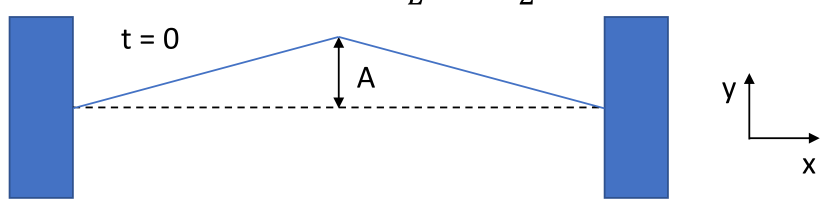
Phys 245 Quantum Computation
Homework 1

1. *[30] Quantization, basis sets, and so on aren't some uniquely quantum phenomena. They appear everywhere, as we'll see in this problem. Notice any similarities to the infinite square well?!* Suppose a string of length L is stretched between two fixed endpoints (think guitar string). The classical wave equation that describes the strings vertical displacement, y, as a function of horizontal displacement, x, is:
$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$
where v is the speed of sound along the string.
   a. [10] Show that a general solution for a standing wave is:
   $$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v}{L}t\right),$$
   where the A_n are constants that depend on the initial condition.
   b. [15] Suppose at t = 0 a string is plucked by pulling it's midpoint away from equilibrium by a distance d and then releasing it, such that:
   $$y(x,0) = \begin{cases} \dfrac{2dx}{L}, & 0 \le x \le \dfrac{L}{2} \\ \dfrac{2d(L-x)}{L}, & \dfrac{L}{2} \le x \le L \end{cases}$$
   

   Find the first three A_n and use them to approximate $y(x,t)$. Plot $y(x,t)$ for a few times to show the evolution.
   c. [5] In words explain why we are able to use the expression in (a) to predict the behavior in (b).

2. *[30] Understanding the RWA.* In lecture, we studied an infinite 1D square well ($V = 0$ for $0 \le x \le L$ and infinite otherwise) that was perturbed by
$$V'(x,t) = \Omega\cos\left(\frac{\pi x}{L}\right)\cos\omega_1 t,$$
where $\hbar\omega_1 = E_2 - E_1$. Here, $E_n = \hbar^2 n^2 \pi^2/(2mL^2)$ is the energy of the nth quantum state, m is the particle mass, and L is the width of the well. During this derivation, we found a set of equations for the evolution of the amplitudes in n = 1 (a_1) and n = 2 (a_2) as:
$$\frac{\Omega}{4}a_2(t)(1 + e^{-2\imath\omega_1 t}) = \imath\hbar\dot{a}_1(t)$$
$$\frac{\Omega}{4}a_1(t)(1 + e^{2\imath\omega_1 t}) = \imath\hbar\dot{a}_2(t)$$
which we approximated as

$$\frac{\Omega}{4}a_2(t) = \imath\hbar\dot{a}_1(t)$$
$$\frac{\Omega}{4}a_1(t) = \imath\hbar\dot{a}_2(t)$$

The reason for this approximation is that the exponential term oscillates quickly between positive and negative values and if $\omega_1 \gg \Omega$ then the term has no effect on the dynamics. As you'll learn, we make this approximation all of the time in the QST when dealing with time-dependent Hamiltonians and for historical reasons it is called the Rotating Wave Approximation or just the RWA. Let's see how good of an approximation this is. In this problem, use your favorite numerical integration tool (e.g. Mathematica or Python) to numerically solve the differential equations above and compare there results. Do the solution for the following parameters:

a.) [10] $\Omega/\hbar$ = 1 and $\omega_1 = 10$.
b.) [10] $\Omega/\hbar$ = 1 and $\omega_1 = 2$.
c.) [10] $\Omega/\hbar$ = 2 and $\omega_1 = 1$.

3. *[20] Embracing the matrix exponential!* Continuing in the same situation as in problem 2 and as we discussed in class, our system of equations after the RWA equation was:

$$\frac{\Omega}{4}a_2(t) = \imath\hbar\dot{a}_1(t)$$
$$\frac{\Omega}{4}a_1(t) = \imath\hbar\dot{a}_2(t)$$

Can be written in matrix form as:

$$\dot{\vec{a}} = M\vec{a}$$

with

$$M = \begin{pmatrix} 0 & -\imath\dfrac{\Omega}{4\hbar} \\ -\imath\dfrac{\Omega}{4\hbar} & 0 \end{pmatrix}$$

As we'll see this quarter, and you've likely seen before, whenever M is a constant matrix, as it is here, the solution of the differential equation is simply:

$$\vec{a}(t) = e^{Mt}\vec{a}(t=0)$$

where

$$e^{Mt} = \sum_{n=0}^{\infty}\frac{1}{n!}(Mt)^n$$

is the matrix exponential of Mt.

a.) [10] Find an analytic expression for $\vec{a}(t)$ by computing the matrix exponential. Hint: writing M in terms of Pauli matrices can be useful.
b.) [10] Use the expression you found in 3(a) for the parameters of 2(a) and compare to the evolution you found from the two approaches (i.e plot them together).

4. *[20] Uncertainty principles in action.* For a particle trapped in a normal infinite square well, calculate:

a. [10] The expectation value of the position and momentum for the nth energy level.
b. [10] The uncertainty in position and momentum for the nth energy level. Do you results seem reasonable? Explain why.

5. *[20] Uncertainty principles in action again.* For a spin-1/2 particle do the following:
    a. For the state $|\psi_1\rangle = |\uparrow\rangle$
        i. [10] Calculate the uncertainty in the observables $\vec{S}^2$, $\vec{S}_z$, and $\vec{S}_x$
    b. For the state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$
        i. [10] Calculate the uncertainty in the observables $\vec{S}^2$, $\vec{S}_z$, and $\vec{S}_x$

6. *[30] QuTip test drive.* Add the QuTip package to your Python installation and use a Jupyter notebook to do the following:
    a. [5] Show the three Pauli matrices
    b. [5] Create the state vector $|\psi_1\rangle = |\uparrow\rangle$ and calculate $\langle\vec{S}_z\rangle$
    c. [5] Create the state vector $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and calculate $\langle\vec{S}_z\rangle$
    d. [5] Use QuTip to redo Problem 5 part a for $\vec{S}_z$ only.
    e. [5] Use Qutip to redo Problem 5 part b for $\vec{S}_z$ only.
    f. [5] Use QuTip to draw $|\psi_1\rangle$ and $|\psi_2\rangle$ on the Bloch sphere. Also, one of these states is commonly written as $|+X\rangle$. Which one do you think it is?