$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x.$$

$$= \exp\left(\lim_{x\to\infty} \log \left(1+\frac{1}{x}\right)^x\right)$$

$$= \exp \lim_{x\to\infty}\left(x \log \left(1+\frac{1}{x}\right)\right)$$

$$\lim_{x\to\infty} x \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

L'Hôpital

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{\cancel{x^2}}}{\cancel{-}\frac{1}{\cancel{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

Therefore, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

Inverse functions

$f: \mathbb{R} \to \mathbb{R}: \ x \to 2x+3.$

$f(x) = 2x+3.$

$f(5) = 13.$

Natural question: If given a y value, can we find an x such that $f(x) = y$.

$x^3 = 7$, To solve this eq, we ~~xx~~ need to find an x value such that $x^3 = 7$.

$f(x) = \cancel{3}\, 2x+3.$

Find a value x such that $f(x) = 10$. . $x = \frac{7}{2}$.

Def: $f: A \to B$. If there is another function $g: B \to A$ such that $f \circ g(x) = \cancel{A}\, x$ and $g \circ f(x) = x$, then g is said to be the inverse of f, written as $f^{-1} = g$.

$f(x) = 2x+3$.

$y = 2x+3$

$g(x) = \frac{x-3}{2}$.

~~f~~ g is the inverse of f.

$f(x) = x^2$.

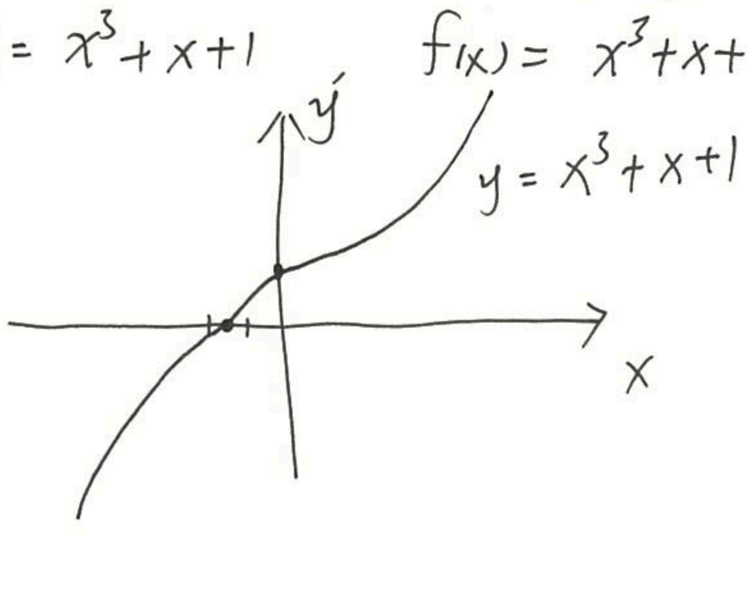
$f: \mathbb{R} \to \mathbb{R}: x \to x^2$.

f has no inverse function on $\mathbb{R}$.

A simple test to determine whether a function f has an inverse or not, is the horizontal line test.

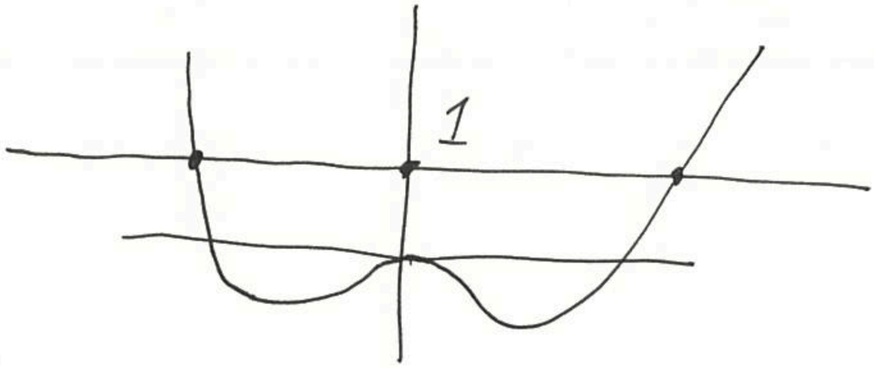
If every horizontal line on the xy plane crosses the graph of $y = f(x)$ at most once, the f has an inverse function.

$y = x^3+x+1$ $\quad f(x) = x^3+x+1$



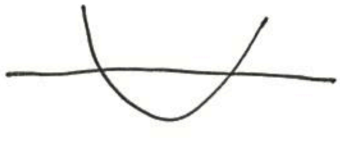
$f(x)$ passes the horizontal line test and hence has an inverse.

$y = x^4 - x^2$. $\quad g(x) = x^4 - x^2$



$g(x)$ has no inverse.

Theorem: Suppose that f is differentiable on (a,b), and $f'(x) \neq 0$ for all $x \in (a,b)$, then f has an inverse on (a,b).



proof. Use MVT.

