MATH1081 2016 S2 Scan Week 2

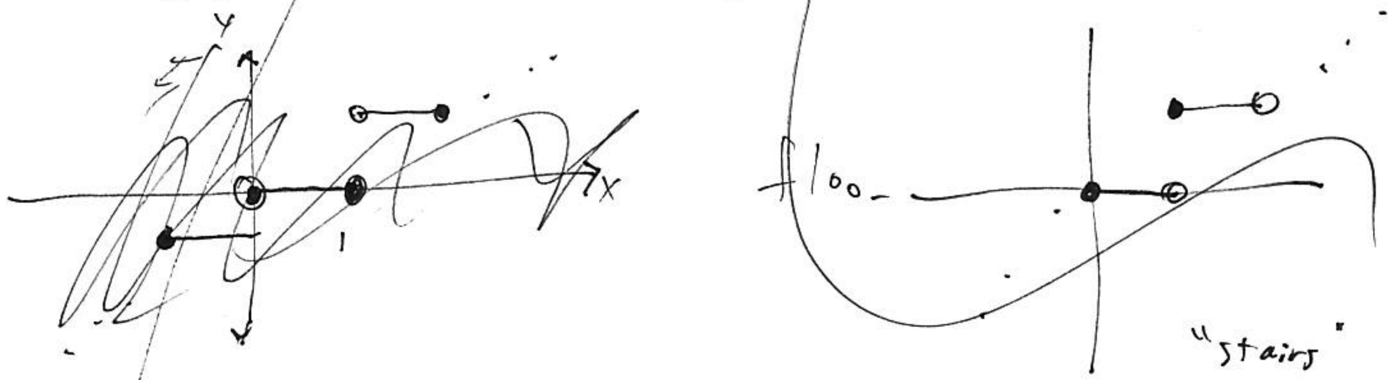
- The *floor* function: (round down)
  for any $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x.
- The *ceiling* function: (round up)
  for any $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x.

Exercise. Evaluate the following:

$\lfloor 3.7 \rfloor = 3$ $\quad \lfloor -3.7 \rfloor = -4$ $\quad \lfloor 3 \rfloor = 3$ $\quad \lfloor -3 \rfloor = -3$

$\lceil 3.7 \rceil = 4$ $\quad \lceil -3.7 \rceil = -3$ $\quad \lceil 3 \rceil = 3$ $\quad \lceil -3 \rceil = -3$

Exercise. What are the ranges of the floor and ceiling functions? $\mathbb{Z}$ for both
Plot the graphs of the floor and the ceiling functions.



Exercise. Determine whether or not each of the following definitions corresponds to a function. If it does, then write down its range.

$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$ N (there's no $f(0)$)

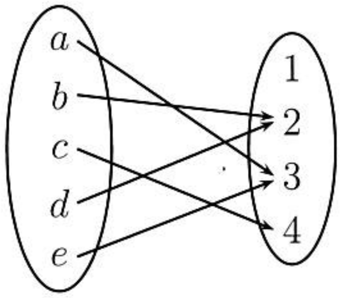
$g : \mathbb{R}^+ \to \mathbb{R}, \quad g(x) = \frac{1}{x}$ Y range: $\mathbb{R}^+ = \{y \in \mathbb{R} \mid y > 0\}$

$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = \lfloor x^2 - x \rfloor$ Y range $\{y \in \mathbb{Z} \mid y \geq -1\}$

$j : \mathbb{R} \to \mathbb{Z} \quad j(x) = 2x$ N (outputs not always integers OR $j(\frac{1}{3}) \notin \mathbb{Z}$

- The *image* of a set $A \subseteq X$ under a function $f : X \to Y$ is
  $$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\} = \{f(x) \mid x \in A\} \subseteq Y.$$
- The *inverse image* of a set $B \subseteq Y$ under a function $f : X \to Y$ is
  $$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X.$$

Example. Let the function f be defined by the arrow diagram



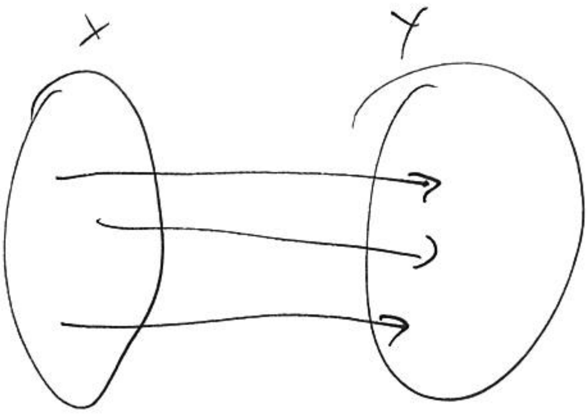
The image of $\{a, b, e\}$ under f is $f(\{a, b, e\}) = \{f(a), f(b), f(e)\} = \{2, 3\}$.
The inverse image of $\{1, 2\}$ under f is $f^{-1}(\{1, 2\}) = \{b, d\}$.

Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Find

(a) The image of the set $\{2, -2, \pi, \sqrt{2}\}$ under f.
(b) The inverse image of the set $\{9, -9, \pi\}$ under f
(c) The inverse image of the set $\{-2, -9\}$ under f.

