MATH1231/1241 Calculus S2 2008 Test 2 v3b

Full Solutions

September 9, 2017

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1. This equation may be tempting to test for exactness, but upon closer inspection, it is separable! Rearranging the equation,

$$\begin{aligned}
y\sqrt{2x^2+3}\,\mathrm{d}y + x\sqrt{4-y^2}\,\mathrm{d}x &= 0\\
\frac{\mathrm{d}y}{\mathrm{d}x}y\sqrt{2x^2+3} + x\sqrt{4-y^2} &= 0\\
\frac{\mathrm{d}y}{\mathrm{d}x}\frac{y\sqrt{2x^2+3}}{x\sqrt{4-y^2}} + 1 &= 0\\
\frac{\mathrm{d}y}{\mathrm{d}x}\frac{y}{\sqrt{4-y^2}} &= -\frac{x}{\sqrt{2x^2+3}}
\end{aligned}$$

$$\int \frac{-y}{\sqrt{4-y^2}}\,\mathrm{d}y = \int \frac{x}{\sqrt{2x^2+3}}\,\mathrm{d}x$$
$$\sqrt{4-y^2} = \frac{1}{2}\sqrt{2x^2+3}+c.$$

Note that these integrals were performed by inspection, mentally applying something known as the "Reverse Chain Rule".
Using the condition of $y=1$ when $x=0$, we get

$$\sqrt{3} = \frac{1}{2}\sqrt{3}+c$$
$$c = \frac{\sqrt{3}}{2}.$$

So, by substituting this in and simplifying, we obtain

$$\begin{aligned}
\sqrt{4-y^2} &= \frac{1}{2}\sqrt{2x^2+3}+\frac{\sqrt{3}}{2} \\
4-y^2 &= \left(\frac{1}{2}\left(\sqrt{2x^2+3}+\sqrt{3}\right)\right)^2 \\
y^2 &= 4-\frac{1}{4}\left(\sqrt{2x^2+3}+\sqrt{3}\right)^2 \\
y &= \sqrt{4-\frac{1}{4}\left(\sqrt{2x^2+3}+\sqrt{3}\right)^2}
\end{aligned}$$

which is the required answer. Note that we took the positive square root for y because from the given initial condition, we need y to be positive at $x=0$.

2. Consider the characteristic equation and solve for the roots:

$$\begin{aligned}
\lambda^2+8\lambda+16 &= 0 \\
(\lambda+4)^2 &= 0 \\
\lambda &= -4.
\end{aligned}$$

Since -4 is a double root of the characteristic equation, the general solution to the differential equation is

$$y(t) = Ae^{-4t}+Bte^{-4t} = (A+Bt)\,e^{-4t}$$

where A and B are arbitrary constants.

3. To evaluate an integral in the form of two polynomials divided by one another (a.k.a. a *rational function*), we see if the denominator can be easily factorised. This would mean partial fractions can be applied.
Since it isn't in this case, we then turn to forming the derivative of the denominator on the numerator to try bring in some potential log terms as $\frac{\mathrm{d}}{\mathrm{d}x}(\log f(x)) = \frac{f'(x)}{f(x)}$. We have

$$
\begin{aligned}
\int \frac{x}{x^2+2x+10}\,\mathrm{d}x &= \int \left(\frac{\frac{1}{2}(2x+2)}{x^2+2x+10} - \frac{1}{x^2+2x+10} \right) \mathrm{d}x \\
&= \frac{1}{2}\ln\left(x^2+2x+10\right) - \int \frac{1}{x^2+2x+10}\,\mathrm{d}x.
\end{aligned}
$$

Next, we can complete the square for the denominator of the second term to obtain an inverse tan integral. We have

$$
\begin{aligned}
\int \frac{x}{x^2+2x+10}\,\mathrm{d}x &= \frac{1}{2}\ln\left(x^2+2x+10\right) - \int \frac{1}{(x+1)^2+9}\,\mathrm{d}x \\
&= \frac{1}{2}\ln\left(x^2+2x+10\right) - \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + c.
\end{aligned}
$$

