Solution Algorithm

Flight to Safety and International Risk Sharing

This document describes the numerical solution method used in the paper. The referenced equations can be found in appendix A.6.

Additional definitions

The numerical solution stores the value functions and consumption choices at each iteration scaled by the representative agents' wealth in that state. In particular, for Home, define

$$
\begin{aligned}
\hat{v}_t &\equiv \tilde{\mu}_t^{-1} \tilde{v}_t \\
\hat{c}_t &\equiv \tilde{\mu}_t^{-1} \tilde{c}_t
\end{aligned}
$$

where the scaling variable $\tilde{\mu}_t$ is an affine transformation of the representative Home agent's wealth in the current state,

$$\tilde{\mu}_t = \frac{\theta_t(\pi_t + (1-\delta)q_t^k)\bar{\tilde{\tilde{k}}}_{t-1} + \bar{a}}{\bar{b}}.$$

The constants $\bar{a}$ and $\bar{b}$ are chosen to ensure that the scaling variable remains positive and centered around 1 while solving the model. We equivalently define $\tilde{\mu}_t^*$, $\hat{v}_t^*$ and $\hat{c}_t^*$.

Following (34) and (37), pricing kernel and certainty equivalent can then be written as

$$\tilde{m}_{t,t+1} = \beta \left(\tilde{\mu}_{t+1}\right)^{-\gamma} \left(\frac{\hat{v}_{t+1}}{\tilde{ce}_t}\right)^{1/\psi-\gamma} \frac{\left(\hat{c}_{t+1}\right)^{-1/\psi} \Phi(\ell_{t+1})^{1-1/\psi}}{\left(\tilde{c}_t\right)^{-1/\psi} \Phi(\ell_t)^{1-1/\psi}}, \tag{N1}$$

$$\tilde{ce}_t = \mathbb{E}_t \left[\exp\left((1-\gamma)\left[\sigma^z \epsilon_{t+1}^z + \varphi_{t+1}\right]\right)\left(\tilde{\mu}_{t+1}\hat{v}_{t+1}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}. \tag{N2}$$

Note that $\tilde{\mu}_{t+1}$ is a function of Home's savings and portfolio choice at t, as

$$
\begin{aligned}
\tilde{\mu}_{t+1} = \frac{1}{\bar{b}} \Bigg[& \frac{1+r_{t+1}}{1-\omega_t}\tilde{\tilde{b}}_{Ht} + \frac{1}{q_{t+1}}(1+r_{t+1}^*)\tilde{\tilde{b}}_{Ft} \\
& + (\pi_{t+1} + (1-\delta)q_{t+1}^k)\tilde{\tilde{k}}_t - \omega_{t+1}\frac{\zeta^* q_{t+1}^{-1}\tilde{c}_{t+1}^*}{\tilde{c}_{t+1} + \zeta^* q_{t+1}^{-1}\tilde{c}_{t+1}^*}\tilde{b}_{Ht+1,s}^g + \bar{a} \Bigg],
\end{aligned} \tag{N3}
$$

and a key idea of the numerical algorithm will be to solve the consumption and portfolio choice problems at each iteration taking as given $\hat{v}_{t+1}$, $\hat{c}_{t+1}$, ℓ_t and ℓ_{t+1}, so that savings and portfolio choices at t are only affecting the pricing kernel and certainty equivalent between t and $t+1$ through $\tilde{\mu}_{t+1}$ and $\tilde{c}_t$, and equivalently for Foreign.

Numerical algorithm

State space and expectations The model is solved over a sparse seven-dimensional Smolyak grid. We form expectations over future values as weighted sums over Gauss-Hermite quadrature nodes for the four normally distributed shocks ϵ^z_{t+1}, ϵ^p_{t+1}, ϵ^ω_{t+1} and ϵ^F_{t+1}, plus one additional node for the disaster shock φ_{t+1}. On a given grid point, each quadrature node is associated with a transition to a new state in the next period. Values of relevant variables at $t+1$ in that state are calculated using Chebyshev interpolation, as this state will generically lie off the grid.

Overview over solution steps We solve the model through backward iteration, while dampening the updating of policies, asset prices and individuals' expectations over the dynamics of the aggregate states. Rather than solving for market clearing prices and agents' savings and portfolio choices jointly, the solution algorithm splits this problem into separate steps. While this approach does not yield the correct solution in one round, it exploits that thousands of iterations are needed to solve the model recursively. As all policy choices are slowly converging (through dampened updates), in the end the correct solution is obtained at which all policies are consistent with each other.

Initialization The algorithm starts from initial guesses for value functions, $\hat{v}$ and $\hat{v}^*$, consumption, $\hat{c}$ and $\hat{c^*}$, labor allocations, ℓ and ℓ^*, prices, i, i^*, q^k, s, P/P_{-1} and P^*/P^*_{-1}, as well as portfolio allocations in the Home and Foreign bond at each grid point. Given an initial guess on the transition dynamics of aggregate states for each state and quadrature node, we use these guesses to form expectations over next period's values through interpolation, which are then held fixed within the current iteration and are taken as given in the steps below.

