MATH1081 DISCRETE MATHEMATICS

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2016

Chapter 4 - Enumeration and Probability.

Introduction
In how many different ways can 10 people line up? If a lotto card has 40 numbers and we have to choose 6, how many possible cards would I need to fill in to guarantee that I win? How many ways can the faces of a cube be coloured using 4 colours so that the colourings are essential different (that is, we cannot turn one cube around so that it is identical to another)? How many different ways can the letters of ABBRACADABRA be re-arranged?

These are all examples of **counting**, or to use a broader and more technical term **combinatorics**. Modern day combinatorics looks at important and difficult questions of enumeration and also of arrangements. In recent years combinatorics has become an important part of our understanding of genomes in biology, not to mention the many applications in modern computing.

In this section we will learn some of the basic concepts of combinatorics and the important concept of recursion.

Ex:
A restaurant menu lists seven different main courses and five different desserts.

(a) If I am going to order a main course or a dessert but not both, how many options do I have?

(b) If I am going to order a main course and a dessert, how many options do I have?

Answer
(a) $7 + 5 = 12$.

(b) $7 \times 5 = 35$.

This problem illustrates two principles which underlie even the most sophisticated counting methods.

Addition and Multiplication Principle
Suppose that a choice can be made from m options, and another choice can be made from n options. Then

(a) the number of ways to make *either* the first choice *or* the second is $m + n$, assuming that all the options for the two choices are different (so a choice of both is not possible);

(b) the number of ways of making the first choice *and* the second is mn, (assuming that all combinations of the two choices are permissible).

Comments
Note that each of the above involves assumptions about connections between the first and second choices.
Here are some examples which violate these assumptions.

In semester 2 last year 172 students studied MATH1081 and 1445 studied MATH1231. The number of ways to choose a student who studied either MATH1081 or MATH1231 is *not* $172 + 1455$. Why not?

There are some students who did both. This is an example of **double counting** - the bane of solving combinatorical problems!

The School of Mathematics offers 8 different first year courses in first semester and 7 in second semester. The number of ways in which a student can do one course from each semester is *not* 8×7. Why not?

Some combinations are not permitted, and you cannot repeat a course which you have already passed.

Ex:
We wish to construct three–letter words from the 26 letters of the English alphabet. Five of these are *vowels* (A, E, I, O, U) and the other 21 are *consonants*. How many possibilities are there, if

(a) any choice of letters is permitted;
(b) the word must contain one vowel;
(c) the word must contain *at least* one vowel;
(d) the word must come before EGG in alphabetical order?

Note that for this course a "*word*" is any string of letters from a given alphabet. It does not matter whether or not the word is in the dictionary, or whether or not it makes sense. For example, if we have an alphabet $\{A, C, T\}$ consisting of just three letters, some of the "words" we can form are

$$\text{CAT} \quad \text{and} \quad \text{TAC} \quad \text{and} \quad \text{CCTTT} \quad \text{and} \quad \text{ATACTCC} \;.$$

Also, usually we will say "one vowel" when we mean "exactly one vowel", but sometimes we will emphasise the "exactly". We will always say "at least one vowel" if that is what we mean.

A more formal way of stating and generalising the principles we gave earlier and an alternative interpretation is:

Theorem
The *addition principle* (the sum rule)

Let $A_1, A_2, \ldots, A_k$ be **disjoint** sets, i.e., $A_i \cap A_j = \emptyset$ **for all** $i \neq j$. Then the cardinality of their union is

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \;=\; |A_1| + |A_2| + \cdots + |A_k|.$$

and
The *multiplication principle* (the product rule) Given sets $A_1, A_2, \ldots, A_k$, the cardinality of their Cartesian product is

$$|A_1 \times A_2 \times \cdots \times A_k| \;=\; |A_1| \times |A_2| \times \cdots \times |A_k|.$$

Ordered selections with repetition.
Suppose that we have n different items and we wish to make r selections from these items, where
(a) the same item may be selected more than once (for example, AABA is a permissible choice of four letters);
(b) the order of the items is significant (for example, ABCA and CBAA are regarded as different and therefore are both counted).

Then the number of possible choices is n^r.

Ex: There are 100 students in a program. Each is to choose one of three elective courses. There are no limits on the number of students taking each elective. In how many different ways can the choices be made?

Subtraction.
If a choice can be made in m different ways altogether, but n of these are not permitted for a particular problem, then the number of permissible ways to make the choice is $m - n$.

Ex:
A student has to choose a semester 1 elective from a list of 12 courses and a semester 2 elective from a list of 15. However these figures include 7 courses which are offered in both semesters, and the student is not allowed to take the same course twice. How many options are open to the student?

Comment.
We have assumed that if there are two courses A and B which can be taken in either semester, then "A in S1, B in S2" and "B in S1, A in S2" are different options and are both counted. If we regarded these as the same option we would not want to count it twice and the problem would be different.

Ex:
How many ten–letter words can be made from the English alphabet which contain at least one X, at least one Y and at least one Z?

Here is an **incorrect** solution - but one which hopefully illustrates what can go wrong if we don't think carefullly.

1. Choose a place for the X .... 10 ways
2. Choose a place for the Y .... 9 ways
3. Choose a place for the Z .... 8 ways
4. Choose another seven letters from 26 possibilities,
with repetitions allowed and order significant in .... 26^7 ways
So the number of words is $10 \times 9 \times 8 \times 26^7$.

Clearly we have counted some words multiple times (for example XYZXYZXYZX), so we should subtract the number of these (if we could work it out).
We shall see later how to solve the preceding problem correctly – you might already be able to work out how from Topic 1 knowledge.

There are two important things that you need to pay attention to when solving combinatorial problems.

- Make sure you count everything!
- Make sure you don't count anything twice!

These may sound completely obvious, but, as the previous example illustrates, they are not always easy to put into practice.

Ex:
How many ten–letter words can be made from the English alphabet which contain exactly one X, exactly one Y and exactly one Z?

Re-arranging distinct objects
In how many ways can 10 people line up?

We will use the notation $n!$, read as n-factorial, defined by

$$n! = n(n-1)(n-2)....3.2.1.$$

Note that we define 0! to be 1.

Arranging non-distinct objects

How many words can be made using letters AABC?

If the letters were different, say we wrote A_1 for the first A and A_2 for the second one, then there would be $4! = 24$ ways. However, in each such word we could swap A_1 and A_2 and remove the subscripts and the two words would be the same. Hence we have counted everything twice so the correct answer is $4!/2 = 12$.

