Intro to Matrices and Linear Systems

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Where are we going?

- We will learn a new angle from which to analyze systems of linear equations (linear systems).
- The methods involve the idea of a MATRIX.

Matrices give us framework with which we can write linear systems in very simple terms.
As we know, linear systems arise frequently, for example, in:

- engineering (mechanical vibrations and control)
- economics (supply / demand dynamics)
- life–sciences (predator–prey models)
- technology (graphics in screens, printing).

Linear Systems: 2 equations, 2 unknowns

These kinds of systems have:-

- No solution – the lines are parallel (but are not identical lines)
- Exactly one solution – the lines touch at one point only
- Infinitely many solutions – the lines are identical.

Linear Systems: 2 equations, 3 unknowns

These kinds of systems have:-

- No solution – the planes are parallel (but are not identical planes)
- Infinitely many solutions – the planes touch along one line
- Infinitely many solutions – the planes are identical.

We can convert a linear system into an equivalent form that will reveal that nature of solutions by:

- Interchanging equations.
- Multipling an equation by a non–zero number.
- Adding a multiple of a equation to another equation.

Example: Solve the linear system

$$x + 2y + 3z = 14 \quad (1)$$

$$x - 3y - 2z = -11. \quad (2)$$

Equation (1) – Equation (2) gives

$$5y + 5z = 25$$

so

$$y + z = 5.$$

Thus, we have the simplified system

$$x + 2y + 3z = 14 \quad (3)$$

$$y + z = 5. \quad (4)$$

Let $y = \lambda$ so (4) becomes

$$z = 5 - y = 5 - \lambda.$$

Backsubstituting $y = \lambda$ and $z = 5 - \lambda$ into (3) we obtain

$$x = 14 - 2y - 3z = -1 + \lambda.$$

Thus, in vector form we have:

Example: Solve the linear system

$$x - 3y + 3z = 2 \tag{5}$$

$$2x - 6y + 6z = 10. \tag{6}$$

Equation (6) – 2* Equation (5) gives

$$0x + 0y + 0z = 6$$

which makes no sense. We conclude that the two planes are parallel (and nonidentical). There is no solution to this problem.

Example: Solve the linear system

$$x - 3y + 3z = 2 \tag{7}$$

$$2x - 6y + 6z = 4. \tag{8}$$

Equation (7) – 2* Equation (8) gives

$$0x + 0y + 0z = 0$$

which is trivial. If we let $y = \lambda$ and $z = \mu$ in (7) then we can obtain a vector parametric form of the solution. From (7) we have

$$x = 2 + 3y - 3z = 2 + 3\lambda - 3\mu.$$

Thus, we see

We conclude that the two planes are identical. There are an infinite number of solutions to this problem.

Linear Systems: 3 equations, 3 unknowns

These kinds of systems have:-

- No solution – the planes might be parallel (but are not identical planes), or they touch but not altogether in the same place
- One solution – the planes intersect at a single point
- Infinitely many solution – the planes touch along one common line
- Infinitely many solutions – the planes are identical.

Why are matrices AWESOME?

There are at least two reasons why matrices are AWESOME:-

1. their real–world applications;
2. their ability simplify simultaneous systems of equations from mathematics.

We shall apply a compact way of writing the linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right. \qquad (*)$$

as an *augmented matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

In addition to writing (*) as an augmented matrix, we can also write it as a *vector equation* —

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

and as a *matrix equation* —

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If we write $\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, and

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ then:-

1. The augmented matrix can be compactly written as $(A|\mathbf{b})$ and the ith row
$$\left(\begin{array}{cccc|c} a_{i1} & a_{i2} & \cdots & a_{in} & b_i \end{array} \right)$$
is denoted by R_i.
2. The vector equation form can written as
$$x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\,\mathbf{a}_n = \mathbf{b}.$$
This follows from the definition of addition and scalar multiplication in $\mathbb{R}^m$.
3. The matrix A is called the coefficient matrix and we can write
$$A\mathbf{x} = \mathbf{b}$$
as the matrix equation form.
4. We will use the subscript ij for the entry in the ith row and jth column of a matrix.

Row Operations

The following operations are called *elementary row operations*. If we apply these operations one one by one then the solution set of the system represented by a matrix will not change. To ensure we don't get confused, we record the operations used.

- **Interchange two rows.** Interchanging row i and row k is recorded by $R_i \leftrightarrow R_k$.
- **Multiply a row by a non–zero number.** Multiplying row i by a non–zero number α is recorded by $R_i = \alpha R_i$.
- **Add a multiple of a row to another row.** Adding α times row k to row i is recorded by $R_i = R_i + \alpha R_k$.

