Homework 1

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1. Let $\mathbb{F}$ be an ordered field and $a, b, \epsilon \in \mathbb{F}$.

(i) Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$

$$
\begin{aligned}
&\mathbb{F} \text{ is ordered} \Leftrightarrow P \ni \mathbb{F} \text{ where } P \text{ is the positive set and } \epsilon \in P && \text{order axiom} \\
&a < b + \epsilon && \\
&a - b < \epsilon && < \text{additivity} \\
&a - b < \epsilon \Rightarrow a - b \neq \epsilon && \\
&a - b \neq \epsilon \Leftrightarrow a - b \notin P && \\
&a - b \notin P \Rightarrow a - b = 0 \vee -(a - b) \in P && \text{trichotomy} \\
&b - a = 0 \vee b - a \in P && \\
&\therefore && \\
&\boxed{a \leq b} &&
\end{aligned}
$$

(ii) Use (i) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$.[1]

$$
\begin{aligned}
&|a - b| < \epsilon && \\
&-\epsilon < a - b < \epsilon && \text{FT abs-value} \\
&-\epsilon < a - b \wedge a - b < \epsilon && \\
&b < a + \epsilon \wedge a < b + \epsilon && \\
&b \leq a \wedge a \leq b && \\
&\therefore && \\
&\boxed{a = b} &&
\end{aligned}
$$

[1] Proof using trichotomy in notes

2. Let $A \subseteq \mathbb{R}$. Define $-A = \{-a : a \in A\}$. Suppose that A is non-empty and bounded below. Show that $\inf(A) = -\sup(-A)$.

$$\forall x \in A : x \geq \inf(A)$$
$$-x \leq -\inf(A)$$
$$\forall x \in A : -x \in -A \Rightarrow -\inf(A) \text{ is upperbound of } -A$$

inf analytic definition lower bound

$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \inf(A) + \epsilon$$
$$-x > -\inf(A) - \epsilon$$
$$\exists\, k \in -A : k = -x, k > -\inf(A) - \epsilon$$
$$\therefore$$
$$\sup(-A) = -\inf(A)$$
$$\boxed{\inf(A) = -\sup(-A)}$$

satisfies both requirements for $\sup(-A)$

3. Let $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Prove that $\sup(A) = 1, \inf(A) = \frac{1}{2}$.

$$A = \{f(n) : n \in \mathbb{N}, f(x) = \frac{x}{x+1}\}$$

$$\frac{n}{n+1} \geq \frac{1}{2}$$
$$2n \geq n+1$$
$$n \geq 1 \Rightarrow \frac{1}{2} \text{ is a lower bound of } A$$

$n \geq 1$ is valid by definition

$$f(1) = \frac{1}{2}$$
$$\frac{1}{2} - f(1) = 0$$
$$\epsilon + \frac{1}{2} - f(1) = \epsilon$$

where $\epsilon > 0$

$$(\frac{1}{2} + \epsilon) - f(1) > 0$$
$$f(1) < \frac{1}{2} + \epsilon$$
$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \frac{1}{2} + \epsilon$$

when $x = f(1)$

$$\therefore$$
$$\boxed{\inf(A) = \frac{1}{2}}$$

$$\frac{n}{n+1} \leq 1$$

$$n \leq n+1$$

$$0 \leq 1 \Rightarrow 1 \text{ is an upper bound of } A$$

$$\frac{n}{n+1} < 1 - \epsilon \qquad \text{where } \epsilon > 0$$

$$\epsilon < 1 - \frac{n}{n+1}$$

$$0 < \epsilon < \frac{1}{n+1}$$

$$\frac{1}{n+1} > 0 \qquad \text{true for } n \in \mathbb{N}$$

$$\text{Given } \epsilon > 0 : \ \exists\, x \in A : x > 1 - \epsilon$$

$$\therefore$$

$$\boxed{\sup(A) = 1}$$

4. Problem 4

5. Problem 5

Notes

1. Question 1-ii proof by trichotomy

$$
\begin{aligned}
&|a-b| < \epsilon && \\
&-\epsilon < a-b < \epsilon && \text{FT abs-value} \\
&-\epsilon < a-b \wedge a-b < \epsilon && \\
&\epsilon > -(a-b) \wedge a-b < \epsilon && < \text{multiplicity} \\
&-(a-b) \notin P \wedge a-b \notin P && \\
&\Rightarrow a-b = 0 && \text{trichotomy} \\
&\therefore && \\
&\boxed{a=b} &&
\end{aligned}
$$