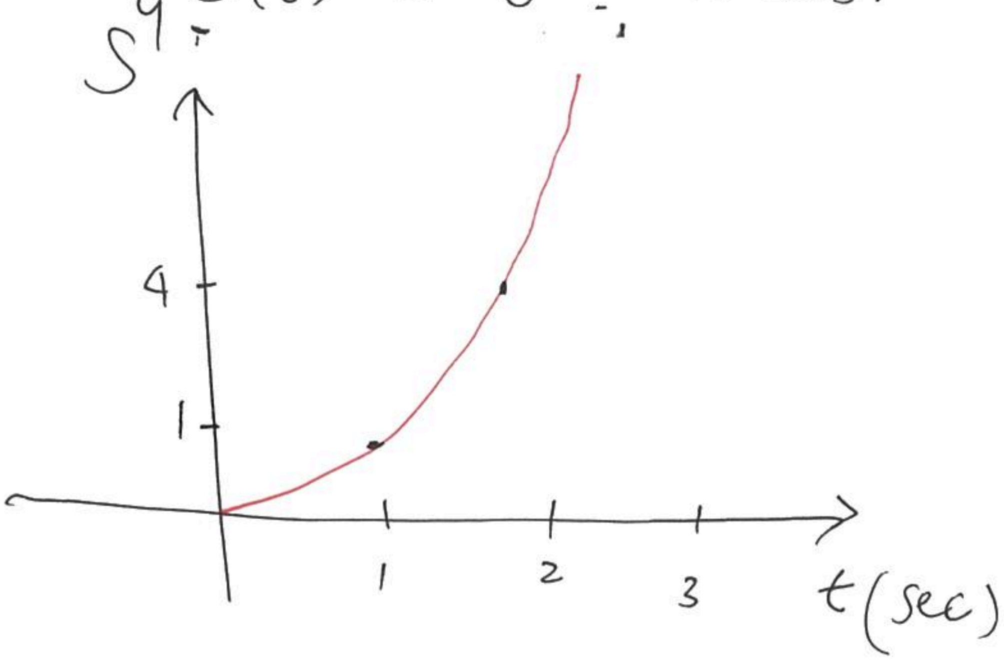
Differentiable functions.

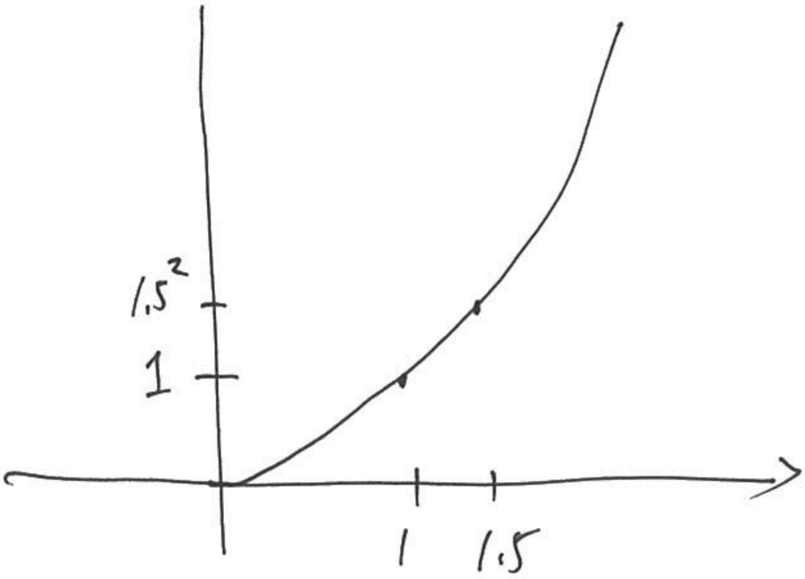
Suppose a particle moving along a straight line. and has the displacement function.

$$S(t) = t^2 \text{ metres.}$$



The average velocity between $t=1$ and $t=2$.

$$\bar{v}_{[1,2]} = \frac{2^2 - 1^2}{2-1} = 3 \text{ m/s.}$$



$$\bar{v}[1, 1.5] = \frac{1.5^2 - 1^2}{1.5 - 1} = 2.5 \text{ m/s}.$$

$$\vec{v}[1, 1.1] = 2.1 \text{ m/s}.$$

$$\vec{v}[1, 1.01] = 2.01 \text{ m/s}$$

$$\vec{v}[1, 1.001] = 2.001 \text{ m/s}.$$

Def: Suppose f is defined on an open interval containing x. We say that f is differentiable at x, if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. In this case, we denote the limit by, $f'(x)$, $\frac{df}{dx}$, $\frac{d}{dx}(f(x))$.

f' is called the derivative of f at x.

