Classroom Activities
Introduction to Linear Algebra

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Contents

1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Name: ______________________

Exercise 1. Solve the system by using elementary row operations on the augmented matrix. Follow the systematic elimination procedure described in this section.

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$$

Exercise 2. The augmented matrix of a linear system has been reduced by row operations to the form shown. Continue the appropriate row operations and describe the solution set of the original system.

(a) $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$

Exercise 3. Solve the system.

$$\begin{aligned} x_1 \qquad\quad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

Exercise 4. In each problem below, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

(a) $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$

Exercise 5 (Extra Time)**.** Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

1.2 Row Reduction and Echelon Forms Name: ____________________

Exercise 1. Row reduce the matrix to reduced echelon form. Identify (circle) the pivot positions in the final matrix and in the original matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

Exercise 2. Find the general solution of the system whose augmented matrix is given below.

$$\begin{bmatrix} 0 & 1 & -6 & 6 \\ 1 & -2 & 9 & -8 \end{bmatrix}$$

Theorem 2. Existence and Uniqueness Theorem
A linear system is consistent if, and only if, the rightmost column of the augmented matrix is *not* a pivot column—that is, if, and only if, an echelon form of the augmented matrix has *no* row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \qquad \text{with } b \text{ nonzero.}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Exercise 3. Suppose each matrix below represents the augmented matrix for a system of linear equations. Determine if the systems are consistent. If the system is consistent, determine if the solution is unique. (Leading entries marked with a $\blacksquare$ may have any nonzero value, and entries marked with a * may have any value including zero.)

(a) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$

(b) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$

Exercise 4. In the system of linear equations below, h and k represent real numbers.

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 3x_1 + 6x_2 &= k \end{aligned}$$

For each part below, choose values for h and k so that the desired property holds.

(a) The system has no solution.

(b) The system has 1 solution.

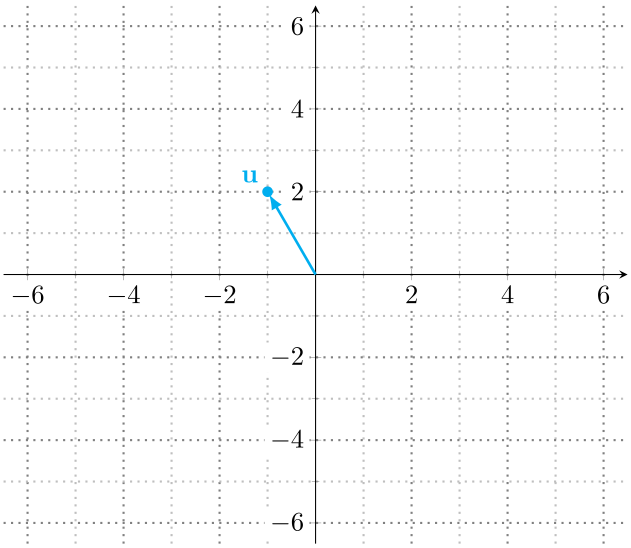
1.3 Vector Equations

Name: ______________________________

Usually vectors are depicted in bold font when in print (e.g. $\mathbf{u}$, $\mathbf{v}$, or $\mathbf{a}_1$). When writing by hand, you may wish to use the "arrow" notation for a vector instead (e.g. $\vec{u}$, $\vec{v}$, or $\vec{a}_1$).

Exercise 1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are given, and $\mathbf{u}$ is displayed in the graph below.

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$



The vector $\mathbf{u}$ is indicated on the graph to the left. Indicate the following vectors in a similar manner.

(a) $\mathbf{v}$

(b) $-2\mathbf{u}$

(c) $\mathbf{u} + \mathbf{v}$

(d) $\mathbf{u} - \mathbf{v}$

(e) $\mathbf{u} - 2\mathbf{v}$

Exercise 2. An augmented matrix for a system is given. For this problem, use the variables x_1 and x_2.

$$\begin{bmatrix} -2 & 0 & -3 \\ 7 & -7 & 6 \\ 3 & 6 & 5 \end{bmatrix}$$

(a) Rewrite the augmented matrix as a system of linear equations. Do not solve the system.

