Stochastic Differential Equations

Midterm Presentation

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Outline

Distribution and Limit Process

$$Y_n = X_0 + X_1 + X_2 + ... + X_n$$

- Characteristic Function

$$\begin{aligned} E[e^{itY_n}] &= \cos(t\triangle x)^{\frac{T}{\triangle t}} && (1) \\ \lim_{n\to\infty} E[e^{itY_n}] &= e^{\frac{-\lambda^2 T}{2}} && (2) \end{aligned}$$

- Equation (2) implies that $\lim_{n\to\infty} Y_n \sim$N(0, T)

Brownian Motion

Properties

i $B(t)$ is continuous

ii $B(t)$ is no where differentiable

iii If $t_1 < t_2 < t_3 < t_4$,
$B(t_1), B(t_2) - B(t_1), B(t_3) - B(t_4)$ are independent random variables

iv If $0 \leq s \leq t$ then $B(t) - B(s) \sim N(0, t - s)$

Characteristic Functions and Step Functions

$$1_{[t_{i-1},t_t)}(t) = \begin{cases} 1 & if \;\; t_{i-1} \leq t < t_i \\ 0 & otherwise \end{cases} \tag{3}$$

$$f_n(t) = \sum_{i=1}^{n} a_i 1_{[t_{i-1},t_t)}(t) \tag{4}$$

Stochastic Integral

$$I(f_n) = \int_a^b f_n(t)\, dB = \sum_{i=1}^{n} a_i[B(t_i) - B(t_{i-1})]$$

- $I(f_n(t))$ is a normally distributed random variable
- $\mathrm{E}[I(f_n(t))] = 0$
- $\mathrm{Var}[I(f_n(t))] = \int_a^b f_n^2(t)\, dt$

Riemann-Stieltjes Stochastic Integration

$$\int_a^b f(t)\,dB(t) \quad = \quad f(t)B(t)\Big|_a^b - (\mathrm{RS})\int_a^b B(t)\,df(t)$$

Second Order Taylor Expansion

$$
\begin{aligned}
f(t,x) - f(a,x_0) \quad = \quad & \frac{\partial f}{\partial t}(a,x_0)(t-a) + \frac{\partial f}{\partial x}(a,x_0)(x-x_0) + \\
& \frac{\partial^2 f}{\partial t^2}(a,x_0)\frac{1}{2!}(t-a)^2 + \\
& \frac{\partial^2 f}{\partial t \partial x}(a,x_0)(t-a)(x-x_0) + \\
& \frac{\partial^2 f}{\partial x^2}(a,x_0)\frac{1}{2!}(x-x_0)^2 + \mathrm{H.O.T.}. \qquad (5)
\end{aligned}
$$

Second Order Taylor Expansion

$$
\begin{aligned}
f(t,B(t)) - f(a,B(a)) \quad = \quad & \frac{\partial f}{\partial t}(a,B(a))(t-a) + \\
& \frac{\partial f}{\partial B}(a,B(a))(B(t)-B(a)) + \\
& \frac{\partial^2 f}{\partial t^2}(a,B(a))\frac{1}{2!}(t-a)^2 + \\
& \frac{\partial^2 f}{\partial t \partial B}(a,B(a))(t-a)(B(t)-B(a)) + \\
& \frac{\partial^2 f}{\partial B^2}(a,B(a))\frac{1}{2!}(B(t)-B(a))^2 + \\
& \text{H.O.T.}. \qquad (6)
\end{aligned}
$$

Itô's Formula

$$
\begin{aligned}
f(t, B(t)) - f(a, B(a)) &= \int_a^t \frac{\partial f}{\partial s}(s, B(s)) + \frac{1}{2}\frac{\partial^2 f}{\partial B^2}(s, B(s))\, ds \\
&\quad + \int_a^t \frac{\partial f}{\partial B}(s, B(s))\, dB \qquad (7)
\end{aligned}
$$

Ordinary Differential Equation

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t g(x, s)\, ds, \\
dX(t) &= g(x, t)dt.
\end{aligned}
$$

Stochastic Differential Equation

$$X(t) = X(0) + \int_0^t f(x,s)\,dB(s) + \int_0^t g(x,s)\,ds,$$

$$dX(t) = f(x,t)\,dB(t) + g(x,t)\,dt.$$

Exponential Decay

$$
\begin{aligned}
dX(t) &= -\beta X(t)\,dt, \\
X(t) &= X(0)e^{-\beta t}.
\end{aligned}
$$

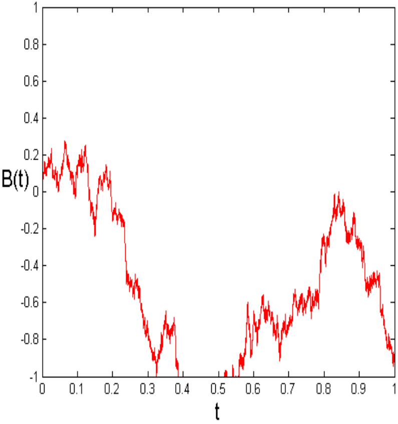
Langevin Stochastic Differential Equation (Additive Noise)

