§1 Sets, Functions, and Sequences

- A **set** is a **well-defined collection of distinct objects**.
- An **element** of a set is any object **in the set**.
  - $\in$ - "belongs to" or "is an element of" or "is in"
  - $\notin$ - "does not belong to" or "is not an element of" or "is not in"
- The **cardinality** of a set S, denoted by $|S|$, is the **number of elements** in S.

Example. Some commonly-used sets in our number system:

- N - the set of *natural numbers* $0, 1, 2, 3, \ldots$
- Z - the set of *integers* (*whole numbers*) $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
- Q - the set of *rational numbers* (*fractions*) $\ldots, -1, 0, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \frac{2}{3}, \ldots$
- R - the set of *real numbers*, which includes all rational numbers as well as *irrational numbers* such as π, e, and $\sqrt{2}$
- R^+ - the set of all *positive real numbers*

Example. We can specify a set by listing its elements between curly brackets, separated by commas:

$$S = \{b, c\}.$$

The elements of S are . Thus $|S| = 2$.
We can write $b \in S$, $c \in S$, and $d \notin S$, for instance.

Example. We can specify a set by some property that all elements must have:

$$S = \{x \in Z \mid x^2 \le 4\}$$
$$(\text{or } S = \{x \in Z : x^2 \le 4\}).$$

The elements of S are -2, -1, 0, 1 and 2. Thus $|S| = 5$.
Also $S = \{-2, -1, 0, 1, 2\}$.
We can write $-2 \in S$, $-1 \in S$, $0 \in S$, $1 \in S$, and $4 \notin S$, for instance.

Exercise. Let $A = \{\{a\}, a\}$. What are the elements of A? What is $|A|$?

- Two sets S and T are **equal**, denoted by $S = T$, if and only if (written iff)

  (i) every element of S is also an element of T, and
  (ii) every element of T is also an element of S.

  i.e., when they have precisely the same elements.

- The **empty set**, denoted by $\emptyset$, is a set which has no elements. $|\emptyset| =$

Exercise. Are any of the following sets equal?

$A = \{2, 3, 4, 5\}$, $\quad C = \{2, 2, 3, 3, 4, 5\}$,
$B = \{5, 4, 3, 2\}$, $\quad D = \{x \in \mathbb{N} \mid 2 \le x \le 5\}$.

Exercise. What is the difference between the sets $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$?

- Loosely speaking, a **subset** is a part of a set. More precisely, a set S is a **subset** of a set T if and only if each element of S is also an element of T.

  $\subseteq$ - "is a subset of", $\quad \not\subseteq$ - "is not a subset of"

  $\Rightarrow$ $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.

  - A set S is a **proper subset** of a set T iff S is a subset of T and $S \neq T$.

    We then write $S \subsetneq T$ (or sometimes $S \subset T$).
    $\Rightarrow$ $\emptyset$ is a proper subset of any non-empty set.
    $\Rightarrow$ Any non-empty set is an improper subset of itself.

- The **power set** $P(S)$ of a set S is the set of all possible subsets of S.

  $\Rightarrow$ For any set S, we have $\emptyset \subseteq S$ and $S \subseteq S$.
  $\Rightarrow$ For any set S, we have $\emptyset \in P(S)$ and $S \in P(S)$.

- The number of subsets of S is $|P(S)| = 2^{|S|}$. $\quad$ (*Why?*)

Example. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Example. Let $S = \{a, b, c\}$. The subsets of S are:

$$\square, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

S has 8 subsets. We can write $\square \subseteq S$, $\{b\} \subseteq S$, $\{a, c\} \subseteq S$, $\{a, b, c\} \subseteq S$, etc.
The power set of S is

$$P(S) = \{\square, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

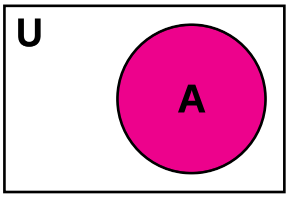
and $|P(S)| = 2^3 = 8$.
We can write $\square \in P(S)$, $\{b\} \in P(S)$, $\{a, c\} \in P(S)$, $\{a, b, c\} \in P(S)$, etc.

Exercise. Let $A = P(P(\{1\}))$. Find A and $|A|$.

Exercise. For $B = \{\square, 0, \{1\}\}$, are the following statements true or false?

