Homework 3

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Exercise 4.1

WTS $\mathbb{E}[I(t)|\mathbb{F}(s)] = I(s)$,set $s < t, \quad s \in [t_l, t_{l+1}) \quad t \in [t_k, t_{k+1})$

$$
\begin{aligned}
I(t) = &\sum_{j=0}^{l-1} \Delta(t_j)(M(t_{j+1}) - M(t_(j))) + \Delta(t_l)(M(t_{l+1} - M(t_l)) \\
&+ \sum_{j=l+1}^{k-1} \Delta(t_j)(M(t_{j+1}) - M(t_(j))) + \Delta(t_k)(M(t_t - M(t_k))
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[I(t)|\mathbb{F}(s)] = &\mathbb{E}[\sum_{j=0}^{l-1} \Delta(t_j)(M(t_{j+1}) - M(t_j))|\mathbb{F}(s)] + \mathbb{E}[\Delta(t_l)(M(t_{l+1}) - M(t_l))|\mathbb{F}(s)] \\
&+ \mathbb{E}[\sum_{j=l+1}^{k-1} \Delta(t_j)(M(t_{j+1}) - M(t_j))|\mathbb{F}(s)] + \mathbb{E}[\Delta(t_k)(M(t_t) - M(t_k))|\mathbb{F}(s)]
\end{aligned}
$$

Using"Taking out what is known"

$$
\mathbb{E}[\sum_{j=0}^{l-1} \Delta(t_j)(M(t_{j+1}) - M(t_j))|\mathbb{F}(s)] = \sum_{j=0}^{l-1} \Delta(t_j)(M(t_{j+1}) - M(t_j))
$$

Because W(t) is martingale, so for $t_{l+1} > s$ $\mathbb{E}[M(t_{l+1}|\mathbb{F}(s)] = M(t_s)$

Using "Taking out what is known" and "Linearity of conditional expectations"

$$
\begin{aligned}
\mathbb{E}[\Delta(t_l)(M(t_{l+1} - M(t_l))|\mathbb{F}(s)] &= \Delta(t_l)\mathbb{E}[M(t_{l+1}) - M(t_l)|\mathbb{F}(s)] \\
&= \Delta(t_l)\mathbb{E}[M(t_{l+1})|\mathbb{F}(s)] - \Delta(t_l)\mathbb{E}[M(t_l)|\mathbb{F}(s)] \\
&= \Delta(t_l)(M(t_s) - M(t_l))
\end{aligned}
$$

For the summands in the third we use "Independence", "Iterated Conditioning" and "Martingale"

$$\begin{aligned}
&\mathbb{E}[\Delta(t_j)(M(t_{j+1}) - M(t_j))|\mathbb{F}(s)] \\
&= \mathbb{E}[\Delta(t_j)] * \mathbb{E}[(M(t_{j+1}) - M(t_j))|\mathbb{F}(s)]] \\
&= \mathbb{E}[\Delta(t_j)] * \mathbb{E}[\mathbb{E}[(M(t_{j+1}) - M(t_j))|\mathbb{F}(j)|\mathbb{F}(s)] \\
&= \mathbb{E}[\Delta(t_j)]\mathbb{E}[M(t_j) - M(t_j)|\mathbb{F}(s)] \\
&= 0
\end{aligned}$$

Using "Independence" and "martingale"

$$\mathbb{E}[\Delta(t_k)(M(t_t) - M(t_k))|\mathbb{F}(s)] = \mathbb{E}[\Delta(t_k)] * \mathbb{E}[(M(t_k) - M(t_k))\mathbb{F}(s)] = 0$$

So

$$\mathbb{E}[I(t)|\mathbb{F}(s)] = \sum_{j=0}^{l-1} \Delta(t_j)(M(t_{j+1}) - M(t_j)) + \Delta(t_l)(M(t_s) - M(t_l)) = I(s)$$

Exercise 4.2

i $0 \le s < t \le T$ set $s \in [t_l, t_{l+1})$, $t \in [t_k, t_{k+1})$

$$\begin{aligned}
I(t) = &\sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1}) - W(t_(j))) + \Delta(t_l)(W(t_{l+1} - W(t_l)) \\
&+ \sum_{j=l+1}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_(j))) + \Delta(t_k)(W(t_t - W(t_k))
\end{aligned}$$

$$I(s) = \sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1}) - W(t_(j))) + \Delta(t_l)(W(t_s) - W(t_l))$$

$$I(t) - I(s) = \Delta(t_l)(W(t_{l+1}) - W(s)) + \sum_{j=l+1}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_k)(W(t_t) - W(t_k))$$