(a) $\{4, \pi^2, 2\}$
(b) $\{3, -3, \sqrt{\pi}, -\sqrt{\pi}\}$
(c) $\emptyset$

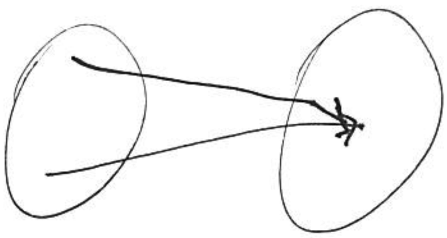
- Recall that if f is a function from X to Y, then
  - for every $x \in X$, there is exactly one $y \in Y$ such that $f(x) = y$.
- We say that a function $f : X \to Y$ is *injective* or *one-to-one* iff
  - for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.
  - OR equivalently, for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
  - OR equivalently, for all $x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.
- We say that a function $f : X \to Y$ is *surjective* or *onto* iff
  - for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.
  - the range of f is the same as the codomain of f.
- We say that a function $f : X \to Y$ is *bijective* iff
  - f is both injective and surjective (one-to-one and onto).
  - for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.



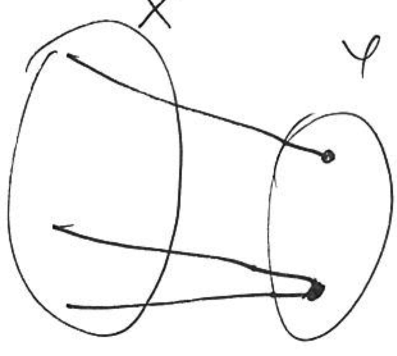
f is injective, one-to-one (as opposed to many-to-one)

[Note: one-to-many is excluded by the definition of function]

not

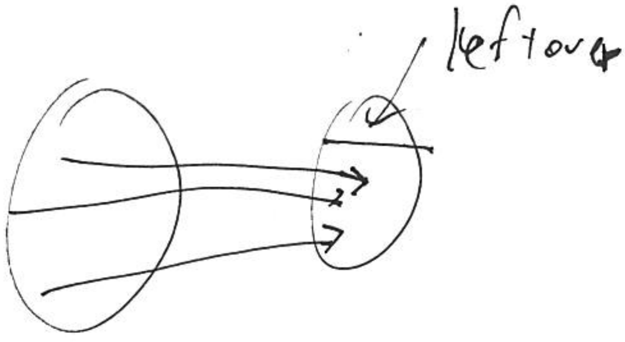


excludes: two inputs give same output



f is surjective (onto) if everything in Y comes from something in X

excludes

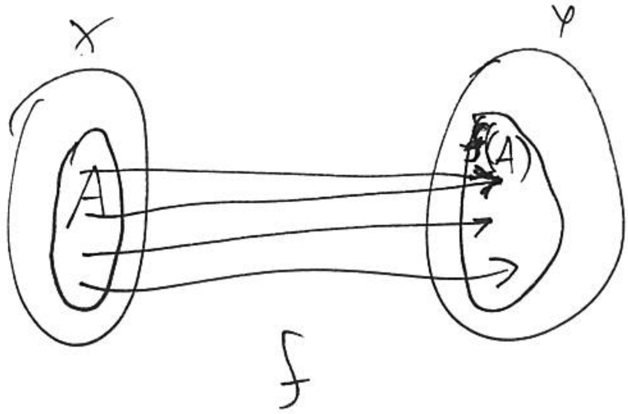


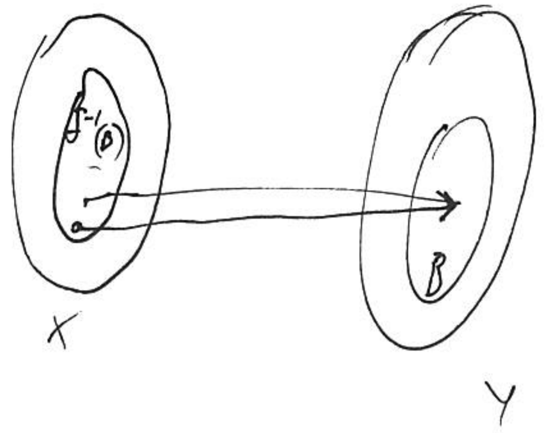
Note: X and Y are part of the <u>definition</u> of the function $f: X \to Y$

So $f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$

is a <u>different</u> function from

$f(x) = x^2$ $f: \mathbb{Z} \to \mathbb{Z}$

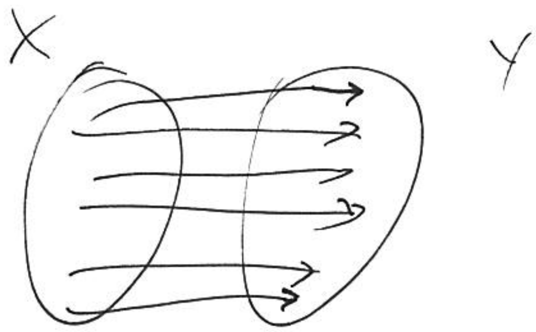




$f^{-1}(B)$ inverse image (or preimage) of B is the set of things that f takes to B

Note: bijection just means a <u>pairing</u>:

The elements of X and Y are paired off by f



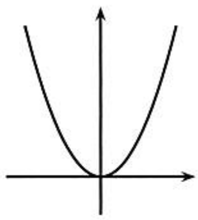
(So X and Y must have the same cardinality if there is a bijective function $f: X \to Y$)

E.g. if the knives and forks on a table are paired off, there must be the same number of knives and forks (without counting)

- In terms of arrow diagrams and graphs...

