Math 310 - HW 6 (Due Wednesday 04/02)

(As always, first try some warm-up problems from the textbook !)

1. Use the $\epsilon - \delta$ definition to show that:

(i) $f(x) = x^2 + 2x + 1$ is continuous on its domain.

(ii) $f(x) = \sqrt{x}$ is continuous on its domain.

(iii) [**Non-Credit**] $f(x) = e^x$ is continuous on its domain. (*Hint*: First show that e^x is continuous at $a = 0$. You may use the fact that $e^x > 1 + x, \forall x$)

2. (i) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that it is discontinuous at $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ but is continuous at every other point. Justify your answer.

(ii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that it is discontinuous at $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ but is continuous at every other point. Justify your answer.

(iii) [**Extra-Credit**] Give an example of a function $f : (0,1) \to (0,1)$ such that it is continuous at every irrational number but discontinuous at every rational number. Justify your answer. (*Hint*: Modify Dirichlet's function. *Remark*: One can then also periodically extend this to $f : \mathbb{R} \to \mathbb{R}$)

(iv) [**Non-Credit**] Show that there cannot be a function which is continuous at every rational number but discontinuous at every irrational number. (*Hint*: If not, then one builds a countable collection of nested closed intervals containing a point of continuity which is a irrational number; a contradiction.)

3. (i) Show that if f_1, f_2 are continuous functions, then $g = \max\{f_1, f_2\}$ and $h = \min\{f_1, f_2\}$ also are.

(ii) Let f be a continuous function. Prove that $f(x)$ can always be written as $f(x) = g(x) - h(x)$, where g,h are continuous functions and non-negative. $\big($*Hint*: Use $(i)\big)$

4. [Applications of IVP; **Do any three**]

(i) If $f : [a,b] \to [a,b]$ is continuous on $[a,b]$, then f has a *fixed point* $\big($that is, $f(c) = c$ for some $c \in [a,b]\big)$.

(ii) Prove that at any given instant, some two diametrically opposite (anti-podal) points on the Equator of our Earth have the same temperature. (Assume Earth to be spherical and thus the Equator to be circular.)

(iii) Prove that there exists NO continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Q}, \forall x \in \mathbb{R} - \mathbb{Q}$ and $f(x) \in \mathbb{R} - \mathbb{Q}, \forall x \in \mathbb{Q}$.

(iv) Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Show that there must exist $x, y \in [0,1]$ with $|x - y| = \frac{1}{2}$ for which $f(x) = f(y)$.

(v) [Hiker's dèjá vu] Show that if a hiker hikes up a mountain from time s_1 to s_2 on one day, and hikes down the same path from time t_1 to t_2 the next day, such that $[s_1, s_2] \cap [t_1, t_2] \neq \emptyset$, then there exists some time instant at which the hiker was at the same spot on both days.

5. [IVP + Monotonicity $\Rightarrow$ Continuity]

Assume f has IVP in $[a,b]$. Show that if f is increasing on $[a,b]$, then f is also continuous on $[a,b]$.

6. [A fixed-point theorem for contraction maps; **Extra-Credit**] Let $f : \mathbb{R} \to \mathbb{R}$ and assume there exists $0 \leq C < 1$ such that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq C|x - y|$$

(i) Prove that f is continuous on $\mathbb{R}$.

(ii) Pick any $x_1 \in \mathbb{R}$ and consider the sequence $x_n = f(x_{n-1})$. Prove that $\{x_n\}$ is a Cauchy sequence.

(iii) Let $\{x_n\} \to c$. Prove that $f(c) = c$.

(iv) Prove that this c is unique for f.