$\Delta(t)$ is nonrandom simple process, so it is independence with $\mathbb{F}(s)$

Because W(t) is a Brownian Motion, for any $s \le i < j$, we have $W(t_j) - W(t_i)$ is independence with $\mathbb{F}(s)$

So $(W(t_{l+1}) - W(s))$ $(W(t_{j+1}) - W(t_j))$ $(W(t_t) - W(t_k))$ are independence with $\mathbb{F}(s)$

So I(t)-I(s) is independence with $\mathbb{F}(s)$

ii

$$\mathbb{E}[I(t) - I(s)] = \sum_{j=s}^{k-1} \Delta(t_j)\mathbb{E}[W(t_{j+1}) - W(t_j)] = 0$$

$$\begin{aligned}
Var[I(t) - I(s)] &= Var[\sum_{j=s}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))] \\
&= \sum_{j=s}^{k-1} (\Delta(t_j))^2 Var[W(t_{j+1}) - W(t_j)] \\
&= \sum_{j=s}^{k-1} (\Delta(t_j))^2 (t_{j+1} - t_j) \\
&= \int_{t_s}^{t_{k-1}} (\Delta(u))^2 du
\end{aligned}$$

iii

$$\begin{aligned}
\mathbb{E}[I(t)|\mathbb{F}(s)] &= \mathbb{E}[I(t) - I(s) + I(s)|\mathbb{F}(s)] \\
&= \mathbb{E}[I(t) - I(s)|\mathbb{F}(s)] + \mathbb{E}[I(s)|\mathbb{F}(s)] \quad (\text{"}Linearity \quad of \quad conditional \quad expectations\text{"}) \\
&= \mathbb{E}[I(t) - I(s)|\mathbb{F}(s)] + I(s) \quad (\text{"}Taking \quad out \quad what \quad is \quad known\text{"}) \\
&= \mathbb{E}[I(t) - I(s)] + I(s) \quad (\text{"}Independence\text{"}) \\
&= 0 + I(s)
\end{aligned}$$

So I(t) is a martingale

iv

$$X(t) = I^2(t) - \int_0^t \Delta^2(u)du$$

$$
\begin{aligned}
\mathbb{E}[X(t)|\mathbb{F}(s)] &= \mathbb{E}[X(t) - X(s) + X(s)|\mathbb{F}(s)] \\
&= \mathbb{E}[X(t) - X(s)|\mathbb{F}(s)] + \mathbb{E}[X(s)|\mathbb{F}(s)] \quad \text{"}Linearity \quad of \quad conditional \quad expectation\text{"} \\
&= \mathbb{E}[X(t) - X(s)|\mathbb{F}(s)] + X(s) \quad \text{"}Taking \quad out \quad what \quad is \quad known\text{"} \\
&= \mathbb{E}[I^2(t) - I^2(s) - \int_s^t \Delta^2(u)du|\mathbb{F}(s)] + X(s) \\
&= \mathbb{E}[I^2(t) - I^2(s)|\mathbb{F}(s)] + X(s) - \int_s^t \Delta^2(u)du \quad \text{"}Taking \quad out \quad what \quad is \quad known\text{"} \\
&= \mathbb{E}[(I(t) - I(s))^2 - 2I^2(s) + 2I(t)I(s)|\mathbb{F}(s)] + X(s) - \int_s^t \Delta^2(u)du \\
&= \mathbb{E}[(I(t) - I(s))^2|\mathbb{F}(s)] - 2\mathbb{E}[I(s)(-I(t) + I(s))|\mathbb{F}(s)] + X(s) - \int_s^t \Delta^2(u)du \\
&= \int_s^t \Delta^2(u)du - 2I(s)\mathbb{E}[(-I(t) + I(s))|\mathbb{F}(s)] + X(s) - \int_s^t \Delta^2(u)du \\
&= \int_s^t \Delta^2(u)du + 0 + X(s) - \int_s^t \Delta^2(u)du \\
&= X(s)
\end{aligned}
$$

