UNSW AUSTRALIA

Never Stand Still | Science | School of Mathematics and Statistics

Course Outline

MATH1081

Discrete Mathematics

School of Mathematics and Statistics

Faculty of Science

Semester 2, 2016

Contents

1. Staff

Position	Name	Email	Contact Details
Course Authority	Dr Jonathan Kress	j.kress@unsw.edu.au	Location: RC-3073
Lecturer	Professor James Franklin	j.franklin@unsw.edu.au	Location: RC-6109
Lecturer	Mr Peter Brown	peter@unsw.edu.au	Location: RC-5106

Staff consultation times are provided on Moodle and in the School of Mathematics and Statistics website for current students, undergraduate, student services, help for students page, at the beginning of each semester.

Tutor details for all courses will be provided at the start of semester in Moodle.

2. Administrative matters

Contacting the Student Services Office

Please visit the School of Mathematics and Statistics web-site for a wide range of information on School Policies, Forms and Help for Students by visiting the "**Student Services"** page.

For information on Courses, please go to "Current Student", "Undergraduate and/or Postgraduate" "**Courses Homepage**" for information on all **course offerings.**

The "Student Notice Board" can be located by going to the "Current Students" page; Notices are posted regularly for your information here. Please familiarise yourself with the information found in these locations. The School web page is found: http://www.maths.unsw.edu.au

If you cannot find the answer to your queries on the web pages you are welcome to contact the Student Services Office directly. The First Year Advisor in the Student Services Office is Mrs Markie Lugton. All administrative enquiries concerning first year Mathematics courses should be sent to M Lugton, either:

- By email to fy.mathsstats@unsw.edu.au
- By phone: 9385 7011
- Or in person to the Red Centre building, level 3, room 3072 (between 9am to 12pm OR 2pm to 4pm)

Change of tutorials, due to timetable clashes or work commitments, permission to take class tests outside your scheduled tutorial, advice on course selection and other administrative matters are handled in the Student Services Office. Constructive comments on course improvement may also be emailed to the Director of First Year Mathematics, Dr Jonathan Kress. Should we need to contact you, we will use your official UNSW email address of Zstudentno@student.unsw.edu.au in the first instance. **It is your responsibility to regularly check your university email account. Please state your student number in all emails to the Student Services Office.**

3. Course information

Units of credit: 6

Pre-requisite(s): The assumed knowledge for this course is equivalent of a combined mark of at least 100 in the HSC Mathematics and HSC Mathematics Extension 1.

Co-requisite: The formal co-requisite is MATH1131 or MATH1141 or MATH1151.

Teaching times and locations: see the link on the Handbook web page:

Timetable for course MATH1081:
http://www.handbook.unsw.edu.au/undergraduate/courses/2016/MATH1081.html

The subject matter of this course is very different from “high school mathematics” and success at high school is no guarantee of success in Discrete Mathematics. In Math1081 emphasis is placed on reasoned argument and clarity of exposition as well as algebraic and computational skills.

Course summary

MATH1081 will enhance your research, inquiry and analytical thinking abilities as it will provide you with the mathematical language and mathematical techniques to unravel many seemingly unrelated problems. The course will engage you in independent and reflective learning through your independent mastery of a wide range of tutorial problems. The mathematical problem solving skills that you will develop are generic problem solving skills, based on logical arguments and mathematical language that can be applied in multidisciplinary work. You will be encouraged to develop your communication skills through active participation in tutorials, and by writing clear, logical arguments when solving problems.

Course aims

The aim of MATH1081 is that by the time you finish the course you should understand the concepts and techniques covered by the syllabus and have developed skills in applying these concepts and techniques to the solution of appropriate problems. Successful completion of the course will give you a good foundation for understanding many problems that arise in computer science.

Course learning outcomes (CLO)

At the successful completion of this course you (the student) should be able to:

- State definitions as specified in the syllabus.
- State and prove appropriate theorems.
- Explain how a theorem applies to specific examples.
- Apply the concepts and techniques of the syllabus to solve appropriate problems.
- Understand and apply appropriate algorithms.
- Use mathematics and other terminology appropriately to communicate information and understanding.

4. Learning and teaching activities

Lecturers in Charge

The Course Authority for MATH1081 is the Director for First Year Mathematics, Dr Jonathan Kress.
Contact details E: j.kress@unsw.edu.au – room RC-3073.

The Lecturers are:

Professor James Franklin
Contact details E: j.frankline@unsw.edu.au – room RC-6109.
Mr Peter Brown
Contact details E: peter@unsw.edu.au – room RC-5106.

Lecture and Tutorial Schedule

	Monday	Tuesday	Wednesday	Thurs	Friday
Lectures	12-1pm **K. Burrows Theatre** Brown/Franklin	11-12pm **K. Burrows Theatre** Brown/Franklin	1-2pm **Physics Theatre** Brown/Franklin		12-1pm **Physics Theatre** Brown/Franklin
Note: Students must enroll in **pairs of tutorials** as follows: **Group A:** Monday 10-11am & Wednesday 2-3pm **Group B:** Monday 4-5pm & Friday 2-3pm **Group C:** Tuesday 9-10am & Wednesday 4-5pm **Group D:** Tuesday 12-1pm & Thursday 5-6pm					

Lectures are given four times per week, commencing in week 1 and running to week 12. Full details of the timetable are shown in your timetable on myUNSW and the online Handbook.

The material presented is divided into five sections, and each part will be presented in 2 or 3 week segments as follows:

Section	(1)	(2)	(3)	(4)	(5)
Weeks	1-2	3-4	5-7	8-10	11-12

Tutorials

Each student enrolled in MATH1081 has been assigned two tutorial time slots as shown in your timetable. Students are able to change their tutorials via myUNSW until the end of week 1. After that time, they can only change tutorials by going to the Student Services Office, Red Centre Building room RC-3072 with evidence of a timetable clash or work commitments. NB: Classroom tutorials commence in week 2 and run until week 13.

Each student will have two tutorials per week with the same tutor, with tutorials starting in week 2 and running until week 13. Attendance at tutorials is compulsory and the roll will be called in tutorials.

UNSW Moodle

The School of Mathematics and Statistics uses the Learning Management System called Moodle. To log into Moodle, use your zID and zPass at the following URL: http://moodle.telt.unsw.edu.au

Here you will find announcements, general information, notes, lecture slide, classroom tutorial and homework problems and links to online tutorial and assessments

Assessment overview

The final mark will be made up as follows:

Class tests	20%
Final examination	80%

Duration of the final exam is 2 hours.

Note:

- You will be able to view your final exam timetable once Exams Central has finalised the timetable. Please visit the web page: https://my.unsw.edu.au/student.unsw.edu.au/exams for details.
- It is very important that you understand the University's rules for the conduct of Examinations and the penalties for **Academic Misconduct Guide**. This information can be accessed through myUNSW at: https://student.unsw.edu.au/exams NB: In recent years there have been cases where severe penalties have been imposed for misconduct in relation to tests and exams in Maths courses.
- Assessment criteria: UNSW assesses students under a standards based assessment policy. For how this policy is applied within the School of Mathematics and Statistics, please visit the web site: http://www.maths.unsw.edu.au/currentstudents/assessment-policies
- If you are unwell / miss your **final examination**, please refer to the Special Consideration Policy by visiting the website: https://student.unsw.edu.au/special-consideration
- As from S1, 2016 students with a **final mark in the range of 45-49** will be permitted to take the Additional Assessment Exam as a Concessional Additional Assessment (AA). There will be no notification to the individual student of the right to take the Concessional AA, but the details of the courses AA exam schedule will be provided on the School's website Notice Board, after the Provisional Results are published (normally 1 week after the exam period ends).

  The final mark after completing the Concessional AA will not increase to a mark higher than 50. Refer to the School Notice Board website: http://www.maths.unsw.edu.au/currentstudents/current-students

Class tests

There will be one test for each of the first four sections outlined in the syllabus above. The best three will count for assessment. Tests for sections (1), (2), (3) and (4) will be held at the beginning of the first tutorial of the weeks given below:

Section	(1)	(2)	(3)	(4)
Week	4	6	9	12

Note:

- You **MUST TAKE EACH TEST IN THE CLASSROOM TUTORIAL TO WHICH YOU HAVE BEEN OFFICIAL ALLOCATED**.
- To each test you **must bring** your **Student ID card**, some blank **A4 writing paper** and a **Stapler** (so that you can staple a cover sheet to your answers).
- Normal exam conditions apply in tests.
- You will **not** be allowed to use a calculator in class tests.
- Your **best three scores** in the four tests will be counted towards your final assessments mark.
- If you miss a **class test** due to illness, please **DO NOT** apply for Special Consideration on-line. You should provide your tutor with a medical certificate in the following week so that your absence can be converted to an M grade to signify a "Medical Absence". The M grade is converted to a mark that is the average result from completed tests, and is recorded towards the end of semester. No more than two "M's" will be accepted in any one semester.
- Tutors are expected to enter class test marks within a fortnight of the test being taken. Your mark will be accessible via the "Maths & Stats Marks" link on the MATH1081 Moodle homepage.
- It is your responsibility to check that these marks are correct and you should keep marked tests until the end of semester in case an error has been made in recording the marks. If there is an error, either speak to your tutor or bring your test paper to the Student Services Office as soon as possible, but no later than Friday in Week 13.

Calculator Information

For end of semester UNSW exams, students must supply their own calculator. Only calculators on the UNSW list of approved calculators may be used in the end of semester exams. Before the exam period, calculators must be given a "UNSW approved" sticker, obtained from the School of Mathematics and Statistics Office, and other student or Faculty centres. The UNSW list of calculators approved for use in end of semester exams is available at: https://student.unsw.edu.au/exams

5. Expectations of students

School Policies

The School of Mathematics and Statistics has adopted a number of policies relating to enrolment, attendance, assessment, plagiarism, cheating, special consideration etc. These are in addition to the Policies of The University of New South Wales. Individual courses may also adopt other policies in addition to or replacing some of the School ones. These will be clearly notified in the Course Initial Handout and on the Course Home Pages on the Maths Stats web site.

Students in courses run by the School of Mathematics and Statistics should be aware of the School and Course policies by reading the appropriate pages on the Maths Stats web site starting at:

http://www.maths.unsw.edu.au/currentstudents/assessment-policies

The School of Mathematics and Statistics will assume that all its students have read and understood the School policies on the above pages and any individual course policies on the Course Initial Handout and Course Home Page. Lack of knowledge about a policy will not be an excuse for failing to follow the procedure in it.

6. Academic integrity, referencing and plagiarism

Academic integrity is fundamental to success at university. Academic integrity can be defined as a commitment to six fundamental values in academic pursuits**:** honesty, trust, fairness, respect, responsibility and courage.[1] At UNSW, this means that your work must be your own, and others' ideas should be appropriately acknowledged. If you don't follow these rules, plagiarism may be detected in your work.

Further information about academic integrity and **plagiarism** can be located at:

- The *Current Students* site https://student.unsw.edu.au/plagiarism**,** and
- The *ELISE* training site http://subjectguides.library.unsw.edu.au/elise/presenting

The *Conduct and Integrity Unit* provides further resources to assist you to understand your conduct obligations as a student: https://student.unsw.edu.au/conduct.

7. Readings and resources

Text Book

S.S. Epp, "Discrete Mathematics with Applications", Fourth Edition, 2011 OR Second (or Third) Edition, PWS 1995.

J Franklin and A. Daoud, "Introduction to Proofs in Mathematics", Prentice Hall, 1988 or "Proof in Mathematics: An Introduction", Quakers Hill Press, 1995.

Reference Books

Any book with "Discrete Mathematics" and many with "Finite Mathematics" in their title should help. Previous texts include K.H. Rosen "Discrete Mathematics and its Application" and K. Kalmanson, "An Introduction to Discrete Mathematics and its Applications". A more advanced reference is "Discrete Mathematics" by K. Ross and C.R.B. Wright.

For interesting applications within Computer Science, try the three part classic – D.E.Knuth, "The Art of Computer Programming".

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[1] International Center for Academic Integrity, 'The Fundamental Values of Academic Integrity', T. Fishman (ed), Clemson University, 2013.

Getting help outside tutorials

Staff Consultations

From week 3 there will be a roster which shows for each hour of the week a list of names of members of staff who are available to help students in the first year mathematics courses, no appointment is necessary. This roster is displayed on the same Notice Board as timetables, near the School Office (room 3070, Red Centre), it is also available from the web page: http://www.maths.unsw.edu.au/currentstudents/consultation-mathematics-staff

Student Support Scheme

The Student Support Scheme (SSS) is a drop-in consultation centre where students can come for free help with certain first and second year mathematics courses. The SSS office is located in RC-3064, and opening times during semester is from 10am – 12pm and 1pm to 3pm from Mondays to Fridays. The SSS services in semester 2 will be MATH1011, MATH1041, MATH1131 and MATH1231. The schedule will be available on the SSS website: http://www.maths.unsw.edu.au/currentstudents/student-support-scheme by the end of week 1. Please note that no appointment is necessary, this is a drop in arrangement to obtain one-on-one help from SSS tutors.

8. Additional support for students

- The Current Students Gateway: https://student.unsw.edu.au/
- Academic Skills and Support: https://student.unsw.edu.au/academic-skills
- Student Wellbeing, Health and Safety: https://student.unsw.edu.au/wellbeing
- Disability Support Services: https://student.unsw.edu.au/disability-services
- UNSW IT Service Centre: https://www.it.unsw.edu.au/students/index.html

Applications for Special Consideration

If you feel that your performance in, or attendance at a final examination has been affected by illness or circumstances beyond your control, or if you missed the examination because of illness or other compelling reasons, you may apply for special consideration. Such an application may lead to the granting of Additional Assessment.

1. Within 3 days of the affected examination, or at least as soon as possible, you must submit a request for Special Consideration to UNSW Student Central **ON-LINE** with supporting documentation attached.

Visit website to Apply for Special Consideration: https://student.unsw.edu.au/special-consideration

2. Please do not expect an immediate response from the School. All applications will be considered together. See the information below.

3. If you miss a **class test** due to illness or other problems, then you should provide the appropriate documentation to your tutor who will record an M. No more than two "M's" will be accepted in any one semester. **DO NOT apply on-line** for Special Consideration for class tests or for on-line or computing tests.

4. If your course involves a MAPLE/MATLAB lab test which you missed, you should contact the lecturer in charge of computing as soon as possible. A resit will be organised for later in the session.

5. **You will NOT be granted Additional Assessment in a course if your performance in the course** (judged by attendance, class tests, assignments and examinations) **does not meet a minimal standard**. A total mark of greater than 40% on all assessment not affected by a request for Special Consideration will normally be regarded as the minimal standard for award of Additional Assessment.

6. It is YOUR RESPONSIBILITY to find out from the School of Mathematics and Statistics, whether you have been granted Additional Assessment and when and where the additional assessment examinations will be held. Do NOT wait to receive official results from the university, as these results are not normally available until after the Mathematics Additional Assessment Exams have started.

Information about award of Additional Assessment and a provisional list of results will be made available on the Maths & Stats Marks page late on Friday 25th of November. A link to the Maths & Stats Marks page is provided on Moodle.

7. **Additional Assessment exam will be on Monday 5th or Tuesday 6th of December.** A link to the Additional Assessment timetable, including locations, will be placed on the Current Students Notice Board under heading "Special Consideration and Additional Assessment" information.

Web link: http://www.maths.unsw.edu.au/currentstudents/current-students

8. If you have two Additional Assessment examinations scheduled for the same time, please consult the Student Services Office either by email or phone (fy.mathsstats@unsw.edu.au or 9385 7011), so that special arrangements can be made.

9. You will need to produce your UNSW Student Card to gain entry to the Additional Assessment examination.

Important Notes

- The Additional Assessment exam may be of a different form to the original exam and must be expected to be at least as difficult.
- If you believe your application for Special Consideration has not been processed, you should immediately consult the Director for First Year Mathematics, Dr Jonathan Kress (Room 3073, Red Centre).
- If you believe that the above arrangements put you at a substantial disadvantage, you should send full documentation of the circumstances to: Director of First Year Mathematics, School of Mathematics and Statistics, University of NSW, Sydney NSW 2052, at the earliest possible time.
- If you suffer from a chronic or ongoing illness that has, or is likely to, put you at a serious disadvantage, then you should contact the Disability Support Services who provide confidential support and advice. Their web site is: https://student.unsw.edu.au/disability
  Disability Support Services (DSS) may determine that your condition requires special arrangements for assessment tasks. Once the School has been notified of these we will make every effort to meet the arrangements specified by DSS.
- Additionally, if you have suffered misadventure during semester then you should provide full documentation to the Director of First Year Mathematics as soon as possible. In these circumstances it may be possible to arrange discontinuation without failure or to make special examination arrangements.

  Professor B. Henry
  Head, School of Mathematics and Statistics

University Statement on Plagiarism

This statement has been adapted from statements by the St James Ethics Centre, the University of Newcastle, and the University of Melbourne.

