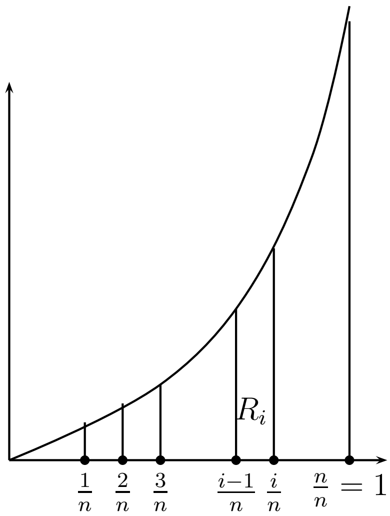
THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 8: - Integration.

We consider the problem of giving meaning to, and finding, the 'area under a curve'. This problem goes back to Archimedes.

We begin by an example:

Ex: $f(x) = x^2$ on $[0, 1]$.



Divide the interval $[0, 1]$ into n equal parts of width $\frac{1}{n}$.

This is called *a partition* of $[0, 1]$.

We now look at the region R bounded by the curve, the x-axis and the line $x = 1$.

Look at a general region R_i bounded by the curve, the x-axis and the lines $x = \frac{i-1}{n}, x = \frac{i}{n}$, where $0 \le i \le n-1$ and construct two rectangles U_i and L_i so that U_i has height $\left(\frac{i}{n}\right)^2$ and L_i has height $\left(\frac{i-1}{n}\right)^2$.

Adding up all the U_i's we have

$$\frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + ... + \left(\frac{n}{n}\right)^2\right] = \frac{1}{n^3}\left(1^2 + 2^2 + ... + n^2\right) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

This is called the *upper Riemann Sum* for f on P and is denoted by $\overline{S_P}$.

Similarly, adding up all the L_i's we have

$$\frac{1}{n}\left[\left(\frac{0}{n}\right)^2+\left(\frac{1}{n}\right)^2+...+\left(\frac{n-1}{n}\right)^2\right]=\frac{1}{n^3}\left(0^2+1^2+...+(n-1)^2\right)=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}.$$

This is called the *lower Riemann Sum* for f on P and is denoted by $\underline{S_P}$.

Clearly $\underline{S_P} \leq$ area of $R \leq \overline{S_P}$, that is, $\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2} \leq$ area of $R \leq \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$.

So, as $n \to \infty$, we have that the area of R is $\frac{1}{3}$.

We can take this as the **definition** of the area of R.

Approximating Areas using Partitions:

Ex: Use the partition $P = \{1, 1.25, 1.5, 1.75, 2\}$ to approximate the area under $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ and hence find an upper and lower bound for $\log 2$.

Now for the more general case:

Suppose we have a function $f(x)$ which is defined on the interval $[a, b]$. We divide this interval into n parts (not necessarily equal), and form the **partition** $P_n = \{x_0 = a, x_1, x_2, ..., x_n = b\}$. In each interval $[x_{i-1}, x_i]$ draw a rectangle of height $f(c_i)$ for some $c_i \in [x_{i-1}, x_i]$.
Let Δx_i be the width of the rectange (i.e. $\Delta x_i = x_i - x_{i-1}$). Now add up the areas of all these rectangles, giving $A = \sum_{i=1}^{n} f(c_i)\Delta x_i$.

In each rectangle, we can choose the c_i so that the height of the rectangle is equal to the maximum value of $f(x)$ for $x \in [x_{i-1}, x_i]$. The corresponding sum will be called the **upper Riemann sum**, $\overline{S_P}$. Similarly, we can take c_i so that the height of the rectangle is equal to the minimum value $f(x)$ for $x \in [x_{i-1}, x_i]$. The corresponding sum will be called the **lower Riemann sum**, $\underline{S_P}$.

Finally, if, for all possible partitions, P, the limit of the lower Riemann sum and the upper Riemann sum both exist and are equal, we say that the function f is **integrable** and we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$$

Ex: $f(x) = 1$ if x is irrational and -1 if x is rational.

If $f(x)$ is positive for all $x \in [a, b]$ and integrable on $[a, b]$, then we define the area bounded by the curve $y = f(x)$, the x axis and the lines $x = a, x = b$ to be the integral $\int_a^b f(x)\,dx$.

Theorem: If there is a partition P such that the upper and lower Riemann sums for f can be made arbitrarily close, then f is Riemann integrable.

Properties of Integrals:

(i) If $f(x) = C$ for all $x \in [a, b]$ then $\displaystyle\int_a^b f(x)\,dx = C(b-a)$.

(ii) If $a < c < b$ and f is integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

(iii) If f and g are both integrable on $[a, b]$ then $\displaystyle\int_a^b (\alpha f(x) + g(x))\,dx = \alpha \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(iv) If $f \geq 0$ and f is integrable on $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx \geq 0$.