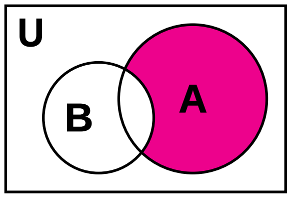
1. $\square \in B$
2. $\square \subseteq B$
3. $\{\square\} \in B$
4. $\{\square\} \subseteq P(B)$
5. $\{0\} \in P(B)$
6. $\{\square\} \square B$
7. $\{\square\} \in P(B)$
8. $\{\{0\}\} \subseteq P(B)$
9. $1 \in B$
10. $\{1\} \subseteq B$
11. $\{1\} \in P(B)$
12. $\{\{1\}\} \subseteq P(B)$
13. $\square \in P(P(P(P(B))))$

- It is often convenient to work inside a specified **universal set**, denoted by **U**, which is assumed to contain **everything that is relevant**.
- **Venn diagrams** are visualizations of sets as regions in the plane.
  For instance, here is a Venn diagram of a universal set U containing a set A:



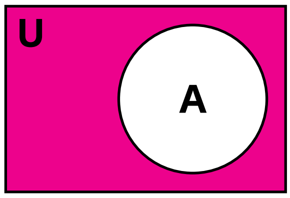
- **Set operations and set algebra:** ∼ illustrated by Venn diagrams ∼
  - **difference** ($-, \setminus$) - "**but not**"

    $$\boxed{\mathbf{A - B = A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\}}}$$



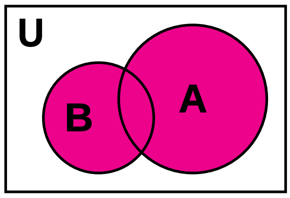
  - **complement** (^{c}, $\overline{}$) - "**not**"

    $$\boxed{\mathbf{A^c = \overline{A} = U \setminus A = \{x \in U \mid x \notin A\}}}$$



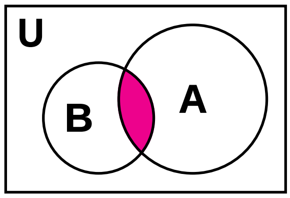
  - **union** ($\cup$) - "**or**" meaning "**one or other or both**"

    $$\boxed{\mathbf{A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}}}$$



  - **intersection** ($\cap$) - "**and**"

    $$\boxed{\mathbf{A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}}}$$



- Two sets A and B are **disjoint** if $\mathbf{A \cap B = \emptyset}$.
- The **Inclusion-Exclusion Principle**: $\mathbf{|A \cup B| = |A| + |B| - |A \cap B|}$.

Example. Set $\mathbf{U = \{1,2,3,4,5,6\}}$, $\mathbf{A = \{1,3,5\}}$, and $\mathbf{B = \{1,2\}}$.
Then

$$\mathbf{A^c = \{2,4,6\} \qquad A \cap B = \{1\} \qquad A \cup B = \{1,2,3,5\} \qquad A - B = \{3,5\}}.$$

Exercise. Given $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$A = \{x \in U \mid x \text{ is odd}\}$
$B = \{x \in U \mid x \text{ is even}\}$
$C = \{x \in U \mid x \text{ is a multiple of } 3\}$
$D = \{x \in U \mid x \text{ is prime}\}$

determine the following sets:

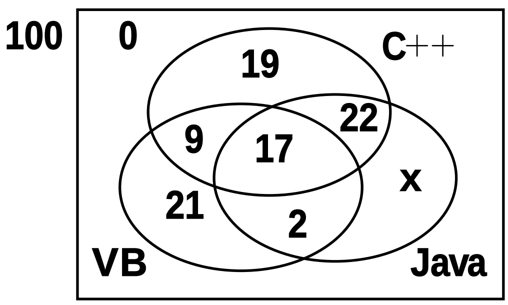
$A \cap C$
$B - D$
$B \cup D$
A^c
$(A \cap C) - D$

Exercise. Determine the sets A and B, where
$A - B = \{a, c\}$, $B - A = \{b, f, g\}$, and $A \cap B = \{d, e\}$.

Example. In a survey of 100 students majoring in computer science, the following information was obtained:

17 can program in C++, Java, and Visual Basic.
22 can program in C++ and Java, but not Visual Basic.
9 can program in C++ and Visual Basic, but not Java.
2 can program in Java and Visual Basic, but not C++.
19 can program in C++, but not Visual Basic or Java.
21 can program in Visual Basic, but not C++ or Java.

Also, all of the 100 students can program in at least one of these three languages. How many students can program in Java, but not C++ or Visual Basic?



$x = 100 - (17 + 22 + 9 + 2 + 19 + 21 + 0) = 10$

Exercise. In a survey of 200 people asked about whether they like apples (A), bananas (B), and cherries (C), the following data was obtained:

$|A| = 112$, $|B| = 89$, $|C| = 71$,
$|A \cap B| = 32$, $|A \cap C| = 26$, $|B \cap C| = 43$,
$|A \cap B \cap C| = 20$.

a) How many people like apples or bananas?
b) How many people like exactly one of these fruit?
c) How many people like none of these fruit?

