Homework 6

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1. Use $\epsilon - \delta$ definition to show that:

(i) $f(x) = x^2 + 2x + 1$ is continuous on its domain.

$$\begin{gathered}
\text{Given } \epsilon > 0, a \in \mathbb{R} \\
|f(x) - f(a)| < \epsilon \\
|x^2 + 2x + 1 - (a^2 + 2a + 1)| < \epsilon \\
|(x+a)(x-a) + 2(x-a)| < \epsilon \\
|x-a||x+a+2| < \epsilon \\
\Leftarrow |x-a|(|x| + |a| + |2|) < \epsilon \qquad \triangle\text{-ineq} \\
\delta < 1 \Rightarrow |x| < |a| + 1 \Rightarrow |x-a|(|x| + |a| + |2|) < |x-a|(2|a| + 3) \qquad \text{capping } \delta \text{ to } 1 \\
\Leftarrow |x-a| < \frac{\epsilon}{2|a| + 3} = \delta \\
\therefore \\
\boxed{\exists \delta : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon}
\end{gathered}$$

(ii) $f(x) = \sqrt{x}$ is continuous on its domain.

$$\begin{gathered}
\text{Given } \epsilon > 0, a \in P \\
|f(x) - f(a)| < \epsilon \\
|\sqrt{x} - \sqrt{a}| < \epsilon \\
|(\sqrt{x} - \sqrt{a}) \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}| < \epsilon \\
|\frac{x-a}{\sqrt{x} + \sqrt{a}}| < \epsilon \\
\Leftarrow |\frac{x-a}{2\sqrt{a+\epsilon}}| < \epsilon \ \because \ a < a + \epsilon \land x \leq a + \epsilon \\
|x-a| < 2\epsilon\sqrt{a+\epsilon} = \delta \\
\therefore \\
\boxed{\exists \delta : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon}
\end{gathered}$$

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that it is discontinuous at S but is continuous at every other point. Justify your answer.

(i) $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

$$g(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\lfloor \frac{1}{x} \rfloor} & x > 0 \end{cases}$$

Case 1: $x < 0$

$$\delta = |x|$$

$$\forall \epsilon > 0 \Rightarrow \exists \delta : \forall a : |x - a| < \delta \Rightarrow |g(x) - g(a)| < \epsilon$$

$\therefore g$ continuous

Case 2: $x = 0$

$$g(0) = 0$$

$$\lim_{x \to 0-} g(x) = \lim_{x \to 0-} 0 = 0$$

$$\lim_{x \to 0+} g(x) = \lim_{x \to 0+} e^{-\lfloor \frac{1}{x} \rfloor} = 0$$

$$\lim_{x \to 0-} g(x) = g(0) = \lim_{x \to 0+} g(x)$$

$\therefore g$ continuous

Case 3: $x \in P \cap S$

$$|e^{-\lfloor \frac{1}{x} \rfloor} - e^{-\lfloor \frac{1}{a} \rfloor}| < \epsilon$$

$$\Leftarrow |x - a| < \min(1 - x \bmod 1, x \bmod 1) = \delta$$

$$\forall \epsilon > 0 \Rightarrow \exists \delta : \forall a : |x - a| < \delta \Rightarrow |g(x) - g(a)| < \epsilon$$

$\therefore g$ continuous

Case 4: $x \in S$

$$\forall a \in (x, \frac{1}{x^{-1} + 1}) \Rightarrow \exists \epsilon > 0 : |e^{-\lfloor \frac{1}{x} \rfloor} - e^{-\lfloor \frac{1}{a} \rfloor}| > \epsilon$$

$$\therefore \lim_{a \to x^+} g(a) \neq g(x) \Rightarrow g \text{ discontinuous}$$

$$\forall x \notin S : g(x) \text{ continuous}$$

$$f(x) = x \sin(x) + g(x)$$

$$\therefore$$

$$\boxed{\exists f \text{ continuous } \forall x \notin S, \text{discontinuous } \forall x \in S}$$

cont.+cont.=cont.,
cont.+disc.=disc.

