§1 Sets, Functions, and Sequences

- A *set* is a well-defined collection of distinct objects.
- An *element* of a set is any object in the set.

  $\in$ - "belongs to" or "is an element of" or "is in"

  $\notin$ - "does not belong to" or "is not an element of" or "is not in"
- The *cardinality* of a set S, denoted by $|S|$, is the number of elements in S.

Example. Some commonly-used sets in our number system:

$\mathbb{N}$ - the set of *natural numbers* $0, 1, 2, 3, \ldots$

$\mathbb{Z}$ - the set of *integers* (*whole numbers*) $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

$\mathbb{Q}$ - the set of *rational numbers* (*fractions*) $\ldots, -1, 0, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \frac{2}{3}, \ldots$

$\mathbb{R}$ - the set of *real numbers*, which includes all rational numbers as well as *irrational numbers* such as π, e, and $\sqrt{2}$

$\mathbb{R}^+$ - the set of all *positive real numbers*

Example. We can specify a set by listing its elements between curly brackets, separated by commas:

$$S = \{b, c\}.$$

The elements of S are b and c. Thus $|S| = 2$.
We can write $b \in S$, $c \in S$, and $d \notin S$, for instance.

Example. We can specify a set by some property that all elements must have:

$$S = \{x \in \mathbb{Z} \,|\, x^2 \leq 4\}$$
$$(\text{or } S = \{x \in \mathbb{Z} : x^2 \leq 4\}).$$

The elements of S are -2, -1, 0, 1 and 2. Thus $|S| = 5$.
Also $S = \{-2, -1, 0, 1, 2\}$.
We can write $-2 \in S$, $-1 \in S$, $0 \in S$, $1 \in S$, and $4 \notin S$, for instance.

Exercise. Let $A = \{\{a\}, a\}$. What are the elements of A? What is $|A|$?

The elements of A are $\{a\}$ and a

$|A| = 2$

- Two sets S and T are *equal*, denoted by $S = T$, if and only if (written iff)
  (i) every element of S is also an element of T, and
  (ii) every element of T is also an element of S.

  i.e., when they have precisely the same elements.
- The *empty set*, denoted by $\varnothing$, is ~~a~~ the set which has no elements. $|\varnothing| = 0$

Exercise. Are any of the following sets equal?

$$A = \{2, 3, 4, 5\}, \qquad C = \{2, 2, 3, 3, 4, 5\},$$
$$B = \{5, 4, 3, 2\}, \qquad D = \{x \in \mathbb{N} \mid 2 \leq x \leq 5\}.$$

Exercise. What is the difference between the sets $\varnothing, \{\varnothing\}$ and $\{\varnothing, \{\varnothing\}\}$?

The cardinalities are 0, 1 and 2 respectively

- Loosely speaking, a *subset* is a part of a set. More precisely, a set S is a *subset* of a set T if and only if each element of S is also an element of T.

  $\subseteq$ - "is a subset of", $\quad \not\subseteq$ - "is not a subset of"

  $\star$ $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.

  - A set S is a *proper subset* of a set T iff S is a subset of T and $S \neq T$.

    We then write $S \subsetneq T$ (or sometimes $S \subset T$).

    $\star$ $\varnothing$ is a proper subset of any non-empty set.

    $\star$ Any non-empty set is an improper subset of itself.
- The *power set* $P(S)$ of a set S is the set of all possible subsets of S.

  $\star$ For any set S, we have $\varnothing \subseteq S$ and $S \subseteq S$.

  $\star$ For any set S, we have $\varnothing \in P(S)$ and $S \in P(S)$.
- The number of subsets of S is $|P(S)| = 2^{|S|}$. (*Why?*)

Example. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Why is $|P(S)| = 2^{|S|}$?

