LEC10 - Continuity Continued

Independent Variable: The variable that does not change in value based on other variables. For example, for $y = x^2$, x is the independent variable, and y is the dependent (y depends on x)

Constant: A value that doesn't change, for example: $\pi, e, 3.7, 2^2$ are all constants

Example 3:

The gravitational force exerted by the Earth on an object of unit mass with center at distance r from the center of the Earth is given by:

$$F(r) = \begin{cases} \frac{GM_E r}{R_E^3} & \text{if } r < R_E \\ \frac{GM_E}{r^2} & \text{if } r \geq R_E \end{cases}$$

Where G is the universal gravitational constant, M_E is the mass of the Earth, and R_E is the radius of the Earth. The domain of F is $(0, \infty)$

1. What is the independent variable? What are the constants?
2. Which piece of the piecewise function is linear, and what is the shape of the other piece?
   - The first piece of the function is linear because the constants effectively act as a slope value for r
   - The second piece of the function is similar to $\frac{1}{x^2}$
3. Is F continuous for all $r \in (0, \infty)$?
   - Yes, it is continuous. We know to the left of RE we have a line (which is always continuous), to the right of the function we have a function similar to $\frac{1}{x^2}$, which is continuous at the transition point

Warm-up Problem:

1. Suppose an airplane takes off from ground level (near 0 altitude), and after 20 minutes it flies at an altitude of 10,000m. Was there a time when the plane was at 7340m altitude? Can we be sure?

   Yes, since its altitude does not skip any values, so there must be a specific time where the altitude was 7340, even if for only a brief period of time

2. Suppose some function f satisfies $f(0) = 0$, and $f(20) = 10000$. Is there some value (lets call it c) between 0 and 20 where $f(c) = 7340$

   There could be, but it isn't guaranteed since we don't know if $f(x)$ is continuous. If it isn't, then the function could be undefined at $y = 7340$, or it might jump past it.

Intermediate Value Theorem - Key Idea: A continuous function can't "skip" values. If f is continuous and goes from $f(a)$ to $f(b)$, then each y-value in between must be crossed at least once.

Intermediate Value Theorem:

Let f be a continuous function defined on a closed, bounded interval $[a, b]$

If z is any real number between $f(a)$ and $f(b)$, then there exists at least one number c in $[a, b]$ satisfying $f(c) = z$

Conditions:	Conclusion:
• $f(x)$ is continuous on the closed interval $[a, b]$	• There exists all real numbers from $f(a)$ to $f(b)$ in $[a, b]$

Example 1:

Show that $f(x) = \ln(x + \frac{1}{2}) + \sin(x)$ has a root (x-intercept, zero etc.) somewhere between 0 and $\pi/2$

$\Rightarrow f(0) = \ln(0 + \frac{1}{2}) + \sin 0 = \ln(\frac{1}{2}) \approx -0.69$

$\Rightarrow f(\frac{\pi}{2}) = \ln(\frac{\pi}{2} + \frac{1}{2}) + \sin(\frac{\pi}{2}) = \ln(\frac{\pi + 1}{2}) + 1 \approx 1.72$

$\Rightarrow$ We know that $\ln(x)$ is continuous when $x > 0$, and $\sin x$ is always continuous, therefore $\ln(x + \frac{1}{2}) + \sin(x)$ is continuous when $x > -\frac{1}{2}$

$\Rightarrow$ Therefore, by the IVT, there exists every real number $\overbrace{y \in [-0.69, 1.72]}^{y}$ from $x \in \overbrace{[0, \frac{\pi}{2}]}^{x}$, which includes $y = 0$

• The intermediate value theorem states that if a function is continuous and goes from a to b, then all real numbers from $f(a)$ to $f(b)$ exist on that interval

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*The IVT is good for showing the existence of a particular y-value on an interval

Example 2:

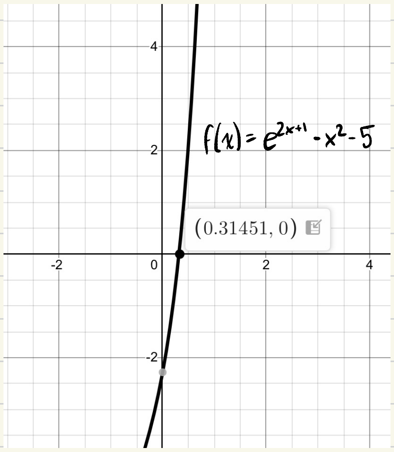
Show that $f(x) = e^{2x+1} - x^2 - 5$ has at least one root

$\Rightarrow f(0) = e^{2(0)+1} - 0^2 - 5 = e - 5$, which is less than 0

$\Rightarrow f(10) = e^{21} - 105$, which is greater than 0

$\Rightarrow f(x)$ is continuous because e^{2x+1}, $-x^2$, and -5 are all continuous for $\{\mathbb{R}\}$

$\Rightarrow$ Therefore, by the IVT, there exists at least one zero on the interval $[0, 10]$



*Recall the 3 point continuity test to solve this problem

Example 3:

Is there a value k such that the function f below is continuous everywhere?

$$f(x) = \begin{cases} x^2 + kx & x \le 1 \\ 3\ln x & x > 1 \end{cases}$$

Will only be continuous if $f(1) = \lim_{x \to 1} f(x)$

$\Rightarrow f(1) = k + 1$

$\Rightarrow \lim_{x \to 1^-} = k + 1$

$\Rightarrow \lim_{x \to 1^+} = 3\ln 1 = 3(0) = 0$

$\Rightarrow 0 = k + 1$

$\Rightarrow k = -1$

Therefore $k = -1$ will make the piecewise function continuous

$f(x) = x^2 - x \ \{x \le 1\}$

$f(x) = 3\ln(x) \ \{x > 1\}$