Homework 5

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Exercise 5.1

(i)

Let $f(t,x) = S(0)e^x$, we have:

$$\frac{\partial f}{\partial t} = 0 \quad \frac{\partial f}{\partial x} = f(t,x) \quad \frac{\partial^2 f}{\partial x^2} = f(t,x)$$

$$\begin{aligned}
&X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s))ds \\
&dX(t) = (\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t) \\
&(dX(t))^2 = \sigma^2(t)dt
\end{aligned}$$

Use Ito formula

$$\begin{aligned}
d(D(t)S(t)) &= d(f(t, X(t))) \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 \\
&= D(t)S(t)dX(t) + \frac{1}{2}D(t)S(t)(dX(t))^2 \\
&= D(t)S(t)(\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t)) + \frac{1}{2}D(t)S(t)\sigma^2(t)dt \\
&= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t)
\end{aligned}$$

(ii)

We have:

$$\begin{aligned}
&dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) \\
&dD(t) = -R(t)D(t)dt \\
&dS(t)dD(t) = 0
\end{aligned}$$

Use Ito product rule

$$\begin{aligned}
d(D(t)S(t)) =& S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\
&= S(t)(-R(t)D(t)dt) + D(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) \\
&= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t)
\end{aligned}$$

Exercise 5.2

For Lemma 5.2.2 and 5.2.30 we have

$$\begin{aligned}
\mathbb{E}[Y|\mathbb{F}(s)] &= \frac{\tilde{1}}{Z(s)}\mathbb{E}[YZ(t)|\mathbb{F}(s)] \\
D(t)V(t) &= \tilde{\mathbb{E}}[D(T)V(T)|\mathbb{F}(t)]
\end{aligned}$$

So

$$\begin{aligned}
&D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathbb{F}(t)] \\
&= \frac{1}{Z(t)}\mathbb{E}[D(T)Z(T)V(T)|\mathbb{F}(t)] \\
&D(t)Z(t)V(t) = \mathbb{E}[D(T)Z(T)V(T)|\mathbb{F}(t)]
\end{aligned}$$

Exercise 5.3

(i)

$$c(0,x) = \mathbb{E}[e^{-rT}(xexp\{\sigma\tilde{W}(T) + (r - \frac{1}{2})T\} - K)^+]$$

By differentiate inside the expected value, we have

$$\begin{aligned}
c_x(0,x) &= \mathbb{E}[e^{-rT}(\mathbb{I}_{\{xexp\{\sigma\tilde{W}(T)+(r-\frac{1}{2}\sigma^2)T>K\}}exp\{\sigma\tilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\}] \\
&= e^{-\frac{1}{2}\sigma^2 T}\tilde{\mathbb{E}}[e^{\sigma\sqrt{T}\frac{\tilde{W}(T)}{\sqrt{T}}}\mathbb{I}_{\{\frac{\sigma\tilde{W}(T)}{\sqrt{T}}-\sigma\sqrt{T}>\frac{1}{\sigma\sqrt{T}}(ln\frac{K}{x}-(r-\frac{1}{2}\sigma^2)T)-\sigma\sqrt{T}\}}] \\
&= e^{-\frac{1}{2}\sigma^2 T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\sigma\sqrt{T})^2}{2}}\mathbb{I}_{\{z-\sigma\sqrt{T}>-d_{+(T,x)}\}}dz \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\sigma\sqrt{T})^2}{2}}\mathbb{I}_{\{z-\sigma\sqrt{T}>-d_{+}(T,x)\}}dz \\
&= N(d_+(T,x))
\end{aligned}$$

(ii)

Set $S(t) = xexp\{\sigma\tilde{W}(t) + (r - \frac{1}{2}\sigma^2)t\}$

Let $\hat{\mathbb{P}}$ be a probability measure equivalent to $\tilde{\mathbb{P}}$ and let Z(t) be a Radon-Nikodym.

