Homework 5

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1. Let $\sum x_n$ be an infinite series of positive terms. (**Do any three**)

(i) Show that $\sum x_n$ is convergent $\Rightarrow \sum x_n^2$ is convergent.

$$\sum x_n = l \Rightarrow \lim_{n\to\infty} x_n = 0$$
$$\exists N_\epsilon : n > N_\epsilon \Rightarrow x_n < \epsilon$$
$$\exists N_1 : n > N_1 \Rightarrow x_n < 1$$

$x_n \in P$ so $|x_n|$ not needed

$$\sum x_n^2 \Leftrightarrow \sum_{n=N_1}^{\infty} x_n^2$$
$$n > N_1 : x_n^2 < x_n \wedge \sum_{n=N_1}^{\infty} x_n \text{ is convergent}$$
$$\sum_{n=N_1}^{\infty} x_n^2 \text{ is convergent}$$
$$\therefore$$
$$\boxed{\sum x_n^2 \text{ is convergent}}$$

(ii) Show that $\sum x_n$ is convergent $\Rightarrow \sum \frac{x_n}{n}$ is convergent.

$$\frac{x_n}{n} < x_n \wedge \sum x_n \text{ is convergent}$$
$$\therefore$$
$$\boxed{\sum \frac{x_n}{n} \text{ is convergent}}$$

(iii) Show that $\sum x_n$ is convergent $\Rightarrow \sum \sqrt{x_n x_{n+1}}$ is convergent.

(iv) Show that $\sum x_n$ is convergent $\Rightarrow \sum \frac{x_n}{1+x_n}$ is convergent.

$$\frac{x_n}{1+x_n} < x_n \wedge \sum x_n \text{ is convergent}$$
$$\therefore$$
$$\boxed{\sum \frac{x_n}{1+x_n} \text{ is convergent}}$$

(v) Let $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Show that $\sum y_n$ is divergent.

2. Test the convergence of the following series (**Do any three**):

(i) $\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \cdots$

$$\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \cdots = \sum x_n : x_n = \frac{1}{1+2^{-n}}$$
$$\lim_{n\to\infty} x_n = 1 \neq 0$$
$$\therefore$$
$$\boxed{\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \cdots \text{ is divergent}}$$

(ii) $\frac{\ln 1}{1^p} + \frac{\ln 2}{2^p} + \frac{\ln 3}{3^p} + \frac{\ln 4}{4^p} + \cdots ; p \in \mathbb{R}$

(iii) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$

$$\begin{aligned} x_n &= \frac{n^2}{n!} \\ \frac{x_{n+1}}{x_n} &= \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} && \forall x_n \to x_n \in P \\ &= \frac{(n+1)^2}{n^2(n+1)} \\ &= \frac{n+1}{n^2} \to 0 \\ 0 &< 1 \end{aligned}$$
$$\therefore$$
$$\boxed{1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots \text{ is convergent}}$$

(iv) $\frac{1^p}{p^1} + \frac{2^p}{p^2} + \frac{3^p}{p^3} + \frac{4^p}{p^4} + \cdots ; p \in \mathbb{R}$

(v) $1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots ; x \in \mathbb{R}$

$$y_n = \frac{x^n}{n}$$

Case 1: $x = 0$

$$y_n = 0$$

$$\sum y_n = 0$$

$$\sum y_n \text{ is convergent}$$

Case 2: $x \in P$

$$\frac{y_{n+1}}{y_n} = \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}$$

$$= x \cdot \frac{n}{n+1} \to x$$

$$\sum y_n \text{ is convergent if } x < 1$$

$x = 1$ is harmonic

Case 3: $-x \in P$

Let $k = |x|$

$$y_n = (-1)^n \frac{k^n}{n}$$

$$= (-1)^n z_n$$

$$z_n = \frac{k^n}{n}$$

$$z_{n+1} \leq z_n$$

when z_n m.d. for AST

$$\Leftarrow \frac{k^{n+1}}{n+1} \leq \frac{k^n}{n}$$

$$\Leftarrow n \cdot k^{n+1} \leq (n+1)k^n$$

$$\Leftarrow nk \leq n+1$$

$$\Leftarrow k \leq \frac{n+1}{n} \to 1$$

$$\sum y_n \text{ is convergent if } x \geq -1$$

$$\therefore$$

$$\boxed{\sum y_n \text{ is convergent if } -1 \leq x < 1}$$

3.