Example: $y = x^3 + x + 1$,

$$y' = 3x^2 + 1 \geq 1 \quad \text{for all } x.$$

Therefore, $y = x^3 + x + 1$ has an inverse.

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example. $f(x) = 2x + \sin x$.

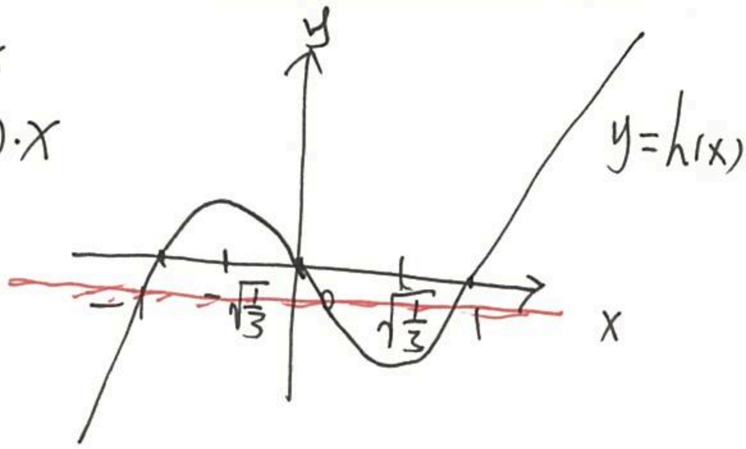
$$f'(x) = 2 + \cos x.$$

$$f'(x) \geq 2 - 1 = 1. \quad \text{for all } x.$$

$$f'(x) \neq 0, \quad \text{for all } x.$$

Therefore $f(x)$ has an inverse on $\mathbb{R}$.

$$h(x) = x^3 - x$$
$$= (x+1)(x-1)\cdot x$$



Find the maximal regions on which $h(x)$ has an inverse.

$h(x)$ is 1-to-1 on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$

$h(x)$ is 1-to-1 on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$

$h(x)$ is 1-to-1 on $\left(\sqrt{\frac{1}{3}}, \infty\right)$

$h(x)$ has an inverse on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$, $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ and $\left(\sqrt{\frac{1}{3}}, \infty\right)$.

$$f(x) = x^3 + x + 1.$$

$f(x)$ has an inverse on $\mathbb{R}$.

Theorem: Suppose f is differentiable on (a,b) and has an inverse $g(x)$ on (a,b), ~~then~~ Then.

$$g'(x) = \frac{1}{f'(g(x))}.$$

proof: $f(g(x)) = x$

differentiate both sides

$\Rightarrow f'(g(x)) \cdot g'(x) = 1$

$\Rightarrow \quad g'(x) = \dfrac{1}{f'(g(x))}$.

$f(x) = x^3 + x + 1$.

Suppose $g(x)$ is the inverse of $f(x)$.