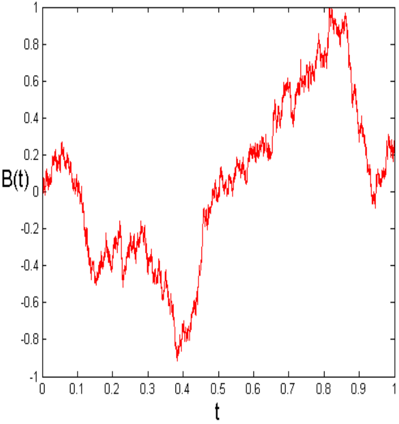
$$
\begin{aligned}
dX(t) &= \alpha\, dB(t) - \beta X(t)\, dt, \\
X(t) &= X(0)e^{-\beta t} + \alpha e^{-\beta t} \int_0^t e^{\beta s}\, dB(s).
\end{aligned}
$$

Geometric Brownian Motion (Multiplicative Noise)

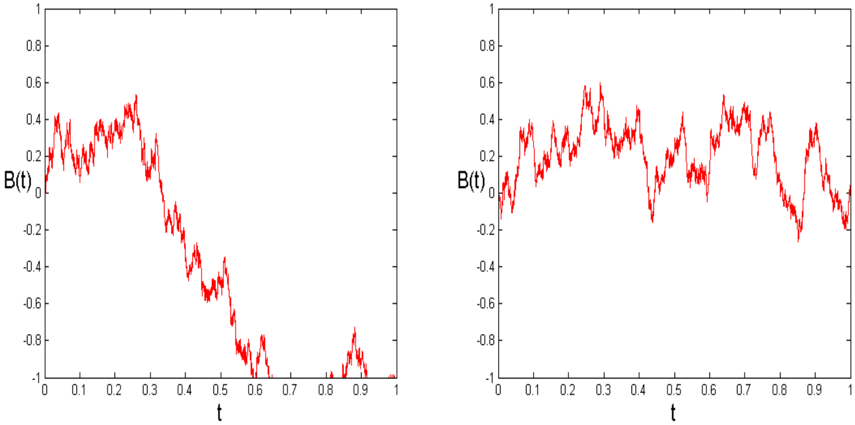
$$
\begin{aligned}
dX(t) &= \alpha X(t)\, dB(t) - \beta X(t)\, dt, \\
X(t) &= X(0) e^{\alpha B(t) - (\beta + \frac{1}{2}\alpha^2) t}.
\end{aligned}
$$