We say f is differentiable on (a,b) if f is differentiable at every $x \in [a,b]$.

example: $f(x) = x^3$.

Find $f'(x)$ by def.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2.$$

$$g(x) = \sqrt{x}$$

$$g'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

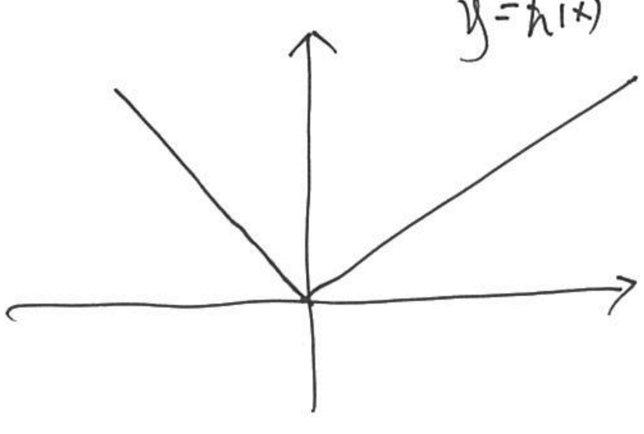
$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$h(x) = |x|.$$

$$= \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$



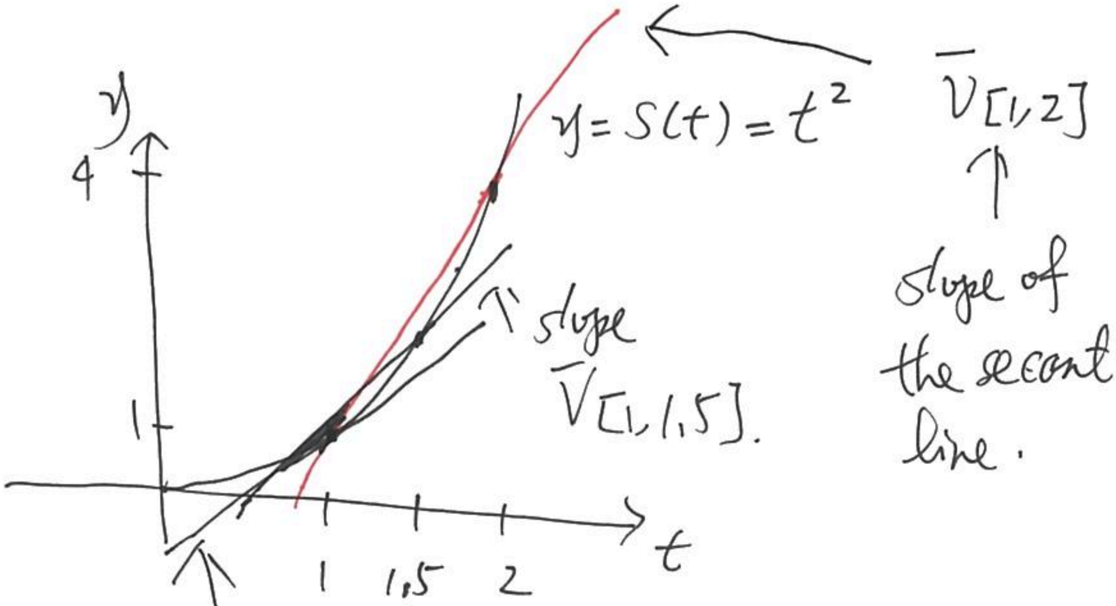
$h'(0) \cancel{\neq \lim_{h \to 0} \frac{h(}{h}}$

$$= \lim_{l \to 0} \frac{h(0+l) - h(0)}{l}$$

$$= \lim_{l \to 0} \frac{|l|}{l}$$

~~de.~~ This limit does not exist.

h is not differentiable at $x=0$.



slope of the tangent line $= \dfrac{dS(t)}{dt}$.

