Homework 2

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September 21, 2015

Exercise 5.2

i Because $a+b \geq 2\sqrt{ab}$ for a,b $\in (0,+\infty)$, so we have

$$f(\sigma) = pe^{\sigma} + qe^{\sigma} \geq 2\sqrt{pe^{\sigma}qe^{-\sigma}} = 2\sqrt{p(1-p)} \quad for \quad p \quad \in (\frac{1}{2}, 1)$$

when $p = q = \frac{1}{2}$ $\quad f(\sigma)$ get the minimum

$Min_{f(\sigma)} = 1$

so $f(\sigma) \geq 1, \quad for \quad all \quad \sigma \leq 0$

ii WTS $\mathbb{E}_n S_{n+1} = S_n$

$$\mathbb{E}_n S_{n+1} = \mathbb{E}_n S_n e^{\frac{\sigma X_{n+1}}{f(\sigma)}} = \frac{S_n}{pe^{\sigma} + qe^{\sigma}} \mathbb{E}_n e^{\sigma X_{n+1}} = \frac{S_n}{pe^{\sigma} + qe^{\sigma}}(pe^{\sigma} + qe^{\sigma}) = S_n$$

"take out what is known" "independence"

iii

For martingale stopped at a stopping time is still a martingale, and thus has constant expectation. The process

$$S_n = e^{\sigma M_n (\frac{1}{f(\sigma)})^n}$$

is a martingale.

so

$$\mathbb{E}[S_0] = \mathbb{E}[S_{n \wedge \tau_1}] = \mathbb{E}[e^{\sigma M_{n \wedge \tau_1} (\frac{1}{f(\sigma)})^{n \wedge \tau_1}}] = 1$$

as for $e^{\sigma M_{n\wedge\tau_1}}$

$$0 \le e^{\sigma M_{n\wedge\tau_1}} \le e^{\sigma m}$$

and for $\tau_m < \infty$

$$\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} = e^{\sigma}$$

as for $(\frac{1}{f(\sigma)})^{n\wedge\tau_1}$

$$\lim_{n\to\infty} = (\frac{1}{f(\sigma)})^n \begin{cases} (\frac{1}{pe^{\sigma}+qe^{-\sigma}})^{\tau_1}, & \tau_1 < \infty \\ 0, & \tau_1 = \infty \end{cases}$$

to sum up

$$\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}(\frac{1}{f(\sigma)})^{n\wedge\tau_1}} = \mathbb{I}_{\{\tau_1<\infty\}} e^{\sigma}(\frac{1}{pe^{\sigma}+qe^{-\sigma}})^{\tau_1}$$

take the limit as n $\to \infty$ in $\mathbb{E}[e^{\sigma M_{n\wedge\tau_1}(\frac{1}{f(\sigma)})^{n\wedge\tau_1}}] = 1$ and obtain

$$\mathbb{E}[\mathbb{I}_{\{\tau_1<\infty\}} e^{\sigma}(\frac{1}{pe^{\sigma}+qe^{-\sigma}})^{\tau_1}] = 1$$

so

$$\mathbb{E}[\mathbb{I}_{\{\tau_1<\infty\}}(\frac{1}{pe^{\sigma}+qe^{-\sigma}})^{\tau_1}] = e^{-\sigma}$$

when we computer the limit of both side as $\sigma \to 0$,we get $\mathbb{P}\tau_m < \infty = 1$

iv

set $\alpha = \frac{1}{pe^{\sigma}+qe^{-\sigma}}$, we have

$$\begin{aligned} &\alpha pe^{\sigma} + \alpha qe^{-\sigma} = 1 \\ &\alpha q(e^{-\sigma})^2 - e^{-\sigma} + \alpha p = 0 \\ &e^{-\alpha} = \frac{1 \pm \sqrt{1-4\alpha^2 pq}}{2\alpha q} \end{aligned}$$