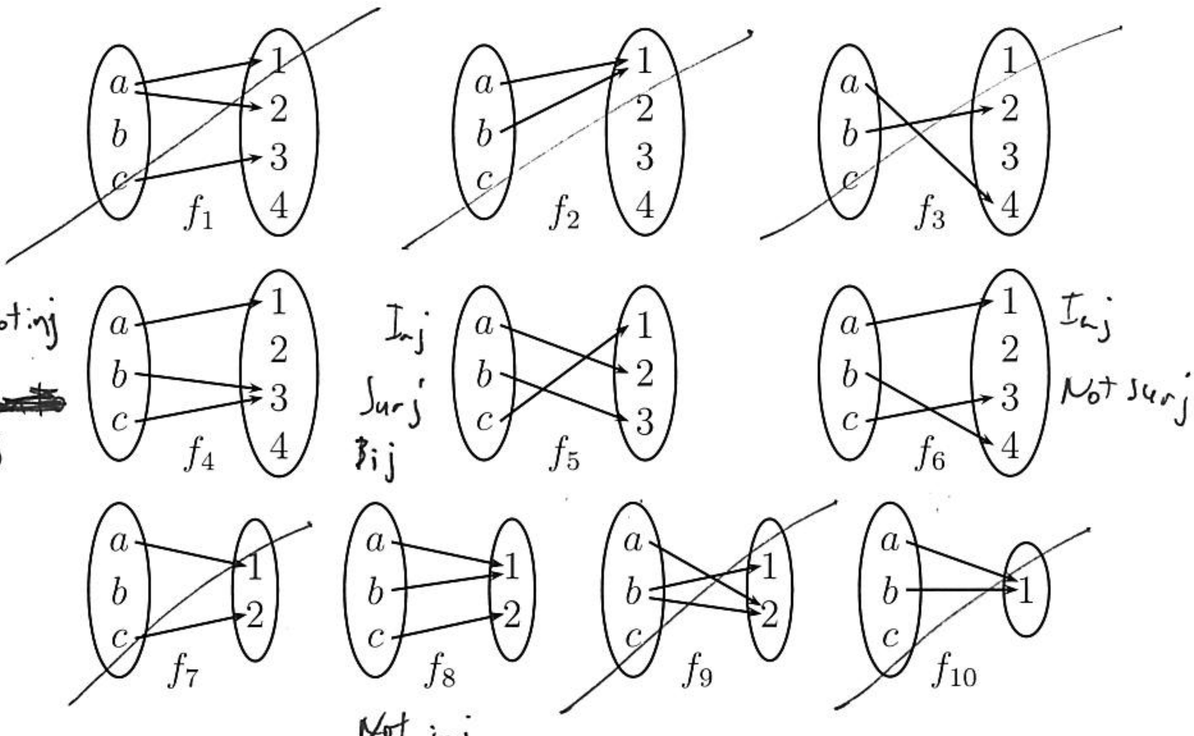
	The arrow diagram for $f : X \to Y$	The graph for $f : \mathbb{R} \to \mathbb{R}$
function	has exactly one outgoing arrow for each element of X	intersects each vertical line in exactly one point
injective one-to-one	has at most one incoming arrow for each element of Y	intersects each horizontal line in at most one point
surjective onto	has at least one incoming arrow for each element of Y	intersects each horizontal line in at least one point
bijective	has exactly one incoming arrow for each element of Y	intersects each horizontal line in exactly one point

Example. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is neither injective nor surjective.



but $f(x) = x$ or $f(x) = x^3$ are both injective and surjective (bijective)

Exercise. Determine whether or not each of the following arrow diagrams corresponds to a function. If it does, then determine whether or not it is injective, surjective, or bijective.



Not inj
Not surj

Inj
Surj
Bij

Inj
Not surj

Not inj.
Surj

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
function										
injective				as shown p.17						
surjective										
bijective										

Exercise. Which of the following definitions correspond to functions? Which of the functions are injective? surjective? bijective?

$$
\begin{array}{ll}
f_1 : \mathbb{R} \to \mathbb{R}, & f_1(x) = \sqrt{x} \\
f_2 : \mathbb{R} \to \mathbb{R}, & f_2(x) = x^2 \\
f_3 : \mathbb{R} \to (\mathbb{R}^+ \cup \{0\}), & f_3(x) = x^2 \\
f_4 : \mathbb{R}^+ \to \mathbb{R}^+, & f_4(x) = x^2 \\
f_5 : (\mathbb{R} - \{0\}) \to \mathbb{R}, & f_5(x) = \frac{1}{x} \\
f_6 : \mathbb{R} \to \mathbb{R}, & f_6(x) = x^2 - 2x - 2
\end{array}
$$

Plot the graph in each case, and give reasons for your answers.

	f_1	f_2	f_3	f_4	f_5	f_6
function	N	Y	Y	Y	Y	Y
injective		N	N	Y	Y	N
surjective		N	Y	Y	N	N
bijective				Y		

Q; Is it possible to pair off (have a bijective function between) a set and a proper subset of itself?

