Homework 3

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February 19, 2025

1. Use the definition of the limit of a sequence to show that:

(i) $\{\frac{n^2+n+1}{3n^2+1}\} \to \frac{1}{3}$

$$\left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon$$
$$\left|\frac{n+\frac{2}{3}}{3n^2+1}\right| < \epsilon$$
$$\frac{n+\frac{2}{3}}{3n^2+1} < \epsilon$$

positive for
$n \in \mathbb{N}$

$$\frac{n+\frac{2}{3}}{\epsilon} < 3n^2+1$$
$$\frac{1}{\epsilon'} < 3n^2+1$$
$$\frac{1-\epsilon'}{3\epsilon'} < n^2$$
$$n \geq \left\lceil\sqrt{\frac{1-\epsilon'}{3\epsilon'}}\right\rceil = N_\epsilon$$
$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow \left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon$$
$$\therefore$$
$$\boxed{\lim_{n\to\infty} \frac{n^2+n+1}{3n^2+1} = \frac{1}{3}}$$

$(ii) \quad \{10 - \frac{1}{\sqrt{n+\sqrt{n+5}}}\} \to 10$

$$\left|10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} - 10\right| < \epsilon$$

$$\left|-\frac{1}{\sqrt{n+\sqrt{n+5}}}\right| < \epsilon$$

$$\frac{1}{\sqrt{n+\sqrt{n+5}}} < \epsilon$$

always positive

$$\frac{1}{\sqrt{n+\sqrt{n+5}}} < \frac{1}{\sqrt{n}}$$

smaller denominator

$$\frac{1}{\sqrt{n}} < \epsilon$$

$$\sqrt{n} > \frac{1}{\epsilon}$$

$$n \geq \left\lceil \frac{1}{\epsilon^2} \right\rceil = N_\epsilon$$

$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow \left|10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} - 10\right| < \epsilon$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} 10 - \frac{1}{\sqrt{n+\sqrt{n+5}}} = 10}$$

2.

(i) Use the definition of the limit of a sequence to show that for a fixed r with $|r| < 1$, $\{nr^n\} \to 0$.

$$\text{Consider } \lim_{n\to\infty}\left|\frac{(n+1)r^{n+1}}{nr^n}\right| = |r|$$

$$\Leftarrow \left|\left|\frac{(n+1)r^{n+1}}{nr^n}\right| - |r|\right| < \epsilon$$

$$\Leftarrow \left|\left|\frac{(n+1)r^{n+1}}{nr^n}\right| - |r|\right| \le \left|\frac{(n+1)r^{n+1}}{nr^n} - r\right| < \epsilon \qquad \triangle\text{-ineq of} <$$

$$\Leftarrow \left|\frac{(n+1)r}{n} - r\right| < \epsilon$$

$$\Leftarrow \left|r + \frac{r}{n} - r\right| < \epsilon$$

$$\Leftarrow \left|\frac{r}{n}\right| < \epsilon$$

$$\Leftarrow n \ge \left\lceil \frac{|r|}{\epsilon} \right\rceil = N_\epsilon$$

$$\text{Almost-geometric sequence: } \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = r : 0 < r < 1, x_n = nr^n$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} nr^n = 0}$$

(ii) Use (i) to show that $\{\frac{\ln(n)}{n}\} \to 0$.

$$\left|\frac{\ln(n)}{n} - 0\right| < \epsilon$$

$$\ln(n) < n\epsilon$$

$$n > \frac{\ln(n)}{\epsilon}$$

$$n \ge \left\lceil \frac{1}{\epsilon'} \right\rceil = N_\epsilon \qquad \epsilon' \text{ arbitrarily small still}$$

$$\exists N_\epsilon : \forall n \ge N_\epsilon \Rightarrow \left|\frac{\ln(n)}{n}\right| < \epsilon$$

