Homework 4

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Exercise 4.10

i

$$dX(t) = \Delta(t) + r(X(t) - \Delta(t)S(t))dt \tag{1}$$

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t) \tag{2}$$

Using Ito product rule

$$dX(t) = d(\Delta(t)S(t)) + d(\Gamma(t)M(t))$$

$$dX(t) = S(t)d\Delta(t) + \Delta(t)dS(t) + d\Delta(t)dS(t) + \Gamma(t)dM(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t) \tag{3}$$

(2) $\rightarrow$ (1)

$$dX(t) = \Delta(t)dS(t) + r\Gamma(t)M(t)dt$$

$$dX(t) = \Delta(t)dS(t) + \Gamma(t)dM(t) \tag{4}$$

(3)-(4)

$$S(t)d\Delta(t) + d\Delta(t)dS(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t) = 0$$

ii

Using Ito product rule

$$\begin{aligned}
N(t) &= \Gamma(t)M(t) \\
dN(t) &= d\Gamma(t)M(t) + \Gamma(t)dM(t) + d\Gamma(t)dM(t) \\
dN(t) &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))dS(t)dS(t) \\
&\quad - \Delta(t)dS(t) - S(t)d\Delta(t) - d\Delta(t)dS(t)
\end{aligned}$$

put last two equations together and use continuous-time self financing condition

$$
\begin{aligned}
\Gamma(t)dM(t) + d\Gamma(t)M(t) + d\Gamma(t)dM(t) + S(t)d\Delta(t) + d\Delta(t)dS(t) \\
= c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) - \Delta(t)dS(t) \\
\Gamma(t)dM(t) = c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) - \Delta(t)dS(t) \\
\frac{N(t)}{M(t)}rM(t)dt = [c_x(t,S(t)) - \Delta(t)]dS(t) + [c_t(t,S(t)) + \frac{1}{2}\sigma^2S^2(t)c_xx(t,S(t))]dt
\end{aligned}
$$

use the delta-hedging formula $\Delta(t) = c_x(t,S(t))$ to cancel out the dS(t)

$$
\begin{aligned}
N(t)dt &= [c_t(t,S(t)) + \frac{1}{2}\sigma^2S^2(t)c_xx(t,S(t))]dt \\
dN(t) &= [c_t(t,S(t)) + \frac{1}{2}\sigma^2S^2(t)c_xx(t,S(t))]dt
\end{aligned}
$$

Exercise 4.12

i

$$
\begin{aligned}
&p(t,x) = c(t,x) - f(t,x) \\
&\Delta : p_x(t,x) = c_x(t,x) - f_x(t,x) = N(d_+(T-t,x)) - 1 \\
&\Gamma : p_{xx} = c_{xx}(t,x) = N'(d_+(T-t,x)\frac{\partial}{\partial x}d_+(T-t,x)) \\
&\Theta : p_t(t,x) = c_t(t,x) - f_t(t,x) \\
&= -rke^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) + rke^{-r(T-t)} \\
&= rke^{-r(T-t)}(1 - N(d_-(T-t,x))) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) + rke^{-r(T-t)} \\
&= rke^{-r(T-t)}(N(-d_-(T-t,x))) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) + rke^{-r(T-t)}
\end{aligned}
$$

ii

For an agent hedging a short position in the put for hold $p_x(t,x)$ shares of stock.

$p_x(t,x) = c_x(t,x) - f_x(t,x) = N(d_+(T-t,x)) - 1$, so $p_x < 0$, so he will short the underlying shock.

He will invest $p(t,St) - p_x(t,S(t))$ in the money account

$$\begin{aligned}
&p(t,St) - S(t)p_x(t,S(t))\\
&= c(t,S(t)) - f(t,S(t)) - S(t)p_x(t,S(t))\\
&= S(t)N(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) - S(t)\\
&+ Ke^{-r(T-t)} - S(t)(N(d_+(T-t,x)) - 1)\\
&= Ke^{-r(T-t)}(1 - N(d_-(T-t,x)))\\
&= Ke^{-r(T-t)}N(-d_-(T-t,x)) > 0
\end{aligned}$$

iii

$$\begin{aligned}
&f(t,S(t)) = S(t) - Ke^{-r(T-t)}\\
&f_t = -Kre^{-r(T-t)} \quad f_x = 1 \quad f_{xx} = 0\\
&\quad f_t + rS(t)f_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x)\\
&= f_t + rS(t) = -Kre^{-r(T-t)} + rS(t) = rf(t,S(t))
\end{aligned}$$

so f(t,S(t)) satisfy Black-Scholes-Merton partial differential equation

Because c(t,S(t)) and f(t,S(t)) satisfy Black-Scholes-Merton partial differential equation, so $p(t,S(t)) = c(t,S(t)) - f(t,S(t))$ satisfy BSM PDE too

