# COMP3411/9414: Artificial Intelligence

Module 5

# Constraint Satisfaction Problems

Russell & Norvig, Chapter 6.1,6.2,6.3,6.4,4.1



Tatjana Zrimec, 2019

# Outline

- Constraint Satisfaction Problems
- CSP examples
- backtracking search and heuristics
- forward checking and arc consistency
- local search
  - hill climbing
  - simulated annealing

# Constraint Satisfaction Problems (CSPs)

- Constraint Satisfaction Problems are defined by a set of variables $X_i$, each with a domain $D_i$ of possible values, and a set of constraints $C$.
- The aim is to find an assignment of the variables $X_i$ from the domains $D_i$ in such a way that none of the constraints $C$ are violated.

# Constraint satisfaction problems (CSPs) *

Standard search problem:

- state is a "black box“ – any data structure that supports successor function, heuristic function, and goal test

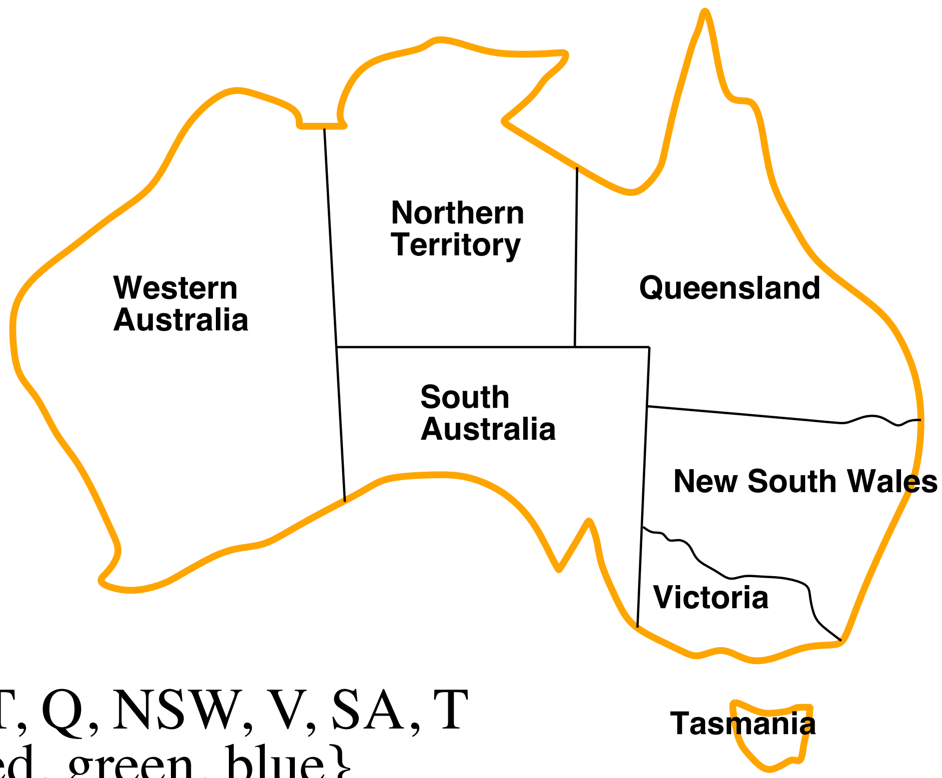
CSP:

- state is defined by variables $X_i$ with values from domain $D_i$
- goal test is a set of constraints specifying allowable combinations of values for subsets of variables

Simple example of a formal representation language

Allows useful general-purpose algorithms with more power than standard search algorithms

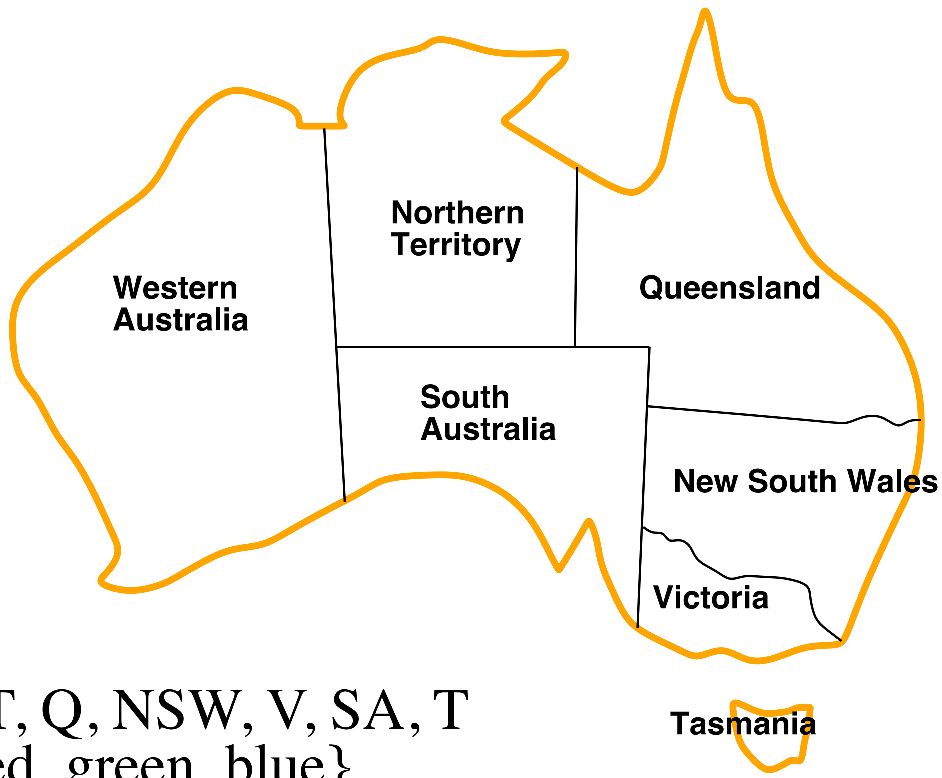
# Example: Map-Coloring




Variables WA, NT, Q, NSW, V, SA, T
Domains $Di$ = {red, green, blue}
Constraints: adjacent regions must have different colors

e.g. WA≠ NT, etc.

# Example: Map-Coloring

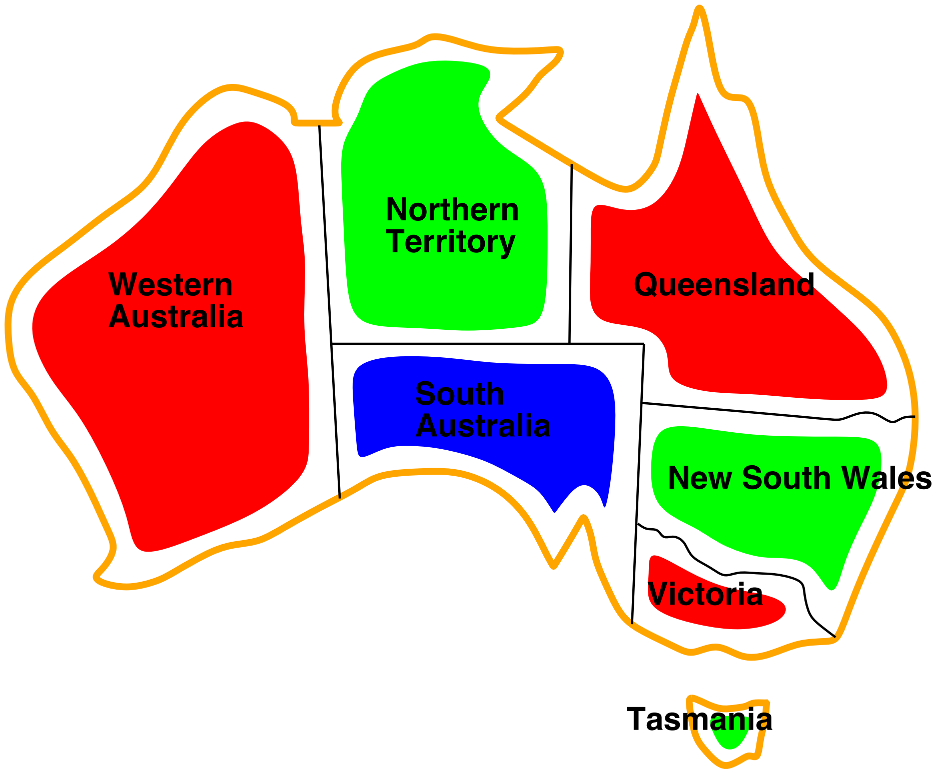



Variables WA, NT, Q, NSW, V, SA, T
Domains $D_i$ = {red, green, blue}
Constraints: adjacent regions must have different colors

e.g. WA≠ NT, etc.