More generally, if have n letters, with n_1 of the first type, n_2 of the second type, and so on to n_k of the kth type, with $n_1 + n_2 + ... + n_k = n$, then the number of distinct re-arrangements of the n letters is given by

$$\frac{n!}{n_1!n_2!...n_k!}.$$

Ex: In how many different ways can the letters of ABRACADDABRA be permuted?

Ex: There are 16 books on a shelf. In how many ways can these be arranged if 12 of them are volumes of a history which must remain in order?

Ex: How many different ways are there to arrange 3 couples into a row without separating the couples?

Distinct Arrangements in a circle

Two circular arrangements will be regarded as the same if the first can be rotated to produce the same arrangement as the second.

In how many ways can 10 people be arranged in a circle?

One can think of this in a variety of ways. Firstly we can fix one position and place one person there. It does not matter who the person is. We can then take one person out of the 9 remaining and place them on the left. Then place another on their left and so on around until the circle is complete. Hence there are 9! such arrangements. Alternatively, we can arrange them in a line in 10! ways and form the line into a circle by getting the people on each end to hold hands. The circle can now be rotated 10 times, producing the same arrangement, so we have to divide by 10. Hence the number is $10!/10 = 9!$.

Thus we can state that the number of ways to arrange n people in a circle is $(n-1)!$.

Ordered selections without repetition.
Suppose we have n different items and we wish to select r of these, where

(a) no item may be selected more than once, (for example, XYZZY is not allowed);
(b) the order of the items is significant, (for example, XYZ, XZY, YXZ, YZX, ZXY and ZYX are all regarded as different and therefore are counted as six different words).

Such a selection is referred to as a *permutation of r objects chosen from n (different) objects*, and the number of possible choices is

$$P(n,r) = n(n-1)(n-2)\cdots(n-(r-1)) \ .$$

We will use the notation $P(n,r)$ for permutations.
Some other notations are

$$P_r^n \quad \text{and} \quad {}^nP_r.$$

but we will generally avoid these.

Ex:
A band of pirates has 13 members, including the captain's parrot. A crew photo is to be taken with seven pirates standing in a row and six sitting on chairs in front of them. Suppose that we don't care who stands where, but we do care who sits on which chair. In how many possible ways can pirates be seated on the chairs? What if the captain and the parrot must sit on the middle two chairs?

A particularly important case of permutations is when we wish to find the number of ordered selections of *all* available objects, taken once each.
If there are n objects, the number of such selections is
$$P(n,n) = n(n-1)(n-2)\cdots 3 \times 2 \times 1 = n!\ .$$

Now, for any n, r, we can write $P(n,r)$ in terms of factorials:
$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-(r-1)) \\ &= \frac{n(n-1)\cdots(n-(r-1)) \times (n-r)\cdots \times 2 \times 1}{(n-r)\cdots \times 2 \times 1} = \frac{n!}{(n-r)!}. \end{aligned}$$
This relation is not helpful for actual calculations (it just means we work out two big products which mostly cancel anyway), but it can be useful for theoretical purposes.

Cards.

A *pack of cards* is very useful for practising combinatorial techniques.
It consists of 52 cards:
comprising thirteen *values* ace (**A**), two (**2**), three (**3**), ..., ten (**10**),
jack (**J**), queen (**Q**) and king (**K**)

in each of four *suits* spades (♠), hearts (♡), diamonds (♢) and clubs (♣).

The spades and clubs are coloured black, while the hearts and diamonds are red.

For example ♡**A** denotes the ace of hearts, while ♣**J** denotes the jack of clubs.

A selection of cards from a pack is referred to as a *hand of cards*.
We shall assume (as is true in most card games) that a hand may not contain the same card twice, and that the order of cards in a hand is not important.

Unordered selections without repetition. Suppose we have n different items and we wish to select r of these, where
(a) no item may be selected more than once
(for example, ♢**K** ♠**7** ♣**A** ♢**K** is not allowed);
(b) the order of the items is not significant
(for example, ♢**K** ♠**7** ♣**A** and ♠**7** ♢**K** ♠**A** are regarded as the same and therefore are not both counted).
(c) Such a selection is called a *combination* of r objects chosen from
n (different) objects, and the number of possible choices is
$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!}\ .$$
(d) In this course we will sometimes use the $C(n,r)$ notation, although the $\binom{n}{r}$ notation is more common in Mathematics.

Ex:
Of all hands containing 13 cards, how many consist of seven cards in one suit and six in another?

Ex
How many thirteen–card hands contain four cards in each of two suits and five in a third?

Two questions. You will notice from the problems we have solved so far that it is frequently important, when faced with a counting problem, to ask yourself two questions concerning the selection of objects.

(a) Is repetition allowed?
(b) Is order important?

We have seen how to handle the cases where repetition is not allowed and order is or is not important; and also the case where repetition is allowed and order is important. The final possibility (repetition allowed and order not important) has a simple answer, but the reasoning behind it is a bit more subtle, so we shall study it later.

More on combinations. The numbers $C(n,r)$ satisfy a huge number of identities, many of which have combinatorial interpretations.
First an easy one. If $0 \le r \le n$ we can write combinations in terms of factorials:

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

or as the following quotient:

$$\begin{aligned} C(n,r) &= \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} \\ &= \frac{n}{1}\,\frac{n-1}{2}\,\frac{n-2}{3}\cdots\frac{n-(r-1)}{r}\,. \end{aligned}$$

Theorem Let n and r be integers with $0 \le r \le n$. We have

(a) $C(n,r) = C(n,n-r)$;
(b) $C(n,r) = C(n-1,r) + C(n-1,r-1)$, provided $r \ne 0$.

Proof:
1. Consider the problem of choosing r different objects from n objects, where order is not significant.
One way of doing this is simply to select the r objects; this can be done in $C(n,r)$ ways.
Alternatively, we could select $n-r$ objects, throw them away, and keep the rest; this can be done in $C(n,n-r)$ ways.
Since we have just solved the same problem by two methods, our two answers must be the same: that is,

$$C(n,r) = C(n,n-r)\ .$$

2. For the second identity, consider choosing r objects (repetitions not allowed, order not important) from n.
Doing this in the obvious way, there are $C(n,r)$ possibilities.
Alternatively, label one of the available items X, and make choices in the following way:

(a) Reject X, and choose r objects
from the remaining $n-1$ in $C(n-1,r)$ ways or
(b) Choose X in one way and then choose $r-1$ objects from the remaining $n-1$ in $C(n-1,r-1)$ ways.

Using this procedure, the number of possible choices is $C(n-1,r) + C(n-1,r-1)$, which is therefore equal to $C(n, r)$.