4. Firstly, divide through by the term in front of $\frac{\mathrm{d}y}{\mathrm{d}x}$ to get

$$
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2}y = \frac{1}{2}e^x.
$$

This is a first-order linear ODE, so an integrating factor can be used. The coefficient of y above is $-\frac{1}{2}$ (don't forget the negative sign present here!), so the integrating factor is given by

$$
I(x) = e^{\int -\frac{1}{2}\,\mathrm{d}x} = e^{-\frac{1}{2}x}.
$$

Multiplying through by this, we obtain

$$
e^{-\frac{1}{2}x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{2}e^{-\frac{1}{2}x} = \frac{1}{2}e^{\frac{1}{2}x}.
$$

The left hand side is simply the product rule (note that the integrating factor is designed so that multiplying through by it will always result in something that is a result of the product rule – that's the whole point of the integrating factor!), so we have

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\frac{x}{2}}y\right) = \frac{1}{2}e^{\frac{x}{2}}.
$$

Now we integrate both sides with respect to x to get

$$
\begin{aligned}
e^{-\frac{x}{2}}y &= \int \frac{1}{2}e^{\frac{x}{2}}\,\mathrm{d}x = e^{\frac{x}{2}} + c \\
y &= e^x + ce^{\frac{x}{2}}.
\end{aligned}
$$

Next, using the given condition of $y=0$ at $x=0$, we have

$$
\begin{aligned}
e^0 + ce^{\frac{0}{2}} &= 0 \\
1 + c &= 0
\end{aligned}
$$

$$c = -1.$$

Therefore, the solution is

$$y = e^x - e^{\frac{x}{2}}.$$

MATH1231/1241 Calculus Test S2 2009 Test 2 v3a

Hints & Worked Sample Solutions

September 9, 2017

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1. **Hints**

   Partial fractions will help. The answer is

$$-\frac{1}{2}\ln|2x+1| + \ln|x-3| + c, \quad \text{where } c \text{ is an arbitrary constant.}$$

   Worked Sample Solution

   We use partial fractions. Write

$$\frac{x+4}{(2x+1)(x-3)} \equiv \frac{A}{2x+1} + \frac{B}{x-3}$$

$$\Rightarrow x + 4 \equiv A(x-3) + B(2x+1).$$

Using the substitution $x = 3$ yields $\boxed{B = 1}$ and $x = -1/2$ yields $\boxed{A = -1}$. Therefore,

$$\int \frac{x+4}{(2x+1)(x-3)}\,\mathrm{d}x = \int \left(\frac{-1}{2x+1} + \frac{1}{x-3}\right)\mathrm{d}x = -\frac{1}{2}\ln|2x+1| + \ln|x-3| + c.$$

2. **Hints**

Use the usual procedure for solving exact differential equations. That is, the ODE is of the form

$$I\,\mathrm{d}x + J\,\mathrm{d}y = 0,$$

where $I \equiv I(x,y)$ and $J \equiv J(x,y)$ are given functions of x and y. So show that $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$ to show exactness. After showing this, find a function $F \equiv F(x,y)$ that satisfies $\frac{\partial F}{\partial x} = I$ and $\frac{\partial F}{\partial y} = J$ (this will require you to do two integrations generally). After finding such an F, the general solution is given by $F(x,y) = C$, where C is a constant.

Worked Sample Solution

The ODE is

$$I\,\mathrm{d}x + J\,\mathrm{d}y = 0,$$

where

$$I = 2x + 3y \quad \text{and} \quad J = 3x + 4y.$$

We see that

$$\frac{\partial I}{\partial y} = 3 = \frac{\partial J}{\partial x}.$$

Hence $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$, so the ODE is exact. To find the general solution, we search for a function $F \equiv F(x,y)$ that satisfies

$$\frac{\partial F}{\partial x} = 2x + 3y \tag{1}$$

$$\text{and} \quad \frac{\partial F}{\partial y} = 3x + 4y. \tag{2}$$

Once we find such an F, the general solution will be given by $F(x,y) = C$ (for some constant C). Partially integrating (1) with respect to x, we find that

