Homework 7

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1. Use the definition of derivative, knowledge of trigonometry and standard limits to find the derivatives of:

(i) $\sin(x)$

$$\begin{aligned}
\sin'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} && \text{angle sum formula} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
\end{aligned}$$

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \text{standard limit}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\frac{\cos(h)-1}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1} \\
&= \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)} \\
&= \lim_{h\to 0}-\frac{\sin^2(h)}{h(\cos(h)+1)} \\
&= -\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\frac{\sin(h)}{\cos(h)+1} \\
&= -\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\lim_{h\to 0}\frac{\sin(h)}{\cos(h)+1} \\
&= 0
\end{aligned}$$

$$\sin'(x) = \sin(x)\cdot 0+\cos(x)\cdot 1$$

$$\therefore$$

$$\boxed{\sin'(x) = \cos(x)}$$

(ii) $\cos(x)$

(iii) $\tan(x)$

2. [Continuous almost everywhere, but differentiable nowhere] Prove that *Thomae's function* $f : [0,1] \to [0,1]$ is not differentiable at any point.

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n}; m, n \in \mathbb{N}; n > 0; \text{ in lowest terms} \\ 0 & x = 0 \text{ and otherwise} \end{cases}$$

$$\text{Case 1: } x \in \mathbb{Q}$$

$$f \text{ disc. @ } x \qquad\qquad \text{shown in hw6}$$

$$\therefore$$

$$f \text{ not diff. @ } x$$

$$\text{Case 2: } x \notin \mathbb{Q}$$

$$\text{Assume } f \text{ diff. @ } x$$

$$\text{Let } g(a) = \frac{f(a) - f(x)}{a - x} = \frac{f(a)}{a - x}$$

$$\exists f' \Rightarrow \exists L : \lim_{a \to x} g = L$$

$$\Rightarrow \forall \epsilon > 0, \exists \delta : \forall b : |a - b| < \delta, |g(b) - L| < \epsilon$$

$$\mathbb{Q} \text{ dense in } [0,1] \Rightarrow \exists b = \frac{m}{n} : m, n \in \mathbb{N}, |a - b| < \delta$$

$$\text{Consider } b : n > \frac{1}{\delta(\epsilon + |L|)}$$

$$|g(b) - L| < \epsilon$$

$$\Leftarrow \left| \frac{f(b)}{b - x} - L \right| < \epsilon$$

$$\Leftarrow \left| \frac{\frac{1}{n}}{b - x} \right| - |L| < \epsilon$$

$$\Leftarrow \left| \frac{1}{n(b - x)} \right| < \epsilon + |L|$$

$$\Leftarrow \frac{1}{n\delta} < \epsilon + |L|$$

$$\Leftarrow \frac{1}{\frac{1}{\delta(\epsilon + |L|)}\delta} < \epsilon + |L|$$

$$\Leftarrow \epsilon + |L| < \epsilon + |L|$$

$$\text{Contradiction}$$

$$\therefore$$

$$f \text{ not diff. @ } x$$

$$\boxed{f \text{ diff. nowhere}}$$

3. Which is greater, e^π or π^e?

4. Prove that in any interval in which the functions f, g, f', g' are continuous and $fg' - f'g \neq 0$, then the roots of f and g 'separate' each other.

$$\begin{aligned}
\text{Let } F(x) &= f(x)g'(x) - f'(x)g(x) \\
R &= \{x : f(x) = 0\} && \text{roots of } f \\
h(x) &= \frac{f(x)}{g(x)} \\
h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \\
&= \frac{-F(x)}{g^2(x)} \\
a, b &\in R : a < b, (a,b) \cap R = \emptyset && a, b \text{ are consecutive roots of } f
\end{aligned}$$

$$\begin{gathered}
\text{Assume } \nexists c \in (a,b) : g(c) = 0 \\
g(x) \neq 0, \forall x \in (a,b) \Rightarrow h(x) \text{ diff. @ } (a,b) \\
F(x) \neq, \forall x \Rightarrow h'(x) \neq 0, \forall x \\
h(a) = h(b) = 0 \wedge h \text{ diff. @ } (a,b) \Rightarrow \exists c \in (a,b) : h'(c) = 0 \qquad \text{Rolle's thm.} \\
h'(x) \neq 0, \forall x \wedge \exists c \in (a,b) : h'(c) = 0 \Rightarrow \text{ Contradiction} \\
\therefore \\
\boxed{\exists c \in (a,b) : g(c) = 0}
\end{gathered}$$

5. [Some inequalities using MVT] Apply MVT to prove the following inequalities