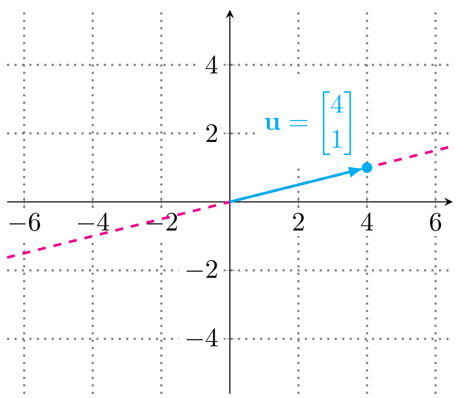
(b) Rewrite the augmented matrix as a vector equation (using each column as a vector). Do not solve the system.

Note that in the first instance, the **rows** of the matrix are the important objects. In the second instance, we instead view the matrix as a linear combination of the **columns**.

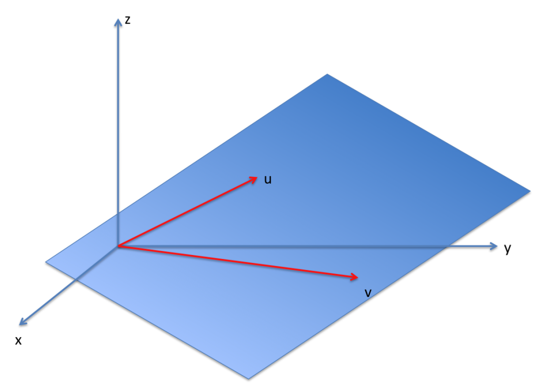
Exercise 3. Determine if $\mathbf{b}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. In other words, determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 4 \\ -5 \\ 14 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 22 \end{bmatrix}$$

Exercise 4. For any nonzero vector $\mathbf{u}$ in $\mathbb{R}^2$, Span$\{\mathbf{u}\}$ is a line passing through $\mathbf{u}$ and the origin, $\mathbf{0}$. See the picture.



If $\mathbf{u}$ and $\mathbf{v}$ are distinct nonzero vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u}, \mathbf{v}\}$ *may* be a plane passing through $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$ (as below)... but this is not always the case!



Give an example of two distinct, nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ so that Span$\{\mathbf{u}, \mathbf{v}\}$ is **not** a plane. What is the geometric interpretation of Span$\{\mathbf{u}, \mathbf{v}\}$ for your example?

1.4 The Matrix Equation $\boldsymbol{Ax = b}$

Name: ______________________________

Exercise 1. A matrix A and column vector $\mathbf{x}$ are given.

$$A = \begin{bmatrix} 6 & 5 \\ -3 & -4 \\ 7 & 4 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

There are two main ways to think about the product $A\mathbf{x}$.

(a) Write the product $A\mathbf{x}$ as a linear combinations of the columns of A using the entries of $\mathbf{x}$ as weights. Compute the product.

(b) Compute the product $A\mathbf{x}$ using the row-vector rule. Show your work clearly.

$$A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -3 & -4 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} =$$

Theorem 3.
If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solutions set as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ which, in turn, has the same solution set as the system of linear equations whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$.

Exercise 2. Write the augmented matrix that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$ and solve the system to find $\mathbf{x}$.

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 1 & 5 & 2 \\ 2 & 3 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -17 \\ 22 \\ 13 \end{bmatrix}$$

Theorem 4.
Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Exercise 3. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all b for which $A\mathbf{x} = \mathbf{b}$ does have a solution.

$$A = \begin{bmatrix} 1 & -4 & -3 \\ -3 & 3 & 0 \\ 4 & 2 & 6 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(Hint: Note that by Theorem 4, to show that $A\mathbf{x} = \mathbf{b}$ does not have a solution for all $\mathbf{b}$, you just need to show that A does not have a pivot in every row. However, to determine $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does have a solution, you will need to put the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ in row echelon form.)