$$F = x^2 + 3xy + c(y),$$

for some function $c(y)$ that does not depend on x. Now partially differentiating this with respect to y, we find that

$$\frac{\partial F}{\partial y} = 3x + c'(y).$$

Comparing with (2), we see that we must have

$$c'(y) = 4y,$$

and so (integrating this with respect to y) it suffices to take $c(y) = 2y^2$. Thus

$$F = x^2 + 3xy + c(y) = x^2 + 3xy + 2y^2,$$

and the general solution to the ODE is

$$x^2 + 3xy + 2y^2 = C.$$

3. **Hints**

(i) Use the characteristic equation as usual.

(ii) Use the initial conditions to find what A and B should be from the general solution obtained in (i).

(iii) Normally we would guess ce^{3t} as a particular solution. But e^{3t} is already a part of the homogeneous solution, so we must adjust our guess (how?).

Worked Sample Solutions

(i) The characteristic equation is

$$\lambda^2 - 7\lambda + 12 = 0.$$

This factors into $(\lambda - 3)(\lambda - 4) = 0$, so the characteristic roots are 3 and 4. Hence the general solution to the differential equation is

$$y = Ae^{3t} + Be^{4t},$$

where A and B are arbitrary constants.

(ii) From part (i), we know that the solution is of the form

$$y = Ae^{3t} + Be^{4t}.$$

Hence

$$y' = 3Ae^{3t} + 4Be^{4t}.$$

Substituting $y = 3$ and $y' = 10$ at $t = 0$ into these (noting that $e^0 = 1$), we see that we must have

$$A + B = 3$$

and

$$3A + 4B = 10.$$

We now solve simultaneously for A and B. Multiplying the first equation by 3 gives $3A + 3B = 9$. Subtracting this from the second equation gives $\boxed{B = 1}$. Now the first equation gives $A + B = 3 \Rightarrow A = 3 - B = 3 - 1 \Rightarrow \boxed{A = 2}$. Thus the desired solution is

$$y = 2e^{3t} + e^{4t}.$$

(iii) Since 3 is a simple root of the characteristic equation, the form of the particular solution we would seek is

$$y_p = cte^{3t},$$

where c is an undetermined coefficient.

Remarks/Tips. Usually we would try ce^{3t} as a particular solution when the right-hand side of the ODE is e^{3t}. But e^{3t} is already a part of the homogeneous solution, so we must adjust our guess. The way to adjust the guess it to *multiply the original guess by* t. This gives us the trial solution of cte^{3t}. Since te^{3t} is no longer a part of the homogeneous solution, this is our final guess. Note however that if te^{3t} were part of the homogeneous solution (which would happen iff 3 were a multiple root of the characteristic equation), we would have to multiply our guess by t again to ct^2e^{3t}. In general, we must keep multiplying our guess by t until the guess no longer has any term in common with the homogeneous solution (and we must not multiply by t any more times than necessary for this to happen). In fact, the correct number of times we should multiply the guess by t by is the multiplicity of the root 3 in the characteristic equation.

MATH1231/1241 Calculus S2 2013 Test 2 v1a

Full Solutions

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1. We use partial fractions. Write

$$\frac{x-3}{(2x-1)(x+2)} \equiv \frac{A}{2x-1} + \frac{B}{x+2}.$$

We apply the Heaviside Cover-Up Method to quickly find A and B:

$$A = \frac{(1/2)-3}{(1/2)+2} = -1; \quad B = \frac{(-2)-3}{2(-2)-1} = 1.$$

Thus, substituting this and integrating, we obtain our answer

$$\begin{aligned}\int \frac{x-3}{(2x-1)(x+2)}\,\mathrm{d}x &= \int \left(-\frac{1}{2x-1}+\frac{1}{x+2}\right)\mathrm{d}x \\ &= -\frac{1}{2}\ln|2x-1| + \ln|x+2| + c.\end{aligned}$$