Computational steps At each iteration, the subroutine `calc_equilibrium_and_update` in `mod_calc.f90` performs the following steps for each grid point:

1. **Current period production** Given aggregate capital, $\tilde{\tilde{\bar{k}}}_{t-1}$, and relative productivity, z_t^F, as well as current guesses on aggregate labor supply in each country, ℓ_t and ℓ_t^*, and the terms of trade, s_t, we calculate the capital allocation across countries, $\tilde{\kappa}_t$ and $\tilde{\kappa}_t^*$, real wages, $\tilde{w}_t$ and $\tilde{w}_t^*$, and profits, π_t, by using equations (54)-(57), (63) and (70)-(72). Given the state variable θ_t, we can now determine the right-hand sides of the budget constraints (41) and (50).

2. **Next period production** As in the previous step, expectations over capital, relative productivity, labor supply and the terms of trade are used to calculate expected profits and wages. Together with expectations over the price of capital, we are now able to form expectations over capital returns following (69). These return expectations will be taken as given in the following steps.

3. **Price of capital** The assumed dynamics of aggregate states imply a value for $\tilde{\bar{k}}_t$, which we use to update the price of capital following (60).

4. **Home bond market clearing** We now solve for the nominal rate, i_t, that clears the Home bond market, as defined in equation (66), while agents' portfolio choice between capital and Home bonds is satisfied, following equations (38), (40), (47) and (49). In this step we hold constant Home and Foreign consumption choices, $\tilde{c}_t$ and $\tilde{c}_t^*$, and their positions in the Foreign bond, while also taking the return to capital and the spread between the Foreign and Home nominal rates, $i_t^* - i_t$, as given. Bond returns in this and the following steps are calculated using (67) and (68). Using the notation in (N1)-(N3), agents' portfolio choice between the Home bond and capital affects the FOC only through changes in $\tilde{\mu}_{t+1}$ and $\tilde{\mu}_{t+1}^*$. Note that solving the portfolio choice problem in that manner will not immediately yield the correct solution as the implied $\tilde{\mu}_{t+1}$ and $\tilde{\mu}_{t+1}^*$ are presumably inconsistent with the equilibrium $\tilde{\mu}_{t+1}$ and $\tilde{\mu}_{t+1}^*$ in future periods, but as the solution converges over many iterations, so will the solution for $\tilde{\mu}_{t+1}$ and $\tilde{\mu}_{t+1}^*$ become consistent with their equilibrium values in the next period.

5. **Foreign bond market clearing** We then solve for the spread between the Foreign and Home nominal rate, $i_t^* - i_t$, that clears the Foreign bond market,

while agents' portfolio choice between the Foreign and Home bonds is satisfied, following equations (38), (39), (47) and (48). In this step we hold constant Home and Foreign consumption choices, $\tilde{c}_t$ and $\tilde{c}_t^*$, and their position in capital (derived as the fraction of their savings not allocated to either the Home or the Foreign bond), while also taking the return to capital and the previously solved Home nominal rate, i_t, as given. As before, agents' portfolio choice between the Home and the Foreign bond affects the FOC only through changes in $\tilde{\mu}_{t+1}$ and $\tilde{\mu}_{t+1}^*$.

6. **Value functions, wealth transition and aggregate savings** We then update the value functions using (33), (34), (42) and (43), storing them scaled by $\tilde{\mu}_t$ and $\tilde{\mu}_t^*$ as described above. To calculate $\tilde{\mu}_t$ and $\tilde{\mu}_t^*$ we use the newly calculated i_t and i_t^* for the bond returns, but hold portfolio allocations in this iteration fixed at their initial levels. Currently assumed savings and portfolio choices as well as returns are then also used to store updates on next period's θ_{t+1} across quadrature nodes using equation (51), as well aggregate capital savings $\tilde{\tilde{k}}_t$ following equation (64).

7. **Terms of trade** For the current guess of consumption and therefore savings and investment choices, we find the terms of trade, s_t, so that the demand for the Home good relative to the Foreign good equals the relative supply of both goods, using (35), (36), (44), (45), (58), (59), (61), (62) and (65).

8. **Consumption** Holding fixed households' portfolio choices at their initial values, we solve for households' consumption-savings choice using equations (37), (40), (41), (46), (49) and (50). Rewriting (37) as in (N1), the savings choice is solved holding fixed all elements but $\tilde{\mu}_{t+1}$ and $\tilde{c}_t$, and equivalently for Foreign.

9. **Labor supply** We then calculate the union's optimal labor supply in Home and Foreign, ℓ_t and ℓ_t^*, using (52) and (53).

10. **Inflation rates** Given the initial guess on i_t and i_t^*, we solve for the implied inflation rates, P_t/P_{t-1} and P_t^*/P_{t-1}^*, using the Taylor rules.

After these steps, we have obtained new values for value functions, $\hat{v}$ and $\hat{v}^*$, consumption, $\hat{c}$ and $\hat{c^*}$, labor allocations, ℓ and ℓ^*, prices, i, i^*, q^k, s, P/P_{-1} and P^*/P^*_{-1}, as well as portfolio allocations in the Home and Foreign bond at each grid point, as well as new state transitions for each state and quadrature node. The assumed values, policies, prices, and transitions are updated using these solutions, dampened for stability. This procedure is repeated until the difference between assumed values and updates is sufficiently small.