$(ii)\quad S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

$$g(x) = \left\lfloor \frac{1}{x} \right\rfloor + \left| \left\lfloor \frac{1}{x} \right\rfloor \right|$$

$$\text{Case 1: } x < 0$$
$$\left\lfloor \frac{1}{x} \right\rfloor \notin P \Rightarrow g(x) = 0$$
$$\delta = |x|$$
$$\forall \epsilon > 0 \Rightarrow \exists \delta : \forall a : |x - a| < \delta \Rightarrow |g(x) - g(a)| < \epsilon$$
$$\therefore g \text{ continuous}$$

$$\text{Case 2: } x = 0$$
$$g(x) \text{ DNE}$$
$$\therefore g \text{ discontinuous}$$

$$\text{Case 3: } x \in P \cap \{\frac{1}{n} : n \in \mathbb{N}\}$$
$$\left| \left\lfloor \frac{1}{x} \right\rfloor + \left| \left\lfloor \frac{1}{x} \right\rfloor \right| - \left\lfloor \frac{1}{a} \right\rfloor - \left| \left\lfloor \frac{1}{a} \right\rfloor \right| \right| < \epsilon$$
$$2 \left| \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{a} \right\rfloor \right| < \epsilon$$
$$\Leftarrow |x - a| < \min(1 - x \bmod 1, x \bmod 1) = \delta$$
$$\forall \epsilon > 0 \Rightarrow \exists \delta : \forall a : |x - a| < \delta \Rightarrow |g(x) - g(a)| < \epsilon$$
$$\therefore g \text{ continuous}$$

$$\text{Case 4: } x \in \{\frac{1}{n} : n \in \mathbb{N}\}$$
$$\forall a \in (x, \frac{1}{x^{-1} + 1}) \Rightarrow 2 \left| \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{a} \right\rfloor \right| = 2 > \epsilon, \forall \epsilon < 2$$
$$\therefore \lim_{a \to x^+} g(a) \neq g(x) \Rightarrow g \text{ discontinuous}$$

$$\forall x \notin S : g(x) \text{ continuous}$$
$$f(x) = x \sin(x) + g(x)$$
$$\therefore$$
$$\boxed{\exists f \text{ continuous } \forall x \notin S, \text{discontinuous } \forall x \in S}$$

cont.+cont.=cont.,
cont.+disc.=disc.

3.

(i) Show that if f_1, f_2 are continuous functions, then $g = \max(f_1, f_2)$ and $h = \min(f_1, f_2)$ also are.

$$\begin{aligned}
\text{Let } L &= \{x : f_1(x) = f_2(x)\} \\
F &= \{f_1, f_2\} \\
x_1 &= \sup(\{a : a < x, a \in L\}) \\
x_2 &= \inf(\{a : a > x, a \in L\}) \\
& x \in L \lor x \notin L
\end{aligned}$$

$$\text{Case 1: } x \notin L$$
$$\exists f_g \in F : x_1 < a < x_2 \Rightarrow f_g(a) = g(a)$$
$$f_g \text{ continuous} \Rightarrow \exists \delta_1 : \forall a : |x - a| < \delta_1 \Rightarrow |f_g(x) - f_g(a)| < \epsilon$$
$$\text{Let } \delta_g = \min(\delta_1, |x - x_1|, |x - x_2|)$$
$$\delta_g \leq \delta_1 \land x_1 \leq x - \delta_g < x + \delta_g \leq x_2 \Rightarrow \forall a : |x - a| < \delta_g \Rightarrow |g(x) - g(a)| < \epsilon$$
$$\therefore \forall x \notin L \Rightarrow g \text{ continuous @ } x$$
$$\exists f_h \in F : x_1 < a < x_2 \Rightarrow f_h(a) = h(a)$$
$$f_h \text{ continuous} \Rightarrow \exists \delta_2 : \forall a : |x - a| < \delta_2 \Rightarrow |f_h(x) - f_h(a)| < \epsilon$$
$$\text{Let } \delta_h = \min(\delta_2, |x - x_1|, |x - x_2|)$$
$$\delta_h \leq \delta_2 \land x_1 \leq x - \delta_h < x + \delta_h \leq x_2 \Rightarrow \forall a : |x - a| < \delta_h \Rightarrow |h(x) - h(a)| < \epsilon$$
$$\therefore \forall x \notin L \Rightarrow h \text{ continuous @ } x$$

