Theorem: If $f(x) = \begin{cases} p(x), & x \geq a \\ q(x), & x < a. \end{cases}$

with $p(x)$ and $q(x)$ differentiable in some interval containing a. Then f is differentiable at a if and only if f is continuous at $x=a$ and $p'(a) = q'(a)$.

---

ex $f(x) = \begin{cases} \sin x & x < \pi \\ ax+b & x \geq \pi. \end{cases}$

Given f is differentiable at π, find the values of ~~a~~ a and b.

We must have

$$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x)$$

$$a\pi + b = \lim_{x \to \pi^+} (ax+b) = \lim_{x \to \pi^-} \sin x = 0$$

$$\left.\begin{aligned} (\sin x)' &= \cos x, \quad \cos \pi = -1. \\ (ax+b)' &= a. \end{aligned}\right\}$$

$$\Rightarrow a = -1$$

$b = \pi$. (as $a\pi + b = 0$ & $a = -1$).

Rules for differentiation.

Suppose f & g are differentiable at $x = a$.
Then at $x = a$; we have

1) $(f \pm g)' = f' \pm g'$.

2) $f(x) = C$, $f'(x) = 0$.

3) $(c f(x))' = c f'(x)$.

4) $(f(x)\, g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$.

$y = x$, $y' = 1$

If n is a positive integer,

$(x^n)' = n x^{n-1}$. (apply the product rule n times,

5) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)\, g(x) - f(x)\, g'(x)}{(g(x))^2}$

$\left(\dfrac{1}{x}\right)' = \dfrac{-1}{x^2}$.

6) $(f \circ g_{(x)})' = f'(g_{(x)}) \cdot g'_{(x)}$.
(the chain rule).

$$y = c \Rightarrow y' = 0.$$

n is positive integer
$$\begin{cases} y = x^n \Rightarrow y' = n x^{n-1} \\ y = \frac{1}{x^n} \Rightarrow y' = \frac{-n}{x^{n+1}} \end{cases}$$

If n is an integer

$$y = x^n ; \quad y' = n x^{n-1}.$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = \left(\sin\left(x + \frac{\pi}{2}\right)\right)'$$

$$= \cos\left(x + \frac{\pi}{2}\right) \cdot 1$$

$$= -\sin x.$$

$$(\tan x)' = \sec^2 x$$

~~csc x~~

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(\csc x)' = -\csc x \cot x.$$

Product rule $\quad (f(x)\, g(x))'$

$$= f'(x)\, g(x) + f(x)\, g'(x).$$

$$(f(x)\, g(x))'$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x+h)\, g(x)}{h}$$

$$+ \lim_{h \to 0} \frac{g(x)\left(f(x+h) - f(x)\right)}{h}$$

$$= \lim_{h \to 0} f(x+h)\; g'(x) + g(x)\, f'(x)$$

$$= f(x)\, g'(x) + g(x)\, f'(x).$$

Suppose $y(t)$ is a function of time t.

The average rate of change of y between time t and $t+h$ is

$$\frac{y(t+h) - y(t)}{h}.$$

If the limit as $h \to 0$ exists, then we say the limit is the instantaneous rate of change of y at time t.

i.e. ~~dy~~ $y' = \frac{dy}{dt}$ represents the instantaneous rate of change at time t.

A particle P moves along the x-axis at the rate of 5 cm/sec. Another particle Q moves along the y-axis at the rate of 10 cm/sec. How fast is the distance between the particles changing when P at at $x=30$ and Q is at $y = 40$

$$D(t) = \sqrt{x^2_{(t)} + y^2_{(t)}}$$

$$\frac{dD(t)}{dt} = \frac{d}{dt}\left(x^2(t) + y^2(t)\right)^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x^2(t)+y^2(t)}} \cdot \left(2x(t)\cdot x'(t) + 2y(t)\right.$$

$$= \frac{x(t)\cdot x'(t) + y(t)\cdot y'(t)}{\sqrt{x^2(t) + y^2(t)}}$$

We are given $x = 30$, $y = 40$
$x' = 5$, $y' = 10$

$$\frac{dD(t)}{dt} = \frac{30\cdot 5 + 40\cdot 10}{\sqrt{30^2 + 40^2}}$$

$$= 11 \text{ cm/sec}.$$

If n is an integer, then $(x^n)' = n x^{n-1}$.