- Hints for proofs:
  - To prove that $S \subseteq T$, we assume that $x \in S$ and show that $x \in T$.
  - To prove that $S = T$, we show that $S \subseteq T$ and $T \subseteq S$.

Example. We prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Proof. Let $A \subseteq C$ and $B \subseteq C$ and suppose that $x \in A \cup B$.
Then either $x \in A$ or $x \in B$ (maybe both).
If $x \in A$, then $x \in C$, because $A \subseteq C$.
Likewise, if $x \in B$, then $x \in C$, since $B \subseteq C$.
In all possible cases, we have $x \in C$, which proves that $A \cup B \subseteq C$.

Exercise. Prove that if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Exercise. **Prove that if $A \subseteq B$, then $A \cap B = A$.**

Exercise. **Prove that if $A \cap B = A$, then $A \subseteq B$.**

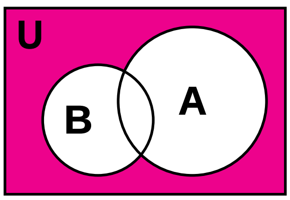
Thus, putting these last two examples together, we can say $A \cap B = A$ if and only if $A \subseteq B$.

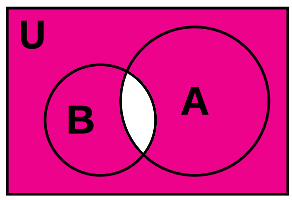
Exercise. **Is the statement $A \cap (B \cup C) = (A \cap B) \cup C$ true (for all sets A, B, C)? Provide a proof if it is true or give a counter example if it is false.**

A wrong answer is "False: because LHS is $(A \cap B) \cup (A \cap C)$ not $(A \cap B) \cup C$."

Exercise. **Is the statement $A - (B - C) = (A - B) - C$ true? Provide a proof if it is true or give a counter example if it is false.**

- **Laws of set algebra:**
  - **Commutative laws** $A \cap B = B \cap A$
    $A \cup B = B \cup A$
  - **Associative laws** $A \cap (B \cap C) = (A \cap B) \cap C$
    $A \cup (B \cup C) = (A \cup B) \cup C$
  - **Distributive laws** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
    $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - **Absorption laws** $A \cap (A \cup B) = A$
    $A \cup (A \cap B) = A$
  - **Identity laws** $A \cap U = U \cap A = A$
    $A \cup \emptyset = \emptyset \cup A = A$
  - **Idempotent laws** $A \cap A = A$
    $A \cup A = A$
  - **Double complement law** $(A^c)^c = A$
  - **Difference law** $A - B = A \cap B^c$
  - **Domination or universal bound laws** $A \cap \emptyset = \emptyset \cap A = \emptyset$
    $A \cup U = U \cup A = U$
  - **Intersection and union with complement** $A \cap A^c = A^c \cap A = \emptyset$
    $A \cup A^c = A^c \cup A = U$
  - **De Morgan's Laws** $(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$





Proof of De Morgan's law $(A \cup B)^c = A^c \cap B^c$:

(i) Suppose that $x \in (A \cup B)^c$. Then we have $x \notin A \cup B$, so $x \notin A$ and $x \notin B$.
Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.
This proves that $(A \cup B)^c \subseteq A^c \cap B^c$.

(ii) Suppose now that $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$.
Thus, $x \notin A \cup B$, so $x \in (A \cup B)^c$.
This proves that $A^c \cap B^c \subseteq (A \cup B)^c$.

Combining (i) and (ii), we conclude that $(A \cup B)^c = A^c \cap B^c$.

Example. We can use the laws of set algebra to simplify $(A^c \cap B)^c \cup B$:

$(A^c \cap B)^c \cup B$	$=$	$((A^c)^c \cup B^c) \cup B$	**De Morgan's law**
	$=$	$(A \cup B^c) \cup B$	**Double complement law**
	$=$	$A \cup (B^c \cup B)$	**Associative law**
	$=$	$A \cup U$	**Union with complement**
	$=$	U	**Domination**

Exercise. Use the laws of set algebra to simplify $(A \cap (A \cap B)^c) \cup B^c$:

Exercise. Use the laws of set algebra to simplify

$$([(A \cup B)^c \cup C] \cup B^c)^c$$

Challenge: **Prove the result (*uniqueness of complement*):**

If $A \cup B = U$ and $A \cap B = \square$ then $B = A^c$.

- ***Principal of Duality***:

  For a set identity involving only unions, intersections and complements, its **dual** is obtained by **replacing $\cap$ with $\cup$, $\cup$ with $\cap$, □ with U, and U with □**.

  As all the relevant laws of set algebra come in dual pairs, then **the dual of any true set identity is also true**.

