Theorem: If $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$ and

$$\int_a^\infty g(x)\, dx$$

converges, then $\int_a^\infty f(x)\, dx$ also converges.

Theorem: If $0 \le f(x) \le g(x)$ on $[a, \infty)$, and $\int_a^\infty f(x)$ diverges,

then $\int_a^\infty g(x)\, dx$ diverges also.

$$\int_2^\infty \frac{1}{\log x}\, dx.$$

We know $\frac{1}{\log x} > \frac{1}{x}$ for all $x > 0$,

and $\int_2^\infty \frac{1}{x}\, dx$ diverges.

Therefore, $\int_2^\infty \frac{1}{\log x}\, dx$ diverges.

Set

$g(x) = x - \log x$.

$g(1) = 1$

$g'(x) = 1 - \frac{1}{x}$.

$g'(x) > 0$ if $x > 1$.

$g(x)$ is increasing $[1, \infty)$.

$g(x) \geq 1$ for all $x \in [1, \infty)$.

$x - \log x \geq 1$ for all $x \in [1, \infty)$

So $x > \log x$ for $x \geq 1$.

$x > \log x$ for $x > 0$.

$\Rightarrow \frac{1}{\log x} > \frac{1}{x}$ for $x > 0$.

(limit comparison test)

Theorem: If f and g are non-negative functions defined on $[a, \infty)$ and both bounded. If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$ exists and is nonzero, then the integral

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either converge together or diverge together.

Example. $$\int_2^\infty \frac{1}{x^4+1}\,dx.$$

$$\lim_{x \to \infty} \frac{\frac{1}{x^4+1}}{\frac{1}{x^4}} = \lim_{x \to \infty} \frac{x^4}{x^4+1}$$

$$= \lim_{x \to \infty} \frac{1}{1+\frac{1}{x^4}} = 1$$

So $\int_2^\infty \frac{1}{x^4}\,dx$ and $\int_2^\infty \frac{1}{x^4+1}\,dx$ are either both convergent or both divergent.

By the p-test, $\int_2^\infty \frac{1}{x^4}\,dx$ converges, then ~~x~~ so does $\int_2^\infty \frac{1}{x^4+1}\,dx$.

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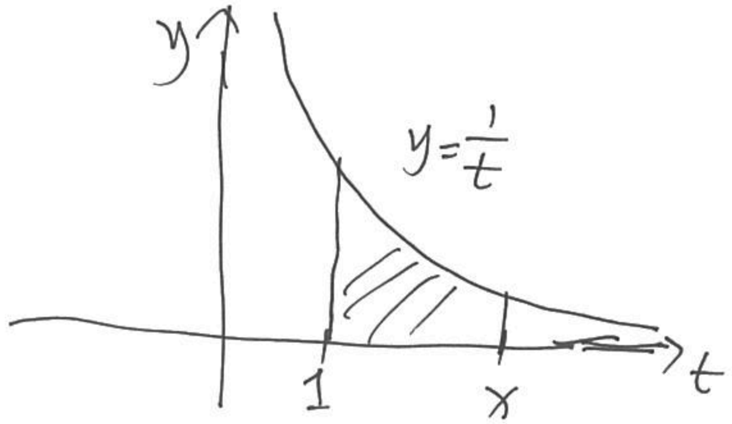
logarithms and exponentials.

$$y = \frac{1}{t}$$

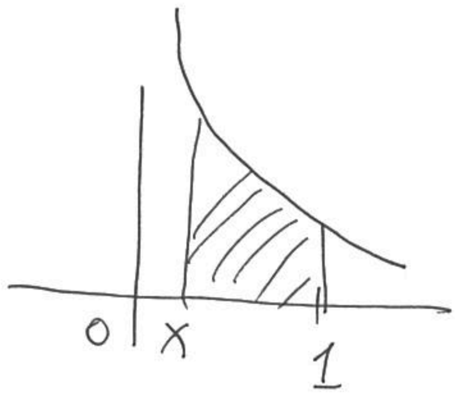
Define $F(x) = \int_1^x \frac{1}{t}\,dt$.



i) $F(1) = 0$

ii) $F(x) > 0$. if $x > 1$.

if $0 < x < 1$,



$$F(x) = \int_1^x \frac{1}{t}\,dt$$

$$= -\int_x^1 \frac{1}{t}\,dt < 0.$$

iii)

$$F(xy) = F(x) + F(y).$$

$$F(xy) = \int_1^{xy} \frac{1}{t}\,dt$$

$$= \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt$$

$$= F(x) + \int_x^{xy} \frac{1}{t}\,dt$$

$u = \dfrac{t}{x} \iff t = ux.$

$$\int_x^{xy} \frac{1}{t}\,dt \qquad\qquad \frac{dt}{du} = x$$

$$= \int_1^{y} \frac{1}{ux}\,x\cdot du \qquad \Rightarrow dt = x\cdot du$$

$$= \int_1^{y} \frac{1}{u}\,du = F(y).$$

Hence, $F(xy) = F(x) + F(y)$.