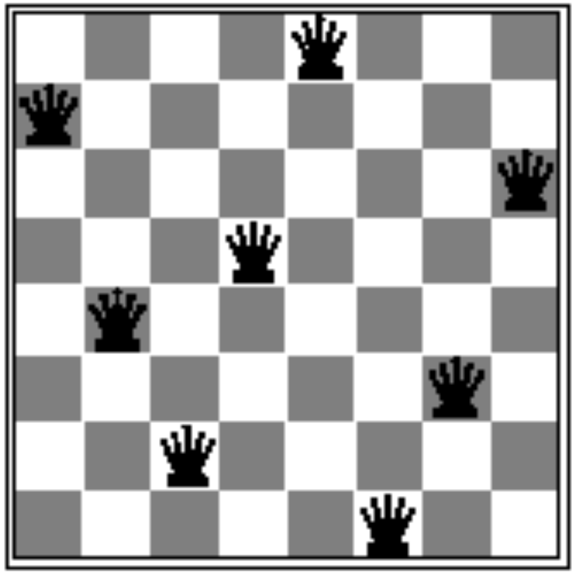
or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

# Example: Map-Coloring



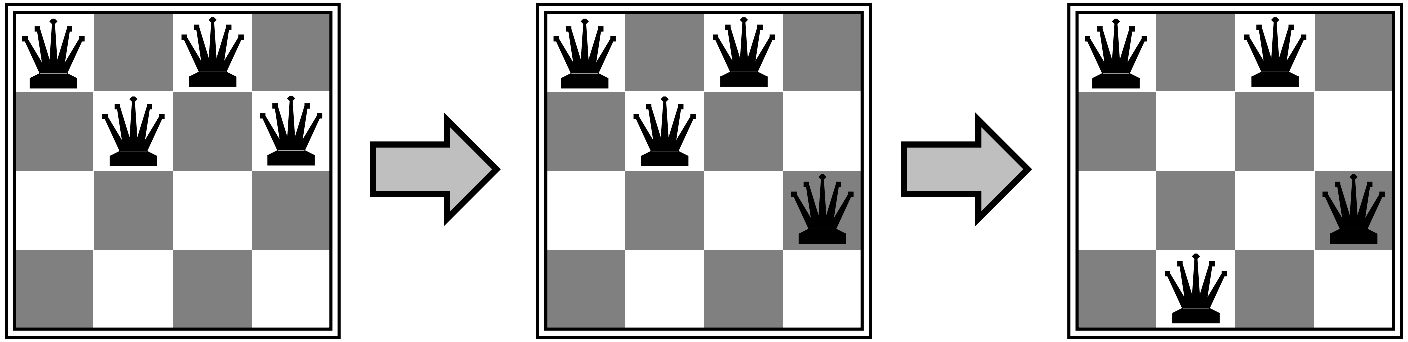

{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}

# Example: n-Queens Puzzle



Put $n$ queens on an $n$-by-$n$ chess board so that no two queens are attacking each other.

Assume one queen in each column. Which row does each one go in?



Variables: $Q1, Q2, Q3, Q4$ Domains: $Di = \{1,2,3,4\}$

Constraints:

$Qi \neq Qj$ (cannot be in same row)

$|Qi - Qj| \neq |i - j|$ (or same diagonal)

# Example: Cryptarithmetic

$$
\begin{array}{cccccc}
 & & S & E & N & D \\
+ & & M & O & R & E \\
\hline
 & M & O & N & E & Y
\end{array}
$$

Variables:
D E M N O R S Y
Domains:
{0,1,2,3,4,5,6,7,8,9}

Constraints:
$M$ /= 0, $S$ /= 0 (unary constraints)
$Y = D+E$ or $Y = D+E-10$, etc. $D$ /= $E$, $D$ /= $M$, $D$ /= $N$, etc.

# Example: Cryptarithmetic

$$
\begin{array}{ccccc}
 & S & E & N & D \\
+ & M & O & R & E \\
\hline
M & O & N & E & Y
\end{array}
$$

Variables: F T U W R O $X1X2X3$

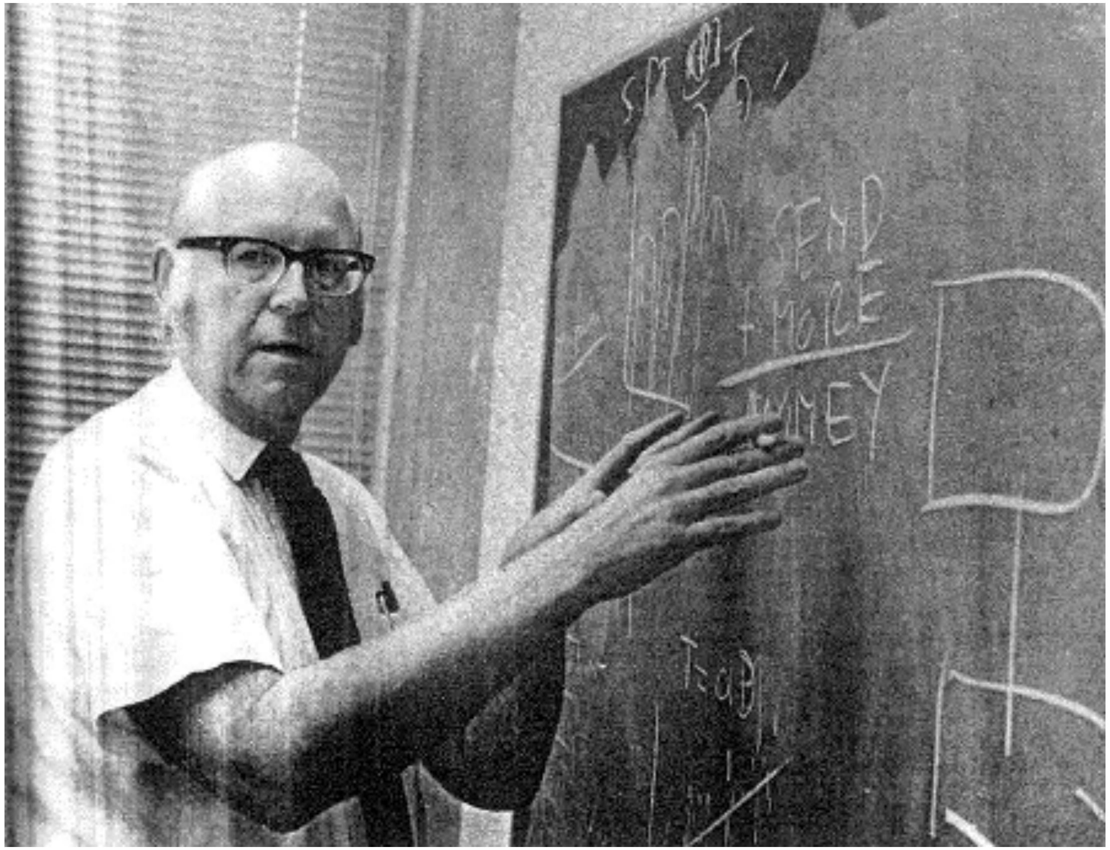
Domains: {0,1,2,3,4,5,6,7,8,9}

Constraints:

AllDifferent(F,T,U,W,R,O)

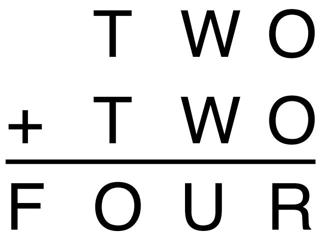
$O + O = R + 10 \cdot X1$, etc.