The duals of the last 3 examples are:

- Generalized set operations:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \text{and} \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Example. If $A_k = \{k, k+1\}$ for every positive integer k, then

$$\bigcup_{k=1}^{3} A_k = A_1 \cup A_2 \cup A_3 = \{1,2\} \cup \{2,3\} \cup \{3,4\} = \{1,2,3,4\}$$

$$\bigcap_{k=1}^{3} A_k =$$

- Let I be an (**index**) set. For each $i \in I$, let A_i be a subset of a given set A.
  - $\bigcup_{i \in I} A_i = \bigcup_{i \in I} A_i = \{a \in A \mid a \in A_i \text{ for some } i \in I\}$
  - $\bigcap_{i \in I} A_i = \bigcap_{i \in I} A_i = \{a \in A \mid a \in A_i \text{ for every } i \in I\}$

Example.
Let I = $\{1, 2, 3, \ldots\}$ be the index set. For each i $\in$ I let
$A_i = [0, \frac{1}{i}] \subseteq \mathrm{R}$ be the set of real numbers between
0 and $\frac{1}{i}$ including 0 and $\frac{1}{i}$.

$\bigcup_{i \in I} A_i = [0, 1] \cup [0, \frac{1}{2}] \cup [0, \frac{1}{3}] \cup \ldots =$

$\bigcap_{i \in I} A_i = [0, 1] \cap [0, \frac{1}{2}] \cap [0, \frac{1}{3}] \cap \ldots =$

Example. (*The Barber Puzzle*) **In a certain town there is a barber**
(*) who shaves all those men, and only those, who do not shave themselves.
Does the barber shave himself?

Problem: **If he shaves himself, (*) $\Longrightarrow$ he doesn't shave himself.**
If he doesn't shave himself, (*) $\Longrightarrow$ he shaves himself.
CONTRADICTION!

Solution:

The paradox occurred because a self-referential statement was used. The "themself" in (*) could also refer to the barber unless our above solution is imposed.

Example. (*Russell's Paradox*)

- Let U be the set of all sets.
- First weird phenomenon: then $U \in U$.
- Even worse, we have Russell's paradox. Let

$$S = \{A \in U \mid A \notin A\}.$$

Is S an element of itself?

i) If $S \in S$, then the definition of S implies that $S \notin S$, a contradiction.

ii) If $S \notin S$, then the definition of S implies that $S \in S$, also a contradiction.

Hence neither $S \in S$ nor $S \notin S$.

Usual Solution:

Key Point: **The notion of set and set theory is very subtle. We will for the most part ignore these subtleties.**

- An ordered pair is a collection of two objects in a specified order.
  We use round brackets to denote ordered pairs; e.g., (a, b) is an ordered pair.
  - Note that (a, b) and (b, a) are different ordered pairs, whereas $\{a, b\}$ and $\{b, a\}$ are the same set.
- An ordered n-tuple is a collection of n objects in a specified order; e.g., $(a_1, a_2, \ldots, a_n)$ is an ordered n-tuple.
  - Two ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$ for all $i = 1, 2, \ldots, n$.
- The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs, the first from A, the second from B:

$$\boxed{A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}}$$

 If $|A| = m$ and $|B| = n$, then we have $|A \times B| =$

- The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A_i$ for all $i = 1, 2, \ldots, n$:

$$\boxed{A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for all } i = 1, 2, \ldots, n\}}$$

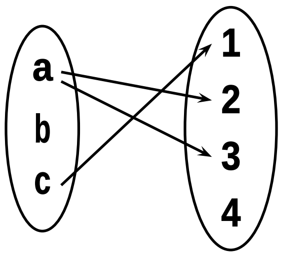
Example. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then

$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}.$$

Exercise. For A in the above example, find $A \times A$.

- When X and Y are small finite sets, we can use an arrow diagram to represent a subset S of $X \times Y$: we list the elements of X and the elements of Y, and then we draw an arrow from x to y for each pair $(x, y) \in S$.

Example. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3, 4\}$, and $S = \{(a, 2), (a, 3), (c, 1)\}$ which is a subset of $X \times Y$, then the arrow diagram for S is



- A function f from a set X to a set Y is a subset of $X \times Y$ such that

  for every $x \in X$ there is exactly one $y \in Y$ for which (x, y) belongs to f.