Exercise 4. The columns of the matrix A span a plane in $\mathbb{R}^3$. Is $\mathbf{u}$ in the plane spanned by the columns of A? If so, write $\mathbf{u}$ as a linear combination of the columns of A.

$$A = \begin{bmatrix} 4 & -6 \\ -3 & 5 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$$

1.5 Solution Sets of Linear Systems

Name: ______________________________

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if, and only if, the equation has at least one free variable.

Exercise 1. Determine if the system has a nontrivial solution.

$$\begin{aligned} -2x_1 + 6x_2 - 6x_3 &= 0 \\ -4x_1 + 8x_2 + 3x_3 &= 0 \end{aligned}$$

To write a solution set in parametric vector form:

(1) Row reduce augmented matrix to RREF

(2) Express basic var. in terms of free var.

(3) Write the solution $\mathbf{x}$ as a vector whose entries depend on the free variables.

(4) Decompose $\mathbf{x}$ into a linear combination of vectors using free variables as parameters.

Example. Solution set: $\begin{cases} x_1 = 3 - 2x_3 \\ x_2 = -4 \\ x_3 \text{ is free} \end{cases}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 2x_3 \\ -4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 2. The matrix A is given below along with its reduced echelon form and the associated system of equations of the matrix equation $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 4 & -3 & 5 \\ 1 & 5 & -5 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & -2 & 2 \end{bmatrix} \xrightarrow{\text{System}} \begin{aligned} x_1 \quad + 5x_3 - 3x_4 &= 0 \\ x_2 - 2x_3 + 2x_4 &= 0 \end{aligned}$$

(a) Describe the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

(b) Circle the best answer.

The solution set of $A\mathbf{x} = \mathbf{0}$ is a: **Single Vector** **Line** **Plane** **None of these**

Exercise 3. Compare $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$
The solution sets of the systems of linear equations below are given.

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 + 6x_3 &= 9 \end{aligned} \qquad \begin{cases} x_1 = 7 - 4x_3 \\ x_2 = -3 + 2x_3 \\ x_3 \text{ is free} \end{cases}$$

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 0 \\ -4x_1 - 4x_2 - 8x_3 &= 0 \\ -3x_2 + 6x_3 &= 0 \end{aligned} \qquad \begin{cases} x_1 = -4x_3 \\ x_2 = 2x_3 \\ x_3 \text{ is free} \end{cases}$$

(a) Write the solution sets in parametric vector form.

(i) First System

(ii) Second System

(b) Provide a geometric comparison between the solution sets of the two systems. (Are the solution sets single vectors? Lines? Planes? Something else? How are they related to one another?)

Exercise 4. A 3×4 matrix A has 3 pivot positions.

(a) Does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? How do you know?

(b) Does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of $\mathbf{b}$ in $\mathbb{R}^3$? How do you know?

1.6 Applications of Linear Systems

Name: ______________________________ Name: ______________________________

Name: ______________________________ Name: ______________________________

Exercise 1. Consider an economy with 3 sectors, Chemicals & Metals, Fuels & Power, and Machinery. Chemicals sells 30% of its output to Fuels and 50% to Machinery and retains the rest. Fuels sells 80% of its output to Chemicals and 10% to Machinery and retains the rest. Machinery sells 40% to Chemicals and 40% to Fuels and retains the rest.

(a) Construct the exchange table for this economy.

Chem	**Fuels**	**Mach**	**Purchased by:**
			Chem
			Fuel
			Mach

(b) Develop a system of equations that leads to prices at which each sector's income matches its expenses using p_C, p_F, and p_M for the price of Chemicals, Fuels, and Machinery outputs, respectively. Then write the augmented matrix that can be row reduced to find these prices.

