# Math 310 - HW 5 (Due Friday 03/07)

(As always, first try some warm-up problems from the textbook !)

**1**. Let $\sum x_n$ be an infinite series of positive terms. (**Do any three**)

$(i)$ Show that $\sum x_n$ is convergent $\Rightarrow \sum x_n^2$ is convergent.

$(ii)$ Show that $\sum x_n$ is convergent $\Rightarrow \sum \frac{x_n}{n}$ is convergent.

$(iii)$ Show that $\sum x_n$ is convergent $\Rightarrow \sum \sqrt{x_n\, x_{n+1}}$ is convergent.

$(iv)$ Show that $\sum x_n$ is convergent $\Rightarrow \sum \frac{x_n}{1+x_n}$ is convergent.

$(v)$ Let $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Show that $\sum y_n$ is divergent.

**2**. Test the convergence of the following series (**Do any three**):

$(i)$ $\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \cdots$

$(ii)$ $\frac{\ln 1}{1^p} + \frac{\ln 2}{2^p} + \frac{\ln 3}{3^p} + \frac{\ln 4}{4^p} + \cdots$ ; $p \in \mathbb{R}$

$(iii)$ $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$

$(iv)$ $\frac{1^p}{p^1} + \frac{2^p}{p^2} + \frac{3^p}{p^3} + \frac{4^p}{p^4} + \cdots$ ; $p \in \mathbb{R}$

$(v)$ $1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ ; $x \in \mathbb{R}$

**3**. $(i)$ Let $\{x_n\}$ be a monotonic decreasing sequence. Prove that:

$$\sum x_n \text{ converges} \Rightarrow \{nx_n\} \to 0$$

(*Hint*: Use Cauchy criterion. *Remark*: This is a strengthening of the Divergence Test under a special case.)

$(ii)$ Show that the converse of $(i)$ does not hold. (*Hint*: A counter-example lies somewhere in this homework.)

$(iii)$ Show that $(i)$ does not hold if *monotonic decreasing* is removed. (*Hint*: Same hint as in $(ii)$.)

**4**. **Cauchy's Condensation Test** says that if $f(n)$ is a monotone decreasing function of positive values, then

$$\sum f(n) \text{ is convergent} \Leftrightarrow \sum 2^n f(2^n) \text{ is convergent}$$

$(i)$ Use Cauchy Condensation Test to discuss the convergence of $\sum \frac{1}{n^p}$ for all values of $p$.

$(ii)$ Use Cauchy Condensation Test to discuss the convergence of $\sum \frac{1}{n(\ln n)^p}$ for all values of $p$.

(*Remark*: You may even try to prove the test yourself or look at other sources. It involves Grouping and Comparison Test similar to what we did for Harmonic series. Part $(ii)$ can also be done using the Integral Test.)

**5**. $(i)$ Show that if $\sum x_n$ is abs. convergent and $\{y_n\}$ is a bounded sequence, then $\sum x_n y_n$ is abs. convergent.

$(ii)$ Use $(i)$ to discuss the convergence of $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots$.

$(iii)$ Use $(i)$ to show that if both $\sum x_n$ and $\sum y_n$ are abs. convergent, then $\sum x_n y_n$ is abs. convergent.

$(iv)$ Do $(i)$ and $(iii)$ still hold if we replace *abs. convergent* by *convergent* ? (*Hint*: Somewhere in this homework. *Remark*: You may later want to look at **Abel's Test** (Problem 6.4.4) in our textbook.)

**6**. [**Extra-Credit**] Consider the **alternating harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

(*i*) Show that $\{s_{2n}\} \to \ln 2$ (*Hint*: Use that if $\gamma_n = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - \ln n$, then by HW 4 #3 $\{\gamma_n\} \to \gamma$)

(*ii*) Show that $\{s_{2n+1}\} \to \ln 2$ and thus the alternating harmonic series converges to $\boxed{\ln 2}$

Now consider a **re-arranged alternating harmonic series**

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$$

(*iii*) Show that $\{t_{3n}\} \to \frac{1}{2} \ln 2$ (*Hint*: Once again you may use the idea of $\gamma_n$ here)

(*iv*) Show that $\{t_{3n+1}\} \to \frac{1}{2} \ln 2$ and $\{t_{3n+2}\} \to \frac{1}{2} \ln 2$ and thus the re-arranged alternating harmonic series converges to $\boxed{\frac{1}{2} \ln 2}$

$\Big($*Remark*: For fun (non-credit) try to show that $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} - \cdots = \boxed{0}$$\Big)$