The mean value theorem (MVT)

If f is a function continuous on $[a, b]$ and differentiable on (a, b) then

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

for some $c \in (a, b)$.

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Leftrightarrow \quad f(b)-f(a) = f'(c)(b-a).$$

$$\Leftrightarrow \quad f(x) - f(a) = f'(c)(x-a)$$

~~$c \in ($~~

c is between a and x.

Theorem: If f and g are two differentiable functions on $\mathbb{R}$. and $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x > a$.

Then $f(x) > g(x)$ for all $x > a$.

Example, Prove $\sin x < x$ for all $x > 0$.

Proof: Let $f(x) = \sin x$, $g(x) = x$

$$f(0) = 0 = g(0)$$

$$f'(x) = \cos x, \quad g'(x) = 1$$

Clearly, $f'(x) \leq g'(x)$

for all $x > 0$.

Therefore, $f(x) < g(x)$ for all $x > 0$. ☐

Def: Suppose f is a function defined on $[a, b]$, $x_0 \in [a, b]$.

i) x_0 is said to be a critical point if $f'(x_0) = 0$ or f is not differentiable at x_0

ii) x_0 is said to be an extreme point if f has a local max or local min at x_0

iii) x_0 is said to be a stationary point if $f'(x_0) = 0$.

If f is differentiable at x_0 and x_0 is an extreme point of f, then x_0 is a stationary point of f.

In practice, to find global max/min, we need to find all critical points, and also the function values at the end points of interval in question.

$$f(x) = x^3 - 3x^2 + 1, \qquad [0,4].$$

Find the global max and min of f on $[0,4]$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

$$f'(x) = 0, \text{ if } x = 0 \text{ or } x = 2.$$

$$f(0) = 1$$

$$f(2) = -3$$

$$f(4) = 64 - 48 + 1 = 17.$$

The global max of f on $[0,4]$ occurs at

$x = 4$.

The global min of f on $[0,4]$ occurs at

$x=2$.

Find local max/min of

$$f(x) = |x-3|\cdot|x|.$$

$$= \begin{cases} (x-3)\cdot x, & \text{if } x \geq 3. \\ -(x-3)\cdot x & \text{if } 0 \leq x < 3. \\ (x-3)\cdot x & \text{if } x < 0 \end{cases}$$

The critical points of f. include

$x=0$, $x=3$.

$$f'(x) = \begin{cases} 2x-3 & x>3 \\ -(2x-3) & 0<x<3 \\ 2x-3 & x<0 \end{cases}$$

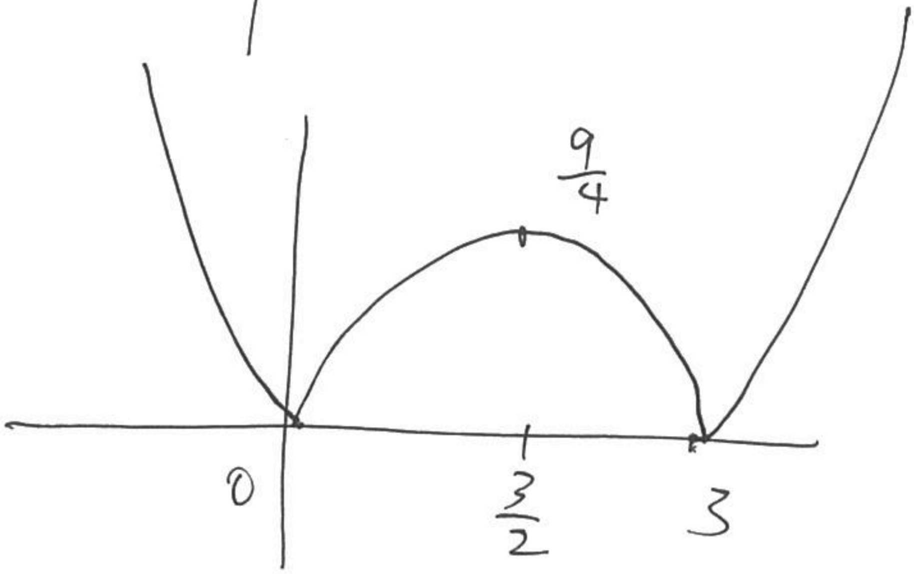
The only stationary point of f is at

$x=\frac{3}{2}$.