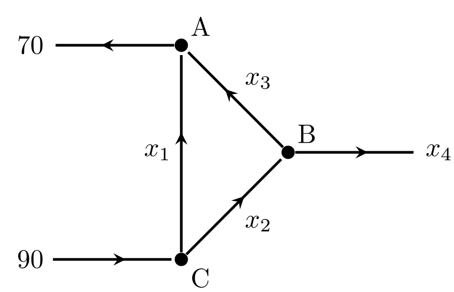
(c) Find a set of equilibrium prices when the price for the Machinery ouput is 100 units.

Exercise 2. Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \longrightarrow Na_3C_6H_5O_7 + H_2O + CO_2$$

Balance the chemical equation. (Hint: there are 5 unknowns—one for each of the 5 molecules. The column vectors you construct will each contain 4 entries—one for each type of atom present, Na, H, C, and O)

Exercise 3. (a) Find the general flow pattern of the network shown in the figure.



(b) Assuming that the flows are all nonnegative, what is the largest possible value for x_3?

1.7 Linear Independence

Name: ______________________________

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}. \tag{†}$$

The equation (†) is called a **linear dependence relation**. (Note that each linear dependence relation corresponds to a nontrivial solution of the vector equation.)

Exercise 1. Determine by inspection if the given vectors are linearly independent. If they are not, write a linear dependence relation to prove they are linearly dependent.

(a)

$$\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -9 \\ 6 \end{bmatrix}$$

(b)

$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The columns of a matrix A are linearly independent if, and only if, the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Exercise 2. A matrix is given along with its reduced echelon form. Determine if the columns of the matrix are linearly independent. Explain your answer.

(a)

$$A = \begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -4 \\ 1 & -4 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & -3 & 5 & 0 \\ 3 & -9 & 6 & 1 \\ 2 & -6 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 3. Given $A = \begin{bmatrix} 2 & 4 & 6 \\ -6 & 1 & -5 \\ -4 & -2 & -6 \\ 4 & 0 & 4 \end{bmatrix}$, observe that the 3rd column is the sum of the 1st and 2nd columns.

(a) Without performing row operations, give a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

(b) Write $A\mathbf{x}$ as a linear combination of the columns of A using the entries of your solution to part (a) as weights. Simplify the expression to show that your solution to part (a) is correct.

> **Theorem 7. Characterization of Linearly Independent Sets (Short Version)**
> An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if, and only if, at least one of the vectors in S is a linear combination of the others.

> **Theorem 8.**
> If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Exercise 4. Determine if the following vectors are linearly independent or linearly dependent. Explain your answer. Be sure to reference the appropriate theorem.

(a)

$$\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 15 \end{bmatrix}$$

Circle one and explain your answer:

Linearly Independent **Linearly Dependent**

(b)

$$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 11 \\ 13 \\ 17 \\ 19 \end{bmatrix}, \begin{bmatrix} 23 \\ 29 \\ 31 \\ 37 \end{bmatrix}, \begin{bmatrix} 41 \\ 43 \\ 47 \\ 53 \end{bmatrix}, \begin{bmatrix} 59 \\ 61 \\ 67 \\ 71 \end{bmatrix}$$

Circle one and explain your answer:

Linearly Independent **Linearly Dependent**

1.8 Intro to Linear Transformations

Name: ______________________________

A **transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns to each vector in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T. For $\mathbf{x} \in \mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the **range** of T.

Exercise 1. Suppose we define a transformation T by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) What is the domain of T?

(b) What is the codomain of T?

(c) Are the range and codomain of T the same? Why or why not?

Hint: think of the range as all possible linear combinations of the columns of A.

Exercise 2. Find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

$$\begin{bmatrix} 1 & -5 & 18 & -3 \\ 0 & 1 & -5 & 2 \\ 4 & -16 & 52 & -4 \end{bmatrix}$$

A transformation T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T.

