Problem:

Initial value problem:

$$y' = -\frac{y^2}{3} - \frac{2}{3x^2}$$

Analytical Solution:

$$\boldsymbol{y = \frac{\frac{2C}{c + \sqrt[3]{x}} - 4}{2x}}$$

Numerical Approximations

Three different numerical approximation methods were used to estimate the solution to the differential equation without solving it analytically: Euler, Improved Euler, and Runge-Kutta. Their errors were then measured to analyze their accuracy and compared to the exact solution. All of them are applied only to the given domain: $x \in (1,10)$ and the initial value problem:

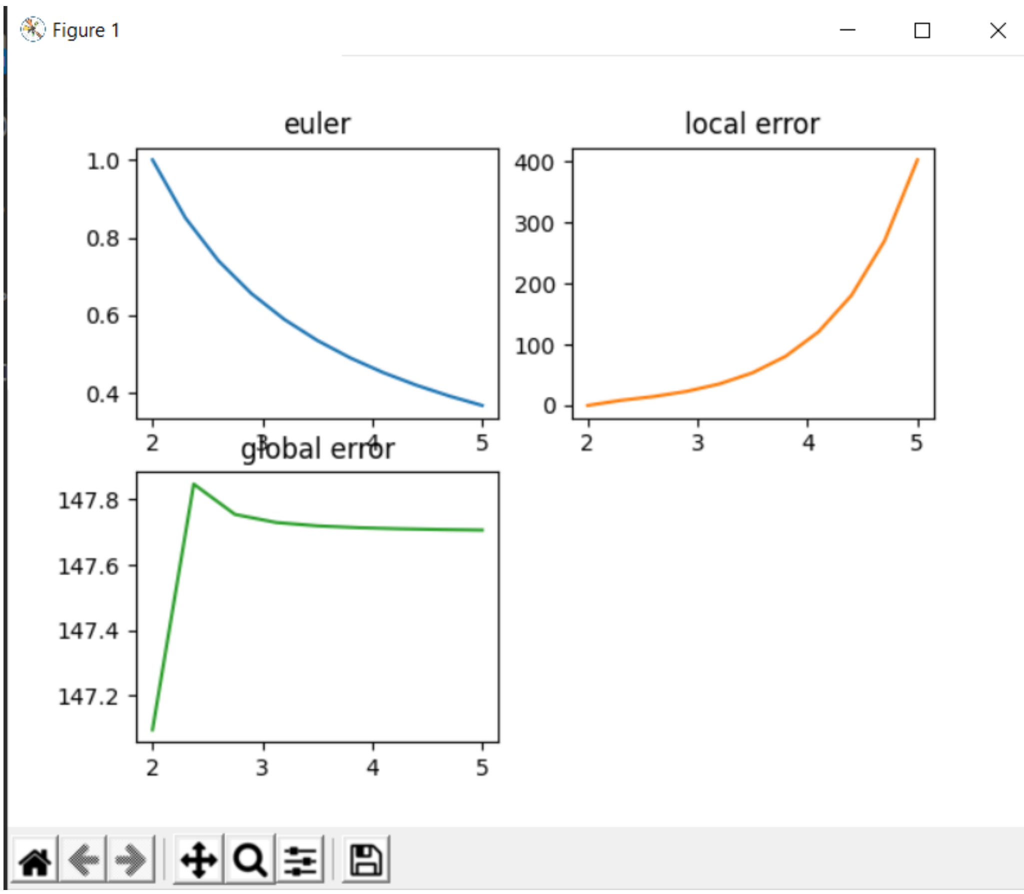
$$y'(x) = f\big(x, y(x)\big), \qquad y(x_0) = y_0$$

Euler's Method

Euler's method is defined as follows:

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$x_n = x_0 + nh$$

Where n is the step number, and h is the step size. The step size, h, affects the resolution accuracy of the approximation. The approximation approaches the exact solution as $h \to 0$. For a particular subdomain of the function $(x0, X)$, we can define the number of steps, N, such that $h = X - x0N$.