Because $\sigma > 0$, we take

$$e^{-\alpha} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

in (iii) we have

$$\mathbb{E}[e^{\sigma}(\frac{1}{f(\sigma)})^{\tau_1}] = 1$$

so we have

$$\mathbb{E}[\alpha^{\tau_1}] = e^{-\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

v

$$\mathbb{E}\alpha^{\tau} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

Differentiating product with respect to α lead to

$$\begin{aligned}
\mathbb{E}\tau_1 \alpha^{\tau_1 - 1} &= \frac{-\frac{1}{2}\sqrt{1 - 4\alpha^2 pq}(-8\alpha qp)(2\alpha q) - 2q(1 - \sqrt{1 - 4\alpha^2 pq})}{4\alpha^2 q^2} \\
&= \frac{8\alpha^2 pq - (\sqrt{1 - 4\alpha^2 pq} - 1 + 4\alpha^2 pq)2q}{4\alpha^2 q^2 \sqrt{1 - 4\alpha^2 pq}} \\
&= \frac{4\alpha^2 p - \sqrt{1 - 4\alpha^2 pq} + 1 - 4\alpha^2 pq}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}} \\
&= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}}
\end{aligned}$$

letting $\alpha \to 1$, we get

$$\mathbb{E}[\tau_1] = \frac{1 - \sqrt{1 - 4pq}}{2q\sqrt{1 - 4pq}} = \frac{1 - \sqrt{(p+q)^2 - 4pq}}{3q\sqrt{(p+q)^2 - 4pq}} = \frac{1 - (p - q)}{2q(p - q)} = \frac{1}{p - q}$$

Exercise 3.2

For $0 \leq s \leq t$, WTS $\mathbb{E}[W^2(t) - t|\mathbb{F}_s] = W^2(s) - s$

$$\mathbb{E}[W^2(t) - t|\mathbb{F}_s] = \mathbb{E}[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t|\mathbb{F}_s]$$

Because $W(s)$ is $\mathbb{F}_s$ measurable, using "take our what is known" and "linearity of conditional expectations"

$$\mathbb{E}[W^2(t) - t|\mathbb{F}_s] = \mathbb{E}[(W(t) - W(s))^2|\mathbb{F}_s] + 2W(s)E[W(t)|\mathbb{F}_s] - W^2(s) - t$$

Because $W(t) - W(s)$ is independent with W(s), using "independence" to drop off the filtration, and W(t) is a martingale, so $\mathbb{E}[W(t)|\mathbb{F}_s] = W(s)$

$$\mathbb{E}[W^2(t) - t|\mathbb{F}_s] = \mathbb{E}[(W(t) - W(s))^2] + 2W^2(s) - W^2(s) - t$$

because$\mathbb{E}[W(t) - W(s)] = 0$ we have

$$\begin{aligned}
\mathbb{E}[W^2(t) - t|\mathbb{F}_s] &= \mathbb{E}[(W(t) - W(s))^2] - \mathbb{E}^2[W(t) - W(s)] + 2W^2(s) - W^2(s) - t \\
&= Var[W(t) - W(s)] + W^2(s) - t \\
&= t - s + W^2(s) - t \\
&= W^2(s) - s
\end{aligned}$$

so $W^2(t) - t$ is a martingale

Exercise 3.3

$$\begin{aligned}
\varphi^{(3)}(u) = \mathbb{E}[(X-\mu)^3 e^{u(X-\mu)}] &= \sigma^4 u e^{\frac{1}{2}\sigma^2u^2} + 2\sigma^4 u e^{\frac{1}{2}\sigma^2u^2} + \sigma^6 u^3 e^{\frac{1}{2}\sigma^2u^2} \\
&= (3\sigma^4 u + \sigma^6 u^3) e^{\frac{1}{2}\sigma^2u^2}
\end{aligned}$$

$$\varphi^{(3)}(0) = \mathbb{E}[(X-\mu)^3] = 0$$

$$\begin{aligned}
\varphi^{(4)}(u) = \mathbb{E}[(X-\mu)^4 e^{u(X-\mu)}] &= 3\sigma^4 e^{\frac{1}{2}\sigma^2u^2} + 3\sigma^6 u^2 e^{\frac{1}{2}\sigma^2u^2} + 3\sigma^6 u^2 e^{\frac{1}{2}\sigma^2u^2} + \sigma^8 u^4 e^{\frac{1}{2}\sigma^2u^2} \\
&= (3\sigma^4 + 6\sigma^6 u^2 + \sigma^8 u^4) e^{\frac{1}{2}\sigma^2u^2}
\end{aligned}$$

$$\mathbb{E}[(X-\mu)^4] = \varphi^{(4)}(0) = 3\sigma^4$$

Exercise 3.4 i

As hint shows

$$\sum_{j=0}^{n-1}(W_{(t_{j+1})} - W_{(t_j)})^2 \leq \max_{0\leq k\leq n-1}|W_{(t_{k+1})} - W_{t_k}| \sum_{j=0}^{n-1}|W_{(t_{(j+1)})} - W_{t_j}|$$