Remark. If you are not familiar with the Heaviside Cover-Up Method, you find more information about it by searching online, or you can just use the usual high-school method of finding A and B:

$$\begin{aligned}\frac{x-3}{(2x-1)(x+2)} &\equiv \frac{A}{2x-1}+\frac{B}{x+2} \\ \Rightarrow x-3 &\equiv A(x+2)+B(2x-1).\end{aligned}$$

Substituting $x=-2$ gives

$$\begin{aligned}-2-3 &= 0+B(2(-2)-1) \\ \Rightarrow -5 &= -5B \\ \Rightarrow &\boxed{B=1}.\end{aligned}$$

Substituting $x=0$ gives

$$\begin{aligned}0-3 &= A(0+2)+1\times(2\times 0-1) \qquad \text{(remember we found } B=1\text{)} \\ \Rightarrow -3 &= 2A-1 \\ \Rightarrow &\boxed{A=-1}.\end{aligned}$$

This agrees with what we found with the Heaviside Cover-Up Method earlier. The Heaviside Cover-Up Method can save you time in exams, so it is a good idea to learn it if you can.

2. Let $I \equiv I(x,y) = y - y\sin xy$ and $J \equiv J(x,y) = x - x\sin xy$. So the differential equation is $I\,\mathrm{d}x + J\,\mathrm{d}y = 0$.
To show that the differential equation is exact, we need to know that $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$.
Computing these partial derivatives (using the product rule where necessary), we find

$$\begin{aligned}\frac{\partial I}{\partial y} &= \frac{\partial}{\partial y}(y - y\sin xy) \\ &= 1 - xy\cos xy - \sin xy.\end{aligned} \qquad \begin{aligned}\frac{\partial J}{\partial x} &= \frac{\partial}{\partial x}(x - x\sin xy) \\ &= 1 - xy\cos xy - \sin xy.\end{aligned}$$

Thus, as $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$, this differential equation is exact.
Since it is exact, it has a solution of the form $H(x,y) = C$ where C is a constant, and $\frac{\partial H}{\partial x} = I(x,y) = y - y\sin xy$ and $\frac{\partial H}{\partial y} = J(x,y) = x - x\sin xy$.

Using $\frac{\partial H}{\partial x} = y - y\sin xy$, we partially integrate both sides with respect to x to obtain

$$H(x, y) = xy + \cos xy + c(y).$$

(We obtain a function of y (and y alone), $c(y)$, as our "constant of integration" is a constant *with respect to the integrating variable*, which is x, but need not be constant with respect to y.) Partially differentiating this with respect to y, we have

$$\frac{\partial H}{\partial y} = x - x\sin xy + c'(y) \equiv x - x\sin xy \quad \left(\text{as } \frac{\partial H}{\partial y} \text{ needs to equal } J(x, y) = x - x\sin xy\right)$$
$$\implies c'(y) = 0$$
$$\implies c(y) = c_0 \text{ where } c_0 \text{ is a constant.}$$

Thus, we have $H(x, y) = xy + \cos xy + c_0 = C$, and as C and c_0 are just constants, the c_0 can be absorbed into the constant C. So the general solution of the differential equation is that y is given implicitly as a function of x by the equation

$$xy + \cos xy = C,$$

where C is a constant.

3. (i) The characteristic equation is

$$\lambda^2 - \lambda - 12 = 0$$
$$\implies (\lambda - 4)(\lambda + 3) = 0$$
$$\implies \lambda = 4 \text{ or } -3.$$

Thus, the general solution to this homogeneous differential equation is given by

$$y(t) = Ae^{4t} + Be^{-3t}, \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

