Homework 7

Keizou Wang

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1. Use the definition of derivative, knowledge of trigonometry and standard limits to find the derivatives of:

(i) $\sin(x)$

$$
\begin{aligned}
\sin'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} && \text{angle sum formula} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}
$$

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \qquad \text{standard limit}$$

$$
\begin{aligned}
\lim_{h\to 0} \frac{\cos(h)-1}{h} &= \lim_{h\to 0} \frac{\cos(h)-1}{h} \cdot \frac{\cos(h)+1}{\cos(h)+1} \\
&= \lim_{h\to 0} \frac{\cos^2(h)-1}{h(\cos(h)+1)} \\
&= \lim_{h\to 0} -\frac{\sin^2(h)}{h(\cos(h)+1)} \\
&= -\lim_{h\to 0} \frac{\sin(h)}{h} \cdot \frac{\sin(h)}{\cos(h)+1} \\
&= -\lim_{h\to 0} \frac{\sin(h)}{h} \cdot \lim_{h\to 0} \frac{\sin(h)}{\cos(h)+1} \\
&= 0
\end{aligned}
$$

$$\sin'(x) = \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$\therefore$$

$$\boxed{\sin'(x) = \cos(x)}$$

$(ii)\quad \cos(x)$

$$\begin{aligned}
\cos'(x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h} \\
&= \cos(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} - \sin(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1
\end{aligned}$$

$\lim_{h\to 0} \frac{\cos(h)-1}{h}$ found in part i

$$\therefore$$

$$\boxed{\cos'(x) = -\sin(x)}$$

$(iii)\quad \tan(x)$

$$\begin{aligned}
\tan'(x) &= \lim_{h\to 0} \frac{\tan(x+h)-\tan(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)} - \tan(x)\right) \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{\tan(x)+\tan(h)-\tan(x)+\tan^2(x)\tan(h)}{1-\tan(x)\tan(h)} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{\tan(h)(1+\tan^2(x))}{1-\tan(x)\tan(h)} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{\tan(h)\sec^2(x)}{1-\tan(x)\tan(h)} \\
&= \sec^2(x)\lim_{h\to 0} \frac{1}{h}\cdot\frac{\tan(h)}{1-\tan(x)\tan(h)} \\
&= \sec^2(x)\lim_{h\to 0} \frac{\sin(h)}{h}\cdot\frac{1}{\cos(h)}\cdot\frac{1}{1-\tan(x)\tan(h)} \\
&= \sec^2(x)\lim_{h\to 0} \frac{\sin(h)}{h}\cdot\frac{1}{\cos(h)}\cdot\frac{1}{1-\tan(x)\tan(h)} \\
&= \sec^2(x)\lim_{h\to 0} \frac{\sin(h)}{h}\cdot\lim_{h\to 0}\frac{1}{\cos(h)}\cdot\lim_{h\to 0}\frac{1}{1-\tan(x)\tan(h)} \\
&= \sec^2(x)\cdot 1\cdot 1\cdot 1
\end{aligned}$$

$$\therefore$$

$$\boxed{\tan'(x) = \sec^2(x)}$$

2. [Continuous almost everywhere, but differentiable nowhere] Prove that *Thomae's function* $f : [0,1] \to [0,1]$ is not differentiable at any point.

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n}; m, n \in \mathbb{N}; n > 0; \text{ in lowest terms} \\ 0 & x = 0 \text{ and otherwise} \end{cases}$$

$$\text{Case 1: } x \in \mathbb{Q}$$

$$f \text{ disc. @ } x \qquad \text{shown in hw6}$$

$$\therefore$$

$$f \text{ not diff. @ } x$$

$$\text{Case 2: } x \notin \mathbb{Q}$$

$$\text{Assume } f \text{ diff. @ } x$$

$$\text{Let } g(a) = \frac{f(a) - f(x)}{a - x} = \frac{f(a)}{a - x}$$

$$\exists f' \Rightarrow \exists L : \lim_{a \to x} g = L$$

$$\Rightarrow \forall \epsilon > 0, \exists \delta : \forall a : |a - x| < \delta, |g(a) - L| < \epsilon$$

$$\mathbb{Q} \text{ dense in } [0,1] \Rightarrow \exists a = \frac{m}{n} : m, n \in \mathbb{N}, |a - x| < \delta$$

