Homework 1

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September 14, 2015

Exercise 1.5

The function $\mathbb{I}_{[0,X(\omega)}(x)$ is a indicator function, so $\mathbb{I}_{[0,X(\omega)}(x) = 1$ when $x \in [0, X(\omega))$

$$\int_\Omega \int_0^\infty \mathbb{I}_{[0,X(\omega)}(x) dx d\mathbb{P}(w) = \int_\Omega \int_0^{X(\omega)} 1 dx d\mathbb{P}(w) = \int_\Omega X(\omega) d\mathbb{P}(w) = \mathbb{E}X$$

The function $\mathbb{I}_{[0,X(\omega)}(x) = 1$, so it is integrable, so the order of the integration can be reversed

$$\int_\Omega \int_0^\infty \mathbb{I}_{[0,X(\omega)}(x) dx d\mathbb{P}(w) = \int_0^\infty \int_\Omega \mathbb{I}_{[0,X(\omega)}(x) dx d\mathbb{P}(w) = \int_0^\infty \int_\Omega 1_{(x<X(\omega))} d\mathbb{P}(w) dx$$

$$= \int_0^\infty P(x < X) dx = \int_0^\infty 1 - P(X \leq x) dx = \int_0^\infty 1 - F(x) dx$$

so

$$\mathbb{E}(X) = \int_0^\infty 1 - F(x) dx$$

Exercise 1.6

i

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Verify that

$$\mathbb{E}(e^{uX}) = e^{u\mu + \frac{1}{2}u^2\mu^2}$$

$$\mathbb{E}(e^{uX}) = \int_{-\infty}^\infty e^{ux} f(x) dx = \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{2\sigma^2 ux - (x^2+\mu^2-2\mu x)}{2\sigma^2}} dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(u\sigma^2+\mu)]^2 - u^2\sigma^4 - 2u\mu\sigma^2}{2\sigma^2}} dx$$

$$= e^{\frac{u^2\sigma^2+2u\mu}{2}} \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(u\sigma^2+\mu)^2]^2}{2\sigma^2}} dx$$

because

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(u\sigma^2+\mu)^2]^2}{2\sigma^2}} dx$$

is cdf of random variable with expectation $u\sigma^2 + \mu$ and deviation σ^2

so

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(u\sigma^2+\mu)^2]^2}{2\sigma^2}} dx = F(\infty) - F(-\infty) = 1$$

So

$$\mathbb{E}(e^{uX}) = e^{\frac{u^2\sigma^2+2u\mu}{2}}$$

Exercise 1.7

i

$$f(x) = \lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}$$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{1}{\sqrt{2n\pi}} = 0 \quad and \quad \lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} e^{-\frac{x^2}{2n}} = 1$$

so

$$f(x) = \lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}} = 0$$

ii

$$\int_{\infty}^{-\infty} \frac{1}{\sqrt{2nx}} e - \frac{x^2}{2n} dx$$

it is the cfd of the normal random variable with expection 0 and deviation n

so it is equal to $F(\infty) - F(-\infty) = 1$

so

$$\lim_{n\to\infty} \int_{\infty}^{-\infty} \frac{1}{\sqrt{2nx}} e - \frac{x^2}{2n} dx = \lim_{n\to\infty} 1 = 1$$

iii

the condition of the Monetone Convergence Theorem is $0 \leq f_1 \leq f_2 \leq f_3 \leq ...$ almost everywhere to a function f

set $t = \frac{1}{\sqrt{2n}}$ and n$\in R$

$$\frac{df_n(x)}{dn} = \frac{df_n(x)}{dt}\frac{dt}{dn} = e^{-x^2t^2}(1-2t^2)(-\frac{1}{2\sqrt{2}}n^{-\frac{3}{2}})$$

$$= e^{-x^2t^2}(1-\frac{1}{n})(-\frac{1}{2\sqrt{2}}n^{-\frac{3}{2}}) < 0$$

so f(n)$\geq f(n+1) \quad (n \in 1,2,3,...)$

so $f_n(x)$ is not in accordance with the condition of the Monetone Convergence Theorem

