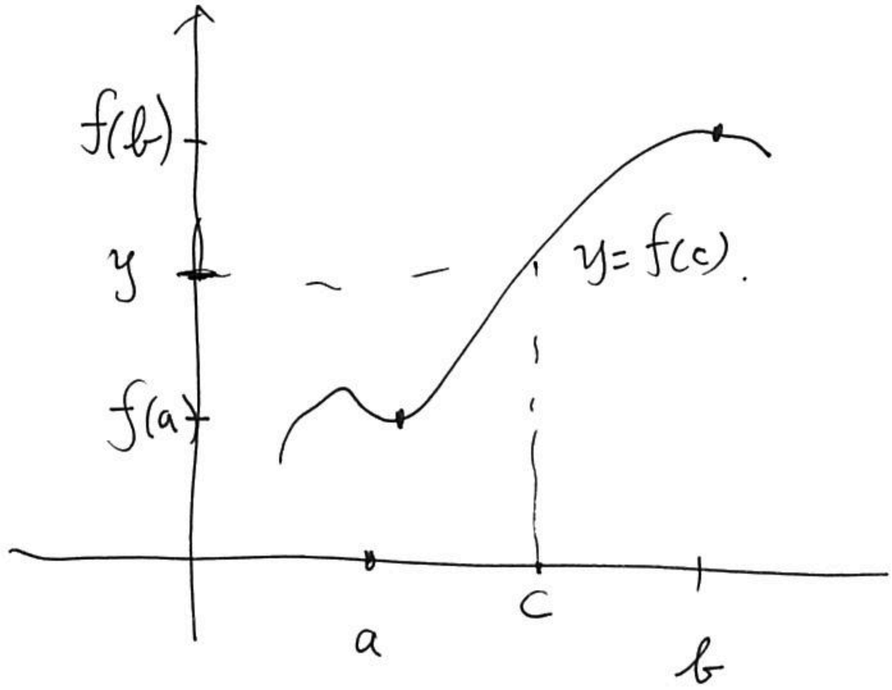


Theorem: (The intermediate value theorem)

Suppose f is continuous on a closed interval $[a, b]$. and y ~~is a~~ lies between $f(a)$ and $f(b)$ ($f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$).

Then there is $c \in [a, b]$ such that

$$f(c) = y.$$

Corollary: Suppose that f is continuous on a closed interval $[a, b]$. and $f(a)$ and $f(b)$ have opposite ~~sg~~ signs. ($f(a) f(b) < 0$).

Then there is $x \in [a, b]$, such that $f(x) = 0$.

example. $f(x) = x^3 - x - 1$. $[1, 2]$.

$f(1) = -1$, $f(2) = 5$

By the intermediate value theorem, there is $c \in [1, 2]$ such that

$$c^3 - c - 1 = 0.$$

$[1, 1.5]$

$f(1) < 0$, $f(1.5) > 0$.

$f(1.25) < 0$.

By IVT, f has a root in $[1.25, 1.5]$.

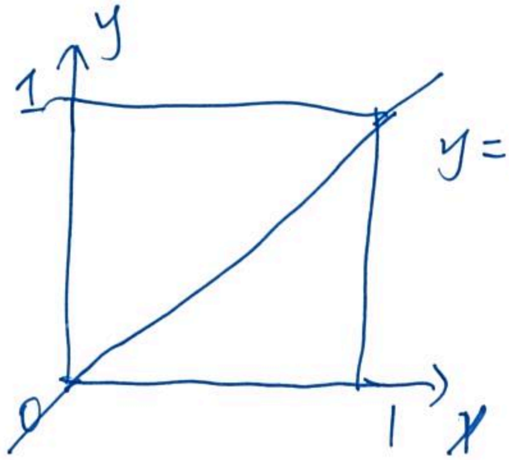
$f(1.324\ldots) = 0$.

Example: Suppose that f is a continuous function on $[0, 1]$. and that the range of f is also in the interval $[0, 1]$.

$$f: [0, 1] \to [0, 1].$$

Prove that there is $c \in [0, 1]$ Such that $f(c) = c$.

f has at least one "fixed point".



