MATH1231/1241 Calculus S2 2009 Test 1 v7a

Full Solutions

August 18, 2017

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1. If $z = xy$, then we can use the chain rule to differentiate it, using following the chain diagram as a mnemonic:

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$
\begin{aligned}
&= (y)(-6\sin 3t) + x(12\cos 3t) \\
&= -6y\sin 3t + 12x\cos 3t \\
&= -6(4\sin 3t)\sin 3t + 12(2\cos 3t)\cos 3t && \text{(Substituting in } x \text{ and } y\text{)} \\
&= -24\sin^2 3t + 24\cos^2 3t
\end{aligned}
$$

2. One way to find the the normal vector is to treat z as $f(x, y)$ and remember that a normal is given by $\begin{pmatrix} \frac{\partial f}{\partial x}(-1,2) \\ \frac{\partial f}{\partial y}(-1,2) \\ -1 \end{pmatrix}$. So:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 6x \\
\frac{\partial f}{\partial y} &= -2y
\end{aligned}
$$

Hence a normal is $\begin{pmatrix} -6 \\ -4 \\ -1 \end{pmatrix}$. Let's make things simpler by using $\mathbf{n} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$ since this is also normal. Once we have found a normal vector, we can use the point-normal form of a plane, where $\mathbf{n}$ is a normal, and $\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ where $x, y, z \in \mathbb{R}$:

$$
\mathbf{n} \cdot \left(\mathbf{x} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \right) = 0
$$

$$
\begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \right) = 0
$$

$$
6x + 4y + z = -1
$$

3.

$$
\begin{aligned}
\int \sec^4\theta \, d\theta &= \int (\tan^2\theta + 1)\sec^2\theta \, d\theta \\
&= \int (u^2 + 1)\, du && \text{(If we let } u = \tan\theta \text{ and } du = \sec^2\theta \, d\theta\text{)} \\
&= \frac{u^3}{3} + u + C \\
&= \frac{\tan^3\theta}{3} + \tan\theta + C
\end{aligned}
$$

4. Since we have $4 - x^2$, the most fitting substitution is $x = 2\sin\theta$ because we foresee that it simplifies most easily.

   From this we get $dx = 2\cos\theta\; d\theta$. So ensuring that we change the limits:

$$\int_0^{\pi/2} \sqrt{4 - 4\sin^2\theta}(2\cos\theta)d\theta = 4\int_0^{\pi/2} \cos^2\theta d\theta$$

   Now we could do this using double angle formulae (i.e. recalling that $\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right)$), but let's have some fun and use some definite integral properties:

$$I = \int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - \theta\right)d\theta = \int_0^{\pi/2} \sin^2\theta d\theta,$$

   where we used the properties $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$ and $\cos(\frac{\pi}{2} - x) = \sin(x)$. Hence, we can add the two different expression for I:

$$2I = \int_0^{\pi/2} (\cos^2\theta + \sin^2\theta)d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

   So

$$I = \int_0^{\pi/2} \cos^2\theta d\theta = \frac{\pi}{4}$$

   So our original integral is equal to $4 \times \frac{\pi}{4} = \pi$.

   Note: We could also recognise that the function is a semicircle, and work this out geometrically – it is just the area of a quadrant of a circle of radius 2 which is equal to $\frac{\pi(2^2)}{4} = \pi$. But the question specifies to use a trigonometric substitution, so just use this fact to check your answer.

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1. The answer is

$$\frac{\partial z}{\partial y} = 2xye^{xy^2}.$$

Note: A common mistake is missing the x in the coefficient, don't forget that! We treat x as a constant.

2. We will calculate the partial derivatives $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ then check if they satisfy the appropriate partial differential equation. And so,

$$\frac{\partial z}{\partial x} = -\sin(x+2y) \implies \frac{\partial^2 z}{\partial x^2} = -\cos(x+2y)$$

and

$$\frac{\partial z}{\partial y} = -2\sin(x+2y) \implies \frac{\partial^2 z}{\partial y^2} = -4\cos(x+2y).$$

Alternatively: Note that these second derivatives could also be obtained very quickly by recalling that if $u = A\sin(\omega t + \varphi)$ for constants A, ω, φ, then

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -\omega^2 u.$$

This was an important result from the Simple Harmonic Motion topic in HSC.

Substituting this into the given partial differential equation gives

$$4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 4 \times -\cos(x+2y) - (-4\cos(x+2y)) = 0,$$

as required.