Exercise 4.13

$$\begin{aligned}
&dB_1(t)dB_2(t) = \rho(t)dt\\
&dW_1(t) = dB_1(t)\\
&dW_2(t) = -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t)
\end{aligned}$$

To show $W_1(t)$ and $W_2(t)$ are independent Brownian motions, we have to use "Levy, two dimensions".WTS $W_1(t), W_2(t)$ satisfy all the conditions of "Levy, two dimensions".

i) Martingale property

$$W_1(t) = B_1(t)$$

$$W_2(t) = -\int_0^t \frac{\rho(u)}{\sqrt{1-\rho^2(u)}}dB_1(u) + \int_0^t \frac{1}{\sqrt{1-\rho^2(u)}}dB_2(u)$$

Because $B_1(t)$ is a Brownian motion, so $W_1(t) = B_1(t)$ is a Brownian motion too, it has martingale property.Because Ito integral is a martingale, so we have $W_2(t)$ has martingale property.

ii) Starting at zero

$$W_1(0) = B_1(0) = 0$$
$$W_2(0) = -\int_0^0 \frac{\rho(u)}{\sqrt{1-\rho^2(u)}} dB_1(u) + \int_0^0 \frac{1}{\sqrt{1-\rho^2(u)}} dB_2(u) = 0$$

so we have $W_1(0) = 0, W_2(t) = 0$

iii) Continuity

$W_1(t)$ is a Brownian motion, so it has continuous paths

Ito integral has continuous paths, so $W_2(t)$ has continuous paths

iv) Unit quadratic variation

$$dW_1(t)dW_1(t) = dB_1(t)dB_1(t)$$

Because $B_1(t)$ is a Brownian motion, so $dB_1(t)dB_1(t) = t$, so $dW_1(t)dW_1(t)$=t

$$\begin{aligned}
&dW_2(t)dW_2(t)\\
&= (-\frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t))^2 + (\frac{1}{\sqrt{1-\rho^2(t)}} dB_2(t))^2 - 2\frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t) \frac{1}{\sqrt{1-\rho^2(t)}} dB_2(t)\\
&= \frac{\rho^2(t)}{1-\rho^2(t)} dt + \frac{1}{1-\rho^2(t)} dt - 2\frac{\rho^2(t)}{1-\rho^2(t)} dt\\
&= \frac{1-\rho^2(t)}{1-\rho^2(t)} dt\\
&= dt
\end{aligned}$$

v) Zero cross variation

$$\begin{aligned}
dW_1(t)dW_2(t) &= dB_1(t)(-\frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}} dB_2(t))\\
&= -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dt + \frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dt\\
&= 0
\end{aligned}$$

Using "Levy, two dimensions", we have $W_1(t), W_2(t)$ are independent Brownian motions.

Exercise 4.14

i

$$Z_j = f''(W(t_j))[(W(t_j) - W(t_j))^2 - (t_{j+1} - t_j)]$$

Because $f''(W(t_j))$, $(W(t_j) - W(t_j))^2$ is $\mathbb{F}(t_{j+1})$-measurable, so $Z_j is\ \mathbb{F}(t_{j+1})$-measurable too.