Comments

(a) This type of argument, where we prove that two expressions are equal by showing that they are both the answer to the same counting problem, is known as a *combinatorial proof.*
(b) Alternatively, both identities can be proved by writing $C(n, r)$ in terms of factorials.

The Binomial Theorem

Lemma
Let $0 \leq r \leq n$. If $(x + y)^n$ is expanded and like terms collected, the coefficient of $x^r y^{n-r}$ is $C(n, r)$.

Proof Imagine $(x + y)^n$ written out as a product of n factors,

$$(x + y)^n = (x + y)(x + y) \cdots (x + y) \ .$$

We obtain a term $x^r y^{n-r}$ by choosing an x from r of these factors. Clearly

(a) we cannot choose two terms from the same factor;
(b) the order of factors is not important;

once we have chosen an x from certain factors, we are forced to choose a y from all the others.

Therefore the number of terms $x^r y^{n-r}$ is $C(n, r)$, and the proof is complete.

Theorem *(The Binomial Theorem)*
If n is a non–negative integer then

$$(x + y)^n = \sum_{r=0}^{n} C(n, r) x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \ .$$

So, the numbers $C(n, r)$ are just those that appear in the Binomial Theorem.

These coefficients can be conveniently displayed using **Pascal's Triangle**.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
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&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

Row Sums:

The most well-known pattern in the triangle is the row sum.

If we put $x = y = 1$ in the binomial theorem, we obtain

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{k} + ... + \binom{n}{n},$$

which tells us that the sum of the numbers in each row is a power of 2.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
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&&&&&1&&3&&3&&1&&&&& \\
&&&&\boxed{1}&&\boxed{4}&&\boxed{6}&&\boxed{4}&&\boxed{1}&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

$$1 + 4 + 6 + 4 + 1 = 16 = 2^4.$$

Putting $x = 1, y = -1$ in the binomial theorem gives

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^k \binom{n}{k} + ... + (-1)^n \binom{n}{n},$$

so the alternating sum across each row is zero.

This is not so surprising for odd numbered rows, since the triangle is symmetric, but it certainly tells us something for the even numbered rows..

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
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&&&&&1&&3&&3&&1&&&&& \\
&&&&\boxed{1}&&\boxed{4}&&\boxed{6}&&\boxed{4}&&\boxed{1}&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

$$1 - 4 + 6 - 4 + 1 = 0$$

A Link to the Fibonacci Numbers:

The (shifted) Fibonacci numbers are: 1,1,2,3,5,8,13,21,34,... and satisfy the recurrence

$a_{n+2} = a_{n+1} + a_n$, with $a_0 = a_1 = 1$. That is, each number is the sum of the previous two.

We can make use of the identity above to derive the following interesting result.

If we let $F(n)$ denote the sum

$$F(n) = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots ,$$

where we keep summing until the lower number exceeds the top one. The sum starts at one place on the far left of the triangle and moves across and up the triangle to the other side. Replacing n with $n+1$ we have

$$F(n+1) = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots$$

and we write

$$F(n) = \qquad\qquad \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots ,$$

Adding the first term in line 1 with the second in line two and so on, and recalling that $\binom{n+1}{0} = \binom{n+2}{0}$ we have

$$F(n+1) + F(n) = \binom{n+2}{0} + \binom{n+1}{1} + \binom{n}{2} + \cdots = F(n+2).$$

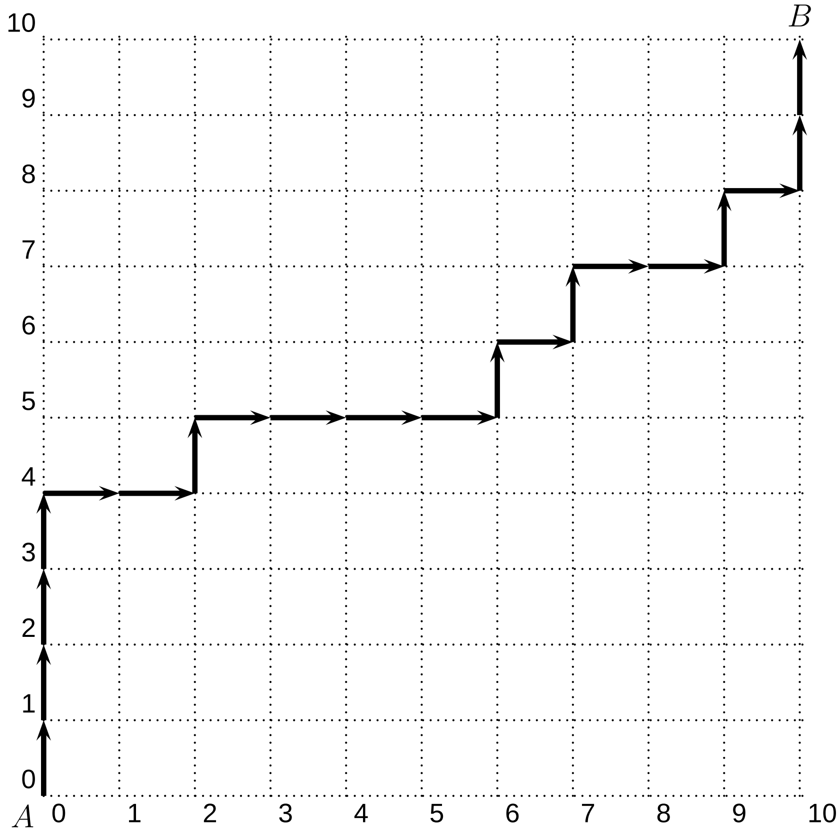
This, combined with $F(0) = F(1) = 1$, shows us that this sum generates the *Fibonacci numbers* since both sequences have the same first two terms and satisfy the same second order recurrence.
This result can be demonstrated in terms of Pascal's triangle.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad \boxed{1} \\
1 \quad 5 \quad 10 \quad \boxed{10} \quad 5 \quad 1 \\
1 \quad 6 \quad \boxed{15} \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad \boxed{7} \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\boxed{1} \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

$1 + 7 + 15 + 10 + 1 = 34$ which is the 9th Fibonacci number.

Grids:

On the grid shown, we wish to move from A to B such that we must either move to the right or upwards. How many paths are there from A to B?



The diagram illustrates one possible path $UUUURRURRRRURURRURUU$.

Observe that to there is a total 20 moves from A to B, 10 up and 10 across. To specify a particular path, we have to say which of the twenty moves are up. Once we have decided which ones are up then the rest must be to the right. There are therefore $\binom{20}{10}$ possible paths.

