Homework 5

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1. Let $\sum x_n$ be an infinite series of positive terms. (**Do any three**)

(*i*) Show that $\sum x_n$ is convergent $\Rightarrow \sum x_n^2$ is convergent.

$$\sum x_n = l \Rightarrow \lim_{n\to\infty} x_n = 0$$

$$\exists N_\epsilon : n > N_\epsilon \Rightarrow x_n < \epsilon \qquad x_n \in P \text{ so } |x_n| \text{ not needed}$$

$$\exists N_1 : n > N_1 \Rightarrow x_n < 1$$

$$\sum x_n^2 \Leftrightarrow \sum_{n=N_1}^{\infty} x_n^2$$

$$n > N_1 : x_n^2 < x_n \wedge \sum_{n=N_1}^{\infty} x_n \text{ is convergent}$$

$$\sum_{n=N_1}^{\infty} x_n^2 \text{ is convergent}$$

$$\therefore$$

$$\boxed{\sum x_n^2 \text{ is convergent}}$$

(*ii*) Show that $\sum x_n$ is convergent $\Rightarrow \sum \frac{x_n}{n}$ is convergent.

$$\frac{x_n}{n} < x_n \wedge \sum x_n \text{ is convergent}$$

$$\therefore$$

$$\boxed{\sum \frac{x_n}{n} \text{ is convergent}}$$

(*iii*) Show that $\sum x_n$ is convergent $\Rightarrow \sum \sqrt{x_n x_{n+1}}$ is convergent.

(*iv*) Show that $\sum x_n$ is convergent $\Rightarrow \sum \frac{x_n}{1+x_n}$ is convergent.

$$\frac{x_n}{1+x_n} < x_n \wedge \sum x_n \text{ is convergent}$$

$$\therefore$$

$$\boxed{\sum \frac{x_n}{1+x_n} \text{ is convergent}}$$

(*v*) Let $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Show that $\sum y_n$ is divergent.