$\mathbb{I}_{S(T)>K}$ is $\mathbb{F}(T)$-measurable and we have

$$\begin{aligned}
\hat{\mathbb{P}}(S(T) > K) &= \hat{\mathbb{E}}[\mathbb{I}_{\{S(T)>K\}}] \\
&= \tilde{\mathbb{E}}[Z(T)\mathbb{I}_{\{S(T)>K\}}] \\
&= \tilde{\mathbb{E}}[exp^{\{\sigma\tilde{W}(T)-\frac{1}{2}\sigma^2T\}}\mathbb{I}_{\{S(T)>K\}}] \\
&= c_x(0,x) \quad we \quad define \quad Z(t) = exp^{\{\sigma\tilde{W}(t)-\frac{1}{2}\sigma^2t\}}
\end{aligned}$$

Use Girsanov, one dimension, set $\theta = -\sigma$ we have

$$\hat{W}(t) = \tilde{W}(t) + \int_0^t (-\sigma)du = \tilde{W}(t) - \sigma t$$

It is a Brownian motion under $\hat{\mathbb{P}}$

iii

$$\begin{aligned}
&S(t) = xexp\{\sigma\tilde{W}(t) + (r - \frac{1}{2}\sigma^2)t\} \\
&\hat{W}(t) = \tilde{W}(t) - \sigma t
\end{aligned}$$

So we have

$$\begin{aligned}
\hat{P}(S(T) > K &= \hat{P}(xe^{\sigma\hat{W}(T)+(r+\frac{1}{2}\sigma^2)T} > K) \\
&= \hat{P}(\frac{\hat{W}(T)}{\sqrt{T}} > -d_+(T,x)) \\
&= N(d_+(T,x))
\end{aligned}$$

Exercise 5.6

For Theorem 5.4.1, we have

$$\begin{aligned}
&\theta(t) = (\theta_1(t), \theta_2(t)) \quad is \quad 2-dimensional \quad adapted \quad process \\
&Z(t) = exp^{\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t ||\theta(u)||^2du\}} \\
&\tilde{W}(t) = W(t) + \int_0^t \theta(u)du
\end{aligned}$$

Use "Levy, two dimensions", WTS that $\tilde{W}(t)$ is a 2-dimensional Brownian motion

i Continuity

$\tilde{W}(t) = W(t) + \int_0^t \theta(u)du$, because Brownian motion W(t) has continuous sample paths and integral is continuous too, so $\tilde{W}(t)$ is continuous.

ii Starting at zero

$$\tilde{W}(0) = W(0) + \int_0^0 \theta(u)du = W(0) = 0$$

iii Unit quadratic and zero cross variation

For i, j =1,2, $j \neq i$

$$\begin{aligned} d\tilde{W}_i d\tilde{W}_i &= (dW_i(t) + \theta_i(t))(dW_i(t) + \theta_i(t)) \\ &= dt \end{aligned}$$

$$\begin{aligned} d\tilde{W}_i d\tilde{W}_j &= (dW_i(t) + \theta_i(t))(dW_j(t) + \theta_j(t)) \\ &= dW_i(t)dW_j(t) \\ &= 0 \end{aligned}$$

iv Martingale property

We define $X(t) = -\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t ||\theta(u)||^2 du$

$$\begin{aligned} dX(t) &= -\theta(t)dW(t) - \frac{1}{2}||\theta(t)||^2 dt \\ dX(t)dX(t) &= (-\theta(t)dW(t) - \frac{1}{2}||\theta(t)||^2 dt)^2 \\ &= \sum_{j=1}^{d}\sum_{k=1}^{d} \theta_j(t)\theta_k(t)dW_j(t)dW_k(t) \\ &= \sum_{j=1}^{d} \theta_j^2(t)dt \\ &= ||\theta(t)||^2 dt \end{aligned}$$

We define $f(t,x) = e^x$

$$f_t = 0, f_x = e^x, f_{xx} = e^x$$

$$\begin{aligned} dZ(t) &= df(X(t)) \\ &= Z(t)dX(t) + \frac{1}{2}Z(t)dX(t)dX(t) \\ &= -Z(t)\theta(t)dW(t) - \frac{1}{2}Z(t)||\theta(t)||^2dt + \frac{1}{2}Z(t)||\theta(t)||^2dt \end{aligned}$$

$= -Z(t)\theta(t)dW(t)$

There is no dt term in dZ(t), so it is a martingale under $\mathbb{P}$. $\mathbb{E}[Z(T)] = Z(0) = 1$, thus it qualifies as a Radon-Nikoym derivative process.