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

This implies the following:

(i) $T(\mathbf{0}) = \mathbf{0}$.

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all scalars c, d and all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T.

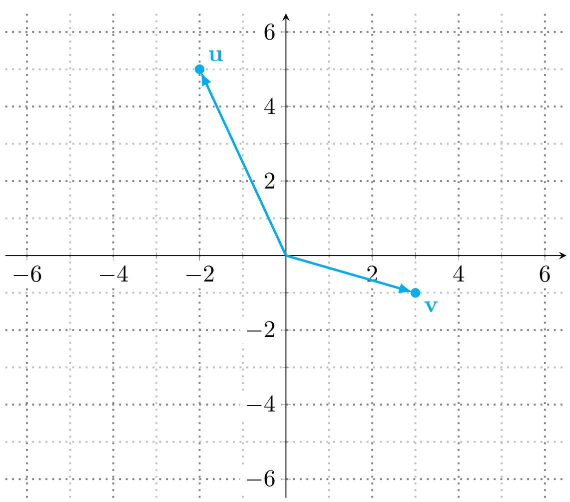
Note: to prove a transformation T is linear, you only need to show $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Exercise 3. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ maps the vector $\mathbf{u} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and maps the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use that fact that T is linear to find the image of $3\mathbf{u} + 4\mathbf{v}$ under T.

Exercise 4. The vectors $\mathbf{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ are plotted on the graph below.

(a) Plot $T(\mathbf{u})$ and $T(\mathbf{v})$ under the given transformation T.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



(b) Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbb{R}^2$. (Is it a rotation? Reflection? Projection? Shear? Dilation? Contraction? Something else?)

1.9 Matrix of a Linear Transformation

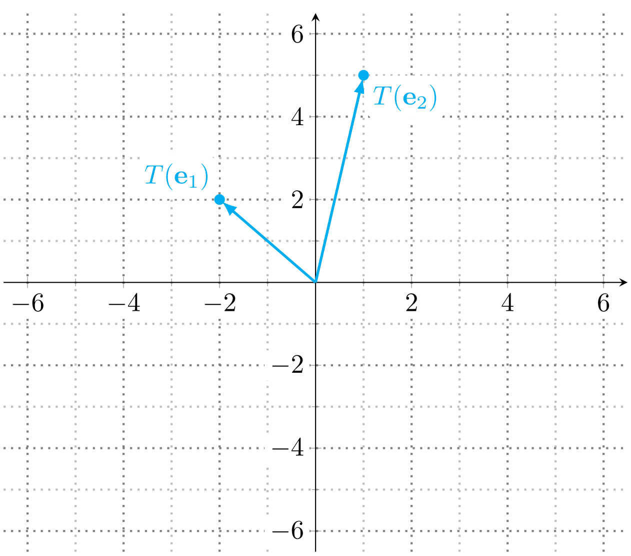
Name: ______________________________

Theorem 10.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}.$$

Exercise 1. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (-5, 3)$, and $T(\mathbf{e}_3) = (3, -8)$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the standard matrix of T.

Exercise 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are the vectors shown in the figure.



(a) Using the figure, sketch the vector $T(-2, 1)$.

(b) Write the standard matrix for the linear transformation.

$$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$. The mapping is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

Theorem 12.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, and only if, the columns of A span $\mathbb{R}^m$;

(b) T is one-to-one if, and only if, the columns of A are linearly independent.

Exercise 3. A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ has standard matrix A. The reduced row echelon form of A is given below.

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Is T one-to-one? Explain.

(b) Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Explain.

Exercise 4. The standard matrix for rotation about the origin by an angle θ is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. This is because the transformation affects $\mathbf{e}_1$ and $\mathbf{e}_2$ in the following way:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Find the standard matrix for the transformation that rotates points by $\dfrac{\pi}{3}$ radians and then reflects them over the horizontal x_1-axis. Simplify any trigonometric expressions.

