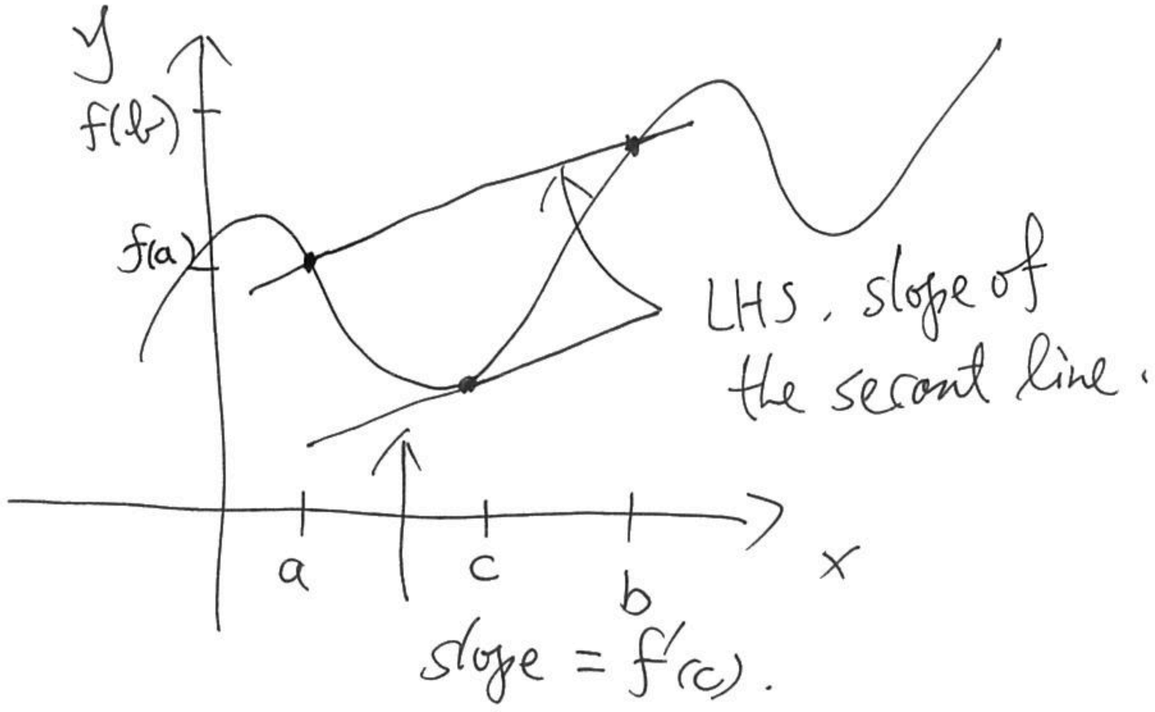
The mean value theorem.

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



$f(x) = 6 - 2x + x^2.$

$[-2, 2].$

$f'(x) = -2 + 2x.$

$f(2) = 6, \quad f(-2) = 14.$

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{-8}{4} = -2.$$

$f'(0) = -2.$

$\sqrt{17} = ?$

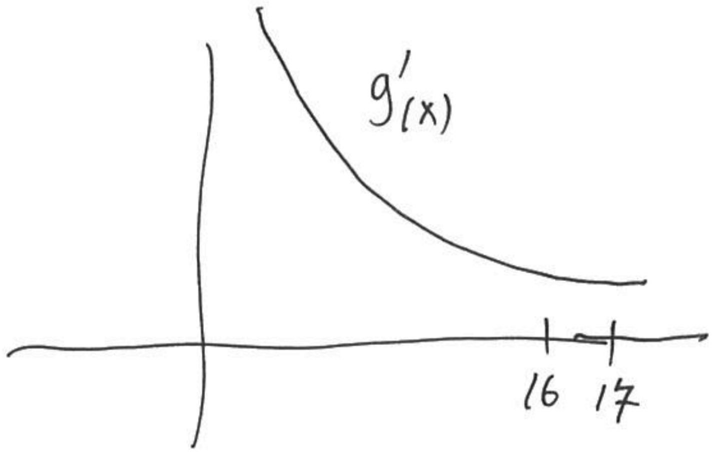
$\cancel{f(x)} = \quad g(x) = \sqrt{x}, \quad g(17) = ?$

$g'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{g(17) - g(16)}{17 - 16} = g'(c) \quad \text{for}$$

some $c \in [16, 17].$

$$g(17) - 4 = \frac{1}{2\sqrt{c}} \quad \text{for some } c \in (16, 17)$$



$$\frac{1}{2\sqrt{c}} \approx \frac{1}{2\sqrt{16}} = \frac{1}{8}.$$

$$\sqrt{17} = g(17) \approx \frac{33}{8}.$$

$f(x) = \log x, \qquad [1, 1.001]$

$\log 1.001 = ?$

$$\frac{\log 1.001 - \log 1}{1.001 - 1} = f'(c)$$

for some $c \in (1, 1.001)$.