Use Ito product rule

$$\begin{aligned} d(\tilde{W}(t)Z(t)) &= \tilde{W}dZ(t) + Z(t)d\tilde{W}(t) + d\tilde{W}(t)dZ(t) \\ &= -\tilde{W}(t)Z(t)\theta(t)dW(t) + Z(t)(dW(t) + \theta(t)dt) + (dW(t) + \theta(t)dt)(-Z(t)\theta(t)dW(t)) \\ &= -\tilde{W}(t)Z(t)\theta(t)dW(t) + Z(t)dW(t) + Z(t)\theta(t)dt - Z(t)\theta(t)dt \\ &= -\tilde{W}(t)Z(t)\theta(t)dW(t) + Z(t)dW(t) \\ &= Z(t)(-\tilde{W}(t)\theta(t) + 1)dW(t) \end{aligned}$$

There is no dt term in $d(\tilde{W}(t)Z(t))$, so it is a martingale under $\mathbb{P}$

Using Lemma 5.2.2

$$\begin{aligned} \tilde{E}[\tilde{W}(t)|\mathbb{F}(s)] &= \frac{1}{Z(s)}\mathbb{E}[\tilde{W}(t)Z(t)|\mathbb{F}(s)] \\ &= \frac{1}{Z(s)}\tilde{W}(s)Z(s) \\ &= \tilde{W}(s) \end{aligned}$$

So $\tilde{W}(t)$ is a 2-dimensional martingale under $\mathbb{P}$

To sum up, $\tilde{W}(t)$ satisfies all the condition of the Levy two dimension, we conclude that $\tilde{W}(t)$ is a 2-dimensional Brownian motion under $\tilde{\mathbb{P}}$

Exercise 6.1

i

$$Z(t) = exp\{\int_t^t \sigma(v)dW(v) + \int_t^t (b(v) - \frac{1}{2}\sigma^2(v))dv\} = 1$$

Let

$$A(u) = \int_t^u \sigma(v)dW(v) + \int_t^u (b(v) - \frac{1}{2}\sigma^2(v))dv$$
$$dA(u) = \sigma(u)dW(u) + (b(u) - \frac{1}{2}\sigma^2(u))du$$

Let $f(u,x) = e^x$

$$\frac{\partial f}{\partial u} = 0, \frac{\partial f}{\partial x} = f, \frac{\partial^2 f}{\partial x^2} = f$$

Use Ito Lemma, we have

$$\begin{aligned} dZ(u) &= df(u, A(u)) \\ &= Z(u)dA(u) + \frac{1}{2}Z(u)dA(u)dA(u) \\ &= (b(u) - \frac{1}{2}\sigma^2(u))Z(u)du + \sigma(u)Z(u)dW(u) + \frac{1}{2}\sigma^2(u)Z(u)du \\ \\ &= b(u)Z(u)du + \sigma(u)Z(u)dW(u) \end{aligned}$$

ii

Using Ito product rule

$$\begin{aligned} dX(u) &= d(Y(u)Z(u)) \\ &= Y(u)dZ(u) + Z(u)dY(u) + dY(u)dZ(u) \\ &= b(u)X(u)du + \sigma(u)X(u)dW(u) + (a(u) - \sigma(u)\gamma(u))du + r(u)dW(u) + \sigma(u)r(u)du \\ &= (a(u) + b(u)X(u))du + (r(u) + \sigma(u)X(u))dW(u) \end{aligned}$$