Interestingly we can now give a geometric proof of our standard identities for the binomial coefficients.

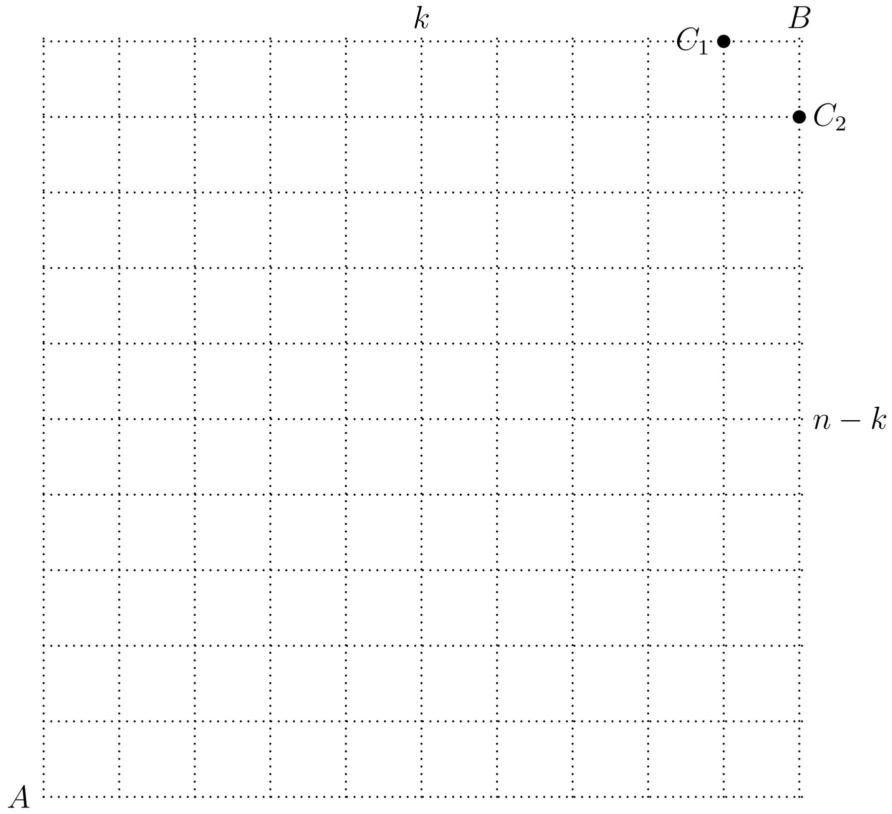
If we have an $k \times (n-k)$ grid, (where $n \geq k$), we count the number of ways to get from A to B. There are n moves in total and we have to choose k of these moves to be towards the right, or alternatively $n-k$ moves in the upwards direction. Since these must be the same we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

We can also prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

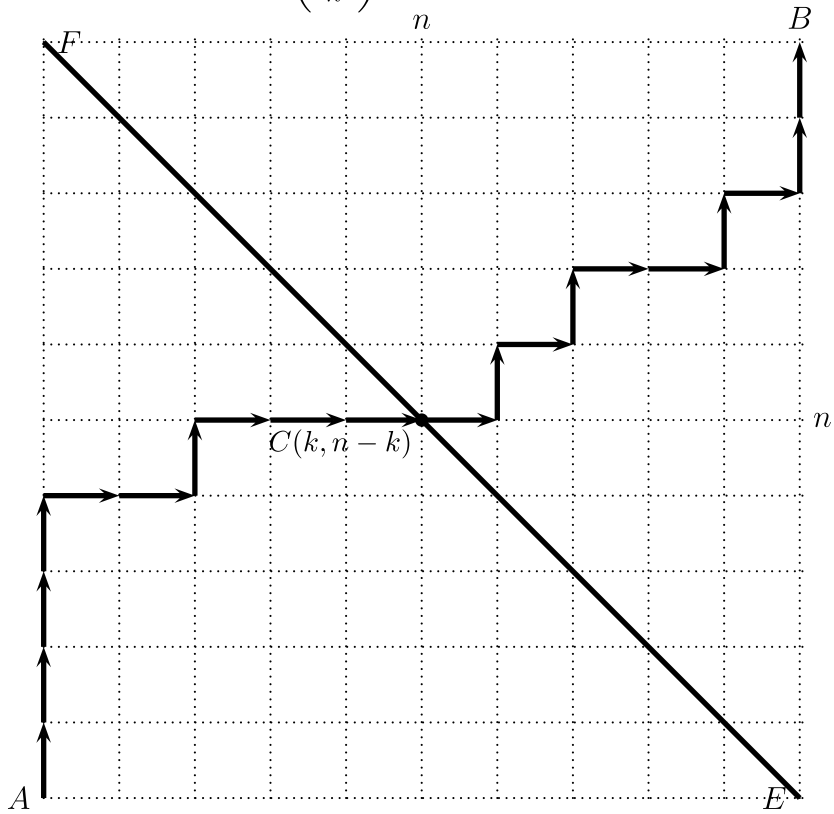
Consider the grid with $n-k$ upward steps and k steps to the right.



Any path from A to B must pass either through the point C_1 or C_2. Once we reach either of these points there is only one way to finish. There are $\binom{n}{k}$ ways to get from A to B, while there are $\binom{n-1}{k-1}$ ways to reach C_1 and $\binom{n-1}{k}$ ways to reach C_2. Hence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Sums of squares:
Furthermore, on the $n \times n$ board, every path must pass through exactly one of the points on the diagonal EF. There are $\binom{2n}{n}$ paths from A to B. Take a general point $C(k, n-k)$ on the diagonal, there are $\binom{n}{k}$ paths from A to C. By symmetry there are also $\binom{n}{k}$ paths from C to B. Thus there are $\binom{n}{k}^2$ paths from A to B passing through C.



Hence, summing over k from 0 to n, we have

$$\begin{aligned}\binom{2n}{n} &= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 \\ &= \sum_{k=0}^{n} \binom{n}{k}^2.\end{aligned}$$

$$
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&&&&\boxed{1}&&\boxed{4}&&\boxed{6}&&\boxed{4}&&\boxed{1}&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&\boxed{70}&&56&&28&&8&&1
\end{array}
$$

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70.$$

Counting functions

We now consider some counting problems involving functions.

Ex:
If S has n elements and T has m elements,
(a) How many functions $f : S \to T$ are there?
(b) How many of these are one–to–one?
(c) How many are onto?
(d) How many are bijections?

