LEC07 - Substitution

Tuesday, January 21, 2025

Section 1.5

Warm Up Problem

If $f(x)=e^{x^2}$, what is $f'(x)$?

$\Rightarrow \frac{d}{dx}[e^{x^2}] = e^{x^2} \cdot 2x + C$

Warm Up Problem

Evaluate the indefinite integral $\int xe^{x^2}$

$\Rightarrow$ Let $u = x^2$, then $du = 2x\,dx$, so $\frac{du}{2x} = dx$

$\Rightarrow$ Then we have $\int \frac{\cancel{x}e^u}{2\cancel{x}} du = \frac{1}{2}\int e^u = \frac{1}{2}e^u = \frac{1}{2}e^{x^2} + C$

Theorem 1.7 - Substitution

Let $u=g(x)$ where $g'(x)$ is continuous over an interval, f continuous over the corresponding range of g. Then:

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

- $\rightarrow$ $f(g(x)) \cdot g'(x)$ looks like the result of a chain rule. This is what happens when you integrate both sides of the chain rule
- $\rightarrow$ If $u=g(x)$, then $du = g'(x)$. Think of this as an abbreviation
- $\rightarrow$ Our hope is that $\int f(u)\,du$ is easier to evaluate

Example

Evaluate $\int x^2 e^{7x^3} dx$

$\Rightarrow$ Let $u = 7x^3$, then $du = 21x^2 dx$, so $\frac{du}{21x^2} = dx$

$\Rightarrow$ Then we have $\int \frac{\cancel{x^2} e^u}{21\cancel{x^2}} du = \frac{1}{21}\int e^u du = \frac{1}{21}e^u + C = \frac{1}{21}e^{7x^3} + C$

Example

Evaluate $\int \cot x\,dx = \int \frac{1}{\tan x} dx = \int \frac{\cos x}{\sin x} dx$

$\Rightarrow$ Let $u = \sin x$, so $du = \cos x\,dx$, so $\frac{du}{\cos x} = dx$

$\Rightarrow$ Then we have $\int \frac{\cancel{\cos x}}{u} \cdot \frac{1}{\cancel{\cos x}} du = \int \frac{1}{u} du = \ln|u| + C = \ln|\sin x| + C$

Theorem 1.8 - Substitution for definite integrals

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \text{ where } u = g(x)$$

Example

Evaluate $\int_e^5 \frac{1}{x(\ln x)^2} dx$

$\Rightarrow$ Let $u = \ln(x)$, so $du = \frac{1}{x}dx$, so $x\,du = dx$

$\Rightarrow$ Then we have $\int_{\ln e}^{\ln 5} \frac{1}{\cancel{x}} \cdot \frac{1}{u^2} \cdot \cancel{x}\,du = \int_1^{\ln 5} \frac{1}{u^2} du = \int_1^{\ln 5} u^{-2} du = -u^{-1}\Big|_1^{\ln 5}$

$\Rightarrow$ Evaluating this gives $-\frac{1}{\ln 5} - 0 = -\frac{1}{\ln 5}$