<u>Proof</u> Let $|S| = n$

Let $S = \{a_1, a_2, a_3, \dots, a_n\}$

Form a subset of S like this:

For each element, decide Y or N : is it in the subset

. " YYNNNN ... $\{a_1, a_2\}$

Each sequence of Y/N choices gives a subset

2 choices at each place

Total choices is $2 \times 2 \times 2 \times \dots \times 2$ (n times)

$$= 2^n$$

$$= |P(S)|$$

(Note: same as number of bit strings of length n)

Example. Let $S = \{a, b, c\}$. The subsets of S are:

$$\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

S has 8 subsets. We can write $\varnothing \subseteq S$, $\{b\} \subseteq S$, $\{a, c\} \subseteq S$, $\{a, b, c\} \subseteq S$, etc. The power set of S is

$$P(S) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

and $|P(S)| = 2^3 = 8$.
We can write $\varnothing \in P(S)$, $\{b\} \in P(S)$, $\{a, c\} \in P(S)$, $\{a, b, c\} \in P(S)$, etc.

Exercise. Let $A = P(P(\{1\}))$. Find A and $|A|$.

$$P(\{1\}) = \{\phi, \{1\}\}$$

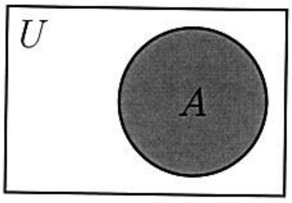
So $$P(P(\{1\})) = \{\phi, \{\phi\}, \{\{1\}\}, \{\phi, \{1\}\}\}$$

$$|A| = 4$$

Exercise. For $B = \{\varnothing, 0, \{1\}\}$, are the following statements true or false?

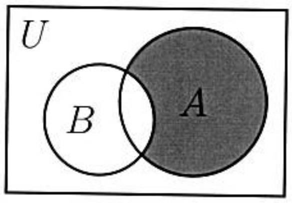
1. $\varnothing \in B$ T
2. $\varnothing \subseteq B$ T
3. $\{\varnothing\} \in B$ F
4. $\{\varnothing\} \subseteq P(B)$ T
5. $\{0\} \in P(B)$ T
6. $\{\varnothing\} \subsetneq B$ T
7. $\{\varnothing\} \in P(B)$ T
8. $\{\{0\}\} \subseteq P(B)$ T
9. $1 \in B$ F
10. $\{1\} \subseteq B$ F
11. $\{1\} \in P(B)$ F
12. $\{\{1\}\} \subseteq P(B)$ F
13. $\varnothing \in P(P(P(P(B))))$ T

- It is often convenient to work inside a specified *universal set*, denoted by U, which is assumed to contain everything that is relevant.
- *Venn diagrams* are visualizations of sets as regions in the plane.
  For instance, here is a Venn diagram of a universal set U containing a set A:



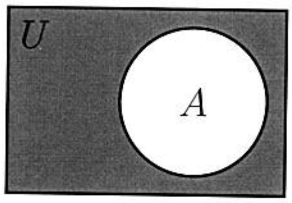
- Set operations and set algebra: ~ illustrated by Venn diagrams ~
  - *difference* $(-, \setminus)$ - "but not"

$$\boxed{A - B = A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\}}$$



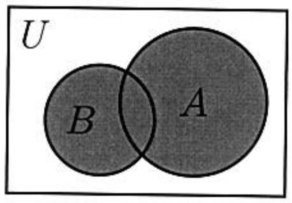
  - *complement* $({}^c, \overline{})$ - "not"

$$\boxed{A^c = \overline{A} = U \setminus A = \{x \in U \mid x \notin A\}}$$



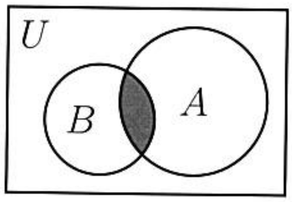
  - *union* $(\cup)$ - "or" meaning "one or other or both"

$$\boxed{A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}}$$



  - *intersection* $(\cap)$ - "and"

$$\boxed{A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}}$$



- Two sets A and B are *disjoint* if $A \cap B = \varnothing$.
- The *Inclusion-Exclusion Principle*: $|A \cup B| = |A| + |B| - |A \cap B|$.

