**THE UNIVERSITY OF NEW SOUTH WALES**
**SCHOOL OF MATHEMATICS AND STATISTICS**
**MATH1131 Calculus**

# MATHEMATICS 1A CALCULUS.
## Section 1: - Functions and Graphs.

### 1. Numbers.

We will use the following notation:
The set of natural numbers, denoted by $\mathbb{N}$, consists of all the whole numbers $\{0, 1, 2, .....\}$.
The set of integers, denoted by $\mathbb{Z}$, consists of all the whole numbers $\{..., -3, -2, -1, 0, 1, 2, 3, .....\}$.
The set of rational numbers, denoted by $\mathbb{Q}$, consists of all numbers of the form $\frac{p}{q}$ where $p, q$ are integers and $q \neq 0$.

The ancient Greeks initially thought that this was all there was (they didn't believe in negative numbers and zero either), until they discovered that $\sqrt{2}$ could not be written as a rational number.

**Theorem:** $\sqrt{2}$ is irrational.

**Proof:**

$\sqrt{2}$ and numbers such as $\pi$ and $e$ are examples of irrational numbers. We think of the set of all real numbers are points which lie on the real line. Giving a formal definition of real numbers is difficult.

We will use the following set notation:

$\{x \in A : P(x)\}$ denotes the set of all elements $x$ of $A$ satisfying property $P$. For example, $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$ denotes all the real numbers between $-1$ and 1 (inclusive).

$A \cap B$ is the intersection of $A$ and $B$ and denotes all the elements that are in both $A$ and $B$.

$A \cup B$ is the union of $A$ and $B$ and denotes all the elements that are in either $A$ or $B$ (or both).

$\emptyset$ is the set which has no elements, for example $\{x \in \mathbb{R} : x^2 < -1\} = \emptyset$.

**Inequalities:**

You are aware of the following facts about inequalities:

For $x, y, z \in \mathbb{R}$ we have
i. if $x > y$ then $x + z > y + z$
ii. if $x > y$ and $z > 0$ then $xz > yz$ and if $z < 0$ we have $xz < yz$.

Note carefully the definition for $|x|$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So for example, $|a - 3|$ is equal to $a - 3$ if $a \geq 3$ and $-(a - 3) = 3 - a$ if $a < 3$.

Note then that $|x| < 3$ means $-3 < x < 3$ and that $|-x| = |x|$. Also note that $\{x : |x-3| < 2\}$ represents the set of all real numbers whose distance from 3 is less that 2.

Finally note that $|xy| = |x||y|$ and that $|x + y| \leq |x| + |y|$. This last result is called the **triangle inequality.** You will see a complex version of this in the algebra strand of the course.

Also of importance is:

**Theorem:** (AM-GM inequality).

If $x, y \geq 0$ are real numbers then

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

(This says that the arithmetic mean of two positive real numbers exceeds their geometric mean.)

**Proof:**

Ex: Prove that for $x > 0$, we have $x + \frac{1}{x} \geq 2$.

Ex: Suppose $a, b, c$ are positive real numbers. Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

**Intervals:** We will use the following notation when dealing with intervals. A round bracket means we do not include the endpoint while we do when a square bracket is used. For example $(3, 9]$ means the interval $3 < x \leq 9$. Note that since infinity is NOT a real number, if we wish to represent the interval from 3 onwards we write this as $[3, \infty)$ (never use a square bracket with infinity.). Here are some further examples:

$$\{x \in \mathbb{R} : x > 3\} \cap \{x \in \mathbb{R} : x < 5\} = (3, 5)$$

$$\{x \in \mathbb{R} : x > 3\} \cup \{x \in \mathbb{R} : x < 5\} = \mathbb{R}$$

$$\{x \in \mathbb{R} : x > 5\} \cup \{x \in \mathbb{R} : x < 3\} = (-\infty, 3) \cup (5, \infty)$$

$$\{x \in \mathbb{R} : x > 5\} \cap \{x \in \mathbb{R} : x < 3\} = \emptyset$$

**Solving Inequalities:**

These are very similar to equations except that we must careful when multiplying by an unknown. You should be familiar with solving quadratic inequations such as
Ex: Find $\{x : x^2 - 2x - 3 > 0\}$.