Inclusion-Exclusion Principle

Ex: Find the number of integers between 1 and 1000 which are divisible either by 3 or 5 or 7.

This is an example of the Inclusion-Exclusion principle. It is best expressed using sets, as you saw back in Chaper 1.

Thus, if we have three sets A_1, A_2, A_3 then

$|A_1 \cup A_2 \cup A_3| = (|A_1| + |A_2| + |A_3|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|.$

We can write this as

$$\sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|,$$

where the sums are understood to be over the range of values $i, j = 1, 2, 3$ and $i \neq j$.

Ex:
How many ten–letter words can be made from the English alphabet which contain at least one X, at least one Y and at least one Z?

This can be written using sets. If we have three sets A_1, A_2, A_3 then

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |U| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

Comments
(a) The principle of inclusion/exclusion is a formula for counting the elements in a union of sets. So, you should always begin an inclusion/exclusion problem by defining a universal set and appropriate subsets.
(b) There is an inclusion/exclusion formula for any (finite) number of sets, and the coefficients are always ± 1. To count the number of elements in a union of sets, add the number in each set, then subtract the number in every possible intersection of two sets, then add the number in every possible intersection of three sets, and so on.

For example,

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| \\ - |A_1 \cap A_2| - \cdots \\ + |A_1 \cap A_2 \cap A_3| + \cdots \\ - |A_1 \cap A_2 \cap A_3 \cap A_4| \ .\end{aligned}$$

This can be be written as

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|,$$

where the sums are understood to be over the range of values $i, j = 1, 2, 3, 4$ and $i \neq j \neq k$.

Inclusion/exclusion is commonly useful where we have to count items which have at least one of various properties, and where an item may have more than one of the properties. This is because to say that an item has one property *and* another gives more definite information about the item than to say that it has one property *or* another, and therefore makes it easier to count the possibilities.

Ex: How may binary strings of length eight are there that either start with 1 or end with 00?

Ex:
How many thirteen–card hands chosen from a standard pack contain exactly five cards in some suit?

The pigeonhole principle

If there are n pigeonholes with more than n pigeons living in them, then there must be a pigeonhole with more than one pigeon.

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing 2 or more objects.

Ex: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Ex: Show that among any group of five numbers, there are two with the same remainder when divided by 4.

There are 4 possible remainders when a number is divided by 4:

$$0, \quad 1, \quad 2, \quad 3.$$

By the pigeonhole principle, in a group of 5 numbers there are two with the same remainder when divided by 4.

Ex: Show that if 4 distinct integers are selected from the first 6 positive integers, there must be a pair of these integers with a sum equal to 7.

Ex: Show that if 7 distinct integers are selected from the first 10 positive integers, there must be 2 pairs of these integers with the sum 11.

Ex: Show that at a party of 20 people, there are 2 people who have the same number of friends.
Solution The number of friends each person at the party can have is a number between 0 and 19. However, 0 and 19 cannot both occur because it is not possible for someone to have 19 friends and someone else to have none. By the pigeonhole principle, since there are 19 possible numbers for 20 people, two people must have the same number of friends.

Ex: Show that in a set of 5 distinct ordered pairs of integers, there are at least 2 ordered pairs (a_1, b_1) and (a_2, b_2) such that a_1 and a_2 have the same parity (both even or both odd), and b_1 and b_2 have the same parity.

The generalized pigeonhole principle

If n objects are placed into k boxes, then there is **at least one** box containing **at least** $\lceil n/k \rceil$ objects.

Ex: Among any group of 50 people, there must be at least $\lceil 50/12 \rceil = 5$ who were born in the same month.

Ex: With phone numbers of the form 0N-NXX-XXXX where N is a non-zero digit and X is any digit, what is the least number of area codes 0N needed to guarantee that 30 million phones have distinct phone numbers?

Ex: Show that in a class of 9 students,
(a) there are at least 5 male students or at least 5 female students.
(b) there are at least 3 male students or at least 7 female students.
(c) there are at least 4 male students or at least 6 female students.

Solution:
(a)
Argument 1: By the generalized pigeonhole principle, there must be a group of at least $\lceil 9/2 \rceil = 5$ students who are all males or all females.

Argument 2: If there were fewer than 5 males and fewer than 5 females, then there would be at most 4 males and at most 4 females, for a total of at most 8 people, contradicting the fact that there are 9 people in total.

(b) If there were fewer than 3 males and fewer than 7 females, then there would be at most 2 males and at most 6 females, for a total of at most 8 people. This contradicts the fact that there are 9 people in total.

(c) If there were fewer than 4 males and fewer than 6 females, then there would be at most 3 males and at most 5 females, for a total of at most 8 people. This contradicts the fact that there are 9 people in total.

Partitions

Ex: How many different solutions are there to the equation $x + y + z = 11$, where x, y, and z are nonnegative integers?

Theorem
Suppose k, n are positive integers.
The number of solutions in the non-negative integers to

$$x_1 + x_2 + ... + x_k = n \qquad\qquad *$$

is given by $\binom{n+k-1}{k-1}$.

Proof

Take $n + k - 1$ crosses and choose $k - 1$ of these to be plus signs. Adding up the number of crosses between each plus sign, we have a solution to (*). Conversely any solution to * can be converted to crosses and plus signs as above, and so this gives a one-to-one correspondence. The result follows.

Ex: How many different solutions are there to the equation $x + y + z = 11$, where x, y, and z are nonnegative integers where

a. $x \geq 1$, $y \geq 2$, and $z \geq 3$ are nonnegative integers?

b. where x, y, and z are integers no less than -5?

c. where $x \leq 5$ and $y \leq 5$ and $z \leq 5$ are nonnegative integers?

d. where $x \leq 7$ and $y \leq 7$ and $z \leq 7$ are nonnegative integers?

Ex: How many different solutions are there to the inequality $x + y + z \leq 11$, where x, y, and z are nonnegative integers?

The Multinomial Theorem

Ex: If the *multinomial*

$$(x + y + z)^{100}$$

is expanded and terms collected,

a. what will the coefficient of $x^{11}y^{49}z^{40}$ be?

b. how many different terms will there be?

In general, if $(x_1 + x + 2 + ... + x_k)^n$ were expanded then the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}....x_k^{\alpha_k}$ is $\dfrac{n!}{\alpha_1!...\alpha_k!}$.

Ex: If $(2a + 4b - 3c + 7d + e)^{1111}$ is expanded,

a. what will the coefficient of $a^{37}\, b^{289}\, c^{4}\, e^{781}$ be?

b. how many different terms will there be?