For Z(t)=1, Y(t)=x, and X(u)=Y(u)Z(u),so X(t)=x

Exercise 6.2

i

The self-financing portfolio process X(t) is given by

$$\begin{aligned} dX(t) = \Delta_1(t)df(t, R(t), T_1) + \Delta_2(t)df(t, R(t), T_2) + R(t)(X(t) \\ - \Delta_1(t)f(t, R(t), T_1) - \Delta_2(t)f(t, R(t).T_2))dt \end{aligned}$$

Using Ito Lemma

$$df(t,R(t),T) = f_t(t,R(t),T)dt + f_r(r,R(t),T)dR(t) + \frac{1}{2}f_{rr}(t,R(t),T)dR(t)dR(t)$$

$$\begin{aligned}
&= f_t(t,R(t),T)dt + \alpha(t,R(t))f_r(t,R(t),T)dt \\
&\quad + \gamma(t,R(t))f_r(t,R(t),T)dW(t) + \frac{1}{2}\gamma^2(t,R(t))f_{rr}(t,R(t),T)dt
\end{aligned}$$

For $dD(t) = -R(t)D(t)dt$

Using Ito product rule

$$\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= \Delta_1(t)D(t)df(t,R(t),T_1) + \Delta_2(t)D(t)df(t,R(t),T_2) \\
&\quad + R(t)D(t)(X(t) - \Delta_1 t f(t,R(t),T_1) - \Delta_2(t)f(t,R(t),T_2))dt - R(t)D(t)X(t)dt \\
&= \Delta_1(t)D(t)[-R(t)f(t,R(t),T_1)dt + df(t,R(t),T1)] \\
&\quad + \Delta_2(t)D(t)[-R(t)f(t,R(t),T_2)dt + df(t,R(t),T_2)] \\
&= \Delta_1(t)D(t)[-R(t)f(t,R(t),T_1) + f_t(t,R(t),T_1) \\
&\quad + \alpha(t,R(t))f_r(t,R(t),T_1) + \frac{1}{2}\gamma^2(t,R(t))f_{rr}(t,R(t),T_1)]dt \\
&\quad + \Delta_2(t)D(t)[-R(t)f(t,R(t),T_2) + f_t(t,R(t),T_2) \\
&\quad + \alpha(t,R(t))f_r(t,R(t),T_2) + \frac{1}{2}\gamma^2(t,R(t))f_r r(t,R(t),T_2)dt \\
&\quad + D(t)\gamma(t,R(t))[\Delta_1(t)f_r t, R(t), T_1 + \Delta_2(t)f_r(t,R(t),T_2)]dW(t) \\
&= \Delta_1(t)D(t)[\alpha(t,R(t)) - \beta(t,R(t),T_1)]f_r(t,R(t),T_1)dt \\
&\quad + \Delta_2(t)D(t)[\alpha(t,R(t)) - \beta(t,R(t),T_2)]f_r(t,R(t),T_2)dt \\
&\quad + D(t)\gamma(t,R(t))[\Delta_1(t)f_r(t,R(t),T_1) + \Delta_2(t)f_r t, R(t), T_2]dW(t)
\end{aligned}$$

ii

$$\begin{aligned}
&\Delta_1(t)f_r(t,R(t),T_1) + \Delta_2(t)f_2(t,R(t),T_2) \\
&= S(t)f_r(t,R(t),T_1)f_r(r,R(t),T_2) - S(t)f_r(t,R(t),T_1)f_r(r,R(t),T_2) \\
&= 0
\end{aligned}$$

So the diffusion term in $d(D(t)X(t))$ vanishes

$$\begin{aligned}
d(D(t)X(t)) &= \Delta_1(t)D(t)[\alpha(t,R(t)) - \beta(t,R(t),T_1)]f_r(t,R(t),T_1)dt \\
&\quad + \Delta_2(t)D(t)[\alpha(t,R(t)) - \beta(t,R(t),T_2)]f_r(t,R(t),T_2)dt \\
&= S(t)D(t)[\beta(t,R(t),T_2) - \beta(t,R(t),T_1)]f_r(t,R(t),T_1)f_r(t,R(t),T_2)dt
\end{aligned}$$