(ii) The conditions we are given are $y(0) = 3$ and $y'(0) = 2$.
Note that the solution is of the form $y(t) = Ae^{4t} + Be^{-3t}$, and so differentiation yields $y'(t) = 4Ae^{4t} - 3Be^{-3t}$. Applying the initial conditions, we find that

$$y(0) = A + B = 3 \tag{1}$$
$$y'(0) = 4A - 3B = 2. \tag{2}$$

We now solve these two equations simultaneously. Observe that $(2) + 3 \times (1)$ gives:

$$4A - 3B + 3A + 3B = 2 + 9$$
$$\implies 7A = 11$$

$$\implies A = \frac{11}{7}.$$

Thus, $B = 3 - A = \frac{21}{7} - \frac{11}{7} = \frac{10}{7}$.
Therefore, $y\left(t\right) = \frac{11}{7} e^{4t} + \frac{10}{7} e^{-3t}$.

(iii) As we already have a e^{-3t} term in our solution of the homogeneous equation, the form the particular solution would take if e^{-3t} was on the RHS would be

$$y_p(t) = Cte^{-3t},$$

where C is an undetermined coefficient (constant). We multiply by a t as $\lambda = -3$ is a root of multiplicity 1 of the characteristic equation.

MATH1231/1241 Calculus S2 2013 Test 2 v1b

Hints and Worked Sample Solutions

September 9, 2017

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1. **Answer**

$$yxe^x + e^y\left(y^2 - 2y + 2\right) = C.$$

Worked Sample Solution
The given differential equation is

$$(x+1)\, ye^x \,\mathrm{d}x + \left(xe^x + y^2e^y\right)\, \mathrm{d}y = 0. \qquad (*)$$

Let $I = (x+1)\,ye^x$ and $J = xe^x + y^2e^y$ (so the differential equation is $I\,\mathrm{d}x + J\,\mathrm{d}y = 0$). Then $\frac{\partial I}{\partial y} = (x+1)\,e^x$ and

$$\begin{aligned}\frac{\partial J}{\partial x} &= e^x + xe^x \quad \text{(product rule)}\\ &= (x+1)\,e^x\\ &= \frac{\partial I}{\partial y}\\ &\Rightarrow (*) \text{ is exact.}\end{aligned}$$

Now, we search for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = (x+1)\,e^x y, \qquad (1)$$

and

$$\frac{\partial F}{\partial y} = xe^x + y^2e^y, \qquad (2)$$

so that y will be implicitly defined as a function of x by $F(x,y) = C$, where C is a constant. Partially integrating (1) with respect to x (so remember y is treated as a constant), we obtain (using Integration by Parts for the integral)

$$\begin{aligned}F(x,y) &= y\int (x+1)\,e^x\,\mathrm{d}x\\ &= y\left(e^x(x+1) - e^x\right) + \phi(y), \text{ for some function } \phi \text{ that depends only on } y\\ \Rightarrow F(x,y) &= yxe^x + \phi(y).\end{aligned}$$

Now partially differentiate this with respect to y:

$$\frac{\partial F}{\partial y} = xe^x + \phi'(y).$$

Comparing with (2), it suffices to find a $\phi(y)$ that satisfies

$$\begin{aligned}\phi'(y) &= y^2e^y\\ \Leftarrow \phi(y) &= \int y^2e^y\,\mathrm{d}y\\ &= e^y y^2 - 2\int ye^y\,\mathrm{d}y \quad \text{(Integration by Parts)}\\ &= e^y y^2 - 2\left(ye^y - e^y\right) \quad \text{(Integration by Parts again)}\\ &= e^y\left(y^2 - 2y + 2\right) \quad \text{(simplifying).}\end{aligned}$$

Thus $F(x,y) = yxe^x + \phi(y) = yxe^x + e^y\left(y^2 - 2y + 2\right)$, and so y is given implicitly as a function of x by the equation

$$yxe^x + e^y\left(y^2 - 2y + 2\right) = C,$$

for some constant C.

2. (i) **Hints**

You will find that the roots of the characteristic equation are $\lambda = 2$ or -3. Answer:

$$\boxed{y(t) = Ae^{2t} + Be^{-3t},\ \text{where } A \text{ and } B \text{ are arbitrary constants.}}$$

Here is a worked sample solution.