For finite sets, it's not possible,

because a set and a proper subset have different cardinalities,

but ~~for~~ infinite sets it is possible

e.g.

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & & \mathbb{N} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & & & \{\text{even numbers}\} \end{array}$$

$f : \mathbb{N} \to \{\text{even numbers}\}$

$f(x) = 2x$

is bijective

- For functions $f : X \to Y$ and $g : Y \to Z$, the *composite* of f and g is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.
- In general, $g \circ f$ and $f \circ g$ are not the same composite functions.
- *Associativity* of composition (assuming they exist): $h \circ (g \circ f) = (h \circ g) \circ f$.

Diagram: see over

Example. Let f and g be functions defined by

$$f : \mathbb{N} \to \mathbb{Z},\ f(x) = x + 3 \qquad \text{and} \qquad g : \mathbb{Z} \to \mathbb{Z},\ g(y) = y^2 .$$

Then the composite function $g \circ f : \mathbb{N} \to \mathbb{Z}$ exists because codomain of $f = \mathbb{Z} =$ domain of g.
It is given by

$$(g \circ f)(x) = g(f(x)) = g(x + 3) = (x + 3)^2 = x^2 + 6x + 9 .$$

Technically, $f \circ g$ is not defined as codomain of $g = \mathbb{Z} \neq \mathbb{N} =$ domain of f.

BUT, range of $g \subseteq \mathbb{N}$ so if we **re-define** g to be closely related function $g : \mathbb{Z} \to \mathbb{N} : y \mapsto y^2$ then,
with this sleight of hand $f \circ g$ is defined and $f \circ g : \mathbb{Z} \to \mathbb{Z}$

$$(f \circ g)(y) = f(g(y)) = f(y^2) = y^2 + 3.$$

Note $f \circ g \neq g \circ f$ and they do not even have the same domains.

Exercise. Let $A = \{1, 2\}$ and $f : A \to A$ be defined by (switch/transposition/toggle)

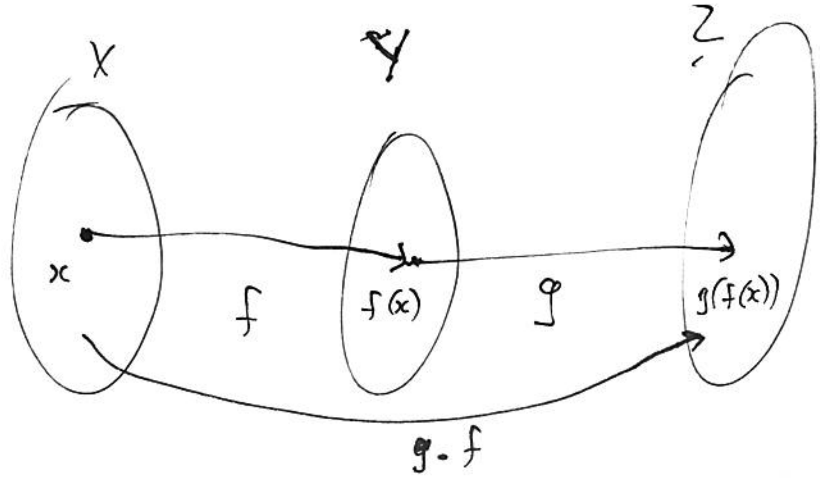
$$f = \{(1, 2), (2, 1)\}.$$ means $f(1) = 2$ and $f(2) = 1$

Find the composite $f \circ f : A \longrightarrow A$.

$(f \circ f)(1) = f(f(1)) = f(2) = 1$
$(f \circ f)(2) = \quad\quad = 2$

Soy: composite is the identity function

Composite:

