LEC25 - Linear Approximations

Warm Up Problem:

Which of the following is the general formula for the tangent line of $f(x)$ to $x=a$

a) $f'(a)x+f(a)$ c) $f'(a)(x-f(a))+a$

b) $f'(a)(x-a)+f(a)$ d) $f'(a)x+a$

Linear Approximations

Let f be a differentiable function, and let L be the tangent line to f at $x=a$. If x is close to a, then $f(x)\approx L(x)$. In other words:

$f(x)\approx f'(a)(x-a)+f(a)$ for values of x near a

*On a test, we wouldn't have a calculator to evaluate the result into a decimal, so we would leave in exact form

Example 1

Approximate $\sqrt{4.1}$ using linear approximation

$\Rightarrow f(x)=\sqrt{x},\ x=4.1,\ a=4$ We chose a value close to x that we can evaluate $f(a)$ for

$\Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$ Take the derivative of $\sqrt{x}$

$\Rightarrow f(4.1)\approx\frac{1}{2\sqrt{4}}(4.1-4)+\sqrt{4}$ Substitute into the linear approximation formula

$\Rightarrow f(4.1)\approx\frac{1}{4}\cdot\frac{1}{10}+2\approx\frac{81}{40}\approx 2.025$ Simplify

Example 2

Approximate $\ln(2.8)$ using linear approximation

$\Rightarrow f(x)=\ln x,\ x=2.8,\ a=e$ We know $e\approx 2.8$, and $\ln(e)=1$. 3 can also be used, it comes down to preferance

$\Rightarrow f'(x)=\frac{1}{x}$ We know that $\frac{d}{dx}[\ln(x)]=\frac{1}{x}$

$\Rightarrow f(2.8)\approx\frac{1}{e}(2.8-e)+\ln(e)$ Substitute into the linear approximation formula

$\Rightarrow f(2.8)\approx\frac{2.8-e}{e}+1$ Simplify

$\Rightarrow f(2.8)\approx\frac{2.8-e+e}{e}\approx\frac{2.8}{e}\approx 1.03$ Calculate

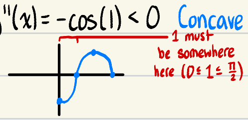
Important idea

When the second derivative is positive at a, local tangent approximations will be underestimates (concave up). Likewise, when the second derivative is negative at a, local tangents approximations will be overestimates (concave down). This was not explicitly taught this lecture, although it is a very convenient way to algebraically find if an approximation is an over/under estimate

Example 3

For which of the following functions is the tangent at $x=1$ an overestimate?

$f(x)=x^2$	$g(x)=\cos x$	$h(x)=e^{1-x}$
$f'(x)=2x$	$g'(x)=-\sin x$	$h'(x)=-e^{1-x}$
$f''(x)=2$	$g''(x)=-\cos x$	$h''(x)=e^{1-x}$
$f''(1)=2>0$ Concave up	$g''(x)=-\cos(1)<0$ Concave down	$h''(1)=e^{1-1}=1>0$ Concave up



- A linear approximation of $f(a)$ uses the tangent line to $x=a$ to approximate values of x close to a
- We can use linear approximations to approximate complicated values, such as $\sqrt{4.1}$, $\ln(2.8)$
- A tangent approximation of $f(a)$ is an overestimate when $f''(a)<0$, and an underestimate when $f''(a)>0$