Worked Sample Solution

The characteristic equation is

$$\begin{aligned} \lambda^2 + \lambda - 6 &= 0 \\ \iff (\lambda - 2)(\lambda + 3) &= 0. \end{aligned}$$

So the characteristic roots are $\lambda_1 = 2$ and $\lambda_2 = -3$. Therefore, the general solution of the given differential equation is

$$y(t) = Ae^{2t} + Be^{-3t},\ \text{where } A \text{ and } B \text{ are arbitrary constants.}$$

(ii) **Hints**

Differentiate the answer in (i) and apply the given initial conditions to solve simultaneously for A and B.

Answer:

$$y(t) = -\frac{1}{5}e^{2t} + \frac{6}{5}e^{-3t}.$$

Worked Sample Solution

From (i), we have $y(t) = Ae^{2t} + Be^{-3t} \Rightarrow y'(t) = 2Ae^{2t} - 3Be^{-3t}$. Substituting the given initial conditions of $y(0) = 1$ and $y'(0) = -4$ (and recalling $e^0 = 1$), we obtain

$$A + B = 1, \qquad (1)$$

and

$$2A - 3B = -4. \qquad (2)$$

Multiplying Equation (1) through by 2, we have

$$2A + 2B = 2. \qquad (1a)$$

Subtracting (2) from ($1a$), we have

$$\begin{aligned}\cancel{2A} + 2B - \cancel{2A} + 3B &= 2 + 4 \\ \Rightarrow 5B &= 6 \\ \Rightarrow \boxed{B = \frac{6}{5}}.\end{aligned}$$

Substituting into (1), we have

$$\begin{aligned}A &= 1 - B \\ &= 1 - \frac{6}{5} \\ \Rightarrow \boxed{A = -\frac{1}{5}}.\end{aligned}$$

So the solution is $y(t) = -\frac{1}{5}e^{2t} + \frac{6}{5}e^{-3t}$.

(iii) **Hints**

Normally we would try $y_p = Ce^{-3t}$. But this forms part of the homogeneous solution found in part (i), so we need to multiply this by t and instead use $y_p = Cte^{-3t}$. This is now no longer part of the homogeneous solution, so this is the answer. (Note that if the homogeneous solution had te^{-3t} in it, which would happen if -3 were a double root of the characteristic equation, we would need to multiply y_p again by t.) In the actual test for this question, you wouldn't need to explain this, you would just need to write down the answer. As such, this is a rare case where the sample solution is actually shorter than the hint.

Worked Sample Solution

We would try

$$y_p(t) = Cte^{-3t}, \text{ where } C \text{ is a constant.}$$

3. **Hints**

You will need to decompose it in to partial fractions, where you'll hopefully find that

$$\frac{2x+6}{(3x+1)(x-1)} \equiv \frac{-4}{3x+1} + \frac{2}{x-1}.$$

You can do this partial fraction decomposition using whatever your favoured method is. The Heaviside Cover-Up Method is a fast way to do this one, but if you don't know this method, you can also do it the standard way of equating coefficients or substituting values etc. The integral of this is then $-\frac{4}{3}\ln|3x+1| + 2\ln|x-1| + c$ (don't forget the $+c$!). Here is a worked sample solution.

Worked Sample Solution

Note that using the *Heaviside Cover-Up Method*, we have

$$\frac{2x+6}{(3x+1)(x-1)} \equiv \frac{A}{3x+1} + \frac{B}{x-1},$$

where

$$\begin{aligned} A &= \frac{2 \times \left(-\frac{1}{3}\right) + 6}{-\frac{1}{3} - 1} \\ &= \frac{-\frac{2}{3} + 6}{-\frac{4}{3}} \times \frac{3}{3} \\ &= \frac{-2+18}{-4} \\ &= \frac{16}{-4} \\ &= -4, \end{aligned}$$

and

$$\begin{aligned} B &= \frac{2 \times 1 + 6}{3 \times 1 + 1} \\ &= \frac{2+6}{3+1} \\ &= \frac{8}{4} \\ &= 2. \end{aligned}$$

So

$$\begin{aligned} \int \frac{2x+6}{(3x+1)(x-1)} \, \mathrm{d}x &= \int \left(\frac{-4}{3x+1} + \frac{2}{x-1} \right) \mathrm{d}x \\ &= -\frac{4}{3} \ln|3x+1| + 2\ln|x-1| + c. \end{aligned}$$

Alternate Method for the Partial Fraction Decomposition

Here is another more standard way for doing the partial fraction decomposition, for those of you who don't know the Heaviside Cover-Up Method.

