Phys 245 Quantum Computation
Homework 3

1. *[50] Single Qubit Gates! We got 'em!* In math, SU(2) is the group of unitary 2x2 matrices with determinant equal to 1. This group is important in quantum mechanics because, for a qubit, all possible quantum operations can be written as an SU(2) matrix. This means that any time evolution operator you find for a qubit will always be in SU(2). Now, SU(2) is closely related to the 3D rotation group, SO(3). SO(3) is just the group of rotations about an origin in Euclidean space. So, we can expect just like in SO(3), where we can define matrices that rotate a vector, we should be able to find matrices in SU(2) that rotate our qubit around the Bloch sphere. Show, using a conveniently chosen initial state, that:
    a. [10] $\hat{R}_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\imath\sin\frac{\theta}{2} \\ -\imath\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ rotates a qubit state vector about the x axis on the Bloch sphere by an angle θ.
    b. [10] $\hat{R}_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ rotates a qubit state vector about the y axis on the Bloch sphere by an angle θ.
    c. [10] $\hat{R}_z(\theta) = \begin{pmatrix} e^{-\imath\frac{\theta}{2}} & 0 \\ 0 & e^{\imath\frac{\theta}{2}} \end{pmatrix}$ rotates a qubit state vector about the z axis on the Bloch sphere by an angle θ.
    d. [10] Show how each of these rotation matrices can be realized from the time-evolution operator we found in class:
    $$\hat{U}_R = \cos\left(\frac{\Omega' t}{2}\right)\hat{I} - \imath\sin\left(\frac{\Omega' t}{2}\right)\left(-\frac{\delta}{\Omega'}\hat{\sigma}_z + \frac{\Omega}{\Omega'}\left(\cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y\right)\right)$$
    e. [10] Show how starting from state $|0\rangle$, $\hat{U}_R$ can be used to create any point on the Bloch sphere – that is any single qubit state can be realized.

2. *[50] Using QuTiP to find the evolution of a real system.* In a real quantum computer, to effect single qubit gates the pulses of radiation are pulsed on and off when an algorithm calls for a certain gate. Due to hardware limitations the pulses do not instantly turn on and off. Therefore, a gate evolution would follow from a Hamiltonian similar to:
$$\hat{H} = \frac{\omega_o}{2}\hat{\sigma}_z + \Omega f(t)\cos\omega t\,\hat{\sigma}_x$$
where $f(t)$ is an 'envelope function' that turns on and off smoothly as dictated by the hardware.
    a.) [10] Transform this Hamiltonian to the interaction picture with respect to the qubit energies and make the RWA.
    b.) [20] Use QuTiP to numerically find the evolution of a system initially in $|0\rangle$ under the Hamiltonian you found in part (a) and with $f(t) = 1$. Here you should use sesolve, mcsolve, or mesolve. Choose the parameters as you wish, but make sure they are appropriate to accomplish a π-pulse.

c.) [20] Use QuTip to numerically solve the problem with $f(t) = \mathrm{A}\,\mathrm{sech}^2 \frac{t}{\Delta t}$. Choose the parameters to accomplish a π-pulse. For the parameters you chose, calculate the integral $\int_{-\infty}^{\infty} A\,\mathrm{sech}^2 \frac{t}{\Delta t}\, dt$.

3. *[40] Composite Pulse Sequences.* Often a quantum computing gate is composed of more pulses than you might expect. For example, as you now know it is possible rotate a qubit from $|0\rangle$ to $|1\rangle$ with a single π-pulse. However, sometimes this is done with more than one pulse. Once such composite Pulse sequence is the Knill sequence. In this sequence instead of one π pulse being applied to the qubit there are five π-pulses applied! Each of those pulses has a different phase. The phases of the pulses are: $\phi_1 = \frac{\pi}{6}$, $\phi_2 = 0$, $\phi_3 = \frac{\pi}{2}$, $\phi_4 = 0$, $\phi_5 = \frac{\pi}{6}$.
   a. [10] Find the time evolution operator that propagates from before the application of the Knill sequence to after it – i.e find the time evolution operator for the Knill sequence.
   b. [10] Now assume that the Rabi frequency can vary away from the π-pulse condition. Assuming the qubit started in $|0\rangle$, plot the population in $|1\rangle$ as a function of Ω for Ω greater and less than the π pulse condition.
   c. [10] Now assume that the drive frequency can vary away from the resonance condition. Assuming the qubit started in $|0\rangle$, plot the population in $|1\rangle$ as a function of δ for δ greater and less than the resonance condition.
   d. [10] Based on (b) and (c) why do you think, despite its extra complexity, that the Knill sequence is used in real devices?