For more difficult inequalities we use the following idea.

Ex: Solve $x > 1 + \frac{2}{x}$.

Ex: Solve $\frac{2}{x} \leq \frac{3}{x-1}$.

**Functions:**

You should be familiar with the function concept from school. Roughly speaking, a function $f : A \to B$ is a rule or formula which associates to each element of a set $A$ (called the domain) **exactly one** element from another set $B$ (called the co-domain). For the most part, we will have $A = B = \mathbb{R}$. The range of the function is the set of values $b$ in $B$ for which there is an $a \in A$ with $f(a) = b$. In less formal terms, the range consists of the output of the function.You will need to be able to find the domain and range of basic functions.

Ex: Find the domain and range of $f(x) = \sqrt{1 - x^2}$.

Ex: Use the AM-GM inequality to find the range of $y = x^2 + \frac{1}{x^2}$ and sketch the graph.

Ex: Find the domain of $f(x) = \sqrt{\cos x}$.

It is often difficult to find the range of a function. For example, what is the range of $2 + \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}$ ?

It is important to be able to draw the graph of a given function. In most Calculus problems this is crucial.

It often helps if the function is even or odd. You will recall that $f$ is **even** if $f(x) = f(-x)$ and $f$ is **odd** if $f(-x) = -f(x)$. Even functions are symmetric about the $y$ axis and odd functions have a central symmetry with respect to the origin.

Thus, if we can draw such a function on the positive half plane we get the rest of the picture for free.
Note that if $f$ is odd and has 0 in its domain, then $f(0) = 0$.

We say that $f$ is periodic of period $T$ if $f(x+T) = f(x)$ for all real $x$ in the domain of $f$.

You have met the trig. functions which are periodic with period $2\pi$.

Ex: Sketch: $f(x) = (x-3)^2 + 4$, and $f(x) = \frac{1}{x^2-1}$.

Ex: Sketch: $f(x) = x$ if $0 \leq x < 1$ and $f(x+1) = f(x)$ for all $x$.

Ex: Sketch $f(x) = x^2$ for $0 < x < 1$, $f$ is periodic of period 2 and $f$ is even.

**Floor and Ceiling Functions:**

Ex: Sketch $f(x) = x - \lfloor x \rfloor$.

**Combining Functions:**
If $f$ and $g$ are two functions, we can add, subtract and multiply them in the obvious way. We can also divide them provided $g$ is not zero. If the range of $g$ equals the domain of $f$ we compose the two functions to form $f \circ g$ which we define to be

$$f \circ g(x) = f(g(x)).$$

$f \circ g$ is called the composite function of $f$ and $g$.

Ex: Find $f \circ g$ and $g \circ f$ if $f(x) = x^3$ and $g(x) = \sqrt{x^2 + 1}$.

Note that some functions cannot be defined by one simple equation. Many functions which occur in the real world are defined piecewise.

Ex: $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \sin x & \text{if } x > 0 \end{cases}$

**Conic Sections:**

An important class of implicitly defined functions arises from the *conic sections* (so called because they are obtained by slicing a cone with various planes.)

You will need to recognise these:

(i) **Circle** $x^2 + y^2 = r^2$

(ii) **Ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ a, b \neq 0.$

(iii) **Hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,\ a, b \neq 0.$

(iv) **Rectangular Hyperbola** $y = \frac{a}{x}, a \neq 0.$

**Other Functions:**

It is assumed that you are familiar with the basic properties of polynomial functions, rational functions, the trigonometric functions, the exponential and logarithmic functions.

Graph $y = \frac{x^2}{x^2-1}$.