2 Matrix Algebra

3.1 Matrix Operations

Name: ______________________

Use the following matrices for the problems on this page.

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & -5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -4 & 2 \\ 2 & -3 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 6 \\ -2 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

Exercise 1. Compute the matrix sum or explain why the expression is undefined.

(a) $A + 2B$

(b) $4C - 2E$

Exercise 2. Compute the matrix product or explain why the expression is undefined.

(a) EB

(b) DB

If A is an $m \times n$ matrix and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$ is an $n \times p$ matrix, then the product AB is the $m \times p$ matrix

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

In other words, each column of AB is a linear combination of the columns of A using the entries of the corresponding columns of B as weights.

Exercise 3. Let A and B be as below. Denote the first and second columns of B by $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively.

$$A = \begin{bmatrix} -1 & 4 \\ 2 & 5 \\ 5 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix}$$

(a) Compute $A\mathbf{b}_1$ and $A\mathbf{b}_2$.

(b) Using part (a), write the matrix AB.

$$A\mathbf{b}_1 = \begin{bmatrix} -1 & 4 \\ 2 & 5 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$A\mathbf{b}_2 =$

Exercise 4. Let $A = \begin{bmatrix} 4 & -8 \\ -4 & 8 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different columns for B.

3.2 The Inverse of a Matrix

Name: ______________________

If A is a matrix, the unique inverse of A, denoted A^{-1}, is defined so that $A^{-1}A = I$ and $AA^{-1} = I$.

Theorem 4. (For 2×2 Matrices)
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $ad - bc = 0$, then A is not invertible.

Exercise 1. Determine if the matrix is invertible by checking to see that $ad - bc \neq 0$. If the matrix is invertible, use Theorem 4 to find the inverse.

(a)

$$A = \begin{bmatrix} 9 & 4 \\ 5 & 2 \end{bmatrix}$$

Circle one. If A is invertible, find A^{-1}.

Invertible **Not Invertible**

(b)

$$B = \begin{bmatrix} 6 & -9 \\ -2 & 3 \end{bmatrix}$$

Circle one. If B is invertible, find B^{-1}.

Invertible **Not Invertible**

Theorem 5.
If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Exercise 2. Suppose A is an invertible 2×2 matrix with inverse $A^{-1} = \begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(a) Use Theorem 5 to solve the equation $A\mathbf{x} = \mathbf{b}$.

(b) Suppose A is the standard matrix for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Is T one-to-one? Explain. (It may be helpful to consider Theorem 7 on the next page and Theorem 12 from section 1.9).

Theorem 7.
An $n \times n$ matrix A is invertible if, and only if, A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Exercise 3. Find the inverse of A, if it exists.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -2 \\ 2 & 3 & -2 \end{bmatrix}$$

Use Theorem 7 and the proceeding comment, i.e., row reduce the following matrix:

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ -3 & 1 & -2 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

Exercise 4. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$.

(a) Show that $CA = I_2$.

(b) Is C the inverse of A? Why or why not?

3.3 Characterizations of Invertible Matrices

Name: ____________________

Theorem 2.8. The Invertible Matrix Theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I$.

(k) There is an $n \times n$ matrix D such that $AD = I$.

(l) A^T is an invertible matrix.

Exercise 1. Using as few calculations as possible, determine which of the following matrices are invertible. Explain your reasoning.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 13 \\ 2 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} -10 & 0 \\ 15 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & 8 & 10 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 5 \\ 3 & -9 & -6 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 0 & 0 \\ -5 & 2 & 0 \\ 11 & -4 & -2 \end{bmatrix}$

Theorem 2.9.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if, and only if, A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique inverse of T.

Exercise 2. A linear transformation T is given below.

$$T(x_1, x_2) = (-2x_1 + 3x_2, 4x_1 - 5x_2)$$

(a) Find the standard matrix, A, for the linear transformation T.

