$$\lim_{x \to \infty} f(x) = L.$$

If for all $\varepsilon > 0$, there is $M \in \mathbb{R}$, such that if $x > M$, then $|f(x) - L| < \varepsilon$.

$$\lim_{x \to \infty} \frac{3x^2 - 4}{x^2 + 1} = 3$$

want $\left| \dfrac{3x^2-4}{x^2+1} - 3 \right| < \varepsilon$ by

make x very large.

$$\frac{3x^2-4}{x^2+1} = 3 - \frac{7}{x^2+1}$$

want $\left| \dfrac{3x^2-4}{x^2+1} - 3 \right| = \left| \dfrac{-7}{x^2+1} \right| = \dfrac{7}{x^2+1} < \varepsilon.$

$$\frac{7}{x^2+1} < \varepsilon, \iff x^2 + 1 > \frac{7}{\varepsilon}$$

$$\iff x > \sqrt{\left(\frac{7}{\varepsilon} - 1\right)}$$

Let $\varepsilon > 0$ be given. ~~Let~~ Set

$$M = \sqrt{\frac{7}{\varepsilon} - 1}$$

Then if $x > M = \sqrt{\frac{7}{\varepsilon} - 1}$

Then
$$\left| \frac{3x^2-4}{x^2+1} - 3 \right|$$
$$= \left| 3 - \frac{7}{x^2+1} - 3 \right|$$
$$= \frac{7}{x^2+1}$$
$$< \frac{7}{\left(\sqrt{\frac{7}{\varepsilon}-1}\right)^2+1}, \quad \text{as } x > M.$$
$$= \varepsilon.$$

Therefore, $\lim_{x \to \infty} \frac{3x^2-4}{x^2+1} = 3.$

Continuity

Def: Let f be defined on an interval containing the point $x=a$.

We say that f is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$.

① a is in the domain of f. ($f(a)$ is defined)

② $\lim_{x \to a} f(x)$ exists.

③ The two quantities are the same.

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

$\lim_{x \to 0} f(x)$ does not exist.

Therefore f is not continuous at 0.

$$g(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

① $g(0)$ is defined. $g(0) = 0$.

② $\lim_{x \to 0} g(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0$.

③ $\lim_{x \to 0} g(x) = 0 = g(0)$.

g is continuous at $x = 0$.

Def: If a function f is defined on an open interval I, we say that f is continuous on I if f is continuous at every point of I.

$$f(x) = \frac{x}{x^2 - 9}$$

Where is f continuous?

Domain of $f = \mathbb{R} \setminus \{\pm 3\}$.

Properties of continuous functions

Suppose $f(x)$ and $g(x)$ are continuous at $x=a$.

Then

i) $f(x) \pm g(x)$ is continuous at $x=a$.

ii) $f(x) \cdot g(x)$ is continuous at $x=a$.

iii) $\frac{f(x)}{g(x)}$ is continuous at provided $g(a) \neq 0$.

All polynomials are continuous on $\mathbb{R}$.

Rational functions are continuous in their respective domains.

iv) $(f(x))^k$ is continuous $x=a$ provide that $k \in \mathbb{Q}$, and $(f(a))^k$ is defined.

v). If $g(x)$ is continuous at $x=a$ and

$f(x)$ is continuous at $x=g(a)$, then

$f \circ g\,(x)$ is continuous at $x=a$.

x, $\sin x$, $\cos x$ are continuous on $\mathbb{R}$.

Types of discontinuity.

i) Removable discontinuity.

f is defined at a

$\lim\limits_{x\to a} f(x)$ exists.

$\lim\limits_{x\to a} f(x) \neq f(a)$.

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3. \\ 8, & x=3. \end{cases}$$

we know $\lim\limits_{x\to 3} f(x) = 6$.

we can redefine f ~~so that its~~ at one point so that the resulting function is continuous everywhere.

$$f_1(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3. \end{cases}$$

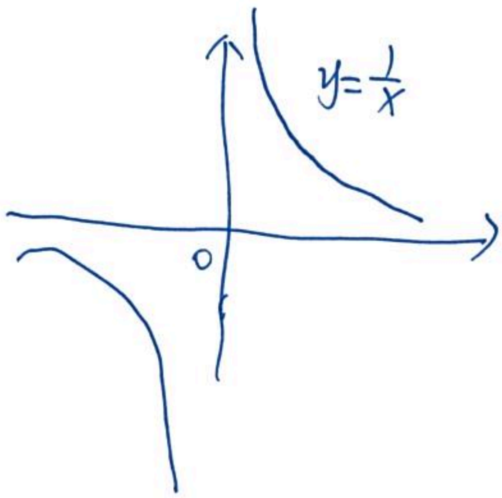
$$f_1(x) = x+3.$$

ii) Essential discontinuity.

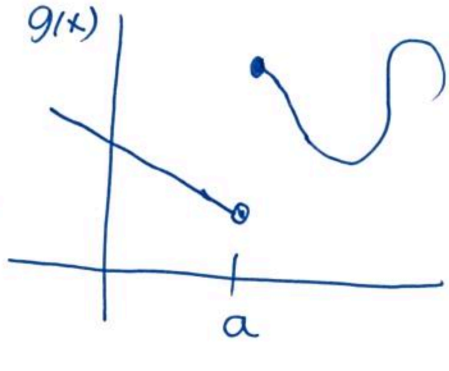
f is said to have an essential discontinuity at $x=a$ if $\lim\limits_{x\to a} f(x)$ does not exist.

$f(x) = \dfrac{1}{x}$,

f has an essential discontinuity at $x=0$.



g has an essential discontinuity at $x=a$.



g has a jump discontinuity at $x=a$.

$$\lim_{x\to a^+} g(x) \neq \lim_{x\to a^-} g(x).$$

Def: If f is a function defined on a closed interval $[a, b]$, then we say that f is continuous at a if $\lim_{x\to a^+} f(x) = f(a)$; similarly, f is said to be continuous at b if $\lim_{x\to b^-} f(x) = f(b)$.

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example: $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$.

$$= (1-x^2)^{\frac{1}{2}}$$

$$g(x) = \begin{cases} x, & x<0 \\ x^2, & x\geq 0. \end{cases}$$

Prove that g is continuous everywhere.

If $x<0$, then $g(x)=x$.

g is continuous on $(-\infty, 0)$.

If $x>0$, then $g(x)=x^2$.

g is continuous on $(0, \infty)$.

If ~~at~~ $x=0$, $g(0)=0$.

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^2 = 0.$$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x = 0.$$

Therefore, $\lim_{x\to 0} g(x) = 0. = g(0)$.

Hence g is continuous at $x=0$.

g is continuous everywhere.