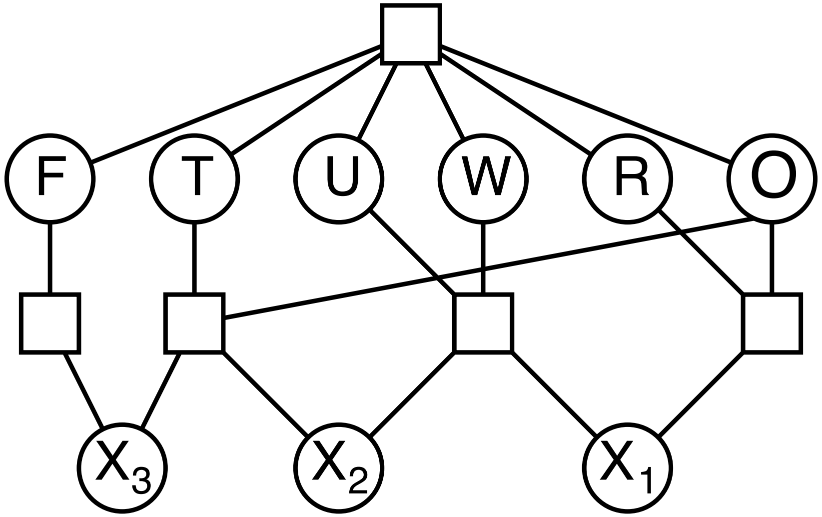
# Cryptarithmetic with Allen Newell



# Cryptarithmetic with Hidden Variables

- We can add “hidden” variables to simplify the constraints.




Variables: F T U W R O *X*1*X*2*X*3

Domains: {0,1,2,3,4,5,6,7,8,9}

Constraints:

AllDifferent(F,T,U,W,R,O)

O + O = R + 10·X1, etc.

# Example: Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9 | | | | 6 | | | | 3 |
| 1 | | 5 | | 9 | 3 | 2 | | 6 |
| | 4 | | | 5 | | | | 9 |
| 8 | | | | | | 4 | 7 | 1 |
| | | 4 | 8 | 7 | | | | |
| 7 | | 2 | 6 | | 1 | | | 8 |
| 2 | | | | | | | | |
| 5 | | | | 3 | 2 | | 9 | 4 |
| | 8 | 7 | | 1 | 6 | 3 | 5 | |

# Real-world CSPs

- Assignment problems (e.g. who teaches what class)
- Timetabling problems (e.g. which class is offered when and where?)
- Hardware configuration
- Transport scheduling
- Factory scheduling

# Varieties of constraints

- Unary constraints involve a single variable
  - $M \neq 0$
- Binary constraints involve pairs of variables
  - $SA \neq WA$
- Higher-order constraints involve 3 or more variables
  - $Y = D + E$ or $Y = D + E - 10$
- Inequality constraints on Continuous variables
  - $\text{EndJob1} + 5 \leq \text{StartJob3}$
- Soft constraints (Preferences)
  - 11am lecture is better than 8am lecture!

# Backtracking Search

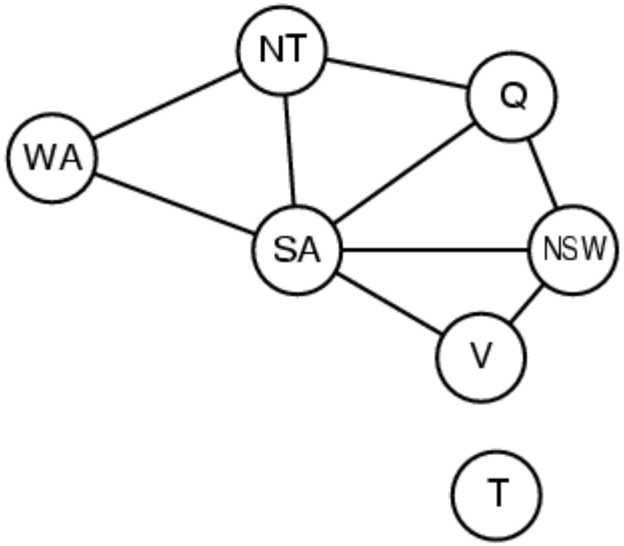
CSPs can be solved by assigning values to variables one by one, in different combinations. Whenever a constraint is violated, we go back to the most recently assigned variable and assign it a new value.

This can be achieved by a Depth First Search on a special kind of state space, where states are defined by the values assigned so far:

- Initial state: the empty assignment.
- Successor function: assign a value to an unassigned variable that does not conflict with previously assigned values of other variables. (If no legal values remain, the successor function fails.)
- Goal test: all variables have been assigned a value, and no constraints have been violated.

# Constraint graph

Constraint graph: nodes are variables, arcs are constraints




Binary CSP: each constraint relates two variables

# Backtracking example

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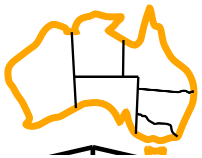

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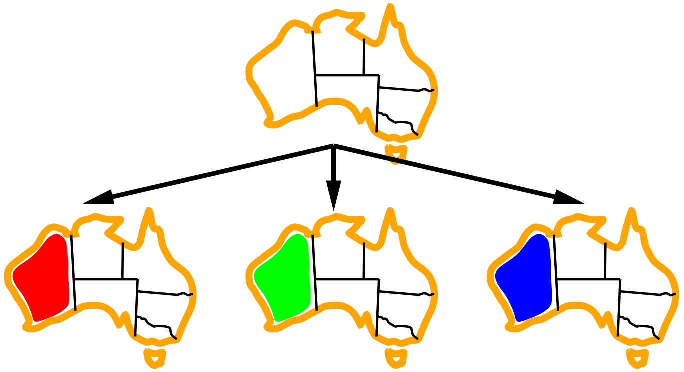
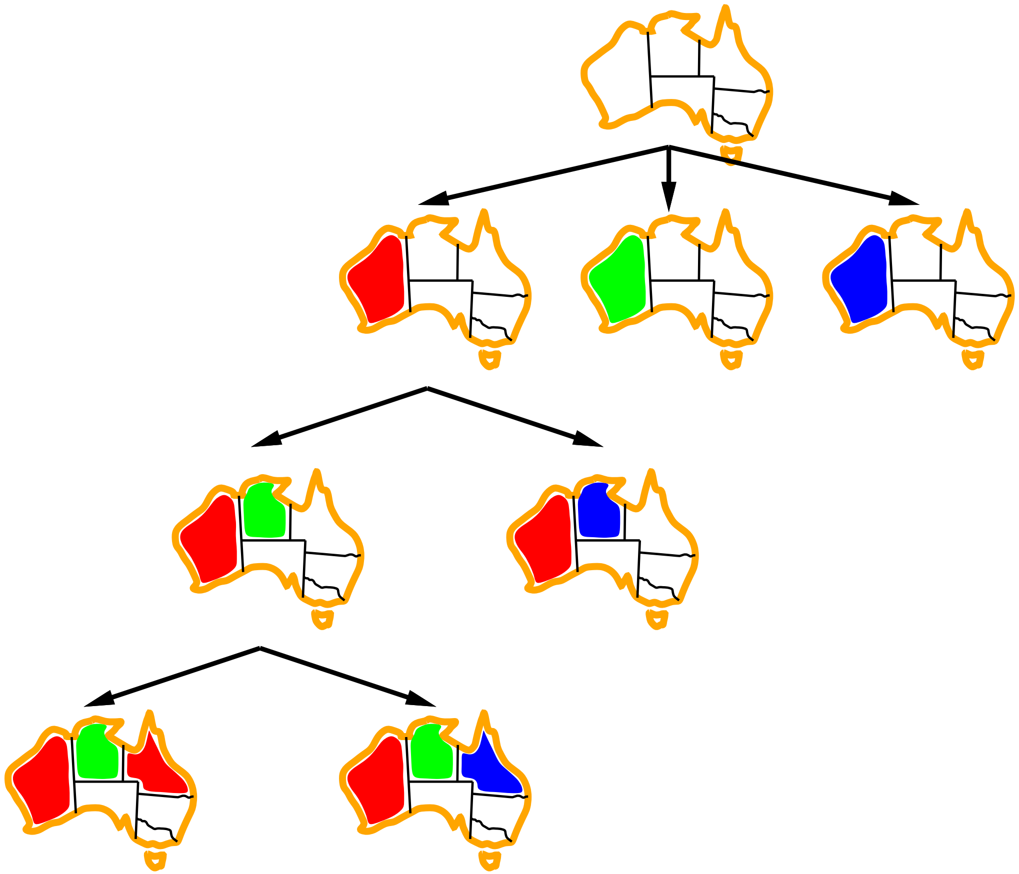