For no-arbitrage to exist, the discounted wealth process of a risk-free portfolio has to be a martingale. So there is no dt term in d(D(t)X(t)), so $\beta(t, R(t), T_2) = \beta(t, R(t), T_1)$, because T_1, T_2 can be any maturities, we conclude that $\beta(t, R(t), T)$ has to be independent of T

iii

We set $T_1 = T, \Delta_1(t) = \Delta(t), \Delta_2(t) = 0$, and $f_r(t, r, T) = 0$, then we have:

$$\begin{aligned} d(D(t)X(t)) = \Delta(t)D(t)[&-R(t)f(t, R(t), T) + f_t(t, R(t), T) + \alpha(t, R(t))f_r(t, R(t), T) \\ &\frac{1}{2}\gamma^2(t, R(t))f_{rr}(t, R(t), T)]dt \end{aligned}$$

For no-arbitrage to exist, the change in the discounted portfolio value must be zero. So:

$$-R(t)f(t, R(t), T) + f_t(tR(t), T) + \frac{1}{2}\gamma^2(t, R(t))f_{rr}(t, R(t), T) = 0$$

Exercise 6.3

i

$$\begin{aligned} \frac{d}{ds}[e^{-\int_0^s b(v)dv}C(s, T)] &= C(s, T)\frac{d}{ds}[e^{-\int_0^s b(v)dv} + e^{-\int_0^s b(v)dv}\frac{d}{ds}(C(s, T)) \\ &= e^{-\int_0^s b(v)dv}[-b(s)C(s, T) + C'(s, T)] \\ &= e^{-\int_0^s b(v)dv}[-b(s)C(s, T) + b(s)C(s, T) - 1] \\ &= -e^{-\int_0^s b(v)dv} \end{aligned}$$

ii

$$\int_t^T \frac{d}{ds}[e^{-\int_0^s b(v)dv}C(s, T)] = e^{-\int_0^T b(v)dv}C(T, T) - e^{-\int_0^t b(v)dv}C(t, T)$$

$$-e^{-\int_0^t b(v)dv}C(t, T) = -\int_t^T e^{-\int_0^s b(v)dv}$$

$$C(t, T) = \int_t^T e^{-\int_0^s b(v)dv}e^{\int_0^t b(v)dv} = \int_t^T e^{\int_s^t b(v)dv}ds$$

iii

$$\begin{aligned} A'(s, T) &= -a(s)C(s, t) + \frac{1}{2}\sigma^2(s)C^2(s, T) \\ A(T, T) - A(t, T) &= -\int_t^T a(s)C(s, T)ds + \frac{1}{2}\int_t^T \sigma^2(s)C^2(s, T)ds \\ A(t, T) &= \int_t^T (a(s)C(s, T) - \frac{1}{2}\sigma^2(s)C^2(s, T))ds \end{aligned}$$

Exercise 11.1

i

$$\begin{aligned}
\mathbb{E}[M^2(t)|\mathbb{F}_s] &= \mathbb{E}[M^2(t) - M^2(s) + M^2(s)|\mathbb{F}_s] \\
&= \mathbb{E}[M^2(t) - M^2(s)|\mathbb{F}_s] + M^2(s) \\
&\text{"}Linearity\text{"} \quad \text{"}Taking \quad out \quad what \quad is \quad know\text{"} \\
&= \mathbb{E}[M^2(t) + M^2(s) - 2M(s)M(t) + 2M^2(s) - 2M(s)M(t)|\mathbb{F}_s] + M^2(s) \\
&= \mathbb{E}[(M(t) - M(s))^2 + 2M(s)(M(s) - M(t))|\mathbb{F}_s] + M^2(s) \\
&= \mathbb{E}[(M(t) - M(s))^2] + \mathbb{E}[2M(s)(M(s) - M(t))|\mathbb{F}_s] + M^2(s) \\
&\text{"}Linearity\text{"} \quad \text{"}Independence\text{"} \\
&\mathbb{E}[(M(t) - M(s))^2] - 2M(s)\mathbb{E}[(M(t) - M(s))] + M^2(s) \\
&\text{"}Taking \quad out \quad what \quad is \quad know\text{"} \quad \text{"}Independence\text{"} \\
&= \lambda(t - s) + M^2(s) \\
&\geq M^2(s)
\end{aligned}$$