  - We write $f : X \to Y$ and say that "f is a function from X to Y".
  - X is the domain of f, Y is the codomain of f.
  - For any $x \in X$, there is a unique $y \in Y$ for which (x, y) belongs to f.
    - We write $f(x) = y$ or $f : x \mapsto y$.
    - We call y "the image of x under f" or "the value of f at x".
  - The range of f is the set of all values of f, that is

    $$f(X) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}.$$

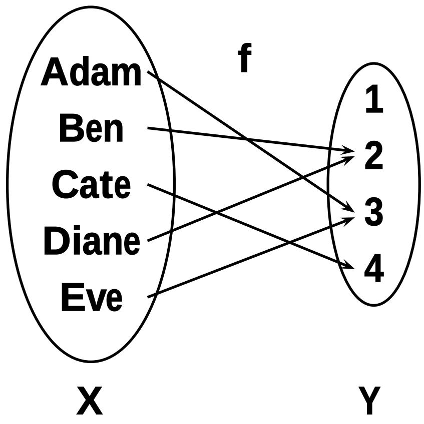
- This definition of a function corresponds to what is normally thought of as the graph of a function, with an x-axis and a y-axis.

How does this relate to the definition of a function given in Calculus in MATH1131/1141/1151?

Example. **Adam, Ben, Cate, Diane, and Eve were each given a mark out of 4. Their marks define a function $f : X \to Y$ as follows:**

$$\begin{aligned} \text{domain} \quad X &= \{\text{Adam, Ben, Cate, Diane, and Eve}\} \\ \text{codomain} \quad Y &= \{1, 2, 3, 4\}, \\ \text{and suppose} \quad f &= \{(\text{Adam}, 3), (\text{Ben}, 2), (\text{Cate}, 4), (\text{Diane}, 2), (\text{Eve}, 3)\}. \end{aligned}$$

The arrow diagram for this function is



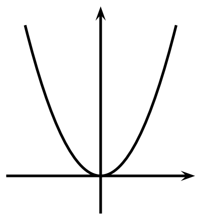
This is a function because every person has exactly one mark.
The range of this function is $\{2, 3, 4\}$.

Exercise. **Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$.**
Determine whether or not each of the following is a function from X to Y.
If it is, then write down its range.

$$\begin{aligned} f &= \{(a, 2), (a, 4), (b, 3), (c, 5)\}, \\ g &= \{(b, 1), (c, 3)\}, \\ h &= \{(a, 5), (b, 2), (c, 2)\}. \end{aligned}$$

Example. **The** *square* **function $f : \mathrm{R} \to \mathrm{R}$ is defined by set of the pairs**

$$\{(x, y) \in \mathrm{R} \times \mathrm{R} \mid y = x^2\}.$$



The function $f : \mathrm{R} \to \mathrm{R}$ can also be specified by

$$f(x) = x^2 \qquad \text{or} \qquad f : x \mapsto x^2.$$

The domain of f is R; the codomain of f is R; and the range of f is

$$\{y \in \mathrm{R} \mid y = x^2 \text{ for some } x \in \mathrm{R}\} = \{y \in \mathrm{R} \mid y \geq 0\} = \mathrm{R}^+ \cup \{0\}.$$

- **The floor function: (round down)**
  for any $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x.
- **The ceiling function: (round up)**
  for any $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x.

Exercise. Evaluate the following:

$\lfloor 3.7 \rfloor =$ $\qquad \lfloor -3.7 \rfloor =$ $\qquad \lfloor 3 \rfloor =$ $\qquad \lfloor -3 \rfloor =$

$\lceil 3.7 \rceil =$ $\qquad \lceil -3.7 \rceil =$ $\qquad \lceil 3 \rceil =$ $\qquad \lceil -3 \rceil =$

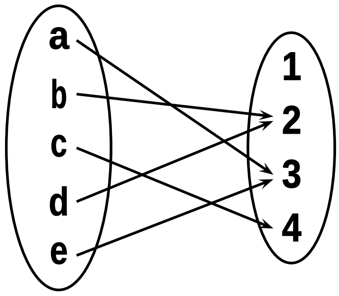
Exercise. What are the ranges of the floor and ceiling functions? Plot the graphs of the floor and the ceiling functions.

Exercise. Determine whether or not each of the following definitions corresponds to a function. If it does, then write down its range.

$$
\begin{aligned}
f : \mathbb{R} \to \mathbb{R}, &\quad f(x) = \tfrac{1}{x} \\
g : \mathbb{R}^{+} \to \mathbb{R}, &\quad g(x) = \tfrac{1}{x} \\
h : \mathbb{R} \to \mathbb{R}, &\quad h(x) = \lfloor x^2 - x \rfloor \\
j : \mathbb{R} \to \mathbb{Z} &\quad j(x) = 2x
\end{aligned}
$$

- The **image** of a set $A \subseteq X$ under a function $f : X \to Y$ is
  $$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\} = \{f(x) \mid x \in A\} \subseteq Y.$$
- The **inverse image** of a set $B \subseteq Y$ under a function $f : X \to Y$ is
  $$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X.$$

Example. Let the function f be defined by the arrow diagram



The image of $\{a, b, e\}$ under f is $f(\{a, b, e\}) = \{f(a), f(b), f(e)\} = \{2, 3\}$.
The inverse image of $\{1, 2\}$ under f is $f^{-1}(\{1, 2\}) = \{b, d\}$.

Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Find

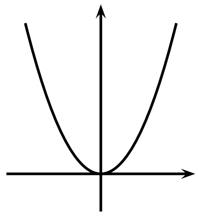
(a) The image of the set $\{2, -2, \pi, \sqrt{2}\}$ under f.
(b) The inverse image of the set $\{9, -9, \pi\}$ under f
(c) The inverse image of the set $\{-2, -9\}$ under f.

- Recall that if f is a function from X to Y, then
  - for every $x \in X$, there is exactly one $y \in Y$ such that $f(x) = y$.
- We say that a function $f : X \to Y$ is **injective** or **one-to-one** iff
  - for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.
  - OR equivalently, for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
  - OR equivalently, for all $x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.
- We say that a function $f : X \to Y$ is **surjective** or **onto** iff
  - for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.
  - the range of f is the same as the codomain of f.
- We say that a function $f : X \to Y$ is **bijective** iff
  - f is both injective and surjective (one-to-one and onto).
  - for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.

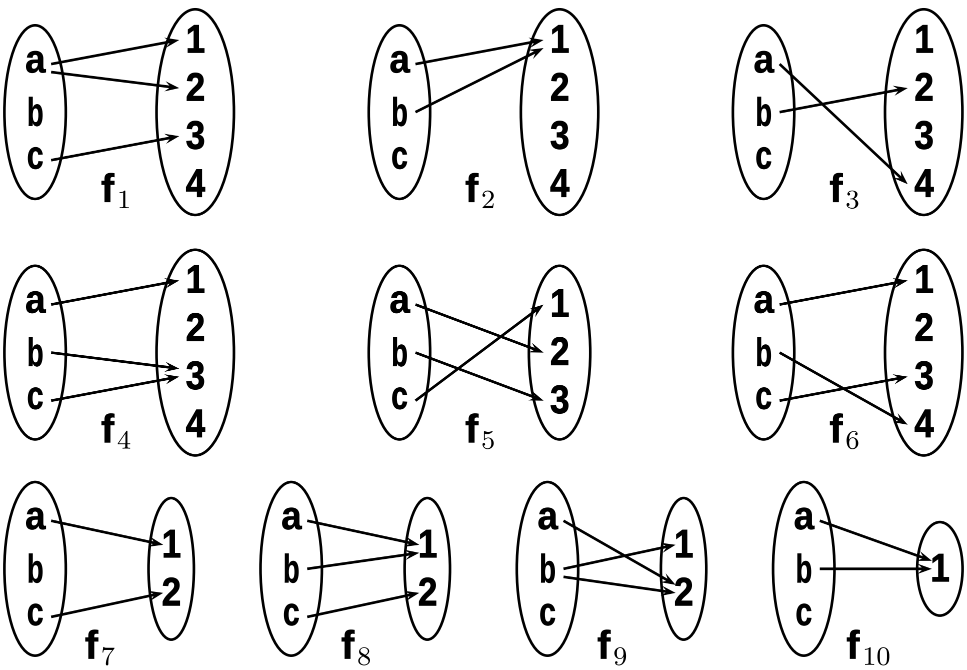
- In terms of arrow diagrams and graphs...

	The arrow diagram for $f : X \to Y$	The graph for $f : \mathbb{R} \to \mathbb{R}$
function	has exactly one outgoing arrow for each element of X	intersects each vertical line in exactly one point
injective one-to-one	has at most one incoming arrow for each element of Y	intersects each horizontal line in at most one point
surjective onto	has at least one incoming arrow for each element of Y	intersects each horizontal line in at least one point
bijective	has exactly one incoming arrow for each element of Y	intersects each horizontal line in exactly one point

Example. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is neither injective nor surjective.



Exercise. Determine whether or not each of the following arrow diagrams corresponds to a function. If it does, then determine whether or not it is injective, surjective, or bijective.



	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
function **injective** **surjective** **bijective**										

Exercise. **Which of the following definitions correspond to functions? Which of the functions are injective? surjective? bijective?**

$$\begin{array}{ll}
f_1 : R \to R, & f_1(x) = \sqrt{x} \\
f_2 : R \to R, & f_2(x) = x^2 \\
f_3 : R \to (R^+ \cup \{0\}), & f_3(x) = x^2 \\
f_4 : R^+ \to R^+, & f_4(x) = x^2 \\
f_5 : (R - \{0\}) \to R, & f_5(x) = \frac{1}{x} \\
f_6 : R \to R, & f_6(x) = x^2 - 2x - 2
\end{array}$$

Plot the graph in each case, and give reasons for your answers.

	f_1	f_2	f_3	f_4	f_5	f_6
function **injective** **surjective** **bijective**						

- For functions $f : X \to Y$ and $g : Y \to Z$, the composite of f and g is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.
- In general, $g \circ f$ and $f \circ g$ are not the same composite functions.
- Associativity of composition (assuming they exist): $h \circ (g \circ f) = (h \circ g) \circ f$.