So X(t) is a martingale

Exercise 4.3

$$
\begin{aligned}
I(t) &= \Delta(t_0)(W(t_1) - W(t_0)) + \Delta(t_1)(W(t_2) - W(t_1)) \\
I(s) &= \Delta(t_0)(W(t_1) - W(t_0)) \\
I(t) - I(s) &= \Delta(t_1)(W(t_2) - W(t_1)) = W(s)(W(t) - W(s))
\end{aligned}
$$

i False

$I(t) - I(s) = \Delta(t_1)(W(t_2) - W(t_1)) = W(s)(W(t) - W(s))$**, W(s)is** $\mathbb{F}(s)$ **measurable**

ii False

$$
\begin{aligned}
\mathbb{E}[(I(t) - I(s))^4] = \mathbb{E}[W^4(s)]\mathbb{E}[(W(t) - W(s))^4] = 3s^2 * 3(t-s)^2 = 9s^2(t-s)^2 \\
3(Var[I(t) - I(s)])^2 3(\mathbb{E}[(I(t) - I(s))^2])^2 = 3(\mathbb{E}[W^2(s)(W(t) - W(s))^2]) = 3s^2(t-s)^2
\end{aligned}
$$

iii True

$$\begin{aligned}\mathbb{E}[I(t)|\mathbb{F}(s)]\mathbb{E}[I(t)-I(s)+I(s)|\mathbb{F}(s)]&\\ &=\mathbb{E}[I(t)-I(s)|\mathbb{F}(s)]+\mathbb{E}[I(s)|\mathbb{F}(s)] \quad ''Linearity \quad of \quad conditional \quad expectation''\\ &=W(s)\mathbb{E}[(W(t)-W(s))|\mathbb{F}(s)]+I(s) \quad ''Taking \quad out \quad what \quad is \quad known''\\ &=I(s)\end{aligned}$$

So I(t) is a martingale

iv True

$$\begin{aligned}&\mathbb{E}[I^2(t)-\int_0^t \Delta^2(u)du|\mathbb{F}(s)]\\ &=\mathbb{E}[I^2(t)-\int_0^t \Delta^2(u)du-I^2(s)+\int_0^s \Delta^2(u)du+I^2(s)-\int_0^s \Delta^2(u)du|\mathbb{F}(s)]\\ &=\mathbb{E}[I^2(s)-\int_0^s \Delta^2(u)du|\mathbb{F}(s)]+\mathbb{E}[I^2(t)-I^2(s)|\mathbb{F}(s)]-\int_s^t \Delta^2(u)du \quad ''Linearity''\\ &=I^2(s)-\int_0^s \Delta^2(u)du+\mathbb{E}[I^2(t)-I^2(s)|\mathbb{F}(s)]-\int_s^t \Delta^2(u)du \quad ''Taking \quad out \quad what \quad is \quad known''\\ &=I^2(s)-\int_0^s \Delta^2(u)du+\mathbb{E}[(I(t)-I(s))^2-2I(s)(I(t)-I(s))|\mathbb{F}(s)]-\int_s^t \Delta^2(u)du\\ &=I^2(s)-\int_0^s \Delta^2(u)du+\mathbb{E}[(I(t)-I(s))^2|\mathbb{F}(s)]-2I(s)\mathbb{E}[(I(t)-I(s)))|\mathbb{F}(s)]-\int_s^t \Delta^2(u)du\\ &=I^2(s)-\int_0^s \Delta^2(u)du-\int_s^t \Delta^2(u)du+\int_s^t \Delta^2(u)du+0\\ &=I^2(s)-\int_0^s \Delta^2(u)du\end{aligned}$$

So $I^2(t)-\int_0^t \Delta^2(u)du$ is a martingale

Exercise 4.5

i set f(t,x)=lnx, so

$$\frac{\partial f}{\partial t}=0, \frac{\partial f}{\partial x}=\frac{1}{x}, \frac{\partial^2 f}{\partial x^2}=-\frac{1}{x^2}$$

$$(dS(t))^2=\sigma^2(t)S^2(t)$$

$$\begin{aligned}
dlnS(t) &= df(t, S(t)) \\
&= \frac{1}{S(t)} dS(t) - \frac{1}{2}\frac{1}{S^2(t)}(dS(t))^2 \\
&= \alpha(t)dt + \sigma(t)dW(t) - \frac{1}{2}\sigma^2(t)dt \\
&= (\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)d(W(t))
\end{aligned}$$

ii

$$\begin{aligned}
lnS(t) &= lnS(0) + \int_0^t \alpha(s) - \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW(s) \\
S(t) &= S(0)exp\{\int_0^t \alpha(s) - \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW(s)\}
\end{aligned}$$

