Math 310 - HW 1 (Due Friday 01/31)

1. Let $\mathbb{F}$ be an ordered field and $a, b, \epsilon \in \mathbb{F}$.

(i) Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

(ii) Use (i) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$. (*Hint*: You may use contradiction.)

2. Let $A \subseteq \mathbb{R}$. Define $-A = \{-a : a \in A\}$. Suppose that A is non-empty and bounded below. Show that

$$inf(A) = -sup(-A)$$

(*Hint*: Use the Analytic definition of *sup,inf*)

3. Let $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. Prove that $sup(A) = 1$, $inf(A) = \frac{1}{2}$.

4. (i) Let $A, B \subseteq \mathbb{R}$ be sets which are bounded above, such that $A \subseteq B$. Show that $sup(A) \leq sup(B)$.

(ii) Let $A, B \subseteq \mathbb{R}$ such that $sup(A) < sup(B)$. Show that there exists $b \in B$ that is an upper bound of A. Show that this result does not hold if we instead assume that $sup(A) \leq sup(B)$.

5. For $A, B \subseteq \mathbb{R}$, define

$$A + B = \{a + b : a \in A, b \in B\}$$

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}$$

(i) Determine $\{3, 1, 0\} + \{2, 0, 2, 3\}$ and $\{3, 1, 0\} \cdot \{2, 0, 2, 3\}$

(ii) Assume that $sup(A)$ and $sup(B)$ exist. Prove that $sup(A + B) = sup(A) + sup(B)$.

(iii) Give an example of sets A, B where $sup(A \cdot B) \neq sup(A) \cdot sup(B)$

Warm-up Problems, Not for credit:

1. Let $\mathbb{F}$ be any field. Prove that both the additive and multiplicative identities in $\mathbb{F}$ are unique.

2. Given an ordered field $\mathbb{F}$, we saw that it has a set of *positive* elements P satisfying certain two conditions.

(i) Give an example of some $P_1 \subseteq \mathbb{R}$ such that it satisfies the first condition, but not the second.

(ii) Give an example of some $P_2 \subseteq \mathbb{R}$ such that it satisfies the second condition, but not the first.

3. Prove the various properties of an ordered field along with those of absolute value function discussed in class.