(v) If f and g are both integrable on $[a, b]$ and $f(x) \geq g(x)$ on $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(vi) (ML Theorem) If f is integrable on $[a, b]$ and m, M are real numbers such that $m \leq f(x) \leq M$ for all $x \in [a, b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Definitions:

Suppose $a < b$ and f is integrable on $[a, b]$ then we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \text{ and } \int_a^a f(x)\,dx = 0.$$

Primitives:

We now begin to relate the problem of integration to that of differentiation.

Definition:
Let f be a continuous function on some interval (a, b).
A function F with the properties that:

(i) F is differentiable on (a, b) and

(ii) $F'(x) = f(x)$ for all $x \in (a, b)$

is called a **primitive** (or antiderivative) of f.

Notice that if F is a primitive of f then so is $F(x) + C$, where C is a constant. In fact, any two primitives for f can only differ by a constant. (For the proof see Calc.notes).

Ex: $f(x) = x^n$ has primitive $F(x) = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.

Note that many functions do not have 'elementary primitives', i.e. we cannot write the primitives in terms of the standard functions. e.g. $f(x) = e^{x^2}$.

The Fundamental Theorem of Calculus:

The great discovery of Newton and Leibniz was to realise that differentiation and integration are in some sense opposite processes. This statement is made more precise by the so called Fundamental Theorem(s) of Calculus. There are two versions of this. The first says that the derivative of the integral gets us back to the function we started with, while the second relates primitives to areas. We cannot 'prove' either of these with modern rigour (but then neither could Leibniz or Newton), but we can give a reasonable argument to explain why these results are true. I am going to combine both into one and then state the results separately afterwards.

Suppose that f is continuous on the interval $[a, b]$.

Take a point $x \in [a, b]$, and consider the area under the curve between a and x, which we denote by

$$A(x) = \int_a^x f(t)\, dt.$$

The function $A(x)$ is continuous, however we will not include the proof of this.

We now make some further restrictions to simplify our analysis.

Suppose that f is integrable, positive and increasing on $[a, b]$ and let F be a primitive of f.

For the moment, consider the area under the curve between x and $x + h$ and observe that we can under and over approximate this area by:

$$f(x)h \leq A(x + h) - A(x) \leq f(x + h)h.$$

Dividing by h, we see that

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h).$$

If we now take a limit as $h \to 0$, then by the Pinching Theorem, we have

$$f(x) = \frac{dA}{dx}.$$

Thus $A(x)$ is a primitive of f. Now since primitives differ at most by a constant, we have $A(x) = F(x) + C$ for some constant C.

Now $A(a) = \displaystyle\int_a^a f(t)\,dt = 0$ so $C = -F(a)$. Also if we set $x = b$ we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The conditions f positive and increasing, can be removed and we have:

Theorem A: (First Fundamental Theorem of Calculus)
Suppose f is cts on $[a, b]$. Then the function F defined by

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b), with $F'(x) = f(x)$ for all $x \in (a, b)$.

Theorem B: (Second Fundamental Theorem of Calculus)
If f is cts on $[a, b]$ and F is a primitive of f, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Ex: Find the area under $y = \frac{1}{\sqrt{1-x^2}}$ between $x = 0$ and $x = \frac{1}{2}$.

Ex: Let $G(x) = \displaystyle\int_0^x e^{t^2}\, dt$. Find $\frac{dG}{dx}$.

Ex: Let $G(x) = \displaystyle\int_{x^2}^{x^4} \sin(t^2)\, dt$. Find $\frac{dG}{dx}$.

Integrals and Series:

We can use integrals to find the value of certain series.

Ex: Use the function $f(x) = \frac{1}{x}$ in $[1, 2]$ with partition $\{\frac{n}{n}, \frac{n+1}{n}, ..., \frac{2n}{n}\}$ to show that $\lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} = \log 2.$

Integration Techniques:

While the rules for differentiation enable us to differentiate a very wide variety of functions, integrals are much mroe difficult and we only have a small number of tools available. Among these are:

Integration by Substitution:

Theorem: Suppose g is a differentiable function then
If $x = g(u)$, then

$$\int f(x)\, dx = \int f(g(u))\, \frac{dg}{du}\, du.$$

Ex: Evaluate $\displaystyle\int xe^{x^2}\,dx$.

Ex: Evaluate $\displaystyle\int_0^2 \frac{1}{2+\sqrt{x}}\,dx$.

Theorem: Suppose f is integrable on $[-a, a]$, where $a \in \mathbb{R}^+$.

(i) If f is odd then $\displaystyle\int_{-a}^{a} f(t)\,dt = 0$.

(i) If f is even then $\displaystyle\int_{-a}^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$

Proof: (of (i))

Recall that f odd means $f(-u) = -f(u)$.

Write the integral as
$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt.$$
Now make the substitution $u = -t$ in the first integral giving,
$$\int_{-a}^{a} f(t)\,dt = \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(t)\,dt = 0.$$

Ex: Find $\displaystyle\int_{-\pi}^{\pi} x^4 \sin(x^3)\,dx$.