so this does not violate the Monetone Convergence Theorem

Exercise 1.10

i

$\mathbb{B}[0,1]$ is Borel $\sigma - algebra$ and A$\in \mathbb{B}[0,1]$

$P(\Omega) = 1 \quad and \quad A$ satisfied with countable additivity

so $S_0(\Omega, \mathbb{A}, \mathbb{P})$ is a probability space

because Z is a nonnegative random variable and $\mathbb{E}(Z) = 1$

defined $\tilde{\mathbb{P}}(A) = \int^A Z(\omega)d\mathbb{P}(w)$

so $\tilde{\mathbb{P}}$ is a probability measure

ii

if $\tilde{\mathbb{P}}(A) = 0$ then A$\notin \mathbb{B}[0,1]$

$$\tilde{\mathbb{P}}(A) = \int_{A\cup[0,\frac{1}{2})} 0d\mathbb{P}(\omega) + \int_{A\cup[\frac{1}{2},1]} 2d\mathbb{P}(\omega) = 2P(A\cup[\frac{1}{2},1]) + 0P(A\cup[0,\frac{1}{2})) = 0$$

iii

$$\tilde{\mathbb{P}}(A) = 2P(A \cup [\frac{1}{2}, 1]) + 0P(A \cup [0, \frac{1}{2}))$$

$\tilde{\mathbb{P}}(A) = 0, P(A) > 0$ so A is the subset of $[0, \frac{1}{2})$

Exercise 1.13

i

$$\frac{1}{\epsilon}\mathbb{P}\{X \in B(x, \epsilon)\} = \frac{1}{\epsilon}\int_{x+\frac{\epsilon}{2}}^{x-\frac{\epsilon}{2}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du$$

according to the Mean Value Theorem

$$\frac{\int_{x+\frac{\epsilon}{2}}^{x-\frac{\epsilon}{2}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du}{(x+\frac{\epsilon}{2}) - (x-\frac{\epsilon}{2})} = \frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}} \qquad \xi \in [x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$$

because x$\in$ $[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$ so

$$\frac{\int_{x+\frac{\epsilon}{2}}^{x-\frac{\epsilon}{2}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du}{(x+\frac{\epsilon}{2}) - (x-\frac{\epsilon}{2})} = \frac{\int_{x+\frac{1}{\epsilon}}^{x-\frac{1}{\epsilon}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du}{\epsilon} = \frac{1}{\epsilon}\int_{x+\frac{1}{\epsilon}}^{x-\frac{1}{\epsilon}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \doteq \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

ii

just use the Mean Value Theorem too

$$\frac{1}{\epsilon}\tilde{\mathbb{P}}\{Y \in B(y, \epsilon)\} = \frac{1}{\epsilon}\int_{y+\frac{\epsilon}{2}}^{y-\frac{\epsilon}{2}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \doteq \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

iii

$$\{X \in B(x, \epsilon)\} \to \{Y - \theta \in B(x, \epsilon)\} \to \{Y \in B(x + \theta, \epsilon)\} \to \{Y \in B(y, \epsilon)\}$$

iv

$$\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \doteq exp\frac{X^2(\overline{\omega}) - Y^2(\overline{\omega})}{2} = exp\frac{X^2(\overline{\omega}) - (X(\overline{\omega}) + \theta)^2}{2} = exp\{\frac{-\theta^2 - 2\theta X(\overline{\omega})}{2}\}$$

Exercise 2.2

i
$$\sigma(X) = \{\emptyset, \Omega, X = 1, X = 0\} = \{\emptyset, \Omega, \{HT, TH\}, \{HH, TT\}\}$$

ii
$$\sigma(S_1) = \{\emptyset, \Omega, S_1 = 8, S_1 = 2\} = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\}$$

iii the condition for A and B is independent under probability measure$\tilde{\mathbb{P}}$ is

$$\tilde{\mathbb{P}}(A \cap B) = \tilde{\mathbb{P}}(A) * \tilde{\mathbb{P}}(B)$$

$$\tilde{\mathbb{P}}(\{HT, TH\} \cap \{HH, HT\}) = \tilde{\mathbb{P}}(\{HT\}) = \frac{1}{4}, \quad \tilde{\mathbb{P}}(\{HT, TH\}) = \frac{1}{2}, \quad \tilde{\mathbb{P}}(\{HH, HT\}) = \frac{1}{2}$$

so
$$\tilde{\mathbb{P}}(\{HT, TH\} \cap \{HH, HT\} = \tilde{\mathbb{P}}(\{HT, TH\}) * \tilde{\mathbb{P}}(\{HH, HT\})$$

in this way, we can get $\tilde{\mathbb{P}}(A \cap B) = \tilde{\mathbb{P}}(A) * \tilde{\mathbb{P}}(B)$ for $A \in \sigma(X) \quad B \in \sigma(S_1)$