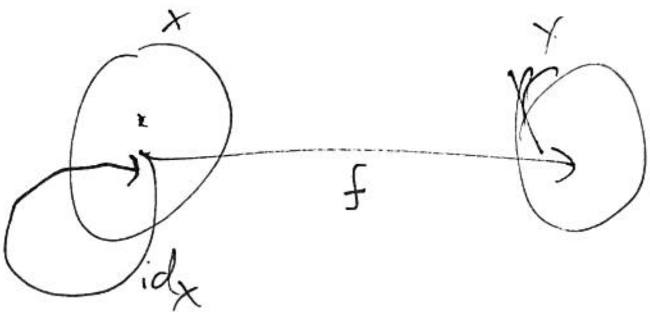


Recall chain rule in calculus

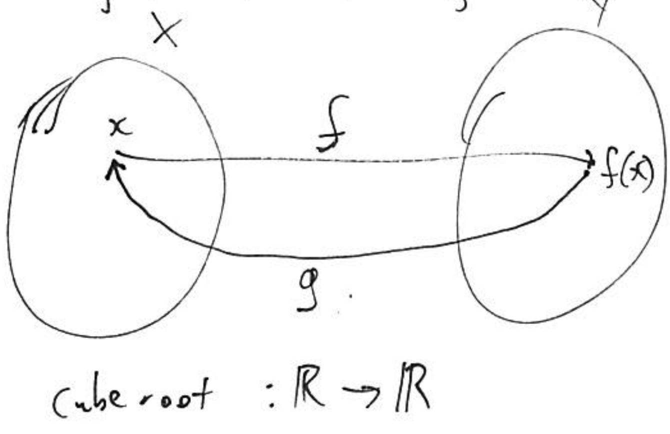
$$\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2)\cdot 2x$$

composite of square and sin functions

Picture of $f: X \to Y$ and $f \circ id_X = f$



Picture of g being inverse of f

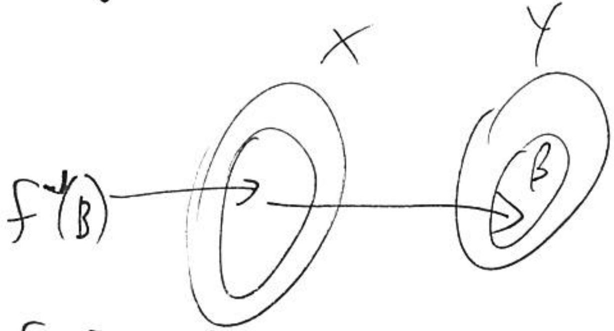


Eg. cube and cuberoot $: \mathbb{R} \to \mathbb{R}$

Warning (ACHTUNG!)

f^{-1} has been used with two meanings

(1.) inverse image of a subset of the codomain of any function



$= \{x \in X \mid f(x) \in B\}$

(2.) For an invertible function (bijective) $f : X \to Y$

f^{-1} means the inverse function of f

$f^{-1}: Y \to X$

- The *identity* function on a set X is the function $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x$.
- For any function $f : X \to Y$, we have $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$.
- A function $g : Y \to X$ is an *inverse* of $f : X \to Y$ if and only if

$$g(f(x)) = x \text{ for all } x \in X, \quad \text{and} \quad f(g(y)) = y \text{ for all } y \in Y,$$

  or equivalently, $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.
- Thus $x = g(y)$ "solves" $f(x) = y$
- **THEOREM:** A function can have at most one inverse. (Check)
- If $f : X \to Y$ has an inverse, then we say that f is *invertible*, and we denote *the* inverse of f by f^{-1}. Thus, $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.
- If g is the inverse of f, then f is the inverse of g. Thus, $(f^{-1})^{-1} = f$.
- **THEOREM:** A function is invertible if and only if it is bijective.
- **THEOREM:** If $f : X \to Y$ and $g : Y \to Z$ are invertible, then so is $g \circ f : X \to Z$, and the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 5$.
To find the inverse f^{-1}, solve the equation $y = f(x)$ with respect to x:

$$y = 2x - 5 \quad \Rightarrow \quad x = \tfrac{y+5}{2}.$$

Thus, $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by $f^{-1}(y) = \frac{y+5}{2}$.

Exercise. For each of the following functions, find its inverse if it is invertible.

$f : \mathbb{R} \to \mathbb{Z}$, $\quad f(x) = \lfloor x \rfloor$ No inverse: not injective

$g : \mathbb{R} \to \mathbb{R}^+$, $\quad g(x) = e^{3x-2}$ Yes: $g^{-1}(y) = (\ln y + 2)/3 \; : \mathbb{R}^+ \to \mathbb{R}$

$h : \{1, 2, 3\} \to \{a, b, c\}$, $\quad h = \{(1, b), (2, c), (3, a)\}$. that is $h(1) = b$, $h(2) = c$, $h(3) = a$

Yes, it's bijective

$h^{-1}(a) = 3$

$h^{-1}(b) = 1$

$h^{-1}(c) = 2$

$h^{-1} : \{a, b, c\} \to \{1, 2, 3\}$

Composite is associative because $(f \circ g) \circ h$ and $f \circ (g \circ h)$ both mean $(f(g(h))(x))$

Example. Prove that a function has at most one inverse.

Proof. Suppose that $f : X \to Y$ has two inverses $g_1 : Y \to X$ and $g_2 : Y \to X$. Then

$$\begin{aligned}
g_1 &= g_1 \circ \mathrm{id}_Y && \text{by property of identity} \\
&= g_1 \circ (f \circ g_2) && \text{by definition of inverse} \\
&= (g_1 \circ f) \circ g_2 && \text{by associativity of composition} \\
&= \mathrm{id}_X \circ g_2 && \text{by definition of inverse} \\
&= g_2 && \text{by property of identity}
\end{aligned}$$

Hence, if f has an inverse, then it is unique.

Exercise. Prove that a function has an inverse if and only if it is bijective.