Recursion

Recursion is a process of defining objects in a self-similar way. It can be used to define sequences, functions, sets and algorithms.

Recursively defined sequence

Let $\{a_n\}_n$ be a sequence of numbers. An equation which gives a_n in terms of $a_{n-1}, a_{n-2}, \dots, a_{n-k}$ is called a *recurrence relation of order k*.

To determine the values of a_n from a recurrence relation of order k, we need to know the values of $a_0, a_1, \dots, a_{k-1}$; these are known as *initial conditions*.

Ex:
The *Fibonacci numbers* F_n are defined by the initial conditions

$$F_0 = 0 \quad \text{and} \quad F_1 = 1,$$

and the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for all } n \geq 2.$$

From the definition we can calculate

$$\begin{aligned}
F_2 &= F_1 + F_0 = 1 + 0 = 1\\
F_3 &= F_2 + F_1 = 1 + 1 = 2\\
F_4 &= F_3 + F_2 = 2 + 1 = 3\\
F_5 &= F_4 + F_3 = 3 + 2 = 5\\
F_6 &= F_5 + F_4 = 5 + 3 = 8\\
&\vdots
\end{aligned}$$

The recurrence relation can also be written as either

$$F_{n+1} = F_n + F_{n-1} \text{ for all } n \geq 1, \qquad \text{or} \qquad F_{n+2} = F_{n+1} + F_n \text{ for all } \ n \geq 0.$$

Ex:
A sequence of numbers a_n are defined by $a_0 = 1$, $a_1 = 3$, $a_2 = 4$, and $a_n = 2a_{n-1} + a_{n-3}$ for $n \geq 3$. Calculate a_n for $n = 3, 4, 5, 6$.

Ex:
When ascending a flight of stairs, an elf can take 1 stair in one stride or 2 stairs in one stride.

Let a_n be the number of different ways for the elf to ascend an n-stair staircase. We wish to obtain a recursive definition for the numbers a_n for $n \geq 1$.

Ex:
An important topic in mathematical computer science is the study of *formal languages*. A language, in the mathematical sense, is any set of words formed from a given alphabet. The study of "natural" languages is difficult because of their lack of precision, so we are thinking rather of things like computer programming languages.
Many languages are defined *recursively* by giving rules for constructing meaningful statements from the simplest possible components.
In studying languages it is often convenient to have a word containing *no* letters! This is referred to as the *empty word* and is denoted by λ.

Ex:
Consider an alphabet $\{a, b\}$, and define a language L as follows:

(*Basis*) λ and a are in L;

(*Recursion*) if w is in L then bw and abw are in L.

Then

$$\begin{aligned} L &= \{\lambda, a,\ b, ba, ab, aba,\ bb, bba, bab, baba, abb, abba, abab, ababa, \ldots\} \\ &= \{\text{ words which contain only } a\text{s and } b\text{s} \\ &\qquad \text{and which have no two consecutive } a\text{s}\}\ . \end{aligned}$$

Now let us find a recursion giving the number of words of length n in this language L.

Let a_n be the number of words of length n in L. For $n \geq 2$, such a word is not one of the basis words and therefore must have been derived by using one of the two recursive rules. That is, such a word is bw, where w is a word in L of length $n-1$; or

abw, where w is a word in L of length $n-2$.

So, the number of such words is $a_{n-1}+a_{n-2}$, and the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \ .$$

This argument, of course, cannot tell us the number of words of length 0 or 1, and these have to be calculated separately to give the initial conditions

$$a_0 = 1 \ , \quad a_1 = 2 \ .$$

From earlier work, we see that

$$a_n = F_{n+2}$$

It takes a bit of thought to be sure that we have avoided double counting in the above argument.

Ex:
Binary strings are constructed from the alphabet $\{0,1\}$ with the restriction that the strings may not contain three consecutive 1's.

Let T_n be the total number of such strings of length n. We wish to obtain a recursive definition for the numbers T_n, $n \geq 0$.

The first few of these numbers are: 1, 2, 4, 7, 13, 24, 44, 81, 149, ...

Ex:
A Vending machine
A vending machine accepts \$1 and \$2 coins.
In how many ways can one insert \$$n$ into the machine?
For example, if $n=5$ there are eight ways:

$$\begin{gathered} 2+2+1\,, \quad 2+1+2\,, \quad 1+2+2\,, \\ 2+1+1+1\,, \quad 1+2+1+1\,, \quad 1+1+2+1\,, \quad 1+1+1+2\,, \\ 1+1+1+1+1\,. \end{gathered}$$

Ex:
Straight lines are drawn on a piece of paper so that every pair of lines intersect but no three lines intersect at a common point.

Let x_n be the number of regions formed by n such lines. Show that $x_0 = 1$, and $x_n = x_{n-1}+n$ for $n \geq 1$.

Solution No line: 1 region, $x_0 = 1$
1 line: 2 regions, $x_1 = 2$
2 lines: 4 regions, $x_2 = 4$ etc.
Suppose we have now $n-1$ such lines, i.e., x_{n-1} regions. If we draw another line, the intersection of this nth line with the other existing lines will divide this nth line into n sections, with each section splitting an existing region into two, creating an additional n regions. Thus $x_n = x_{n-1} + n$.

We can then see (as we proved by induction earlier):

$$x_n = x_{n-1} + n = x_{n-2} + (n-1) + n = \cdots = 1 + \sum_{k=1}^{n} k = 1 + \frac{1}{2}n(n+1) = \frac{1}{2}(n^2 + n + 2).$$

Ex:
In the Tower of Hanoi game, we have three pegs A, B, and C, with a pile of n rings on peg A such that each ring has a smaller diameter than the ring immediately below it and thus the largest ring is at the bottom of the pile. The aim of the game is to move the rings individually between pegs, at no stage placing a larger ring on top of a smaller ring, so that the whole pile is ultimately shifted to another peg, say B.

Let H_n be the minimal number of moves required to shift a pile of n rings from one peg to another.

a. Find H_1, H_2, and H_3.

b. Obtain a recurrence relation for H_n.

Solving recurrences
The *solution* of a recurrence relation is an explicit formula for the sequence.

