Math 310 - HW 2 (Due Wednesday 02/12)

1. (*i*) Show that given finitely many countable sets $A_1, A_2, \cdots, A_n$, the set $A_1 \times A_2 \times \cdots \times A_n$ is also countable.

(*ii*) Is it true that given countably many countable sets $A_1, A_2, \cdots$, the set $A_1 \times A_2 \times \cdots$ is also countable ? Justify your answer.

2. (*i*) Let $\mathcal{F}$ be the collection of all functions $f : \{0, 1\} \to \mathbb{N}$. Is $\mathcal{F}$ countable ? Justify your answer.

(*ii*) Let $\mathcal{F}$ be the collection of all functions $f : \mathbb{N} \to \{0, 1\}$. Is $\mathcal{F}$ countable ? Justify your answer.

(*iii*) Let $\mathcal{F}$ be the collection of all functions $f : \mathbb{N} \to \mathbb{N}$. Is $\mathcal{F}$ countable ? What about $f : \mathbb{R} \to \mathbb{R}$? Justify your answers.

3. (*i*) Let $(0, 1)$ be the open interval from 0 to 1. Show that $|(0, 1)| = |\mathbb{R}^+|$ by finding an explicit bijection.

(*ii*) [**Extra-Credit**] Show that $|(0, 1)| = |\mathbb{R}|$, by finding an explicit bijection.

(*iii*) Show that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}^+|$ (*Hint*: Use (*i*) and binary representation of numbers in $(0, 1)$.)

4. A real number x is said to be *algebraic* (over $\mathbb{Q}$) if it satisfies some polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$, where each $a_i \in \mathbb{Q}, a_n \neq 0$. If x is not algebraic, it is called *transcendental*.
(*Remark*: All rational numbers are obviously algebraic. $\sqrt{2}$ and golden-ratio ϕ are two irrational numbers which are also algebraic, while e and π are not, and hence transcendental.)

(*i*) Show that the set of all polynomials over $\mathbb{Q}$ (as mentioned above) is countable.

(*ii*) Prove that there are countably many algebraic numbers. (*Hint*: Use (*i*) and here you may use the Fundamental theorem of Algebra, that any real polynomial has finitely many roots in $\mathbb{R}$.)

(*iii*) Prove that there are uncountably many transcendental numbers.

Not for Credit

1. Let X, Y be sets with a function $f : X \to Y$. Prove that the following are equivalent:

(*a*) f is $1-1$.
(*b*) $f(A - B) = f(A) - f(B)$ for all subsets A and B of X.
(*c*) $f^{-1}(f(E)) = E$ for all subsets E of X.
(*d*) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of X.

2. Prove that the set of points on the unit circle in $\mathbb{R}^2$, that is $\{(x, y) : x^2 + y^2 = 1\}$ is uncountable.

3. Is the subset of rational numbers $\{\frac{m}{n} : m, n \in \mathbb{Z}, 1 \leq n \leq 100\}$ is dense in $\mathbb{R}$?

4. Prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite by showing that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^{n-1}(2m - 1)$ is a bijection.