$$\sum_{j=0}^{n-1}|W_{(t_{(j+1)})} - W_{t_j}| \geq \frac{\sum_{j=0}^{n-1}(W_{(t_{j+1})} - W_{(t_j)})^2}{\max_{0\leq k\leq n-1}|W_{(t_{k+1})} - W_{t_k}|}$$

we compute the limit of both sides as $||\pi|| \to 0$, we get

$$\lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W_{(t_{(j+1)})} - W_{t_j}| \geq \frac{\lim_{||\pi||\to 0} \sum_{j=0}^{n-1} (W_{(t_{j+1})} - W_{(t_j)})^2}{\lim_{||\pi||\to 0} \max_{0\leq k\leq n-1} |W_{(t_{k+1})} - W_{t_k}|}$$

As for numerator

$$\begin{aligned}
\lim_{||\pi||\to 0} \sum_{j=0}^{n-1} (W_{(t_{j+1})} - W_{(t_j)})^2 &= \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} Var[W_{(t_{j+1})} - W_{(t_j)}] \\
&= \lim_{||\pi||\to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j) \\
&= t_n \leq \infty
\end{aligned}$$

As for denominator

$$\lim_{||\pi||\to 0} \max_{0\leq k\leq n-1} |W_{(t_{k+1})} - W_{t_k}| \doteq \lim_{||\pi||\to 0} \mathbb{E}[W(t_{j+1}) - W(t_j)] = 0$$

so

$$\lim_{||\pi||\to 0} \sum_{j=0}^{n-1} |W_{(t_{(j+1)})} - W_{t_j}| \geq \frac{\lim_{||\pi||\to 0} \sum_{j=0}^{n-1} (W_{(t_{j+1})} - W_{(t_j)})^2}{\lim_{||\pi||\to 0} \max_{0\leq k\leq n-1} |W_{(t_{k+1})} - W_{t_k}|} = \infty$$

ii

from hint, we get

$$\sum_{j=0}^{n-1} (W_{(t_{j+1})} - W_{(t_j)})^3 \leq \max_{0\leq k\leq n-1} |W_{(t_{k+1})} - W_{t_k}| \sum_{j=0}^{n-1} (W_{(t_{(j+1)})} - W_{t_j})^2$$

as show in (i)

$$\lim_{||\pi||\to 0} \max_{0\leq k\leq n-1} |W_{(t_{k+1})} - W_{t_k}| \doteq \lim_{||\pi||\to 0} \mathbb{E}[W(t_{j+1}) - W(t_j)] = 0$$

$$\lim_{||\pi||\to 0}\sum_{j=0}^{n-1}(W_{(t_{j+1})}-W_{(t_j)})^2 = \lim_{||\pi||\to 0}\sum_{j=0}^{n-1}Var[W_{(t_{j+1})}-W_{(t_j)}]$$
$$= \lim_{||\pi||\to 0}\sum_{j=0}^{n-1}(t_{j+1}-t_j)$$
$$= t_n \leq \infty$$

so

$$\sum_{j=0}^{n-1}(W_{(t_{j+1})}-W_{(t_j)})^3 \leq \max_{0\leq k\leq n-1}|W_{(t_{k+1})}-W_{t_k}|\sum_{j=0}^{n-1}(W_{(t_{(j+1)})}-W_{t_j})^2 = t_n * 0 = 0$$

Exercise 3.6

i Want to show :

$$\mathbb{E}[f(X(t))|\mathbb{F}(s)] = g(X(s))$$

$$\mathbb{E}[f(X(t)-X(s)+X(s)|\mathbb{F}(s)]$$

Because X(s) is $\mathbb{F}(s)$ measurable, we set X(s) as a dummy variable x

And set

$$g(x) = \mathbb{E}[f(X(t)-X(s)+x|\mathbb{F}(s)]$$

$$X(t)-X(s) = W(t)-W(s)+\mu(t-s)$$

is normally distributed with mean $\mathbb{E} = \mu(t-s)$ and variance $t-s$

So

$$g(x) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(w-\mu(t-s))^2}{2(t-s)^2}}f(w+x)dw$$