Example. Set $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, and $B = \{1, 2\}$.
Then

$$A^c = \{2, 4, 6\} \quad A \cap B = \{1\} \quad A \cup B = \{1, 2, 3, 5\} \quad A - B = \{3, 5\}.$$

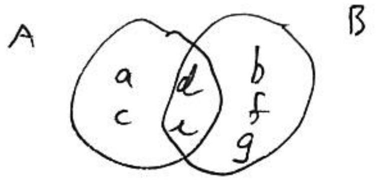
Exercise. Given $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$A = \{x \in U \mid x \text{ is odd}\}$
$B = \{x \in U \mid x \text{ is even}\}$
$C = \{x \in U \mid x \text{ is a multiple of } 3\}$
$D = \{x \in U \mid x \text{ is prime}\}$

determine the following sets:

$A \cap C \quad = \{3, 9\}$
$B - D$
$B \cup D$
A^c
$(A \cap C) - D \quad = \{9\}$

Exercise. Determine the sets A and B, where
$A - B = \{a, c\}$, $B - A = \{b, f, g\}$, and $A \cap B = \{d, e\}$.

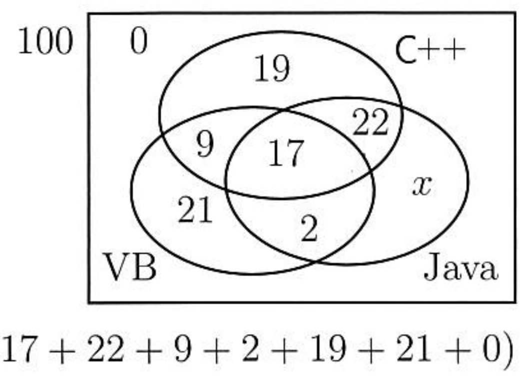


$A = \{a, c, d, e\}$
$B = \{d, e, b, f, g\}$

Example. In a survey of 100 students majoring in computer science, the following information was obtained:

17 can program in C++, Java, and Visual Basic.
22 can program in C++ and Java, but not Visual Basic.
9 can program in C++ and Visual Basic, but not Java.
2 can program in Java and Visual Basic, but not C++.
19 can program in C++, but not Visual Basic or Java.
21 can program in Visual Basic, but not C++ or Java.

Also, all of the 100 students can program in at least one of these three languages. How many students can program in Java, but not C++ or Visual Basic?



$$x = 100 - (17 + 22 + 9 + 2 + 19 + 21 + 0) = 10$$

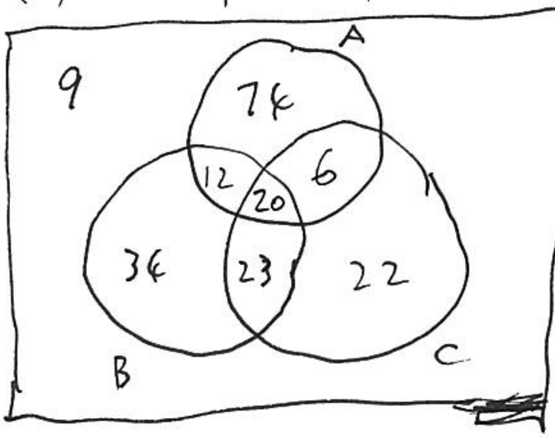
Exercise. In a survey of 200 people asked about whether they like apples (A), bananas (B), and cherries (C), the following data was obtained:

$$|A| = 112, \qquad |B| = 89, \qquad |C| = 71,$$
$$|A \cap B| = 32, \qquad |A \cap C| = 26, \qquad |B \cap C| = 43,$$
$$|A \cap B \cap C| = 20.$$

a) How many people like apples or bananas?
b) How many people like exactly one of these fruit?
c) How many people like none of these fruit?

(a.) $|A \cup B| = |A| + |B| - |A \cap B| = 112 + 89 - 32 = 169$

(b)



Plan: Start in the middle (triple intersection) and work outwards to find numbers in individual regions

Answer: $74 + 34 + 22 = 130$

(c.) 9

- **Hints for proofs:**
  - To prove that $S \subseteq T$, we assume that $x \in S$ and show that $x \in T$.
  - To prove that $S = T$, we show that $S \subseteq T$ and $T \subseteq S$.