# Backtracking example



# Path Search vs. Constraint Satisfaction

Important difference between Path Search Problems and CSP's:

- Constraint Satisfaction Problems (e.g. n-Queens)
  - difficult part is knowing the final state
  - how to get there is easy
- Path Search Problems (e.g. Rubik's Cube)
  - knowing the final state is easy
  - difficult part is how to get there

# Backtracking search

The search space for this Depth First Search has certain very specific properties:

- if there are $n$ variables, every solution will occur at exactly depth $n$.
- variable assignments are commutative

  [WA = red then NT = green] same as [NT = green then WA = red]

Backtracking search can solve $n$-Queens for $n \approx 25$

# Improvements to Backtracking search

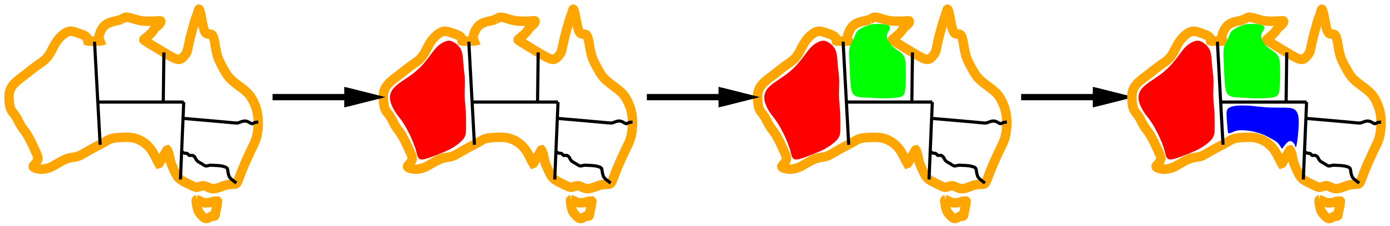
General-purpose methods can give huge gains in speed:

1. which variable should be assigned next?
2. in what order should its values be tried?
3. can we detect inevitable failure early?

Improving backtracking efficiency

# Most constrained variable

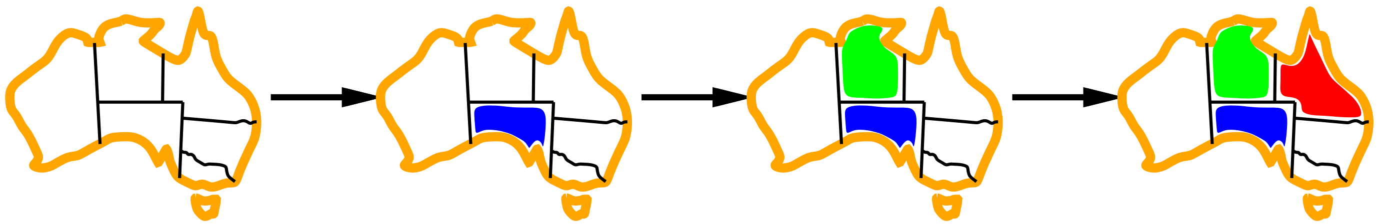
- Most constrained variable:
- choose the variable with the fewest legal values



- a.k.a. minimum remaining values (MRV) heuristic

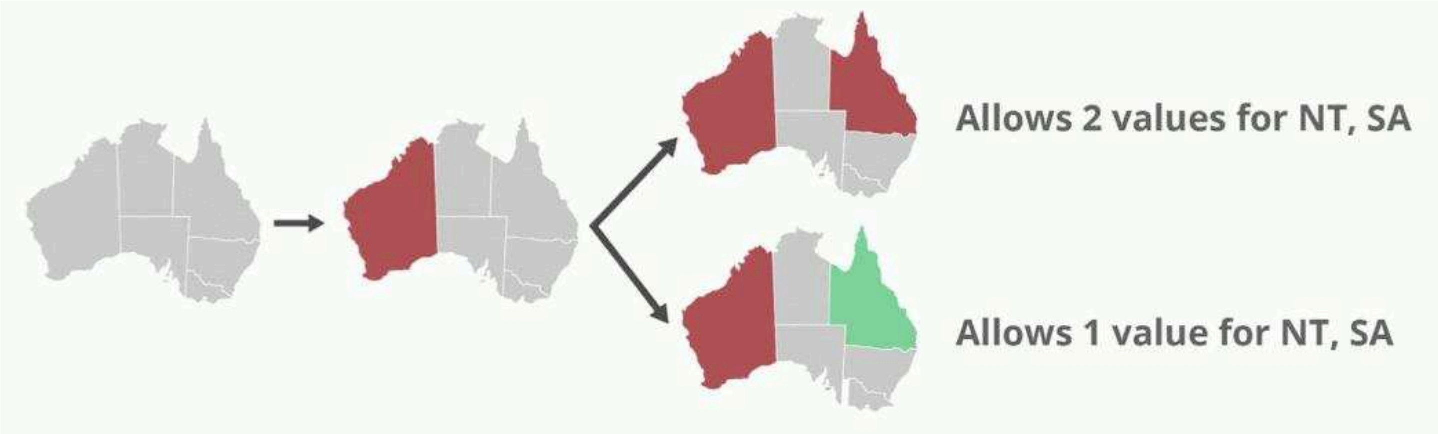
# Most constrained variable

- Tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on



# Least Constraining Value

- Given a variable, choose the least constraining value: the one that rules out the fewest values in the remaining variables

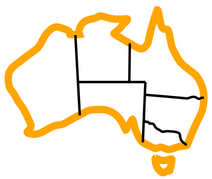



- (More generally, 3 allowed values would be better than 2, etc.) Combining these heuristics makes 1000 queens feasible.