so $\sigma(X) \quad and \quad \sigma(S_1)$ are independent under measure $\tilde{\mathbb{P}}$

iv use the same method as in **iii**

$$\tilde{\mathbb{P}}(\{HT, TH\} \cap \{HH, HT\}) = \tilde{\mathbb{P}}(\{HT\}) = \frac{2}{9}, \quad \tilde{\mathbb{P}}(\{HT, TH\}) = \frac{4}{9}, \quad \tilde{\mathbb{P}}(\{HH, HT\}) = \frac{2}{3}$$

so
$$\tilde{\mathbb{P}}(\{HT, TH\} \cap \{HH, HT\} \neq \tilde{\mathbb{P}}(\{HT, TH\}) * \tilde{\mathbb{P}}(\{HH, HT\})$$

so $\sigma(X) \quad and \quad \sigma(S_1)$ are not independent under measure $\mathbb{P}$

v because $\sigma(X) \quad and \quad \sigma(S_1)$ are not independent under measure $\mathbb{P}$, under the condition of X=1, the distribution of S_1 can be changed

Exercise 2.6

i $\sigma(X) = \{\emptyset, \Omega, X = 1, X = 2\} = \{\emptyset, \Omega, \{a, b\}, \{c, d\}\}$

ii

$$\mathbb{E}[Y|X](a) = \mathbb{E}[Y|X](b) = \frac{Y(a)\mathbb{P}(a) + Y(b)\mathbb{P}(b)}{\mathbb{P}(a) + \mathbb{P}(b)} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

$$\mathbb{E}[Y|X](c) = \mathbb{E}[Y|X](d) = \frac{Y(c)\mathbb{P}(c) + Y(d)\mathbb{P}(d)}{\mathbb{P}(c) + \mathbb{P}(d)} = 0$$

$$\int_\Omega \mathbb{E}[Y|X](\omega)d\mathbb{P}(\omega) = -\frac{1}{3} * (\frac{1}{6} + \frac{1}{3}) = -\frac{1}{6}$$

$$\int_\Omega X(\omega)d\mathbb{P}(\omega) = \frac{1}{6} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} = -\frac{1}{6} = \int_\Omega \mathbb{E}[Y|X](\omega)d\mathbb{P}(\omega)$$

so the partial-averaging property is satisfied

iii

$$\int_\Omega \mathbb{E}[Z|X](\omega)d\mathbb{P}(\omega) = \mathbb{E}[Z|X] = \mathbb{E}[X+Y|X] = X+\mathbb{E}[Y|X] = X_{(-1,1)} - \frac{1}{6} = -\frac{1}{6}$$

$$\int_\Omega Z(\omega)d\mathbb{P}(\omega) = \mathbb{E}[Z] = \frac{1}{3} * (-2 * \frac{1}{4}) = -\frac{1}{6}$$

so the partial-averaging property is satisfied

iv

$$\mathbb{E}[Z|X] - \mathbb{E}[Y|X] = -\frac{1}{6} + X - (-\frac{1}{6}) = X$$

according to Taking out what is known in Theorem 2.3.2

Exercise 2.8

set α is one of the $\sigma(X)$ measurable r.v.

$$\mathbb{E}[Y_2|\alpha] = \mathbb{E}[Y - \mathbb{E}[Y|X]|\alpha] = \mathbb{E}[Y|\alpha] - \mathbb{E}[\mathbb{E}[Y|X]|\alpha]$$

because α is $\sigma(X)$ measurable

so

$$\mathbb{E}[Y_2|\alpha] = \mathbb{E}[Y|\alpha] - \mathbb{E}[Y|\alpha] = 0$$

according to Iterated conditioning

so Y_2 and X are uncorrelated

Exercise 2.10

$$\int_A g(X)d\mathbb{P} = \int_{-\infty}^{\infty} g(x)f_X(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{yf_{x,y}(x,y)}{f_X(x)}f_X(x)dydx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf_{x,y}(x,y)dydx = \mathbb{E}[Y] = \int_A Yd\mathbb{P}$$

extra problem

A

$$E[M_1] = p - q = M_0 = 0 \qquad p + q = 1$$

so $q = p = 0.5$

B

$$E[M_{302}|M_{300} = 60] = E[M_{300}] + E[M_2] = 60 + 2 * \frac{1}{4} + 0 * \frac{1}{2} - 2 * \frac{1}{4} = 60$$