set y=w+x so dy=dw

$$g(x)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(y-x-\mu(t-s))^2}{2(t-s)^2}}f(y)dy$$

set

$$\tau = t - s \quad and \quad \mathbb{P}(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{\frac{(y-x-\mu\tau)^2}{2\tau}}$$

so

$$g(x) = \int_{-\infty}^{+\infty} f(y)\mathbb{P}(\tau, x, y)dy$$

that is

$$\mathbb{E}[f(X(t))|\mathbb{F}(s)] = g(X(s))$$

hence X has the Markov property

ii

$$\mathbb{E}[f(S(t) * S(s)/S(s))\mathbb{F}(s)]$$

S(s) is $\mathbb{F}(s)$ measurable, so set S(s) as a dummy variable x, and set

$$g(x) = \mathbb{E}[f(\frac{S(t)}{S(s)}x)]$$

$$\frac{S(t)}{S(s)} = e^{\sigma[W(t)-W(s)]+\nu(t-s)}$$

$\sigma[W(t) - W(s)] + \nu(t - s)$ is normally distributed with mean $\nu(t - s)$ and variance $\sigma^2(t - s)$

So $\frac{S(t)}{S(s)}$ is log-normally distributed

$$g(x) = \int_0^{+\infty} \frac{1}{w\sqrt{2\pi\sigma^2(t-s)}} e^{-\frac{(ln(w)-\nu(t-s))^2}{2\sigma^2(t-s)}} f(xw)dw$$

set y=xw, so dy=xdw, $\tau = t - s$

$$g(x) = \int_0^{+\infty} \frac{1}{w\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(ln(\frac{y}{x})-\nu\tau)^2}{2\sigma^2\tau}} f(y)dy$$

set

$$\mathbb{P}(\tau, x, y) = \frac{1}{\sigma y\sqrt{2\pi\tau}} e^{-\frac{(ln(y/x)-\nu\tau)^2}{2\sigma^2\tau}}$$

so

$$g(x) = \int_0^{+\infty} f(y)\mathbb{P}(\tau, x, y)dy$$

that is

$$\mathbb{E}[f(S(t))|\mathbb{F}(s)] = g(S(s))$$

hence S has the Markov property

Exercise 3.7 i

$$\mathbb{E}[Z(t)|\mathbb{F}(s)] = \mathbb{E}[Z(s)e^{(\sigma(\mu(t-s)+W(t)-W(s))-(\sigma\mu+\frac{1}{2}\sigma)(t-s)}|\mathbb{F}(s)]$$

Z(s) is $\mathbb{F}(s)$ measurable, use "take out what is known"

$$\mathbb{E}[Z(t)|\mathbb{F}(s)] = Z(s)\mathbb{E}[e^{\sigma(W(t)-W(s))-\frac{1}{2}(t-s)}|\mathbb{F}(s)]$$

$W(t) - W(s)$ is independent with $W(s)$, use "independence" to drop off $\mathbb{F}(s)$

$$\mathbb{E}[Z(t)|\mathbb{F}(s)] = Z(s)\mathbb{E}[e^{(\sigma(W(t)-W(s))-\frac{1}{2}\sigma^2(t-s))}]$$

use "take out what is known" we have

$$\mathbb{E}[Z(t)|\mho(s)] = \frac{Z(s)e^{(\sigma(W(t)-W(s)))}}{e^{-\frac{1}{2}\sigma^2(t-s)}}$$

$W(t) - W(s)$ is normally distributed with mean 0 and variance t-s

Because $\mathbb{E}[e^{ux}] = e^{\frac{1}{2}u^2t}$, x with mean 0 and variance t

so

$$\mathbb{E}[e^{\sigma(W(t)-W(s))}] = e^{\frac{1}{2}\sigma^2(t-s)}$$

so

$$\mathbb{E}[Z(t)|\mathbb{F}(s)] = Z(s)$$

is a martingale

ii

Z(t) is martingale, $0 \le t < \infty$, $\tau_m = Min\{t \ge 0; X(t) = m\}$

for martingale that is stopped at stopping time is still a martingale, thus has constant expectation

$$1 = Z(0) = \mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}[e^{\sigma X(t\wedge\tau_m)-(\sigma\mu+\frac{1}{2}\sigma^2)(t\wedge\tau_m)}] = 1$$

iii

for $m > 0$ and $\sigma > 0$, Brownian motion is always at or below level m for $t \geq \tau_m$