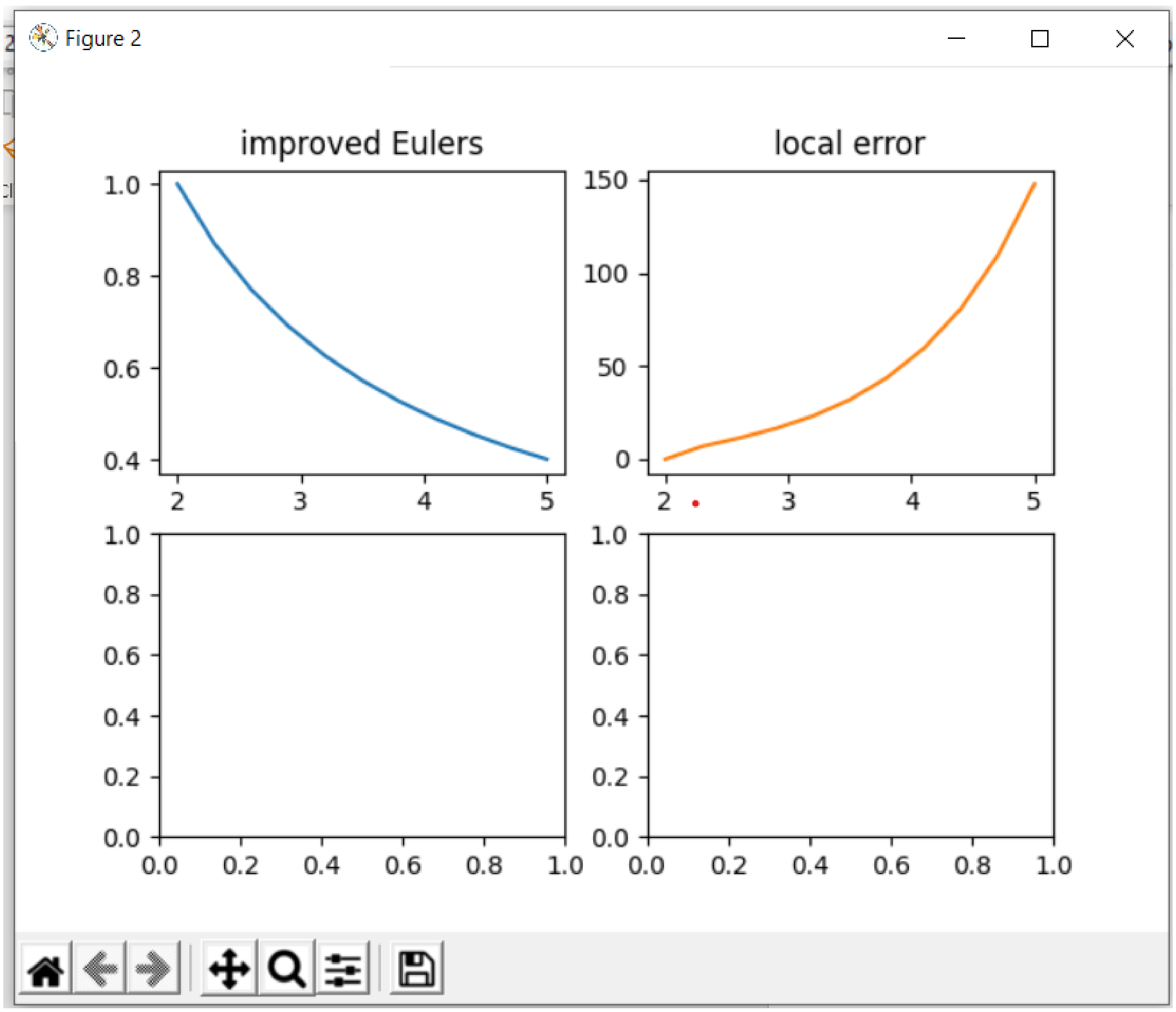


The global error for a particular N is the difference between the approximation and the solution at the last point of the domain. It is proportional to the step size h, so it approaches 0 as h approaches

Improved Euler method:

This method (also called modified Euler's method, Heun's method, or explicit trapezoidal method) is an improvement over Euler's method by calculating an intermediate value.

$$\tilde{y}_{n+1} = y_n + hf(x_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})]$$
$$x_{n+1} = x_n + h$$



Runge-Kutta Method:

The Runge-Kutta method is defined by adding more intermediate steps to the calculation:

