Extended Algorithms Courses COMP3821/9801

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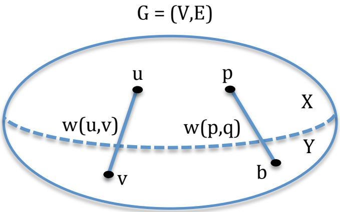
More randomized algorithms: Karger's MinCut Algorithm

Karger's MinCut Algorithm

- Assume you are given an undirected, connected weighted graph $G = (V, E)$, with weights of all edges positive reals.
- A *cut* $T = (X, Y)$ in G is any partition of the set of vertices V into two non empty disjoint subsets X and Y such that $V = X \cup Y$.
- The capacity of a *cut* $T = (X, Y)$ in G is the total sum of weights of all edges which have one end in X and the other in Y.



Cut T = (X,Y);

Capacity C(T) of T is:

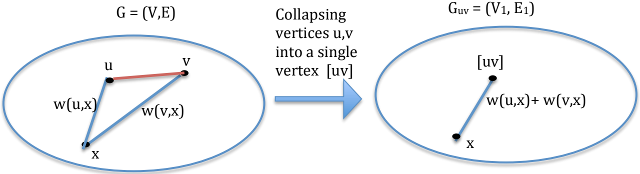
$C(T) = \Sigma\{w(u,v) : u \in X \,\&\, v \in Y\}$

- A cut $T = (X, Y)$ in G is a *minimal cut* if it has the lowest capacity among all cuts in G.
- We say that an edge $e(u, v)$ belongs to a cut $T = (X, Y)$ if one of its vertices belongs to X and the other belongs to Y.

Karger's MinCut Algorithm

- We design a randomised algorithm in two stages, refining in the second stage the algorithm designed in the first stage.

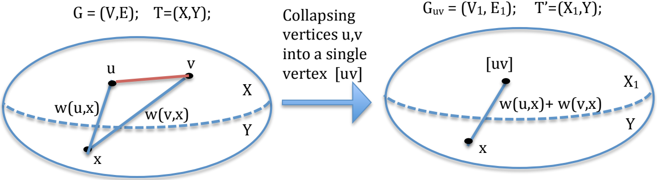
- The basic operation: contracting an edge $e(u, v)$ by fusing the two vertices u and v into a single vertex $[uv]$ and replacing edges $e(u, x)$ and $e(v, x)$ into a single edge $e([uv], x)$ of weight $\mathrm{w}([uv], x) = \mathrm{w}(u, x) + \mathrm{w}(v, x)$:



- We denote thus obtained graph as G_{uv}

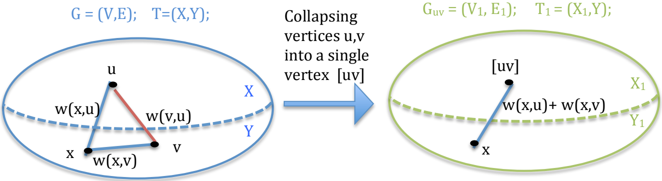
Karger's MinCut Algorithm

- **Claim1:** If two vertices u and v belong to the same side of a minimal cut (X, Y) then after collapsing u and v into a single vertex the capacity of the minimal cut in G_{uv} is the same as the capacity of the minimal cut in G.

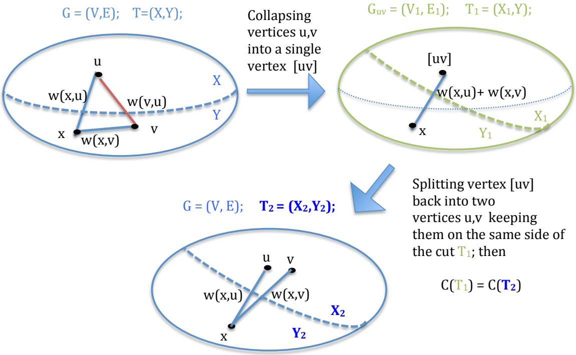


Karger's MinCut Algorithm

- **Claim2:** If two vertices u and v belong to the opposite sides of a minimal cut (X, Y) in G then after collapsing u and v into a single vertex the capacity of the minimal cut in G_{uv} is larger or equal to the capacity of the minimal cut in G.