Perform the following row operations, in order, on the matrix below:
$R_2 = R_2 - 5R_1$; $R_3 \to R_3 - 3R_1$; $R_3 \to R_3 - 2R_2$.

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 3 \\ 5 & -11 & 3 & -7 & 13 \\ 3 & -8 & 1 & -7 & 5 \end{array}\right)$$

Row–echelon and Reduced row–echelon forms

For a matrix A:

- a *leading row* of A is a row which is not all zeros;
- the *leading entry* in a leading row of A is the first (i.e. leftmost) non–zero entry in that row;
- a *leading column* of A is a column which contains the leading entry for some row.

The following forms are ones from which we can determine the solution set to a linear system.

Row–echelon form

A matrix is said to be in *row-echelon form* if

(1) in every leading row, the leading entry is further to the right than the leading entry in any rows higher up in the matrix, and

(2) all non-leading rows are at the bottom of the matrix.

Reduced–row echelon form

A matrix is said to be in *reduced row-echelon form* if it is in row-echelon form and

(3) every leading entry is 1, and

(4) every leading entry is the only non-zero entry in its column.

Example

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

Leading rows:
Leading columns:
Non-leading columns:

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & 0 & 4 & 8 \\ 0 & -3 & 2 & 7 \end{array}\right)$$

Leading rows:
Leading columns:
Non-leading columns:

Gaussian Elimination

Gaussian Elimination is an algorithm for reducing a matrix to a row–echelon matrix.

1. Select a *pivot element*: from the leftmost column which is not all zeros choose a non-zero entry as the pivot entry. The column containing the pivot entry is called the *pivot column* and the row containing the pivot entry is called the *pivot row*.
2. Swap the pivot row and the top row if necessary.
3. Eliminate (i.e., reduce to 0) all entries in the pivot column below the pivot element.
4. Repeat steps (1) to (3) on the submatrix of rows and columns strictly to the right of and below the pivot element and stop when the augmented matrix is in row-echelon form.

- The process of reducing a matrix to a row-echelon form is called *row reduction*.
- To keep calculations (by hand) simple, choose a 1 to be the pivot element to avoid fractions.
- Gaussian elimination does not need the row operation “multiplying a row by a number”. However, if a row consists of integers which have a common factor, then we apply this operation to divide the whole row by the highest common factor.
- Besides using the “multiply a row by a number”, we *do* follow the Gaussian elimination.
- Different choices of pivots and different order of applying row operations may end up with *different* row-echelon form matrices. However, all these row-echelon matrices represent systems with the same solution set.

Reduce $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 8 \\ 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 8 \end{array}\right)$ to a row-echelon form.

Reduce $\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & -9 \end{array}\right)$ to a row-echelon form.

Solving Linear Systems

Linear systems have: no solution; a unique solution; or infinitely many solutions. Form the equivalent row-echelon form matrix we can determine the solvability of a system. A system is said to be *consistent* if it has solutions (one or infinite). If the system is consistent, we shall use *back substitution* to find the solution set.
Suppose that the augmented matrix of a system of equations is reduced to row-echelon form. If there is a row

$$(0\ 0\ \cdots\ 0 \,|\, \alpha) \qquad \text{where } \alpha \neq 0,$$

then the system has no solution. In other words, if the right hand column of a row-echelon matrix is leading, then the system is inconsistent.

Case 1. All left hand columns are leading.

If, after row reduction, the augmented matrix of a system of equations in x_1, x_2, x_3 is reduced to $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & -1 & -5 & -7 \\ 0 & 0 & -3 & -3 \end{array}\right)$ then we start from the last non-zero row. In this case, it is the third row which represents the equation

$$-3x_3 = -3, \qquad \text{i.e.} \quad x_3 = 1.$$

Then we substitute the result into the row immediately above. In this case, it is the second row. We have

$$-x_2 - 5(1) = -7. \qquad \text{So} \quad x_2 = 2.$$

Once again, we substitute the result into the row immediately above. In this cases, it is the first row. We have

$$x_1 + 2(2) + 3(1) = 8. \qquad \text{So} \quad x_1 = 1.$$

Hence the system has a unique solution $x_1 = 1$, $x_2 = 2$, $x_3 = 1$.

Case 2. Some left hand columns are non-leading. (Don't forget that we are still discussing the case that the right hand column is non-leading). In this case, the variables corresponding to the non-leading columns are called *non-leading variables*. We solve the system from its row-echelon matrix also by back-substitution, but before that we have to assign each non-leading variable a (different) parameters.

Example. Solve the following linear system

$$\left\{\begin{array}{ccccccccccc} x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 1 \\ & & & & 2x_3 & + & x_4 & + & 3x_5 & = & 3 \\ & & & & & & & & x_4 & = & 1. \end{array}\right.$$

From row-echelon form to reduced row-echelon form.