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

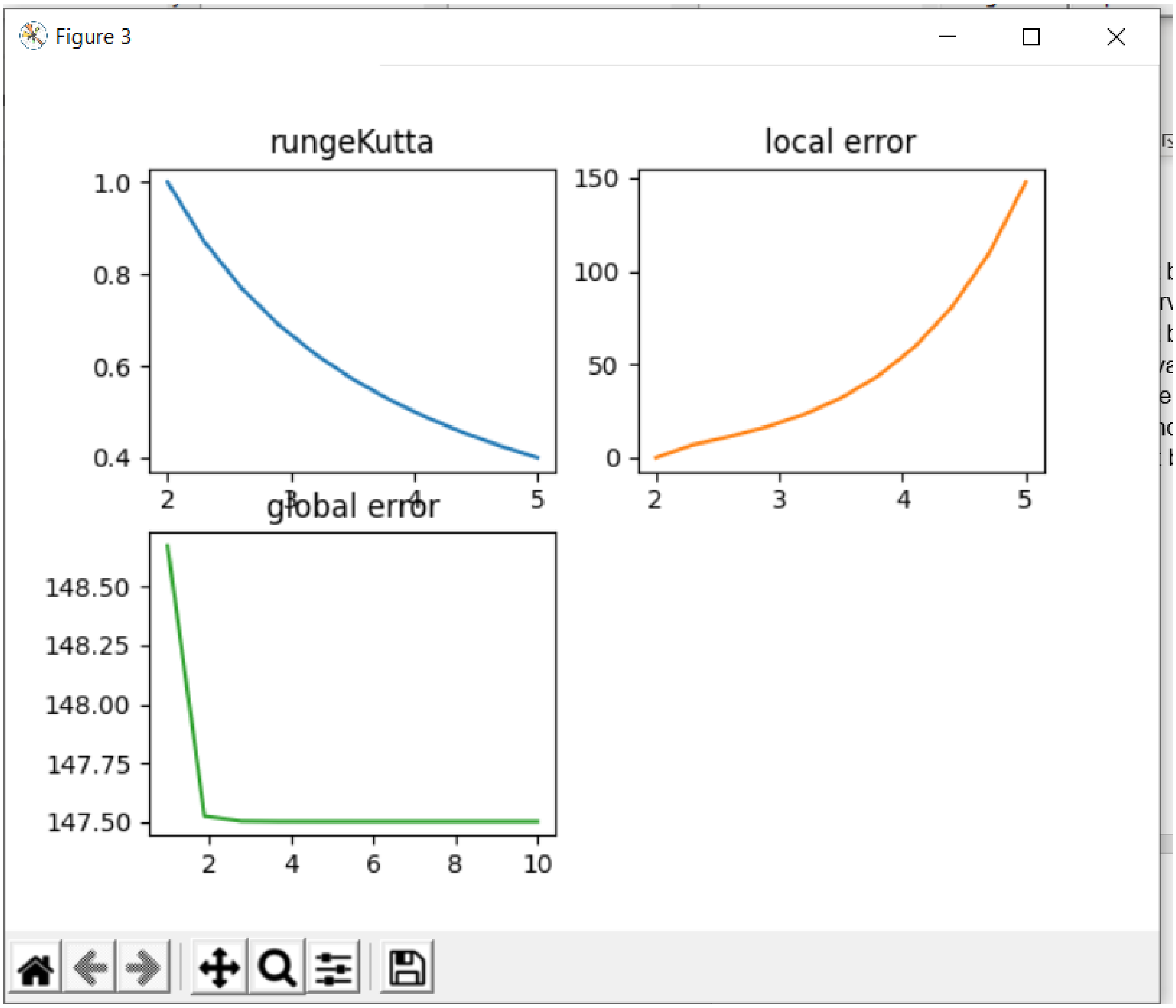
$$k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

Where, according to Wikipedia,

- $k1$ is the increment based on the slope at the beginning of the interval, using y;
- $k2$ is the increment based on the slope at the midpoint of the interval, using y and $k1$;
- $k3$ is again the increment based on the slope at the midpoint, but now using y and $k2$;
- $k4$ is the increment based on the slope at the end of the interval, using y and $k3$.



It can be observed that this method approximates the solution much better than the previous two, even for a small number of steps (big step size – in this graph h=1)