Karger's MinCut Algorithm



Proof:

- Let $T_1 = (X_1, Y_1)$ be a minimal cut in G_{uv} (T_1 can be completely unrelated to the minimal cut T in G).
- Split vertex $[uv]$ back into two vertices u and v but keep them on the same side of the minimal cut T_1. This produces a cut T_2 in G of the same capacity as the minimal cut T_1 in G_{uv}. Thus, the capacity of the minimal cut in G can only be smaller than the capacity of the minimal cut T_1 in G_{uv}.

Karger's MinCut Algorithm - first attempt

Algorithm 1:

- Pick an edge to contract with a probability proportional to the weight of that edge:

$$P(e(u,v)) = \frac{w(u,v)}{\sum_{e(p,q)\in E} \mathrm{w}(p,q)}$$

- Continue until only one edge is left (we are assuming that the graph is connected).

- Take the capacity of that last edge to be the estimate of the capacity of the minimal cut in G.

Karger's MinCut Algorithm - first attempt

- **Theorem 1:** Let G_{uv} the graph obtained from a graph G with n vertices by contracting an edge $e(u, v) \in E$. Then the probability that the capacity of a minimal cut in G_{uv} is larger than the capacity of a minimal cut in G is smaller than $2/n$:

$$P\Big(\textsc{min-cut-capacity}(G_{uv}) > \textsc{min-cut-capacity}(G)\Big) < \frac{2}{n} \tag{1}$$

- **Proof:** As we have shown, the capacity of the min cut can increase only if the vertices collapsed are on the opposite sides of every min cut in G.
- Let also $M = (X, Y)$ be a min cut in G; then clearly

$$P\Big(\textsc{min-cut-capacity}(G_{uv}) > \textsc{min-cut-capacity}(G)\Big) \leq P\Big(e(u, v) \in M\Big) \tag{2}$$

Note that

$$P\Big(e(u, v) \in M\Big) = \frac{\sum\{\mathrm{w}(p, q) \ : \ e(p, q) \in M\}}{\sum\{\mathrm{w}(u, v) \ : \ e(u, v) \in E\}} \tag{3}$$

Karger's MinCut Algorithm - first attempt

- **Claim:**

$$2\sum_{e\in E} \mathrm{w}(e) = \sum_{v\in V} \sum_{u\,:\,e(v,u)\in E} \mathrm{w}(v,u) \tag{4}$$

- **Proof:** In the sum on the right every edge is counted twice, once for each of its vertices.

- **Claim:** For every $v \in V$,

$$\sum_{u\,:\,e(v,u)\in E} \mathrm{w}(v,u) \geq \text{MIN-CUT-CAPACITY}(G) \tag{5}$$

- **Proof:** If we let $X = \{v\}$ and $Y = V \backslash \{v\}$ we get a cut $T = (X, Y)$ whose capacity must be larger or equal to the capacity of the minimal cut M.

- Since $|V| = n$, from (4) and (5) we now obtain

$$\sum_{e\in E} \mathrm{w}(e) > \frac{n}{2} \cdot \text{MIN-CUT-CAPACITY}(G) \tag{6}$$

Karger's MinCut Algorithm - first attempt

- From (3) and (6) we now obtain

$$\begin{aligned} P\Big(e(u,v) \in T\Big) &= \frac{\sum\{\mathrm{w}(p,q)\ :\ e(p,q) \in T\}}{\sum\{\mathrm{w}(u,v)\ :\ e(u,v) \in E\}} \\ &\leq \frac{\text{MIN-CUT-CAPACITY}(G)}{\frac{n}{2} \cdot \text{MIN-CUT-CAPACITY}(G)} \\ &= \frac{2}{n} \end{aligned}$$

Thus, (2) and the above imply the claim, i.e., that

$$P\Big(\text{MIN-CUT-CAPACITY}(G_{uv}) > \text{MIN-CUT-CAPACITY}(G)\Big) < \frac{2}{n} \tag{7}$$

Karger's MinCut Algorithm - first attempt

- **Theorem 2:** If we run edge contraction procedure until we get a single edge, then the probability π that the capacity of that final edge is equal to the capacity of a minimal cut in G is $\Omega\left(\frac{1}{n^2}\right)$.