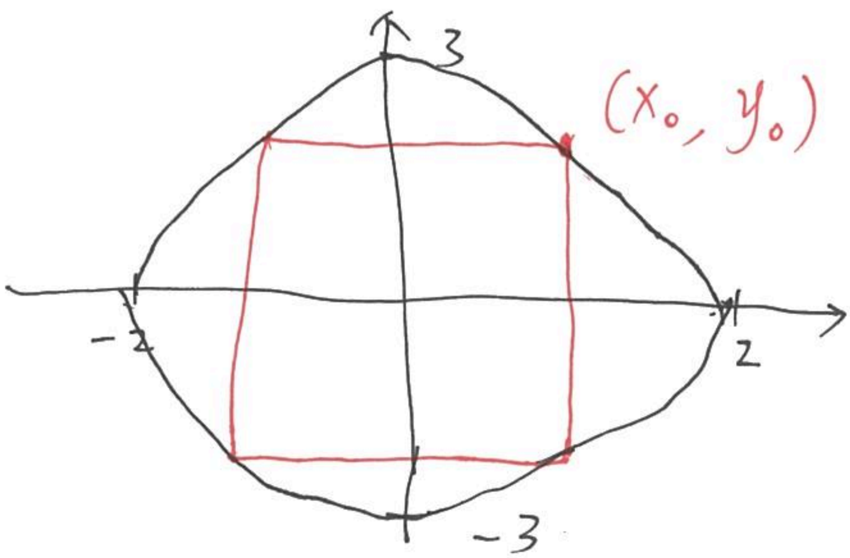
$f(0)=0$. $f(3)=0$. $f(\frac{3}{2}) = \frac{9}{4}$.

f has local mins at $x=0$ and $x=3$.

f has local max at $x=\frac{3}{2}$.



Find the dimension of the rectangle of maximum area that can be ~~inscrib~~ inscribed in the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



$$\frac{x_0^2}{4} + \frac{y_0^2}{9} = 1.$$

$$\Rightarrow y_0^2 = 9 - \frac{9}{4} x_0^2 \Rightarrow y_0 = \sqrt{9 - \frac{9}{4} x_0^2}$$

Area of the rectangle is

$$A = 4 x_0 y_0$$

$$A^2 = 16 x_0^2 y_0^2$$

$$= 16 x_0^2 \left(9 - \frac{9}{4} x_0^2\right)$$

$$0 \le x_0 \le 2$$

$$\cancel{\frac{d}{dx_0} x^2}$$

$$\frac{d}{dx_0}\left(A^2\right) = 288 x_0 - 144 x_0^3$$

$$= 144 x_0 \left(2 - x_0^2\right)$$

The stationary pts are $x_0 = 0$, $x_0 = \pm\sqrt{2}$.

When $x_0 = 0$, $A = 0$

When $x_0 = \sqrt{2}$, $y_0 = \sqrt{\frac{9}{2}} \Rightarrow A = 4 \cdot x_0 y_0 = 12$.

When $x_0 = 2$, $A = 0$

Theorem: (L'Hôpital's Rule).

Suppose f and g are differentiable functions on $\mathbb{R}$ (except possibly at a). and

$$f(a) = g(a) = 0, \text{ or}$$

$$\lim_{x \to a} f(x) = \pm\infty = \lim_{x \to a} g(x).$$

~~Then~~ If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\cancel{\lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Suppose $f(a) = g(a) = 0$.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \frac{\dfrac{f(x) - f(a)}{x-a}}{\dfrac{g(x) - g(a)}{x-a}}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

example:

$$\lim_{x \to 0} \frac{e^x - 1}{\sin 2x}$$

$$= \lim_{x \to 0} \frac{e^x}{2\cos 2x}$$

$$= \frac{1}{2}.$$

$$\lim_{x \to 1} \frac{1 - x + \log x}{1 + \cos \pi x}.$$

$$= \lim_{x \to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x}$$

$$= \lim_{x \to 1} \frac{-\frac{1}{x^2}}{-\pi^2 \cos \pi x}$$

$$= \frac{-1}{\pi^2}$$

Under the same conditions as before

if $\lim_{x \to \infty} f(x)$, and $\lim_{x \to \infty} g(x)$

are both zero or both $\pm \infty$,

then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

Note that you must check that the conditions of L'Hôpital's Rule are satisfied before apply it.

$$\lim_{x\to 1} \frac{2x^2+5}{3x^2+1} \overset{L'H}{=} \lim_{x\to 1} \frac{4x}{6x} = \frac{2}{3}.$$

But this is wrong! L'H's rule is not applicable as we don't have $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

$$\lim_{x\to 1} \frac{2x^2+5}{3x^2+1} = \frac{7}{4}.$$

$$\begin{aligned}
\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x &= \exp\left(\lim_{x\to\infty} \log\left(1+\frac{1}{x}\right)^x\right) \\
&= \exp\left(\lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right)\right) \\
&= \exp\left(\lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right) \\
&= \exp\left(\lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^{-1}\cdot\frac{-1}{x^2}}{-\frac{1}{x^2}}\right) \\
&= \exp\left(\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}\right) = \exp(1) = e.
\end{aligned}$$