Homework 4

Keizou Wang

February 26, 2025

1. Consider the following continued fraction: $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$.

(i) Write the above continued fraction as the limit of a sequence. Then write a (first-order) recurrence relation between the terms of the sequence.

$$x_1 = 1 + \frac{1}{2} = \frac{3}{2}$$
$$x_n = 1 + \frac{1}{1 + x_{n-1}} = \frac{2 + x_{n-1}}{1 + x_{n-1}}$$
$$\lim_{n\to\infty} x_n = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\cdots}}}$$

(ii) Show that the sequence is bounded.

$$1 + \frac{1}{1 + x_{n-1}} < 1 + \frac{1}{1+0} \qquad x_{n-1} \in P$$
$$\therefore$$
$$\boxed{x_n < 2}$$

(iii) Show that the subsequence of odd-indexed terms and even-indexed terms are monotonic.

$$\begin{aligned} x_{n+2} - x_n &= 1 + \frac{1}{1 + x_{n+1}} - (1 + \frac{1}{1 + x_{n-1}}) \\ &= \frac{1}{1 + x_{n+1}} - \frac{1}{1 + x_{n-1}} \\ &= \frac{x_{n-1} - x_{n+1}}{(1 + x_{n+1})(1 + x_{n-1})} \qquad (1) \\ &= \frac{x_{n-1}}{(1 + x_{n+1})(1 + x_{n-1})} - \frac{x_{n+1}}{(1 + x_{n+1})(1 + x_{n-1})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2+x_{n-2}}{(1+x_{n-2})(1+x_{n+1})(1+x_{n-1})} - \frac{2+x_n}{(1+x_n)(1+x_{n+1})(1+x_{n-1})} \\
&= \frac{(2+x_{n-2})(1+x_n)-(2+x_n)(1+x_{n-2})}{(1+x_{n-2})(1+x_n)(1+x_{n+1})(1+x_{n-1})} \\
&= \frac{x_n - x_{n-2}}{(1+x_{n-2})(1+x_n)(1+x_{n+1})(1+x_{n-1})} \qquad (2)
\end{aligned}
$$

(2) shows $x_n \gtreqless x_{n-2} \Rightarrow x_{n+2} \gtreqless x_n \therefore$ odd and even terms separately monotonic
(1) shows if one is monotonic increasing the other is monotonic decreasing

(iv) Show that the above continued fraction converges and find the limit.

$$
\begin{gathered}
\text{By MCT, let } \lim_{n\to\infty} x_{2n+1} = l_1 \wedge \lim_{n\to\infty} x_{2n} = l_2 \\
l_1 = 1 + \frac{1}{1+l_2} \wedge l_2 = 1 + \frac{1}{1+l_1} \\
l_1 = l_2 = l \\
l = 1 + \frac{1}{1+l} \\
(l-1)(l+1) = 1 \\
l^2 - 1 = 1 \\
l^2 = 2 \\
l = \sqrt{2} \\
\therefore \\
\boxed{\lim_{n\to\infty} x_n = \sqrt{2}}
\end{gathered}
$$