Plagiarism is the presentation of the thoughts or work of another as one's own. Examples include:

- Direct duplication of the thoughts or work of another, including by copying work, or knowingly permitting it to be copied. This includes copying material, ideas or concepts from a book, article, report or other written document (whether published or unpublished), composition, artwork, design, drawing, circuitry, computer program or software, web site, Internet, other electronic resource, or another person's assignment without appropriate acknowledgement
- Paraphrasing another person's work with very minor changes keeping the meaning, form and/or progression of ideas of the original;
- Piecing together sections of the work of others into a new whole;
- Presenting an assessment item as independent work when it has been produced in whole or part in collusion with other people, for example, another student or a tutor; and,
- Claiming credit for a proportion a work contributed to a group assessment item that is greater than that actually contributed.
- Submitting an assessment item that has already been submitted for academic credit elsewhere may also be considered plagiarism.
- The inclusion of the thoughts or work of another with attribution appropriate to the academic discipline does not amount to plagiarism.

Students are reminded of their Rights and Responsibilities in respect of plagiarism, as set out in the University Undergraduate and Postgraduate Handbooks, and are encouraged to seek advice from academic staff whenever necessary to ensure they avoid plagiarism in all its forms.

The Learning Centre website is the central University online resource for staff and student information on plagiarism and academic honesty. It can be located at: www.lc.unsw.edu.au/plagiarism

The Learning Centre also provides substantial educational written materials, workshops, and tutorials to aid students, for example, in:

- Correct referencing practices;
- Paraphrasing, summarising, essay writing, and time management;
- Appropriate use of, and attribution for, a range of materials including text, images, formulae and concepts.

Individual assistance is available on request from The Learning Centre.

Students are also reminded that careful time management is an important part of study and one of the identified causes of plagiarism is poor time management. Students should allow sufficient time for research, drafting, and the proper referencing of sources in preparing all assessment items.

Syllabus

References are to the textbook by Epp, unless marked otherwise. F indicates the textbook by Franklin and Daoud and R indicates the book *Discrete Mathematics with Applications* by K.H. Rosen (6th edition). The UNSW Library has multiple copies of Rosen numbered P510/482A,B,C, etc.

The references shown in the righthand column are *not* intended to be a definition of what you will be expected to know. They are just intended as a guide to finding relevant material. Some parts of the course are not covered in the textbooks and some parts of the textbooks (even in sections mentioned in the references below) are not included in the course.

In the Reference column below, column A refers to Epp 3rd edition, and Rosen 2nd edition, while column B to Epp 4th edition and Rosen 6th edition.

Within sections of the course, the topics may not be covered in exactly the order in which they are listed below.

Topic	References A	References B
1. Sets, functions and sequences		
Sets, subsets, power sets. Equality, cardinality.	5.1, 5.3	1.2,6.1,6.3
Set operations: union, intersection, difference, cartesian product. Universal sets, complements.	5.1 5.2	6.1 6.2
Russell's paradox.	5.4	6.4
Functions. Domain, codomain and range. Arrow diagrams. Ceiling and floor functions. Images and inverse images of sets.	7.1, 3.5	1.3, 7.1,4.5
Injective (one-to-one), surjective (onto) and bijective functions.	7.3	7.2
Composition of functions	7.4	7.3
Inverse functions.	7.2	7.2
Sequences, sums and products. Notation. Change of variable in a sum. Telescoping sums.	4.1	5.1
2. Integers, Modular Arithmetic and Relations		
Prime numbers and divisibility	3.1, 3.3	4.1,4.3
Fundamental Theorem of Arithmetic	3.3	4.3
Euclidean Algorithm	3.8	4.8
Modular Arithmetic	3.4	4.4, 8.4
Solving Linear Congruences	R2.5	R3.7
General Relations	10.1	8.1
Reflexivity, symmetry and transitivity	10.2	8.2
Equivalence Relations	10.3	8.3
Partially ordered sets and Hasse diagrams	10.5	8.5

Topic	References A	References B
3. Logic and Proofs		
Proof versus intuition. Direct proof.	F1	F1
Propositions, connectives, compound propositions.	1.1	2.1
Truth tables. Tautology, contingency, logical equivalence.	1.1	2.1
Implication, converse, inverse, biconditional.	1.2	2.2
Rules of inference.	1.3	2.3
Contrapositive, indirect proof, proof by contradiction.	1.2, 3.6, F6,3.7	2.2,4.6,4.7, F6
Quantifiers	2.1	3.1
Proof of universal statements, exhaustion, proof by cases.	2.1, F2, F3	3.1, F2, F3
Proof of existential statements. Constructive and non-constructive proofs. Counterexamples.	2.1, 3.1, F4, F6	3.1,4.1,F4,F6
Negation of quantified statements.	2.1	3.2
Statements with multiple quantifiers.	2.2, 2.3, F5	3.2,3.3,F5
Common mistakes in reasoning. Converse and inverse fallacies. Begging the question, tacit assumption, etc.	2.3, 3.1	3.3,3.4,4.1
Mathematical induction	4.2-4.4, F8	5.2-5.4,F8
Note: In addition to the sections of Epp mentioned above, sections 4.2-4.5 and 4.7 (3.2-3.5,3.7 for edition 3) provide many useful worked examples of constructing proofs in elementary number theory.		
4. Enumeration and Probability		
Counting and Probability	6.1	9.1
Multiplication Rule	6.2	9.2
Addition Rule	6.3	9.3
Principle of Inclusion-Exclusion	6.3	9.3
Pigeonhole Principle	7.3	9.4
Permutations and Combinations	6.4, 6.5	9.5,9.6
Binomial and Multinomial Theorem	6.7, R4.6	9.7, R5.4
Discrete Probability	R4.4,6.1	R6.1,9.1
Recurrence Relations	8.2, 8.3	5.6,5.7,5.8
Recursively Defined Sets and Functions	8.1	5.9
5. Graphs		
Basic terminology. simple graphs, K_n. Directed graphs. Subgraphs, complementary graphs.	11.1	10.1
Degree, the Handshaking Theorem (Epp Theorem 10.1.1 (11.1.1 in ed. 3))	11.1	10.1
Bipartite graphs, $K_{m,n}$.	11.1	10.1
Adjacency and incidence matrices.	11.3	10.3
Isomorphism, isomorphism invariants.	11.4	10.4
Walks, paths and circuits. Euler and Hamilton paths. Connected graphs, connected components.	11.2	10.2
Planar graphs. Euler's formula. Dual graphs. Necessary conditions for planarity. Kuratowski's Theorem.	R7.7	R9.7
Trees, spanning trees.	11.5, 11.6	10.5,10.7
Weighted graphs. Minimal spanning trees. Kruskal and Dijkstra algorithms.	11.6	10.6,10.7

PROBLEM SETS

Recommended Problems: It is strongly recommended that you attempt all questions marked by †. You should regard these questions as the minimum that you should attempt if you are to pass this course. However, the more practice in solving problems you get the better you are likely to do in class tests and exams, and so you should aim to solve as many of the problems on this sheet as possible. Ask your tutor about any problems you cannot solve.
Problems marked by a star ($*$) are more difficult, and should only be attempted after you are sure you can do the unstarred problems.

PROBLEM SET 1

Basic Set Theory

1. Are any of the sets $A = \{1, 1, 2, 3\}$, $\quad B = \{3, 1, 2, 2\}$, $\quad C = \{1, 2, 1, 2, 4, 2, 3\}$ equal?

†2. Show that
$$A = \{x \in \mathbb{R} \mid \cos x = 1\}$$
is a subset of
$$B = \{x \in \mathbb{R} \mid \sin x = 0\}.$$
Is the first a proper subset of the second? Give reasons.

3. a) List all the subsets of the set $A = \{a, b, c\}$.
   b) List all the elements of $A \times B$ where $A = \{a, b, c\}$ and $B = \{1, 2\}$.

4. Given the sets $X = \{24k + 7 \,|\, k \in \mathbb{Z}\}$, $Y = \{4n + 3 \,|\, n \in \mathbb{Z}\}$, $Z = \{6m + 1 \,|\, m \in \mathbb{Z}\}$, prove that $X \subseteq Y$ and $X \subseteq Z$ but $Y \not\subseteq Z$.

†5. If $S = \{0, 1\}$, find
   a) $|P(S)|$,
   b) $\big|P\big(P(S)\big)\big|$,
   c) $\Big|P\Big(P\big(P(S)\big)\Big)\Big|$.

†6. Determine whether the following are true or false
   a) $a \in \{a\}$,
   b) $\{a\} \in \{a\}$,
   c) $\{a\} \subseteq \{a\}$,
   d) $a \subseteq \{a\}$,
   e) $\{a\} = \{a, \{a\}\}$,
   f) $\{a\} \in \{a, \{a\}\}$,
   g) $\{a\} \subseteq \{a, \{a\}\}$,

7. If A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.

8. Is it true that if $P(A) = P(B)$ for two sets A, B then $A = B$?

Set Operations and Algebra

9. If $A = \{$ letters in the word *mathematics* $\}$ and

   $B = \{$ letters in the words *set theory* $\}$, list the elements of the sets

   a) $A \cup B$,
   b) $A \cap B$,
   c) $A - B$,
   d) $B - A$.

10. Define the sets R, S and T by

$$\begin{aligned} R &= \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\} \\ S &= \{x \in \mathbb{Z} \mid x \text{ is divisible by } 3\} \\ T &= \{x \in \mathbb{Z} \mid x \text{ is divisible by } 6\}. \end{aligned}$$

   a) Is $S = T$?
   b) Is $R \subseteq T$?
   c) Is $T \subseteq R$?
   d) Is $T \subseteq S$?
   e) Find $R \cap S$.

†11. In a class of 40 people studying music: 2 play violin, piano and recorder, 7 play at least violin and piano, 6 play at least piano and recorder, 5 play at least recorder and violin, 17 play at least violin, 19 play at least piano, and 14 play at least recorder. How many play none of these instruments?

†12. Prove the following statements if they are true and give a counter-example if they are false.

   a) For all sets A, B and C, if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ then $A \subseteq B$.
   b) For all sets A, B and C, $(A \cup B) \cap C = A \cup (B \cap C)$.

†13. Let A and B be general sets. Determine the containment relation ($\subseteq, \supseteq, =$, none) that holds between

   a) $P(A \cup B)$ and $P(A) \cup P(B)$,
   b) $P(A \cap B)$ and $P(A) \cap P(B)$.

†14. Let A and B be general sets. Determine the containment relation ($\subseteq, \supseteq, =$, none) that holds between

$$P(A \times B) \quad \text{and} \quad P(A) \times P(B)$$

15. Show that $A - B = A \cap B^c$ and hence simplify the following using the laws of set algebra.

   a) $A \cap (A - B)$.
   b) $(A - B) \cup (A \cap B)$.
   c) $(A \cup B) \cup (C \cap A) \cup (A \cap B)^c$.

†16. Use the laws of set algebra to simplify

$$(A - B^c) \cup (B \cap (A \cap B)^c).$$

†17. Simplify

$$[A \cap (A \cap B^c)] \cup [(A \cap B) \cup (B \cap A^c)]\,,$$

and hence simplify

$$[A \cup (A \cup B^c)] \cap [(A \cup B) \cap (B \cup A^c)]\,.$$

18. Draw a Venn diagram for the general situation for three sets A, B and C. Use it to answer the following:

    a) If $B - A = C - A$, what subregions in your diagram must be empty?

    b) Prove or produce a counter-example to the following statement:

$$\text{If } B - A = C - A \quad \text{then} \quad B = C.$$

19. Use the laws of set algebra to prove that

$$(R - P) - Q = R - (P \cup Q)$$

for any sets R, P and Q.

*20. Define the *symmetric difference* $A \oplus B$ of two sets A and B to be

$$A \oplus B = (A - B) \cup (B - A).$$

   a) Draw a Venn diagram illustrating $A \oplus B$.

   b) If $A = \{$ even numbers strictly between 0 and 20 $\}$,

      $B = \{$ multiples of 3 strictly between 0 and 20 $\}$, write down the set $A \oplus B$.

   c) Explain why $A \oplus B$ can also be written as $(A \cup B) - (A \cap B)$.

   d) Suppose A, B and C are sets such that $A \oplus C = B \oplus C$. Prove that $A = B$.

      (Hint: You may use a Venn diagram to assist your argument.)

†21. Prove if true or give a counter example if false:

For all sets A, B and C, $\quad A \times (B \cup C) = (A \times B) \cup (A \times C)$.

†22. Let

$$A_k = \{n \in \mathbb{N} \mid k \leq n \leq k^2 + 5\}$$

for $k = 1, 2, 3, \ldots$. Find

   a) $\displaystyle\bigcup_{k=1}^{4} A_k$;

   b) $\displaystyle\bigcap_{k=10}^{90} A_k$;

   c) $\displaystyle\bigcap_{k=1}^{\infty} A_k$

23. Repeat the previous question if

$$A_k = \{x \in \mathbb{R} \mid 1 - \frac{1}{k} < x \leq k\}.$$

Functions

24. Which of the following are functions?

    a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{x^2 - 1}$.
    b) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2x + 1$.
    c) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$.
    d) $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = q$ where $x = \frac{p}{q}$, p, q integers.
    e) $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = q$ where $x = \frac{p}{q}$, p, q integers with $q > 0$ and no common factor except 1.

25. Recall from lectures that $\lfloor x \rfloor$ is the largest integer less than or equal to x, and that $\lceil x \rceil$ is the smallest integer greater than or equal to x. Evaluate

    a) $\lfloor \pi \rfloor$,
    b) $\lceil \pi \rceil$,
    c) $\lfloor -\pi \rfloor$,
    d) $\lceil -\pi \rceil$.

*26. Prove that if n is an integer, then

$$n - \left\lfloor \frac{1}{3} n \right\rfloor - \left\lfloor \frac{2}{3} n \right\rfloor$$

equals either 0 or 1.

(Hint: Write n as $3k, 3k + 1$ or $3k + 2$, where k is an integer.)

†27. Determine which of the following functions are one-to-one, which are onto, and which are bijections.

    a) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2x$.
    b) $f : \mathbb{Q} \to \mathbb{Q}, \quad f(x) = 2x + 3$.
    c) $f : \mathbb{R} \to \mathbb{Z} \quad f(x) = \lceil x \rceil$.
    *d) $f : \mathbb{R} \to \mathbb{R} \quad f(x) = x - \lfloor x \rfloor$.

†28. a) Let S be the set $\{ n \in \mathbb{N} \mid 0 \leq n \leq 11 \}$ and define $f : S \to S$ by letting $f(n)$ be the remainder when $5n + 2$ is divided by 12. Is f one-to-one? Is f onto?
    b) Repeat part (a) with $5n + 2$ replaced by $4n + 2$.

†29. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 - 4x + 6.$$

    a) What is the range of f?
    b) Is f onto? Explain.
    c) Is f one-to-one? Explain.

30. Repeat the above question with $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = x^3 - x + 1$$

and $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = x^3 + x + 1$$

(Hint: You may use differentiation.)

†31. Let $\mathbb{Z}^+$ be the set of all positive integers and $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function defined by $f(m,n) = mn$ for all $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Determine whether f is one-to-one or onto.

32. Find $g \circ f$ for each of the following pairs of functions

a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x - 3 \qquad g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \sqrt{x^2 + 2},$

b) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2x + 1 \qquad g : \{\text{odd integers}\} \to \mathbb{Z}, \quad g(x) = \frac{x-1}{2}.$

†33. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions.

a) Show that if f and g are both onto then $g \circ f$ is also onto.

b) Is it true that if f and g are both one-to-one then $g \circ f$ is also one-to-one?

*34. If $f, g\ : \mathbb{N} \to \mathbb{N}$ are functions defined by $f(n) = 2n$ and

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

show that $g \circ f = \iota$ but $f \circ g \neq \iota$ where ι is the identity function.

35. For each of the following bijections find the inverse and the domain and range of the inverse.

a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 5x + 3.$

b) * $g : \mathbb{Z} \to \mathbb{N}$

$$g(x) = \begin{cases} 2|x| - 1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

†36. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x^2 - 1$. Find

a) $f(A)$ if $A = \{x \mid -2 \leq x \leq 3\}$,

b) $f^{-1}(B)$ if $B = \{y \mid 1 \leq y \leq 17\}$.

†37. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function given by

$$f(m,n) = m^2 - n^2.$$

a) Show that f is not onto.

b) Find $f^{-1}(\{8\})$.

†38. Suppose f is a function from X to Y and A, B are subsets of X, and suppose that S, T are subsets of Y.

a) What containment relation (if any) is there between

i) $f(A) \cup f(B)$ and $f(A \cup B)$,

ii) $f(A) \cap f(B)$ and $f(A \cap B)$?

b) Show that $f^{-1}(S) \cup f^{-1}(T) = f^{-1}(S \cup T)$.

c) Show that $f^{-1}(S) \cap f^{-1}(T) = f^{-1}(S \cap T)$.