Example. We prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Proof. Let $A \subseteq C$ and $B \subseteq C$ and suppose that $x \in A \cup B$.
Then either $x \in A$ or $x \in B$ (maybe both).
If $x \in A$, then $x \in C$, because $A \subseteq C$.
Likewise, if $x \in B$, then $x \in C$, since $B \subseteq C$.
In all possible cases, we have $x \in C$, which proves that $A \cup B \subseteq C$.

see next page for fuller version

Exercise. Prove that if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$. see p. 6b

To prove "If p then q"

Suppose p

$\Big\downarrow$ (work)

q

Let $A \subseteq C$ and $B \subseteq C$

Suppose $x \in A \cup B$

$x \in A$ or $x \in B$

Case 1: If $x \in A$

then $x \in C$ because $A \subseteq C$

Case 2: If $x \in B$

then $x \in C$ because $B \subseteq C$

So in either case, $x \in C$

$$A \cup B \subseteq C$$

To prove: if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$

Suppose $A \subseteq B$ and $A \subseteq C$

Let $a \in A$

$a \in B$ since $A \subseteq B$

and $a \in C$ since $A \subseteq C$

So $a \in B \cap C$

So $A \subseteq B \cap C$

Exercise. Prove that if $A \subseteq B$, then $A \cap B = A$.

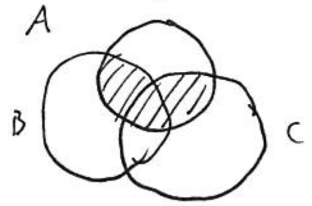
See over: 7a – 7b

Exercise. Prove that if $A \cap B = A$, then $A \subseteq B$.

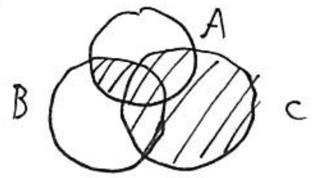
Thus, putting these last two examples together, we can say
$A \cap B = A$ if and only if $A \subseteq B$.

Exercise. Is the statement $A \cap (B \cup C) = (A \cap B) \cup C$ true (for all sets A, B, C)? Provide a proof if it is true or give a counter example if it is false.

LHS



RHS



Venn diagram gives insight: statement is false

One counterexample:
$A = \{1,2\}$, $B = \emptyset$, $C = \{2,3\}$. $A \cap (B \cup C) = \{2\}$ but $(A \cap B) \cup C = \{2,3\}$

A wrong answer is "False: because LHS is $(A \cap B) \cup (A \cap C)$ not $(A \cap B) \cup C$."

Exercise. Is the statement $A - (B - C) = (A - B) - C$ true?
Provide a proof if it is true or give a counter example if it is false.

Again, draw a Venn diagram to get insight. It's false
Then take particular sets which make the two sides unequal
e.g. $A = \{1,2\}$, $B = \{3\}$, $C = \{2,4\}$

Suppose $A \subseteq B$

Let $x \in A \cap B$

So $x \in A$ and $x \in B$

So $x \in A$

So $A \cap B \subseteq A$

Also let $x \in A$

So $x \in B$ since $A \subseteq B$

$x \in A \cap B$

So $A \subseteq A \cap B$

Therefore $A = A \cap B$

Suppose $A \cap B = A$

Let $a \in A$

So $a \in A \cap B$ since $A \cap B = A$

So $x \in A$ and $x \in B$

So $x \in B$

So $A \subseteq B$

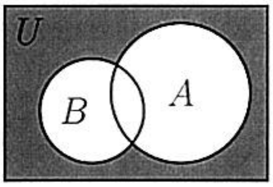
"p if and only if q"

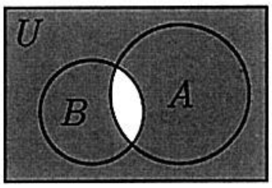
means "If p then q and if q then p"

To prove an "all" statement true, give a general proof

To prove an "all" statement <u>false</u>, give <u>one</u> counterexample

- **Laws of set algebra:**
  - *Commutative* laws $A \cap B = B \cap A$, $A \cup B = B \cup A$
  - *Associative* laws $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$
  - *Distributive* laws $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - *Absorption* laws $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$
  - *Identity* laws $A \cap U = U \cap A = A$, $A \cup \varnothing = \varnothing \cup A = A$
  - *Idempotent* laws $A \cap A = A$, $A \cup A = A$
  - *Double complement* law $(A^c)^c = A$
  - *Difference* law $A - B = A \cap B^c$
  - *Domination* or *universal bound* laws $A \cap \varnothing = \varnothing \cap A = \varnothing$, $A \cup U = U \cup A = U$
  - *Intersection and union with complement* $A \cap A^c = A^c \cap A = \varnothing$, $A \cup A^c = A^c \cup A = U$
  - *De Morgan's* Laws $(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$





