THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 3: - Properties of Continuous Functions.

Having defined what we mean by continuity, we wish to see what the practical and theoretical implications of this definition are.

Functions can be defined globally on all of $\mathbb{R}$ or locally on some closed interval $[a, b]$ and in the latter case, we need to extend our concept of continuity.

Definition: If f is a function defined on a closed interval $[a, b]$, then we say that f is cts at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$ and similarly, f is cts at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$.

Ex: $f(x) = \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Split Functions:

A function may be defined *piece-wise* using a split definiton.

Ex:

$$f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Prove that f is continuous everywhere.

The Intermediate Value Theorem:

The following theorem is very useful in locating zeros of functions. Although it appears intuitively obvious, it is somewhat hard to prove and the proof will not be given.

Theorem: (IVT)

Suppose that f is cts on a closed interval $[a, b]$ and y lies between $f(a)$ and $f(b)$, then there is at least one $x \in [a, b]$ such that $f(x) = y$.

The theorem is more often used in the following form:

Corollary: Suppose that f is cts on a closed interval $[a, b]$ and that $f(a)$ and $f(b)$ have opposite sign. Then there exists at least one $x \in [a, b]$ such that $f(x) = 0$.

Ex: Let $f(x) = x^3 - x - 1$ in the interval $[1, 2]$.

Note that a polynomial of even degree has an even (possibly zero) number of roots (counting multiplicity) and a polynomial of odd degree has an odd number of roots (counting multiplicity).

Ex: Suppose that f is a cts function defined on the interval $[0, 1]$ which has its range also in the interval $[0, 1]$. Prove that there is a real number $c \in [0, 1]$ such that $f(c) = c$.

Maxima and Minima:

Suppose f is a function defined on an interval $[a, b]$. We say that f has a **global maximum** at $x = c$ (also called *absolute maximum*), if $f(c) \geq f(x)$ for all $x \in [a, b]$.

We similarly define the term **global minimum**.

For example, the function $f(x) = x(x-1)$ defined on $[0, 2]$ has

Theorem: (Min-Max Theorem)

Suppose that f is cts on a closed interval $[a, b]$ then f has a global maximum and a global minimum on $[a, b]$.

In other words, there are real numbers c and d in the interval $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d) \qquad \text{for all } x \in [a, b].$$

Notice that the result is no longer true if we use open intervals or if the function is not cts.

For example $f(x) = \frac{1}{x}$ on the interval $[-1, 1]$ or on the interval $(0, 1)$.

Again, while this result seems intuitively obvious, it is not trivial to prove and we shall delete the proof.

You should be able to state precisely, both of the above theorems.

Ex: Find the min and max of $f(x) = |x||x+1|$ on $[-2, 4]$.

Ex: Repeat for $f(x) = \sin x$ on $(\pi/2, \pi)$.