Sequences and Summation

†39. Use the formulae

$$\sum_{k=1}^{N} k = \frac{N}{2}(N+1) \quad \text{and} \quad \sum_{k=1}^{N} k^2 = \frac{N}{6}(N+1)(2N+1)$$

to evaluate

$$\sum_{k=3}^{22}(3k+4)^2.$$

40. a) Make use of changes of summation index to show that

$$\sum_{k=1}^{N}(k+1)^3 - \sum_{k=1}^{N}(k-1)^3 = (N+1)^3 + N^3 - 1.$$

b) Hence show that

$$\sum_{k=1}^{N} k^2 = \frac{N}{6}(N+1)(2N+1).$$

†41. Show that

$$\frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2}$$

and hence, that for $N \geq 1$

$$\sum_{k=1}^{N} \frac{2}{k(k+2)} = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}.$$

†42. Show that

$$\frac{5k-2}{k(k-1)\,(k-2)} = \frac{4}{k-2} - \frac{3}{k-1} - \frac{1}{k}$$

and hence evaluate

$$\sum_{k=3}^{n} \frac{5k-2}{k(k-1)\;(k-2)}.$$

43. By writing out the terms, show that for $N > 0$

$$\prod_{k=1}^{N} \frac{k}{k+2} = \frac{2}{(N+1)(N+2)}.$$

44. Using the fact that

$$1 - \frac{1}{k^2} = \frac{(k-1)(k+1)}{k^2}$$

find an expression for $\displaystyle\prod_{k=2}^{N}\left(1 - \frac{1}{k^2}\right)$ in terms of N.

PROBLEM SET 2

Integers and Modular Arithmetic

†1. Find the quotient and (non-negative) remainder when

a) 19 is divided by 7,
b) -111 is divided by 11,
c) 1001 is divided by 13.

2. Are the following true or false?

$7 \mid 161, \quad 7 \mid 162, \quad 17 \mid 68, \quad 17 \mid 1001.$

3. Which of the following are prime?

17, 27, 37, 111, 1111, 11111.

†4. Find the prime factorization of the following

117, 143, 3468, 75600.

†5. Find the gcd and lcm of the following pairs

a) $2^2 \cdot 3^5 \cdot 5^3$ and $2^5 \cdot 3^3 \cdot 5^2$,
b) $2^2 \cdot 3 \cdot 5^3$ and $3^2 \cdot 7$,
c) 0 and 3.

†6. Evaluate 13 **mod** 3, 155 **mod** 19, (-97) **mod** 11.

†7. Prove for $a, b, c, d \in \mathbb{Z}$, $k, m \in \mathbb{Z}^+$ that

a) if $a \mid c$ and $b \mid d$ then $ab \mid cd$,
b) if $ab \mid bd$ and $b \neq 0$, then $a \mid d$,
c) if $a \mid b$ and $b \mid c$ then $a \mid c$,
d) if $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ then $a - b \equiv c - d \pmod{m}$,
e) if $k \mid m$ and $a \equiv b \pmod{m}$ then $a \equiv b \pmod{k}$,
f) if $d = \gcd(a, b)$ then $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$,
*g) if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.

†8. a) Find the least positive integer n for which

$$3^n \equiv 1 \pmod{7}.$$

Hence evaluate $3^{100} \pmod{7}$.

b) Find the least positive integer n for which

$$5^n \equiv 1 \pmod{17} \quad \text{or} \quad 5^n \equiv -1 \pmod{17}.$$

Hence evaluate $5^{243} \pmod{17}$.

c) Evaluate $2^4 \pmod{18}$ and hence evaluate $2^{300} \pmod{18}$.

†9. For each of the following, use the Euclidean Algorithm to find $d = \gcd(a, b)$ and $x,\ y \in \mathbb{Z}$ with $d = ax + by$

a) $\gcd(12, 18)$,
b) $\gcd(111, 201)$,
c) $\gcd(13, 21)$,
d) $\gcd(83, 36)$,
e) $\gcd(22, 54)$,
f) $\gcd(112, 623)$.

†10. Solve, or prove there are no solutions. Give your answer in terms of the original modulus and also, where possible, in terms of a smaller modulus.

a) $151x - 294 \equiv 44 \pmod{7}$,
b) $45x + 113 \equiv 1 \pmod{20}$,
c) $25x \equiv 7 \pmod{11}$,
d) $2x \equiv 3 \pmod{1001}$,
e) $111x \equiv 75 \pmod{321}$,
f) $1215x \equiv 560 \pmod{2755}$,
g) $182x \equiv 481 \pmod{533}$.

*11. Let a, b be integers, not both zero, let S be the set of integers defined by

$$S = \{ax + by \mid x, y \in \mathbb{Z}\},$$

and let d_0 be the smallest positive integer in the set S.

The aim of this question is to use the Division Algorithm and the definition of greatest common divisor (gcd) to show that $d_0 = \gcd(a, b)$.

Prove the following

a) If $s \in S$, then d_0 is a divisor of s. (Hint: Write $s = d_0 q + r$ and show $r \in S$).
b) d_0 is a divisor of both a and b.
c) If d is a divisor of both a and b, then d is a divisor of d_0.
d) $d_0 = \gcd(a, b)$, and hence there exist integers x, y such that $ax + by = \gcd(a, b)$.

12. a) Prove that if a, b and m are integers with the properties $\gcd(a, b) = 1$ and $a \mid m$ and $b \mid m$, then $ab \mid m$. (Hint: use 11(d).)
b) Prove that if a and b are coprime integers and $a \mid bc$, then $a \mid c$. (Hint: use 11(d).)

Relations

†13. List the ordered pairs in the relations R_i, for $i = 1, 2, 3$, from $A = \{2, 3, 4, 5\}$ to $B = \{2, 4, 6\}$ where

a) $(m, n) \in R_1$ iff $m - n = 1$,

b) $(m, n) \in R_2$ iff $m \mid n$,

c) $(m, n) \in R_3$ iff $\gcd(m, n) = 1$.

†14. Represent each relation R_i of Question 13 by:

a) an arrow diagram,

b) a matrix M_{R_i}.

15. Construct arrow diagrams representing relations on $\{a, b, c\}$ that have the following properties.

a) Reflexive, but neither transitive nor symmetric.

b) Symmetric, but neither transitive nor reflexive.

c) Transitive, but neither symmetric nor reflexive.

d) Symmetric and transitive, but not reflexive.

e) Transitive and reflexive, but not symmetric.

f) Symmetric and antisymmetric and reflexive.

16. A relation R on $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is defined by $a\,R\,b$ iff $a - b$ has either 2 or 3 as a divisor. Show that R is reflexive and symmetric, but not transitive.

†17. Define an equivalence relation $\sim$ on the set $S = \{0, 1, 2, 3, 4, 5, 6\}$ by $x \sim y$ if and only if $x \equiv y \pmod 3$. Partition S into equivalence classes.

†18. Define a relation $\sim$ on the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ by $x \sim y$ if and only if $x^2 \equiv y^2 \pmod 5$.

a) Prove that $\sim$ is an equivalence relation on S.

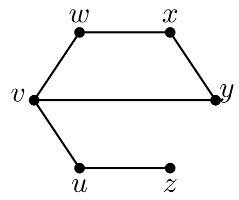
b) Partition S into equivalence classes.

19. a) Let $A_1 = \{0\}$, $A_2 = \{1, 2\}$, $A_3 = \{3, 4\}$ be subsets of $S = \{0, 1, 2, 3, 4\}$. Define a relation $\sim$ on the set S by $x \sim y$ if and only if $x, y \in A_i$ for some $i \in \{1, 2, 3\}$. Show that $\sim$ is an equivalence relation on S.

b) Let $B_1 = \{0\}$, $B_2 = \{1, 2\}$, $B_3 = \{3\}$ be subsets of $S = \{0, 1, 2, 3, 4\}$. Explain why the relation $\sim$ on S defined by $x \sim y$ if and only if $x, y \in B_i$ for some $i \in \{1, 2, 3\}$ is **not** an equivalence relation on S.

c) Let $C_1 = \{0, 1\}$, $C_2 = \{1, 2\}$, $C_3 = \{3, 4\}$ be subsets of $S = \{0, 1, 2, 3, 4\}$. Explain why the relation $\sim$ on S defined by $x \sim y$ if and only if $x, y \in C_i$ for some $i \in \{1, 2, 3\}$ is **not** an equivalence relation on S.

20. The following diagram represents a set $V = \{u, v, x, y, z\}$ of six cities and direct flights between them.



   a) Define a relation $\sim$ on V by $a \sim b$ if and only if it is possible to fly from a to b using an even number of flights (including 0 flights).
      i) Prove that $\sim$ is an equivalence relation on V.
      ii) Partition the set V into equivalence classes.
   b) Define a relation R on V by aRb if and only if it is possible to fly from a to b using an odd number of flights. Prove that R is *not* an equivalence relation.

†21. Consider the set $S = \{0, 1, 2, \ldots, 11\}$ of integers modulo 12. Define the relation $\sim$ on S by $x \sim y$ iff $x^2 \equiv y^2 \bmod 12$. Given that $\sim$ is an equivalence relation, partition S into equivalence classes.

†22. Let a and b be two fixed real numbers. Define a relation $\sim$ on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ iff $ax_1 + by_1 = ax_2 + by_2$. Prove that $\sim$ is an equivalence relation and give a geometric description of the equivalence class of $(1, 1)$.

23. Answer the following questions for the Poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$:
   a) Draw a Hasse diagram for the Poset.
   b) Find the maximal elements.
   c) Find the minimal elements.
   d) Is there a greatest element?
   e) Is there a least element?
   f) Find all upper bounds of $\{\{2\}, \{4\}\}$.
   g) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.
   h) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.
   i) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ if it exists.

24. Answer the following questions for the Poset $(\{2, 4, 6, 9, 12, 27, 36, 54, 60, 72\}, |)$:
   a) Draw a Hasse diagram for the Poset.
   b) Find the maximal elements.
   c) Find the minimal elements.
   d) Is there a greatest element?
   e) Is there a least element?
   f) Find the set of upper bounds of $\{2, 9\}$.
   g) Find the least upper bound of $\{2, 9\}$, if it exists.
   h) Find the set of lower bounds of $\{60, 72\}$.
   i) Find the greatest lower bound of $\{60, 72\}$ if it exists.

25. Give an example of a poset that

  a) Has a minimal element but no least element.
  b) Has a maximal element but no greatest element.

†26. A relation $\mid$ is defined on $A = \{\,1, 2, 4, 6, 8, 9, 12, 18, 36, 72, 108\,\}$ by $a \mid b$ iff a divides b.

  a) Show that $(A, \mid)$ is a poset.
  b) Construct its Hasse diagram.
  c) Which members of A are minimal elements? Which are maximal elements? Which are greatest elements? Which are least elements?

†27. Let $S = \{0, 1, 2, 3\}$ and $P(S)$ denote the power set of S.
Define the relation $A \preceq B$ for $A,\ B \in P(S)$ by $A \preceq B$ iff $A \subseteq B$.

  a) Show that $\preceq$ is a partial order.
  b) Construct the Hasse diagram of $\big(P(S),\ \preceq\big)$.
  c) Which members of $P(S)$ are minimal elements? Which are maximal elements? Which are greatest elements? Which are least elements?

PROBLEM SET 3

Note that the subject matter of this section of the course is mathematical proof itself, and not the particular results proved in classes or posed in the following problem set. You should be prepared to prove results from any area of school or first-year university mathematics.

In this problem set, F&D refers to the book by Franklin and Daoud, and the problems have been printed here with the permission of the author. Note the solutions to some of the F&D questions are available in the book "Introduction to Proofs in Mathematics" by J. Franklin and A. Daoud.

Basic Proofs

1. (F&D Chapter 1 Q1) Show that: $\dfrac{1}{1,000} - \dfrac{1}{1,002} < \dfrac{2}{1,000,000}$

†2. (F&D Chapter 1 Q3) Show that: $\sqrt{1,001} - \sqrt{1,000} < \dfrac{1}{2\sqrt{1,000}}$
Hint: multiply $\sqrt{1,001} - \sqrt{1,000}$ by $\dfrac{\sqrt{1,001} + \sqrt{1,000}}{\sqrt{1,001} + \sqrt{1,000}}$

†3. (F&D Chapter 1 Q9) Prove that: $\sqrt[7]{7!} > \sqrt[6]{6!}$

†4. [V] (F&D Chapter 1 Q12) Show that: $\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}} < 2\sqrt{2}$

5. (F&D Chapter 1 Q14)

   a) Prove that: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

   b) Hence show that $\cos\dfrac{2\pi}{3}$ is a root of the equation $4x^3 - 3x - 1 = 0$

6. a) Suppose that n is a positive integer. Use the Binomial theorem and appropriate inequalities to prove that
$$0 < \left(1 + \frac{1}{n}\right)^n < 3.$$

   b) (F&D Chapter 1 Q20) Prove that: $99^{100} > 100^{99}$

7. (F&D Chapter 1 Q21) Show that: $\dfrac{\pi}{4} = \tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right)$

Generalisation; "all" statements

†8. Prove by exhaustion of cases that for any real number x we have
$$|x(x-2)| \leq 2x^2 + 1\ .$$

†9. (F&D Chapter 2 Q3) Prove that the product of any two odd numbers is an odd number.

†10. (F&D Chapter 2 Q11) Prove that $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$ for any whole number n (imitate the proof in Chapter 1).

11. (F&D Chapter 2 Q12) Prove that the product of three consecutive whole numbers, of which the middle one is odd, is divisible by 24.

12. (F&D Chapter 2 Q19) Find a generalisation of:

$$\frac{1}{1000} - \frac{1}{1002} < \frac{2}{(1000)^2}$$

and prove it.

13. (F&D Chapter 2 Q21) Prove that for all whole numbers n, $(n+1)(n+2)\ldots(2n-1)(2n) = 2^n.1.3.5\ldots(2n-1)$

†14. (F&D Chapter 5 Q9) Consider the following statement concerning a positive integer $x \geq 2$.

"If x is not divisible by any positive integer n satisfying $2 \leq n \leq \sqrt{x}$ then x is a prime number."

a) Show that the above statement is true.

b) Is the statement still true if the condition on n is replaced by $2 \leq n < \sqrt{x}$?

*15. a) Let a and n be integers greater than 1. Prove that $a^n - 1$ is prime only if $a = 2$ and n is prime. Is the converse of this statement true?

b) [V] Show that $2^n + 1$ is prime only if n is a power of 2.

*16. Consider the following statement concerning a positive integer n:

"For all $a, b \in \mathbb{Z}$, if $n \mid ab$ then either $n \mid a$ or $n \mid b$."

a) Prove that if n is prime then the statement is true.

b) Prove that the statement is false when $n = 30$.

c) Prove that if n is composite then the statement is false.

17. a) Use the result of part (a) of the previous question to prove that if p is prime, a is an integer and $p \mid a^2$, then $p \mid a$.

*b) Are there any integers n other than primes for which it is true that for all integers a, if $n \mid a^2$ then $n \mid a$? If so, describe all such n.

*c) Prove that if p is prime then $\sqrt{p}$ is irrational.

18. a) Show that 2 is a multiplicative inverse for 4 (mod 7) and 3 is a multiplicative inverse for 5 (mod 7). Hence determine the value of 5! (mod 7)

*b) [V] Prove that if p is prime then

$$(p-2)! \equiv 1 \pmod{p}.$$

Hint: What is the multiplicative inverse of $p - 1 \pmod{p}$ when p is prime?

Writing proofs

†19. In the following questions you are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow; and any necessary algebraic details.

a) **Theorem**. If $x^3+x^2+x+2 \ = 0 \mod 5$ then $x \ = 1 \mod 5$.
Basic ideas: if $x \ = 0,2,3,4 \mod 5$ then $x^3+x^2+x+2 \ = 2,1,1,1 \mod 5$.

b) **Theorem**. Let x, y and m be integers. If $m \mid (4x+y)$ and $m \mid (7x+2y)$ then $m \mid x$ and $m \mid y$.
Basic ideas: $2(4x+y)-(7x+2y)=x$ and $m \mid$ LHS.

c) **Theorem**. $\log_2 7$ is irrational.
Basic ideas: if $\log_2 7 = p/q$ then $2^p = 7^q$, but then LHS is even and RHS is odd.

d) **Theorem**. If n is a non–negative integer then 11 is a factor of $2^{4n+3}+3\times 5^n$.
Basic ideas: if $2^{4n+3} = 11k - 3\times 5^n$ then $2^{4n+7} = 11(16k-3\times 5^n) - 3\times 5^{n+1}$.

Converse; if and only if

†20. Prove the following statement, then write down its converse. Is the converse true or false? Prove your answer.

"For all $x \in \mathbb{Z}$, if $x \equiv -1 \pmod 7$ then $x^3 \equiv -1 \pmod 7$."

21. a) Prove that the following proposition is true.

$$\text{If } x \equiv 3 \pmod 4 \text{ then } x^3+2x-1 \equiv 0 \pmod 4.$$

b) Write down the contrapositive of the proposition in part (a). Is it true? Explain.

c) Write down the converse of the proposition in part (a). Is it true? Explain.

†22. (F&D Chapter 3 Q6) Prove that a whole number is odd if and only if its square is odd.

†23. (F&D Chapter 3 Q10) Prove that a triangle is isosceles if and only if two of its angles are equal. (An isosceles triangle is by definition a triangle with two equal sides.)

24. (F&D Chapter 3 Q13) Show that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

25. (F&D Chapter 3 Q18) Prove that a real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

*26. For integers x and y, show that $7|x^2+y^2$ if and only if $7|x$ and $7|y$.