$$\text{Consider } a : n > \frac{1}{\delta(\epsilon + |L|)}$$

$$|g(a) - L| < \epsilon$$

$$\Leftarrow \left| \frac{f(a)}{a - x} - L \right| < \epsilon$$

$$\Leftarrow \left| \frac{\frac{1}{n}}{a - x} \right| - |L| < \epsilon$$

$$\Leftarrow \left| \frac{1}{n(a - x)} \right| < \epsilon + |L|$$

$$\Leftarrow \frac{1}{n\delta} < \epsilon + |L|$$

$$\Leftarrow \frac{1}{\frac{1}{\delta(\epsilon + |L|)} \delta} < \epsilon + |L|$$

$$\Leftarrow \epsilon + |L| < \epsilon + |L|$$

$$\text{Contradiction}$$

$$\therefore$$

$$f \text{ not diff. @ } x$$

$$\boxed{f \text{ diff. nowhere}}$$

3. Which is greater, e^π or π^e?

$$\text{Let } f(x) = \ln(x), f'(x) = \frac{1}{x}$$
$$g(x) = x, g'(x) = 1$$
$$\exists c \in (\pi^e, e^\pi) : \frac{f'(c)}{g'(c)} = \frac{f(e^\pi) - f(\pi^e)}{g(e^\pi) - g(\pi^e)}$$
$$f'(c)(g(e^\pi) - g(\pi^e)) = g'(c)(f(e^\pi) - f(\pi^e))$$
$$\frac{e^\pi - \pi^e}{c} = \ln(e^\pi) - \ln(\pi^e)$$
$$\frac{e^\pi - \pi^e}{c} = \pi - e\ln(\pi)$$
$$e^\pi - \pi^e > \pi - e\ln(\pi)$$

$$\text{Consider } h(x) = x - e\ln(x), h'(x) = 1 - \frac{e}{x}$$
$$h(e) = e - e\ln(e) = 0$$
$$\forall x > e, \frac{e}{x} < 1 \Rightarrow h'(x) > 0$$
$$\pi > e \Rightarrow h(\pi) > 0$$

$$e^\pi - \pi^e > 0$$
$$\therefore$$
$$\boxed{e^\pi > \pi^e}$$

4. Prove that in any interval in which the functions f, g, f', g' are continuous and $fg' - f'g \neq 0$, then the roots of f and g 'separate' each other.

$$\text{Let } F(x) = f(x)g'(x) - f'(x)g(x)$$

$$R = \{x : f(x) = 0\} \qquad \text{roots of } f$$

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$= \frac{-F(x)}{g^2(x)}$$

$$a, b \in R : a < b, (a, b) \cap R = \emptyset \qquad a, b \text{ are consecutive roots of } f$$

$$\text{Assume } \nexists c \in (a, b) : g(c) = 0$$

$$g(x) \neq 0, \forall x \in (a, b) \Rightarrow h(x) \text{ diff. @ } (a, b)$$

$$F(x) \neq, \forall x \Rightarrow h'(x) \neq 0, \forall x$$

$$h(a) = h(b) = 0 \wedge h \text{ diff. @ } (a, b) \Rightarrow \exists c \in (a, b) : h'(c) = 0 \qquad \text{Rolle's thm.}$$

$$h'(x) \neq 0, \forall x \wedge \exists c \in (a, b) : h'(c) = 0 \Rightarrow \text{ Contradiction}$$

$$\therefore$$

$$\boxed{\exists c \in (a, b) : g(c) = 0}$$

5. [Some inequalities using MVT] Apply MVT to prove the following inequalities

(i) $x - \frac{x^3}{6} < \sin(x) < x; \forall 0 < x < \frac{\pi}{2}$

$$\sin'(x) = \cos(x)$$

$$\forall x, \exists c \in (0, x) : \sin'(c) = \frac{\sin(x) - \sin(0)}{x - 0}$$

$$\cos(c) = \frac{\sin(x)}{x}$$

$$x \cos(c) = \sin(x)$$

$$\forall x \in \left(0, \frac{\pi}{2}\right), \cos(x) < 1 \Rightarrow x > \sin(x)$$