$$\begin{aligned}
\mathbb{E}[Z_j|\mathbb{F}(t_j)] &= \mathbb{E}[f''(W(t_j))[(W(t_j) - W(t_j))^2 - (t_{j+1} - t_j)]|\mathbb{F}(t_j)] \\
&= f''(W(t_j))\mathbb{E}[[(W(t_j) - W(t_j))^2 - (t_{j+1} - t_j)]|\mathbb{F}(t_j)] \\
&(''Taking \quad out \quad what \quad is \quad known'') \\
&= f''(W(t_j))\mathbb{E}[(W(t_j) - W(t_j))^2 - (t_{j+1} - t_j)] \\
&(''Independence'') \\
&= -(t_{j+1} - t_j)f''(W(t_j)) + (Var[W(t_j) - W(t_j)] - (\mathbb{E}[W(t_j) - W(t_j)])^2)f''(W(t_j)) \\
&= -(t_{j+1} - t_j)f''(W(t_j)) + (t_{j+1} - t_j + 0)f''(W(t_j)) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[(Z_j)^2|\mathbb{F}(t_j)] &= \mathbb{E}[(f''(W(t_j)))^2[(W(t_j) - W(t_j))^4 + (t_{j+1} - t_j)^2 - 2(W(t_j) - W(t_j))^2(t_{j+1} - t_j)]|\mathbb{F}(t_j)] \\
&= (f''(W(t_j)))^2\mathbb{E}[[(W(t_j) - W(t_j))^4 + (t_{j+1} - t_j)^2 - 2(W(t_j) - W(t_j))^2(t_{j+1} - t_j)]|\mathbb{F}(t_j)] \\
&(''Taking \quad out \quad what \quad is \quad known'') \\
&= (f''(W(t_j)))^2\mathbb{E}[(W(t_j) - W(t_j))^4 + (t_{j+1} - t_j)^2 - 2(W(t_j) - W(t_j))^2(t_{j+1} - t_j)] \\
&(''Independence'') \\
&= (f''(W(t_j)))^2\mathbb{E}[3(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 - 2((t_{j+1} - t_j)^2] \\
&= 2(f''(W(t_j)))^2(t_{j+1} - t_j)^2
\end{aligned}$$

(The fourth moment of a normal random variable with zero mean is three times its variance squared)

ii

$$\begin{aligned}
\mathbb{E}[\sum_{j=0}^{n-1} Z_j] &= \sum_{j=0}^{n-1} \mathbb{E}[Z_j] \\
&= \sum_{j=0}^{n-1} \mathbb{E}[\mathbb{E}[Z_j|\mathbb{F}(t_j)]] \\
&= n * 0 \\
&= 0
\end{aligned}$$

iii

$$
\begin{aligned}
Var[\sum_{j=0}^{n-1} Z_j] &= \mathbb{E}[(\sum_{j=0}^{n-1} Z_j)^2] - (\mathbb{E}[(\sum_{j=0}^{n-1} Z_j)])^2 \\
&= \mathbb{E}[(\sum_{j=0}^{n-1} Z_j)^2] \\
&= \mathbb{E}[\sum_{j=0}^{n-1}(Z_j)^2 + 2\sum_{0\le i<j\le n-1} Z_i Z_j] \\
&= \sum_{j=0}^{n-1}\mathbb{E}[(Z_j)^2] + 2\sum_{0\le i<j\le n-1}\mathbb{E}[Z_i Z_j] \\
&= \sum_{j=0}^{n-1}\mathbb{E}[\mathbb{E}[(Z_j)^2|\mathbb{F}t_j]] + 2\sum_{0\le i<j\le n-1}\mathbb{E}[\mathbb{E}[Z_i|\mathbb{F}(t_i)]\mathbb{E}[Z_j|\mathbb{F}(t_j)]] \\
&= \sum_{j=0}^{n-1}\mathbb{E}[2(f''(W(t_j)))^2(t_{j+1}-t_j)^2] \\
&= \sum_{j=0}^{n-1}\mathbb{E}[2(f''(W(t_j)))]^2(t_{j+1}-t_j)^2 \\
&\le \max_{0\le j\le n-1}|t_{j+1}-t_j|\sum_{j=0}^{n-1}\mathbb{E}[2(f''(W(t_j)))]^2(t_{j+1}-t_j) \\
&= \lim_{\pi\to 0} 2\pi\sum_{j=0}^{n-1}\mathbb{E}[(f''(W(t_j)))]^2(t_{j+1}-t_j) \\
&= 0
\end{aligned}
$$