$$\text{Case 2: } x \in L$$
$$\exists f_{g1} \in F : x_1 < a < x \Rightarrow f_{g1}(a) = g(a)$$
$$\exists f_{g2} \in F : x < a < x_2 \Rightarrow f_{g2}(a) = g(a)$$
$$f_{g1} \text{ continuous} \Rightarrow \exists \delta_{g1} : \forall a : x - a < \delta_{g1} \Rightarrow |g(x) - g(a)| < \epsilon$$
$$f_{g2} \text{ continuous} \Rightarrow \exists \delta_{g2} : \forall a : a - x < \delta_{g2} \Rightarrow |g(x) - g(a)| < \epsilon$$
$$\text{Let } \delta_g = \min(\delta_{g1}, \delta_{g2})$$
$$\forall a : x - a < \delta_g \lor a - x < \delta_g \Rightarrow |g(x) - g(a)| < \epsilon$$
$$\exists \delta_g : \forall a : |x - a| < \delta_g \Rightarrow |g(x) - g(a)| < \epsilon$$
$$\therefore \forall x \in L \Rightarrow g \text{ continuous @ } x$$
$$\exists f_{h1} \in F : x_1 < a < x \Rightarrow f_{h1}(a) = h(a)$$
$$\exists f_{h2} \in F : x < a < x_2 \Rightarrow f_{h2}(a) = h(a)$$
$$f_{h1} \text{ continuous} \Rightarrow \exists \delta_{h1} : \forall a : x - a < \delta_{h1} \Rightarrow |h(x) - h(a)| < \epsilon$$
$$f_{h2} \text{ continuous} \Rightarrow \exists \delta_{h2} : \forall a : a - x < \delta_{h2} \Rightarrow |h(x) - h(a)| < \epsilon$$
$$\text{Let } \delta_h = \min(\delta_{h1}, \delta_{h2})$$

$$\forall a : x - a < \delta_h \vee a - x < \delta_h \Rightarrow |h(x) - h(a)| < \epsilon$$
$$\exists \delta_h : \forall a : |x - a| < \delta_h \Rightarrow |h(x) - h(a)| < \epsilon$$
$$\therefore \forall x \in L \Rightarrow h \text{ continuous @ } x$$

$$\therefore$$
$$\boxed{g \text{ and } h \text{ continuous}}$$

(ii) Let f be a continuous function. Prove that $f(x)$ can always be written as $f(x) = g(x) - h(x)$, where g, h are continuous functions and non-negative.

$$\text{Let } g(x) = |\max(f(x), 2f(x))|$$
$$h(x) = |\min(f(x), 2f(x))|$$
$$g(x) \text{ continuous } \wedge h(x) \text{ continuous } \because \ \mathbf{3}(i)$$
$$g(x) - h(x) = f(x)$$

4. [Applications of IVP; **Do any three**]

(i) If $f : [a, b] \to [a, b]$ is continuous on $[a, b]$, then f has a fixed point (that is, $f(c) = c$ for some $c \in [a, b]$)

$$\text{Let } g(x) = f(x) - x$$
$$\text{Note } g(x) = 0 \Leftrightarrow f(x) = x$$
$$f(a) \geq a \Rightarrow g(a) \geq 0$$
$$f(b) \geq b \Rightarrow g(b) \leq 0$$
$$\operatorname{sgn}(g(a)) \neq \operatorname{sgn}(g(b)) \Rightarrow \exists c \in [a, b] : g(c) = 0$$

