Math 310 - HW 7 (Due Monday 04/14)

(As always, first try some warm-up problems from the textbook !)

1. Use the definition of derivative, knowledge of trigonometry and standard limits to find the derivatives of:

$$\sin x\ ,\ \ \cos x\ ,\ \ \tan x$$

2. [Continuous almost everywhere, but differentiable nowhere] Prove that following *Thomae's function* $f : [0,1] \to [0,1]$ (from HW 6) is not differentiable at any point.

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n};\ m,n \in \mathbb{N};\ n > 0;\ \text{in lowest terms} \\ 0 & x = 0 \text{ and otherwise} \end{cases}$$

(*Hint*: Given an irrational number a, consider two approximations of it: a (non-recurring) decimal representation of a using rational numbers and another approximation using irrational numbers.

Remark: You may also want to look up functions like Weierstrass function, which is continuous everywhere, but differentiable nowhere !)

3. Which is greater, e^π or π^e ? (Don't quote your calculator. *Hint*: Consider the function $\frac{\ln x}{x}$ and use MVT)

4. (i) Prove that in any interval in which the functions f, g, f', g' are continuous and $fg' - f'g \neq 0$, then the roos of f and g 'separate' each other (that is, between any two roots of f lies a root of g and vice-versa). (*Hint*: Use Rolle's Theorem.)

(ii) Verify (i) for $\sin x$ and $\cos x$.

5. [Some inequalities using MVT] Apply MVT to prove the following inequalities (**Do any two**):

(i) $x - \frac{x^3}{6} < \sin x < x\ ;\ \forall\, 0 < x < \frac{\pi}{2}$

(ii) $x - \frac{x^2}{2} < \ln(1 + x) < x\ ;\ \forall\, 0 < x$

(iii) $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}\ ;\ \forall\, 0 < x < 1$

(iv) $0 < \frac{1}{\ln(1+x)} - \frac{1}{x} < 1\ ;\ \forall - 1 < x, x \neq 0$

(v) $0 < \frac{1}{x} \ln(\frac{e^x-1}{x}) < 1\ ;\ \forall\, x \neq 0$

6. [Leibniz's General Product Rule for Derivatives, **Extra credit**] Let f, g have n^{th} order derivatives on (a, b), where $f^{(k)}(c)$, $g^{(k)}(c)$ denotes the k^{th} order derivative of f and g at c, respectively. Also let $h = f \cdot g$. Show that for any $c \in (a, b)$,

$$h^{(n)}(c) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(c) \cdot g^{(n-k)}(c)$$

(*Hint*: Use induction)