$$\therefore$$

$$\boxed{\lim_{n\to\infty} \frac{\ln(n)}{n} = 0}$$

3. Find the limits of the following sequences, if they exist:

(i) $x_n = \sqrt{n^2+n} - n$

$$|\sqrt{n^2+n} - n - \frac{1}{2}| < \epsilon$$
$$\left|\frac{n^2+n-(n+\frac{1}{2})^2}{\sqrt{n^2+n}+(n+\frac{1}{2})}\right| < \epsilon$$
$$\left|\frac{n^2+n-(n^2+n+\frac{1}{4})}{\sqrt{n^2+n}+n+\frac{1}{2}}\right| < \epsilon$$
$$\left|\frac{-\frac{1}{4}}{\sqrt{n^2+n}+n+\frac{1}{2}}\right| < \epsilon$$
$$\frac{\frac{1}{4}}{\sqrt{n^2+n}+n+\frac{1}{2}} < \epsilon$$
$$\Leftarrow \frac{1}{4n} < \epsilon$$
$$n \geq \left\lceil \frac{1}{4\epsilon} \right\rceil = N_\epsilon$$
$$\exists N_\epsilon : \forall n \geq N_\epsilon \Rightarrow |\sqrt{n^2+n} - n - \frac{1}{2}| < \epsilon$$
$$\therefore$$
$$\boxed{\lim_{n\to\infty} x_n = \frac{1}{2}}$$

(ii) $x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$

$$x_n = \sum_{x=1}^{n} \frac{1}{(n+x)^2}$$
$$0 < \sum_{x=1}^{n} \frac{1}{(n+x)^2} < \sum_{x=1}^{n} \frac{1}{x^2}$$
$$\lim_{n\to\infty} 0 = 0$$
$$\lim_{n\to\infty} \sum_{x=1}^{n} \frac{1}{x^2} = 0$$

proven in lecture notes

$$\therefore$$
$$\boxed{\lim_{n\to\infty} x_n = 0 \text{ by squeeze theorem}}$$

4. Discuss the convergence of the sequence $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}$ by proving:

(i) Show that $\{x_n\}$ is bounded.

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}=\sum_{x=1}^{n}\frac{1}{n+x}$$
$$\sum_{x=1}^{n}\frac{1}{n+x}<\sum_{x=1}^{n}\frac{1}{n}=1$$
$$\therefore$$
$$\boxed{\{x_n\}\text{ is upper bounded by 1}}$$

(ii) Show that $\{x_n\}$ is monotonic increasing.

$$x_{n+1}-x_n>0$$
$$\frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\cdots+\frac{1}{2(n+1)}-(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})>0$$
$$\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+2}-(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})>0$$
$$\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}>0$$
$$\frac{n+1}{4x^3+10x^2+8x+2}>0$$
$$n+1>0 \qquad \text{true for } n\in\mathbb{N}$$
$$\therefore$$
$$\boxed{\{x_n\}\text{ is monotonic increasing}}$$

(iii) Find the limit of $\{x_n\}$ by comparing it to an integral.

$$\int_1^2\frac{1}{x}dx=\lim_{n\to\infty}\sum_{x=1}^{n}\frac{2-1}{n}\cdot\frac{1}{1+x\frac{2-1}{n}}$$
$$=\lim_{n\to\infty}\sum_{x=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\frac{x}{n}}$$
$$=\lim_{n\to\infty}\sum_{x=1}^{n}\frac{1}{n+x}$$
$$\therefore$$
$$\boxed{\{x_n\}=\int_1^2\frac{1}{x}dx=\ln(2)}$$

5. Consider sequence $\{x_n\}$ such that $0 \leq x_1 < x_2$ and $x_n = \frac{x_{n-1}+x_{n-2}}{2}, \forall n \geq 3$. Show that $\{x_n\} \to \frac{x_1+2x_2}{3}$.

$$\begin{aligned}
x_n - x_{n-1} &= \frac{x_{n-1}+x_{n-2}}{2} - x_{n-1} \\
&= \frac{x_{n-2}-x_{n-1}}{2} \\
&= -\frac{1}{2}(x_{n-1} - x_{(n-1)-1}) \\
\Delta x_n &= -\frac{\Delta x_{n-1}}{2} \\
&= (x_2 - x_1)(-\frac{1}{2})^n && \text{recursive to } \Delta x_2 = x_2 - x_1 \\
x_n &= x_1 + \sum_{k=1}^{n-2}(x_2 - x_1)(-\frac{1}{2})^n \\
\lim_{n\to\infty} x_n &= x_1 + \frac{x_2 - x_1}{1 + \frac{1}{2}} && \text{geometric series to } \infty \\
&= x_1 + \frac{2x_2 - 2x_1}{3} \\
&= \boxed{\frac{x_1 + 2x_2}{3}}
\end{aligned}$$