Math 310 - HW 4 (Due Wednesday 02/26)

(All problems are Extra Credit)

1. Consider the following *continued fraction*:

$$1+\cfrac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$$

(i) Write the above continued fraction as the limit of a sequence. Then write a (first-order) recurrence relation between the terms of the sequence.

(ii) Show that the sequence is bounded.

(iii) Show that the subsequence of odd-indexed terms and even-indexed terms are monotonic.

(iv) Show that the above continued fraction converges and find the limit.

2. Given $x_1 = 2\sqrt{3}$, $y_1 = 3$, two sequences $\{x_n\}, \{y_n\}$ are defined by:

$$x_{n+1} = \frac{2x_n y_n}{x_n + y_n} \quad , \quad y_{n+1} = \sqrt{x_{n+1} y_n}$$

(i) Prove that

$\{x_n\}$ is monotonic decreasing and bounded below

$\{y_n\}$ is monotonic increasing and bounded above

$y_n < x_n$ for all n

(*Hint*: First try Not-for-credit #1. Here you may use (loaded) induction.)

(ii) Show that they both converge to the same limit (say ℓ), by showing that $0 < x_{n+1} - y_{n+1} < \frac{x_1 - y_1}{2^n}$

(iii) Find any good bounds for ℓ and guess what its exact value could be.

3. Consider the two sequences:

$$u_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln n \quad , \quad v_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) - \ln n$$

(i) Show that $\frac{1}{n+1} < \ln(1 + \frac{1}{n}) < \frac{1}{n}$ for all n (*Hint*: Use your knowledge of Taylor expansions of e^x and $\ln(1 \pm x)$ from Calculus)

(ii) Use (i) to show that $\{u_n\}$ is bounded from below and $\{v_n\}$ is bounded from above.

(iii) Use (i) to show that $\{u_n\}$ is monotonic decreasing and $\{v_n\}$ is monotonic increasing.

(iv) Show that $\{u_n\}$ and $\{v_n\}$ both converge to the same limit. Find any good numerical bounds for this limit. Explain this graphically. What is this limit called in Math literature ?

4. (i) Given $x_n > 0, \forall n$ and $\left\{\frac{x_{n+1}}{x_n}\right\} \to \ell$ $(\ell \neq 0)$, show that $\left\{\sqrt[n]{x_n}\right\} \to \ell$ (*Hint*: Use one of Cauchy's theorems from class.)

(ii) Use (i) to find the limit of $\left\{\frac{\sqrt[n]{n!}}{n}\right\}$ (*Remark*: Note that gives us some idea about the growth of the *factorial* function. To know even more, one has to look into *Stirling's Approximation*)

Not for Credit:

1. Given $0 < y_1 < x_1$, two sequences $\{x_n\}, \{y_n\}$ are defined by:

$$x_{n+1} = \frac{x_n + y_n}{2} \quad , \quad y_{n+1} = \sqrt{x_n y_n}$$

Prove that both sequences are monotonic and then show that they both converge to the same limit.