- **Proof:** Let G_i for $0 \leq i \leq n-2$, be the sequence of graphs obtained by successive edge contractions, starting from $G_0 = G$. The probability π that the capacity of the final edge is equal to the capacity of a minimal cut in G is greater or equal to the probability that we never contracted an edge belonging to M.
- Thus, (7) implies

$$\begin{aligned}
\pi &= P\Big(\text{MIN-CUT-CAPACITY}(G) = \text{MIN-CUT-CAPACITY}(G_{n-2})\Big) \\
&= \prod_{i=1}^{n-2} P\Big(\text{MIN-CUT-CAPACITY}(G_i) = \text{MIN-CUT-CAPACITY}(G_{i-1})\Big) \\
&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right)\cdots\left(1 - \frac{2}{3}\right) \\
&= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} \cdot \ldots \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \\
&= \frac{2}{n(n-1)}, \quad \text{which implies the claim of the theorem.}
\end{aligned}$$

Karger's MinCut Algorithm - refinement

- However, $\pi = \Omega\left(\frac{1}{n^2}\right)$ is a too small probability; somehow we have to boost it.
- Let us run our contraction algorithm only until the number of edges is $\lfloor \frac{n}{2} \rfloor$.
- Then

$$
\begin{aligned}
\pi &= P\Big(\text{MIN-CUT-CAPACITY}(G) = \text{MIN-CUT-CAPACITY}(G_{n/2})\Big) \\
&= \prod_{i=1}^{n/2} P\Big(\text{MIN-CUT-CAPACITY}(G_i) = \text{MIN-CUT-CAPACITY}(G_{i-1})\Big) \\
&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right)\ldots\left(1 - \frac{2}{n/2+1}\right)\cdot\left(1 - \frac{2}{n/2}\right) \\
&= \frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\frac{n-4}{n-2}\cdot\frac{n-5}{n-3}\cdot\ldots\cdot\frac{n/2}{n/2+2}\cdot\frac{n/2-1}{n/2+1}\cdot\frac{n/2-2}{n/2} \\
&= \frac{(n/2-1)(n/2-2)}{n(n-1)} \\
&\approx \frac{1}{4}
\end{aligned}
$$

- Run time: $O(n^2)$

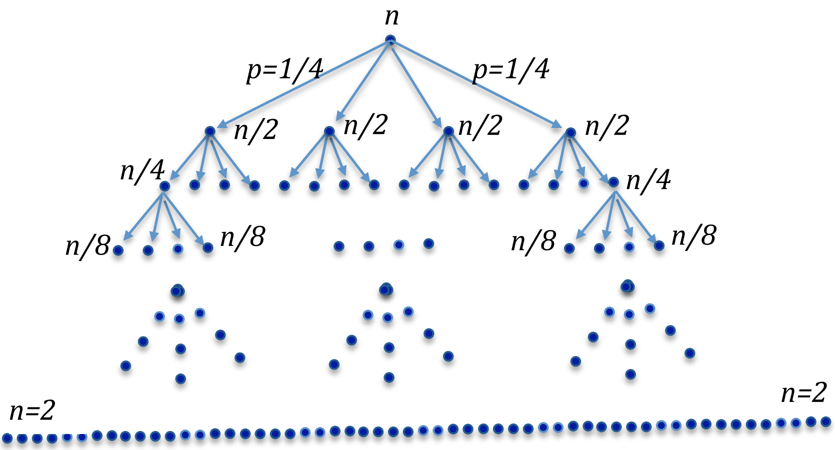
Karger's MinCut Algorithm - refinement

- This shows that the probability of not picking an edge which belongs to a min cut M is fairly large after $n/2$ many contractions, but drops fast afterwards. This suggests the following algorithm:

4-CONTRACT(G)

$G_0 = (V_0, E_0) \longleftarrow G = (V, E)$
while $|V_0| > 2$
    for $i = 1$ to 4
        run the randomised edge contraction algorithm on G_0 until you get a graph $G_i = (V_i, E_i)$ with $|V_i| = |V_0|/2$ many vertices;
    end for
    4-CONTRACT(G_1)
    4-CONTRACT(G_2)
    4-CONTRACT(G_3)
    4-CONTRACT(G_4)
end while
return the smallest capacity among the capacities of all thus produced single edges.

Karger's MinCut Algorithm - refinement



- Run time: $T(n) = 4T(n/2) + O(n^2)$
- By the Master Theorem (case 2), $T(n) = O(n^2 \log n)$.

Karger's MinCut Algorithm - refinement

- What is the probability that at least one of the edges will have the capacity of the min cut of G, and thus that the algorithm will produce the correct value of $\textsc{min-cut-capacity}(G)$??

$$
\begin{aligned}
P(\text{success for a graph of size } n) &= 1 - P(\text{failure on all 4 branches}) \\
&= 1 - P(\text{failure on one branch})^4 \\
&= 1 - (1 - P(\text{success on one branch}))^4 \\
&= 1 - \left(1 - \frac{1}{4} P\left(\text{success for a graph of size } \frac{n}{2}\right)\right)^4
\end{aligned}
$$

Let $p(n) = P(\text{success for a graph of size } n)$; then

$$
p(n) = 1 - \left(1 - \frac{1}{4} p\left(\frac{n}{2}\right)\right)^4
$$

Karger's MinCut Algorithm - refinement

$$\begin{aligned} p(n) &= 1 - \left(1 - \frac{1}{4}p\left(\frac{n}{2}\right)\right)^4 \\ &= p\left(\frac{n}{2}\right) - \frac{3}{8}p\left(\frac{n}{2}\right)^2 + \frac{1}{16}p\left(\frac{n}{2}\right)^3 - \frac{1}{256}p\left(\frac{n}{2}\right)^4 \\ &> p\left(\frac{n}{2}\right) - \frac{3}{8}p\left(\frac{n}{2}\right)^2 \end{aligned}$$

- One can now show by induction of type $\phi(1)\&\forall n(\phi(n/2) \to \phi(n)) \to \forall n\phi(n)$ that the assumption $p(n/2) > \frac{1}{\log(n/2)}$ implies

$$\begin{aligned} p(n) &> p\left(\frac{n}{2}\right) - \frac{3}{8}p\left(\frac{n}{2}\right)^2 \\ &> \frac{1}{\log\frac{n}{2}} - \frac{3}{8}\frac{1}{(\log\frac{n}{2})^2} \\ &= \frac{1}{\log n - 1} - \frac{3}{8}\frac{1}{(\log n - 1)^2} \\ &> \frac{1}{\log n} \quad \text{(by multiplying both sides by } \log n(\log n - 1)^2. \text{)} \end{aligned}$$

Karger's MinCut Algorithm - refinement

- Thus, if we run our 4-Contract(G) algorithm $(\log n)^2$ many times and take the smallest capacity estimate produced, probability π that this estimate will be correct is

$$\pi = 1 - \left(1 - \frac{1}{\log n}\right)^{(\log n)^2}$$

- We now use the fact that for all reasonably large k we have $(1 - 1/k)^k \approx e^{-1}$
- Thus,

$$\pi \approx 1 - e^{-\log n} = 1 - 1/n$$

- So, for large n (which is when other algorithms for min cut are slow) we get the correct value with probability $1 - 1/n$, i.e., almost certainly!
- To run our algorithm $(\log n)^2$ times it takes in total $n^2 \log n \times (\log n)^2 = n^2(\log n)^3$.
- The fastest deterministic algorithms for the same task (based on max flow algorithms which we study next and which runs in time $O(n^3)$) run in time $\Theta(n^4)$ which is much, much slower!!