27. A parallelogram is defined to be a quadrilateral with both pairs of opposite sides parallel. Use properties of congruent and similar triangles to prove that a quadrilateral is a parallelogram if and only if two opposite sides are equal and parallel.

"Some" statements

†28. (F&D Chapter 4 Q11) Show that there is a solution of, $x^{100} + 5x - 2 = 0$ between $x = 0$ and $x = 1$.

29. [V] (F&D Chapter 4 Q4) A perfect number is one which equals the sum of its factors (counting 1 as a factor, but not the number itself). Show that there exists a perfect number.

†30. (F&D Chapter 4 Q13) Consider the infinite geometric progression,

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \ldots$$

Prove that there exists an integer N such that the sum of the first N terms of the above series differs from 1 by less than 10^{-6}.

31. (F&D Chapter 4 Q16) A formula for e^x is,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$$

Show that $e^3 = 20.1$, correct to 1 decimal place, by showing:

a) For any $n \geq 4$,
$$\frac{3^n}{n!} < \frac{9}{2}\left(\frac{3}{4}\right)^{n-3}$$

b) Hence, that there exists N such that,
$$\frac{3^{N+1}}{(N+1)!} + \frac{3^{N+2}}{(N+2)!} + \cdots < \frac{1}{20}$$

c) Hence, that $e^3 = 20.1$ correct to 1 decimal place.

*32. (F&D Chapter 5 Q11) Every one of six points is joined to every other one by either a red or a blue line. Show that there exist three of the points joined by lines of the same colour.

Multiple quantifiers

*33. [V] (F&D Chapter 5 Q8) Prove that between any two irrational numbers there is a rational number.

†34. Show that the sequence $\{u_n\}$ given by

$$u_n = n^2 \quad \text{for all } n \in \mathbb{N}$$

diverges to infinity, by showing that

$$\forall M \in \mathbb{N} \quad \exists N \in \mathbb{N} \quad \forall n > N \quad u_n > M.$$

35. A function $f(x)$ is called continuous at $x = a$ iff:

$$\forall \epsilon > 0 \quad \exists\, \delta > 0 \quad \forall x \in (a - \delta, a + \delta) \quad |f(x) - f(a)| < \epsilon.$$

a) Complete: "A function $f(x)$ is not continuous at $x = a$ iff: ......"

b) * Show that the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 3$.

†36. The definition of $\lim_{x\to\infty} f(x) = \ell$ is

$$\text{“}\forall\epsilon > 0 \;\; \exists M \in \mathbb{R} \;\; \forall x > M \;\; |f(x) - \ell| < \epsilon.\text{”} \qquad (**)$$

a) Write down the negation of $(**)$ (that is, $\lim_{x\to\infty} f(x) \neq \ell$) and simplify it so that the negation symbol does not appear.

b) By working directly from the definition prove that $\lim_{x\to\infty} \dfrac{2x^2+1}{x^2+2} = 2$.

Negation; proof by contradiction

†37. (F&D Chapter 6 Q16) Prove that each of the following is irrational:

a) $\sqrt[3]{4}$

b) $1+\sqrt{3}$

c) $\sqrt{3}-\sqrt{2}$

†38. Prove that $\log_{10} 3$ is irrational.

39. Prove that $\log_5 15$ is irrational.

†40. (F&D Chapter 6 Q1) What is the common feature in showing:

a) a “not all” statement to be true?

b) an “all” statement to be false?

Give an example in each case.

41. (F&D Chapter 6 Q11) Are the following statements true or false? Prove your answers.

a) Let a, b, c be three integers. If a divides c and b divides c, then either a divides b or b divides a.

b) If a, b, c, d are real numbers with $a < b$ and $c < d$, then $ac < bd$.

42. (F&D Chapter 5 Q5)

a) Prove that if a and b are rational numbers with $a \neq b$ then,

$$a + \frac{1}{\sqrt{2}}(b-a)$$

is irrational.

b) Hence prove that between any two rational numbers there is an irrational number.

43. (F&D Chapter 7 Q14) A set of real numbers is called bounded if it does not “go to infinity”. More precisely, S is bounded if there exist real numbers M, N such that for all $s \in S, M < s < N$.
(For example, the set of real numbers x such that $1 < x^3 < 2$ is bounded, since for all $x \in S, 1 < x < 1.5$.)

a) Give an example of a set which is not bounded.

b) Prove that if S is bounded and $T \subset S$, then T is bounded.

c) Prove that any finite set of real numbers is bounded.

d) Prove that if T is not bounded and $T \subset S$, then S is not bounded.

e) Prove that if S and T are bounded, then $S \cap T$ is bounded.

f) If S and T are bounded, is $S \cup T$ always bounded? Prove your answer.

g) Let S and T be bounded, Let,

$$U = \{u \in \mathbb{R} : u = s + t \quad \text{for some} \quad s \in S, t \in T\}$$

Show that U is bounded. It might help to calculate some examples first, say,

$$S = [0, 1], T = [2, 3]$$

(The set U is sometimes denotes $S + T$, since it is the set of all sums of something in S with something in T.)

44. (F&D Chapter 7 Q15) A region in the plane is called convex if the line segment joining any two points in the region lies wholly inside the region. For example, an ellipse, a parallelogram, a triangle and a straight line are convex, but on annulus and a star-shaped region are not. In symbols, R is convex if, for all (x_1, y_1) and (x_2, y_2) in R, $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in R$ for all $\lambda \in [0, 1]$.

a) Prove that if R and S are convex, then $R \cap S$ is convex.

b) If R and S are convex, is $R \cup S$ always convex? Prove your answer.

c) Prove that if R is convex, then the reflection of R in the x-axis is convex.

d) If R is convex, is the set,

$$2R = \{(x, y) : (x, y) = (2x', 2y') \text{ for some } (x', y') \in R\}$$

always convex? Prove your answer and illustrate with a diagram.

†45. (F&D Chapter 6 Q24) Show that there do not exist three consecutive whole numbers such that the cube of the greatest equals the sum of the cubes of the other two.

Mathematical induction

46. (F&D Chapter 8 Q2) Show that, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots + (-1)^{n-1}\frac{1}{n}$ is always positive.

†47. (F&D Chapter 8 Q5)

a) Prove, by mathematical induction, that if n is a whole number then,

$$n^3 + 3n^2 + 2n \quad \text{is divisible by 6.}$$

b) Prove the same result without mathematical induction by first factorising $n^3 + 3n^2 + 2n$.

†48. (F&D Chapter 8 Q7)

a) Calculate,

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$$

for a few small values of n.

b) Make a conjecture about a formula for this expression.

c) Prove your conjecture by mathematical induction.

49. (F&D Chapter 8 Q8) The following is a famous fallacy that uses the method of mathematical induction. Explain what is wrong with it.
Theorem
Everything is the same colour.
Proof
Let $P(n)$ be the statement, "In every set of n things, all the things have the same colour".
We will show that $P(n)$ is true for all $n = 1, 2, 3, \ldots$, so that every set consists of things of the same colour.
Now, $P(1)$ is true, since in every set with only one thing in it, everything is obviously of the same colour.
Now, suppose $P(n)$ is true.
Consider any set of $n+1$ things.
Take an element of the set, a. The n things other than a form a set of n things, so they are all the same colour (since $P(n)$ is true).
Now take a set of n things out of the $n+1$ which does include a.
These are also all the same colour, so a is the same colour as the rest.
Therefore $P(n+1)$ is true.

50. Show that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ whenever $n \in \mathbb{Z}^+$.

51. Show that $1^2 + 3^2 + \cdots + (2n+1)^2 = \frac{1}{3}(n+1)(2n+1)(2n+3)$ whenever $n \in \mathbb{N}$.

†52. Show that for all $n \in \mathbb{N}$, $64 \mid 7^{2n} + 16n - 1$.

53. Prove that for all $n \in \mathbb{Z}^+$, $21 \mid 4^{n+1} + 5^{2n-1}$.

*54. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \geq \frac{2}{3} n \sqrt{n}.$$

55. Prove by induction that if the sequence (u_n) is defined by

$$\begin{cases} u_0 & = & 0 \\ u_1 & = & 1 \\ u_n & = & u_{n-1} + u_{n-2} \text{ for all } n \geq 2 \end{cases}$$

then

$$u_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

for all $n \geq 0$.

†56. Prove, using induction, that if the sequence $\{u_n\}$ is defined by:

$$\begin{cases} u_1 & = & 12 \\ u_2 & = & 30 \\ u_n & = & 5u_{n-1} - 6u_{n-2} \quad \text{for } n \geq 3, \end{cases}$$

then $u_n = 3 \times 2^n + 2 \times 3^n$, for all $n \geq 1$.

57. a) Prove by induction that for $n \in \mathbb{N}$, the nth derivative of e^{x^2} is a polynomial times e^{x^2}. Can you say anything about these polynomials?
(**Note:** By convention, the $0'$th derivative of $f(x)$ is just $f(x)$.)

b) Guess and prove a similar result concerning the derivatives of the function $e^{1/x}$.

†58. [V] Prove that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ we have $3^n < n!$.

Logic and truth tables

59. Which of the following are propositions?

a) The moon is made of green cheese.
b) Read this carefully.
c) Two is a prime number.
d) Will the game be over soon?
e) Next year interest rates will rise.
f) Next year interest rates will fall.
g) $x^2 - 4 = 0$

60. Using letters for the component propositions, translate the following compound statements into symbolic notation:

a) If prices go up, then housing will be plentiful and expensive; but if housing is not expensive, then it will still be plentiful.
b) Either going to bed or going swimming is a sufficient condition for changing clothes; however, changing clothes does not mean going swimming.
c) Either it will rain or it will snow but not both.
d) If Janet wins or if she loses, she will be tired.
e) Either Janet will win or, if she loses, she will be tired.

61. Write a statement that represents the negation of each of the following:

a) If the food is good, then the service is excellent.
b) Either the food is good or the service is excellent.
c) Either the food is good and the service is excellent, or else the price is high.
d) Neither the food is good nor the service excellent.
e) If the price is high, then the food is good and the service is excellent.

†62. Construct a truth table for each of the following propositions.

a) $\sim \big((p \wedge q) \to (p \wedge q)\big)$;
b) $p \to (p \to q)$;
c) $\sim p \to (p \to q)$;
d) $\big(p \to (q \to r)\big) \to \big((p \to q) \to (p \to r)\big)$.

Which of the above propositions are tautologies? Are any contradictions?

†63. Show, by using truth tables, that the following pairs of propositions are logically equivalent.

a) $\sim (p \vee q),\ \sim p \wedge \sim q$;
b) $\sim p \vee q,\ p \to q$;
c) $\sim p \to (q \vee r),\ \sim q \to (\sim r \to p)$;
d) $p \vee (p \wedge q), p$.

64. Use standard logical equivalences to simplify each of the following logical expressions.

a) $(p \vee \sim q) \wedge \sim (p \wedge q)$
b) $[p \to (q \vee \sim p)] \to (p \wedge q)$
c) $(p \wedge q) \wedge\ \sim (\sim p \wedge q)\ \wedge\ (q \wedge r)$
d) $(p \leftrightarrow q)\ \vee\ (\sim p \wedge q)$

65. Use standard logical equivalences to show that $\sim (p \vee \sim q) \to (q \to r)$ is logically equivalent to each of the following propositions

a) $q \to (p \vee r)$;
b) $(q \to p) \vee (q \to r)$;
c) $\sim (q \to p) \to (\sim q \vee r)$.

Valid and invalid arguments

66. Suppose that

> "If I do not do my homework, I will not pass.
> If I study hard, I will pass.
> I passed."

Did I do my homework or not? Did I study hard or not? Explain.

†67. "Watson, I have uncovered the following facts:

- If Mrs Smith is lying then Moriarty has not escaped.
- Either Moriarty is dead or he is really Jones.
- If Moriarty is really Jones then he has escaped.
- I am convinced Mrs Smith is lying."

"Good Lord Holmes," replied Dr Watson, "what can you make of all this?"
"Elementary my dear Watson, Moriarty is dead!"

Is Holmes correct? Justify your answer.

†68. Let e denote "Einstein is right", b denote "Bohr is right" q denote "Quantum mechanics is right", w denote "The world is crazy" and consider the sentences:

(1) "Either Einstein or Bohr is right, but they are not both right".
(2) "Einstein is right only if quantum mechanics is wrong, and the world is crazy if quantum mechanics is right".

a) Write the two sentences (1) and (2) as symbolic expressions involving e, b, q and w.

b) Suppose the world is not crazy. From the truth of (1) and (2) is it valid to deduce that Einstein is right? Explain.

69. Consider the following two statements:

(1) "If either Peta or Queenie has passed then either Roger and Peta have both passed, or Roger and Queenie have both passed".

and

(2) "If either Peta has passed or Queenie has failed then Roger has passed."

a) Write the two statements (1) and (2) in symbolic form by letting p stand for "Peta has passed," q stand for "Queenie has passed" and r stand for "Roger has passed".

b) Suppose that the statement (1) is <u>false</u>. Deduce that Roger has failed.

c) Suppose that the statement (1) is false and that the statement (2) is true. Decide whether or not Peta has passed.

PROBLEM SET 4

Enumeration and Probability

†1. How many strings of six non-zero decimal digits

a) begin with two odd digits?

b) consist of one even digit, followed by two odd digits, followed by three digits less than 7?

c) have no digit occurring more than once?

d) contain exactly three nines?

e) contain fewer than three nines?

f) contain exactly three nines, with no other digit repeated?

g) have their last digit equal to twice their first digit?

†2. a) Express in terms of factorials $P(21, 8)$, $C(21, 8)$, $\binom{321}{123}$, $C(2n, n)$.

b) Calculate explicitly $P(7, 4)$, $\binom{7}{4}$, $P(6, 3)$, $C(4, 2)$, $C(201, 199)$.

†3. How many seven-letter words can be made from the English alphabet which contain

a) exactly one vowel,

b) exactly two vowels,

c) exactly three vowels,

d) at least three vowels?

4. Repeat the previous question if repeated letters are not permitted.

†5. a) A set of eight Scrabble® tiles can be arranged to form the word SATURDAY. How many three-letter "words" can be formed with these tiles if no tile is to be used more than once?

b) How many ten-letter words can be formed from the letters of PARRAMATTA? How many nine-letter words? *How many eight-letter words?

c) How many four-letter "words" come before "UNSW" if all four-letter words are listed in alphabetical order?

†6. How many eight-letter words constructed from the English alphabet have

a) exactly two Ls?

b) at least two Ls?

7. Consider the following.

"*Problem*: How many eight-card hands chosen from a standard pack have at least one suit missing?

Solution: Throw out one entire suit (4 possibilities), then select 8 of the remaining 39 cards. The number of hands is $4C(39, 8)$."

a) What is wrong with the given solution?

b) Solve the problem correctly.

†8. a) Find the number of positive integers less than or equal to 200 which are divisible *neither* by 6 *nor* by 7.

b) Find the number of positive integers less than or equal to 200 which are divisible neither by 6, nor by 7, nor by 8.

9. What is the probability that a hand of eight cards dealt from a shuffled pack contains:

a) exactly three cards of the same value and the remaining cards all from the remaining suit (*for example,* $\heartsuit 4, \diamondsuit 4, \spadesuit 4$ *and five clubs not including the* $\clubsuit 4$);

b) exactly three cards in at least one of the suits;

c) exactly three cards in exactly one of the suits. (*Hint.* First find the number of ways in which five cards can be chosen from three specified suits, with none of the three represented by three cards.)

10. For any positive integer n we define $\phi(n)$ to be the number of positive integers which are less than or equal to n and relatively prime to n. Given that p, q, r are primes, all different, use the inclusion/exclusion principle to find $\phi(pq)$, $\phi(p^2q)$ and $\phi(pqr)$.

†11. A hand of thirteen cards is dealt from a shuffled pack. Giving reasons for your answers, determine which of the following statements are *definitely true* and which are *possibly false*.

a) The hand has at least four cards in the same suit.

b) The hand has exactly four cards in some suit.

c) The hand has at least five cards in some suit.

d) The hand has *exactly one* suit containing four or more cards.

†12. a) Prove that if nine rooks (castles) are placed on a chessboard in any position whatever, then at least two of the rooks attack each other.

b) Prove that if fifteen bishops are placed on a chessboard then two of them attack each other.

†13. Let A be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Prove that if 5 integers are selected at random from A, then at least one pair of these integers has sum 9.

14. In a square with sides of length 3cm, 10 points are chosen at random. Prove that there must be at least two of these points whose distance apart is less than or equal to $\sqrt{2}$ cm.

†15. If 31 cards are chosen from a pack, prove that there must be at least 3 of the same value, and there must be at least 8 in the same suit.

†16. a) To each integer n we assign an ordered pair $p(n)$ whose members are the remainders when n is divided by 3 and 4 respectively. For example, $p(5)$ and $p(17)$ are both equal to $(2, 1)$. If ten thousand integers are chosen at random, how many can you say for certain must have the same value for p?

*b) Repeat part (a) with the divisors 3 and 4 replaced by 4 and 6.

17. Let S and T be finite sets with $|S| > |T|$, and let f be a function from S to T. Show that f is not one-to-one.

†18. Prove that there were two people in Australia yesterday who met exactly the same number of other people in Australia yesterday.

