Homework 1

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1. Let $\mathbb{F}$ be an ordered field and $a, b, \epsilon \in \mathbb{F}$.

(i) Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

$$\begin{aligned}
&\mathbb{F} \text{ is ordered} \Leftrightarrow P \ni \mathbb{F} \text{ where } P \text{ is the positive set and } \epsilon \in P && \text{order axiom} \\
&a < b + \epsilon \\
&a - b < \epsilon && < \text{additivity} \\
&a - b < \epsilon \Rightarrow a - b \neq \epsilon \\
&a - b \neq \epsilon \Leftrightarrow a - b \notin P \\
&a - b \notin P \Rightarrow a - b = 0 \vee -(a-b) \in P && \text{trichotomy} \\
&b - a = 0 \vee b - a \in P \\
&\therefore \\
&\boxed{a \leq b}
\end{aligned}$$

(ii) Use (i) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$.[1]

$$\begin{aligned}
&|a - b| < \epsilon \\
&-\epsilon < a - b < \epsilon && \text{FT abs-value} \\
&-\epsilon < a - b \wedge a - b < \epsilon \\
&b < a + \epsilon \wedge a < b + \epsilon \\
&b \leq a \wedge a \leq b \\
&\therefore \\
&\boxed{a = b}
\end{aligned}$$

[1] Proof using trichotomy in notes

2. Let $A \subseteq \mathbb{R}$. Define $-A = \{-a : a \in A\}$. Suppose that A is non-empty and bounded below. Show that $\inf(A) = -\sup(-A)$.

$$x \in A \Rightarrow x \geq \inf(A)$$
$$-x \leq -\inf(A)$$
$$x' \in -A \Rightarrow x' = -x, x \in A \Rightarrow x' \leq -\inf(A)$$
$$-\inf(A) \text{ is an upper bound of } -A$$

inf analytic definition lower bound

$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \inf(A) + \epsilon$$
$$-x > -\inf(A) - \epsilon$$
$$\exists\, k \in -A : k = -x, k > -\inf(A) - \epsilon$$
$$\therefore$$
$$\sup(-A) = -\inf(A)$$
$$\boxed{\inf(A) = -\sup(-A)}$$

satisfies both requirements for $\sup(-A)$

3. Let $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Prove that $\sup(A) = 1, \inf(A) = \frac{1}{2}$.

$$A = \{f(n) : n \in \mathbb{N}, f(x) = \frac{x}{x+1}\}$$

$$\frac{n}{n+1} \geq \frac{1}{2}$$
$$2n \geq n + 1$$
$$n \geq 1 \Rightarrow \frac{1}{2} \text{ is a lower bound of } A$$

$n \geq 1$ is valid by definition

$$f(1) = \frac{1}{2}$$
$$\frac{1}{2} - f(1) = 0$$
$$\epsilon + \frac{1}{2} - f(1) = \epsilon$$

where $\epsilon > 0$

$$(\frac{1}{2} + \epsilon) - f(1) > 0$$
$$f(1) < \frac{1}{2} + \epsilon$$
$$\text{Given } \epsilon > 0,\ \exists\, x \in A : x < \frac{1}{2} + \epsilon$$

when $x = f(1)$

$$\therefore$$
$$\boxed{\inf(A) = \frac{1}{2}}$$

$$\frac{n}{n+1} \leq 1$$
$$n \leq n+1$$
$$0 \leq 1 \Rightarrow 1 \text{ is an upper bound of } A$$
$$\frac{n}{n+1} < 1 - \epsilon \qquad \text{where } \epsilon > 0$$
$$\epsilon < 1 - \frac{n}{n+1}$$
$$0 < \epsilon < \frac{1}{n+1}$$
$$\frac{1}{n+1} > 0 \qquad \text{true for } n \in \mathbb{N}$$
$$\text{Given } \epsilon > 0: \ \exists\, x \in A : x > 1 - \epsilon$$
$$\therefore$$
$$\boxed{\sup(A) = 1}$$

4. Let $A, B \supseteq \mathbb{R}$:

(i) $\exists \sup(A), \exists \sup(B), A \subseteq B$. Show that $\sup(A) \leq \sup(B)$.

$$\exists b \in B : b > \sup(A) \vee \neg\exists b \in B : b > \sup(A) \qquad \text{a tautology } (a \vee \neg a)$$

Case 1

$$\exists b \in B : b > \sup(A)$$
$$\sup(A) < b \leq \sup(B)$$
$$\sup(A) < \sup(B)$$

Case 2

$$\neg\exists b \in B : b > \sup(A)$$
$$\Leftrightarrow b \in B \Rightarrow b \leq \sup(A)$$
$$\Rightarrow \sup(A) \text{ is an upperbound of } B$$
$$\text{Given } \epsilon > 0 : \exists a \in A : a > \sup(A) - \epsilon$$
$$B \supset A \Rightarrow a \in B$$
$$a \in B : a > \sup(A) - \epsilon$$
$$\sup(A) = \sup(B)$$

$$\sup(A) < \sup(B) \vee \sup(A) = \sup(B)$$
$$\therefore$$
$$\boxed{\sup(A) \leq \sup(B)}$$

