Homework 3

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1. Use the definition of the limit of a sequence to show that:

(i) $\{\frac{n^2+n+1}{3n^2+1}\} \to \frac{1}{3}$

$$\left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon$$
$$\left|\frac{n+\frac{2}{3}}{3n^2+1}\right| < \epsilon$$
$$\frac{n+\frac{2}{3}}{3n^2+1} < \epsilon \qquad \text{positive for } n \in \mathbb{N}$$
$$\frac{n+\frac{2}{3}}{\epsilon} < 3n^2+1$$
$$\frac{1}{\epsilon'} < 3n^2+1$$
$$\frac{1-\epsilon'}{3\epsilon'} < n^2$$
$$n \geq \left\lceil \sqrt{\frac{1-\epsilon'}{3\epsilon'}} \right\rceil = N_\epsilon$$
$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow \left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon$$
$$\therefore$$
$$\boxed{\lim_{n\to\infty} \frac{n^2+n+1}{3n^2+1} = \frac{1}{3}}$$

$(ii)\quad \{10 - \frac{1}{\sqrt{n+\sqrt{n+5}}}\} \to 10$

$$\left|10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} - 10\right| < \epsilon$$

$$\left|-\frac{1}{\sqrt{n+\sqrt{n+5}}}\right| < \epsilon$$

$$\frac{1}{\sqrt{n+\sqrt{n+5}}} < \epsilon$$

always positive

$$\frac{1}{\sqrt{n+\sqrt{n+5}}} < \frac{1}{\sqrt{n}}$$

smaller denominator

$$\frac{1}{\sqrt{n}} < \epsilon$$

$$\sqrt{n} > \frac{1}{\epsilon}$$

$$n \geq \left\lceil \frac{1}{\epsilon^2} \right\rceil = N_\epsilon$$

$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow \left|10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} - 10\right| < \epsilon$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} 10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} = 10}$$

2.

(i) Use the definition of the limit of a sequence to show that for a fixed r with $|r| < 1$, $\{nr^n\} \to 0$.

$$\text{Consider } \lim_{n\to\infty}\left|\frac{(n+1)r^{n+1}}{nr^n}\right| = |r|$$

$$\Leftarrow \left|\left|\frac{(n+1)r^{n+1}}{nr^n}\right| - |r|\right| < \epsilon$$

$$\Leftarrow \left|\left|\frac{(n+1)r^{n+1}}{nr^n}\right| - |r|\right| \le \left|\frac{(n+1)r^{n+1}}{nr^n} - r\right| < \epsilon \qquad \triangle\text{-ineq of} <$$

$$\Leftarrow \left|\frac{(n+1)r}{n} - r\right| < \epsilon$$

$$\Leftarrow \left|r + \frac{r}{n} - r\right| < \epsilon$$

$$\Leftarrow \left|\frac{r}{n}\right| < \epsilon$$

$$\Leftarrow \frac{r}{n} < \epsilon \qquad \text{always positive}$$

$$\Leftarrow n \ge \left\lceil\frac{r}{\epsilon}\right\rceil = N_\epsilon$$

$$\text{Almost-geometric sequence: } \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = r : 0 < r < 1, x_n = nr^n$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} nr^n = 0}$$

3.

4.

(i)

(ii) Show that $\{x_n\}$ is monotonic increasing

$$x_{n+1} - x_n > 0$$

$$\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{2(n+1)} - (\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}) > 0$$

$$\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2} - (\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}) > 0$$

$$\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > 0$$

$$\frac{n+1}{4x^3 + 10x^2 + 8x + 2} > 0$$

$$n + 1 > 0 \qquad \text{true for } n \in \mathbb{N}$$

$$\therefore$$

$$\boxed{\{x_n\} \text{ is monotonic increasing}}$$