19. Twenty hotel management students all guess the answers on the final examination, so it can be taken that all orders of students on the list of results are equally likely. The top student is given a mark of 100, the next 95, and so on, down to 5 for the last student and no two students get the same mark. Find the probability that Polly gets an HD, both Manuel and Sybil get a CR or better, and Basil fails.

†20. A die is rolled 21 times. Find the probability of obtaining a 1, two 2s, … and six 6s.

†21. For this problem assume that the 365 dates of the year are equally likely as birthdays.

a) Find the probability that two people chosen at random have the same birthday.
b) Find the probability that in a group of n people, at least two have the same birthday.
c) How large does n have to be for the probability in (b) to be greater than $\frac{1}{2}$?
d) Criticise the assumption made at the beginning of this question.

†22. Twenty cars to be bought by a company must be selected from up to four specific models. In how many ways may the purchase be made if

a) no restrictions apply?
b) at least two of each model must be purchased?
c) at most three different models must be purchased?

23. How many outcomes are possible from the roll of four dice

a) if the dice are distinguishable (for example, they are of different colours)?
b) if the dice are not distinguishable?

†24. a) Find the coefficient of $w^2x^5y^7z^9$ in $(w+x+y+z)^{23}$.
b) When $(w+x+y+z)^{23}$ is expanded and terms collected, how many different terms will there be?

†25. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 40$$

if $x_1, x_2, x_3, x_4, x_5, x_6$ are non-negative integers,

a) with no further assumptions?
b) with each $x_j \geq 3$?
c) with each $x_j \leq 10$?
*d) with each $x_j \leq 8$?
e) if each x_j is even?
f) if at least one x_j is odd?
*g) if every x_j is odd?
*h) with $x_1 \leq 9$, $5 \leq x_2 \leq 14$ and $10 \leq x_3 \leq 19$?

*26. By counting in two ways the number of non-negative integer solutions of the inequality $x_1 + x_2 + \cdots + x_r \leq n$, prove that

$$\binom{n+r}{r} = \binom{n+r-1}{r-1} + \binom{n+r-2}{r-1} + \cdots + \binom{r}{r-1} + \binom{r-1}{r-1}.$$

Interpret this result in Pascal's triangle.

Recurrence Relations

27. Write down the first four terms of the sequences defined recursively by

    a) $a_n = \frac{3}{2}a_{n-1}, \quad a_0 = 16;$

    b) $a_n = 3a_{n-1} - 2a_{n-2}, \quad a_0 = 2, \quad a_1 = 5;$

    c) $a_n = 2(a_{n-1} + a_{n-2} + \cdots + a_1 + a_0), \quad a_0 = 1.$

28. Write down a recurrence and initial conditions to describe each of the following sequences.

    a) $\{2, 4, 8, 16, ...\}$

    b) $\{1, 3, 5, 7, ...\}$

    c) $\{3, -6, 12, -24, ...\}$

    d) $\{1, 3, 6, 10, 15, 21, 28, ...\}$

    e) $\{2, 2, 2, 2, 2, ...\}$

    f) $\{1, 2, 3, 5, 8, 13, 21, 34, ...\}$

†29. Let a_n be the number of ways to climb n steps, if the person climbing the stairs can only take two steps or three steps at a time.

    a) Write down a recurrence relation with initial conditions for a_n.

    b) Find a_4 and a_8.

†30. Let a_n denote the number of bit strings of length n where no two consecutive zeros are allowed.

    a) Find a_1, a_2, a_3.

    b) Show that a_n satisfies the recurrence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

31. A bank lends me $50,000 at 18% per year interest, compounded monthly, and I pay back $900 per month. (So at the end of each month the amount I owe is increased by $\left(\frac{18}{12}\right)$% and then reduced by $900.)

    If $$u_n$ is the amount still owing after n months, write down a recurrence relation for u_n.

†32. a) Find the general solution of the first order recurrence $a_n = 5a_{n-1}$.

    b) Find the solution of the first order recurrence $a_n + 4a_{n-1} = 0$ subject to the initial condition $a_1 = -12$.

33. Find the general solution of the following recurrence relations, each holding for $n \geq 2$.

    a) $a_n + a_{n-1} - 6a_{n-2} = 0.$

    b) $a_n = 3a_{n-1} - 2a_{n-2}.$

    c) $a_n = 6a_{n-1} - 9a_{n-2}.$

    d) $a_n - 2a_{n-1} - 4a_{n-2} = 0.$

†34. Find the solution of the following recurrence relations (defined for $n \geq 2$) subject to the given initial conditions

    a) $a_n + 2a_{n-1} - 15a_{n-2} = 0, \qquad a_0 = 7, \quad a_1 = -3.$

    b) $a_n = 5a_{n-1} - 6a_{n-2}, \qquad a_0 = 5, \quad a_1 = 13.$

c) $a_n + 4a_{n-1} + 4a_{n-2} = 0, \qquad a_0 = 2, \quad a_1 = 4.$

d) $a_n - 4a_{n-1} - 6a_{n-2} = 0, \qquad a_0 = 2, \quad a_1 = 4.$

†35. Find the general solution of the following recurrence relations (defined for $n \geq 2$).

a) $a_n + a_{n-1} - 6a_{n-2} = 4^n$

b) $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$

c) $a_n = 6a_{n-1} - 9a_{n-2} + 8n + 4$

d) $a_n = 6a_{n-1} - 9a_{n-2} + 3^n$

36. Find the solution of the recurrence

$$a_n - 3a_{n-1} - 4a_{n-2} = 5(-1)^n \quad \text{for } n \geq 2, \text{ given that } \; a_0 = 1, \quad a_1 = 8.$$

†37. Suppose we wish to tile a $2 \times n$ rectangular board with smaller tiles of size 1×2 and 2×2.

Let a_n be the number of ways in which this can be done.

a) Show that, for $n > 2$, $\quad a_n = a_{n-1} + 2a_{n-2}$ and find a_1, a_2.

b) Solve the recurrence to find a closed formula for a_n.

*38. An n-digit *quaternary sequence* is a string of n digits chosen from the numbers $0, 1, 2, 3$.

Let a_n be the number n–digit of quaternary sequences with an even number of 0's.

a) Show that, for $n \geq 1$,

$$a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$$

and

$$a_0 = 1, a_1 = 3.$$

b) Find a closed formula for a_n.

†39. Define a set S of words on the alphabet $\{x, y, z\}$ by

(B) $\quad x, y \in S$

(R) $\quad$ If $w \in S$ then $wx, wy, wzx, wzy \in S$.

Let a_n be the number of words of length n in S.

a) Find the first few values of a_n.

b) Find a recurrence relation for a_n.

c) Solve the recurrence to find a closed formula for a_n.

†40. We call a word on the alphabet $\{x, y, z\}$ *z-abundant* if the letter z appears at least once in any two successive letters. (So, for example, the empty word is z-abundant and so is $xzzxzy$ but $xzyxzy$ is not.)

Let a_n be the number of z-abundant words of length n.

a) Show that $a_1 = 3$ and that $a_2 = 5$.

b) Explain carefully why, for $n \geq 2$,

$$a_n = a_{n-1} + 2a_{n-2}.$$

(Hint: Consider the possible ways in which a z-abundant word can begin.)

c) Find an explicit formula for a_n.

*41. a) Show that the set $\{1, 2, 3, ..., n\}$ can be partitioned into two non-empty sets in precisely $2^{n-1} - 1$ ways.

b) Let s_n be the number of ways in which the set $\{1, 2, 3, ..., n\}$ can be partitioned into three non-empty sets.

Show that $s_1 = 0, \quad s_2 = 0, \quad s_3 = 1$ and write down the six such partitions for $n = 4$.

c) Show that, for $n \geq 2$, the sequence s_n defined above satisfies $s_n = 3s_{n-1} + 2^{n-2} - 1$ and find a closed formula for s_n.

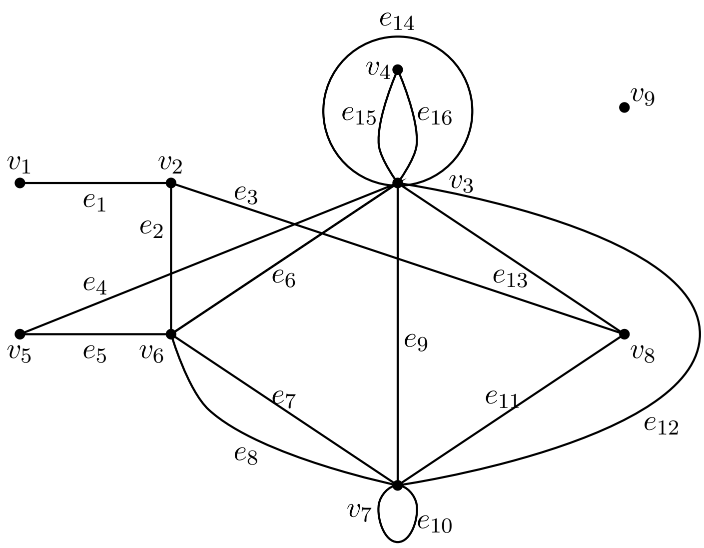
PROBLEM SET 5

Graphs

1. Draw the graph $G = (V, E, f)$ with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and edge–endpoint or incidence function $f : E \to \big\{\{x, y\} \,|\, x, y \in V\big\}$

e	$f(e)$
e_1	$\{v_3, v_4\}$
e_2	$\{v_1\}$
e_3	$\{v_1, v_5\}$
e_4	$\{v_2, v_5\}$
e_5	$\{v_2\}$
e_6	$\{v_5, v_2\}$
e_7	$\{v_4, v_2\}$

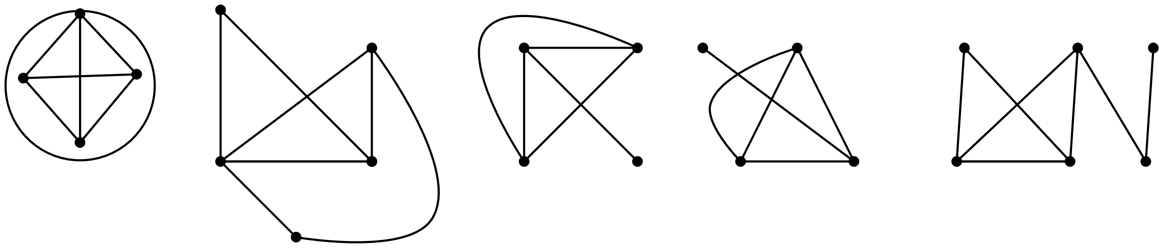
2. For the graph



Find

a) the edge–endpoint function (in a table),
b) the vertex degrees (in a table),
c) total vertex degree and total number of edges,
d) all loops,
e) all parallel edges,
f) edges incident on v_3,
g) vertices adjacent to v_8,
h) all isolated vertices.

†3. Determine with reasons which of the following graphs are simple.



G_1 G_2 G_3 G_4 G_5

†4. How many edges does a graph have if it has vertices of degree $4, 3, 3, 2, 2$? Draw a simple graph with these vertex degrees. (Is it the only example?)

†5. Determine whether or not there is a graph or simple graph for each of the following sequences of vertex degrees. Draw examples of those that exist. (Try to minimise the number of loops or parallel edges in non–simple examples.)

i) $4, 4, 3, 2, 2, 1$. ii) $4, 4, 3, 3, 2, 1$. iii) $5, 5, 3, 2, 2, 1$.

iv) $5, 4, 3, 2, 2$. v) $6, 5, 4, 4, 2, 2, 1$. vi) $6, 5, 4, 3, 2, 1, 1$.

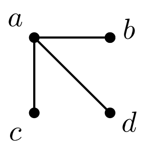
6. Draw the graphs

i) K_6. ii) $K_{2,3}$.

†7. Find the total number of vertices and edges for the special simple graphs

a) K_n
b) $K_{m,n}$
c) C_n, (cyclic graph with n vertices)
d) Q_n (n-cube graph)

†8. How many subgraphs are there of the graph G :



G

9. Recall that two vertices in the complement $\overline{G}$ are neighbours iff they are not neighbours in G. Find the following

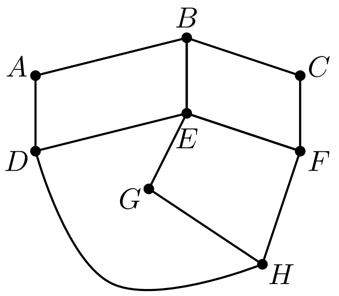
i) $\overline{K_n}$ ii) $\overline{K_{m,n}}$ iii) $\overline{C_n}$ for $n = 5, 6$

10. If a simple graph G has n vertices and m edges how many edges does $\overline{G}$ have?

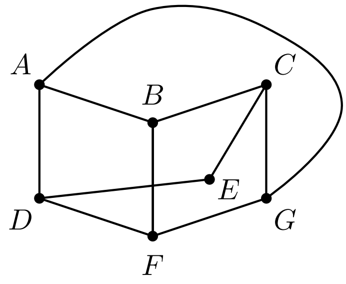
Bipartite Graphs

†11. Determine, with reasons, which of the following graphs is bipartite.

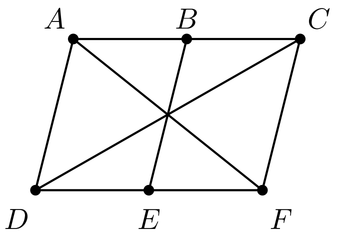
a)



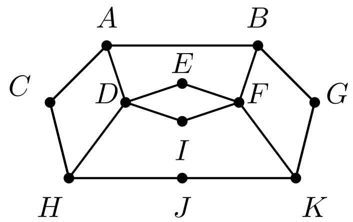
b)



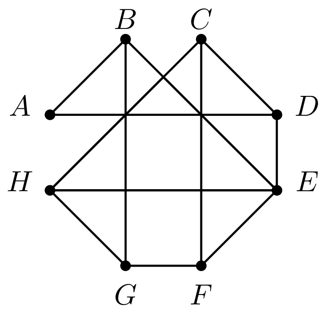
c)



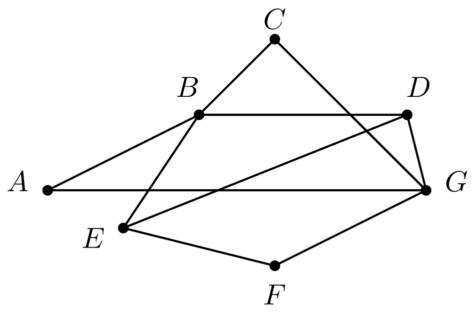
d)



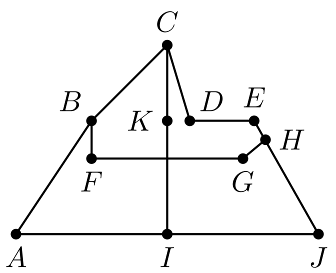
e)



f)



g)



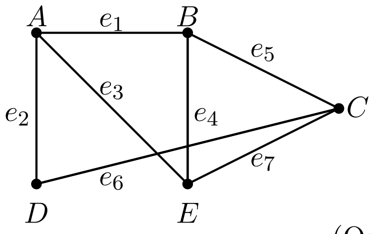
12. For what values of n are the following graphs bipartite?

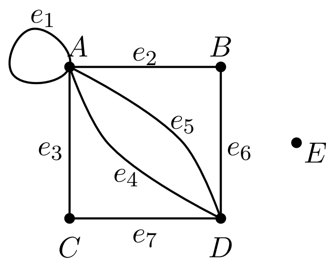
i) K_n ii) C_n iii) Q_n

Adjacency Matrices

†13. Represent the following graphs by adjacency matrices

a)





(Order vertices alphabetically)

b) K_4

c) $K_{2,3}$.