Proof of De Morgan's law $(A \cup B)^c = A^c \cap B^c$:

(i) Suppose that $x \in (A \cup B)^c$. Then we have $x \notin A \cup B$, so $x \notin A$ and $x \notin B$.
Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.
This proves that $(A \cup B)^c \subseteq A^c \cap B^c$.

(ii) Suppose now that $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$.
Thus, $x \notin A \cup B$, so $x \in (A \cup B)^c$.
This proves that $A^c \cap B^c \subseteq (A \cup B)^c$.

Combining (i) and (ii), we conclude that $(A \cup B)^c = A^c \cap B^c$.

Example. We can use the laws of set algebra to simplify $(A^c \cap B)^c \cup B$:

$$\begin{aligned}
(A^c \cap B)^c \cup B &= ((A^c)^c \cup B^c) \cup B && \text{De Morgan's law}\\
&= (A \cup B^c) \cup B && \text{Double complement law}\\
&= A \cup (B^c \cup B) && \text{Associative law}\\
&= A \cup U && \text{Union with complement}\\
&= U && \text{Domination}
\end{aligned}$$

Exercise. Use the laws of set algebra to simplify $(A \cap (A \cap B)^c) \cup B^c$:

$= (A \cap (A^c \cup B^c)) \cup B^c$ De Morgan's

$= ((A \cap A^c) \cup (A \cap B^c)) \cup B^c$ Distributive

$= (\phi \cup (A \cap B^c)) \cup B^c$ ~~Domination (Intersection~~ Intersection

$= (A \cap B^c) \cup B^c$ Identity

$= B^c \cup (B^c \cup A) = B^c$ Commutative Absorption

Exercise. Use the laws of set algebra to simplify

$([(A \cup B)^c \cup C] \cup B^c)^c$

You do

Challenge: Prove the result (*uniqueness of complement*):

If $A \cup B = U$ and $A \cap B = \varnothing$ then $B = A^c$.

Suppose $A \cup B = U$ and $A \cap B = \phi$

Let $x \in B$. So $x \notin A$ (as $A \cap B = \phi$). So $x \in A^c$. Thus $B \subseteq A^c$

Let $x \in A^c$. So $x \notin A$. So $x \in B$ (otherwise, $x \notin A$ and $x \notin B$ so $x \notin A \cup B$. But $A \cup B = U$ so $x \in U$)

Thus $A^c \subseteq B$.

Therefore $B = A^c$

- ***Princi~~pal~~ple of Duality*:**

  For a set identity involving only unions, intersections and complements, its *dual* is obtained by replacing $\cap$ with $\cup$, $\cup$ with $\cap$, $\varnothing$ with U, and U with $\varnothing$.

  As all the relevant laws of set algebra come in dual pairs, then the dual of any true set identity is also true.

The duals of the last 3 examples are:

Dual of $(A^c \cap B)^c \cup B = U$

is $(A^c \cup B)^c \cap B = \varnothing$

- **Generalized set operations:**

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{and} \quad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Example. If $A_k = \{k, k+1\}$ for every positive integer k, then

$$\bigcup_{k=1}^{3} A_k = A_1 \cup A_2 \cup A_3 = \{1,2\} \cup \{2,3\} \cup \{3,4\} = \{1,2,3,4\}$$

$$\bigcap_{k=1}^{3} A_k = A_1 \cap A_2 \cap A_3 = \{1,2\} \cap \{2,3\} \cap \{3,4\} = \varnothing$$

- Let I be an (*index*) set. For each $i \in I$, let A_i be a subset of a given set A.
  - $\bigcup_{i\in I} A_i = \bigcup_{i\in I} A_i = \{a \in A \,|\, a \in A_i \text{ for some } i \in I\}$
  - $\bigcap_{i\in I} A_i = \bigcap_{i\in I} A_i = \{a \in A \,|\, a \in A_i \text{ for every } i \in I\}$

Example.
Let $I = \{1, 2, 3, \ldots\}$ be the index set. For each $i \in I$ let
$A_i = [0, \frac{1}{i}] \subseteq \mathbb{R}$ be the set of real numbers between
0 and $\frac{1}{i}$ including 0 and $\frac{1}{i}$.