3. As z is a function of x and y and as x and y are both functions of t by the chain rule we have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= 4t\frac{\partial z}{\partial x} + 3t^2\frac{\partial z}{\partial y}.\end{aligned}$$

Now as

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x \\ &= 4t^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial y} &= 6y \\ &= 6t^3\end{aligned}$$

we have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= 4t \times 4t^2 + 3t^2 \times 6t^3 \\ &= 16t^3 + 18t^5.\end{aligned}$$

4. The vector $\mathbf{n}$ is given by $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)^T$ evaluated at $(-1,2,3)^T$. Firstly, we calculate the

partial derivatives $\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial x}$. We find that

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{4y}{2\sqrt{x^2+2y^2}}\\ &= \frac{2y}{\sqrt{x^2+2y^2}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{2x}{2\sqrt{x^2+2y^2}}\\ &= \frac{x}{\sqrt{x^2+2y^2}}.\end{aligned}$$

Hence, by substituting in the point $(-1,2,3)^T$ we find that the required vector is

$$\mathbf{n} = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{(-1)^2+2(2)^2}} \\ \frac{2\times 2}{\sqrt{(-1)^2+2(2)^2}} \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ -1 \end{pmatrix}.$$

To find the tangent plane at this point, we use the point-normal form of a plane, i.e.

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{x_1}) = 0$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, $\mathbf{x_1}$ is a point on the plane, and $\mathbf{n}$ is the normal to the plane. We use a scalar multiple of $\mathbf{n}$ to make things easier, so we use $\mathbf{n} = (-1,4,-3)^T$, and let $\mathbf{x_1} = (-1,2,3)^T$. Substituting these values in and simplifying,

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x_1}) &= 0 \\ \implies \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right) &= 0 \\ (-x+4y-3z)-(1+8-9) &= 0 \\ x-4y+3z &= 0. \end{aligned}$$

5. We make the substitution $x = 2\tan\theta$. Then

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec^2\theta$$

and

$$x = 0 \implies \theta = 0, \qquad x = 2 \implies \theta = \frac{\pi}{4}.$$

Hence we have

$$\begin{aligned}
\int_0^2 \frac{1}{(4+x^2)^{3/2}}\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{(4+4\tan^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{8(1+\tan^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{4}[\sin\theta]_0^{\frac{\pi}{4}} \\
&= \frac{1}{4\sqrt{2}}.
\end{aligned}$$

MATH1231/1241 Calculus S2 2011 Test 1 v1b

Answers & Worked Sample Solutions

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1. **Answer**

   $\frac{\mathrm{d}z}{\mathrm{d}t} = 8t^3 + 96t^5$

   Worked Sample Solution

   Let $f(x, y) = 2x^2 + y^2$, so $z = f(x, y)$. Then $\frac{\partial f}{\partial x} = 4x$ and $\frac{\partial f}{\partial y} = 2y$. Also, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 12t^2$. So by the multivariate chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= 4x \times 2t + 2y \times 12t^2 \\ &= 4t^2 \times 2t + 2 \times 4t^3 \times 12t^2 \quad \text{(using the given expressions for } x, y \text{ as functions of } t\text{)}\end{aligned}$$

$$= 8t^3 + 96t^5.$$

2. **Answer**

$$\mathbf{n} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}, \qquad 5x - 3y - z = 0$$

Worked Sample Solution

Recall that if $z = f(x, y)$ is a smooth surface, then a normal to the surface at a point $\mathbf{a}$ on the surface is $\mathbf{n} = \begin{pmatrix} f_x \\ f_y \\ -1 \end{pmatrix}$, where the partial derivatives are evaluated at the x and y values of the point $\mathbf{a}$. In this particular question, $f(x, y) = x^2y - 2y^2 + 3x$, and the point is $\mathbf{a} = (1, 1, 2)$ (so we evaluate the partials at $x = 1, y = 1$). So $f_x(x, y) = 2xy + 3 \Rightarrow f_x(1, 1) = 2 + 3 = 5$. Also, $f_y(x, y) = x^2 - 4y \Rightarrow f_y(1, 1) = 1 - 4 = -3$. So a normal to the surface is $\mathbf{n} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$.

An equation for the tangent plane can be found using the point-normal form

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a},$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Since $\mathbf{n} \cdot \mathbf{a} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 5 - 3 - 2 = 0$, the plane has equation $5x - 3y - z = 0$. (Remember, the equation of a plane or its normal is not unique; you could have ended up with a non-zero scalar multiple of the normal given in this answer and it would still be correct. But if the normal you obtained is *not* a scalar multiple of the one in this answer, it is not correct.)

