THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 5: - Mean Value Theorem.

Mean Value Theorem:

Suppose f is cts on $[a, b]$ and diffble on (a, b). Then there is a real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Ex: Demonstrate the Mean Value Theorem for the function, $f(x) = 6 - 2x + x^2$, on $[-2, 2]$.

We can use the MVT to do a range of problems.

Ex: Use the MVT to find an approximate value of $\sqrt{17}$.

Ex: Give a precise estimate of $\log 1.001$.

By MVT, with $f(x) = \log x$, on $[1, 1.001]$ we have

$$\frac{\log(1.001) - \log 1}{1.001 - 1} = f'(c)$$

for some $c \in [1, 1.001]$.

Hence $\frac{1}{1.001} < f'(c) < 1$ so $\frac{1}{1.001} < \frac{\log 1.001}{.001} < 1$. Thus $0.00099 < \log 1.001 < 0.001$ so $\log 1.001 = 0.000995 \pm 0.000005$.

Ex: Use the MVT to prove that $\tan x \geq x$ for all $x \in [0, \frac{\pi}{2})$.

Ex: Prove that for all real x and y, $|\sin x - \sin y| \leq |x - y|$.

Error Estimates:
Suppose I measure an angle in radians to be 0.7^c and I take the sine of that angle. If the error involved in my measurement is approximately 0.01^c what is the worst error involved in taking the sine of this number?

That is, if $f(x) = \sin x$ and $\Delta x = \pm 0.01$, we want a bound on the size of

$$|\Delta f(x)| = |f(x + \Delta x) - f(x)|.$$

Theorem: If $f'(x)$ exists, then

$$|\Delta f(x)| = |f(x + \Delta x) - f(x)| \approx f'(x)\Delta x.$$

Ex: In the above example, $\Delta f(x) \approx \cos 0.7 \times 0.01 \approx 7.65 \times 10^{-3}$.

Here are some consequences of the MVT:

Definition: A function f defined on $[a, b]$ is said to be **increasing** if $f(x) > f(y)$ whenever $x > y$, and **decreasing** when $f(x) < f(y)$ whenever $x > y$.

Theorem: Suppose f is diffble on (a, b),

(i) If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing on (a, b).

(ii) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on (a, b)

(iii) If $f'(x) < 0$ for all $x \in (a, b)$ then f is decreasing on (a, b).

Proof: The proof of all of these comes from applying the MVT to f on (x, y), any subset of (a, b) giving

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

In the first case we have $f(y) > f(x)$ whenever $y > x$ so f is increasing. Similarly for (iii). For (ii), we have $f(x) = f(y)$, for all x and y so f is a constant.

Theorem: Suppose that f is cts on $[a, b]$ and diffble on (a, b) and that $f(a)$ and $f(b)$ have opposite signs. If $f'(x) > 0$ for all $x \in (a, b)$ (or $f'(x) < 0$ for all $x \in (a, b)$), then f has **exactly** one real zero in (a, b).

Ex: $f(x) = x^3 + x + 1$ on $[-1, 1]$.

Ex: Show that $5x^5 + 2x + 1 = 0$ has exactly one real solution.

Theorem: Suppose that f, g are differentiable functions such that $f(a) = g(a)$ and for all $x > a$, we have $f'(x) > g'(x)$.
Then $f(x) > g(x)$ for all $x > a$.

Ex: Prove that $\sin x < x$ for all $x > 0$.

Types of points:

We wish to classify all the sorts of interesting points a function can have.

Definition:

Suppose that f is a function defined on an interval $[a, b]$ and let $x_0 \in [a, b]$.

(i) x_0 is called a **critical point** if $f'(x_0) = 0$ or if f is not differentiable at x_0.

(ii) x_0 is called an **extreme point** if x_0 is a local maximum or local minimum.

(iii) x_0 is called a **stationary point** if $f'(x_0) = 0$.

In practise, to find the (global) maximum and minimum, we need to find the stationary points and check their y values and also check the y values at the end points.

Ex: Find the global max and min of $f(x) = x^3 - 3x^2 + 1$ on the interval $[0, 4]$.

Ex: Find the local max and min of $f(x) = |x-3||x|$

Ex: Find the dimensions of the rectangle (with vertical and horizontal sides) of maximum area which can be inscribed in the ellipse, $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

L'Hôpital's Rule:

We return to the problem of calculating limits.

Theorem: (L'Hôpital's Rule)

Suppose that f and g are differentiable functions (except possibly at a) and that $f(a)$ and $g(a)$ are both equal to 0, or both tend to ∞ as $x \to a$.

If $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof: (Outline). Suppose we have the case $f(a) = g(a) = 0$. Apply the MVT to f and g on the interval (a, x), where $x > a$, so that for some $c, d \in (a, x)$ we have $\frac{f(x)-0}{x-a} = f'(c)$ and $\frac{g(x)-0}{x-a} = g'(d)$.

Hence

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \frac{f'(c)}{g'(d)}.$$

Hence as $x \to a^+$ we have $c \to a^+$ and $d \to a^+$, so that if the limit of $\frac{f'(x)}{g'(x)}$ exists as $x \to a$, we have $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Ex: $\lim_{x \to 0} \frac{e^x - 1}{\sin 2x}$.

Ex: $\lim_{x \to 1} \frac{1 - x + \log x}{1 + \cos \pi x}$.

When dealing with limits to infinity, we need the following version of L'Hôpital's rule.

Theorem: Suppose f and g are differentiable. Suppose further that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$ (or $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$).
If $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

Ex: $\lim\limits_{x\to\infty} \dfrac{\log x}{x}$.

Ex: $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$.