$\bigcup_{i \in I} A_i = [0, 1] \cup [0, \frac{1}{2}] \cup [0, \frac{1}{3}] \cup \ldots = [0,1] = A_1$

$\bigcap_{i \in I} A_i = [0, 1] \cap [0, \frac{1}{2}] \cap [0, \frac{1}{3}] \cap \ldots = \{0\}$ (not $\emptyset$)

Example. (*The Barber Puzzle*) In a certain town there is a barber
(*) who shaves all those men, and only those, who do not shave themselves.
Does the barber shave himself?

Problem: If he shaves himself, (*) $\Longrightarrow$ he doesn't shave himself.
If he doesn't shave himself, (*) $\Longrightarrow$ he shaves himself.
CONTRADICTION!

Solution: Avoid self-reference

The paradox occurred because a self-referential statement was used. The "them-self" in (*) could also refer to the barber unless our above solution is imposed.
?? But some self-reference is OK, e.g. if the gas bill says "This is our first and final demand."

Example. *(Russell's Paradox)*

- Let U be the set of all sets.
- First weird phenomenon: then $U \in U$.
- Even worse, we have *Russell's paradox*. Let

$$S = \{A \in U \mid A \notin A\}.$$

Is S an element of itself?

i) If $S \in S$, then the definition of S implies that $S \notin S$, a contradiction.

ii) If $S \notin S$, then the definition of S implies that $S \in S$, also a contradiction.

Hence neither $S \in S$ nor $S \notin S$.

Usual Solution: (1.) Avoid selfreference. (2.) Avoid $X \in X$
(3.) Build up from individuals, sets of individuals, power sets...

Key Point: The notion of set and set theory is very subtle. We will for the most part ignore these subtleties.

- An *ordered pair* is a collection of two objects in a specified order.
We use round brackets to denote ordered pairs; e.g., (a, b) is an ordered pair.
  - Note that (a, b) and (b, a) are different ordered pairs,
  whereas $\{a, b\}$ and $\{b, a\}$ are the same set.
- An *ordered n-tuple* is a collection of n objects in a specified order;
e.g., $(a_1, a_2, \ldots, a_n)$ is an ordered n-tuple.
  - Two ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are *equal*
  if and only if $a_i = b_i$ for all $i = 1, 2, \ldots, n$.
- The *Cartesian product* of two sets A and B, denoted by $A \times B$,
is the set of all ordered pairs, the first from A, the second from B:

$$\boxed{A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}}$$

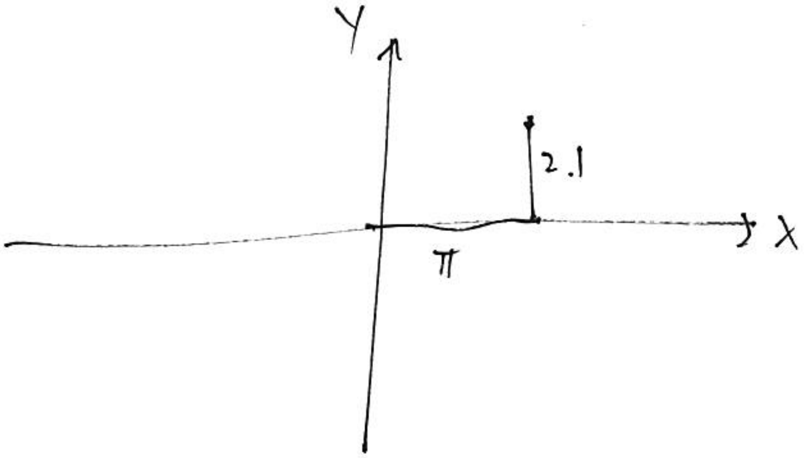
  $\star$ If $|A| = m$ and $|B| = n$, then we have $|A \times B| = mn = |A| \times |B|$