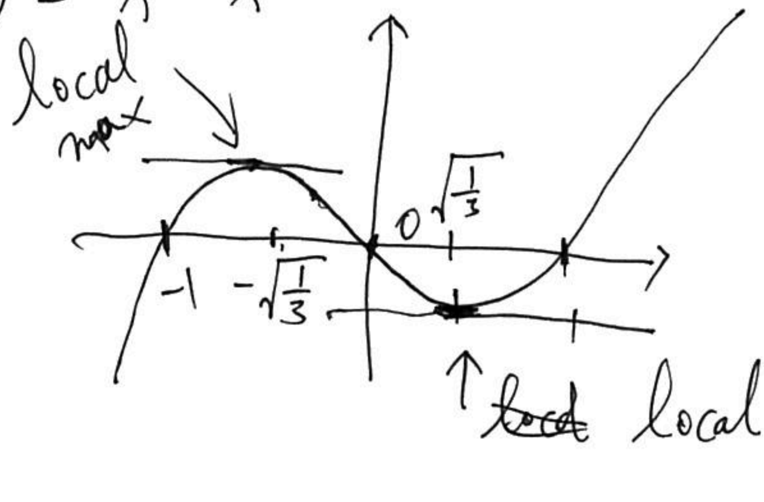
The rule also works if n is a rational number.

~~Loc~~ local max and min.

Suppose f is a function and x_0 is a pt in the domain of f. If there is $\delta > 0$, such $f(x_0) \geq f(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$, then we say that f has a local maximum at x_0.

local minimums are defined similarly.

$f(x) = x^3 - x$

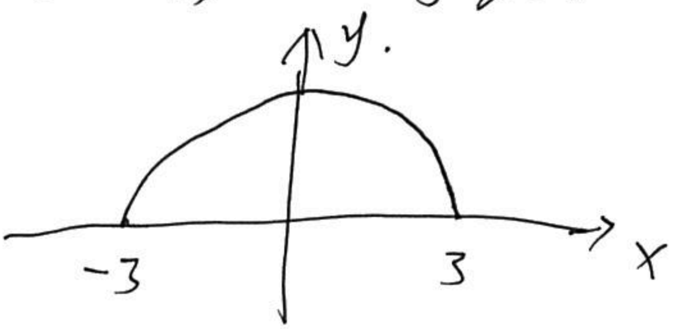


Theorem: Suppose f is defined on (a,b). and f has a local max or min at $c \in (a,b)$. If f is differentiable at $x=c$, then

$$f'(c) = 0.$$

Implicit differentiation.

$x^2+y^2=9, \quad y \geq 0.$



$y' = ?$ $\quad y = \sqrt{9-x^2}.$

$$y' = \frac{-2x}{2\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}}$$

$y^5 + xy = 3.$

$y' = ?$ if $y^5 + xy = 3$.

$$\frac{d}{dx}\left(y^5 + xy\right) = \frac{d}{dx}(3).$$

$$5y^4 \cdot y' + 1 \cdot y + x \cdot y' = 0$$

$$y' = \frac{-y}{5y^4 + x}.$$

$x^4 + y^4 = 1$

$y' = ?$

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx} 1$$

$$4x^3 + 4y^3 \cdot y' = 0$$

$$y' = -\frac{x^3}{y^3}.$$

Suppose α is a rational number.

$$y = x^{\alpha}. \qquad y' = ?$$

α is a rational number.

$$\alpha = \frac{p}{q}, \quad p, q \text{ are integers. } q \neq 0.$$

$$y = x^{\frac{p}{q}} \Rightarrow y^{q} = x^{p}$$

$$\frac{d}{dx}\left(y^{q}\right) = \frac{d}{dx}\left(x^{p}\right)$$

$$q\,y^{q-1} \cdot y' = p\,x^{p-1}$$

$$y' = \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}}$$

$$= \frac{p}{q}\,\frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}} = \frac{p}{q}\,x^{\frac{p}{q}-1}$$

$$y' = \alpha\,x^{\alpha-1}.$$