# Forward checking

Idea:

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



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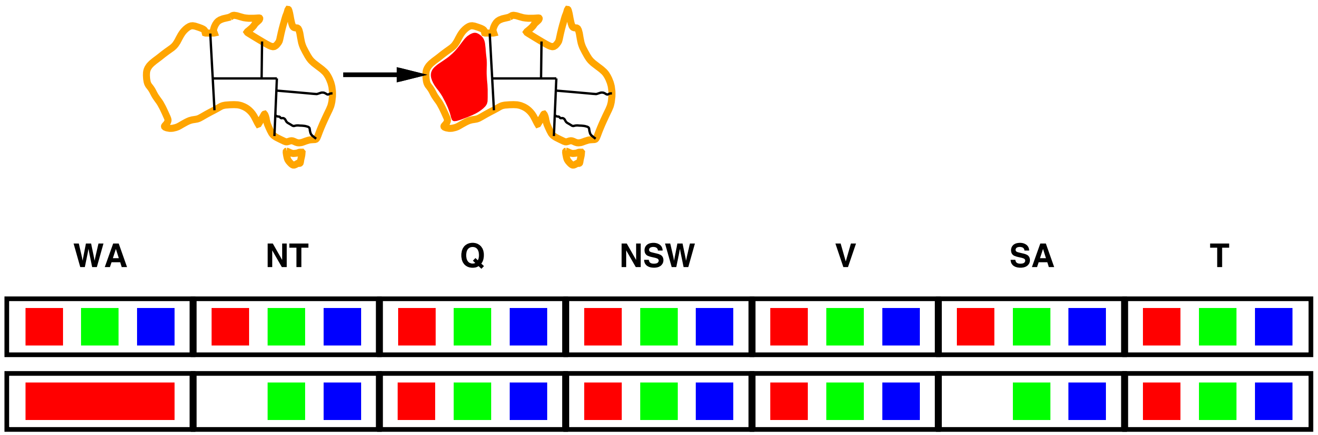

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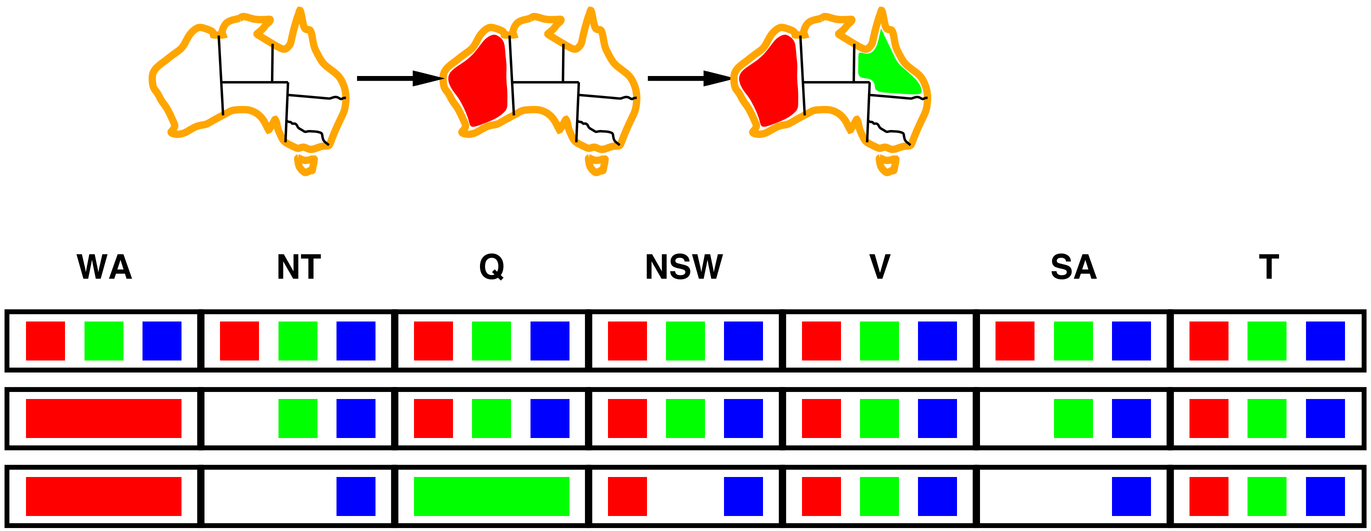

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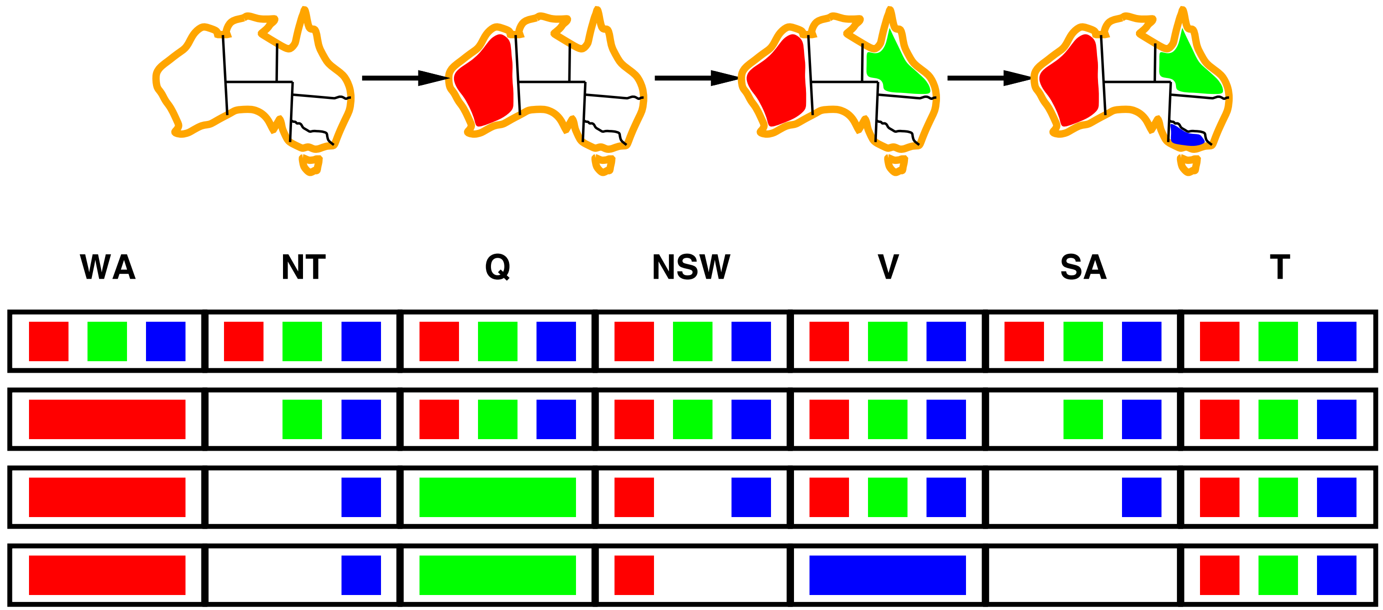
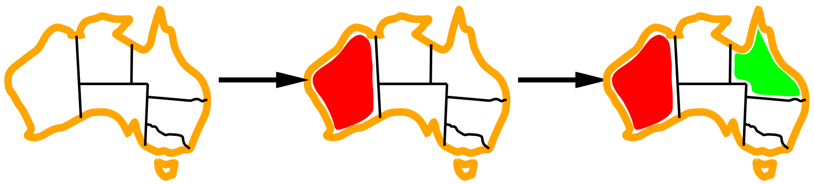


# Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:




NT and SA cannot both be blue!

