LEC02 – The Definite Integral

Friday, January 10, 2025

Section 1.1, 1.2

Warm Up Problem

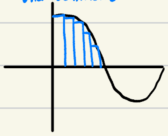
We want to find a Riemann sum of $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$ using 5 subintervals. Suppose we want to have an underestimate of the area. Should we use a left or right endpoint?

Left

Overestimate

Right

Underestimate



Therefore, a right Riemann sum should be used

Sigma Notation

- Σ is the capital Greek letter sigma and means "sum"
- Sigma notation is used to abbreviate long sums that follow a pattern. If $a_1, a_2, \dots, a_n$ is a list of n numbers, then $\sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n$

Example

Evaluate the following sums

1) $\sum_{i=1}^{5} \frac{i}{11} = \frac{1}{11} + \frac{2}{11} + \frac{3}{11} + \frac{4}{11} + \frac{5}{11} = \frac{15}{11}$ 2) $\sum_{i=3}^{6} (-1)^i (2i+1) = -7 + 9 - 11 + 13 = 4$ 3) $\sum_{i=1}^{3} 2 = 2 + 2 + 2 = 6$

Properties of Sigma Notation

1. $\sum_{i=1}^{n} c = nc$ 2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ 3. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

A) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ B) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ C) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example

Which of the following is equal to $\sum_{i=1}^{100} (2i + i^2 - 3)$?

a) $2\sum_{i=1}^{100} i + \sum_{i=1}^{100} i^2 - \sum_{i=1}^{100} 3$ b) $2\sum_{i=1}^{100} i + \underbrace{\left(\sum_{i=1}^{100} i\right)^2}$ $- \sum_{i=1}^{100} 3$ c) $\underbrace{\left(\sum_{i=1}^{100} 2\right)\left(\sum_{i=1}^{100} i\right)} + \sum_{i=1}^{100} i^2 + \sum_{i=1}^{100} (-3)$

no property guarantees these are always true

Example

Compute the value of $\sum_{i=1}^{100} (2i + i^2 - 3)$

$$\sum_{i=1}^{100} (2i + i^2 - 3) = 2\sum_{i=1}^{100} i + \sum_{i=1}^{100} i^2 + \sum_{i=1}^{100} (-3)$$

$$= 2\,\frac{100(100+1)}{2} + \frac{100(100+1)(2(100)+1)}{6} - 3(100)$$

$$= 100(101) + \frac{100(101)(201)}{6} - 300$$

$$= 348150$$

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<u>Area Under a Curve</u>

- Let f be a continuous, non-negative function on $[a,b]$. Then, the area under the curve $y=f(x)$ on $[a,b]$ is given by

$$A = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right] = \int_a^b f(x)\,dx \quad \text{(assuming the limit exists)}$$

- Where $\Delta x = \frac{b-a}{n}$, $x_i^* = a + i\Delta x$, $x_i^* \in [x_{i-1}, x_i]$
- If the limit exists, we say f is integrable on $[a,b]$

<u>Example</u>

Evaluate $\int_0^4 (x^2+3)\,dx$ using the limit definition and right endpoints ($x_i^* = x_i$)

$\Rightarrow \Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$, $x_i = a + i\Delta x = 0 + i\left(\frac{4}{n}\right) = \frac{4i}{n}$

$$\Rightarrow \int_0^4 (x^2+3) = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n}\right.$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\left(\frac{4i}{n}\right)^2 + 3\right)\left(\frac{4}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{64i^2}{n^3} + \frac{12}{n}\right)\right.$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{64}{n^3}i^2 + \sum_{i=1}^{n}\frac{12}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\sum_{i=1}^{n} i^2 + \frac{12}{n}\cdot n\right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\frac{n(n+1)(2n+1)}{6} + 12\right] \leftarrow \text{largest term will be } 128n^3$$

$$= \frac{128}{6} + 6 = \frac{100}{3}$$

<u>Example</u>

What is the Riemann sum for $\int_2^9 x^3\,dx$ using right endpoints and 10 subintervals

$\Delta x = \frac{9-2}{10} = \frac{7}{10}$, $x_i = a + i\Delta x = 2 + \frac{7i}{10}$, $f(x_i) = \left(2 + \frac{7i}{10}\right)^3$

$A \approx \sum_{i=1}^{10}\left(2 + \frac{7i}{3}\right)^3\left(\frac{7}{10}\right)$

<u>Example</u>

Which of the following integrals can be approximated by this Riemann sum? $\sum_{i=1}^{n}\sqrt{3+\frac{12i}{n}}\cdot\frac{12}{n}$

a) $\int_0^{12}\sqrt{x}\,dx$ b) $\int_3^{15}\sqrt{x}\,dx$ c) $\int_3^{15}\sqrt{x}\cdot x\,dx$ d) $\int_3^{12}\sqrt{x}\cdot x\,dx$

$\Rightarrow \Delta x = \frac{12}{n}$, so $b-a=12$, $a=3$ is the only remaining choice

$\Rightarrow f(a+\Delta x i) = f\left(3+\frac{12i}{n}\right) = \sqrt{3+\frac{12i}{n}}$, so $f(x)=\sqrt{x}$

$\Rightarrow$ Therefore, it must be $\int_3^{15}\sqrt{x}\,dx$