Let

$$\frac{2x+6}{(3x+1)(x-1)} \equiv \frac{A}{3x+1} + \frac{B}{x-1}.$$

Multiplying out the denominator of the L.H.S. gives us the identity

$$2x + 6 \equiv A(x-1) + B(3x+1). \qquad (\dagger)$$

Substitute $x = 1$:

$$
\begin{aligned}
2 + 6 &= 0 + B \times 4 \\
\Rightarrow 4B &= 8 \\
\Rightarrow B &= 2.
\end{aligned}
$$

Now equate coefficients of x on either side of the identity (†):

$$
\begin{aligned}
A + 3B &= 2 \\
\Rightarrow A + 6 &= 2 \quad (\text{as } B = 2) \\
\Rightarrow A &= -4.
\end{aligned}
$$

So

$$\frac{2x+6}{(3x+1)\,(x-1)} \equiv \frac{-4}{3x+1} + \frac{2}{x-1},$$

as before.

MATH1231/1241 Calculus S2 2013 Test 2 v2a

Hints & Worked Sample Solutions

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We cannot guarantee that our working is error-free, or that it would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are **NOT** in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

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1. **Hints**

   Complete the square. You should get the answer

$$\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c.$$

Worked Sample Solutions

Completing the square, we see that $x^2+4x+8=(x+2)^2+4$. Thus

$$\int \frac{1}{x^2+4x+8}\,\mathrm{d}x=\int \frac{1}{(x+2)^2+4}\,\mathrm{d}x$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c.$$

2. **Hints**

   Do the usual procedure for solving exact differential equations (refer to solutions to other exact ODE questions earlier in this document, such as the solution to question 2 on page 6 of this document). You should find the general solution is (up to a constant multiple)

$$\left(2x - \frac{1}{2}y - \frac{1}{4}\right)e^{2y} = C.$$

   Worked Sample Solutions

   The ODE is

$$I\,\mathrm{d}x + J\,\mathrm{d}y = 0,$$

   where

$$I = 2e^{2y} \quad \text{and} \quad J = (4x - y - 1)\,e^{2y}.$$

   We see that

$$\frac{\partial I}{\partial y} = 4e^{2y}$$

   and

$$J = 4xe^{2y} - (y+1)\,e^{2y} \Rightarrow \frac{\partial J}{\partial x} = 4e^{2y}.$$

   Hence $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$, so the ODE is exact. To find the general solution, we search for a function $F \equiv F(x,y)$ that satisfies

$$\frac{\partial F}{\partial x} = 2e^{2y} \tag{1}$$

$$\text{and} \quad \frac{\partial F}{\partial y} = (4x - y - 1)\,e^{2y} = 4xe^{2y} - (y+1)\,e^{2y}. \tag{2}$$

   Once we find such an F, the general solution will be given by $F(x,y) = C$ (for some constant C). Partially integrating (1) with respect to x, we find that

$$F = 2xe^{2y} + c(y),$$

   for some function $c(y)$ that does not depend on x. Now partially differentiating this with respect to y, we find that

$$\frac{\partial F}{\partial y} = 4xe^{2y} + c'(y).$$

   Comparing with (2), we see that we must have

$$c'(y) = -(y+1)\,e^{2y},$$

and so (integrating this with respect to y) it suffices to take

$$\begin{aligned}
c(y) &= \int -(y+1)e^{2y}\,\mathrm{d}y \\
&= -\frac{1}{2}(y+1)e^{2y} + \int \frac{1}{2}e^{2y}\,\mathrm{d}y && \text{(using integration by parts)} \\
&= -\frac{1}{2}(y+1)e^{2y} + \frac{1}{4}e^{2y} && \text{(don't need to worry about a ``}+C\text{'' here)} \\
&= e^{2y}\left(-\frac{1}{4} - \frac{1}{2}y\right).
\end{aligned}$$