Proof: over

Let $f: X \to Y$ be a function

Suppose f has an inverse,

so there is a function $f^{-1}: Y \to X$ such that $f^{-1}(f(x)) = x$ for all $x \in X$

and $f(f^{-1}(y)) = y$ for all $y \in Y$

Let $f(x_1) = f(x_2)$

Take f^{-1} $\quad f^{-1}(f(x_1)) = f^{-1}(f(x_2))$

$x_1 = x_2$ by definition of inverse

f is injective (one-to-one)

Now let $y \in Y$

Then $y = f(f^{-1}(y))$ by definition of inverse

f is surjective (onto)

f is bijective

So, if f has an inverse, then f is bijective

Conversely, suppose f is bijective

Take any $y \in Y$

y is f of <u>some</u> element $x \in X$, because f is surjective

and y comes from <u>only</u> one element $x \in X$, because f is injective

So define $f^{-1}(y)$ to be <u>the</u> $x \in X$ such that $f(x) = y$

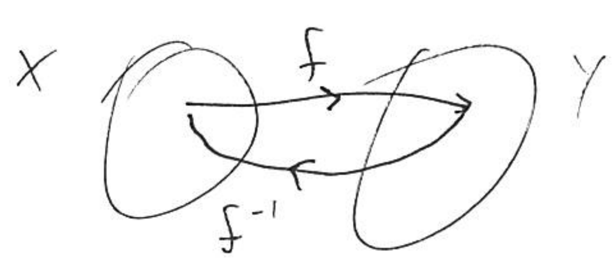
So $f^{-1}(f(x)) = x$ because x is the only $x \in X$ such that $f(x) = f(x)$

and $f(f^{-1}(y)) = y$ because $f^{-1}(y)$ is <u>the</u> element of X such that $f(f^{-1}(y)) = y$

f has an inverse

So, if f is bijective then f has an inverse

Therefore, f has an inverse if and only if f is bijective



'Some / there exists' statements

There exists an irrational number greater than 3
Some irrational number is greater than 3

Proof: $\pi > 3$.

Every natural number has an irrational number greater than it

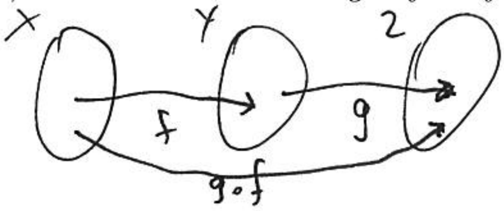
For every $n \in \mathbb{N}$, there exists an irrational number greater than n

Proof: Let $n \in \mathbb{N}$

then $\pi + n$ is an irrational number greater than n

(The logic of the proof of surjectivity on the previous page works the same.)

Exercise. Prove that if $f : X \to Y$ and $g : Y \to Z$ are invertible, then so is $g \circ f : X \to Z$, and the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.



Suppose $f: X \to Y$ and $g: Y \to Z$ are invertible with inverses $f^{-1}: Y \to X$ and $g^{-1}: Z \to Y$

Then $f^{-1} \circ g^{-1}$ satisfies the conditions for being the inverse of $f \circ g$,

because $(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ id_Y \circ f = f^{-1} \circ f = id_X$

and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ id_Y \circ g^{-1} = g \circ g^{-1} = id_Z$

So $g \circ f$ is invertible, with inverse $f^{-1} \circ g^{-1}$.

Final question on functions, see over

New topic:

- Informally speaking, a *sequence* is an ordered list of objects,

$$a_0, a_1, a_2, \ldots, a_k, \ldots,$$

where each object a_k is called a *term*, and the subscript k is called an *index* (typically starting from 0 or 1). We denote the sequence by $\{a_k\}$.

- If all terms a_k lie in a set A, we can think of the sequence as a function $a : \mathbb{N} \to A : k \mapsto a_k$.

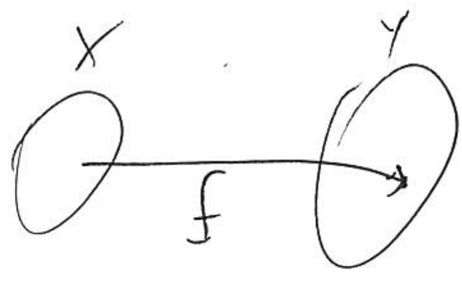
Example.

- An *arithmetic progression* is a sequence $\{b_k\}$ where $b_k = a + kd$ for all $k \in \mathbb{N}$ for some fixed numbers $a \in \mathbb{R}$ and $d \in \mathbb{R}$. Its terms are

$$a, a + d, a + 2d, a + 3d, \ldots .$$

- A *geometric progression* is a sequence $\{c_k\}$ defined by $c_k = ar^k$ for all $k \in \mathbb{N}$ for some fixed numbers $a \in \mathbb{R}$ and $r \in \mathbb{R}$. Its terms are

$$a, ar, ar^2, ar^3, \ldots .$$



Q: if $|X| = n$ and $|Y| = m$, how many functions are there from X to Y?