Similarly, $F\left(\frac{x}{y}\right) = F(x) - F(y)$.

iv) $F(x^n)$ if n is a positive integer

$$= F(\underbrace{x\cdot x \cdots x}_{n \text{ times}})$$

$$= n\,F(x).$$

We re-label $F(x)$ by $\log x$.

$$F(x) = \log x = \log_e x = \ln x.$$

$$\log x = \int_1^x \frac{1}{t}\,dt$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

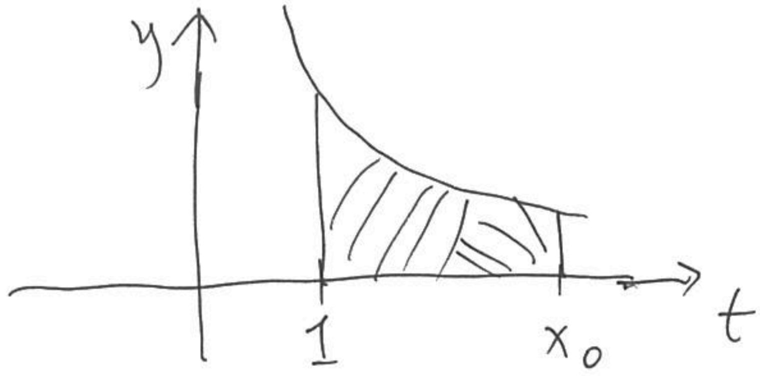
---

example. $\frac{d}{dx}\left(\log\left(\log x\right)\right)$

$$\lim_{x\to\infty} \log(\log x) = \infty.$$

$$\frac{d}{dx}\left(\log\left(\log x\right)\right)$$

$$= \frac{1}{\log x}\cdot\frac{1}{x}.$$



There is a value $x_0 \in [1, \infty)$, such that $\log x_0 = 1$.

We denote this number by e, and

$$e \approx 2.718 \ldots$$

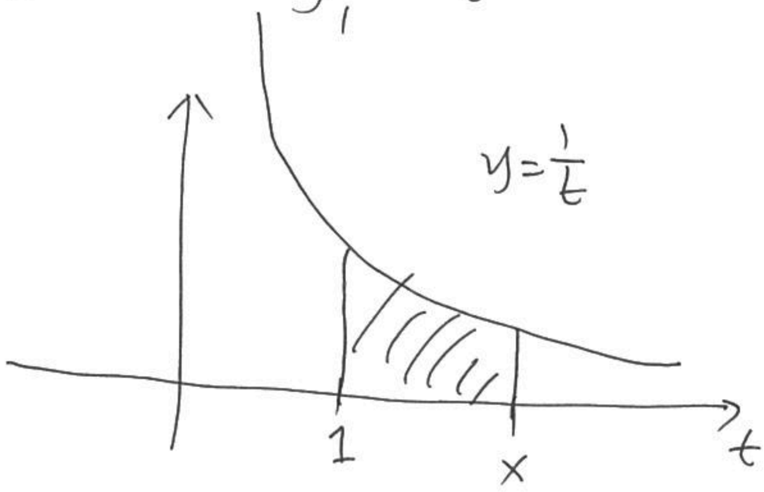
We know that

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}.$$

$$\log x = \int_1^x \frac{1}{t}\, dt.$$



$\log x$ is increasing for $x > 0$.

Therefore $\log x$ is one-to-one for $x > 0$.

Then $\log x$ has an inverse on $(0, \infty)$.

Since $\log x$ is the logarithic function with base e, its inverse is ~~an~~ the exponential function e^x.

$$e^x = \exp(x).$$

we get

$$\log(e^x) = x, \text{ for all } x.$$

$$\exp(\log x) = x, \text{ for all } x > 0.$$

If $y = e^x$, then $y' = ?$

$$\log(e^x) = x$$

$\frac{d}{dx}$: $$\frac{1}{e^x} \cdot \left(\frac{d}{dx} e^x\right) = 1$$

$$\frac{d}{dx} e^x = e^x.$$

If a is a positive real number,

then $$a^x = e^{x \log a}.$$

$$= \exp(\log a \cdot x).$$