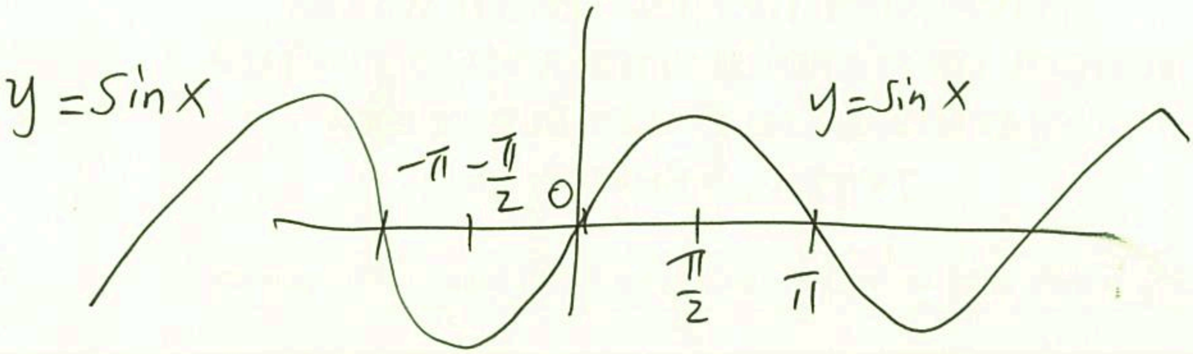
$g'(1) = ?$

$$g'(1) = \frac{1}{f'(g(1))}$$

$f'(x) = 3x^2 + 1$, $g(1) = 0$

$$= \frac{1}{f'(0)}$$

$$= 1.$$

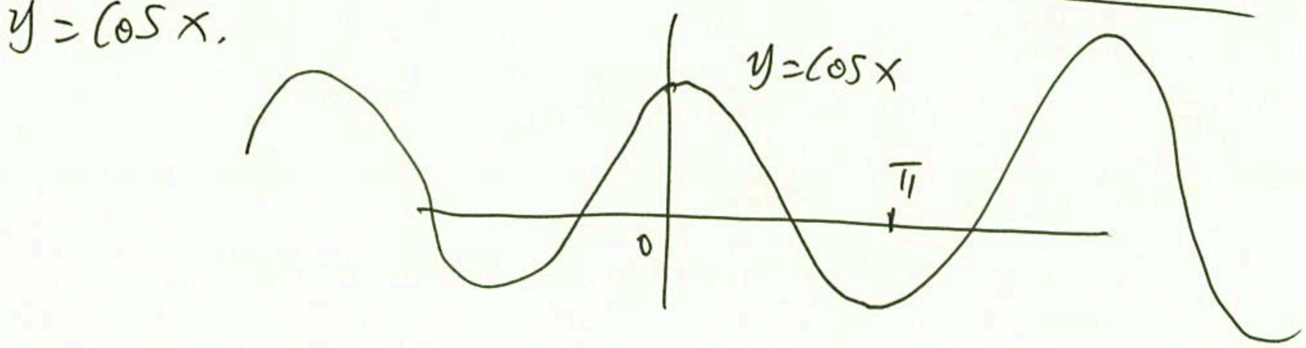


On $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin x$ is 1-to-1.

Therefore $\sin x$ has an inverse on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

We call this function $\arcsin x$

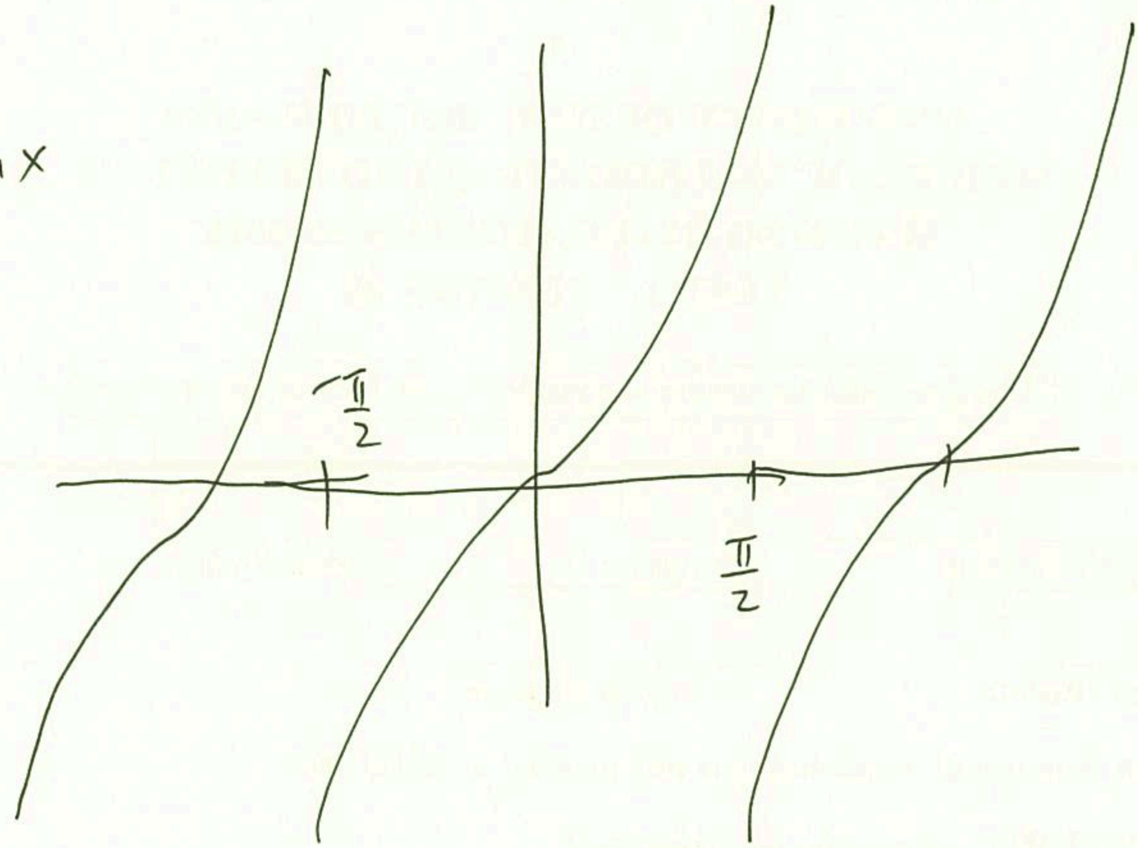
$$\arcsin x : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}].$$



$\arccos x$ is the inverse of $\cos x$ on $[0, \pi]$.

$$\arccos x : [-1, 1] \longrightarrow [0, \pi].$$

$\tan x$



$\arctan x$ is the inverse of $\tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The domain of $\arctan x$ is $(-\infty, \infty)$.

The range of $\arctan x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\operatorname{arccot} x$, $\operatorname{arcsec} x$ and $\operatorname{arccsc} x$ are defined similarly.