so $0 \leq e^{\sigma W(t\wedge\tau_m)} \leq e^{\sigma m}$

if $\tau_m < \infty$ and large enough,

$$e^{-(\sigma\mu+\frac{1}{2})(t\wedge\tau_m)} = e^{-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m}$$

if $\tau_m = \infty$,

$$e^{-(\sigma\mu+\frac{1}{2})(t\wedge\tau_m)} = e^{-(\sigma\mu+\frac{1}{2}\sigma^2)t}$$

as $t \to \infty$, it converges to 0

so

$$\lim_{t\to\infty} e^{-(\sigma\mu+\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = \mathbb{I}_{\{\tau_m<\infty\}} e^{-(\sigma\mu+\frac{1}{2}\sigma)\tau_m}$$

when

$$\tau_m < \infty, \quad e^{\sigma W(t\wedge\tau_m)} = e^{\sigma} m$$

when

$$\tau_m = \infty, \quad e^{\sigma W(t\wedge\tau_m)} \leq e^{\sigma} m < \infty$$

so

$$\lim_{t\to\infty} e^{-(\sigma\mu+\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = 0$$

so we have

$$e^{\sigma W(t\wedge\tau_m)-(\sigma\mu+\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = \mathbb{I}_{\{\tau_m<\infty\}} e^{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m}$$

for Z(t) has constant expectation, so

$$1 = \mathbb{E}[Z(t \wedge \tau)] = \mathbb{E}[\mathbb{I}_{\{\tau_m < \infty\}} e^{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m}]$$

$$\mathbb{E}[\mathbb{E}[\mathbb{I}_{\{\tau_m < \infty\}} e^{-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m}]] = e^{-\sigma m}$$

take limit as both side $\sigma to 0$

$$\mathbb{E}[\mathbb{I}_{\{\tau_m < \infty\}}] = 1$$

so

$$\mathbb{P}\{\mathbb{I}_{\tau_m < \infty}\} = 1$$

set $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$

$$\frac{1}{2}\sigma^2 + \sigma\mu - \alpha = 0 \qquad \sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$$

because $\sigma > 0$ so $\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}$

so

$$\mathbb{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha^2 + \mu^2}}$$

for all $\sigma > 0$

iv

$$\mathbb{E}e^{-\alpha\tau_m} = e^m\mu - m\sqrt{2\alpha + \mu^2}$$

Differentiate it with respect to x

$$\mathbb{E} - \tau_m e^{-\alpha\tau_m} = -\frac{2m}{2\sqrt{2\alpha + \mu^2}} e^{m\mu - m\sqrt{2\alpha + \mu^2}}$$

$$\mathbb{E}\tau_m e^{-\alpha\tau_m} = \frac{m}{\sqrt{2\alpha + \mu^2}} e^{m\mu - m\sqrt{2\alpha + \mu^2}}$$

take both side as $\alpha \to 0$

$$\mathbb{E}[\tau_m] = \frac{m}{\mu}$$

v

for $\sigma > -2\mu$, $\sigma\mu + \frac{1}{2}\sigma^2 > \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2 = 0$

so
$$\lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m)-(\sigma\mu+\frac{1}{2})(t\wedge\tau_m)} = \mathbb{I}_{\{\tau_m<\infty\}}e^{\sigma m-(\sigma\mu-\frac{1}{2}\sigma^2)\tau}$$
is still valid

so
$$\mathbb{E}[e^{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau}\mathbb{I}_{\{\tau_m<\infty\}}] = 1$$
is still hold

take the limit for $\sigma \to -2\mu$, we have
$$\mathbb{E}[\mathbb{I}_{\{\tau_m<\infty\}}] = e^{-\sigma m} = e^{2\mu m}$$

for $m > 0$, $\mu < 0$, so $e^{2\mu m} < 1$

set$\alpha = \sigma\mu + \frac{1}{2}\sigma^2$, so we have $\frac{1}{2}\sigma^2 + \mu\sigma - \alpha = 0$
$$\sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$$

because $\sigma > 0$, we choose $\sigma = mu + \sqrt{\mu^2 + 2\alpha}$
$$\mathbb{E}[e^{-\alpha\tau_m}\mathbb{I}_{\{\tau_m<\infty\}}] = e^{m\mu - m\sqrt{\mu^2+2\alpha}}$$

when $\tau_m = \infty$, $e^{\alpha\tau_m} = 0$

so drop the condition and get
$$e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{\mu^2+2\sigma}}$$
for all α

