ATCT

Winter 2024

Contents

Measure and Integration

Exercise A.4

Part a

Wts.

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \tag{1.1}$$

Solution

$$\begin{aligned} \mu(A) &= \mu(A \setminus B \cup (A \cap B)) \\ &= \mu(A \setminus B) + \mu(A \cap B) \end{aligned} \tag{1.2}$$

$$\begin{aligned} \mu(A \cup B) &= \mu((A \setminus B) \cup (B \setminus A) \cup (A \cap B)) \\ &= \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B) \end{aligned} \tag{1.3}$$

Combining equations (1.2) and (1.3) we get

$$\begin{aligned} \mu(A) + \mu(B) &= \mu(A \setminus B) + \mu(B \setminus A) + 2\mu(A \cap B) \\ &= \mu(A \cup B) + \mu(A \cap B) \end{aligned} \tag{1.4}$$

So it follows that

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \tag{1.5}$$

Part b

Wts.

$$B \subseteq A \implies \mu(A \cap B^c) = \mu(A) - \mu(B) \tag{1.6}$$

Solution

$$\begin{aligned}\mu(A) &= \mu(A \setminus B \cup B) \\ &= \mu(A \setminus B) + \mu(B) \\ &= \mu(A \cap B^c) + \mu(B)\end{aligned} \tag{1.7}$$

So it follows that

$$\mu(A) - \mu(B) = \mu(A \cap B^c) \tag{1.8}$$

Exercise A.5

Part a

Wts.

$$\int_X f(x)\, d\mu(x) = \sum_{i=1}^{k} f(x_i)\, p_i \tag{2.1}$$

Solution

Definition A.17

$$\int_X f(x)\, d\mu(x) = \sup_{\varphi} \int_X \varphi(x)\, d\mu(x) \tag{2.2}$$

Where $0 \leq \varphi \leq f$ are simple functions. The integral of a simple function is defined as

$$\int_X \varphi(x)\, d\mu(x) = \sum_{0<y<\infty} y\mu\left(g^{-1}(\{y\})\right) \tag{2.3}$$

g is a function so the pullback of each y is a disjoint set A_y

$$\begin{aligned}\sum_{0<y<\infty} y\mu\left(g^{-1}(\{y\})\right) &= \sum_{0<y<\infty} y\mu(A_y) \\ &= \sum_{0<y<\infty} y \sum I_{A_i}(x_i)\, \mu(\{x_i\}) \\ &= \sum_{0<y<\infty} \sum g(x_i)\, I_{A_i}(x_i)\, p_i \\ &= \sum_i g(x_i)\, p_i\end{aligned} \tag{2.4}$$

Where in the last line we note that A_y partitions X. But by definition

$$\tag{2.5}$$

Exercise A.6

Part a

Wts. f and g are not measurable.

Solution

f and g are measurable iff for every interval $I \subseteq \mathbb{R}$ the pullback $f^{-1}(I) \in \mathcal{F}$.

$$f^{-1}([0,1]) = g^{-1}([1,2]) = [0,1] \notin \mathcal{F} \tag{3.1}$$

So f and g are not measurable.

Part b

We want to compute the following

$$\int_X f d\mu, \int_X g d\mu, \text{ and } \int_X (f+g) d\mu$$

Solution

Exercise A.7