Some recurrence relations can be solved by

- listing the terms of the sequence until a pattern emerges
- guessing a formula based on the pattern
- proving the formula by induction

Ex:
In the Tower of Hanoi example, we have $H_1 = 1$ and $H_n = 2H_{n-1} + 1$ for $n \geq 2$. Thus

$$\begin{aligned}
&H_1 = 1\\
&H_2 = 2 \times 1 + 1 = 2 + 1\\
&H_3 = 2(2+1) + 1 = 2^2 + 2 + 1\\
&H_4 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1.
\end{aligned}$$

Therefore we guess that

$$H_n = 1 + 2 + 2^2 + \cdots + 2^{n-1} = \frac{2^{n-1+1} - 1}{2 - 1} = 2^n - 1,$$

where we used the geometric series formula.

Now we prove this formula by (informal) induction.
For $n = 1$, the formula gives $H_1 = 1$ as required.
Suppose that the formula holds for some particular $n = k \geq 1$, that is, $H_k = 2^k - 1$. Then, by the recurrence relation and the induction hypothesis, we have

$$H_{k+1} = 2H_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1,$$

which shows that the formula also holds for $n = k + 1$. Hence, by induction, the formula is correct.

Ex:
By listing the terms of the Fibonacci numbers (see page 1), can you **guess** that the solution is (as we proved by induction in Topic 3.3)

$$F_n \;=\; \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \;???$$

Note: It is not always possible to solve a recurrence relation by pattern spotting. Below we discuss the technique for solving one type of recurrence relations.

A *linear recurrence relation of order k with constant coefficients* takes the form

$$a_n \;=\; c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n),$$

or equivalently,

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} \;=\; f(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers independent of n, and $c_k \neq 0$.

The case $f(n) = 0$ is referred to as the *homogeneous* case.

The case $f(n) \neq 0$ is referred to as the *non-homogeneous* case.

To find the solution of a recurrence relation of order k, we need to know the *initial conditions* $a_0, a_1, \ldots, a_{k-1}$. Without them we can obtain only the *general solution* of the recurrence relation; this will involve k unknown constants.

The technique for solving linear recurrence relations with constant coefficients is very similar to the technique for solving linear differential equations with constant coefficients which is done in Maths 1B.

Substituting $a_n = r^n$ into the *homogeneous equation*

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} \;=\; 0\,, \tag{1}$$

and cancelling out the factor r^{n-k} (assuming that $r \neq 0$), we obtain the *characteristic equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-1} - \cdots - c_k \;=\; 0\,, \tag{2}$$

which is a polynomial of degree k.

Theorem (proof omitted). If the characteristic equation (2) has k distinct real roots $r_1, r_2, r_3 \ldots, r_k$, then the *general solution of the homogeneous equation* (1) is

$$a_n \;=\; A_1 r_1^n + A_2 r_2^n + A_3 r_3^n + \cdots + A_k r_k^n\,, \tag{3}$$

where $A_1, A_2, A_3, \ldots, A_k$ are arbitrary constants.

If the characteristic equation (2) has repeated roots, then we need to " multiply certain terms in (3) by powers of n": for example, if $r_1 = r_2 = r_3$, then (3) becomes

$$a_n \;=\; A_1 r_1^n + A_2 n r_1^n + A_3 n^2 r_1^n + A_4 r_4^n + \cdots + A_k r_k^n\,.$$

First Order Equations

Theorem (*first order homogeneous equation*). The first order recurrence relation $a_n = r\,a_{n-1}$ with $a_0 = A$ has solution

$$a_n = A\,r^n.$$

Proof. By induction. This is left for exercise.

Second order equations

Theorem (*second order homogeneous equation*). Consider the recurrence

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} = 0\,. \tag{4}$$

Let α and β denote the roots of the characteristic equation

$$r^2 - c_1 r - c_2 \;=\; 0\,.$$

- If α, β are real and $\alpha \neq \beta$, then the general solution of the recurrence is

$$a_n = A\alpha^n + B\beta^n. \tag{5}$$

- If $\alpha = \beta$ is real, then the general solution of the recurrence is

$$a_n = A\alpha^n + B\,n\,\alpha^n.$$

- (Not examinable) If α, β are complex conjugates $re^{\pm i\theta}$, then the general solution of the recurrence is

$$a_n = r^n(A\cos(n\theta) + B\sin(n\theta)).$$

In the above, A and B are arbitrary constants.

Proof: We prove the case with distinct roots, leaving the other case as an exercise.

For any arbitrary A and B, with a_n defined by (5) for $n \geq 0$, we have

$$\begin{aligned}
&a_n - c_1 a_{n-1} - c_2 a_{n-2}\\
&= (A\alpha^n + B\beta^n) - c_1(A\alpha^{n-1} + B\beta^{n-1}) - c_2(A\alpha^{n-2} + B\beta^{n-2})\\
&= A\alpha^{n-2}(\alpha^2 - c_1\alpha - c_2) + B\beta^{n-2}(\beta^2 - c_1\beta - c_2)\\
&= A\alpha^{n-2}\cdot 0 + B\beta^{n-2}\cdot 0\\
&= 0.
\end{aligned}$$

Hence the formula (5) is a solution of the homogenous recurrence relation (4).

It can be shown that this is in fact the general solution, that is, any solution of the equation (4) where $\alpha \neq \beta$ has the form (5). In this case, given initial values a_0 and a_1, we find that

$$\begin{cases} a_0 = A + B \\ a_1 = A\alpha + B\beta \end{cases} \quad\Longrightarrow\quad \begin{cases} A = \frac{a_1 - a_0\beta}{\alpha-\beta} \\ B = \frac{a_0\alpha - a_1}{\alpha-\beta} \end{cases}$$

Ex:
Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 4$.

Ex:
Solve the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 1$.

Ex:
Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 4$ and $a_1 = 11$.

Ex:
Find a general formula for the Fibonacci numbers F_n by solving the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$.

Non-homogeneous equations

Theorem. The general solution of the *non-homogeneous equation*

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} \;=\; f(n) \tag{6}$$

is of the form

$$a_n = h_n + p_n\,,$$

where h_n is the *general solution of the homogeneous equation*

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} \;=\; 0\,, \tag{7}$$

p_n is a *"particular solution" of the non-homogeneous equation* (6).

Proof Suppose that $a_n = h_n + p_n$ for all $n \geq 0$, where h_n is the general solution of the homogeneous equation (7) and p_n is a particular solution of the non-homogeneous equation (6). Then we have

$$\begin{aligned} h_n - c_1 h_{n-1} - c_2 h_{n-2} - \cdots - c_k h_{n-k} \;&=\; 0\,,\\ p_n - c_1 p_{n-1} - c_2 p_{n-2} - \cdots - c_k p_{n-k} \;&=\; f(n)\,. \end{aligned}$$

Adding the two equations together yields

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} \;=\; f(n)\,.$$

This shows that a_n is a solution to the non-homogeneous equation (6).