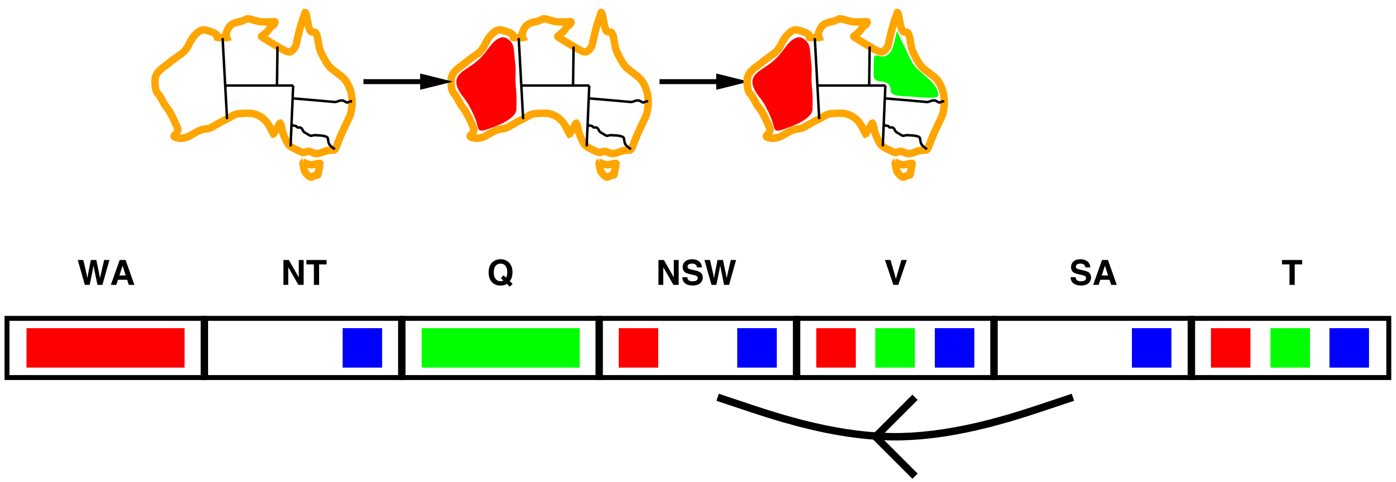
Constraint propagation repeatedly enforces constraints locally

# Arc consistency

Simplest form of propagation makes each arc consistent

$X \rightarrow Y$ is consistent iff

for every value $x$ of $X$ there is some allowed $y$



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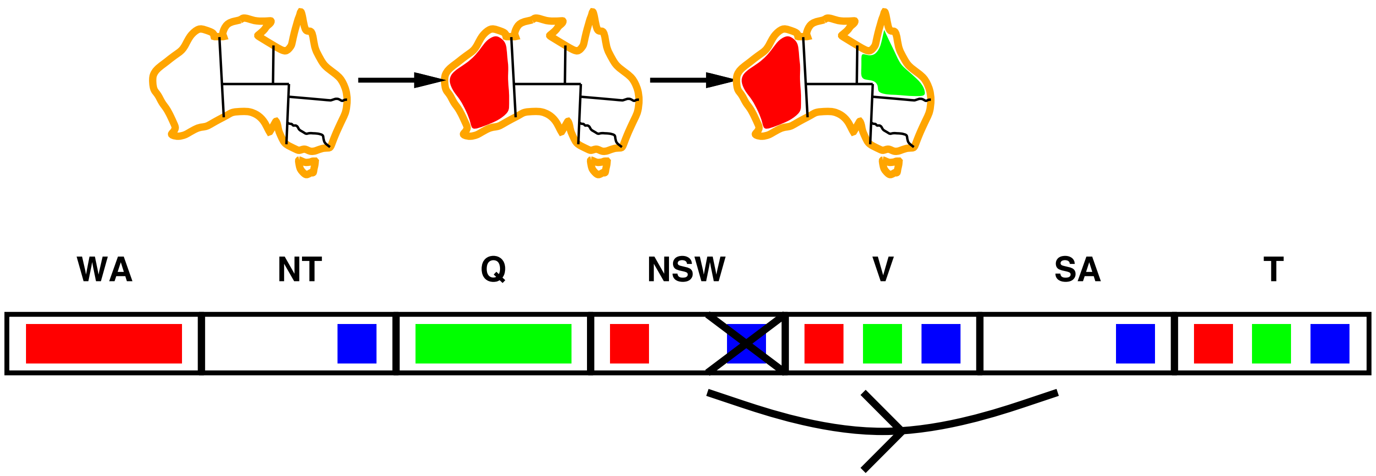

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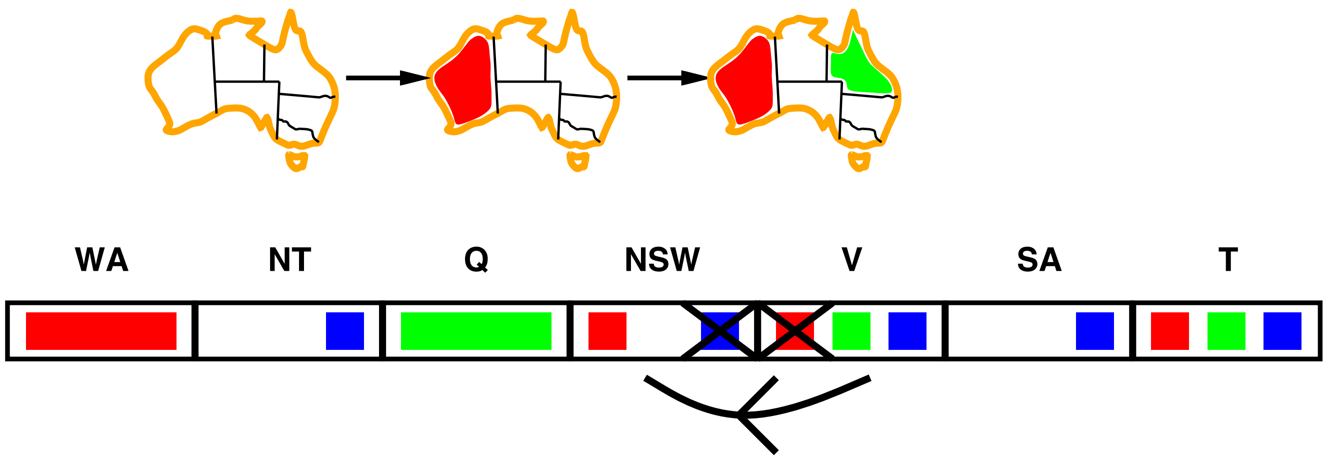


If $X$ loses a value, neighbors of $X$ need to be rechecked.

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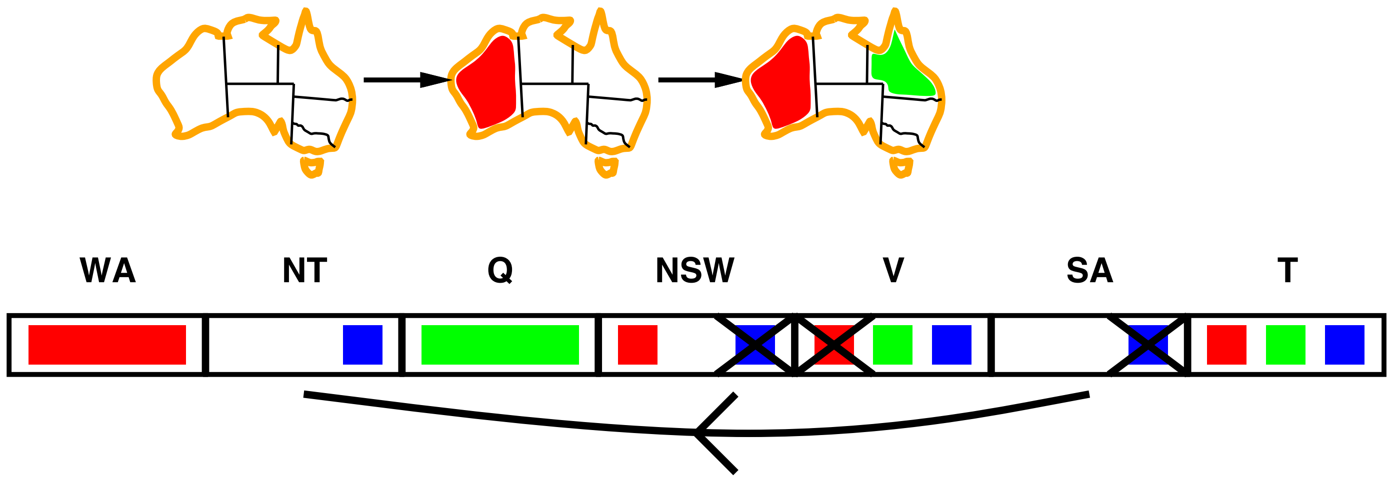


Arc consistency detects failure earlier than forward checking.
For some problems, it can speed up the search enormously.
For others, it may slow the search due to computational overheads.