(ii) $\sup(A) < \sup(B)$. Show that there exists $b \in B$ that is an upper bound of A. Show that this result does not hold if we instead assume that $\sup(A) \leq \sup(B)$.

$$\begin{gathered}
\sup(A) \leq \sup(B) \\
\Diamond(\sup(A) = \sup(B) = k, k \in \mathbb{R}) \\
b \in B \Rightarrow b \leq k \\
a \in A \Rightarrow a \leq k \\
\Diamond(\max(b) < k \wedge \max(a) = k) \\
\max(b) < \max(a) \\
\therefore \\
\boxed{\sup(A) \leq \sup(B) \Rightarrow \Diamond\neg\exists b \in B : b \text{ is an upper bound of } A}
\end{gathered}$$

$$\begin{gathered}
\sup(A) < \sup(B) \\
\text{Given } \epsilon > 0 : \exists b \in B : b > \sup(B) - \epsilon \\
b + \epsilon > \sup(B) \Rightarrow b + \epsilon > \sup(A) \\
b + \epsilon - \sup(A) > 0 \\
b - \sup(A) > -\epsilon \\
-(b - \sup(A)) < \epsilon \\
-(b - \sup(A)) \notin P \\
b - \sup(A) \in P \vee b - \sup(A) = 0 \qquad \text{trichotomy} \\
b \geq \sup(A) \geq a \\
\therefore \\
\boxed{\sup(A) < \sup(B) \Rightarrow \exists b \in B : b \text{ is an upper bound of } A}
\end{gathered}$$

5. For $A, B \supseteq \mathbb{R}$, define

$$A + B = \{a + b : a \in A, b \in B\}$$
$$A \cdot B = \{a \cdot b : a \in A, b \in B\}$$

(i) Determine $\{3, 1, 0\} + \{2, 0, 2, 3\}$ and $\{3, 1, 0\} \cdot \{2, 0, 2, 3\}$.

$$\begin{aligned}
A &= \{3, 1, 0\} \\
B &= \{2, 0, 2, 3\} \\
A + B &= \{3+2, 3+0, 3+2, 3+3, 1+2, 1+0, 1+2, 1+3, 0+2, 0+0, 0+2, 0+3\} \\
&= \{5, 3, 6, 1, 4, 2, 0\} \\
A \cdot B &= \{3 \cdot 2, 3 \cdot 0, 3 \cdot 2, 3 \cdot 3, 1 \cdot 2, 1 \cdot 0, 1 \cdot 2, 1 \cdot 3, 0 \cdot 2, 0 \cdot 0, 0 \cdot 2, 0 \cdot 3\} \\
&= \{6, 0, 9, 2, 3\}
\end{aligned}$$

(*ii*) Assume that $\sup(A)$ and $\sup(B)$ exist. Prove that $\sup(A+B) = \sup(A) + \sup(B)$.

$$a \in A \Rightarrow a \le \sup(A)$$
$$b \in B \Rightarrow b \le \sup(B)$$
$$a + b \le \sup(A) + \sup(B)$$
$$c \in A + B \Rightarrow c = a + b, a \in A, b \in B$$
$$c \le \sup(A) + \sup(B)$$
$$\sup(A) + \sup(B) \text{ is an upper bound of } A + B$$
$$\text{Given } \epsilon > 0 :$$
$$\exists a \in A : a > \sup(A) - \epsilon$$
$$\exists b \in B : b > \sup(B) - \epsilon$$
$$a + b > \sup(A) + \sup(B) - 2\epsilon$$
$$2\epsilon > 0 \Rightarrow a + b > \sup(A) + \sup(B) - \epsilon$$
$$a + b \in A + B \Rightarrow \exists c \in A + B : c > (\sup(A) + \sup(B)) - \epsilon$$
$$\therefore$$
$$\boxed{\sup(A + B) = \sup(A) + \sup(B)}$$

(*iii*) Give an example of sets A, B where $\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)$.

$$A = \{-1\}, B = \{1, 2\}$$
$$\sup(A) = -1$$
$$\sup(B) = 2$$
$$\sup(A) \cdot \sup(B) = -2$$
$$A \cdot B = \{-1, -2\}$$
$$\sup(A \cdot B) = -1$$
$$-1 \neq -2$$
$$\boxed{\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)}$$

Notes

1. Question 1-ii proof by trichotomy

$$
\begin{aligned}
&|a-b| < \epsilon && \\
&-\epsilon < a-b < \epsilon && \text{FT abs-value} \\
&-\epsilon < a-b \land a-b < \epsilon && \\
&\epsilon > -(a-b) \land a-b < \epsilon && < \text{multiplicity} \\
&-(a-b) \notin P \land a-b \notin P && \\
&\Rightarrow a-b = 0 && \text{trichotomy} \\
&\therefore && \\
&\boxed{a=b} &&
\end{aligned}
$$