- The *Cartesian product* of n sets $A_1, A_2, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A_i$ for all $i = 1, 2, \ldots, n$:

$$\boxed{A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for all } i = 1, 2, \ldots, n\}}$$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ Cartesian plane

What about $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$



Cartesian plane

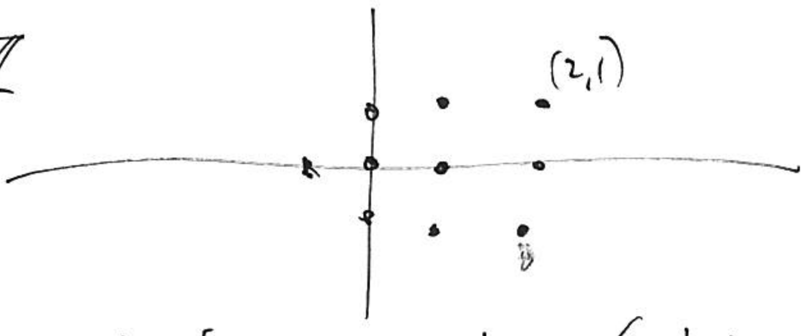
~~A = {(0)a~~ $A = \{0, a\}$

$A \times A = \{(0,0), (0,a), (a,0), (a,a)\}$

$A = \{\{1\}, 2\}$

$A \times A = \{(\{1\},\{1\}), (\{1\},2), \dots\}$

$\mathbb{Z} \times \mathbb{Z}$



Grid of integer points on Cartesian plane
Lattice

Example. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then

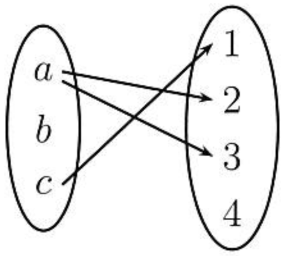
$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}.$$

Exercise. For A in the above example, find $A \times A$.

$A \times A = \{(a,a),(a,b),(b,a),(b,b)\}$

- When X and Y are small finite sets, we can use an *arrow diagram* to represent a subset S of $X \times Y$: we list the elements of X and the elements of Y, and then we draw an arrow from x to y for each pair $(x, y) \in S$.

Example. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3, 4\}$, and $S = \{(a, 2), (a, 3), (c, 1)\}$ which is a subset of $X \times Y$, then the arrow diagram for S is



- A *function* f from a set X to a set Y is a subset of $X \times Y$ such that

  for every $x \in X$ there is exactly one $y \in Y$ for which (x, y) belongs to f.

  - We write $f : X \to Y$ and say that "*f is a function from X to Y*".
  - X is the *domain* of f, Y is the *codomain* of f.
  - For any $x \in X$, there is a unique $y \in Y$ for which (x, y) belongs to f.
    - We write $f(x) = y$ or $f : x \mapsto y$.
    - We call y "the *image* of x under f" or "the *value* of f at x".
  - The *range* of f is the set of all values of f, that is

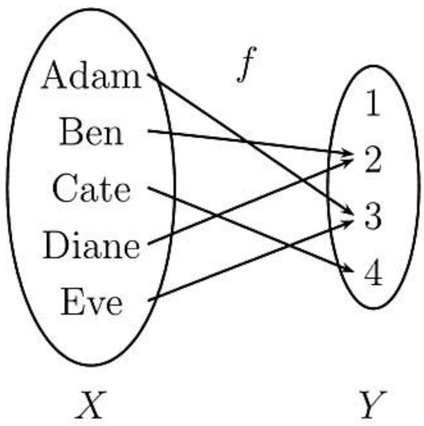
    $f(X) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$.
- This definition of a function corresponds to what is normally thought of as the *graph* of a function, with an x-axis and a y-axis.

