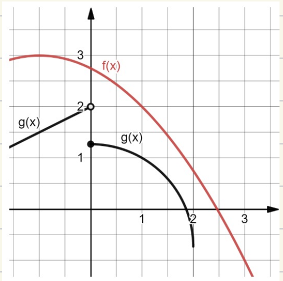
LEC7 & LEC8 - The Limit Laws

Warm-Up Problems

*Use the limit properties to convert one hard problem into multiple easier problems

Use the graphs to find the limits:



$$\lim_{x\to 1} \frac{f(x)g(x)}{2}$$
$$\Rightarrow \frac{1}{2}\left(\lim_{x\to 1} f(x)\right)\left(\lim_{x\to 1} g(x)\right)$$
$$\Rightarrow \frac{1}{2}(2)(1)$$
$$\Rightarrow 1$$

$$\lim_{x\to 0^-} \sqrt{g(x)}$$
$$\Rightarrow \text{DNE}$$
because graph DNE to the left of 0

$$\lim_{x\to -1} [3f(x) + xg(x)]$$
$$\Rightarrow 3\left(\lim_{x\to -1} f(x)\right) + \left(\lim_{x\to -1} x\right)\left(\lim_{x\to -1} g(x)\right)$$
$$\Rightarrow 3(3) + (-1)(1.5)$$
$$\Rightarrow 9 - 1.5$$
$$\Rightarrow 7.5$$

Match each limit with the most appropriate technique for computing it:

*2 polynomials suggests you should try factoring & cancelling

*Sum of square root suggests you should try multiplying by the conjugate

*Nested fractions suggest you should try simplifying the denominators to get one fraction

*Always try direct substitution before attempting any more complicated techniques

a) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4}$

b) $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2}$

c) $\lim_{x\to -3} \frac{\frac{1}{3}+\frac{1}{x}}{3+x}$

d) $\lim_{x\to 4} \frac{x^2+2x+c}{x-3}$ (c is a real number)

1) Multiply by the conjugate

2) Factor and cancel

3) Direct Substitution

4) Use Common Denominator

Solutions

a) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4}$

$$\Rightarrow \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$
$$\Rightarrow \frac{(x+3)}{(x+2)} \Rightarrow \frac{5}{4}$$

b) $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2}$

$$\Rightarrow \lim_{x\to 2} \frac{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}{(x-2)(\sqrt{x+2}+2)}$$
$$\Rightarrow \lim_{x\to 2} \frac{x+2-4}{(x-2)(\sqrt{x-2}-2)}$$
$$\Rightarrow \lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x-2}-2)}$$
$$\Rightarrow -\frac{1}{2}$$

c) $\lim_{x\to 2} \frac{\frac{1}{3}+\frac{1}{x}}{3+x}$

$$\Rightarrow \lim_{x\to 2} \frac{\frac{x}{3x}+\frac{3}{3x}}{3+x}$$
$$\Rightarrow \lim_{x\to 2} \frac{\cancel{x+3}}{\cancel{(3+x)}(3x)}$$
$$\Rightarrow \frac{1}{3(2)} \Rightarrow \frac{1}{6}$$

d) $\lim_{x\to 4} \frac{x^2+2x+c}{x-3}$

$$\Rightarrow 20+c$$

- There are many techiques to evaluate limits of indeterminate form. Knowing when to use which ones will allow you to solve problems much more efficiently

LEC7 & LEC8 - The Limit Laws

The Squeeze Theorem

Let f, g, and h be functions defined for at least for all $x \neq a$ in an open interval containing a.

IF $f(x) \leq g(x) \leq h(x)$

For all $x \neq a$ in an open interval containing a and

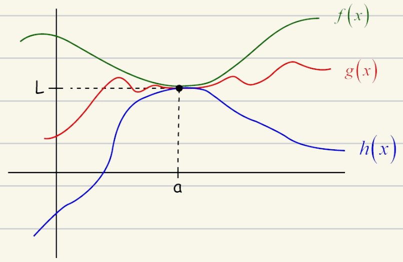
$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

Where L is some real number

THEN $\lim_{x \to a} g(x) = L$

If $h(x) \leq g(x) \leq f(x)$ when x is near a, except possibly at a,

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.