Extra a If W and B are independent Brownian Motions then the average of W and B given by $X_t = (\frac{1}{2})(W_t + B_t)$ is again a Brownian Motion

False WTS:If X is a Brownian Motion, then for all $0 = t_0 < t_1 < ... < t_m$, the increments $X(t_1) = X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_m) - X(t_{m-1})$ are independent. And each of these increments is normally distributed with mean 0 and variance $t_{i+1} - t_i$

set $0 < t_i < t_j < t_k$

because W and B are independent Brownian Motions, so $W(t_j) - W(t_i)$ and $B(t_j) - B(t_i)$ is $\mathbb{F}(t_j)$ measurable, so $X(t_j) - X(t_i)$ is $\mathbb{F}(t_j)$ measurable also we have $W(t_k) - W(t_j)$ and $B(t_k) - B(t_j)$ is not $\mathbb{F}(t_j)$ measurable, that is $X(t_k) - X(t_j)$ is not $\mathbb{F}(t_j)$ measurable

so we have prove the increments of X are independent

$$\mathbb{E}[X(t_k) - X(t_j)] = \mathbb{E}[\frac{1}{2}(W(t_k) - W(t_j) + B(t_k) - B(t_j))]$$

because W and X are independent, we have:

$$\mathbb{E}[X(t_k) - X(t_j)] = \frac{1}{2}\mathbb{E}[W(t_k) - W(t_j) + \frac{1}{2}\mathbb{E}[B(t_k) - B(t_j)] = \frac{1}{2}(0+0) = 0$$

$$\begin{aligned} Var[X(t_k) - X(t_j)] &= \mathbb{E}[(X(t_k) - X(t_j))^2] - (\mathbb{E}[X(t_k) - X(t_j)])^2 \\ &= \mathbb{E}[(X(t_k) - X(t_j))^2] \\ &= \mathbb{E}[(\frac{1}{2}(W(t_k) - W(t_j)) - \frac{1}{2}(B(t_k) - B(t_j)))^2] \\ &= \frac{1}{4}(\mathbb{E}[(W(t_k) - W(t_j))^2] + \mathbb{E}[(B(t_k) - B(t_j))^2] \\ &\quad - 2\mathbb{E}[(W(t_k) - W(t_j))]\mathbb{E}[(B(t_k) - B(t_j))]) \\ &= \frac{1}{4}(Var[W(t_k) - W(t_j)] + Var[B(t_k) - B(t_j)]) \\ &= \frac{1}{2}(t_k - t_j) \neq t_k - t_j \end{aligned}$$

so X is not a Brownian Motion

Extra b If X and Y are martingales then the average of X and Y give by $Z_t = \frac{1}{2}(X_t + Y_t)$ is again a martingale

True set $s < t$, WTS: for $Z_t = \frac{1}{2}(X_t + Y_t)$ $\mathbb{E}[Z_t|\mathbb{F}(s)] = Z_s$

$$\mathbb{E}[Z_t|\mathbb{F}(s)] = \mathbb{E}[\frac{1}{2}(X_t + Y_t)|\mathbb{F}(s)]$$

use"Linearity of conditional expectation"

$$\mathbb{E}[Z_t|\mathbb{F}(s)] = \mathbb{E}[\frac{1}{2}X_t|\mathbb{F}(s)] + \mathbb{E}[\frac{1}{2}Y_t|\mathbb{F}(s)]$$

Because X and Y are martingales, so we have

$$\mathbb{E}[Z_t|\mathbb{F}(s)] = \frac{1}{2}X_s + \frac{1}{2}Y_s = Z_s$$

So, Z is a martingale

Extra c If X has finite, non-zero quadratic variation:i.e. $0 < [X, X] < \infty$ then X has infinite first variation:i.e. $FV(X) = \infty$

False

$$f(x) = \begin{cases} 1, & ,0 < x < 1 \\ 2, & 1 \leq x \end{cases}$$

its quadratic variation:

$$[f, f] = \lim_{||\pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

when $t_{j+1} \to 1 + \epsilon$ **and**$t_j \to 1 - \epsilon$**,** $\epsilon \to 0$**, we have** $[f, f] = 1 < \infty$

also we can have its first variation $FV(f) = \lim_{||\pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)] = 1 < \infty$