Random Walk Positions

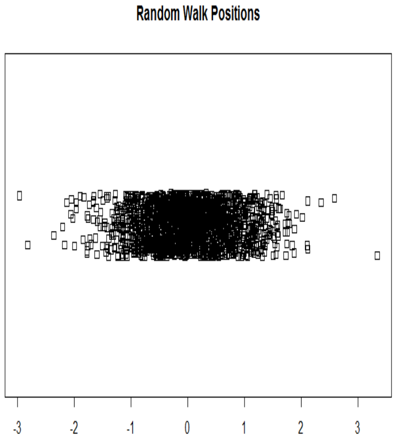


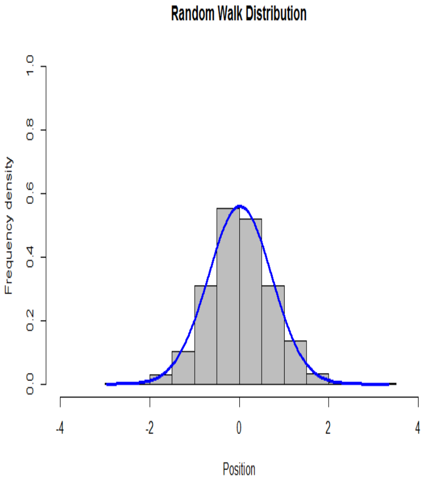


Random Walk Positions



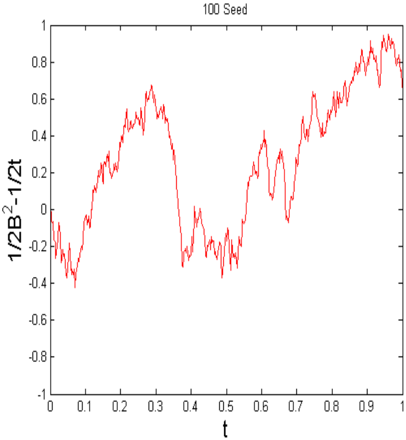
Random Walks Distribution

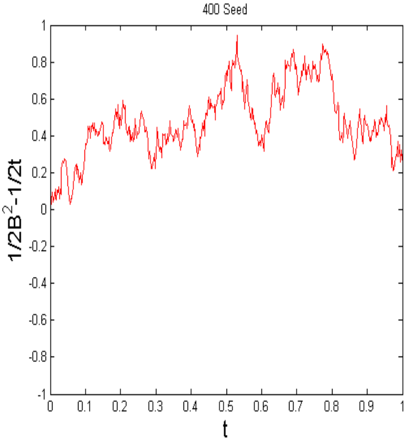




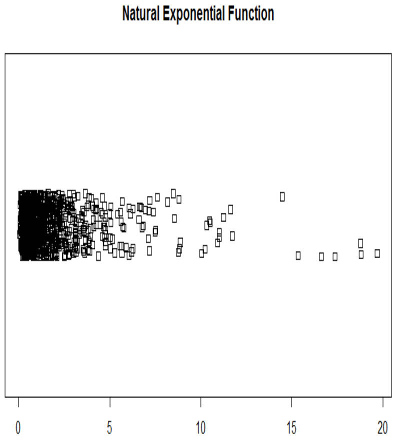
Stochastic Integrals

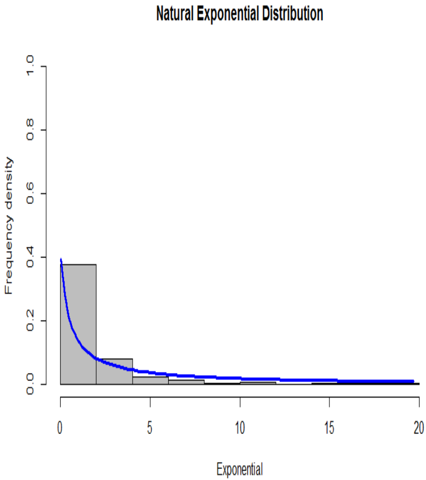
$$\int_0^t B(s)\,dB(s) = \frac{1}{2}B^2(t) - \frac{1}{2}t$$





Exponential Functions of $B(t)$





Euler-Maruyama Method vs Milstein Method

$$dX(t) = f(x,t)dB(t) + g(x,t)dt.$$

Definition (Euler-Maruyama Method)

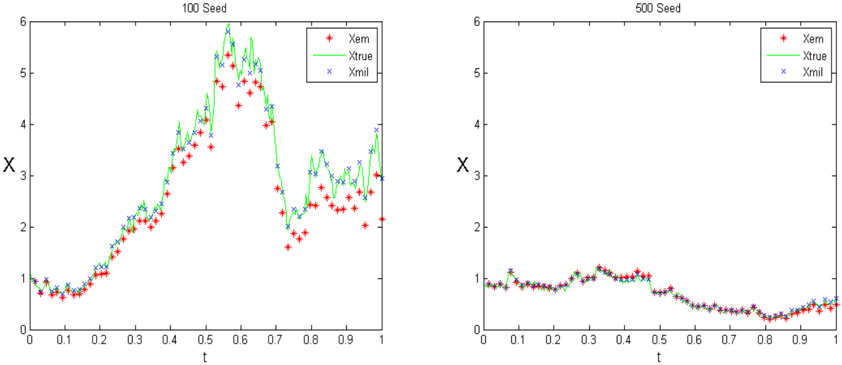
$$\begin{aligned} X_j &= X_{j-1} + f(X_{j-1})\triangle t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})) \\ j &= 1, 2, ..., L \end{aligned}$$

Definition (Milstein Method)

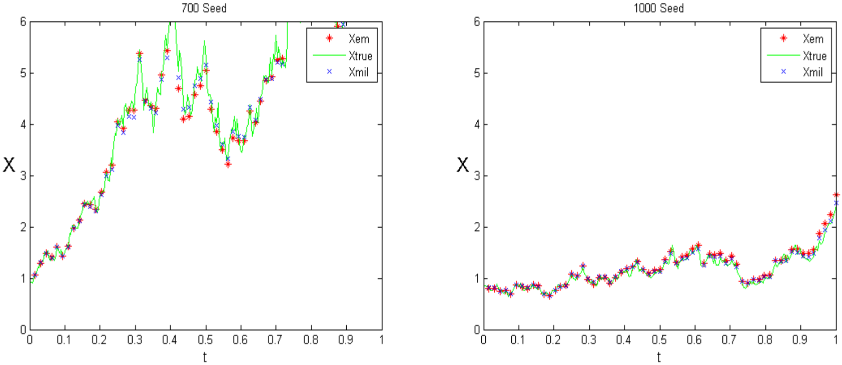
$$\begin{aligned} X_j &= X_{j-1} + \triangle t f(X_{j-1}) + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})) \\ &\quad + \frac{1}{2} g(X_{j-1}) g'(X_{j-1})((W(\tau_j) - W(\tau_{j-1}))^2 - \triangle t) \\ j &= 1, 2, ..., L \end{aligned}$$

Euler-Maruyama Method vs Milstein Method

$$dX(t) \quad = \quad X(t)\, dB(t) + 2X(t)\, dt$$



Euler-Maruyama Method vs Milstein Method



Possible Goals/Questions?

- Actuarial Applications
- An evaluation of a mitigation strategy for deer-vehicle collisions (Bissonette, Rosa, 2012)
- Designing Index-Based Livestock Insurance for Managaing Asset Risk in Northern Kenya (Chantarat, *et al.*, 2013)

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