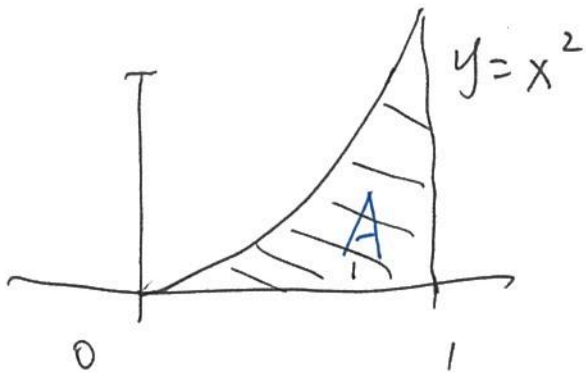


$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq A \leq \underbrace{\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}}$$

Divide $[0,1]$ into n ~~equal~~ equal pieces ~~each~~ each with width $\frac{1}{n}$. pick ~~x~~ C_i in each of the subintervals corresponding to the ~~the~~ max of x^2 on the subinterval.

$$\sum_{i=1}^{n} \frac{1}{n} (C_i)^2$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right]$$

A must be $\frac{1}{3}$.

$0.6345 \le \log 2 \le 0.7595$

$f(x)$ is a function defined on $[a, b]$.

We divide $[a, b]$ into n subintervals.

and form a partition $P_n = \{x_0 = a, x_1, x_2, \cdots, x_n = b$

Choose $c_i \in [x_{i-1}, x_i]$.

c_i is call a sample pt.

widths of the rectangles: $\Delta x_i = x_i - x_{i-1}$

heights of of the rectangles: $f(c_i)$

$$A = \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$$

A is an approximation of the area under the curve $y = f(x)$ from a to b.

$\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called the Riemann sum of f over $[a,b]$ with respect to the partition P_n and the sample point $\{c_i\}$.

If we choose c_i to correspond to the max of f on $[x_{i-1}, x_i]$, then we get the "upper Riemann sum", $\overline{S_P}$.

If we choose c_i to correspond to the min of f on $[x_{i-1}, x_i]$, then we get the "lower Riemann sum", $\underline{S_P}$.

If

$$\lim_{\max \Delta x_i \to 0} \overline{S_P} = \lim_{\max \Delta x_i \to 0} \underline{S_P} = L,$$

then we say that f is <u>integrable</u> on $[a,b]$, and

$$\int_a^b f(x)\,dx = L = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i .$$

Example:
$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}. \\ -1, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Consider $f(x)$ on $[0, 1]$.

$$\overline{S_P} = \sum_{i=1}^{n} 1 \cdot \Delta x_i$$

$$= 1$$

$$\underline{S_P} = \sum_{i=1}^{n} (-1)\, \Delta x_i$$

$$= -1.$$

f is not integrable on $[0, 1]$.

If $f(x)$ is positive for all $x \in [a, b]$, then we define the area bounded by the curve $y = f(x)$, $x = a$, $x = b$, $y = 0$, to be $\int_a^b f(x)\, dx$.

Theorem: If there is a partition P such that the upper and lower Riemann sum for f can be made arbitrarily close, then f is integrable.

For all $\varepsilon > 0$, there exists a partition P such that

$$\overline{S}_P - \underline{S}_P < \varepsilon.$$

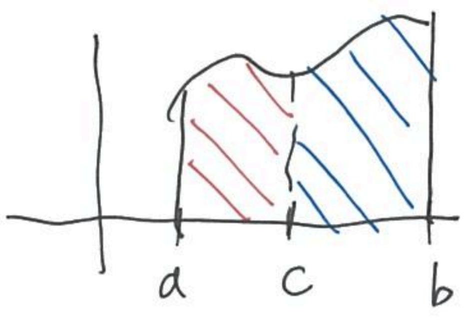
Properties of integrals.

① If $f(x) = C$ for all $x \in [a,b]$,

then $\int_a^b f(x)\,dx = C\cdot(b-a)$.

② If $a < c < b$ and f is integrable on $[a,b]$,

then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



③. If f and g are both integrable on $[a,b]$, and α is a constant, then

$$\int_a^b (\alpha f(x) + g(x))\,dx = \alpha \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

④ If $f(x) \geq 0$ for all $x \in [a,b]$ and f is integrable, then $\int_a^b f(x)\,dx \geq 0$.

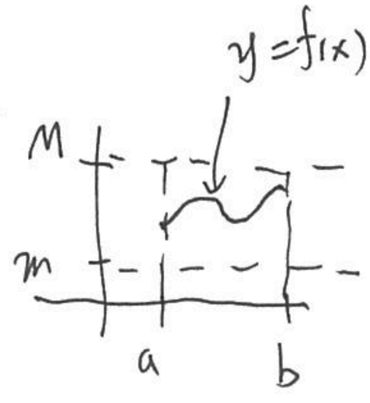
⑤ If $f(x) \geq g(x)$ for all $x \in [a,b]$ and f and g are integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

⑥. If f is integrable on $[a,b]$ and

$$m \leq f(x) \leq M$$

for all $x \in [a,b]$, then



$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Suppose $a < b$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

and $$\int_a^a f(x)\,dx = 0.$$

Def: Let f be a continuous function on $[a, b]$.

A function F with the properties

1) F is differentiable on (a, b).
2) $F'(x) = f(x)$ for all $x \in [a, b]$.

is called a primitive (or anti-derivative) of f.

If F is a primitive of f, then

$F(x) + C$ is also a primitive of f for any constant C.

Example. If $f(x) = x^n$, then $F(x) = \dfrac{x^{n+1}}{n+1} + C$

is an anti-derivative of f, for all $n \neq -$