Exercise 4.6

Let $f(t, x) = S(0)e^x$

$$\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = f(x), \frac{\partial^2 f}{\partial x^2} = f(x)$$

define $X(t) = (\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)$

So $dX(t) = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dW(t)$

$(dX(t))^2 = \sigma^2 dt$

$$\begin{aligned}
d(S(t)) &= df(t, X(t)) \\
&= S(t)dX(t) + \frac{1}{2}S(t)(dX(t))^2 \\
&= S(t)dX(t) + \frac{1}{2}S(t)\sigma^2 dt \\
&= S(t)\alpha dt - \frac{1}{2}\sigma^2 S(t)dt + \sigma S(t)dW(t) + \frac{1}{2}S(t)\sigma^2 dt \\
&= S(t)\alpha dt + \sigma S(t)dW(t)
\end{aligned}$$

Let $f(t, x) = x^p$

$$\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = px^{p-1}, \frac{\partial^2 f}{\partial x^2} = p(p-1)x^{p-2}$$

$$(dS(t))^2 = \sigma^2 S(t)dt$$

$$
\begin{aligned}
d(S^p(t)) &= d(t, S(t)) \\
&= pS^{p-1}(t)dS(t) + \frac{1}{2}p(p-1)S^{p-2}(t)(dS(t))^2 \\
&= pS^{p-1}(t)dS(t) + \frac{1}{2}p(p-1)S^{p-2}(t)\sigma^2 S(t)dt \\
&= pS^{p-1}(t)(S(t)\alpha dt + \sigma S(t)dW(t)) + \frac{1}{2}p(p-1)S^{p-2}(t)\sigma^2 S(t)dt \\
&= (\sigma + \frac{1}{2}(p-1))pS^p(t)dt + \sigma pS^p(t)dW(t)
\end{aligned}
$$

Exercise 4.7

i

Let $f(t,x) = x^4$

$$
\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = 4x^3, \frac{\partial^2 f}{\partial x^2} = 12x^2
$$

$$
\begin{aligned}
d(W^4(t)) &= d(t, W(t)) \\
&= 4W^3(t)dW(t) + 6W^2(t)dt
\end{aligned}
$$

Integration of with both sides

$$
W^4(T) = W^4(0) + 4\int_0^T W^3(t)dW(t) + 6\int_0^T W^2(t)dt
$$

$$
W^4(T) = 4\int_0^T W^3(t)dW(t) + 6\int_0^T W^2(t)dt
$$

ii

$$
\mathbb{E}[W^4(T)] = \mathbb{E}[4\int_0^T W^3(t)dW(t)] + \mathbb{E}[6\int_0^T W^2(t)dt]
$$

The expectation of an Ito integral is zero, so $\mathbb{E}[4\int_0^T W^3(t)dW(t)] = 0$

$$
\mathbb{E}[W^4(T)] = \mathbb{E}[6\int_0^T W^2(t)dt] = 6\int_0^T \mathbb{E}[W^2(t)dt] = 6\int_0^T tdt = 3T^2
$$

iii

Let $f(t,x) = x^6$

$$\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = 6x^5, \frac{\partial^2 f}{\partial x^2} = 30x^4$$

$$\begin{aligned} d(W^4(t)) &= d(t, W(t)) \\ &= 6W^5(t)dW(t) + 30W^4(t)dt \end{aligned}$$

Integration of with both sides

$$W^6(T) = W^6(0) + 6\int_0^T W^5(t)dW(t) + 15\int_0^T W^4(t)dt$$

$$W^6(T) = 6\int_0^T W^5(t)dW(t) + 15\int_0^T W^4(t)dt$$

$$\mathbb{E}[W^6(T]) = \mathbb{E}[6\int_0^T W^5(t)dW(t)] + \mathbb{E}[15\int_0^T W^4(t)dt]$$