Ex: Find $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\,dx$.

Theorem: Suppose that f is integrable on $\mathbb{R}$.
If f is periodic with period T, then for any $a \in \mathbb{R}$ we have
$$\int_{a}^{a+T} f(t)\,dt = \int_{0}^{T} f(t)\,dt.$$

Proof: Again we split the integral as
$$\int_{a}^{a+T} f(t)\,dt = \int_{a}^{0} f(t)\,dt + \int_{0}^{a+T} f(t)\,dt$$
and put $t = u - T$ in the first integral giving
$$\int_{a}^{a+T} f(t)\,dt = \int_{a+T}^{T} f(u)\,du + \int_{0}^{a+T} f(t)\,dt = \int_{0}^{T} f(t)\,dt.$$

Ex: $\displaystyle\int_{\frac{\pi}{17}}^{2\pi+\frac{\pi}{17}} \sin^7 x\,dx.$

Integration by Parts:

From the product rule, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Re-arranging, integrating and using the FTC we have

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}.$$

This formula is called *integration by parts* and is very useful when integrating the product of two functions.

In each given problem we have to decide which term in the product we will choose to be u and which we choose to be $\frac{dv}{dx}$. We normally make our choice so that $\frac{dv}{dx}$ is easy to integrate and u has a simple derivative.

Ex: Find $\displaystyle\int xe^x\,dx.$

Ex: Find $\displaystyle\int \log x \, dx$.

Ex: Find $\displaystyle\int e^x \sin x \, dx$.

Improper Integrals:

Thus far, our integrals have been defined for functions which are bounded on bounded intervals. Can we push the boundaries?

Ex: Suppose we take the area under the curve $y = \frac{1}{x}$ from 1 to N and see what happens as $N \to \infty$.

$$\int_1^N \frac{1}{x} \, dx = \log N.$$

Clearly the area just keeps increasing as N increases, so we could say that

$$\lim_{N \to \infty} \int_1^N \frac{1}{x} \, dx \to \infty.$$

In this case, we say that the improper integral $\displaystyle\int_1^\infty \frac{1}{x} \, dx$ diverges to infinity.

On the other hand, suppose we find the volume of the solid generated when we rotate this curve about the x axis between 1 and N. Then

$$V = \pi \int_1^N \frac{1}{x^2}\, dx = \pi(1 - \frac{1}{N})$$

and this clearly has the limit π as $N \to \infty$.
In this case we say that the improper integral $\int_1^\infty \frac{1}{x^2}\, dx$ converges to 1.

Definition: Suppose that

$$\int_a^N f(x)\, dx \to L, \quad \text{as} \quad N \to \infty$$

then we say that the improper integral $\int_a^\infty f(x)\, dx$ converges to L, and that f is *integrable* on $[a, \infty)$.

On the other hand if the integral $\int_a^N f(x)\, dx$ does NOT have a (finite) limit as $N \to \infty$ then we say that the improper integral diverges to infinity.

One can similarly define improper integrals at $-\infty$.

Ex: Look at $\int_0^\infty xe^{-x^2}\, dx$.

Ex: Look at $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$.

Ex: Look at $\displaystyle\int_3^\infty \frac{x^2}{1+x^3}\,dx$.

Note that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges if and only if $\displaystyle\lim_{M\to\infty}\lim_{N\to\infty}\int_{-M}^N f(x)\,dx$ exists, where the two limits are done separately.

Hence for example, $\displaystyle\int_{-\infty}^\infty x\,dx$ does not exist even though $\displaystyle\int_{-M}^M f(x)\,dx = 0$ for every real number M.

Theorem: (p-test) The integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.

Proof:

This result can be referred to as the p-test.

Comparison Test:

In many applications, we are often not interested (or are unable to find) the value to which a convergent improper integral converges. We often are only interested in whether or not a given improper integral converges. To deal with this question, we try to develop some tests for convergence that do not rely on us actually finding the limit of the integral. This is done by trying to bound the integral by other integrals which we can easily evaluate.

Theorem: Suppose that $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$, and $\displaystyle\int g(x)\,dx$ converges. Then

$$\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx,$$

i.e. $\displaystyle\int_a^\infty f(x)\,dx$ converges.

On the other hand, if $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$, and $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Ex: Look at $\displaystyle\int_2^\infty \frac{1}{x^4+1}\,dx$

Ex: Look at $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x^4+2x+1}}\, dx$

Ex: Look at $\displaystyle\int_2^{\infty} \frac{1}{\log x}\, dx$.

There is an equivalent result called the **limit form of the comparison test** which says:

Theorem: Suppose that f and g are non-negative and bounded on $[a, \infty)$. If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is NON-ZERO then the integrals $\displaystyle\int_a^{\infty} f(x)\, dx$ and $\displaystyle\int_a^{\infty} g(x)\, dx$ either BOTH converge OR BOTH diverge.

Ex: Look at $\displaystyle\int_2^{\infty} \frac{1}{x^4+1}\, dx$.