Thus

$$F = 2xe^{2y} + c(y) = 2xe^{2y} + e^{2y}\left(-\frac{1}{4} - \frac{1}{2}y\right) = \left(2x - \frac{1}{2}y - \frac{1}{4}\right)e^{2y},$$

and the general solution to the ODE is

$$\left(2x - \frac{1}{2}y - \frac{1}{4}\right)e^{2y} = C.$$

3. **Hints**

(i) Note that if you find the roots take the form of $\lambda = a \pm ib$ (note that they will always be conjugates by the Conjugate Root Theorem), then your general solution to the homogeneous differential equation is

$$y(t) = e^{at}\left(A\sin bt + B\cos bt\right), \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

You will find the answer to this particular question is

$$y(t) = A\cos 2t + B\sin 2t, \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

(ii) Use the initial conditions to find the values of A and B. You will end up with a system of simultaneous equations to solve for A and B, which you can solve however you like (e.g. by inspection, row-reduction, high-school methods, etc.). You should end up finding that the solution is

$$y(t) = 2\cos 2t + 3\sin 2t.$$

(iii) Since there is already a $\cos 2t$ term in the solution to the homogeneous equation, and the roots of the characteristic equation are of multiplicity 1, we append a t to this.

$$y_p(t) = Ct\cos 2t, \text{ where } C \text{ is a constant.}$$

Worked Sample Solutions

(i) The characteristic equation is

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i.$$

Hence the general solution to the ODE is

$$y(t) = A\cos 2t + B\sin 2t.$$

(ii) The solution is of the form

$$y(t) = A\cos 2t + B\sin 2t.$$

Hence

$$y'(t) = -2A\sin 2t + 2B\cos 2t.$$

Therefore, remembering that $\cos 0 = 1$ and $\sin 0 = 0$, the initial condition $y(0) = 2$ gives $A = 2$. The initial condition $y'(0) = 6$ gives $2B = 6 \Rightarrow B = 3$. Thus the solution is

$$y(t) = 2\cos 2t + 3\sin 2t.$$

(iii) Normally we would try $C\cos 2t$. But since there is already a $\cos 2t$ term in the homogeneous solution, and the roots of the characteristic equation are of multiplicity 1, we would try a particular solution of the form

$$y_p(t) = Ct\cos 2t, \text{ where } C \text{ is a constant.}$$

Remark. Generally, for second order constant coefficient ODE's of the form

$$a_2y'' + a_1y' + a_0y = \cos at \quad (a > 0),$$

we would actually try $C_1\cos at + C_2\sin at$ as our particular solution (where $a = 2$ in this test question), assuming this is not a part of the homogeneous solution (if it is, we would try $t\,(C_1\cos at + C_2\sin at)$). However, *since the question's ODE has no y' term, i.e. is of the form*

$$y'' + \omega^2 y = \cos\omega t, \quad \omega > 0,$$

it can easily be shown that there necessarily exists a unique particular solution of the form $Ct\cos\omega t$ (and also that if the ODE were instead $y'' + \omega^2 y = \cos at$, where $a \neq \pm\omega$, then there exists a unique solution of the form $C\cos at$). That is, when the constant coefficient ODE has no y' term (only y'' and y terms), and just has a $\cos at$ term on the RHS, our particular solution does not need to include a $\sin at$ term. If you did not know this, you could just say for your answer to the question that we would try $y_p(t) = t\,(C_1 \cos 2t + C_2 \sin 2t)$. This is still correct, but you will end up finding that $C_2 = 0$ if you try to find the C_1 and C_2.