$$f(x) = \sin(x) - x + \frac{x^3}{6}$$

$$f'(x) = \cos(x) - 1 + \frac{x^2}{2}$$
$$\forall x, \exists c \in (0, x) : f'(c) = \frac{f(x) - f(0)}{x - 0}$$
$$\frac{\sin(x) - x + \frac{x^3}{6}}{x} = \cos(c) - 1 + \frac{c^2}{2}$$
$$\sin(x) - x + \frac{x^3}{6} = x\left(\cos(c) + \frac{c^2}{2}\right) - x$$
$$\sin(x) + \frac{x^3}{6} = x\left(\cos(c) + \frac{c^2}{2}\right)$$
$$\text{Let } g(x) = \cos(x) + \frac{x^2}{2}$$
$$g(0) = 1, g'(x) = x - \sin(x)$$
$$\forall x \in (0, c), \sin(x) < x \Rightarrow \forall x \in (0, c), g'(x) > 0$$
$$\Rightarrow g(c) > 1$$
$$\sin(x) + \frac{x^3}{6} > x$$
$$\sin(x) > x - \frac{x^3}{6}$$

$$\therefore$$
$$\boxed{x - \frac{x^3}{6} < \sin(x) < x; \forall 0 < x < \frac{\pi}{2}}$$

(ii) $\quad x - \frac{x^2}{2} < \ln(1 + x) < x; \forall x > 0$

$$f(x) = \ln(1 + x)$$
$$f'(x) = \frac{1}{1 + x}$$
$$\forall x, \exists c \in (0, x) : f'(c) = \frac{f(x) - f(0)}{x - 0}$$
$$\frac{1}{1 + c} = \frac{\ln(1 + x) - \ln(1)}{x}$$
$$\frac{x}{1 + c} = \ln(1 + x)$$
$$\forall c \in (0, x), \frac{1}{1 + c} < 1 \Rightarrow \ln(1 + x) < x$$

$$f(x) = \ln(1 + x) - x + \frac{x^2}{2}$$

$$f'(x) = \frac{1}{1+x} + x - 1$$
$$\exists c \in (0,x) : f'(c) = \frac{f(x) - f(0)}{x - 0}$$
$$\frac{1}{1+c} + c - 1 = \frac{\ln(1+x) - x + \frac{x^2}{2}}{x}$$
$$x(\frac{1}{1+c} + c) - x = \ln(1+x) - x + \frac{x^2}{2}$$
$$x(\frac{1}{1+c} + c) = \ln(1+x) + \frac{x^2}{2}$$
$$\frac{1}{1+c} + c > 1 \Rightarrow \ln(1+x) + \frac{x^2}{2} > x$$
$$\ln(1+x) > x - \frac{x^2}{2}$$

$$\therefore$$
$$\boxed{x - \frac{x^2}{2} < \ln(1+x) < x; \forall x > 0}$$

6. [Leibniz's General Product Rule for Derivatives] Let f, g have n^{th} order derivatives on (a, b), where $f^{(k)}(c), g^{(k)}(c)$ denotes the k^{th} order derivative of f and g at c, respectively. Also let $h = f \cdot g$. Show that for any $c \in (a, b)$,

$$h^{(n)}(c) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(c) \cdot g^{(n-k)}(c)$$

Base case: $n = 1$
$$h^{(1)}(c) = f'(c)g(c) + f(c)g'(c)$$
$$\sum_{k=0}^{1} \binom{1}{k} f^{(k)}(c) \cdot g^{(1-k)}(c) = f'(c)g(c) + f(c)g'(c)$$
$$h^{(1)}(c) = \sum_{k=0}^{1} \binom{1}{k} f^{(k)}(c) \cdot g^{(1-k)}(c)$$

Inductive case:
$$\text{Let } d(F, G) = (FG)' = \sum_{k=0}^{1} \binom{1}{k} f^{(k)}(c) \cdot g^{(1-k)}(c)$$

$$
\begin{aligned}
h^{(n+1)}(c) &= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(c) \cdot g^{(n-k)}(c)\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)}(c) \cdot g^{(n-k)}(c)\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} (f^{(k)}(c) \cdot g^{(n-k)}(c))' \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{k'=0}^{1} \binom{1}{k'} f^{(k+k')}(c) \cdot g^{(n-k+1-k')}(c)\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(c) \cdot g^{(n-k+1)}(c) + \binom{n}{k} f^{(k+1)}(c) \cdot g^{(n-k)}(c) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(c) \cdot g^{((n+1)-k)}(c)
\end{aligned}
$$

By Induction

$\therefore$

$$
\boxed{h^{(n)}(c) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(c) \cdot g^{(n-k)}(c)}
$$