If you're not sure how to begin, try writing $T(x_1, x_2)$ as a column vector or in parametric vector form.

(b) If A is invertible, then T is invertible by Theorem 2.9. Find the inverse of A (if it exists), and use it to determine the formula for T^{-1}.

3.5 Matrix Factorizations

Name: ____________________

Algorithm for an LU Factorization

(1) Reduce A to a row echelon form U by a sequence of row replacement operations, if possible.

(2) Place entries in L such that the same sequence of row operations reduces L to I. (If you used the row operation $R_i' = R_i + kR_j$ in step (1), then the entry of L in the ith row and jth column is $L_{ij} = -k$.)

Exercise 1. Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$.

(a) Find an LU factorization of A.

(b) Use multiplication to verify that $LU = A$.

(c) Solve $L\mathbf{y} = \mathbf{b}$.

(d) Solve $U\mathbf{x} = \mathbf{y}$.

(e) Verify that $A\mathbf{x} = \mathbf{b}$.

3.8 Subspaces of $\mathbb{R}^n$

Name: ____________________

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

(i) The zero vector is in H.

(ii) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

(iii) For each $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H.

Exercise 1. Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Determine if $\mathbf{w} = \begin{bmatrix} -2 \\ 6 \\ -1 \end{bmatrix}$ lies in the subspace of $\mathbb{R}^3$ that is generated by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

The **Column Space** of a matrix A is the set $\operatorname{Col} A$ of all linear combinations of the columns of A.

The **Null Space** of a matrix A is the set $\operatorname{Nul} A$ of all solutions of the equation $A\mathbf{x} = \mathbf{b}$.

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

To find a basis for $\operatorname{Nul} A$, write the general solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form. The vectors in the solution form a basis for $\operatorname{Nul} A$ (whenever $\operatorname{Nul} A \neq \{\mathbf{0}\}$)

Theorem 12.3.
The pivot columns of a matrix A form a basis for $\operatorname{Col} A$.

Exercise 2. Assume that A is row equivalent to B.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -6 & 1 \\ -3 & 8 & 5 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 2 & 8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $\operatorname{Nul} A$.

(b) Find a basis for $\operatorname{Col} A$.

Exercise 3. Determine if the set is linearly independent, if it spans $\mathbb{R}^3$, and if it is a basis for $\mathbb{R}^3$. Show work or explain your answer.

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Circle all that apply:

Linearly Independent?	**Spans $\mathbb{R}^3$?**	**Is a Basis for $\mathbb{R}^3$?**
Yes **No**	**Yes** **No**	**Yes** **No**

3.9 Dimension and Rank

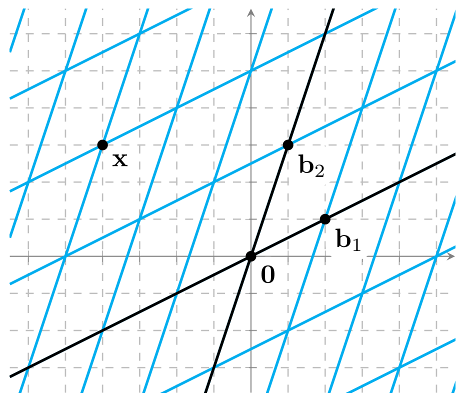
Name: ______________________

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace H and $\mathbf{x}$ is in H of $\mathbb{R}^n$. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

We call $[\mathbf{x}]_{\mathcal{B}}$ the **coordinate vector of x (relative to $\mathcal{B}$)** or the **$\mathcal{B}$-coordinate vector of x**.

Exercise 1. The vectors $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are shown in the figure. The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ provide a basis for $\mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. This basis provides a new "coordinate system" as shown in the figure.



Using the figure, find $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Theorem
If a vector space H has a basis of n vectors, then every basis for H must consist of exactly n vectors.

The **dimension** of a vector space H, denoted $\dim H$, is the number of vectors in a basis for H.