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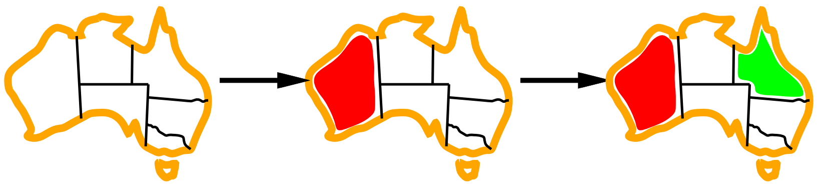
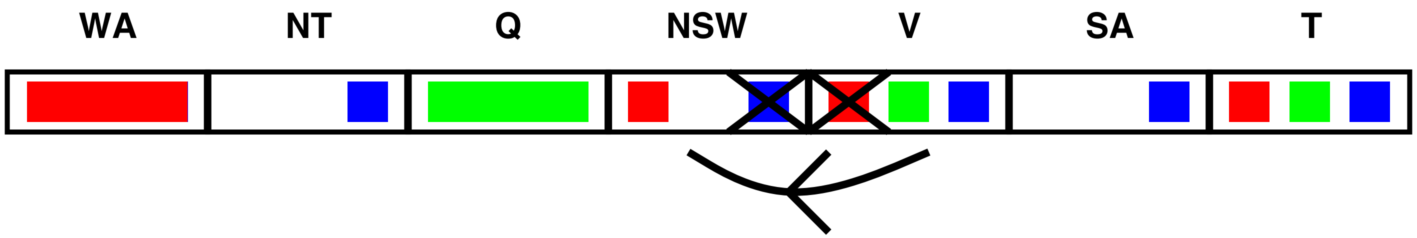
$X \rightarrow Y$ is consistent iff

for every value $x$ of $X$



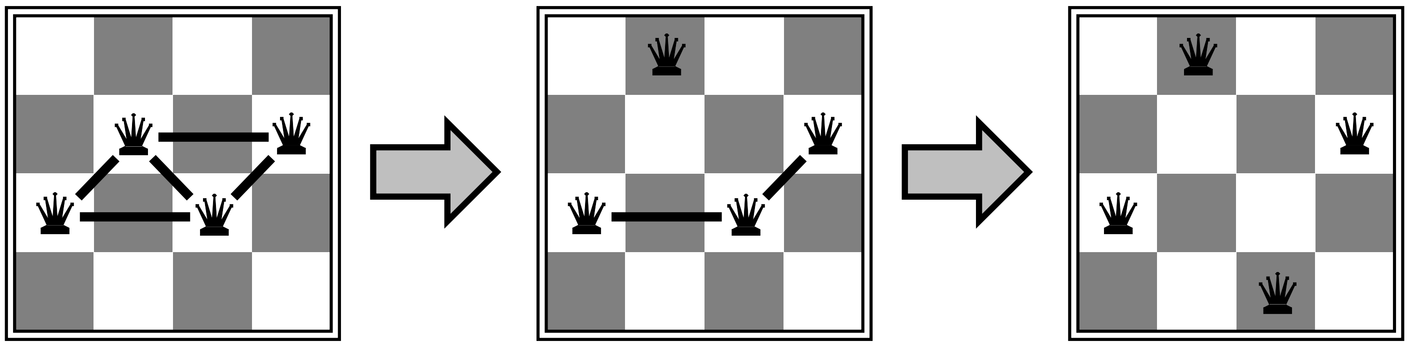



- If $X$ loses a value, neighbors of $X$ need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

# Local Search

There is another class of algorithms for solving CSP's, called "Iterative Improvement" or "Local Search".

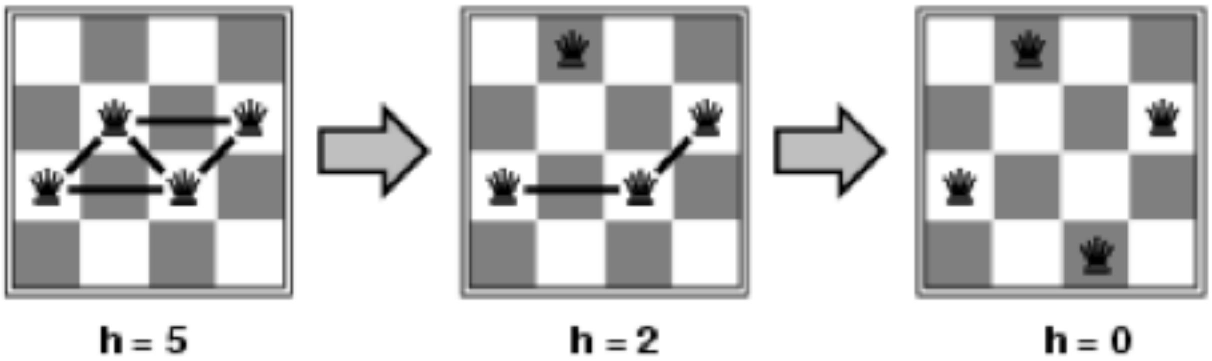
These algorithms assign all variables randomly in the beginning (thus violating several constraints), and then change one variable at a time, trying to reduce the number of violations at each step.



h = 5 h = 2 h = 0

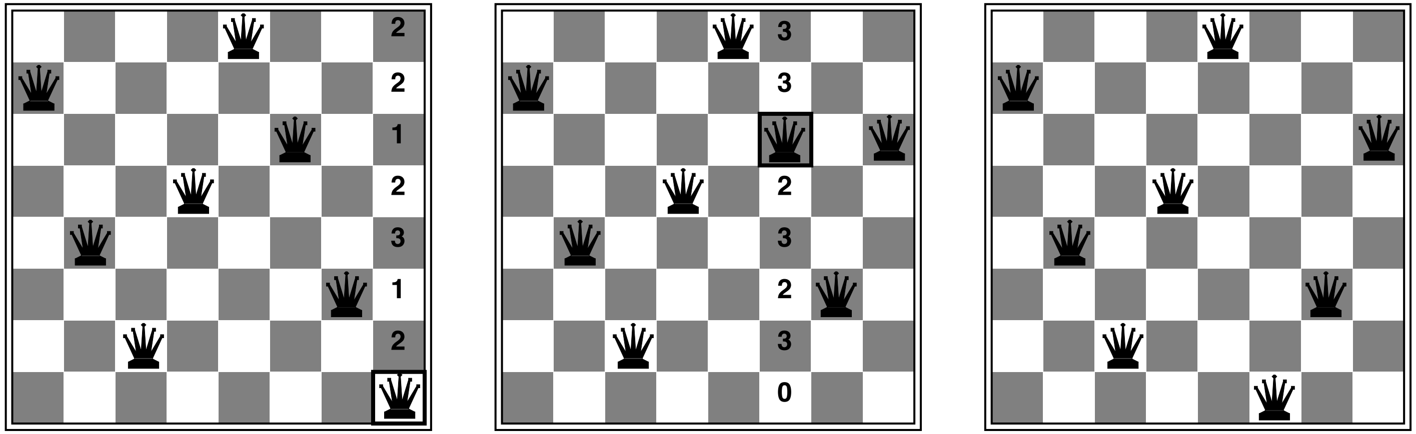
# Example: 4-Queens *

- States: 4 queens in 4 columns ($4^4$ = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of att



