Math 310 - HW 3 (Due Wednesday 02/19)

1. (i) Use the definition of the limit of a sequence to show that

$$\left\{\frac{n^2+n+1}{3n^2+1}\right\} \to \frac{1}{3}$$

(ii) Use the definition of the limit of a sequence to show that

$$\left\{10 - \frac{1}{\sqrt{n+\sqrt{n+5}}}\right\} \to 10$$

2. (i) Use the definition of the limit of a sequence to show that for a fixed r with $|r| < 1$,

$$\{nr^n\} \to 0$$

(ii) Use (i) to show that (*Hint*: Start by considering $m \in \mathbb{N}$, such that $m \leq \ln n < m+1$)

$$\left\{\frac{\ln n}{n}\right\} \to 0$$

3. (i) Find the limit of the following sequence, if it exists

$$x_n = \sqrt{n^2+n} - n$$

(ii) Find the limit of the following sequence, if it exists

$$x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}$$

4. Discuss the convergence of the sequence $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ by proving:

(i) Show that $\{x_n\}$ is bounded.

(ii) Show that $\{x_n\}$ is monotonic increasing.

(iii) Find the limit of $\{x_n\}$ by comparing it to an integral (*Hint*: Requires the knowledge of Riemann-Sum definition of an integral from Calc I.)

5. Consider sequence $\{x_n\}$ such that $0 \leq x_1 < x_2$ and $x_n = \frac{x_{n-1}+x_{n-2}}{2}, \forall n \geq 3$. Show that

$$\{x_n\} \to \frac{x_1 + 2x_2}{3}$$

(*Hint*: Consider $x_n - x_{n-1}$ in terms of $x_2 - x_1$)

6. [**Extra-Credit**; $e \notin \mathbb{Q}$] In class we saw that the sequence $\left\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right\}$ converges to a number called e. Show that this e is irrational. (*Hint*: If e were rational and $= \frac{p}{q}$, then consider $(q!)e$, and obtain a contradiction.)

Not for Credit

1. Use the definition of the limit of a sequence to show that given any fixed $r > 0$, $\{r^{1/n}\} \to 1$. (*Hint*: Break into three cases: $r = 1, 0 < r < 1, r > 1$.)