3. **Answer**

$$\int_0^3 \frac{1}{(9 + x^2)^{\frac{3}{2}}} \, \mathrm{d}x = \frac{1}{9\sqrt{2}}$$

Worked Sample Solution

Let

$$I = \int_0^3 \frac{1}{(9 + x^2)^{\frac{3}{2}}} \, \mathrm{d}x.$$

Substitute $x = 3\tan\theta$, $\theta \in \left[0, \frac{\pi}{4}\right]$. So $\mathrm{d}x = 3\sec^2\theta \, \mathrm{d}\theta$, and $9 + x^2 = 9\left(1 + \tan^2\theta\right) = 9\sec^2\theta \Rightarrow \left(9 + x^2\right)^{\frac{3}{2}} = \left(9\sec^2\theta\right)^{\frac{3}{2}} = 3^3\sec^3\theta$. When $x = 0$, $\theta = 0$, and when $x = 3$, $\theta = \frac{\pi}{4}$.

So

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \frac{3\sec^2\theta}{3^3\sec^3\theta}\,\mathrm{d}\theta \\
&= \frac{1}{3^2}\int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta}\,\mathrm{d}\theta \\
&= \frac{1}{9}\int_0^{\frac{\pi}{4}} \cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{9}\left(\sin\frac{\pi}{4} - \sin 0\right) \quad \text{(Fundamental Theorem of Calculus)} \\
&= \frac{1}{9}\left(\frac{1}{\sqrt{2}} - 0\right) \\
&= \frac{1}{9\sqrt{2}}.
\end{aligned}$$

4. By simple calculus (differentiation of a logarithm and chain rule),

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}.$$

5. **Worked Sample Solution**

Since[1]

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x+2y)$$

and

$$\frac{\partial^2 z}{\partial y^2} = -4\sin(x+2y),$$

we have

$$\frac{\partial^2 z}{\partial y^2} = 4\frac{\partial^2 z}{\partial x^2} \Rightarrow 4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0,$$

and the claim is proved.

[1]These second derivatives can be obtained very quickly by recalling that if $u = A\sin(\omega t + \varphi)$ for constants A, ω, φ, then

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -\omega^2 u.$$

This was an important result from the Simple Harmonic Motion topic in HSC.

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1. We find

$$
\begin{aligned}
z &= e^{x^2y^3} \\
\implies \frac{\partial z}{\partial x} &= 2xy^3e^{x^2y^3} \\
\implies \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(2xy^3e^{x^2y^3}\right) = 2x\frac{\partial}{\partial y}\left(y^3e^{x^2y^3}\right) \\
&= 2x\left[y^3\left(3x^2y^2e^{x^2y^3}\right) + 3y^2\left(e^{x^2y^3}\right)\right] \quad \text{(product rule)}
\end{aligned}
$$

$$= 6xy^2e^{x^2y^3}\left(x^2y^3+1\right) \quad \text{(cleaning up).}$$

2. Recall that using the *total differential approximation*, the maximum error is given by

$$|\Delta z|_{\text{max.}} \approx \left|\frac{\partial z}{\partial x}\right| |\Delta x| + \left|\frac{\partial z}{\partial y}\right| |\Delta y| \qquad (*)$$

where the partial derivatives are evaluated at the measured values of x and y.
Note $\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x}} \times \frac{1}{y+1}$ and $\frac{\partial z}{\partial y} = -\frac{\sqrt{x}+1}{(y+1)^2}$.
So, at $x = 9, y = 1$, we have $\frac{\partial z}{\partial x} = \frac{1}{2\times 3} \times \frac{1}{2} = \frac{1}{12}$ and $\frac{\partial z}{\partial y} = -\frac{3+1}{2^2} = -\frac{4}{4} = -1$.
It is also given the measurement of each x and y have an error with absolute value at most 0.03 cm. This implies that $\Delta x = \Delta y = 0.03$ for the formula $(*)$.
Hence, the maximum error in the calculated value of z is found by the following:

$$\begin{aligned}
|\Delta z|_{\text{max.}} &\approx \left|\frac{\partial z}{\partial x}\right| |\Delta x| + \left|\frac{\partial z}{\partial y}\right| |\Delta y| \\
&= \left|\frac{1}{12}\right| |0.03| + |-1|\,|0.03| \\
&= \frac{1}{12} \times 0.03 + 1 \times 0.03 \\
&= \frac{13}{12} \times 0.03 = \frac{13}{12} \times \frac{3}{100} \\
&= 0.0325 \text{ cm} = \frac{13}{400} \text{ cm.}
\end{aligned}$$