Theorem: If $x \in \mathbb{R}$.

$$\frac{d}{dx}(\sin x) = \cos x.$$

proof:

$$\frac{d}{dx}(\sin x)$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h/2}$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{h/2}$$

$$= \cos x.$$

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

f is continuous on $\mathbb{R}$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

This limit does not exist.

Therefore, f is not differentiable at $x=0$.

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

g is differentiable at $x=0$.

Equivalent def: We say that f is differentiable at $x=a$, if

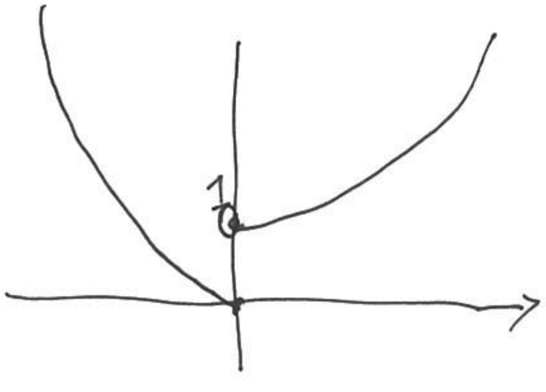
$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

exists.

Theorem: If f is differentiable at $x=a$, then f is continuous at $x=a$.

Corollary: If f is not continuous at $x=a$, then f is not differentiable at $x=a$.

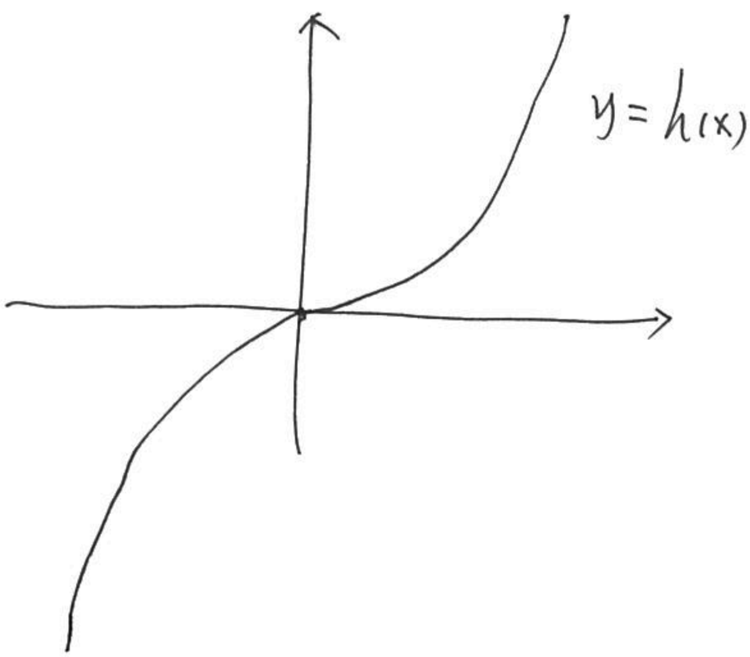
$$f(x) = \begin{cases} e^x, & \text{if } x > 0 \\ x^2, & \text{if } x \leq 0. \end{cases}$$



f is not continuous at $x=0$, so f is not differentiable there either.

$$h(x) = x|x|.$$

$$= \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0. \end{cases}$$



$$h'(0) = \lim_{l \to 0} \frac{h(0+l) - h(0)}{l}$$

$$= \lim_{l \to 0} \frac{l|l|}{l}$$

$$= \lim_{l \to 0} |l|$$

$$= 0$$

h is differentiable ~~the~~ at $x=0$.

Theorem: $f(x) = \begin{cases} p(x), & x \geq a \\ q(x), & x < a. \end{cases}$

with $p(x)$ and $q(x)$ differentiable in some interval containing a. Then f is differentiable at $x=a$ if and only if f is continuous at $x=a$, and

$$p'(a) = q'(a).$$