Homework 1

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1. Let $\mathbb{F}$ be an ordered field and $a, b, \epsilon \in \mathbb{F}$.

(i) Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$

$$
\begin{array}{cr}
\mathbb{F} \text{ is ordered} \Leftrightarrow P \ni \mathbb{F} \text{ where } P \text{ is the positive set and } \epsilon \in P & \text{order axiom} \\
a < b + \epsilon & \\
a - b < \epsilon & < \text{additivity} \\
a - b < \epsilon \Rightarrow a - b \neq \epsilon & \\
a - b \neq \epsilon \Leftrightarrow a - b \notin P & \\
a - b \notin P \Rightarrow a - b = 0 \vee -(a - b) \in P & \text{trichotomy} \\
b - a = 0 \vee b - a \in P & \\
\therefore & \\
\boxed{a \leq b} &
\end{array}
$$

(ii) Use (i) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$.

$$
\begin{array}{cr}
|a - b| < \epsilon & \\
-\epsilon < a - b < \epsilon & \text{FT abs-value} \\
-\epsilon < a - b \wedge a - b < \epsilon & \\
\epsilon > -(a - b) \wedge a - b < \epsilon & < \text{multiplicity} \\
-(a - b) \notin P \wedge a - b \notin P & \\
\Rightarrow a - b = 0 & \text{trichotomy} \\
\therefore & \\
\boxed{a = b} &
\end{array}
$$

2. Let $A \subseteq \mathbb{R}$. Define $-A = \{-a : a \in A\}$. Suppose that A is non-empty and bounded below. Show that $\inf(A) = -\sup(-A)$.

$$\neg\exists x \in A : x < \inf(A)$$
$$-x > -\inf(A)$$
$$\neg\exists k \in -A : k = -x, k > -\inf(A) \Rightarrow -\inf(A) \text{ is upperbound of } -A$$

inf analytic definition lower bound

$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \inf(A) + \epsilon$$
$$-x > -\inf(A) - \epsilon$$
$$x \in A \Rightarrow -x \in -A$$
$$\exists\, k \in -A : k = -x, k > -\inf(A) - \epsilon$$
$$\therefore$$
$$\sup(-A) = -\inf(A)$$
$$\boxed{\inf(A) = -\sup(-A)}$$

satisfies both requirements for $\sup(-A)$

3. Let $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Prove that $\sup(A) = 1, \inf(A) = \frac{1}{2}$.

$$A = \{f(n) : n \in \mathbb{N}, f(x) = \frac{x}{x+1}\}$$

$$\text{Prove } \min A = \frac{1}{2}:$$
$$\min A = \min f(x)$$
$$f(k+1) - f(k) = \frac{k+1}{(k+1)+1} - \frac{k}{k+1}$$
$$\frac{k+1}{k+2} - \frac{k}{k+1}$$
$$\frac{k^2 + 2k + 1 - (k^2 + 2k)}{k^2 + 3k + 2}$$
$$\frac{1}{k^2 + 3k + 2} > 0$$

given $k \in \mathbb{N}$

$$f(k+1) - f(k) > 0 \Leftrightarrow f(k+1) > f(k)$$
$$\min f(x) = f(\min \mathbb{N})$$
$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$
$$\therefore$$
$$\min A = \frac{1}{2}$$

$$x^2$$

4. Problem 4

5. Problem 5