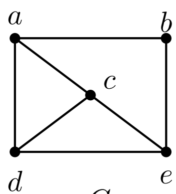
†14. Draw the graph with adjacency matrices

i) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & 1 & 3 & 2 \end{bmatrix}$

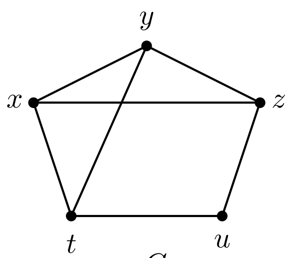
Isomorphism

†15. Determine, with reasons, whether or not the following pairs of graphs are isomorphic.

a)

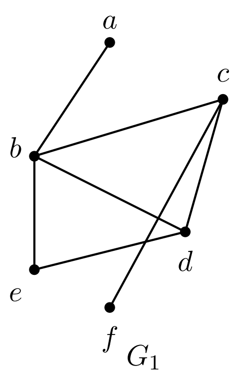


G_1

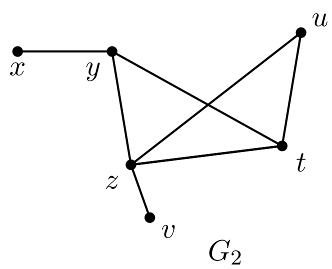


G_2

b)

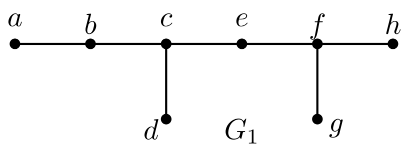


G_1

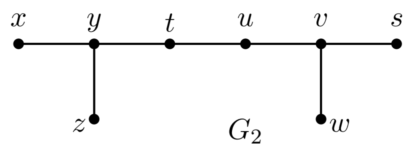


G_2

c)

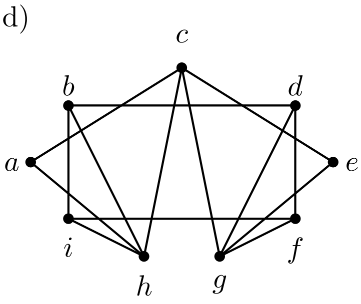


G_1

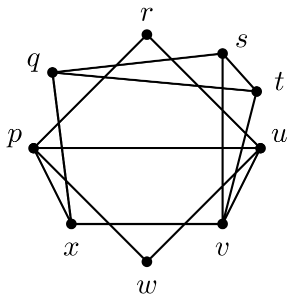


G_2

d)

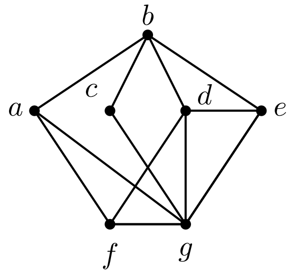


G_1

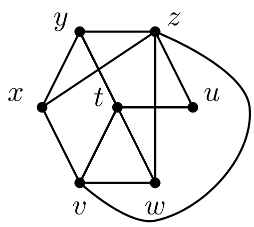


G_2

e)

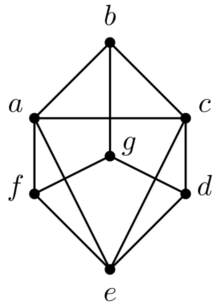


G_1

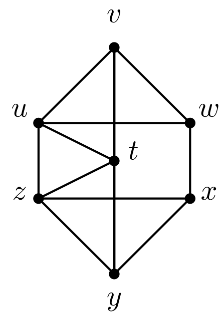


G_2

f)



G_1



G_2

16. A simple graph G is *self–complementary* if it is isomorphic to its complement $\overline{G}$ (see Q. 9).

    a) Show if G is self–complementary then G has $4k$ or $4k + 1$ vertices for some integer k. Is the converse true?

    *b) Find all self–complementary graphs with four or five vertices.

    ** c) (Challenge) Find a self–complementary graph with 8 vertices.

*17. Find all non–isomorphic simple graphs with

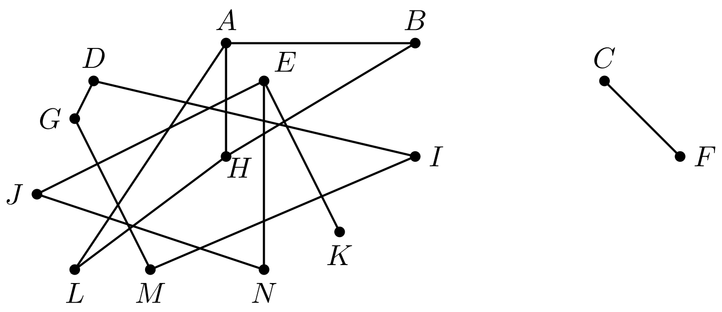
    a) 2 vertices,

    b) 3 vertices,

    c) 4 vertices.

Connectivity

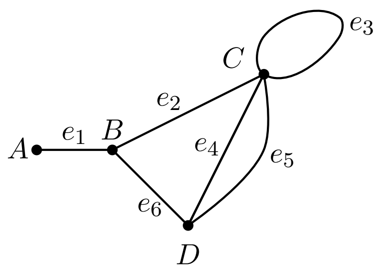
18. Find the connected components (draw separately) of the graph



Adjacency Matrix and Paths

19. What is the number of walks of length n between any two adjacent vertices of K_4 for $n = 2, 3, 4, 5$?

†20. a) Find the adjacency matrix M of the graph G



(ordering the vertices as A, B, C, D).

b) Find M^2, M^3.

c) How many walks of length 2 and 3 are there between B and C? Write them down.

d) Let $N = I_4 + M + M^2 + M^3$.

i) The $(3, 4)$ entry of N is 20. What does this mean in terms of walks?

ii) No entries of N are zero. What does this mean?

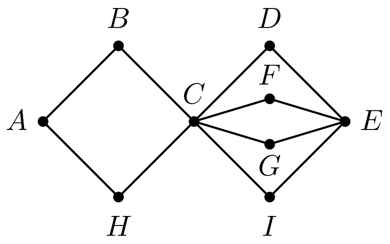
Euler and Hamilton Paths and Circuits.

†21. Determine, with reasons, whether or not there is

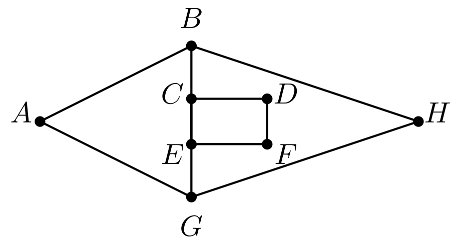
α) an Euler path, which is not a circuit,
β) an Euler circuit,
γ) a Hamilton path, which is not a circuit,
δ) a Hamilton circuit,

in the following graphs. Give an example, if one exists, in each case.

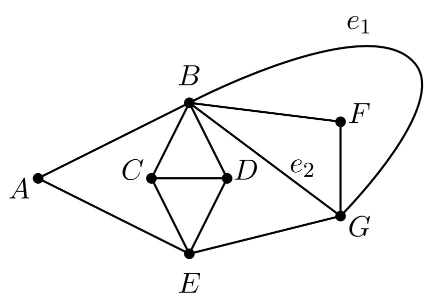
a)



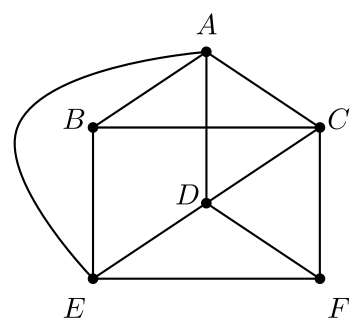
b)



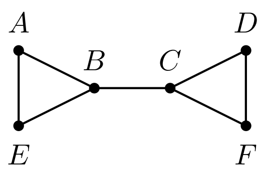
c)



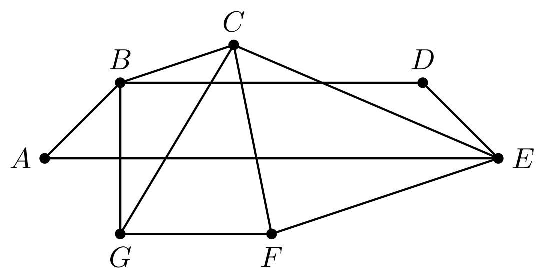
d)



e)



f)



†22. For what values of n does K_n, C_n, Q_n have an Euler circuit?

23. For what values of n does K_n, C_n, Q_n have an Euler path which is not a circuit?

†24. For what values of m, n does $K_{m,n}$ have

a) an Euler circuit,

b) an Euler path which is not a circuit,

c) a Hamilton circuit.

*25. Show each Q_n has a Hamilton circuit.

[Hint: Construct one recursively, constructing Hamiltonian paths between adjacent vertices of Q_n. Construct such a path for Q_{n+1} from one such path for Q_n].

*26. The **Knight's Tour Puzzle** asks if it is possible to find a sequence of 64 knight's moves so that a knight on a chessboard visits all the different squares and ends up on the starting point.

a) Formulate the problem in terms of graphs.

b) Can the puzzle be solved?

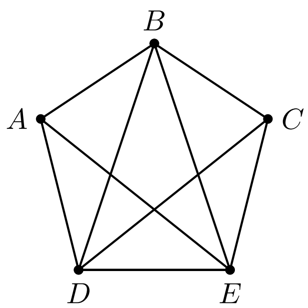
c) Can you find an explicit solution?

d) What happens on a 3×3 chessboard?

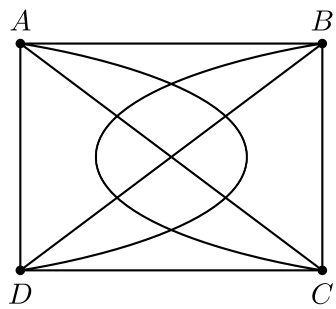
Planar Graphs

†27. Show that the following graphs are planar by redrawing them as planar maps.

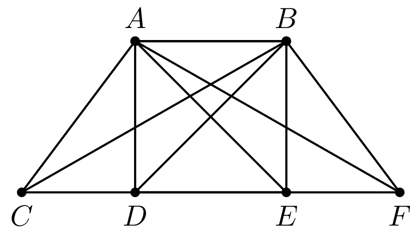
a)



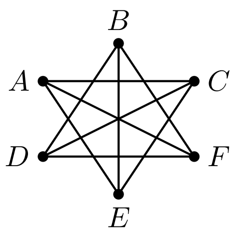
b)



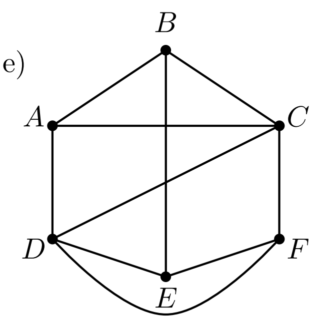
c)



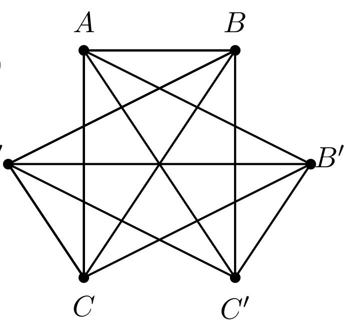
d)



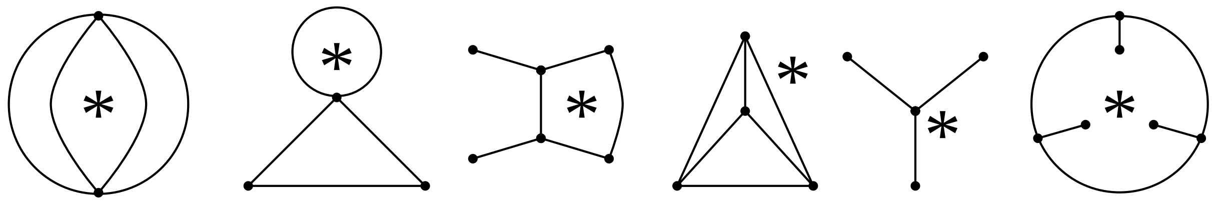
e)



*f)



†28. For each of



a) Find the degree of each of the regions indicated by an asterisk in each map.

b) What is the sum of the degrees of the regions in each map?

c) Give the dual of each of the planar maps, drawing it on a separate diagram.

d) Verify Euler's formula in the maps.

†29. A connected planar graph has 11 vertices; 5 have degree 1, 5 have degree 4, and 1 has degree 5.

a) How many edges are there?

b) How many regions are there?

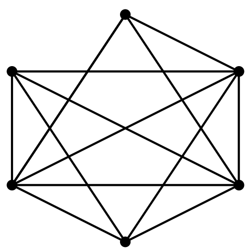
c) Give an example of such a graph, drawing it as a planar map.

30. a) Show if G is a connected planar simple graph with v vertices and e edges with $v \geq 3$ then $e \leq 3v - 6$.

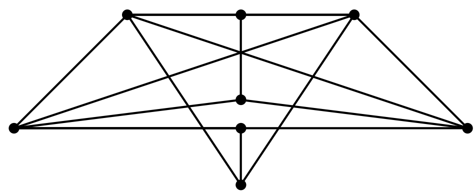
b) Further show if G has no circuits of length 3 then $e \leq 2v - 4$.

31. Apply the last question to show the following graphs are non–planar.

a)



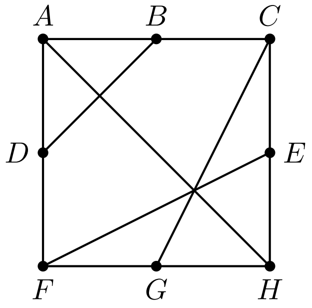
b)



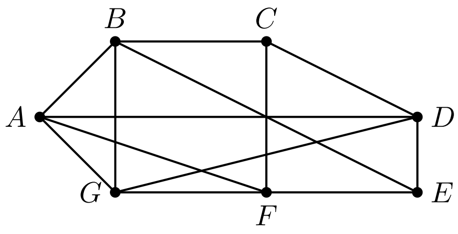
32. Use the results of Q30 to decide which Q_n are planar.

†33. Use Kuratowski's Theorem to show that the following graphs are not planar.

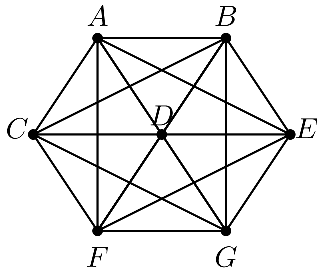
a)



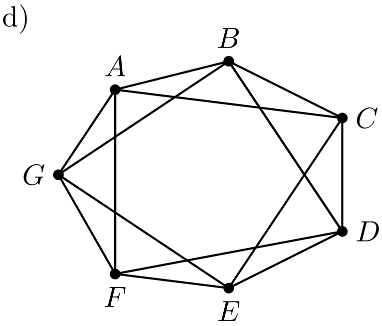
b)



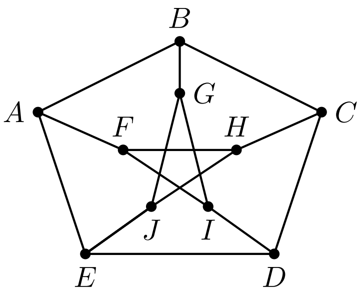
c)



d)

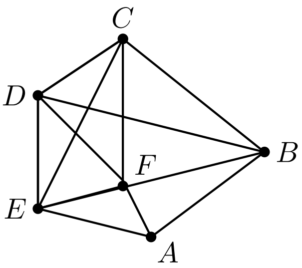


*e)

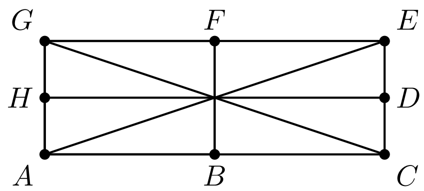


34. Show the converses of Q30 a) and b) are false by considering the following examples. (Hint: Kuratowski's Theorem.)

a)



b)



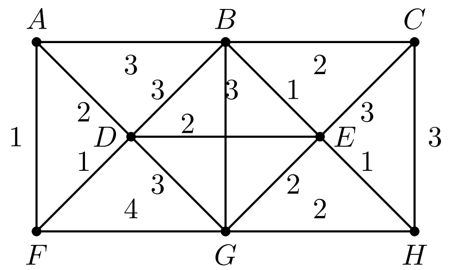
Trees

35. A tree T has 8 vertices, **at least** two of which have degree 3.
    a) How many edges are there?
    b) What are the possible vertex degrees for T in non–increasing order?
    c) What are the possible forms for T up to isomorphism?

*36. Prove that a tree has at least one vertex of degree 1.

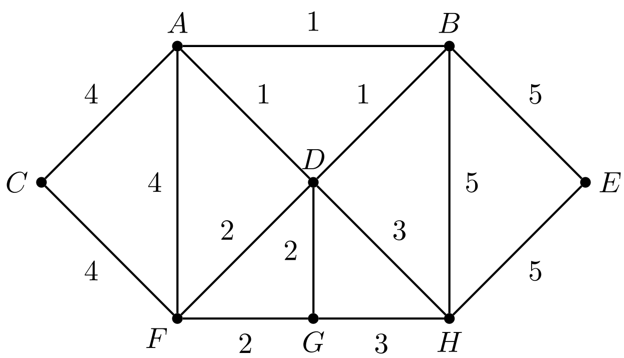
*37. Use Q36 to prove by induction that a tree with n vertices has $n-1$ edges.

†38. a) Use Kruskal's algorithm to find a minimal spanning tree for the following weighted simple graph.



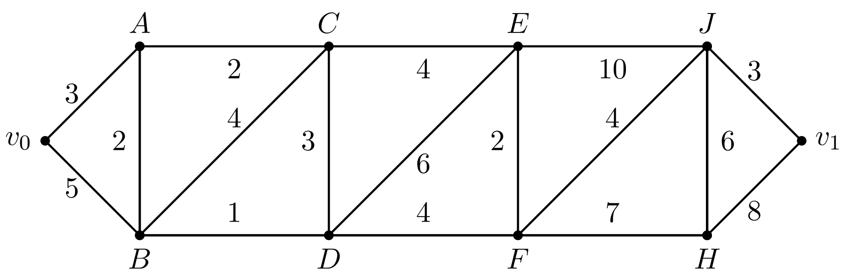
b) Use Dijkstra's algorithm to construct a tree giving shortest paths from A to each of the other vertices in the weighted graph in part (a).

39. a) Use Kruskal's algorithm to find a minimal spanning tree for the following weighted graph. Then find a second minimal spanning tree for the graph. How many other minimal spanning trees are there?



b) Use Dijkstra's algorithm to construct a tree giving shortest paths from A to each of the other vertices in the weighted graph in part (a).

