LEC28 - Mean Value Theorem Continued

Friday, November 15, 2024
Section 4.4, 4.5

Warm Up Problem

What does the MVT say when $f(a)=f(b)$?

⇒ MVT states that there is a $c\in(a,b)$ such that $f(c)=m_{sec}$

⇒ $m_{sec}=\frac{f(a)-f(b)}{b-a}=0$

⇒ There is a place where the tangent is horizontal

Rolle's Theorem

Let f be continuous over the interval $[a,b]$, and differentiable over the interval (a,b). Suppose $f(a)=f(a)$. Then, there exists at least one point $c\in(a,b)$ such that $f'(c)=0$

Example 1

*We are using a proof-by-contradiction here (i.e. we assume there are more than 1 real roots, and show that something bad happens)

Show that $f(x)=4x^5+x^3+2x+1$ has at most one real root

⇒ Suppose f has 2 roots, a and b

⇒ We know f is continuous for all $x\in[a,b]$, and that $f(a)=f(b)=0$

⇒ By Rolle's theorem, there must exist a $c\in(a,b)$ such that $f'(c)=0$

⇒ We know $f'(x)=20x^4+3x^2+2$, we know that $20x^4, 3x^2, 2$ are non-negative, so the minimum of $f'(x)=2$

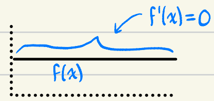
⇒ Therefore, there is no $c\in\mathbb{R}$ such that $f'(c)=0$ (This is a contradiction because by the theorem, if the requirements are true, then there should exist a c)

⇒ Therefore, there is no a, b such that $f(a)=f(b)=0$ (The requirements must not be met)

- The Mean Value theorem can be used to prove many important mathematical results

Theorem 4.6: Corollary 1 of the MVT

Let f be differentiable over an open interval I. If $f'(x)=0$ for all $x\in I$, then $f(x)$ is constant over I



Theorem 4.7: Corollary of corollary 1

If f and g are differentiable over an open interval I and $f'(x)$ and $g'(x)$ for all $x\in I$, then $f(x)=g(x)+C$, for some constant C

Theorem 4.8: Corollary 3 of the MVT

Let f be continuous over $[a,b]$, and differentiable over (a,b)

(i) If $f'(x)>0$ for all $x\in(a,b)$ then f is increasing over $[a,b]$

(ii) If $f'(x)<0$ for all $x\in(a,b)$ then f is decreasing over $[a,b]$

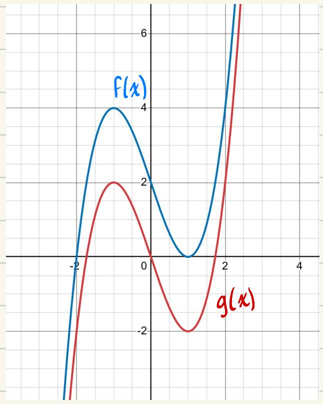
$f(x)=x^3-3x+2$

$f'(x)=3x^2-3$

$g(x)=x^3-3x$

$g'(x)=3x^2-3$

equal derivatives (they differ by 2 in this case)



Example 2

*Recall that functions are increasing when $f'(x)>0$, and decreasing when $f'(x)<0$

Find the intervals where $f(x)=x^3+\frac{48}{x^2}$ is increasing and decreasing. Find all local maxima and minima of f

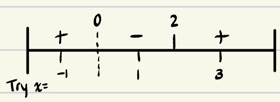
⇒ $f'(x)=3x^2-\frac{96}{x^3}$

$=\frac{3x^5-96}{x^3}$

⇒ $0=3x^5-96$

$2=x$

⇒ f is undefined at $x=0$



Increasing for $(-\infty,0)\cup(2,\infty)$

Decreasing for $(0,2)$

- Rolles theorem is a corollary of the MVT that states if $f(a)=f(b)$, then there is a horizontal tangent on the interval
- We can use a proof by contradiction paired with Rolles theorem to prove that functions have 1 root (assuming they actually have 1 root)
- There are 3 more corollaries that can be proven from the MVT