The expectation of an Ito integral is zero, so $\mathbb{E}[6\int_0^T W^5(t)dW(t)] = 0$

$$\mathbb{E}[W^4(T)] = \mathbb{E}[15\int_0^T W^4(t)dt] = 15\int_0^T \mathbb{E}[W^4(t)dt] = 15\int_0^T 3t^2dt = 15T^3$$

Exercise 4.8

i

let $f(t,x) = e^{\beta t}x$

$$\frac{\partial f}{\partial t} = \beta f(t,x), \frac{\partial f}{\partial x} = e^{\beta t}, \frac{\partial^2 f}{\partial x^2} = 0$$

$$(dR(t))^2 = \sigma^2 dt$$

$$\begin{aligned} d(e^{\beta t}R(t)) &= df(t, R(t)) \\ &= \beta e^{\beta t}R(t)dt + e^{\beta t}d(R(t)) \\ &= \beta e^{\beta t}R(t)dt + e^{\beta t}((\alpha - \beta R(t))dt + \sigma dW(t)) \\ &= \alpha e\beta t R(t)dt + \sigma e^{\beta t}dW(t) \end{aligned}$$

ii

$$e^{\beta t}R(t) = R(0) + \int_0^t \alpha e^{\beta s}ds + \int_0^t \sigma e^{\beta t}dW(s) = R(0) + \frac{\alpha}{\beta}e^{\beta t}\Big|_0^t + \sigma\int_0^t e^{\beta u}dW(u)$$

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{\beta t}) + e^{-\beta t}\sigma\int_0^t e^{\beta u}dW(u)$$

Addition a

$$X_t = e^{tW_t}$$

let $f(t,x) = e^{tx}$

$$\frac{\partial f}{\partial t} = xf(t,x), \frac{\partial f}{\partial x} = tf(t,x), \frac{\partial^2 f}{\partial x^2} = t^2 f(t,x)$$

$$\begin{aligned}
d(X_t) &= df(t, W(t)) \\
&= W(t)f(t,x)dt + tf(t,x)dW(t) + \frac{1}{2}t^2 f(t,x)d^2(W(t)) \\
&= W(t)f(t,x)dt + tf(t,x)dW(t) + \frac{1}{2}t^2 f(t,x)dt \\
&= (W(t) + \frac{1}{2}t^2)e^{tW(t)}dt + te^{tW(t)}dW(t)
\end{aligned}$$

b

let $F(t,x) = e^{\gamma t} f(W(t))$

$$\frac{\partial F}{\partial t} = \gamma F(t,x), \frac{\partial F}{\partial x} = e^{\gamma t} f'(W(t)), \frac{\partial^2 F}{\partial x^2} = e^{\gamma t}\lambda f(W(t)) = \lambda F(t,x)$$

$$\begin{aligned}
d(X_t) &= df(t, W(t)) \\
&= \gamma F(t,x)dt + e^{\gamma t} f'(W(t))dW(t) + \frac{1}{2}\lambda F(t,x)d^2 W(t) \\
&= \gamma F(t,x)dt + e^{\gamma t} f'(W(t))dW(t) + \frac{1}{2}\lambda F(t,x)d(t) \\
&= e^{\gamma t} f(w(t))(\gamma + \frac{1}{2}\lambda)dt + e^{\gamma t} f'(W(t))dW(t)
\end{aligned}$$

Integration of the both sides

$$X(t) = X(0) + (\gamma + \frac{1}{2}\lambda)\int_0^t e^{\gamma z} f(W(z))dz + \int_0^t e^{\gamma z} f'(W(z))dW(z)$$

The expectation of an Ito integral is zero, so $\mathbb{E}[\int_0^t e^{\gamma z} f'(W(z))dW(z)] = 0$

$$\mathbb{E}[X(t)] = 1 + (\gamma + \frac{1}{2}\lambda)\mathbb{E}[\int_0^t e^{\gamma z} f(W(z))dz]$$

$$\mathbb{E}[e^{\gamma t} f(W(t))] = 1 + (\gamma + \frac{1}{2}\lambda)\mathbb{E}[\int_0^t e^{\gamma z} f(W(z))dz]$$

Set $\gamma = -\frac{1}{2}\lambda$

$$\mathbb{E}[f(W(t))] = e^{\frac{1}{2}}\lambda t$$