That is, the maximum error in the calculated value of z is 0.0325 cm $= \frac{13}{400}$ cm.
Tip. Since in the class tests you are not allowed a calculator, it should be fine to leave the answer in fraction form here (which is probably slightly easier/faster to obtain without a calculator than the decimal form). Here is a way to calculate the decimal form from the fraction without a calculator if you wanted to give the answer as a decimal.
We want to express $\frac{13}{400}$ as a decimal. Let's express $\frac{13}{4}$ as a decimal, and then divide by 100 (i.e. shift the decimal point two spaces left). Note that $\frac{13}{4} = 3\frac{1}{4} = 3.25$. So shifting the decimal point two spaces left, we have $\frac{13}{400} = 0.0325$.

3. Note that $\sin^3\theta\cos^2\theta = \sin\theta\sin^2\theta\cos^2\theta = \left(1-\cos^2\theta\right)\cdot\cos^2\sin\theta$. Also $\sin\theta\,\mathrm{d}\theta = -\mathrm{d}\left(\cos\theta\right)$. Since $\cos 0 = 1$ and $\cos\frac{\pi}{2} = 0$, we obtain using the substitution $u = \cos\theta$:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,\mathrm{d}\theta &= -\int_1^0 \left(1-u^2\right)u^2\,\mathrm{d}u && \text{(where we substituted } u = \cos\theta) \\
&= \int_0^1 u^2 - u^4\,\mathrm{d}u && \left(\text{as } -\textstyle\int_a^b f(x)\,\mathrm{d}x = \int_b^a f(x)\,\mathrm{d}x\right) \\
&= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 \\
&= \frac{1}{3} - \frac{1}{5}
\end{aligned}$$

$$= \frac{2}{15}.$$

Alternatively, a slightly quicker (but equivalent) method is to just skip the substitution and do steps in your head:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta \,\mathrm{d}\theta &= -\int_0^{\frac{\pi}{2}} \left(1-\cos^2\theta\right)\cos^2\theta \,\mathrm{d}\left(\cos\theta\right) \\
&= -\int_0^{\frac{\pi}{2}} \left(\cos^2\theta - \cos^4\theta\right)\mathrm{d}\left(\cos\theta\right) \\
&= \frac{1}{3} - \frac{1}{5} \qquad (*) \\
&= \frac{2}{15}.
\end{aligned}$$

Remarks/Tips. To get line $(*)$, we essentially did a mental substitution of $u = \cos\theta$ in the previous line and noted that this becomes integrating powers of u from 0 to 1, and used the fact that $\int_0^1 u^n \,\mathrm{d}u = \frac{1}{n+1}$ for $n > 0$ (in fact $n > -1$). Also, since the integrand of the original integral is positive throughout the domain of integration, we know the final answer must be positive, so we could say it had to be $\frac{1}{3} - \frac{1}{5}$ rather than $\frac{1}{5} - \frac{1}{3}$ without needing to think about how the minus sign and the bounds of the integral affected anything. If you feel uncomfortable doing so many steps in your head, feel free to use the first method with all steps shown.

4. Let $I_n = \int_0^1 \left(1-x^2\right)^n \,\mathrm{d}x$.
Then, we have $I_n = \frac{2n}{2n+1} I_{n-1}$ from the given reduction formula.
Solving the given integral,

$$\begin{aligned}
\int_0^1 \left(1-x^2\right)^4 \,\mathrm{d}x &= I_4 \\
&= \frac{2\,(4)}{2\,(4)+1} I_3 \\
&= \frac{8}{9}\cdot\frac{2\,(3)}{2\,(3)+1} I_2 \\
&= \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{2\,(2)}{2\,(2)+1} I_1 \\
&= \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2\,(1)}{2\,(1)+1}\cdot I_0 \\
&= \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot I_0.
\end{aligned}$$

As $I_0 = \int_0^1 1 \,\mathrm{d}x = 1$, we have

$$\int_0^1 \left(1-x^2\right)^4 \,\mathrm{d}x = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

$$= \frac{8 \times 2 \times 4 \times 2}{9 \times 7 \times 5}$$

(simplifying the blue factors to help since we do not have a calculator)

$$= \frac{128}{315}.$$

(no more common factors above, so just calculate numerator and denominator)