Example. Let f and g be functions defined by

$$f : \mathbb{N} \to \mathbb{Z},\ f(x) = x + 3 \qquad \text{and} \qquad g : \mathbb{Z} \to \mathbb{Z},\ g(y) = y^2.$$

Then the composite function $g \circ f : \mathbb{N} \to \mathbb{Z}$ exists because
codomain of $f = \mathbb{Z} =$ domain of g.
It is given by

$$(g \circ f)(x) = g(f(x)) = g(x + 3) = (x + 3)^2 = x^2 + 6x + 9.$$

Technically, $f \circ g$ is not defined as codomain of $g = \mathbb{Z} \neq \mathbb{N} =$ domain of f.

BUT, range of $g \subseteq \mathbb{N}$ so if we re-define g to be closely related function $g : \mathbb{Z} \to \mathbb{N} : y \mapsto y^2$ then,
with this sleight of hand $f \circ g$ is defined and

$$(f \circ g)(y) = f(g(y)) = f(y^2) = y^2 + 3.$$

Note $f \circ g \neq g \circ f$ and they do not even have the same domains.

Exercise. Let $A = \{1, 2\}$ and $f : A \to A$ be defined by

$$f = \{(1, 2), (2, 1)\}.$$

Find the composite $f \circ f : A \longrightarrow A$.

- The **identity** function on a set X is the function $id_X : X \to X$, $id_X(x) = x$.
- For any function $f : X \to Y$, we have $f \circ id_X = f = id_Y \circ f$.
- A function $g : Y \to X$ is an **inverse** of $f : X \to Y$ if and only if

$$g(f(x)) = x \text{ for all } x \in X, \quad \text{and} \quad f(g(y)) = y \text{ for all } y \in Y,$$

  or equivalently, $g \circ f = id_X$ and $f \circ g = id_Y$.
- Thus $x = g(y)$ "solves" $f(x) = y$
- THEOREM: A function can have at most one inverse.
- If $f : X \to Y$ has an inverse, then we say that f is **invertible**, and we denote the inverse of f by f^{-1}. Thus, $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$.
- If g is the inverse of f, then f is the inverse of g. Thus, $(f^{-1})^{-1} = f$.
- THEOREM: A function is invertible if and only if it is bijective.
- THEOREM: If $f : X \to Y$ and $g : Y \to Z$ are invertible, then so is $g \circ f : X \to Z$, and the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.

Example. Let $f : R \to R$ be defined by $f(x) = 2x - 5$.
To find the inverse f^{-1}, solve the equation $y = f(x)$ with respect to x:

$$y = 2x - 5 \quad \Rightarrow \quad x = \frac{y+5}{2}.$$

Thus, $f^{-1} : R \to R$ is given by $f^{-1}(y) = \frac{y+5}{2}$.

Exercise. For each of the following functions, find its inverse if it is invertible.

$f : R \to Z$, $\quad f(x) = \lfloor x \rfloor$
$g : R \to R^+$, $\quad g(x) = e^{3x-2}$
$h : \{1, 2, 3\} \to \{a, b, c\}$, $\quad h = \{(1, b), (2, c), (3, a)\}$.

Example. **Prove that a function has at most one inverse.**

Proof. **Suppose that $f : X \to Y$ has two inverses $g_1 : Y \to X$ and $g_2 : Y \to X$. Then**

$$\begin{aligned}
g_1 &= g_1 \circ \mathrm{id}_Y && \text{by property of identity} \\
&= g_1 \circ (f \circ g_2) && \text{by definition of inverse} \\
&= (g_1 \circ f) \circ g_2 && \text{by associativity of composition} \\
&= \mathrm{id}_X \circ g_2 && \text{by definition of inverse} \\
&= g_2 && \text{by property of identity}
\end{aligned}$$

Hence, if f has an inverse, then it is unique.

Exercise. **Prove that a function has an inverse if and only if it is bijective.**

Exercise. Prove that if $f : X \to Y$ and $g : Y \to Z$ are invertible, then so is $g \circ f : X \to Z$, and the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.

- Informally speaking, a **sequence** is **an ordered list of objects**,

$$a_0, a_1, a_2, \ldots, a_k, \ldots ,$$

  where each object a_k is called a **term**, and the subscript k is called an **index** (typically starting from 0 or 1). We denote the sequence by $\{a_k\}$.
- If all terms a_k lie in a set A, we can think of the sequence as a function $a : \mathbb{N} \to A : k \mapsto a_k$.

