Homework 3

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1. Use the definition of the limit of a sequence to show that:

(i) $\{\frac{n^2+n+1}{3n^2+1}\} \to \frac{1}{3}$

$$\left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon$$

$$\left|\frac{n+\frac{2}{3}}{3n^2+1}\right| < \epsilon$$

$$\frac{n+\frac{2}{3}}{3n^2+1} < \epsilon$$

positive for $n \in \mathbb{N}$

$$\frac{n+\frac{2}{3}}{\epsilon} < 3n^2+1$$

$$\frac{1}{\epsilon'} < 3n^2+1$$

$$\frac{1-\epsilon'}{3\epsilon'} < n^2$$

$$n \geq \left\lceil \sqrt{\frac{1-\epsilon'}{3\epsilon'}} \right\rceil = N_\epsilon$$

$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow \left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} \frac{n^2+n+1}{3n^2+1} = \frac{1}{3}}$$

$(ii)\quad \{10 - \frac{1}{\sqrt{n+\sqrt{n+5}}}\} \to 10$

$$\left|10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} - 10\right| < \epsilon$$

$$\left|-\frac{1}{\sqrt{n+\sqrt{n+5}}}\right| < \epsilon$$

$$\frac{1}{\sqrt{n+\sqrt{n+5}}} < \epsilon$$

always positive

$$\frac{1}{\sqrt{n+\sqrt{n+5}}} < \frac{1}{\sqrt{n}}$$

smaller denominator

$$\frac{1}{\sqrt{n}} < \epsilon$$

$$\sqrt{n} > \frac{1}{\epsilon}$$

$$n \geq \left\lceil \frac{1}{\epsilon^2} \right\rceil = N_\epsilon$$

$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow \left|10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} - 10\right| < \epsilon$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} 10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} = 10}$$

2.

(i) Use the definition of the limit of a sequence to show that for a fixed r with $|r| < 1$, $\{nr^n\} \to 0$.

$$\text{Consider } \lim_{n\to\infty} \left| \frac{(n+1)r^{n+1}}{nr^n} \right| = |r|$$

$$\Leftarrow \left| \left| \frac{(n+1)r^{n+1}}{nr^n} \right| - |r| \right| < \epsilon$$

$$\Leftarrow \left| \left| \frac{(n+1)r^{n+1}}{nr^n} \right| - |r| \right| \leq \left| \frac{(n+1)r^{n+1}}{nr^n} - r \right| < \epsilon \qquad \triangle\text{-ineq of} <$$

$$\Leftarrow \left| \frac{(n+1)r}{n} - r \right| < \epsilon$$

$$\Leftarrow \left| r + \frac{r}{n} - r \right| < \epsilon$$

$$\Leftarrow \left| \frac{r}{n} \right| < \epsilon$$

$$\Leftarrow \frac{r}{n} < \epsilon \qquad \text{always positive}$$

$$\Leftarrow n \geq \left\lceil \frac{r}{\epsilon} \right\rceil = N_\epsilon$$

$$\text{Almost-geometric sequence: } \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = r : 0 < r < 1, x_n = nr^n$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} nr^n = 0}$$

3.

(i) Find the limit of the following sequence, if it exists $x_n = \sqrt{n^2+n} - n$

$$x_{n+1} - x_n \geq 0$$
$$\sqrt{(n+1)^2 + (n+1)} - (n+1) - (\sqrt{n^2+n} - n) \geq 0$$
$$\sqrt{n^2+3n+2} - n - 1 - \sqrt{n^2+n} + n \geq 0$$
$$\sqrt{n^2+3n+2} - \sqrt{n^2+n} \geq 1$$
$$\frac{n^2+3n+2-(n^2+n)}{\sqrt{n^2+3n+2}+\sqrt{n^2+n}} \geq 1$$
$$\frac{2n+2}{\sqrt{n^2+3n+2}+\sqrt{n^2+n}} \geq 1$$
$$2n+2 \geq \sqrt{n^2+3n+2}+\sqrt{n^2+n}$$
$$\Leftarrow 2n+2 \geq \sqrt{n^2+3n+(\frac{3}{2})^2} + \sqrt{n^2+n+(\frac{1}{2})^2}$$
$$2n+2 \geq \sqrt{(n+\frac{3}{2})^2} + \sqrt{(n+\frac{1}{2})^2}$$
$$2n+2 \geq n+\frac{3}{2}+n+\frac{1}{2}$$
$$2n+2 \geq 2n+2$$
$$x_n \text{ is monotonic increasing}$$

$$\sqrt{n^2+n} - n \leq \frac{1}{2}$$
$$\frac{1}{2} - (\sqrt{n^2+n} - n) \geq 0$$
$$\frac{1}{2} - \frac{n}{\sqrt{n^2+n}+n} \geq 0$$
$$\Leftarrow \frac{1}{2} - \frac{n}{\sqrt{n^2}+n} \geq 0$$
$$\frac{1}{2} - \frac{n}{n+n} \geq 0$$
$$0 \geq 0 \Rightarrow x_n \leq \frac{1}{2}$$
$$\frac{1}{2} \text{ is an upperbound of } x_n$$

$$\sqrt{n^2+n} - n > \frac{1}{2} - \epsilon$$
$$\frac{1}{2} - \sqrt{n^2+n} + n < \epsilon$$
$$\text{Given } \epsilon > 0 : \exists n \in \mathbb{N} : x_n \geq \frac{1}{2} - \epsilon$$

(ii) Find the limit of the following sequence, if it exists $x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$

$$\begin{gathered} x_n = \sum_{x=1}^{n} \frac{1}{(n+x)^2} \\ 0 < \sum_{x=1}^{n} \frac{1}{(n+x)^2} < \sum_{x=1}^{n} \frac{1}{x^2} \\ \lim_{n\to\infty} 0 = 0 \\ \lim_{n\to\infty} \sum_{x=1}^{n} \frac{1}{x^2} = 0 \qquad \text{proven in lecture notes} \\ \therefore \\ \boxed{\lim_{n\to\infty} x_n = 0 \text{ by squeeze theorem}} \end{gathered}$$

4. Discuss the convergence of the sequence $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ by proving:

(i) Show that $\{x_n\}$ is bounded.

$$\begin{gathered} \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} = \sum_{x=1}^{n} \frac{1}{n+x} \\ \sum_{x=1}^{n} \frac{1}{n+x} < \sum_{x=1}^{n} \frac{1}{n} = 1 \\ \therefore \\ \boxed{\{x_n\} \text{ is upper bounded by } 1} \end{gathered}$$

(ii) Show that $\{x_n\}$ is monotonic increasing.

$$
\begin{aligned}
x_{n+1} - x_n &> 0 \\
\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{2(n+1)} - (\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}) &> 0 \\
\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2} - (\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}) &> 0 \\
\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} &> 0 \\
\frac{n+1}{4x^3 + 10x^2 + 8x + 2} &> 0 \\
n + 1 &> 0 \qquad \text{true for } n \in \mathbb{N}
\end{aligned}
$$

$$\therefore$$

$$\boxed{\{x_n\} \text{ is monotonic increasing}}$$

(iii) Find the limit of $\{x_n\}$ by comparing it to an integral.

$$
\begin{aligned}
\int_1^2 \frac{1}{x} dx &= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{2-1}{n} \cdot \frac{1}{1 + x\frac{2-1}{n}} \\
&= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{1}{n} \cdot \frac{1}{1 + \frac{x}{n}} \\
&= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{1}{n+x}
\end{aligned}
$$

$$\therefore$$

$$\boxed{\{x_n\} = \int_1^2 \frac{1}{x} dx = \ln(2)}$$