LEC11 - Volumes By Slicing

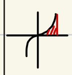
Warm-Up Problem

What is a solid of revolution?

A 3D solid made by rotating a region around a line

Example

Find the volume of the solid given by rotating the region under the curve $f(x)=x^3$ from 0 to 1, along the x-axis



$r=f(x)$, $A=\pi r^2=\pi(f(x))^2$

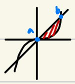
$V=\pi\int_0^1 (x^3)^2\,dx=\pi\int_0^1 x^6\,dx=\pi\cdot\frac{1}{7}(1)^7-\frac{1}{7}(0)^7=\frac{\pi}{7}$

Volumes by Slicing

1. Draw a picture and determine how you should slice the solid
2. Find a formula for the areas of the cross sections [$A(x)$ or $A(y)$]
3. Integrate the area formula to get a volume $V=\int_a^b A(x)\,dx$ or $V=\int_c^d A(y)\,dy$

Example

Let R be the region in the first quadrant between $f(x)=x^3$ and $g(x)=x$. Find the volume obtained by rotating R about the x-axis



1. Find a and b (points of intersection)

$\Rightarrow x^3=x$

$\Rightarrow x(x^2-1)=0$

$\Rightarrow x=-1,0,1$

2. Find the formula for volume

$\Rightarrow$ Circular cross sections (Washers)

$\Rightarrow A=\pi r_1^2-\pi r_2^2$, where r_1 is the outer radius $(g(x))$, and r_2 is the inner radius $(f(x))$

$\Rightarrow A=\pi((g(x))^2-(f(x))^2)$

$\Rightarrow V=\int_0^1 \pi((g(x))^2-(f(x))^2)\,dx=\pi\int_0^1[(x)^2-(x^3)^2]\,dx=\pi\int_0^1[x^2-x^6]\,dx$

3. Evaluate the integral

$\Rightarrow V=\pi\left(\frac{1}{3}x^3-\frac{1}{7}x^7\right)\Big|_0^1=\pi\left(\frac{1}{3}-\frac{1}{7}\right)=\pi\left(\frac{7}{21}-\frac{3}{21}\right)=\frac{4\pi}{21}$

Example

Let R be the region in the first quadrant between $f(x)=x^3$ and $g(x)=x$. Now, find the volume of the solid obtained by rotating R about the y-axis

$y=x^3 \Rightarrow x=\sqrt[3]{y}$

$y=x \Rightarrow x=y$

$A=\pi(\sqrt[3]{y}^2-y^2)$

$V=\pi\int_0^1[y^{\frac{2}{3}}-y^2]\,dy$

$=\pi\left[\frac{3}{5}y^{5/3}-\frac{1}{3}y^3\right]\Big|_0^1$

$=\pi\left(\frac{3}{5}-\frac{1}{3}\right)=\pi\left(\frac{9}{15}-\frac{5}{15}\right)=\frac{4\pi}{15}$

Solids of revolution

For us, the cross-sections of solids of revolutions will always be disks or washers

- → Disks: $A=\pi r^2$
- → Washers: $A=\pi(\text{outer})^2-\pi(\text{inner})^2$

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<u>Example</u>

Let R be the region enclosed by $f(x)=\sqrt{x-1}$, $g(y)=3y+1$. To find the volume of the solid obtained by rotating R about the y-axis, what is the inner radius?

$\Rightarrow y=\sqrt{x-1} \Rightarrow y^2+1=x$

Find the volume obtained by rotating R around the y-axis

$\Rightarrow 3y+1=y^2+1$

$\to y^2-3y=0$

$\to y(y-3)=0$

$\to y=0,3$

$\Rightarrow A=\pi(y^2+1)^2-\pi(3y+1)^2$

$\Rightarrow V=\pi\int_0^3[(3y+1)^2-(y^2+1)^2]dy$

$\Rightarrow V=\pi\int_0^3[-y^4+7y^2+6y]dy$

$\Rightarrow V=\pi(-\frac{1}{5}y^5+\frac{7}{3}y^3+3y^2)\Big|_0^3=\frac{207\pi}{5}$