How does this relate to the definition of a function given in Calculus in MATH1131/1141/1151?: bottom of next page

Example. Adam, Ben, Cate, Diane, and Eve were each given a mark out of 4. Their marks define a function $f : X \to Y$ as follows:

$$\begin{aligned} \text{domain} \quad & X = \{\text{Adam, Ben, Cate, Diane, and Eve}\} \\ \text{codomain} \quad & Y = \{1, 2, 3, 4\}, \\ \text{and suppose} \quad & f = \{(\text{Adam}, 3), (\text{Ben}, 2), (\text{Cate}, 4), (\text{Diane}, 2), (\text{Eve}, 3)\}. \end{aligned}$$

The arrow diagram for this function is



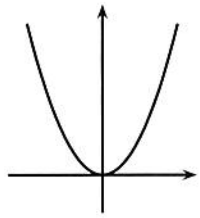
This is a function because every person has exactly one mark.
The range of this function is $\{2, 3, 4\}$.

Exercise. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$.
Determine whether or not each of the following is a function from X to Y.
If it is, then write down its range.

$f = \{(a, 2), (a, 4), (b, 3), (c, 5)\}$, N (two values for a)
$g = \{(b, 1), (c, 3)\}$, N (there's no g(a))
$h = \{(a, 5), (b, 2), (c, 2)\}$. Y: range is {2, 5}

Example. The *square* function $f : \mathbb{R} \to \mathbb{R}$ is defined by set of the pairs

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, y = x^2\}.$$



The function $f : \mathbb{R} \to \mathbb{R}$ can also be specified by

$$f(x) = x^2 \qquad \text{or} \qquad f : x \mapsto x^2.$$

The domain of f is $\mathbb{R}$; the codomain of f is $\mathbb{R}$; and the range of f is

$$\{y \in \mathbb{R} \,|\, y = x^2 \text{ for some } x \in \mathbb{R}\} = \{y \in \mathbb{R} \,|\, y \geq 0\} = \mathbb{R}^+ \cup \{0\}.$$

Function: each vertical line intersects the graph once

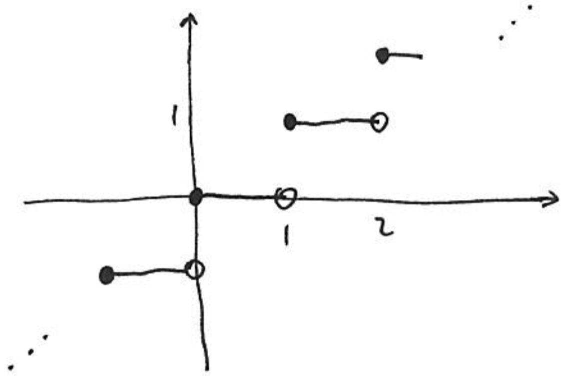
- The *floor* function: (round down)
  for any $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x.
- The *ceiling* function: (round up)
  for any $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x.

Exercise. Evaluate the following:

$\lfloor 3.7 \rfloor = 3$ $\quad \lfloor -3.7 \rfloor = -4$ $\quad \lfloor 3 \rfloor = 3$ $\quad \lfloor -3 \rfloor = -3$

$\lceil 3.7 \rceil = 4$ $\quad \lceil -3.7 \rceil = -3$ $\quad \lceil 3 \rceil = 3$ $\quad \lceil -3 \rceil = 3$

Exercise. What are the ranges of the floor and ceiling functions? $\mathbb{Z}$ (for both)
Plot the graphs of the floor and the ceiling functions.



floor

Exercise. Determine whether or not each of the following definitions corresponds to a function. If it does, then write down its range.

$$
\begin{aligned}
f : \mathbb{R} \to \mathbb{R}, &\quad f(x) = \tfrac{1}{x} \\
g : \mathbb{R}^+ \to \mathbb{R}, &\quad g(x) = \tfrac{1}{x} \\
h : \mathbb{R} \to \mathbb{R}, &\quad h(x) = \lfloor x^2 - x \rfloor \\
j : \mathbb{R} \to \mathbb{Z} &\quad j(x) = 2x
\end{aligned}
$$

- The *image* of a set $A \subseteq X$ under a function $f : X \to Y$ is
  $f(A) = \{y \in Y \mid y = f(x)$ for some $x \in A\} = \{f(x) \mid x \in A\} \subseteq Y$.
- The *inverse image* of a set $B \subseteq Y$ under a function $f : X \to Y$ is
  $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$.