Phys 245 Quantum Computation
Homework 7

1. *[60] RWA too!* In class we found the time evolution operator under the Jaynes-Cummings Hamiltonian to be:

$$\hat{U} = \begin{pmatrix} \cos\left(\frac{\Omega' t}{2}\right) - i\frac{\delta}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right) & -i\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right) \\ -i\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right) & \cos\left(\frac{\Omega' t}{2}\right) + i\frac{\delta}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right) \end{pmatrix}$$

on the basis of $|\psi\rangle = a|1, n+1\rangle + b|0, n\rangle$, where the first label in the ket is the qubit state and the second the harmonic oscillator state. Here $\Omega = \frac{g}{2}\sqrt{n+1}$, $\delta = \omega - \omega_o$, and $\Omega' = \sqrt{\Omega^2 + \delta^2}$. When deriving this time-evolution operator, we originally started with the Hamiltonian

$$\frac{H}{\hbar} = \frac{\omega_o}{2}\hat{\sigma}_z + \omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right) + \frac{g}{2}\hat{\sigma}_x(\hat{a} + \hat{a}^\dagger)$$

before using the RWA to drop the counter rotating terms. Use a numerical solver (e.g QuTIP) to find the evolution under this original Hamiltonian and compare it to that predicted by our time evolution operator for the cases:

   a. [15] $|\psi(t=0)\rangle = |0{,}0\rangle$, $\omega = \omega_o = 2\pi$ and $g = \omega/100$
   b. [15] $|\psi(t=0)\rangle = |0{,}0\rangle, \omega = \omega_o = 2\pi$ and $g = \frac{\omega}{2}$
   c. [15] $|\psi(t=0)\rangle = |0, 10\rangle, \omega = \omega_o = 2\pi$ and $g = \frac{\omega}{2}$
   d. [15] $|\psi(t=0)\rangle = |00\rangle, \omega = \omega_o = 2\pi$ and $g = \omega$

2. [60] *Jayne says too!* Suppose that two qubits coupled to the same harmonic oscillator can described by the Hamiltonian:

$$\frac{H}{\hbar} = \frac{\omega_o}{2}\hat{\sigma}_z^{(1)} + \frac{\omega_o}{2}\hat{\sigma}_z^{(2)} + \omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right) + \frac{g}{2}\left((\hat{\sigma}_+^{(1)} + \hat{\sigma}_+^{(2)})\hat{a} + (\hat{\sigma}_-^{(1)} + \hat{\sigma}_-^{(2)})\hat{a}^\dagger\right)$$

Implement this Hamiltonian in QuTip and use it to find the time dynamics of the following situations. For simplicity assume $\omega_o = 2\pi$, and $g = \omega_o/100$. Since you cannot use an infinite dimensioned Hilbert space in QuTip you'll need to truncate the basis at some maximum $|n\rangle$. Make sure and choose that maximum n large enough to not affect your answer. For $\omega = 2\omega_o$ find the time evolution of the population in the initial state for the following initial states over one period of oscillation:

   i. [10] $|\psi(t=0)\rangle = |110\rangle$ -- for this problem our ordering in the ket is $|qubit\ 1, qubit\ 2, QHO\rangle$.
   ii. [10] $|\psi(t=0)\rangle = |010\rangle$
   iii. [10] $|\psi(t=0)\rangle = |100\rangle$
   iv. [10] $|\psi(t=0)\rangle = |000\rangle$
   v. [10] $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle)$
   vi. [10] $|\psi(t=0)\rangle = |019\rangle$
   vii. [10] Compare your answer in part (vi) to the analytical expression we derived in class. Comment on similarities and differences.

3. *[50] Quantum computing runs on effective Hamiltonians.* As we saw in class, two qubits can be coupled by a harmonic oscillator to produce an effective Hamiltonian that reproduces their evolution but does not contain the harmonic oscillator. For example, we saw that two qubits coupled by a harmonic oscillator on resonance with the qubit transitions produced an effective Hamiltonian of the form

$$\frac{\hat{H}_{eff}}{\hbar} = \frac{g}{4}\left(\hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)} + \hat{\sigma}_y^{(1)}\hat{\sigma}_y^{(2)}\right).$$

It turns out that using very similar schemes these system can be engineered to produce other effective Hamiltonians. In this problem we will explore the time-evolution due to these effective Hamiltonians. Suppose an effective Hamiltonian is created between two qubits of the form

$$\frac{\hat{H}_{eff}}{\hbar} = \frac{g}{2}\hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)}.$$

a. [20] Calculate the time evolution operator of this Hamiltonian.

b. [30] A controlled-NOT gate can be constructed from an XX gate by sandwiching it between some single qubit rotations. The paper:

   D. Maslov, *New J. Phys.* **19** 023035 (2017)

   shows one way of doing this in Figure 1 – incidentally, this is how IonQ implements a CNOT gate on their hardware. Show that this prescription indeed produces a CNOT gate up to a global phase. A few tips:
   1. Remember that to apply the unitaries from left to right means that you operate first with the leftmost operator.
   2. Choose $\frac{gt}{2}$ to reproduce the needed angle χ for the XX gate – χ is defined in the text above Fig. 1.
   3. Remember that to apply e.g. a Y rotation to just qubit 1 means you are also applying the identity to qubit 2 and therefore the appropriate operator is the tensor produce of the Y rotation and the identity operator.