The graph of f must cross the graph of $y = x$ at least once.

---

proof: $f(c) = c \iff f(c) - c = 0$.

$\iff f(x) - x$ has a zero in $[0,1]$.

Let $g(x) = f(x) - x$.

$g(0) = f(0) - 0 = f(0) \geq 0$,

Since the range of f is in $[0,1]$.

$g(1) = f(1) - 1 \leq 1 - 1 = 0$.

We have $g(0) \geq 0$, $g(1) \leq 0$.

Therefore, by IVT, there is $c \in [0,1]$ such that $g(c) = 0$.

or equivalently,

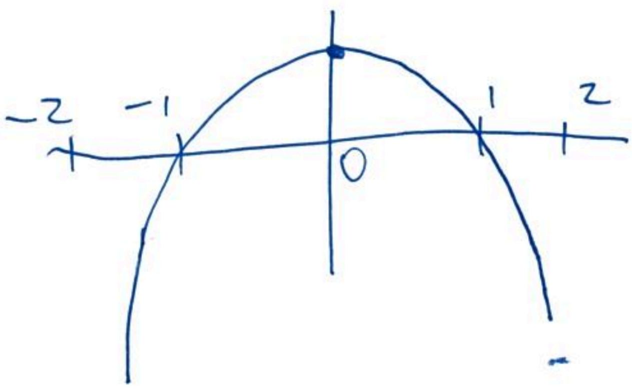
$$f(c) - c = 0. \quad \boxed{\checkmark}$$

---

max & min.

Suppose that f is a function defined on an interval $[a, b]$. We say that f has a global ~~ms~~ maximum at $x = c$, if ~~d~~ $f(c) \geq f(x)$ for all $x \in [a, b]$.

$$f(x) = 1 - x^2 \qquad [-2, 2]$$

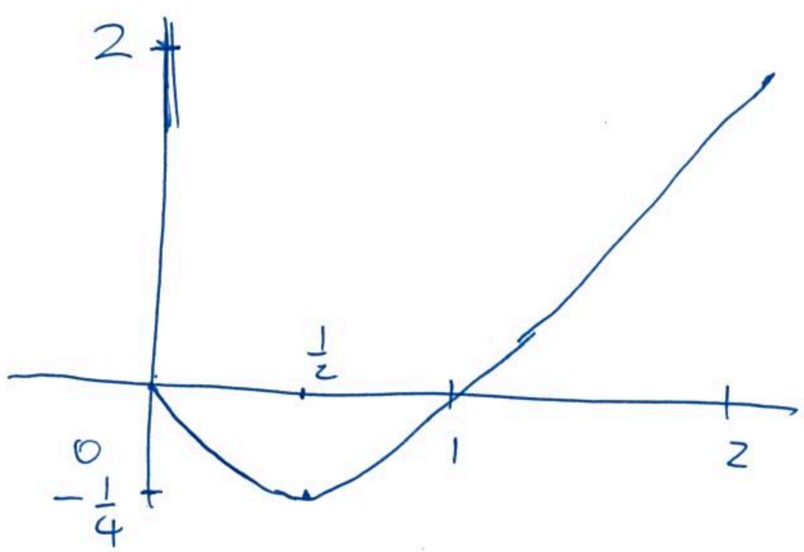


f has a global max on $[-2, 2]$ at $x = 0$.

Global minimum is defined similarly.

$$g(x) = x(x-1), \qquad [0,2].$$

$$= x^2 - x + \tfrac{1}{4} - \tfrac{1}{4} = \left(x-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}.$$