So $M^2(t)$ is a submartingale

ii

$$\begin{aligned}
\mathbb{E}[M^2(t) - \lambda t|\mathbb{F}_s] &= \mathbb{E}[M^2(t) - M^2(s) + M^2(s) - \lambda t|\mathbb{F}_s] \\
&= \mathbb{E}[M^2(t) - M^2(s)|\mathbb{F}_s] + M^2(s) - \lambda t \\
&\text{"}Linearity\text{"} \quad \text{"}Taking \quad out \quad what \quad is \quad know\text{"} \\
&= \mathbb{E}[M^2(t) + M^2(s) - 2M(s)M(t) + 2M^2(s) - 2M(s)M(t)|\mathbb{F}_s] + M^2(s) - \lambda t \\
&= \mathbb{E}[(M(t) - M(s))^2 + 2M(s)(M(s) - M(t))|\mathbb{F}_s] + M^2(s) - \lambda t \\
&= \mathbb{E}[(M(t) - M(s))^2] + \mathbb{E}[2M(s)(M(s) - M(t))|\mathbb{F}_s] + M^2(s) - \lambda t \\
&\text{"}Linearity\text{"} \quad \text{"}Independence\text{"} \\
&\mathbb{E}[(M(t) - M(s))^2] - 2M(s)\mathbb{E}[(M(t) - M(s))] + M^2(s) - \lambda t \\
&\text{"}Taking \quad out \quad what \quad is \quad know\text{"} \quad \text{"}Independence\text{"} \\
&= -\lambda(s) + M^2(s)
\end{aligned}$$

So $M^2(t) - \lambda(t)$ is a martingale

Exercise 11.2

For the increment of Poisson process has stationary property

And for Lemma 11.2.2, we have

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

i

$$\begin{aligned}
\mathbb{P}\{N(s+t) = k | N(s) = k\} &= \mathbb{P}\{N(s+t) - N(s) = 0 | N(s) = k\} \\
&= \mathbb{P}\{N(t) = 0\} \\
&= e^{-\lambda t} \\
&= 1 - \lambda t + O(t^2)
\end{aligned}$$

ii

$$\begin{aligned}
\mathbb{P}\{N(s+t) = k+1 | N(s) = k\} &= \mathbb{P}\{N(s+t) - N(s) = 1 | N(s) = k\} \\
&= \mathbb{P}\{N(t) = 1\} \\
&= \frac{(\lambda t)^1}{1!} e^{-\lambda t} \\
&= \lambda t(1 - \lambda t + O(t^2)) \\
&= \lambda t + O(t^2)
\end{aligned}$$

iii

$$\begin{aligned}
\mathbb{P}\{N(s+t) \geq k+2 | N(s) = k\} &= \mathbb{P}\{N(s+t) - N(s) \geq 2 | N(s) = k\} \\
&= \mathbb{P}\{N(t) \geq 2\} \\
&\sum_{k=2}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= O(t^2)
\end{aligned}$$

Additional Problem

The self-financing portfolio process X(t) is given by

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= \Delta(t)(rS(t)dt + \sigma S(t)d\tilde{W}(t)) + r(X(t) - \Delta(t)S(t))dt \\ &= \Delta(t)\sigma S(t)d\tilde{W}(t) + rX(t)dt \end{aligned}$$