40. Use Dijkstra's algorithm to find a shortest path from the vertex v_0 to the vertex v_1 in the following weighted graph.



ANSWERS TO SELECTED PROBLEMS

Important Note

Here are some answers (not solutions) and some hints to the problems. These are NOT intended as complete solutions and rarely are any reasons given. To obtain full marks in test and examination questions FULL reasoning must be given, and your work should be clearly and logically set out.

Problem Set 1.

1. $A = B$.

2. Yes.

5. a) 4; b) 16; c) 65536.

6. a) T; b) F; c) T; d) F; e) F; f) T; g) T

8. Yes.

10. a) No; b) No; c) Yes; d) Yes; e) T.

11. 6.

12. a) True; b) False.

13. a) $P(A) \cup P(B) \subseteq P(A \cup B)$; b) The sets are equal.

14. There is no containment relation.

15. a) $A - B$; b) A; c) U.

16. B.

17. $A \cup B$; $A \cap B$.

20. b) $\{2, 3, 4, 8, 9, 10, 14, 15, 16\}$.

22. a) $\{1, 2, \ldots, 21\}$; b) $\{90, \ldots, 105\}$; c) $\emptyset$.

23. a) $(0, 4]$; b) $(\frac{89}{90}, 10]$; c) $\{1\}$.

24. a) No; b) Yes; c) No; d) No; e) Yes.

25. a) 3; b) 4; c) -4; d) -3.

27. a) $1-1$, not onto; b) bijection; c) onto, not $1-1$; d) not $1-1$ and not onto.

28. a) One-to-one and onto; b) Neither one-to-one nor onto.

29. a) $f(x) \geq 2$; b) No; c) No, eg. $f(0) = f(4) = 6$.

31. f is not $1-1$ but is onto.

32. a) $\sqrt{4x^2 - 12x + 11}$; b) $g \circ f(x) = x$.

33. b) Yes.

35. a) $f^{-1}(x) = \frac{x-3}{5}$.

36. a) $[-1, 17]$; b) $[-3, -1] \cup [1, 3]$.

37. b) $(m, n) = (\pm 3, \pm 1)$.

38. a) i) The sets are equal. ii) $f(A \cap B) \subseteq f(A) \cap f(B)$.

39. 40430.

42. $\frac{9}{2} - \frac{4}{n-1} - \frac{1}{n}$.

44. $\frac{N+1}{2N}$.

Problem Set 2.

1. a) 2, 5; b) −11, 10; c) 77, 0.

2. T, F, T, F.

3. T, F, T, F, F, F.

4. $3^2 \cdot 13$, $11 \cdot 13$, $2^2 \cdot 3 \cdot 17^2$, $2^4 \cdot 3^3 \cdot 5^2 \cdot 7$.

5. a) $2^2 \cdot 3^3 \cdot 5^2$, $2^5 \cdot 3^5 \cdot 5^3$; b) 3, $2^2 \cdot 3^2 \cdot 5^3 \cdot 7$; c) 3, not defined.

6. 1, 3, 2.

8. a) $n = 6$, 4; b) $5^8 \equiv -1$, 6; c) −2, 10.

9. a) $6 = -1 \cdot 12 + 1 \cdot 18$; b) $3 = 29 \cdot 11 - 16 \cdot 201$; c) $1 = -8 \cdot 13 + 5 \cdot 21$;

   d) $1 = -13 \cdot 83 + 30 \cdot 36$; e) $2 = 5 \cdot 22 - 2 \cdot 54$; f) $7 = 39 \cdot 112 - 7 \cdot 623$.

10. a) $x \equiv 4 \pmod 7$; b) no solution; c) $x \equiv 6 \pmod{11}$;

    d) $x \equiv 502 \pmod{1001}$; e)$x \equiv 99, 206, 313 \pmod{321}$ or $x \equiv 99 \pmod{107}$

    f) $x \equiv 200, 751, 1302, 1853, 2404 \pmod{2755}$ or $x \equiv 200 \pmod{551}$

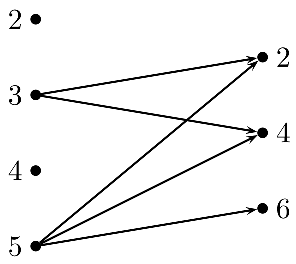
    g)$x \equiv 29, 70, 111, 152, 193, 234, 275, 316, 357, 398, 439, 480, 521 \pmod{533}$ or

    $x \equiv 29 \pmod{41}$

13. c) $R_3 = \{(3, 2),\ (3, 4),\ (5, 2),\ (5, 4),\ (5, 6)\}$.

14. R_3

a)



b) $M_{R_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

17. $\{\{0,3,6\},\{1,4\},\{2,5\}\}$

18. b) $\{\{0,5\},\{1,4,6\},\{2,3,7,8\}\}$

21. $S = \{0,6\} \cup \{1,5,7,11\} \cup \{2,4,8,10\} \cup \{3,9\}$.

24. b) 54, 60, 72 c) 2, 9 d) no e) no

f) $\{36,54,72\}$ g) none h) $\{2,4,6,12\}$ i) 12

Problem Set 3.

15. a) Converse is false. It is known that $2^p - 1$ is prime for the primes $p = 2,3,5,7,13,17,19,31,61,89,107,127$ and more than 20 other (larger) values of p. These are called Mersenne primes.

20. Converse is false.

21. b) True.
c) True.

59. a, c, e and f

60. a) $p =$"prices go up",
$q =$"housing will be plentiful";
$r =$"housing will be expensive"
$[p \to q \wedge r] \wedge [\sim r \to q]$

e) $p =$"Janet wins"; $q =$"Janet loses"; $r =$ "Janet will be tired"
$p \vee (q \to r)$.

61. a) The food is good but the service is poor.
b) The food is poor and so is the service.

62. a) Contradiction.

b) Contingency.

c) Tautology.

d) Tautology

64. a) $\sim q$

b) p

c) $p \wedge q \wedge r$

d) $\sim p \vee q$.

66. I did my homework, but there is no way of deciding whether or not I studied.

67. Holmes is correct.

68. We cannot deduce whether Einstein is right or wrong.

69. c) Peta must have failed.

Problem Set 4.

1. a) $5^2 \times 9^4$; b) $4 \times 5^2 \times 6^3$; c) $P(9,6) = 9 \times 8 \times 7 \times 6 \times 5 \times 4$; d) $C(6,3) \times 8^3$; e) $8^6 + 6 \times 8^5 + C(6,2)8^4$; f) $C(6,3)P(8,3)$; g) 4×9^4.

2. a) $21!/13!$, $21!/8!\,13!$, $321!/123!\,198!$, $(2n)!/(n!)^2$; b) 840, 35, 120, 6, 20100.

3. a) $7 \times 5 \times 21^6$; b) $C(7,2) \times 5^2 \times 21^5$; c) $C(7,3) \times 5^3 \times 21^4$; d) $26^7 - 21^7 - 7 \times 5 \times 21^6 - C(7,2) \times 5^2 \times 21^5$.

4. a) $C(7,1)P(5,1)P(21,6)$; b) $C(7,2)P(5,2)P(21,5)$; c) $C(7,3)P(5,3)P(21,4)$; d) $C(7,3)P(5,3)P(21,4) + C(7,4)P(5,4)P(21,3) + C(7,5)P(5,5)P(21,2)$.

5. a) 228; b) 37800; 37800; 22260; c) $20 \times 26^3 + 13 \times 26^2 + 18 \times 26 + 22 = 360798$.

6. a) $C(8,2) \times 25^6$; b) $26^8 - 25^8 - 8 \times 25^7$.

7. b) $4C(39,8) - 6C(26,8) + 4C(13,8)$.

8. a) 143; b) 128.

9. a) $13C(4,3)C(12,5)/C(52,8)$;

b) $(4C(13,3)C(39,5) - 6C(13,3)^2C(26,2))/C(52,8)$;

c) $4C(13,3) \times (C(39,5) - 3C(13,3)C(26,2))/C(52,8)$.

10. $\phi(pq) = (p-1)(q-1)$, $\phi(p^2q) = p(p-1)(q-1)$, $\phi(pqr) = (p-1)(q-1)(r-1)$;

16. a) 834; b) 834 again (NOT 417).

19. $4 \times 7 \times 6 \times 9 \times 16!/20!$.

20. $21!/(1!\,2!\,3!\,4!\,5!\,6!\,6^{21})$.

21. a) $\frac{1}{365}$; b) $1 - P(365,n)/365^n$; c) $n \geq 23$.

22. a) $C(23,3)$; b) $C(15,3)$; c) $C(23,3)-C(19,3)$.

23. a) 6^4; b) $C(9,5)$.

24. a) $23!/2!\,5!\,7!\,9!$; b) $C(26,3)$.

25. a) $C(45,5)$; b) $C(27,5)$;

c) $C(45,5)-C(6,1)C(34,5)+C(6,2)C(23,5)-C(6,3)C(12,5)$

d) $C(13,5)$; e) $C(25,5)$; f) $C(45,5)-C(25,5)$;

g) $C(22,5)$; h) $C(30,5)-3C(20,5)+3C(10,5)$.

28. (There are other correct answers.)

a) $a_n=2a_{n-1},\ a_0=2$; b) $a_n=a_{n-1}+2,\ a_0=1$; c) $a_n=-2a_{n-1},\ a_0=3$;
d) $a_n=a_{n-1}+n,\ a_1=1$; e) $a_n=a_{n-1}, a_0=2$; f) $a_n=a_{n-1}+a_{n-2},\ a_0=1,\ a_1=2$.

29. a) $a_n=a_{n-2}+a_{n-3}$ for $n\geq 4$, $a_1=0$, $a_2=1$, $a_3=1$; b) 1, 4.

30. 2, 3, 5.

31. $u_n=\left(1+\frac{18}{1200}\right)\ u_{n-1}-900,$ for $n\geq 1$, $u_0=50000$.

32. a) $a_n=A(5^n)$; b) $a_n=3(-4)^n$.

33. a) $a_n=A(2^n)+B(-3)^n$; b) $a_n=A+B(2^n)$; c) $a_n=A(3^n)+Bn(3^n)$;
d) $a_n=A(1+\sqrt{5})^n+B(1-\sqrt{5})^n$.

34. a) $a_n=4(3^n)+3(-5)^n$; b) $a_n=2^{n+1}+3^{n+1}$; c) $a_n=2(-2)^n-4n(-2)^n$;
d) $a_n=(2+\sqrt{10})^n+(2-\sqrt{10})^n$.

35. a) $a_n=A(2^n)+B(-3)^n+\frac{8}{7}\,4^n$; b) $a_n=A+B(2^n)+n(2^{n+1})$;
c) $a_n=A(3^n)+Bn(3^n)+2n+7$; d) $a_n=A(3^n)+Bn(3^n)+\frac{1}{2}\,n^2 3^n$.

36. $a_n=2(4^n)+(-1)^{n+1}+n(-1)^n$.

37. a) $a_1=1,\ a_2=3$; b) $a_n=\frac{1}{3}\left(2^{n+1}+(-1)^n\right)$.

38. b) $a_n=2^{n-1}+\frac{1}{2}\,4^n$

39. a) $0,2,4,12,32,88$; b) $a_n=2a_{n-1}+2a_{n-2}$; c) $\frac{1}{\sqrt{3}}\left((1+\sqrt{3})^n-(1-\sqrt{3})^n\right)$.

40. c) $a_n=\frac{1}{3}\left(2^{n+2}+(-1)^{n+1}\right)$

41. c) $\frac{1}{2}\left(3^{n-1}+1-2^n\right)$

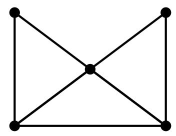
Problem Set 5.

2. c) Total Vertex degree = 32, total number of edges = 16.
d) Loops are e_{10}, e_{14}.
e) Sets of parallel edges are $\{e_7,e_8\}$, $\{e_9,e_{12}\}$, $\{e_{15},e_{16}\}$.

f) Edges incident to v_3 are $e_4, e_6, e_9, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.
g) Vertices adjacent to v_8 are v_2, v_3, v_7.
h) v_9 is the only isolated vertex.

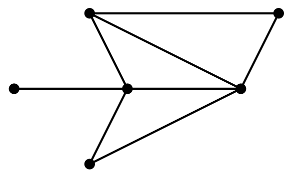
3. G_1, G_3, G_4 are nonsimple. The rest are simple.

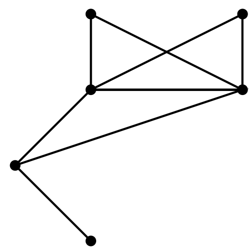
4. Number of edges = 7.



It is the only simple graph with these degrees. (The two degree 2 vertices cannot be adjacent.)

5. a) $\exists$ simple examples





b) There is no **graph** with these vertex degrees.

c) d) e) f) There is no **simple** graph with these vertex degrees.

7. $|V(G)|,\ |E(G)|$ are respectively

   a) $n,\ \binom{n}{2}$

   b) $m+n,\ mn$

   c) $n,\ n$

   d) $2^n,\ n2^{n-1}$

8. 35 $\quad (= 15 + 12 + 6 + 1 + 1)$

9. a) $\overline{K}_n \cong E_n$, ie, the graph with n vertices and no edges.

   b) $\overline{K}_{m,n} \cong K_m \oplus K_n$ (disjoint union)

10. $\dfrac{n(n-1)}{2} - m$

11. Y =Bipartite, N =Not Bipartite.

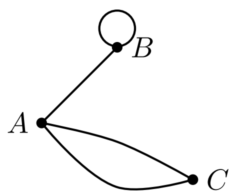
    a) Y b) N c) Y d) N e) Y f) N g) N.

12. a) $n = 2$ b) $n \geq 3$ and even c) all $n \geq 0$.

13. a) $\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

    b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

14. a)



15. a) Isomorphic eg. $f : V(G_1) \to V(G_2)$

$f:$	a	b	c	d	e
	t	u	y	x	z

    b) Isomorphic eg. $f : V(G_1) \to V(G_2)$

$f:$	a	b	c	d	e	f
	v	z	y	t	u	x

c) Not isomorphic.

d) Not isomorphic.

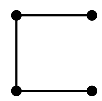
e) Isomorphic eg. $f : V(G_1) \to V(G_2)$

$$\begin{array}{c|ccccccc} f: & a & b & c & d & e & f & g \\ \hline & y & t & u & v & w & x & z \end{array}$$

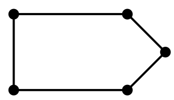
f) Not isomorphic.

16. b) Up to isomorphism there is/are

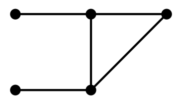
α) Only 1 self–complementary graph on 4 vertices



β) 2 self–complementary graphs on 5 vertices



and



17. a) $n = 2 : 2$ possible

b) $n = 3 : 4$ possible

c) $n = 4 : 11$ possible

18. There are 4 components with vertex sets $\{A, B, H, L\}$, $\{C, F\}$, $\{D, G, I, M\}$, $\{E, J, K, N\}$.

19. a) 2 b) 7 c) 20 d) 61.

20. a) $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$

b) $M^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 2 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 3 & 5 \end{pmatrix}$, $M^3 = \begin{pmatrix} 0 & 3 & 3 & 2 \\ 3 & 5 & 10 & 9 \\ 3 & 10 & 15 & 15 \\ 2 & 9 & 15 & 8 \end{pmatrix}$

c) 3 and 10 respectively.

Walks of length 2 B to C are

$$BDe_4C$$
$$BDe_5C$$
$$BCe_3C$$

Walks of length 3 B to C are

$$\begin{gathered} BABC \\ BDBC \\ BCBC \\ BCe_4De_4C \\ BCe_4De_5C \\ BCe_5De_4C \\ BCe_5De_5C \\ BDe_4Ce_3C \\ BDe_5Ce_3C \\ BCe_3Ce_3C \end{gathered}$$

d) i) This means there are 20 walks of length ≤ 3 from C to D.
   ii) This means G is connected.

21. a) α) N
   β) Y eg. $ABCDEFCGEICHA$
   γ) N
   δ) N

b) α) N
   β) N
   γ) Y
   δ) N

c) α) Y eg $CBAECDEGe_1BFGe_2BD$.
   β) N
   γ) Y eg $DCEABFG$
   δ) N

d) α) Y eg $BAEBCADCFDEF$
   β) N
   γ) Y eg $ABCDFE$
   δ) Y eg $ABCDFEA$.

e) α) Y eg $BAEBCDFC$
   β) N
   γ) Y eg $EABCDF$
   δ) N

f) α) Y eg $GBAEDBCGFCEG$
   β) N
   γ) Y eg $ABDECGF$
   δ) N.

22. a) $n \geq 1$ and odd
   b) All $n \geq 3$
   c) All $n \geq 0$ and even.

23. a) $n = 2$
   b) Never
   c) $n = 1$.

24. a) m, n both even

    b) $m = 2$ and n odd or $n = 2$ and m odd

    c) $m = n \geq 2$.