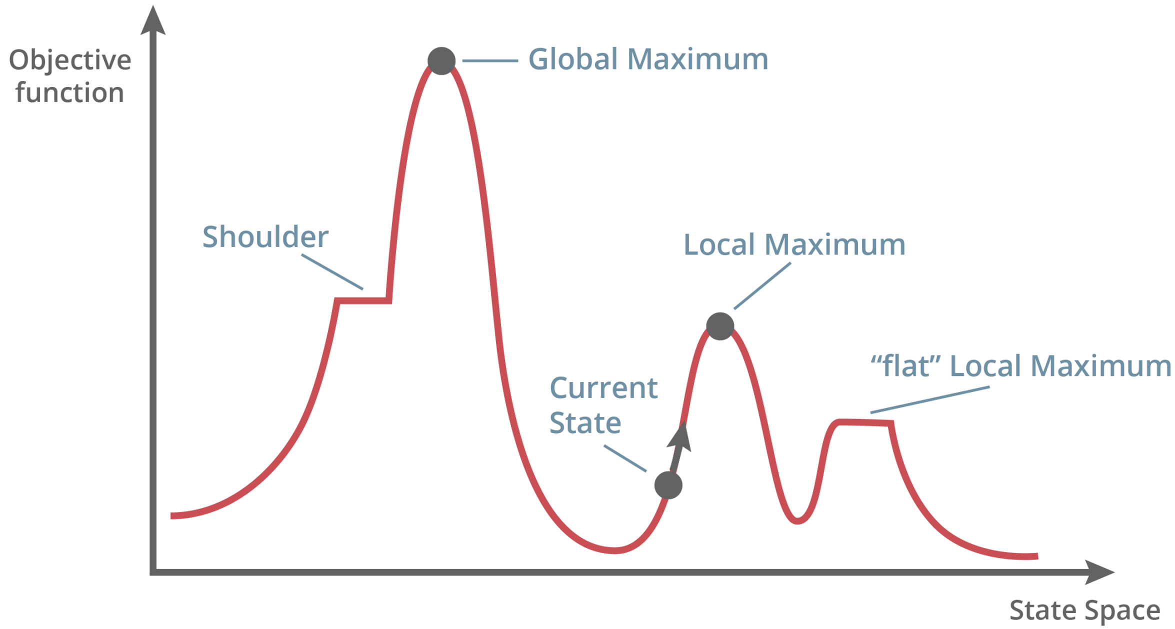

- Given random initial state, can solve $n$-queens in almost constant time for arbitrary $n$ with high probability (e.g., $n$ = 10,000,000)

# Hill-climbing by min-conflicts



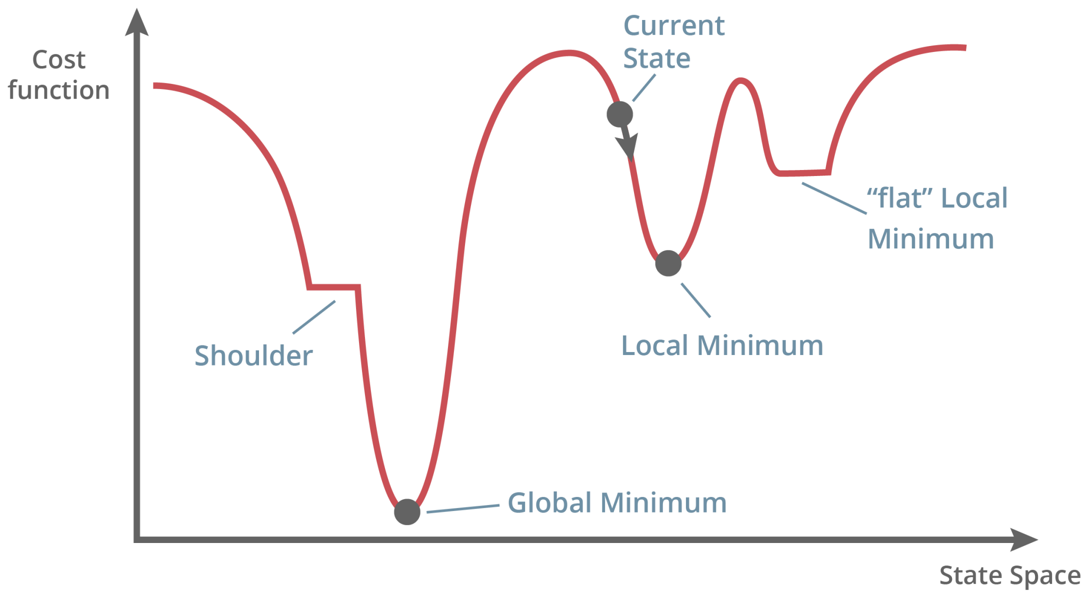

- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic
  - choose value that violates the fewest constraints

# Flat regions and local optima




- Sometimes, have to go sideways or even backwards in order to make progress towards the actual solution.

# Inverted View




When we are minimizing violated constraints, it makes sense to think of starting at the top of a ridge and climbing down into the valleys.

# Simulated Annealing

- stochastic hill climbing based on difference between evaluation of previous state ($h_0$) and new state ($h_1$).
- if $h_1 < h_0$, definitely make the change
- otherwise, make the change with probability

$$e^{-(h1 - h0)/T}$$

where $T$ is a "temperature" parameter.

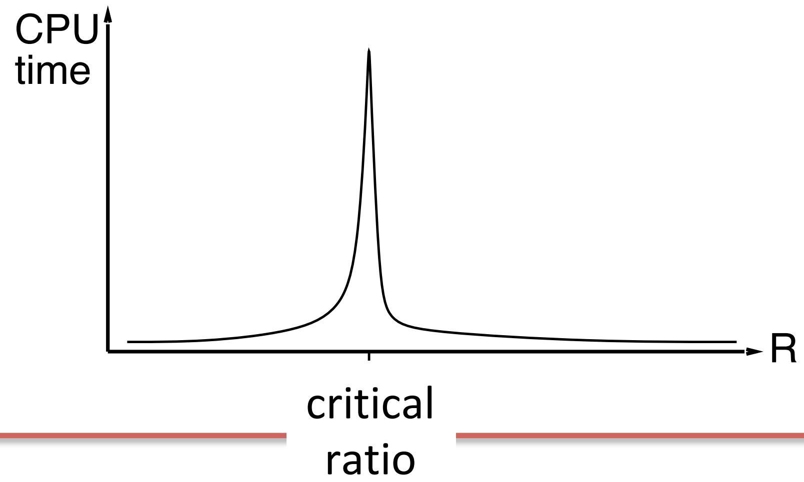
- reduces to ordinary hill climbing when $T = 0$
- becomes totally random search as $T \rightarrow \infty$
- sometimes, we gradually decrease the value of $T$ during the search

# Phase transition in CSP's

Given random initial state, hill climbing by min-conflicts with random restarts can solve $n$-queens in almost constant time for arbitrary $n$ with high probability (e.g., $n$ = 10,000,000).

In general, randomly-generated CSP's tend to be easy if there are very few or very many constraints. They become extra hard in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



# Summary

- Much interest in CSP’s for real-world applications
- Backtracking = depth-first search with one variable assigned per node
- Variable and Value ordering heuristics help significantly
- Forward Checking helps by detecting inevitable failure early
- Hill Climbing by min-conflicts often effective in practice
- Simulated Annealing can help to escape from local optima
- Which method(s) are best? It varies from one task to another!