < \frac{1}{n+1}$$

$$\frac{1}{n+1} > 0$$

true for $n \in \mathbb{N}$

$$\text{Given } \epsilon > 0 : \exists x \in A : x > 1 - \epsilon$$

\therefore

$$\boxed{\sup(A) = 1}$$

4. Let $A, B \subseteq \mathbb{R}$:

(i) $\exists \sup(A), \exists \sup(B), A \subseteq B$. Show that $\sup(A) \leq \sup(B)$.

$$\exists b \in B : b > \sup(A) \vee \neg \exists b \in B : b > \sup(A)$$

a tautology
($a \vee \neg a$)

Case 1

$$\exists b \in B : b > \sup(A)$$

$$\sup(A) < b \leq \sup(B)$$

$$\sup(A) < \sup(B)$$

Case 2

$$\neg \exists b \in B : b > \sup(A)$$

$$\Leftrightarrow b \in B \Rightarrow b \leq \sup(A)$$

$$\Rightarrow \sup(A) \text{ is an upperbound of } B$$

$$\text{Given } \epsilon > 0 : \exists a \in A : a > \sup(A) - \epsilon$$

$$B \supset A \Rightarrow a \in B$$

$$a \in B : a > \sup(A) - \epsilon$$

$$\sup(A) = \sup(B)$$

$$\sup(A) < \sup(B) \vee \sup(A) = \sup(B)$$

\therefore

$$\boxed{\sup(A) \leq \sup(B)}$$

- (ii) $\sup(A) < \sup(B)$. Show that there exists $b \in B$ that is an upper bound of A . Show that this result does not hold if we instead assume that $\sup(A) \leq \sup(B)$.

$$\begin{aligned}
& \sup(A) \leq \sup(B) \\
& \Diamond(\sup(A) = \sup(B) = k, k \in \mathbb{R}) \\
& b \in B \Rightarrow b \leq k \\
& a \in A \Rightarrow a \leq k \\
& \Diamond(\max(b) < k \wedge \max(a) = k) \\
& \max(b) < \max(a) \\
& \therefore
\end{aligned}$$

$\sup(A) \leq \sup(B) \Rightarrow \Diamond \neg \exists b \in B : b \text{ is an upper bound of } A$

$$\begin{aligned}
& \sup(A) < \sup(B) \\
& \text{Given } \epsilon > 0 : \exists b \in B : b > \sup(B) - \epsilon \\
& b + \epsilon > \sup(B) \Rightarrow b + \epsilon > \sup(A) \\
& b + \epsilon - \sup(A) > 0 \\
& b - \sup(A) > -\epsilon \\
& -(b - \sup(A)) < \epsilon \\
& -(b - \sup(A)) \notin P \\
& b - \sup(A) \in P \vee b - \sup(A) = 0 \\
& b \geq \sup(A) \geq a \\
& \therefore
\end{aligned}$$

trichotomy

$\sup(A) < \sup(B) \Rightarrow \exists b \in B : b \text{ is an upper bound of } A$

5. For $A, B \subseteq \mathbb{R}$, define

$$\begin{aligned}
A + B &= \{a + b : a \in A, b \in B\} \\
A \cdot B &= \{a \cdot b : a \in A, b \in B\}
\end{aligned}$$

- (i) Determine $\{3, 1, 0\} + \{2, 0, 2, 3\}$ and $\{3, 1, 0\} \cdot \{2, 0, 2, 3\}$.

$$\begin{aligned}
A &= \{3, 1, 0\} \\
B &= \{2, 0, 2, 3\} \\
A + B &= \{3 + 2, 3 + 0, 3 + 2, 3 + 3, 1 + 2, 1 + 0, 1 + 2, 1 + 3, 0 + 2, 0 + 0, 0 + 2, 0 + 3\} \\
&= \{5, 3, 6, 1, 4, 2, 0\} \\
A \cdot B &= \{3 \cdot 2, 3 \cdot 0, 3 \cdot 2, 3 \cdot 3, 1 \cdot 2, 1 \cdot 0, 1 \cdot 2, 1 \cdot 3, 0 \cdot 2, 0 \cdot 0, 0 \cdot 2, 0 \cdot 3\} \\
&= \{6, 0, 9, 2, 3\}
\end{aligned}$$

- (ii) Assume that $\sup(A)$ and $\sup(B)$ exist. Prove that $\sup(A+B) = \sup(A) + \sup(B)$.

$$\begin{aligned}
a \in A &\Rightarrow a \leq \sup(A) \\
b \in B &\Rightarrow b \leq \sup(B) \\
a + b &\leq \sup(A) + \sup(B) \\
c \in A + B &\Rightarrow c = a + b, a \in A, b \in B \\
c &\leq \sup(A) + \sup(B) \\
\sup(A) + \sup(B) &\text{ is an upper bound of } A + B \\
\text{Given } \epsilon > 0 : & \\
\exists a \in A : a &> \sup(A) - \epsilon \\
\exists b \in B : b &> \sup(B) - \epsilon \\
a + b &> \sup(A) + \sup(B) - 2\epsilon \\
2\epsilon > 0 &\Rightarrow a + b > \sup(A) + \sup(B) - \epsilon \\
a + b \in A + B &\Rightarrow \exists c \in A + B : c > (\sup(A) + \sup(B)) - \epsilon \\
&\therefore \\
\boxed{\sup(A + B) = \sup(A) + \sup(B)}
\end{aligned}$$

- (iii) Give an example of sets A, B where $\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)$.

$$\begin{aligned}
A &= \{-1\}, B = \{1, 2\} \\
\sup(A) &= -1 \\
\sup(B) &= 2 \\
\sup(A) \cdot \sup(B) &= -2 \\
A \cdot B &= \{-1, -2\} \\
\sup(A \cdot B) &= -1 \\
-1 &\neq -2 \\
\boxed{\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)}
\end{aligned}$$

Notes

1. Question 1-ii proof by trichotomy

$$\begin{aligned} &|a - b| < \epsilon \\ &-\epsilon < a - b < \epsilon && \text{FT abs-value} \\ &-\epsilon < a - b \wedge a - b < \epsilon \\ &\epsilon > -(a - b) \wedge a - b < \epsilon && < \text{multiplicity} \\ &-(a - b) \notin P \wedge a - b \notin P \\ &\Rightarrow a - b = 0 && \text{trichotomy} \\ &\therefore \\ &\boxed{a = b} \end{aligned}$$