Example.

- An *arithmetic progression* is a sequence $\{b_k\}$ where $b_k = a + kd$ for all $k \in \mathbb{N}$ for some fixed numbers $a \in \mathbb{R}$ and $d \in \mathbb{R}$. Its terms are

$$a, a + d, a + 2d, a + 3d, \ldots .$$

- A *geometric progression* is a sequence $\{c_k\}$ defined by $c_k = ar^k$ for all $k \in \mathbb{N}$ for some fixed numbers $a \in \mathbb{R}$ and $r \in \mathbb{R}$. Its terms are

$$a, ar, ar^2, ar^3, \ldots .$$

- Summation notation: for $m \leq n$,

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n .$$

- Properties of summation:

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k \quad \text{and} \quad \sum_{k=m}^{n} (\lambda a_k) = \lambda \sum_{k=m}^{n} a_k ,$$

but in general

$$\sum_{k=m}^{n} a_k b_k \neq \left(\sum_{k=m}^{n} a_k \right) \left(\sum_{k=m}^{n} b_k \right) .$$

Example. The sum of the first $n+1$ terms of the arithmetic progression $\{a+kd\}$ is

$$\sum_{k=0}^{n} (a+kd) = a + (a+d) + (a+2d) + \cdots + (a+nd) = \frac{(2a+nd)(n+1)}{2} .$$

Why?

We find a formula for the sum of the first n positive integers, by setting $a = 0$ and $d = 1$:

$$1 + 2 + \cdots + n = 0 + 1 + 2 + \cdots + n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2} .$$

Example. The sum of the first $n + 1$ terms of the geometric progression $\{ar^k\}$ is

$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} .$$

Why?

Exercise. **Given the formulas**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

evaluate

$$\sum_{k=1}^{10} (k-3)(k+2)$$

Exercise. **Use the formula for the geometric progression to evaluate**

$$\sum_{k=11}^{40} (3^k + 2)^2$$

Example. (Change of summation index)
The sum

$$\sum_{k=1}^{5} \frac{1}{k+2}$$

can be transformed by a change of variable like $j = k + 2$ as follows:

Lower limit: when $k = 1$, we have $j = 1 + 2 = 3$.
Upper limit: when $k = 5$, we have $j = 5 + 2 = 7$.
General term: we have $\frac{1}{k+2} = \frac{1}{j}$.

Thus, we obtain

$$\sum_{k=1}^{5} \frac{1}{k+2} = \sum_{j=3}^{7} \frac{1}{j}.$$

We could now replace the variable j by the variable k (if this is preferred):

$$\sum_{k=1}^{5} \frac{1}{k+2} = \sum_{k=3}^{7} \frac{1}{k}.$$

More generally, for any sequence $\{a_k\}$ and any integer d we have

$$\boxed{\sum_{k=m}^{n} a_k = \sum_{k=m+d}^{n+d} a_{k-d}}.$$

For example,

$$a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k = \sum_{k=2}^{4} a_{k-1} = \sum_{k=0}^{2} a_{k+1} = \cdots.$$

Exercise. Simplify

$$\sum_{k=2}^{n+1} x^{k-2} - \sum_{k=1}^{n-1} x^k + \sum_{k=0}^{n-1} x^{k+1}$$

Example. (A telescoping sum)

Using the identity $\frac{3}{k(k+3)} = \frac{1}{k} - \frac{1}{k+3}$ for $k \geq 1$, we can write

$$\sum_{k=1}^{n} \frac{3}{k(k+3)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+3} \right)$$
$$= \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+3}\right).$$

This is an example of a *telescoping sum*: $\sum a_k$, where $a_k = b_k - b_{k+d}$. By changing the summation index, we see that

$$\sum_{k=1}^{n} \frac{3}{k(k+3)} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+3} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=4}^{n+3} \frac{1}{k}$$
$$= \left(\sum_{k=1}^{3} \frac{1}{k} + \sum_{k=4}^{n} \frac{1}{k} \right) - \left(\sum_{k=4}^{n} \frac{1}{k} + \sum_{k=n+1}^{n+3} \frac{1}{k} \right)$$
$$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}.$$

Exercise. Use the identity $\frac{2}{k(k+1)(k+2)} = \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}$ for $k \geq 1$ to simplify

$$\sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)}$$

- **Product notation:** for $m \le n$,

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n .$$

- **Properties of product:**

$$\prod_{k=m}^{n} a_k b_k = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right) \quad \text{but} \quad \prod_{k=m}^{n} (a_k + b_k) \neq \prod_{k=m}^{n} a_k + \prod_{k=m}^{n} b_k .$$

Exercise. **Simplify**

$$\prod_{k=1}^{n} \frac{k}{k+3}$$