* The question states that R(x) lies between two other functions, meaning it's a good idea to try using the Squeeze Theorem

Example: Suppose R is a function such that $3x \leq R(x) \leq x^3 + 2$ for all $0 < x < 2$. What is $\lim_{x \to 1} R(x)$?

$\lim_{x \to 1} 3x = 3$ & $\lim_{x \to 1} x^3 + 2 = 3 \Rightarrow$ By the squeeze theorem, $\lim_{x \to 1} R(x) = 3$

* Since $5x^3$ is a polynomial: $\lim_{x \to 0^+} = \lim_{x \to 0^-} = \lim_{x \to 0}$

Example: Evaluate $\lim_{x \to 0^+} 5x^3 \cos(\frac{1}{x})$

Bounds are $\pm 5|x^3|$

$\lim_{x \to 0} 5|x^3| = 0$ $\quad$ $\lim_{x \to 0} -5|x^3| = 0$

$\Longrightarrow$ Therefore, by the squeeze theorem, $\lim_{x \to 0^+} 5x^3 \cos(\frac{1}{x}) = 0$

LEC8 - The Limit Laws

* Vertical asymptotes occur at division by zero and cause the function to tend toward positive or negative infinity

Warm-Up Problems: For which values of x does $f(x)$ have a vertical asymptote

$f(x) = \frac{5}{x-4} \Rightarrow x \neq 4 \Rightarrow f(x)$ has a vertical asymptote at $x = 4$

Key Observation:

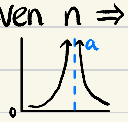
Dividing by numbers around $0 = \pm\infty$

↳ Small negative $= -\infty$

↳ Small positive $= +\infty$

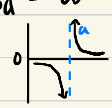
Theorem 2.3

even $n \Rightarrow \lim_{x \to a} = +\infty$



odd n

$\lim_{x \to a^-} = -\infty$ $\quad$ $\lim_{x \to a^+} = +\infty$



Recall Theorem 2.3

$n > 0$ Even integer $\quad \lim_{x \to a} \frac{1}{(x-a)^n} = +\infty$

$n > 0$ Odd integer $\quad \lim_{x \to a^+} \frac{1}{(x-a)^n} = -\infty$

$n > 0$ Odd integer $\quad \lim_{x \to a^-} \frac{1}{(x-a)^n} = +\infty$

- The squeeze theorem states that if you have 3 functions f(x), g(x), h(x) and you know that one is between the other two in terms of y-values, then if the limits at a particular x-value for the upper and lower functions approach the same value, the middle function must also approach it

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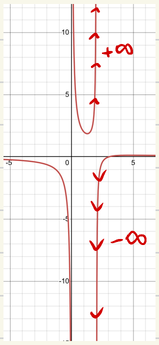
Example: Evaluate $\lim_{x \to 2^-} \frac{x-3}{x^2-2x}$

$$\Rightarrow \lim_{x \to 2^-} \frac{x-3}{x(x-2)}$$

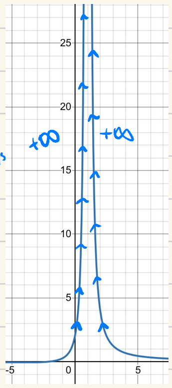
$$\Rightarrow \frac{-1}{2(-\text{Small})} \Rightarrow +\infty$$



Example: Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$

square makes always positive

$$\Rightarrow \frac{3}{+\text{Small}} \Rightarrow +\infty$$



- We can evaluate limits that approach infinity by observing one sided limits and observing what will happen to individual parts of a function when approaching a specific value