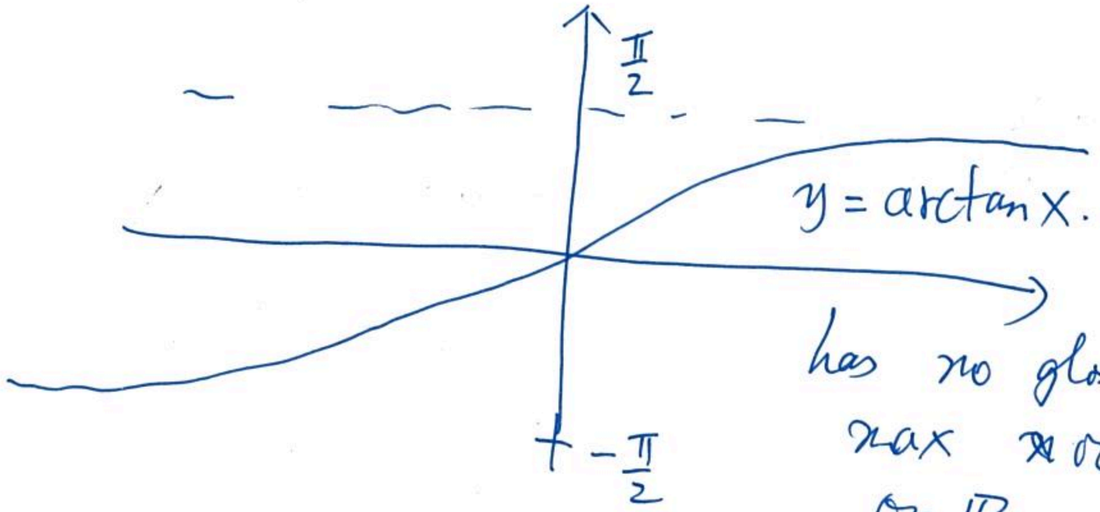
The global max of g is at $x=2$. $g(2)=2$

The global min of g is at $x=\frac{1}{2}$. $g(\frac{1}{2}) = -\frac{1}{4}$

Theorem: Suppose that f is continuous on a closed interval $[a,b]$. then f has a global max and a global min on $[a,b]$.

$\{ 1, \frac{1}{2}, \frac{1}{3} \cdots \}$.

max's & min's must be attained by f.



has no global max or min on $\mathbb{R}$.

If f is continuous on $[a, b]$, then there are $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$.

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

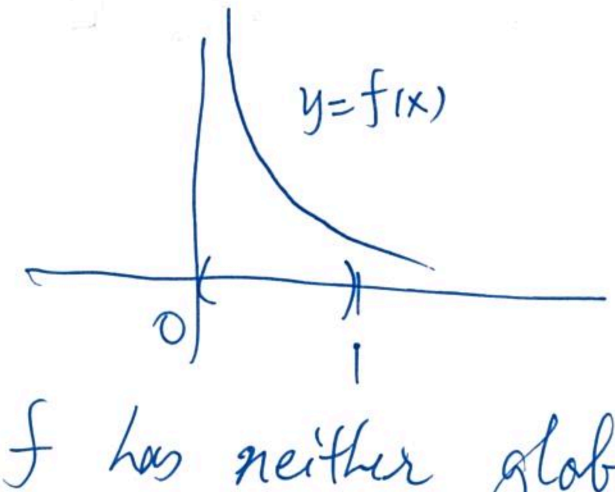
$[-1, 1]$.

f has neither global max nor min on $[-1,1]$.

Can't use theorem if f is not continuous.

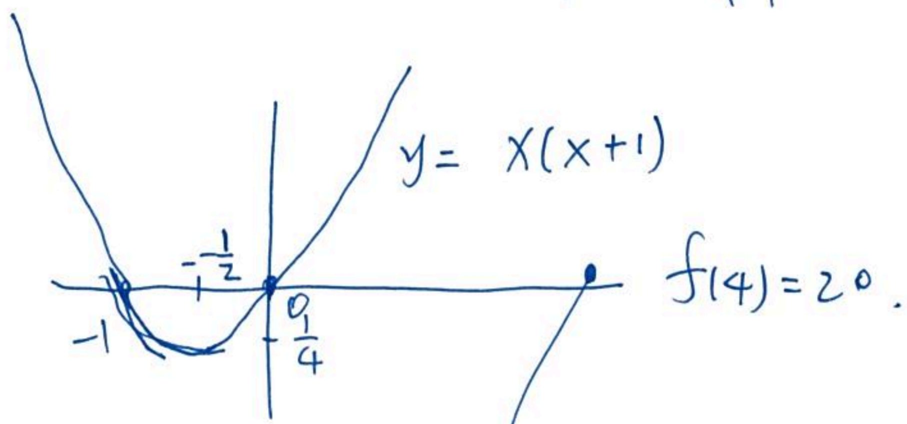
Note $f(x)$ on $(0, 1)$.