(i) Let $\{x_n\}$ be a m.d. sequence. Prove that: $\sum x_n$ converges $\Rightarrow \{nx_n\} \to 0$.

$$\text{Given } \epsilon > 0$$

$$\exists N_\epsilon : \forall n > N_\epsilon \Rightarrow |x_n| < \epsilon$$

$$\exists N : \forall n > N, \forall p \in \mathbb{N} \Rightarrow |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \epsilon$$

$$\left| \sum_{k=1}^{p} x_{n+k} \right| < \epsilon$$

$$p|x_n| < \epsilon \qquad x_n \text{ m.d.}$$

$$|x_n| < \frac{\epsilon}{p}$$

$$|nx_n| < \epsilon$$

$$\Leftarrow \frac{n\epsilon}{p} < \epsilon$$

$$\Leftarrow p > n \qquad \text{p over all } \mathbb{N}, \text{ can} > n$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} nx_n = 0}$$

(ii) Show that the converse of (i) does not hold.

$$\boxed{x_n = \frac{1}{n\ln(n)}}$$

(iii) Show that (i) does not hold if *monotonic decreasing* is removed.

$$\boxed{x_n = \frac{(-1)^n}{n}}$$

$$\because$$

$$\sum x_n \text{ is convergent} \qquad \mathbf{2.}(v)$$

$$\lim_{n\to\infty} nx_n \text{ diverges}$$

4. **Cauchy's Condensation Test** says that if $f(n)$ is a monotone decreasing function of positive values, then $\sum f(n)$ is convergent $\Leftrightarrow$ $\sum 2^n f(2^n)$ is convergent.

(i) Use Cauchy Condensation Test to discuss the convergence of $\sum \frac{1}{n^p}$ for all values of p.

$$
\begin{aligned}
\sum \frac{1}{n^p} \text{ converges} &\Leftrightarrow \sum \frac{2^n}{2^{np}} \text{ converges} \\
&\Leftarrow \sum 2^{n-np} \\
&\Leftarrow \sum \left(\frac{1}{2}\right)^{n(p-1)} \\
&\Leftarrow \sum \left(\frac{1}{2}^{p-1}\right)^{n} \\
&\Leftarrow \frac{1}{2}^{p-1} < 1 && r \text{ of geo. series} \\
&\Leftarrow p-1 > 0 \\
&\Leftarrow p > 1 \\
&\therefore
\end{aligned}
$$

$$\boxed{\sum \frac{1}{n^p} \text{ is convergent if } p > 1}$$

(ii) Use Cauchy Condensation Test to discuss the convergence of $\sum \frac{1}{n(\ln n)^p}$ for all values of p.

$$
\begin{aligned}
\sum \frac{1}{n(\ln n)^p} \text{ converges} &\Leftrightarrow \sum \frac{2^n}{2^n(\ln(2^n))^p} \text{ converges} \\
&\Leftarrow \sum \frac{1}{(n \ln 2)^p} \\
&\Leftarrow \ln^{-p}(2) + \sum \frac{1}{n^p} \\
&\Leftarrow \sum \frac{1}{n^p} && \text{p-series} \\
&\therefore
\end{aligned}
$$

$$\boxed{\sum \frac{1}{n(\ln n)^p} \text{ is convergent if } p > 1}$$

5.

(*i*) Show that if $\sum x_n$ is abs. convergent and $\{y_n\}$ is a bounded sequence, then $\sum x_n y_n$ is abs. convergent.

$$\exists \alpha : \forall y_n \Rightarrow y_n \leq \alpha$$

$$\sum |x_n y_n| \leq \sum |x_n \alpha| = \alpha \sum |x_n|$$ constant times conv. series

$$\therefore$$

$$\boxed{\sum |x_n y_n| \text{ is abs. convergent}}$$

(*ii*) Use (*i*) to discuss the convergence of $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots$.

$$\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots = \sum \frac{1}{n^2} \cdot \frac{n+1}{n}$$

$$x_n = \frac{1}{n^2} \text{ is abs. convergent}$$ p-series

$$y_n = \frac{n+1}{n} \text{ bounded within } (1, 2]$$

$$\therefore$$

$$\boxed{\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots \text{ is abs. convergent}}$$

(*iii*) Use (*i*) to show that if both $\sum x_n$ and $\sum y_n$ are abs. convergent, then $\sum x_n y_n$ is abs. convergent.

$$\sum y_n \text{ abs. convergent} \Rightarrow |y_n| \text{ bounded}$$

$$\exists \alpha : \forall y_n \Rightarrow |y_n| \leq \alpha$$

$$\sum |x_n y_n| \leq \alpha \sum |x_n|$$

$$\therefore$$

$$\boxed{\sum x_n y_n \text{ is abs. convergent}}$$

(iv) Do (i) and (iii) still hold if we replace *abs. convergent* with *convergent*?

For part (i): Consider $x_n = \sum \frac{(-1)^n}{n}, y_n = (-1)^n$

x_n is convergent and y_n is bounded between [-1, 1]

$\sum x_n y_n$ diverges harmonic

$\therefore$

$\boxed{(i) \text{ does not hold}}$

For part (iii): $\exists \alpha, \beta : \forall y_n \Rightarrow \beta \leq y_n \leq \alpha$

$\exists a \in \{\alpha, \beta\} : \sum x_n y_n \leq \alpha \sum x_n$

$\alpha \sum x_n$ is convergent

$\therefore$

$\boxed{\sum x_n y_n \text{ is convergent}}$