Tuesday, February 25, 2025 Section 3.9

LEC19 - Improper Integrals

Warm-up Problem

Is the following attempt to evaluate $\int_1^\infty \frac{6}{x^3}\,dx$ correct? If so, finish evaluating the integral

$$\int_1^\infty \frac{6}{x^3}\,dx = \lim_{t\to\infty}\left[\int_1^t \frac{6}{x^3}\,dx\right] = \lim_{t\to\infty}\left[-3x^{-2}\Big|_1^t\right] = \lim_{t\to\infty}\left[-\frac{3}{t^2}+3\right] = 3$$

converges to 3

Improper Integrals

If f is continuous over $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\left[\int_a^t f(x)\,dx\right]$, if the limit exists. In this case, we say the integral converges.

If the limit does not exist, we say the limit diverges.

Example

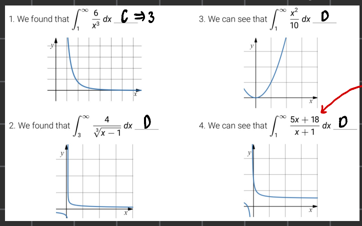
Does the following improper integral converge or diverge? If it converges, evaluate it.

$$\int_3^\infty \frac{4}{\sqrt[3]{x-1}}\,dx = \lim_{t\to\infty}\left[\underbrace{\int_3^t \frac{4}{\sqrt[3]{x-1}}\,dx}_{4(x-1)^{-1/3} \overset{AD}{=} 6(x-1)^{2/3}}\right] = \lim_{t\to\infty}\left[6(x-1)^{2/3}\Big|_3^t\right] = \lim_{t\to\infty}\left[6(t-1)^{2/3} - 6(2)^{2/3}\right] = \lim_{t\to\infty}\left[6(t-1)^{2/3}\right] - \lim_{t\to\infty}\left[6(2)^{2/3}\right] = \infty$$

Diverges

Example

Fill in converges or diverges. When can we easily see convergence/divergence


1. We found that $\int_1^\infty \frac{6}{x^3}\,dx$ C ⇒ 3
2. We found that $\int_3^\infty \frac{4}{\sqrt[3]{x-1}}\,dx$ D
3. We can see that $\int_1^\infty \frac{x^2}{10}\,dx$ D
4. We can see that $\int_1^\infty \frac{5x+18}{x+1}\,dx$ D

Cant converge due to HA at $y=5$

Comparison Theorem

Let f and g be continuous over $[a,\infty)$, assume $0 \le f(x) \le g(x)$ for all $x \in [a,\infty)$, then...

1. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges
2. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

Follow-Up Questions

3. If $\int_a^\infty g(x)\,dx$ diverges, what can we say about $f(x)$? Nothing.
4. If $\int_a^\infty f(x)\,dx$ converges, what can we say about $g(x)$? Nothing.

Example

Let f be positive and continuous on $[1,\infty)$. Suppose that $\int_1^\infty f(x)\,dx$ diverges. What (if anything) can we say about $\int_1^\infty (\cos^2 x+1)\cdot f(x)\,dx$?

⇒ $1 \le \cos^2 x + 1 \le 2$, so $f(x) \le (\cos^2 x+1)f(x)$ for all $x \in [1,\infty)$

⇒ Therefore, by the comparison theorem $\int_1^\infty (\cos^2 x+1)f(x)\,dx$ must diverge

Example

Let f be continuous on $[a,\infty)$. Which of the following statements are true?

1. If $\lim_{x\to\infty} f(x) = 5$, then $\int_a^\infty f(x)\,dx$ diverges. True from (2)
2. If $\lim_{x\to\infty} f(x) = L \ne 0$, then $\int_a^\infty f(x)\,dx$ diverges. True, since all converging integrals must have their area approach 0, otherwise the area is infinite
3. If $\lim_{x\to\infty} f(x) = 0$, then $\int_a^\infty f(x)\,dx$ converges. False. Consider $f(x) = \frac{1}{x}$, which is continuous on $[1,\infty)$, and $\lim_{x\to\infty} = 0$, but $\int_1^\infty f(x)\,dx$ diverges