A: For each of the n inputs in X, there are m choices for $f(x)$

So the total number of different choices (functions) is $m \times m \times \ldots \times m$ (n times)

$$= m^n$$

$$= |Y|^{|X|}$$

Eg. Number of bit strings of length $n = 2^n$

because a bit string just <u>is</u> a function from $\{1, \ldots n\} \to \{0, 1\}$

(for each of the n places, there's a choice of 0 or 1)

- **Summation notation:** for $m \le n$,

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n \,.$$

- **Properties of summation:**

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k \quad \text{and} \quad \sum_{k=m}^{n} (\lambda\, a_k) = \lambda \sum_{k=m}^{n} a_k \,,$$

but in general

$$\sum_{k=m}^{n} a_k b_k \ne \left(\sum_{k=m}^{n} a_k\right)\left(\sum_{k=m}^{n} b_k\right) .$$

Example. The sum of the first $n{+}1$ terms of the arithmetic progression $\{a{+}kd\}$ is

$$\sum_{k=0}^{n} (a{+}kd) = a + (a{+}d) + (a{+}2d) + \cdots + (a{+}nd) = \frac{(2a{+}nd)(n{+}1)}{2} .$$

Why? $= \sum_{k=0}^{n} a + d \sum_{k=0}^{n} k$

$= (n+1)a + d\,\frac{n(n+1)}{2} = \nearrow$

(by below)

We find a formula for the sum of the first n positive integers, by setting $a = 0$ and $d = 1$:

$$1 + 2 + \cdots + n = 0 + 1 + 2 + \cdots + n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2} .$$ over

Example. The sum of the first $n+1$ terms of the geometric progression $\{ar^k\}$ is

$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} .$$

Why? $(r-1)(a + ar + ar^2 + \cdots + ar^n)$

$= ar + ar^2 + ar^3 + \cdots + ar^{n+1}$
$\quad -a - ar - ar^2 - \cdots - ar^n$ $= -a + ar^{n+1}$

Then divide by $r-1$ to get result

$$\sum_{k=1}^{n} k =$$

$$\left.\begin{array}{l} 1 + 2 + 3 + \dots + (n-1) + n \\ + n + (n-1) + \dots \quad 2 \quad + 1 \end{array}\right\} \text{twice answer}$$

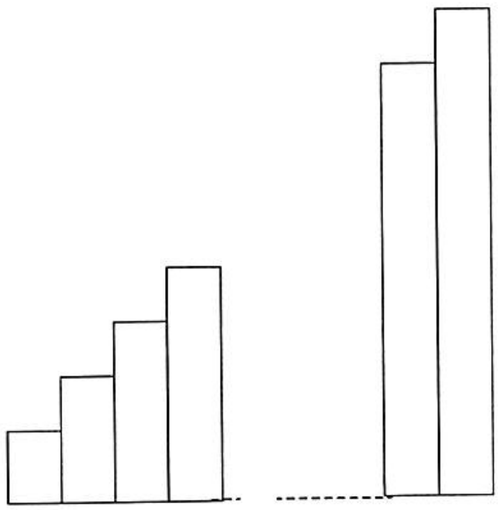
$$= \underbrace{(n+1) + (n+1) + \dots (n+1) + (n+1)}_{n \text{ times}}$$

Twice answer $= n(n+1)$

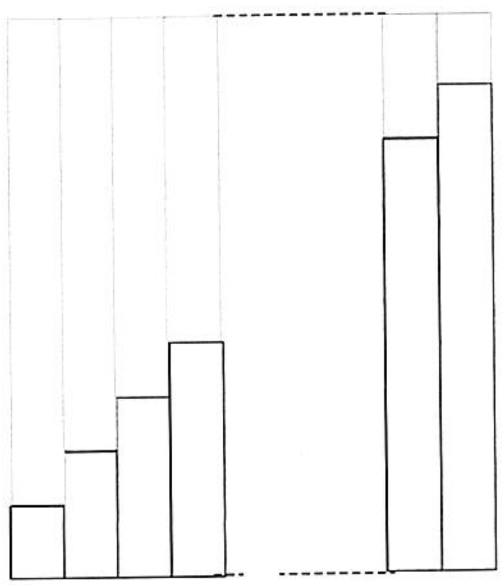
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

See over for picture

role of diagrams in understanding mathematics, especially in grasping proof. Let's go back to the example of adding the numbers 1 to n. It is natural to draw the problem thus:



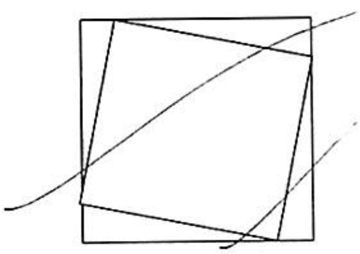
Now, imagine taking a copy of this diagram out of the page, turning it over, and placing it above the original diagram, in the position pictured below with the dotted boundary:



It is clear that the total area of the rectangle formed is $n\ (n + 1)$, so that the sum of the numbers 1 to n is half this.

It is also clear that this proof is just a geometrical version of the symbolic proof above: it might be easier to grasp the picture intuitively, but it would be simple to translate it into words and symbols, if desired.

