Homework 1

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1. Let $\mathbb{F}$ be an ordered field and $a, b, \epsilon \in \mathbb{F}$.

(i) Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

$$
\begin{aligned}
&\mathbb{F} \text{ is ordered} \Leftrightarrow P \ni \mathbb{F} \text{ where } P \text{ is the positive set and } \epsilon \in P && \text{order axiom} \\
&a < b + \epsilon \\
&a - b < \epsilon && < \text{additivity} \\
&a - b < \epsilon \Rightarrow a - b \neq \epsilon \\
&a - b \neq \epsilon \Leftrightarrow a - b \notin P \\
&a - b \notin P \Rightarrow a - b = 0 \vee -(a-b) \in P && \text{trichotomy} \\
&b - a = 0 \vee b - a \in P \\
&\therefore \\
&\boxed{a \leq b}
\end{aligned}
$$

(ii) Use (i) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$.[1]

$$
\begin{aligned}
&|a - b| < \epsilon \\
&-\epsilon < a - b < \epsilon && \text{FT abs-value} \\
&-\epsilon < a - b \wedge a - b < \epsilon \\
&b < a + \epsilon \wedge a < b + \epsilon \\
&b \leq a \wedge a \leq b \\
&\therefore \\
&\boxed{a = b}
\end{aligned}
$$

[1] Proof using trichotomy in notes

2. Let $A \subseteq \mathbb{R}$. Define $-A = \{-a : a \in A\}$. Suppose that A is non-empty and bounded below. Show that $\inf(A) = -\sup(-A)$.

$$\forall x \in A : x \geq \inf(A)$$
$$-x \leq -\inf(A)$$
$$\forall x' \in -A : -x' \in A \Rightarrow \forall x' \in -A : x' \leq -\inf(A)$$
$$-\inf(A) \text{ is an upper bound of } -A$$

inf analytic definition lower bound

$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \inf(A) + \epsilon$$
$$-x > -\inf(A) - \epsilon$$
$$\exists\, k \in -A : k = -x, k > -\inf(A) - \epsilon$$
$$\therefore$$
$$\sup(-A) = -\inf(A)$$
$$\boxed{\inf(A) = -\sup(-A)}$$

satisfies both requirements for $\sup(-A)$

3. Let $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Prove that $\sup(A) = 1, \inf(A) = \frac{1}{2}$.

$$A = \{f(n) : n \in \mathbb{N}, f(x) = \frac{x}{x+1}\}$$

$$\frac{n}{n+1} \geq \frac{1}{2}$$
$$2n \geq n+1$$
$$n \geq 1 \Rightarrow \frac{1}{2} \text{ is a lower bound of } A$$

$n \geq 1$ is valid by definition

$$f(1) = \frac{1}{2}$$
$$\frac{1}{2} - f(1) = 0$$
$$\epsilon + \frac{1}{2} - f(1) = \epsilon$$

where $\epsilon > 0$

$$(\frac{1}{2} + \epsilon) - f(1) > 0$$
$$f(1) < \frac{1}{2} + \epsilon$$
$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \frac{1}{2} + \epsilon$$

when $x = f(1)$

$$\therefore$$
$$\boxed{\inf(A) = \frac{1}{2}}$$

$$\frac{n}{n+1} \leq 1$$
$$n \leq n+1$$
$$0 \leq 1 \Rightarrow 1 \text{ is an upper bound of } A$$
$$\frac{n}{n+1} < 1 - \epsilon \qquad \text{where } \epsilon > 0$$
$$\epsilon < 1 - \frac{n}{n+1}$$
$$0 < \epsilon < \frac{1}{n+1}$$
$$\frac{1}{n+1} > 0 \qquad \text{true for } n \in \mathbb{N}$$
$$\text{Given } \epsilon > 0 : \ \exists\, x \in A : x > 1 - \epsilon$$
$$\therefore$$
$$\boxed{\sup(A) = 1}$$

4. Let $A, B \supseteq \mathbb{R}$:

(i) $\exists \sup(A), \exists \sup(B), A \subseteq B$. Show that $\sup(A) \leq \sup(B)$.

$$\exists b \in B : b > \sup(A) \vee \neg \exists b \in B : b > \sup(A) \qquad \text{a tautology } (a \vee \neg a)$$

Case 1

$$\exists b \in B : b > \sup(A)$$
$$\sup(A) < b \leq \sup(B)$$
$$\sup(A) < \sup(B)$$

Case 2

$$\neg \exists b \in B : b > \sup(A)$$
$$\forall b \in B : b \leq \sup(A)$$
$$\Rightarrow \sup(A) \text{ is an upperbound of } B$$
$$\text{Given } \epsilon > 0 : \exists a \in A : a > \sup(A) - \epsilon$$
$$B \supset A \Rightarrow a \in B$$
$$a \in B : a > \sup(A) - \epsilon$$
$$\sup(A) = \sup(B)$$

$$\sup(A) < \sup(B) \vee \sup(A) = \sup(B)$$
$$\therefore$$
$$\boxed{\sup(A) \leq \sup(B)}$$

(*ii*) $\sup(A) < \sup(B)$. Show that there exists $b \in B$ that is an upper bound of A. Show that this result does not hold if we instead assume that $\sup(A) \leq \sup(B)$.

5. For $A, B \supseteq \mathbb{R}$, define

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ A \cdot B &= \{a \cdot b : a \in A, b \in B\} \end{aligned}$$

(*i*) Determine $\{3, 1, 0\} + \{2, 0, 2, 3\}$ and $\{3, 1, 0\} \cdot \{2, 0, 2, 3\}$.

$$\begin{aligned} A &= \{3, 1, 0\} \\ B &= \{2, 0, 2, 3\} \\ A + B &= \{3 + 2, 3 + 0, 3 + 2, 3 + 3, 1 + 2, 1 + 0, 1 + 2, 1 + 3, 0 + 2, 0 + 0, 0 + 2, 0 + 3\} \\ &= \{5, 3, 5, 6, 3, 1, 3, 4, 2, 0, 2, 3\} \\ &= \{5, 3, 6, 1, 4, 2, 0\} \\ A \cdot B &= \{3 \cdot 2, 3 \cdot 0, 3 \cdot 2, 3 \cdot 3, 1 \cdot 2, 1 \cdot 0, 1 \cdot 2, 1 \cdot 3, 0 \cdot 2, 0 \cdot 0, 0 \cdot 2, 0 \cdot 3\} \\ &= \{6, 0, 6, 9, 2, 0, 2, 3, 0, 0, 0, 0\} \\ &= \{6, 0, 9, 2, 3\} \end{aligned}$$

(*ii*) Assume that $\sup(A)$ and $\sup(B)$ exist. Prove that $\sup(A+B) = \sup(A) + \sup(B)$.

(*iii*) Give an example of sets A, B where $\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)$.

Notes

1. Question 1-ii proof by trichotomy

$$\begin{aligned}
|a-b| &< \epsilon \\
-\epsilon < a-b &< \epsilon && \text{FT abs-value} \\
-\epsilon < a-b &\wedge a-b < \epsilon \\
\epsilon > -(a-b) &\wedge a-b < \epsilon && < \text{multiplicity} \\
-(a-b) \notin P &\wedge a-b \notin P \\
&\Rightarrow a-b = 0 && \text{trichotomy} \\
&\therefore \\
&\boxed{a=b}
\end{aligned}$$