Next we assume that a_n is an arbitrary solution to the non-homogeneous equation, and that p_n is another solution to the non-homogeneous equation (6). Then we have

$$\begin{aligned} a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} \;&=\; f(n)\,,\\ p_n - c_1 p_{n-1} - c_2 p_{n-2} - \cdots - c_k p_{n-k} \;&=\; f(n)\,. \end{aligned}$$

Subtracting the second equation from the first and defining $h_n = a_n - p_n$ for $n \geq 0$, we obtain

$$h_n - c_1 h_{n-1} - c_2 h_{n-2} - \cdots - c_k h_{n-k} \;=\; 0\,,$$

which shows that h_n is a solution of the homogeneous equation (7).
Hence any solution a_n of the non-homogeneous equation can be written as a sum of a solution h_n of the homogeneous equation and another "particular" solution p_n of the non-homogeneous equation.

How do we obtain a particular solution of the non-homogeneous equation?

- " Guess" a formula for the solution with unknown coefficients based on the form of $f(n)$.

- Substitute the formula into the non-homogeneous equation to determine the coefficients.

This is called the *Method of undetermined coefficients.*
I call it simply " intelligent guessing"

How do we "guess" a formula for the particular solution of the non-homogeneous equation? Here is a list of intelligent guesses:

If $f(n)$ is a constant, try a constant

$$p_n = c\,.$$

If $f(n)$ is a polynomial in n of degree s, try a polynomial of degree s

$$p_n = b_s n^s + b_{s-1} n^{s-1} + \cdots + b_1 n + b_0\,.$$

If $f(n)$ is of the exponential form α^n, try the exponential form

$$p_n = c\,\alpha^n\,.$$

If $f(n)$ is a product of the two previous forms, try a product

$$p_n = (b_s n^s + b_{s-1} n^{s-1} + \cdots + b_1 n + b_0)\,\alpha^n\,.$$

HOWEVER, if by some choice of the coefficients in our guess of p_n we get a function that is already a non-zero solution of the homogeneous equation ($h(n)$ for suitable choices of A and B), then we need to multiply our whole guess by n; and multiply by another factor of n or more if necessary.

Notes: The above strategy only works for certain types of functions $f(n)$. There is no general method which covers every function $f(n)$.

Ex:

Solve the recurrence $a_n - a_{n-1} - 6a_{n-2} = 12$ subject to the initial conditions $a_0 = 1$ and $a_1 = 2$.

Ex:

Given that the general solution of the homogeneous recurrence relation $a_n - a_{n-1} - 6a_{n-2} = 0$ is $h_n = A(-2)^n + B3^n$, find the general solution of the following recurrence relations.

a. $a_n - a_{n-1} - 6a_{n-2} = 36n$.

b. $a_n - a_{n-1} - 6a_{n-2} = 3 \cdot 2^n$.

c.$a_n - a_{n-1} - 6a_{n-2} = 2n3^n$.

Ex:
Given that the general solution of the homogeneous recurrence relation $a_n - 7a_{n-1} + 12a_{n-2} = 0$ is $h_n = A3^n + B4^n$, guess a formula for a particular solution of the following recurrence relations:

Recurrence relation	Guess a particular solution
(a) $a_n - 7a_{n-1} + 12a_{n-2} = 30$	$p_n = c$
(b) $a_n - 7a_{n-1} + 12a_{n-2} = 30n$	$p_n = cn + d$
(c) $a_n - 7a_{n-1} + 12a_{n-2} = 3 \cdot 2^n$	$p_n = c2^n$
(d) $a_n - 7a_{n-1} + 12a_{n-2} = 3n2^n$	$p_n = (cn + d)2^n$
(e) $a_n - 7a_{n-1} + 12a_{n-2} = 2 \cdot 3^n$	$p_n = cn3^n$

Ex:
Given that the general solution of the homogeneous recurrence relation $a_n + 2a_{n-1} - 3a_{n-2} = 0$ is $h_n = A + B(-3)^n$, guess a formula for a particular solution of the following recurrence relations:

Recurrence relation	Guess a particular solution
(a) $a_n + 2a_{n-1} - 3a_{n-2} = 10$	$p_n = cn$
(b) $a_n + 2a_{n-1} - 3a_{n-2} = 10n$	$p_n = (cn + d)n$
(c) $a_n + 2a_{n-1} - 3a_{n-2} = 2 \cdot 3^n$	$p_n = c3^n$
(d) $a_n + 2a_{n-1} - 3a_{n-2} = 2n3^n$	$p_n = (cn + d)3^n$
(e) $a_n + 2a_{n-1} - 3a_{n-2} = 2n(-3)^n$	$p_n = (cn + d)n(-3)^n$

Ex:

Given that the general solution of the homogeneous recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 0$ is $h_n = A2^n + Bn2^n$, guess a formula for a particular solution of the following recurrence relations:

Recurrence relation	Guess a particular solution
(a) $a_n - 4a_{n-1} + 4a_{n-2} = 5$	$p_n = c$
(b) $a_n - 4a_{n-1} + 4a_{n-2} = 5n$	$p_n = cn + d$
(c) $a_n - 4a_{n-1} + 4a_{n-2} = 5 \cdot 2^n$	$p_n = cn^2\, 2^n$
(d) $a_n - 4a_{n-1} + 4a_{n-2} = 5n \cdot 2^n$	$p_n = (cn + d)n^2\, 2^n$
(e) $a_n - 4a_{n-1} + 4a_{n-2} = 5n(-2)^n$	answer$p_n = (cn + d)(-2)^n$

Ex:

Solve the recurrence

$$a_n + 5a_{n-1} - 6a_{n-2} = (-6)^n$$

subject to the initial conditions $a_0 = 12$ and $a_1 = -\frac{1}{7}$.

Recursively defined sets

Recursive definitions can also be used to define functions or sets.

Ex:
Recursively defined functions:
Factorials

$$0! = 1, \quad \text{and} \quad n! = n \times (n-1)! \quad \text{for} \quad n \geq 1.$$

Powers

$$a^0 = 1, \quad \text{and} \quad a^n = a \times a^{n-1} \quad \text{for} \quad n \geq 1.$$

Binomial coefficients

$$\binom{n}{0} = \binom{n}{n} = 1 \text{ for } n \geq 0, \text{ and } \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ for } n > k \geq 1.$$

String length

$$\ell(\lambda) = 0, \quad \text{and} \quad \ell(wx) = \ell(w) + 1 \quad \text{for any string } w \text{ and any alphabet } x.$$