In the following example, though, it is not clear how to translate into symbols. This is an ancient Indian proof of Pythagoras' theorem:



Exercise. Given the formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

evaluate

$$\sum_{k=1}^{10} (k-3)(k+2)$$

$$= \sum_{k=1}^{10} k^2 - k - 6$$

$$= \sum_{k=1}^{10} k^2 - \sum_{k=1}^{10} k - 6 \sum_{k=1}^{10} 1$$

$$= \frac{10.11.21}{6} - \frac{10.11}{2} - 6 \times 10 \qquad \text{by formulas above}$$

= whatever

Exercise. Use the formula for the geometric progression to evaluate

$$\sum_{k=11}^{40} (3^k + 2)^2$$

$$= \sum_{k=11}^{40} \left(3^{2k} + 4.3^k + 4\right)$$

$$= \sum_{k=11}^{40} 3^{2k} + 4.\sum_{k=11}^{40} 3^k + 4.\sum_{k=11}^{40} 1$$

= GP with $a = 3^{22}$, $r = 9$ GP with $a = 3^{11}$, $r = 3$ 4×30

Apply GP formula

Example. **(Change of summation index)**
The sum

$$\sum_{k=1}^{5} \frac{1}{k+2}$$

can be transformed by a change of variable like $j = k+2$ as follows:

Lower limit: when $k = 1$, we have $j = 1 + 2 = 3$.
Upper limit: when $k = 5$, we have $j = 5 + 2 = 7$.
General term: we have $\dfrac{1}{k+2} = \dfrac{1}{j}$.

Thus, we obtain

$$\sum_{k=1}^{5} \frac{1}{k+2} = \sum_{j=3}^{7} \frac{1}{j}.$$

We could now replace the variable j by the variable k (if this is preferred):

$$\sum_{k=1}^{5} \frac{1}{k+2} = \sum_{k=3}^{7} \frac{1}{k}.$$

More generally, for any sequence $\{a_k\}$ and any integer d we have

$$\boxed{\sum_{k=m}^{n} a_k = \sum_{k=m+d}^{n+d} a_{k-d}}.$$

For example,

$$a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k = \sum_{k=2}^{4} a_{k-1} = \sum_{k=0}^{2} a_{k+1} = \cdots .$$

Exercise. Simplify

$$\sum_{k=2}^{n+1} x^{k-2} - \sum_{k=1}^{n-1} x^k + \sum_{k=0}^{n-1} x^{k+1}$$

$$= \sum_{k=0}^{n-1} x^k - \sum_{k=1}^{n-1} x^k + \sum_{k=1}^{n} x^k$$

$$= x^0 + \cancel{\sum_{k=1}^{n-1} x^k} - \cancel{\sum_{k=1}^{n-1} x^k} + \sum_{k=1}^{n} x^k$$

$$= \sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$$ (geometric progression

Example. (A telescoping sum)
Using the identity $\dfrac{3}{k(k+3)} = \dfrac{1}{k} - \dfrac{1}{k+3}$ for $k \geq 1$, we can write

$$\begin{aligned}\sum_{k=1}^{n} \frac{3}{k(k+3)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+3}\right) \\ &= \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+3}\right).\end{aligned}$$

This is an example of a *telescoping sum:* $\sum a_k$, where $a_k = b_k - b_{k+d}$.
By changing the summation index, we see that

$$\begin{aligned}\sum_{k=1}^{n} \frac{3}{k(k+3)} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+3} &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=4}^{n+3} \frac{1}{k} \\ &= \left(\sum_{k=1}^{3} \frac{1}{k} + \sum_{k=4}^{n} \frac{1}{k}\right) - \left(\sum_{k=4}^{n} \frac{1}{k} + \sum_{k=n+1}^{n+3} \frac{1}{k}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}.\end{aligned}$$

Exercise. Use the identity $\dfrac{2}{k(k+1)(k+2)} = \dfrac{1}{k} - \dfrac{2}{k+1} + \dfrac{1}{k+2}$ for $k \geq 1$ to simplify

$$\sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)}$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right)$$

$$= \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \left(\frac{1}{4} - \cdots\right) \cdots$$

$$+ \left(\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

- **Product notation:** for $m \leq n$,

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n \,.$$

- **Properties of product:**

$$\prod_{k=m}^{n} a_k b_k = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right) \quad \text{but} \quad \prod_{k=m}^{n} (a_k + b_k) \neq \prod_{k=m}^{n} a_k + \prod_{k=m}^{n} b_k \,.$$

Exercise. Simplify

$$\prod_{k=1}^{n} \frac{k}{k+3}$$

$$= \frac{1}{\cancel{4}} \cdot \frac{2}{\cancel{5}} \cdot \frac{3}{6} \cdot \frac{\cancel{4}}{7} \cdot \frac{\cancel{5}}{8} \cdots \frac{\cancel{n-3}}{\cancel{n}} \cdot \frac{\cancel{n-2}}{n+1} \cdot \frac{\cancel{n-1}}{n+2} \cdot \frac{\cancel{n}}{n+3} \quad (\text{for } n > 3)$$

$$= \frac{6}{(n+1)(n+2)(n+3)}$$

(for $n \leq 3$, would need to check separately if the formula holds)