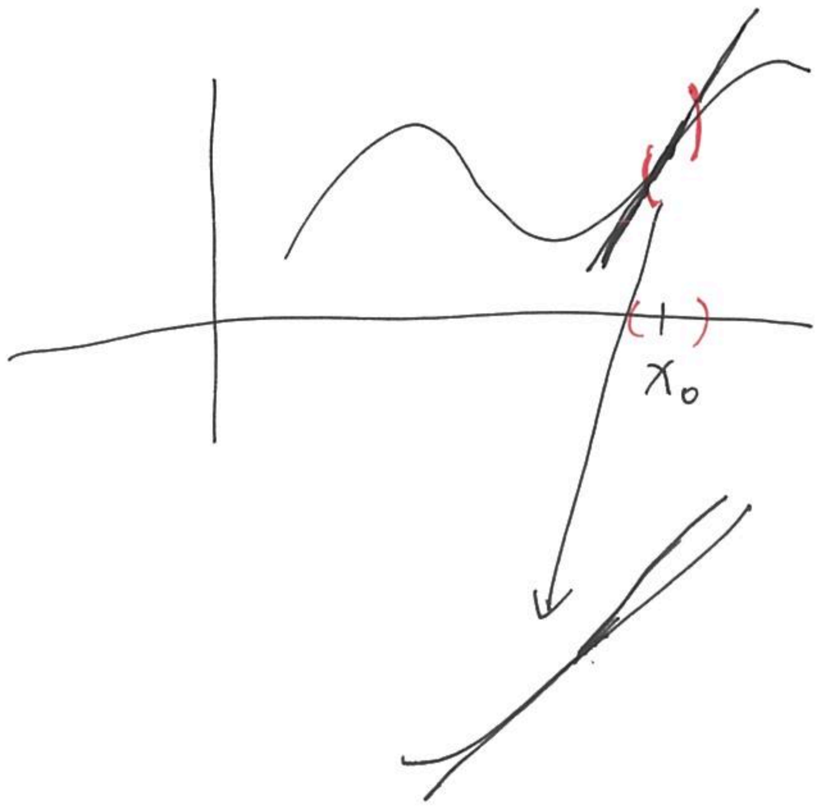
$$f'(x) = \frac{1}{x}.$$

$$\frac{\log 1.001}{0.001} = f'(c) = \frac{1}{c}.$$

$$1 < c < 1.001 \Rightarrow \frac{1}{1.001} < \frac{1}{c} < 1$$

$$\frac{1}{1.001} < \frac{\log 1.001}{0.001} < 1$$

$$\Rightarrow 0.00099 < \log 1.001 < 0.001$$



Example. use the MVT to prove

that $\tan x \geq x$ for all $x \in [0, \frac{\pi}{2})$.

Let $g(x) = \tan x - x$.

It suffices to show that $g(x) \geq 0$

for $x \in [0, \frac{\pi}{2})$.

$g(0) = 0$. MVT.

$$g(x) = g(x) - g(0) \overset{\downarrow}{=} (x-0)\, g'(c) \text{ for some } c \in (0, x).$$

$$= x \cdot g'(c)$$

$g'(x) = \sec^2 x - 1. \geq 0$, for all $x \in [0, \frac{\pi}{2})$.

Therefore, $g(x) \geq 0$, for all $x \in [0, \frac{\pi}{2})$.

Example: $x, y \in \mathbb{R}$,

Prove
$$|\sin x - \sin y| \le |x-y|.$$

If $x = y$, then the inequality is trivial.

If $x \neq y$,

$$|\sin x - \sin y|$$

$$= |x-y| \left|(\sin' c)\right|, \text{ for some } c \text{ between } x \text{ and } y$$

$$= |x-y| \, |\cos c|$$

$$\le |x-y|.$$

Theorem: If $f'(x)$ exists, then

$$|\Delta f(x)| = |f(x+\Delta x) - f(x)|$$
$$\approx f'(x) \cdot \Delta x.$$

An ~~a~~ angle is measured to be 0.7 Radian. We compute the sine of the angle. Suppose that the error in the angle measurement is at most 0.01 rad.
What is the worst error involved in taking the sine of the angle?

$f(x) = \sin x, \quad f'(x) = \cos x.$

Error
$$= |\Delta f(x)| \approx \cos(0.7)\,\Delta x$$
$$= \cos 0.7 \cdot 0.01 \approx 7.65 \times 10^{-3}.$$

Def: A function f on $[a,b]$ is said to be increasing if $f(x) > f(y)$ for all $x > y$.
and decreasing if $f(x) < f(y)$ for all $x > y$.

Theorem: Suppose that f is differentiable on (a, b).

i) If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on (a,b).

ii) " < 0 " "
" decreasing on (a,b)

ii) " $= 0$ " then f is constant on (a,b).

If $x > y$, then

$$\frac{f(x) - f(y)}{x - y} = f'(c), \text{ for some } c$$

between x and y.

So $f(x) - f(y)$ has the same sign as $f'(c)$.

Then the result of the theorem follows.

Theorem: Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a)$ and $f(b)$ have opposite signs.

If $f'(x) > 0$ for all $x \in (a, b)$

$\left(\text{or } f'(x) < 0 \text{ for all } x \in (a, b) \right)$;

then f has exactly one real zero in (a, b).

Show that

$5x^5 + 2x + 1 = 0$ has

exactly one real sol.

$f(x) = 5x^5 + 2x + 1$

$f'(x) = 25x^4 + 2. \geq 2. > 0.$

$f(0) = 1.$

$f(-1) = -5 - 2 + 1 = -6.$

Therefore, f has ~~at least~~ exactly

one real root.