Bolzano's thm.

$$\therefore$$
$$\boxed{\exists c \in [a, b] : f(c) = c}$$

(ii) Prove that at any given instant, some two diametrically opposite points on the Equator of our Earth have the same temperature.

$$T(x) : [0, 2\pi] \to \mathbb{R}$$

temp. model func.

$$\text{Let } f(x) : [0, \pi] = T(x)$$
$$g(x) : [0, \pi] = T(x + \pi)$$
$$a = f(0) = g(\pi)$$
$$b = f(\pi) = g(0)$$
$$\exists \text{ 2 anti-podal points with same temp.} \Rightarrow \exists c : f(c) = g(c)$$

Case 1: $a = b$

$$\therefore f(0) = g(0)$$

$$\text{Case 2: } a \neq b$$
$$\text{Let } g(x) = f(x) - g(x)$$
$$\text{Note } g(x) = 0 \Leftrightarrow f(x) = g(x)$$
$$h(0) = a - b \wedge h(\pi) = b - a \Rightarrow h(0) = -h(\pi)$$ opposite signs
$$\therefore \exists c : h(c) = 0 \Rightarrow \exists c : f(c) = g(c)$$ Bolzano's thm.

$$\therefore$$
$$\boxed{\exists \text{ 2 anti-podal points with same temp.}}$$

(iv) Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Show that there must exist $x, y \in [0,1]$ with $|x - y| = \frac{1}{2}$ for which $f(x) = f(y)$.

$$\text{Let } g(x) : \left[0, \frac{1}{2}\right] = f(x)$$
$$h(x) : \left[0, \frac{1}{2}\right] = f(x + \frac{1}{2})$$
$$a = g(0) = h(\frac{1}{2})$$
$$b = g(\frac{1}{2}) = h(0)$$
$$\text{Note } \exists x, y \in [0,1] : |x - y| = \frac{1}{2} \wedge f(x) = f(y) \Leftrightarrow \exists c : g(c) = h(c)$$

$$\text{Case 1: } a = b$$
$$\therefore g(0) = h(0)$$

$$\text{Case 2: } a \neq b$$
$$\text{Let } j(x) = g(x) - h(x)$$
$$\text{Note } j(x) = 0 \Leftrightarrow g(x) = h(x)$$
$$j(0) = a - b \wedge j(\frac{1}{2}) = b - a \Rightarrow j(0) = -j(\frac{1}{2})$$ opposite signs
$$\therefore \exists c : j(c) = 0 \Rightarrow \exists c : g(c) = h(c)$$ Bolzano's thm.

$$\therefore$$
$$\boxed{\exists x, y \in [0,1] : |x - y| = \frac{1}{2} \wedge f(x) = g(x)}$$

5. [IVP + Monotonicity $\Rightarrow$ Continuity]
Assume f has IVP in $[a, b]$. Show that if f is increasing on $[a, b]$, then f is also continuous on $[a, b]$.

$$\text{Assume the negation: } \exists c \in [a, b] : f(c) \neq \lim_{x \to c} f(x)$$

$$f(c) \neq \lim_{x \to c} f(x) \Rightarrow \neg(\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x))$$

$$f \text{ incr.} \Rightarrow \lim_{x \to c^-} f(x) \leq f(c) < \lim_{x \to c^+} f(x) \vee \lim_{x \to c^-} f(x) < f(c) \leq \lim_{x \to c^+} f(x)$$

$$\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x)$$

$$\forall y \in [\lim_{x \to c^-} f(x), f(c)) \cup (f(c), \lim_{x \to c^+} f(x)] \Rightarrow y \in [f(a), f(b)] \wedge \nexists x : f(x) = y$$

$$\text{IVP contradiction}$$

$$\therefore$$

$$\boxed{f \text{ continuous on } [a, b]}$$