28. i) a) 2 b) 8
    ii) a) 1 b) 8
    iii) a) 4 b) 12
    iv) a) 3 b) 12
    v) a) 6 b) 6
    vi) a) 9 b) 12

29. a) 15 b) 6

32. Only for $n = 0, 1, 2, 3$.

36. a) 7 b) $3, 3, 3, 1, 1, 1, 1, 1$ and $3, 3, 2, 2, 2, 1, 1, 1, 1$ c) There are six possible answers.

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MATH1081 DISCRETE MATHEMATICS S1 2008
TEST 1 VERSION 1A

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note: To obtain full marks for this test **you must give reasons** for the steps you take and the conclusions you draw.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Prove that $\{10k+7 \mid k \in \mathbb{Z}\}$ is a proper subset of $\{5m-8 \mid m \in \mathbb{Z}\}$.

2. (*3 marks*)
   Let f be a function from A to B and g a function from B to C. Show that if the composite function $g \circ f$ is one–to–one (injective), then f is one–to–one (injective).

3. (*4 marks*)
   Prove that if $k > 1$ then
   $$\frac{1}{(k-1)^2} - \frac{1}{(k+1)^2} = \frac{4k}{(k^2-1)^2} \ .$$
   Hence simplify
   $$\sum_{k=2}^{n} \frac{k}{(k^2-1)^2} \ .$$

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MATH1081 DISCRETE MATHEMATICS S2 2008
TEST 1 VERSION 1B

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Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note: To obtain full marks for this test **you must give reasons** for the steps you take and the conclusions you draw.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Let
   $$A_1 = \{ a \} , \quad A_2 = A_1 \cup \{ A_1 \} \quad \text{and} \quad A_3 = A_2 \cup \{ A_2 \} .$$
   (i) List the elements of A_3.
   (ii) Are the following statements true or false? Give clear reasons.
   (i) $\{ \{ a \} \} \in A_3$; (ii) $\{ \{ a \} \} \subseteq A_3$.

2. (*3 marks*)
   Let f be a function from A to B and g a function from B to C. Show that if the composite function $g \circ f$ is one–to–one (injective), then f is one–to–one (injective).

3. (*4 marks*)
   Prove that if $k > 1$ then
   $$\frac{1}{k-1} + \frac{6}{k} - \frac{7}{k+1} = \frac{8k-6}{k(k^2-1)} .$$
   Hence simplify
   $$\sum_{k=2}^{n} \frac{4k-3}{k(k^2-1)} .$$

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TEST 1 VERSION 1B

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Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note: To obtain full marks for this test **you must give reasons** for the steps you take and the conclusions you draw.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
Let
$$A_1 = \{ a, b, c \} \ , \quad A_2 = P(A_1) \quad \text{and} \quad A_3 = P(A_2) \ .$$
(i) How many elements are there in A_3?
(ii) Are the following statements true or false? Give clear reasons.
(i) $A_2 \in A_3$; (ii) $A_2 \subseteq A_3$.

2. (*3 marks*)
A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$f(x) = 2x^3 + 3x^2 - 4 \ .$$
Find the range of f. Is f one–to–one (injective)? Is f onto (surjective)? Is f a bijection? Give reasons for all your answers.

3. (*4 marks*)
Prove that if $k > 1$ then
$$\frac{5}{k-1} - \frac{3}{k} - \frac{2}{k+2} = \frac{9k+6}{(k-1)k(k+2)} \ .$$
Hence simplify
$$\sum_{k=2}^{n} \frac{3k+2}{(k-1)k(k+2)} \ .$$

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MATH1081 DISCRETE MATHEMATICS S2 2009
TEST 1 VERSION 1A

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Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note: To obtain full marks for this test **you must give reasons** for the steps you take and the conclusions you draw.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Let $A = \{ n, s, w \}$.
   (i) Write in full the set $\mathcal{P}(A)$.
   (ii) How many elements are there in the set $\mathcal{P}(A \cup \mathcal{P}(A))$?

2. (*3 marks*)
   Let f be a function from A to B and let g be a function from B to C.
   (i) State what it means to say that f is a one–to–one function, and what it means to say that g is a one–to–one function.
   (ii) Prove that if both f and g are one–to–one functions, then the composite function $g \circ f$ is also one–to–one.

3. (*4 marks*)
   Use the laws of set algebra to simplify

$$(A \cap B^c) \cup (A^c \cap B^c)^c \; .$$

   Show your working and give a reason for each step.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2010
TEST 1 VERSION 2B

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note: To obtain full marks for this test **you must give reasons** for the steps you take and the conclusions you draw.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   If $A = \{ a \}$, $B = \{ b, c \}$ and $C = \{ d, e, f, g, h, i, j \}$, find
   (i) $P(A \times B)$;
   (ii) $|P(B \times C)|$.

2. (*3 marks*)
   Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by
   $$f(x) = x^2 \quad \text{and} \quad g(x) = e^x \ .$$
   (i) Write down formulae for the composite functions $f \circ g$ and $g \circ f$.
   (ii) Is $f \circ g$ a one-to-one function? Is $g \circ f$ a one-to-one function? Give reasons for your answers.

3. (*4 marks*)
   Use the formula
   $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$
   to prove that
   $$\tan k \tan(k-1) = \frac{\tan k - \tan(k-1)}{\tan 1} - 1 \ .$$
   Hence simplify
   $$\sum_{k=1}^{n} \tan k \tan(k-1) \ .$$

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2008
TEST 2 VERSION 2A

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Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Solve the following congruences, or explain why they have no solution:
   (i) $28x \equiv 3 \pmod{66}$;
   (ii) $29x \equiv 3 \pmod{67}$.

2. (*3 marks*)
   Consider the divisibility relation on the set
   $$S = \{\, 2, 6, 7, 14, 15, 30, 70, 105, 210 \,\} \ .$$
   It is given that this relation is a partial order on S (do not prove it!).
   (i) Draw the Hasse diagram for this partial order.
   (ii) Find all maximal elements and all minimal elements of S.
   (iii) Does S have a greatest element? Does S have a least element? If so, write them down; if not, explain why not.

3. (*4 marks*)
   A relation $\sim$ is defined on $\mathbb{Z}^+$ by
   $$x \sim y \quad \text{if and only if} \quad y = 3^k x \ \text{ for some integer } k$$
   (note that k may be positive, negative or zero). Prove that $\sim$ is an equivalence relation.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S2 2008
TEST 2 VERSION 2A

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Student's Family Name	Initials	Student Number
Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Solve the congruence $20x \equiv 16 \pmod{92}$. Give your answer as
   (i) a congruence to the smallest possible modulus;
   (ii) a congruence modulo 92.

2. (*3 marks*)
   Let a, b and c be integers. Prove that if $a^2 \mid b$ and $b^3 \mid c$ then $a^4b^5 \mid c^3$. Be sure to set out your answer clearly and logically.

3. (*4 marks*)
   A relation $\sim$ is defined on the set of all real numbers by

$$x \sim y \quad \text{if and only if} \quad \sin x = \sin y\ .$$

   You may assume that $\sim$ is an equivalence relation.
   (i) Find the equivalence class of 0.
   (ii) For any $a \in \mathbb{R}$, find the equivalence class of a.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2009
TEST 2 VERSION 2A

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Student's Family Name	Initials	Student Number
Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)

   Solve the congruence $45x \equiv 15 \pmod{78}$. Give your answer as

   (i) a congruence to the smallest possible modulus;

   (ii) a congruence modulo 78.

2. (*3 marks*)

   Let a and m be integers. Prove that if $a \mid m$ and $a+1 \mid m$ then $a(a+1) \mid m$. Be sure to set out your answer clearly and logically.

3. (*4 marks*)

   A relation $\preceq$ is defined on the set of all positive integers by

   $$x \preceq y \quad \text{if and only if} \quad y = 3^k x \text{ for some non–negative integer } k.$$

   Prove that $\preceq$ is a partial order.

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S2 2009
TEST 2 VERSION 2B

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Solve the following congruences, or explain why they have no solution:
   (i) $79x \equiv 5 \pmod{98}$;
   (ii) $78x \equiv 5 \pmod{99}$.

2. (*3 marks*)
   Let x, y and m be integers. Prove that if $m \mid 4x + y$ and $m \mid 7x + 2y$ then $m \mid x$ and $m \mid y$.

3. (*4 marks*)
   Let F be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. A relation $\preceq$ is defined on F by
   $$f \preceq g \quad \text{if and only if} \quad f(x) \leq g(x) \;\text{ for all } x \in \mathbb{R}\ .$$
   Prove that $\preceq$ is a partial order.

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2010
TEST 2 VERSION 1A

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Solve the following congruences, or explain why they have no solution:
   (i) $25x \equiv 3 \pmod{109}$;
   (ii) $25x \equiv 3 \pmod{110}$.

2. (*3 marks*)
   (i) Find the prime factorisation of 6500, and of 1120.
   (ii) Hence write down, in factorised form, $\gcd(6500, 1120)$ and $\operatorname{lcm}(6500, 1120)$.

3. (*4 marks*)
   We write $\mathbb{R}^+$ for the set of positive real numbers. A relation $\sim$ is defined on $\mathbb{R}^+$ by

$$x \sim y \quad \text{if and only if} \quad x - y \text{ is an integer} .$$

   Prove that $\sim$ is an equivalence relation.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2008
TEST 3 VERSION 3A

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number
Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   (i) Construct truth tables for the two propositional calculus formulae
   $$\sim p \rightarrow (q \wedge p) \quad \text{and} \quad (p \wedge q) \rightarrow q\ .$$
   (ii) Giving reasons for your answer, determine whether the first formula logically implies the second, or the second logically implies the first, or both, or neither.

2. (*3 marks*)
   Prove that $\log_6 11$ is irrational.

3. (*4 marks*)
   Show that
   $$q(n) = 11n^2 + 32n$$
   is a prime number for **two** integer values of n, and is composite for all other integer values of n.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S2 2008
TEST 3 VERSION 1A

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number
Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   (i) Construct truth tables for the two propositional calculus formulae

$$(p \to (\sim q)) \wedge r \quad \text{and} \quad q \to ((\sim p) \wedge r) \ .$$

   (ii) Giving reasons for your answer, determine whether the first formula logically implies the second, or the second logically implies the first, or both, or neither.

2. (*3 marks*)
   You are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow and any necessary algebraic details.

   Theorem. If m and n are positive integers then $m!\,n! < (m+n)!$.

   Basic ideas: $m! = 1 \times 2 \times \cdots \times m$, and $1 < m+1$, $2 < m+2, \ldots, n < m+n$.

3. (*4 marks*)
   Prove that if a is any positive real number then the equation

$$ax = \cos \pi x$$

   has exactly one solution x such that $0 \le x \le 1$.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2009
TEST 3 VERSION 2A

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Use standard logical equivalences to show that

   $$p \rightarrow (\sim(q \wedge (\sim p)))$$

   is a tautology. Show all your working and give a reason for every step. You may assume that $u \rightarrow v$ is logically equivalent to $(\sim u) \vee v$.

2. (*3 marks*)
   You are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow and any necessary algebraic details.

   Theorem. If n is a positive integer then

   $$(1 \times 2) + (2 \times 5) + (3 \times 8) + \cdots + n(3n-1) = n^2(n+1) \ .$$

   Basic ideas: $n^2(n+1) + (n+1)(3n+2) = (n+1)^2(n+2)$.

3. (*4 marks*)
   Prove that if x is a real number and $2x^2 - 3 = 0$ then x is irrational.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S2 2009
TEST 3 VERSION 1A

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number
Tutorial Code	Tutor's Name	Mark

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Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   (i) Express the following argument in symbolic form using logical connectives. Be careful to define any notation you introduce.

   "If I earn some money then I will go for a holiday this summer.
   I will either go for a holiday or work this summer.

   Therefore, if I don't go for a holiday this summer then I will not have earned any money and will be working."

   (ii) Show that the above argument **is** logically valid.

2. (*3 marks*)
   You are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow and any necessary algebraic details.

   Theorem. Between any two different rational numbers there is another rational number.

   Basic ideas: if x and y are rational then so is $\dfrac{x+y}{2}$.

3. (*4 marks*)
   Prove that if n is a positive integer then $4^{2n} + 10n - 1$ is a multiple of 25.

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2010
TEST 3 VERSION 2B

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number
Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*3 marks*)
   Use standard logical equivalences to show that
   $$(p \to q) \wedge (q \to (\sim p \vee r))$$
   is logically equivalent to $p \to (q \wedge r)$. Show all your working and give a reason for every step. You may assume that $u \to v$ is logically equivalent to $(\sim u) \vee v$.

2. (*3 marks*)
   Prove that if n is a positive integer then
   $$(n+1)(n+2)\cdots(2n) = 2^n \times 1 \times 3 \times 5 \times \cdots \times (2n-1)\ .$$

3. (*4 marks*)
   Prove that if n is any positive integer than $\sqrt{4n-2}$ is irrational.

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SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2008
TEST 4 VERSION 3B

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and P and C notation where appropriate.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*2 marks*)
   How many eight–letter words can be formed from the English alphabet
   (i) if no letter may be used twice?
   (ii) if one letter is to be used exactly twice and each other letter is to be used at most once? (DISCRETE is an example of such a word.)

2. (*2 marks*)
   An ordinary six–sided die is rolled ten times. Find the probability that five different faces come up twice each.

3. (*3 marks*)
   Find a particular solution of the recurrence relation
$$a_n - 8a_{n-1} + 15a_{n-2} = 21 \times 2^n \ .$$

4. (*3 marks*)
   How many hands of 13 cards can be chosen from a standard pack which have exactly five cards in some suit?

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S2 2008
TEST 4 VERSION 1B

This sheet must be filled in and stapled to the front of your answers

Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and P and C notation where appropriate.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*1 mark*)

   Find the number of seven-letter words which can be constructed from the English alphabet and which contain at least six consonants.

2. (*2 marks*)

   Find the probability that a hand of eight cards dealt from a standard 52-card pack contains exactly three kings.

3. (*3 marks*)

   (i) How many solutions has the equation

   $$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

   in which x_1, x_2, x_3, x_4, x_5 are non-negative integers? Explain your reasoning.

   (ii) How many of these solutions satisfy the conditions $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 2$ and $x_5 \geq 1$? Explain.

4. (*4 marks*)

   Find the general solution of the recurrence relation

   $$a_n - 16a_{n-2} = 9 \times 2^n \ .$$

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2009
TEST 4 VERSION 1A

This sheet must be filled in and stapled to the front of your answers

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Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and P and C notation where appropriate.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*2 marks*)
   How many solutions has the equation
   $$x_1 + x_2 + x_3 + x_4 + x_5 = 66 \ ,$$
   (i) if x_1, x_2, x_3, x_4, x_5 are non–negative integers?
   (ii) if x_1, x_2, x_3, x_4, x_5 are even numbers?

2. (*2 marks*)
   In a survey, viewers are given a list of 20 TV programmes. They are asked to label their three favourites $1, 2$ and 3, and to put a tick against those they have heard of (if any) from the remaining 17. In how many ways can the form be filled out? (Assume that everyone has three favourite programmes to nominate.)

3. (*3 marks*)
   Find the solution of the recurrence relation
   $$a_n - 10a_{n-1} + 21a_{n-2} = 0$$
   subject to the initial conditions $a_0 = -1$ and $a_1 = 1$.

4. (*3 marks*)
   There are 50 houses along one side of Discrete St. A survey shows that 26 of these houses have MATH1081 students living in them (what a coincidence). Prove that there are two MATH1081 students who live *exactly* five houses apart in Discrete St.

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S2 2009
TEST 4 VERSION 2B

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Student's Family Name	Initials	Student Number

Tutorial Code	Tutor's Name	Mark

Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and P and C notation where appropriate.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*2 marks*)

   How many ten–letter words can be constructed from the English alphabet which have no repeated letters and contain at least eight consonants?

2. (*2 marks*)

   How many thirteen–card hands can be selected from a standard pack which contain exactly four spades or exactly four diamonds?

3. (*3 marks*)

   Find the general solution of the recurrence relation

   $$a_n + 3a_{n-1} - 10a_{n-2} = 2^n \ .$$

4. (*3 marks*)

   A course has seven elective topics, and students must complete exactly three of them in order to pass the course. Show that if 200 students passed the course, at least six of them must have completed the same electives as each other.

UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1081 DISCRETE MATHEMATICS S1 2010
TEST 4 VERSION 2A

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Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and P and C notation where appropriate.

QUESTIONS (*Time allowed: 20 minutes*)

1. (*2 marks*)
   Students in a certain subject have 6 class hours per week. The timetable schedules 4 possible timeslots each day (Monday to Friday). In how many ways can a student timetable classes for this subject
   (i) if any choice of 6 class hours is allowed?
   (ii) if the student must choose two hours on one day and one hour on every other day?

2. (*2 marks*)
   How many solutions has the equation
   $$x_1 + x_2 + x_3 + x_4 + x_5 = 55 ,$$
   if x_1, x_2, x_3, x_4, x_5 are non–negative integers
   (i) with no further restrictions?
   (ii) and every x_k is an odd number?

3. (*3 marks*)
   Find the solution of the recurrence relation
   $$a_n - 7a_{n-1} + 10a_{n-2} = 0$$
   which satisfies the initial conditions $a_0 = 5$ and $a_1 = 1$.

4. (*3 marks*)
   How many eleven–letter words (constructed from the English alphabet) contain the subword FRED?