LEC27 - The Mean Value Theorem

Monday, November 13, 2024
Section 4.4

Warm Up Problem

Suppose you drive from UTM to Buffalo (NY) and it takes you 2 hours. The distance is about 140km. What is your average speed?

⇒ $v_{avg} = \frac{140km}{2h}$

⇒ $v_{avg} = 70km/h$

*We can use the formula for m_{sec} to find the average rate of change over the interval $[0,2]$

Mean Value Theorem

Let f be continuous over the interval $[a,b]$, and differentiable over the interval (a,b). Then, there exists at least one $c \in (a,b)$ such that $f'(c) = m_{sec}(a,b)$

Example 1

Suppose f is a function such that $1 \leq f'(x) \leq 4$ for all $x \in \mathbb{R}$, and $f(3) = 1$. Use the MVT to find the smallest possible value for $f(5)$.

*We know f is continuous and differential because the derivative exists for all x, and differentiability implies continuity

⇒ Since $1 \leq f'(x) \leq 4$ for all $x \in \mathbb{R}$, f must be continuous and differentiable for all $x \in \mathbb{R}$

⇒ Consider the interval $[3,5]$

⇒ By the mean value theorem, we know there is a $3 < c < 5$ such that $f'(c) = \frac{f(5) - f(3)}{5-3}$

⇒ $f(5)$ will be minimized when $f'(c)$ is also minimized $[2f'(c) + 1 = f(5)]$

⇒ The minimum value of $f'(c)$ is 1, so the minimum value of $f(5) = 3$

Example 2

Suppose the temperature T (in °C) as a function of altitude h (in km) is continuous and differentiable. We have the following measurements: $T(3.4) = 8.1°C$ and $T(5.8) = -12.9°C$

If the instantaneous rate at which temperature falls with altitude (lapse rate) is more than 7°C/km, then the risk of a thunderstorm is high. Given the data, can we conclude a thunderstorm is likely.

⇒ We know $T(h)$ is continuous and differentiable

⇒ By the MVT, we know there exists a c such that $f'(c) = m_{sec}$

⇒ $m_{sec} = \frac{-12.9 - 8.1}{5.8 - 3.4} = \frac{-21}{2.4} \approx 8.75$

⇒ Therefore, a thunderstorm is likely

Example 3

Does there exist a function f such that all the following are true?

- $f'(x) \leq 2$ for all x
- $f(1) = -1$
- $f(3) = 5$

⇒ Since $f'(x) \leq 2$ for all x, f must be continuous for all x

⇒ Consider the interval $[1,3]$. The MVT states that there exists a c such that $f'(c) = m_{sec}$

⇒ Then we have $f'(c) = \frac{f(3) - f(1)}{3-1} = \frac{5-(-1)}{2} = 3$

⇒ Since $f'(c)$ is more than 2, such a function is impossible

- The slope of the secant line over an interval is the average rate of change over the interval
- The mean value theorem states that for any function f which is continuous over $[a,b]$ and differentiable (a,b), then there is a $c \in (a,b)$ such that $f'(c) = m_{sec}$