Homework 2

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1.

(i) Show that given finitely many countable sets $A_1, A_2, \cdots, A_n$, the set $A_1 \times A_2 \times \cdots \times A_n$ is also countable.

$$|A_k| = |\mathbb{N}| \Rightarrow \exists f_k : \mathbb{N} \to A_k \text{ is bijective}$$
$$f(x_1, x_2, \cdots, x_n) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n)) : x_1, x_2, \cdots, x_n \in \mathbb{N}$$
$$f : \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \to A_1 \times A_2 \times \cdots \times A_n \text{ is bijective}$$

$$h_2(x_1, x_2) = 2^{n-1}(2m-1) : x_1, x_2 \in \mathbb{N}$$
$$h_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ is bijective}$$
$$h_3(x_1, x_2, x_3) = h_2(h_2(x_1, x_2), x_3) : x_1, x_2, x_3 \in \mathbb{N} h_2(x_1, x_2)$$
$$h_3 : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ is bijective}$$
$$h_k(x_1, x_2, \cdots, x_k) = h_{k-1}(h_{k-1}(x_1, x_2, \cdots, x_{k-1}), x_k) : x_1, x_2, \cdots, x_k \in \mathbb{N}$$
$$h_k : \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \to \mathbb{N} \text{ is bijective}$$
$$\text{Let } g_n = h_n^{-1} \Rightarrow g_n : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \text{ is bijective}$$

$$f \circ g : \mathbb{N} \to A_1 \times A_2 \times \cdots \times A_n \text{ is bijective}$$
$$\therefore$$
$$\boxed{|A_1 \times A_2 \times \cdots \times A_n| = |\mathbb{N}|}$$

(ii) Is it true that given countably many countable sets $A_1, A_2, \cdots$, the set $A_1 \times A_2 \times \cdots$ is also countable? Justify your answer.

Assume bijection, all output pairs can be represented as: f_{nm} as in section (i)

$$\begin{matrix} (f_{11}(x_{11}) & f_{12}(x_{12}) & \cdots & f_{1m}(x_{1m}) & \cdots) \\ (f_{21}(x_{21}) & f_{22}(x_{22}) & \cdots & f_{2m}(x_{2m}) & \cdots) \\ (f_{n1}(x_{n1}) & f_{n2}(x_{n2}) & \cdots & f_{nm}(x_{nm}) & \cdots) \end{matrix} \quad \vdots$$

Consider the pair: ()