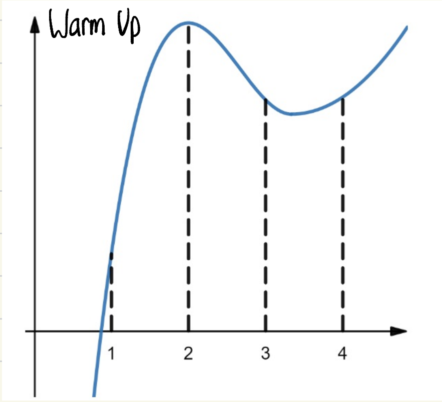
LEC12 – Derivatives

Warm-up Problem:

Arrange the following derivatives from smallest to largest

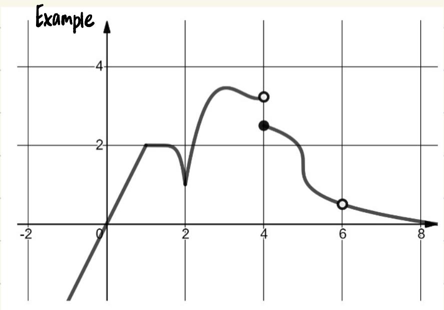
$f'(3) < f'(2) < f'(4) < f'(1)$



Example:

Where is $f(x)$ not differentiable: $x = 1, 2, 4, 5, 6$

Where is $f(x)$ not continuous: $x = 4, 6$



Example:

Which of the following are true?

1. If f is continuous at a then f is differentiable at a
   False because of cusps, corners, etc.

2. If f is NOT continuous at a then f is NOT differentiable at a
   TRUE since a function must be continuous to be differential

3. If f is NOT differentiable at a then f is NOT continuous at a
   FALSE because a function doesn't need to be differentiable to be continuous

* If a function is not continuous, it is guaranteed to not be differentiable. However, this is not nesessarily true the other way around

Summary:

What are all the ways a function can fail to be differentiable

- Corners $\Rightarrow |x|$
- Cusps (Vertical Tangents) $\Rightarrow \sqrt[3]{x}$
- Infinite Discontinuities $\Rightarrow \frac{1}{x}$
- Removable Discontinuities $\Rightarrow \frac{(x+2)(x-2)}{(x-2)}$
- Jump Discontinuity $\Rightarrow \begin{cases} 1 & x \geq 0 \\ 3 & x < 0 \end{cases}$
- Infinite Oscillations $\Rightarrow \sin\left(\frac{1}{x}\right)$

Example:

Find all a such that the tangent line to the graph of $f(x) = \frac{x}{x+2}$ at $x = a$ is parallel to the line $y = 2x+3$

$\Rightarrow f'(x) = \frac{1(x+2) - x(1)}{(x+2)^2}$

$\Rightarrow f'(x) = \frac{2}{(x+2)^2}$

$\Rightarrow 2 = \frac{2}{(x+2)^2}$

$\Rightarrow (x+2)^2 = 1$

$\Rightarrow x+2 = \pm 1$

$\Rightarrow x = \pm 1 - 2$

$\Rightarrow x = -1, -3$

Therefore, the tangent line of $f(x)$ is parallel to $y = 2x+3$ at $a = -1, -3$

- A function must be continuous to be differentiable, but it does have to be differential to be continuous
- There are many ways for a function to fail to be differentiable

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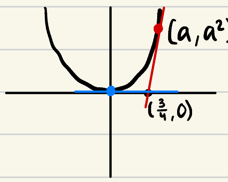
Problem:

There are 2 tangent lines to the curve $y=x^2$ that pass through the point $(\frac{3}{4},0)$. One of them is the x-axis. Find the equation of the other tangent line.



- We know the unknown point is (a,a^2), since $y=x^2$
- We know the derivative of x^2 is $2x$, so the slope at $x=a$ is $2a$
- We know the tangent line passes through $(\frac{3}{4},0)$

Solution

$\Rightarrow$ Recall point-slope form of a line

$(y-y_1)=m(x-x_1)$ for two points $(x,y),(x_1,y_1)$

$\Rightarrow$ Substitute $(a,a^2),(\frac{3}{4},0)$, and $m=2a$ into this equation

$\Rightarrow (a^2-0)=(2a)(a-\frac{3}{4})$

$\Rightarrow a^2 = 2a(a-\frac{3}{4})$

$\Rightarrow a^2 = 2a^2-\frac{3}{2}a$

$\Rightarrow -a^2 = -\frac{3}{2}a$

$\Rightarrow a^2-\frac{3}{2}a=0$

$\Rightarrow a(a-\frac{3}{2})=0$

$\Rightarrow a = 0, \frac{3}{2}$

We get 2 solutions, which makes sense because there are 2 tangent lines. 0 is the x-value of the x-axis tangent line, and $\frac{3}{2}$ is the x-value of the point we are solving for

$\Rightarrow$ Therefore, the unknown point is $(\frac{3}{2},\frac{9}{4})$

$\Rightarrow$ Therefore, the slope at this point is 3 (since $2(\frac{3}{2})=3$)

$\Rightarrow$ Arrange into slope intercept form

$y=mx+b \rightarrow \frac{9}{4}=3(\frac{3}{2})+b \rightarrow b=-\frac{9}{4}$

$\Rightarrow$ Therefore, the slope of the missing tangent is $y=3x-\frac{9}{4}$