We define discount process $D(t) = e^{-tr}$

Because of $dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$

We use Ito product rule

$$\begin{aligned} d(X(t)D(t)) &= X(t)dD(t) + D(t)dX(t) + dX(t)dD(t) \\ &= \Delta(t)\sigma D(t)S(t)d\tilde{W}(t) \end{aligned}$$

Because there is no dt term in dS(t)D(t), so X(t)D(t) is a martingale. So it has constant expectation

$$\begin{aligned} &D(0)X(0) = \mathbb{E}[D(T)X(T)] \\ &X(0) = \mathbb{E}[e^{-rT}S^2(T)] \end{aligned}$$

Because $dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$, so S(t) is a generalized geometric Brownian motion. So $S(t) = S(0)exp\{\int_0^t \sigma d\tilde{W}(t) + \int_0^t (r - \frac{1}{2}\sigma^2)dt\}$

$$\begin{aligned} X(0) &= \mathbb{E}[e^{-rT}S^2(T)] \\ &= \mathbb{E}[e^{-rT}S^2(0)e^{2\int_0^T (r-\frac{1}{2}\sigma^2)dt + 2\int_0^T \sigma(s)d\tilde{W}(t)}] \\ &= e^{-rT}S^2(0)e^{2T(r-\frac{1}{2}\sigma^2}\mathbb{E}[e^{2\int_0^T \sigma(s)d\tilde{W}(t)}] \\ &= e^{-rT}S^2(0)e^{2T(r-\frac{1}{2}\sigma^2}\mathbb{E}[e^{2\int_0^T \sigma^2(s)d\tilde{W}(t)}] \\ &\text{"}Use \quad moment \quad generating \quad function \quad\text{"} \\ &= S^2(0)e^{rT+\sigma^2 T} \end{aligned}$$

Let V(T) be an $\mathbb{F}(t)$ measurable random variable, represents the pay off at time T of a derivative security.That is

$$X(T) = V(T) \quad almost \quad surely$$

According to 5.2.30, $D(t)V(t) = \mathbb{E}[D(T)V(T)|\mathbb{F}(t)]$, so it is a martingale

$$\begin{aligned} D(t)V(t) &= \mathbb{E}[D(T)V(T)|\mathbb{F}(t)] \\ &= \mathbb{E}[D(T)S^2(T)|\mathbb{F}(t)] \\ &= S^2(0)e^{(r+\sigma^2)T}\mathbb{E}[e^{-2\sigma^2 T+2\sigma\tilde{W}(T)}|\mathbb{F}(t)] \end{aligned}$$

Set $f(t,x) = e^{-2\sigma^2 t+2\sigma x}$

Using Ito Lemma

$$\begin{aligned} f_t = -2\sigma^2 f \quad f_x &= 2\sigma f \quad f_{xx} = 4\sigma^2 f \\ df =& -2\sigma^2 f dt + 2\sigma f d\tilde{W}(t) + \frac{1}{2}4\sigma^2 f dt \\ =& \, 2\sigma f d\tilde{W}(t) \end{aligned}$$

So there is no dt term in df, so f(t,x) is a martingale

$$\begin{aligned} D(t)V(t) &= S^2(0)e^{(r+\sigma^2)T}\mathbb{E}[e^{-2\sigma^2 T+2\sigma\tilde{W}(T)}|\mathbb{F}(t)] \\ &= S^2(0)e^{(r+\sigma^2)T}e^{-2\sigma^2 t+2\sigma\tilde{W}(t)} \\ &= S^2(0)e^{(r+\sigma^2)T}f(t,\tilde{W}(t)) \end{aligned}$$

Because d(D(t)V(t))=d(D(t)X(t)), we have

$$\begin{aligned} &S^2(0)e^{(r+\sigma^2)T}2\sigma f(t,\tilde{W}(t))d\tilde{W}(t) = e^{-rt}\sigma\Delta(t)S(t)d\tilde{W}(t) \\ &\Delta(t) = 2S(0)e^{(r+\sigma^2)T-\frac{3}{2}\sigma^2 t+\sigma\tilde{W}(t)} \end{aligned}$$