Continue from a row-echelon matrix.

(5) Start with the lowest row which is not all zeros. Multiply it by a suitable constant to make its leading entry 1.

(6) Add multiples of this row to higher rows to get all zeros in the column above the leading entry of this row.

(7) Repeat steps (5) and (6) with the second lowest non-zero row, and so on until the matrix is in reduced row-echelon form.

Reduce $\left(\begin{array}{rrr|r} 1 & 2 & 3 & 8 \\ 0 & -1 & -5 & -7 \\ 0 & 0 & -3 & -3 \end{array}\right)$ to reduced row-echelon form.

Reduce the augmented matrix for

$$\left\{\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 1 \\ & & & & 2x_3 & + & x_4 & + & 3x_5 & = & 3 \\ & & & & & & & & x_4 & = & 1 \end{array}\right.$$

to reduced row-echelon form and solve the system of equations.

As a summary, to solve a linear system, we first write the system as an augmented matrix and reduce it to a row-echelon form.

1. If the right hand column is leading then the system is inconsistent, i.e. it has no solution.
2. If the right hand column is non-leading then the system is consistent. We have the following possibilities:
    1. all columns on the left are leading and so the system has a unique solution;
    2. some columns on the left are non-leading and so the system has infinitely many solutions.

    Secondly, we set a parameter to each of the non-leading variables if there is any and get the solution(s) by back substitution.

Each of the following row-echelon matrices represents a linear system of equations. Determine whether each system has no solution, a unique solution or infinitely many solutions.

1. $$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

2. $$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3. $$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 11 \\ 0 & 0 & 2 & 2 & 7 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

4. $$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 1 & 11 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right)$$

When the right hand sides of all equations in a system of linear equations are zero, the system is called *homogeneous*.
A homogeneous system always has at least one solution. Why?

Does the point $(0, 1, 2)$ lie on the plane through $(-5, 1, 3)$ parallel to
$\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$?

Find the intersection of the planes

$$\mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\3\\5\end{pmatrix}, \quad \text{and}$$

$$\mathbf{x} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \mu_1 \begin{pmatrix}1\\3\\4\end{pmatrix} + \mu_2 \begin{pmatrix}1\\4\\6\end{pmatrix}.$$

Consider the system of equations given in augmented matrix form by

$$\left(\begin{array}{ccc|c} 1 & 2 & -a & 1 \\ 2 & a & 3 & 3 \\ a & 4a-8 & 3a & 5 \end{array}\right)$$

For which value(s) of a does the associated linear system have:

a) a unique solution?

b) no solution?

c) infinitely many solutions.

Do the lines $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$

intersect?

Determine the general solution to the following linear system by applying Gaussian elmination on the associated augmented matrix and performing back–substitution

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 4x_4 &= 2 \\ 2x_1 + 4x_2 - 5x_3 + 9x_4 &= 0 \\ x_1 - x_2 - 4x_3 + 6x_4 &= -6. \end{aligned}$$

Linear systems with indeterminate right hand sides

Solve

$$\left\{\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & b_1, \\ 2x_1 & + & x_2 & - & x_3 & = & b_2, \\ x_1 & + & x_2 & + & 2x_3 & = & b_3. \end{array}\right.$$

The above linear system may or may not have solutions. It depends on the vector **b**.

If the augmented matrix $(A|\mathbf{b})$ is reduced to a row-echelon form $(U|\mathbf{y})$ and U has no non-leading columns then what does this say about the solutions of the system of equations? What if U has neither non-leading columns nor non-leading rows?

Find the condition that $\mathbf{v} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ is in span $\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \\ 4 \end{pmatrix} \right\}$.

General properties of the solution of $A\mathbf{x} = \mathbf{b}$

The following are true for a homogeneous system of linear equations.

1. $\mathbf{0}$ is a solution.
2. If $\mathbf{v}$, $\mathbf{w}$ are solutions and $\lambda \in \mathbb{R}$ then $\mathbf{v} + \mathbf{w}$ and $\lambda\mathbf{v}$ are solutions.

Suppose we write the system in matrix form $A\mathbf{x} = \mathbf{0}$. The coefficient matrix A is reduced to a row-echelon form U which has k non-leading columns. Then the general solution of the homogeneous system can be written in form of

$$\mathbf{x} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k,$$

where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are non-zero solutions.

Furthermore, if $\mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x}_p + \mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Conversely, any solution to $A\mathbf{x} = \mathbf{b}$ is of this form. Hence the general solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{x}_p + \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k.$$

We call $\mathbf{x}_p$ a particular solution to $A\mathbf{x} = \mathbf{b}$.