$(\sum_{j=0}^{n-1}\mathbb{E}[(f''(W(t_j)))]^2(t_{j+1}-t_j) < \infty)$

Exercise 4.15

i

Use "Levy, one dimension" to prove B_i is a Brownian motion

Because $(W_1(t), ..., W_d(t))$ is a d-dimensional Brownian motion, so $(W_1(0), ..., W_d(0)) = 0$, so $\sigma_{1j} = 0$,j=1,...,d, so $B_i(0) = 0$

$B_i(t)$ is a sum of Ito integrals, so it has the continuity and the martingale property.

$$dB_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t)$$

$$dB_i(t)dB_i(t) = \frac{1}{\sigma_i^2(t)} \sum_{j=1}^{d} \sigma_{ij}^2(t) d(t) = \frac{\sum_{j=1}^{d} \sigma_{ij}^2(t)}{\sum_{j=1}^{d} \sigma_{ij}^2(t)} dt = dt$$

SO B_i satisfies all condition of "Levy, one dimension", so it is a Brownian motion

ii

For $W_i(t), W_j(t), i \neq j$,they are independent, so $dW_i(t)dW_j(t) = 0, i \neq j$

$$\begin{aligned}
dB_i(t)dB_i(t) &= [\sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t)][\sum_{l=1}^{d} \frac{\sigma_{kl}(t)}{\sigma_k(t)} dW_l(t)] \\
&= [\sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t)][\frac{\sigma_{kj}(t)}{\sigma_k(t)} dW_j(t)] \\
&= \frac{1}{\sigma_i(t)\sigma_k(t)} \sum_{j=1}^{d} \sigma_{ij}\sigma_{kj} dt \\
&= \rho_{ik}(t) dt
\end{aligned}$$

Exercise 4.18

i

Let $\zeta(t,x) = exp\{-\theta x - (r + \frac{1}{2}\theta^2)t\}$. We have

$$\frac{\partial \zeta}{\partial t} = -(r + \frac{1}{2}\theta^2)\zeta(t,x), \quad \frac{\partial \zeta}{\partial x} = -\theta\zeta(t,x), \quad \frac{\partial^2 \zeta}{\partial x^2} = \theta^2 \zeta(t,x)$$

$$\begin{aligned}
d\zeta(t) &= d\zeta(t, W(t)) \\
&= \frac{\partial \zeta}{\partial t} dt + \frac{\partial \zeta}{\partial x} dW(t) + \frac{1}{2} \frac{\partial^2 \zeta}{\partial x^2} dW^2(t) \\
&= -(r + \frac{1}{2}\theta^2)\zeta(t)dt - = -\theta\zeta(t)dW(t) + \frac{1}{2}\theta^2\zeta(t)dt \\
&= -r\zeta(t)dt - \theta\zeta(t)dW(t)
\end{aligned}$$

ii

$$\begin{aligned}
d(\zeta(t)X(t)) &= \zeta(t)dX(t) + X(t)d\zeta(t) + dX(t)d\zeta(t) \\
&= rX(t)\zeta(t)dt + \delta(t)(\alpha - r)S(t)\zeta(t)dt + \delta(t)\sigma S(t)\zeta(t)dW(t) \\
&\quad - X(t)r\zeta(t)dt - X(t)\theta\zeta(t)dW(t) - \theta\zeta(t)\delta(t)\sigma S(t)dt \\
&= \zeta(t)(\delta(t)\sigma S(t) - \theta X(t))dW(t)
\end{aligned}$$

So we have

$$\zeta(t)X(t) = \zeta(0)X(0) + \int_0^t \zeta(u)(\delta(u)\sigma S(u) - \theta X(u))dW(u)$$

So $\zeta(t)X(t)$ is Ito integral, it is a martingale

iii

Let $\theta(t)$ be an adapted portfolio process satisfy X(t)=V(t)

$$\zeta(0)X(0) = exp\{-\theta W(0) - (r + \frac{1}{2}\theta^2)0\}X(0) = X(0)$$

Because $\zeta(t)X(t)$ is a martingale, hence it has constant expectation.

$$\begin{aligned}
\zeta(0)X(0) &= \mathbb{E}[\zeta(t)X(t)] \\
X(0) &= \mathbb{E}[\zeta(t)V(t)]
\end{aligned}$$