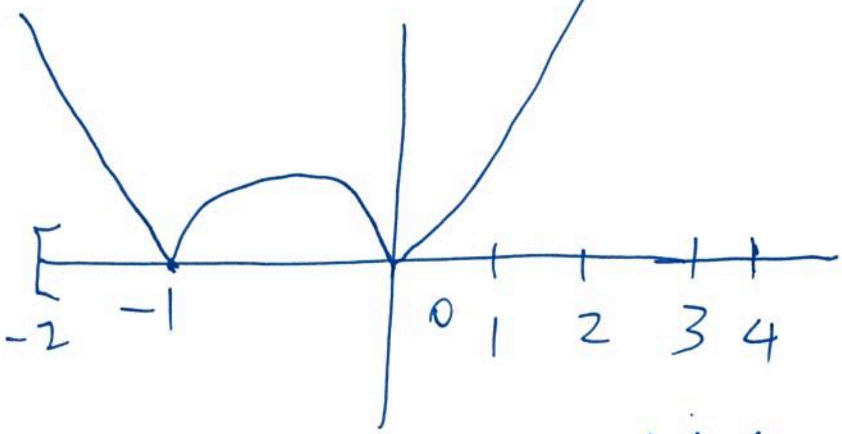
f is continuous on $(0,1)$.



f has neither global max nor global min on $(0,1)$.

$$f(x) = |x|\,|x+1|, \quad [-2,4].$$
$$= |x(x+1)|$$
$$= |x^2+x+\tfrac{1}{4}-\tfrac{1}{4}|$$
$$= |(x+\tfrac{1}{2})^2-\tfrac{1}{4}|$$

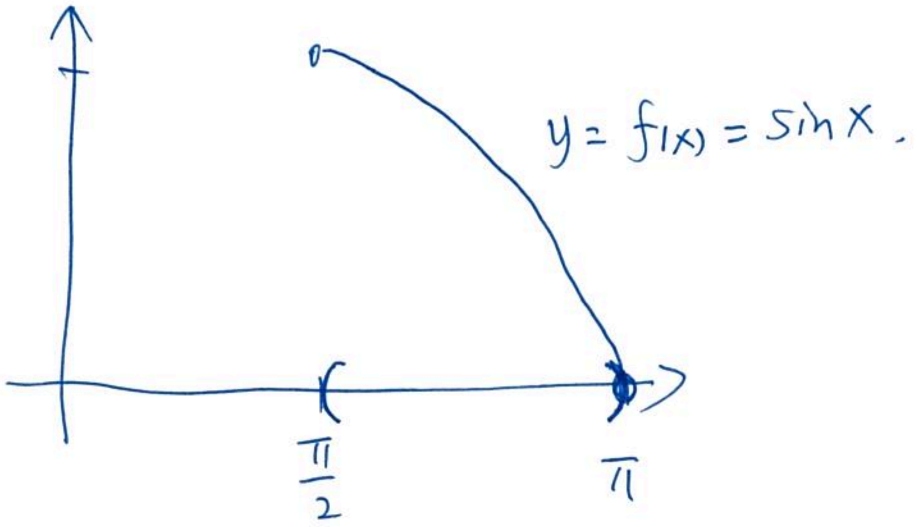




The global min of f on $[-2,4]$ occurs at $x=-1$ & $x=0$.

The global max of f on $[-2,4]$ occurs at $x=4$.

$f(x) = \sin x \qquad \left(\frac{\pi}{2}, \pi\right).$



f has no global max. on $\left(\frac{\pi}{2}, \pi\right)$.

f has no global min on $\left(\frac{\pi}{2}, \pi\right)$.

max & min of f on an interval need to be attained on that interval.