The dimension of $\operatorname{Nul} A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

The **rank** of A is the dimension of the column space of A.

Theorem 2.14. The Rank Theorem
The dimension of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Exercise 2. Determine the dimensions of the null space and column space for A.

(a)

$$A = \begin{bmatrix} 1 & -5 & -5 & 3 & -6 \\ 0 & 1 & 7 & -2 & 2 \\ 0 & 0 & 0 & -6 & -9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{dim\,Nul}\,\boldsymbol{A} =$

$\mathbf{rank}\,\boldsymbol{A} = \mathbf{dim\,Col}\,\boldsymbol{A} =$

Theorem
If a subspace H of $\mathbb{R}^n$ has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$, then any set in H containing more than p vectors must be linearly dependent.

Theorem 2.15. The Basis Theorem
Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of vectors of exactly p elements in H is automatically a basis for H. Any set of exactly p elements that spans H is automatically a basis for H.

Exercise 3. Use the above theorems to answer the questions here.

(a) The set of vectors below span $\mathbb{R}^5$.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Is this set a basis for $\mathbb{R}^5$? Explain.

(b)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Is this set a basis for $\mathbb{R}^3$? Explain.

4 Matrix Operations

5.1 Introduction to Determinants

Name: ______________________________

Given a matrix A, we define a submatrix, A_{ij}, by deleting the ith row and jth column.
Example: Given A below, you get A_{23} by deleting the 2nd row and 3rd column from A.

$$A = \begin{bmatrix} 3 & 0 & 3 \\ 3 & 4 & 2 \\ 0 & 5 & -1 \end{bmatrix} \xrightarrow[\text{\& 3rd Column}]{\text{Delete 2nd Row}} A_{23} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Exercise 1. Given $A = \begin{bmatrix} 6 & 7 & -5 \\ 2 & 0 & 4 \\ 5 & 2 & 3 \end{bmatrix}$, determine the following.

(a) $\det A_{11}$

(b) $\det A_{12}$

(c) $\det A_{13}$

The (i, j)-cofactor of A is defined as.

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example: (uses A_{23} from previous example)

$$\begin{aligned} C_{23} &= (-1)^{2+3} \det A_{23} \\ &= (-1)^5 \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} \\ &= (-1)(3 \cdot 5 - 0) = -15 \end{aligned}$$

The factor $(-1)^{i+j}$ determines the following pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Exercise 2. Using Exercise 1 and the above definition, compute the determinant of $A = \begin{bmatrix} 6 & 7 & -5 \\ 2 & 0 & 4 \\ 5 & 2 & 3 \end{bmatrix}$.

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= (6) \det A_{11} - (7) \det A_{12} + (-5) \det A_{13} \\ &= \end{aligned}$$

Theorem 3.1.
The determinant of a $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors $C_{ij} = (-1)^{i+j} \det A_{ij}$ is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Exercise 3. Compute the determinant by cofactor expansion. At each step, try to choose a row or column that involves the least amount of computation.

$$\begin{vmatrix} 1 & 6 & 3 & 9 \\ 0 & 7 & 0 & 3 \\ 0 & 2 & 0 & 1 \\ 4 & -7 & 0 & 7 \end{vmatrix}$$

Theorem 3.2.
If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Exercise 4. Compute the following determinants.

(a) $\begin{vmatrix} 5 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{vmatrix}$

(b) $\begin{vmatrix} 3 & -1 & 7 & 5 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -2 \end{vmatrix}$

(c) $\begin{vmatrix} 4 & 7 & 4 & 6 & 1 & 6 \\ 0 & 1 & 6 & 4 & 4 & 1 \\ 0 & 0 & 5 & 3 & 1 & 4 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$

5.2 Properties of Determinants

Name: ______________________________

Read the Properties of Determinants Section outline from a different book and answer the following questions. This reading assignment is due Wednesday, February 26, 2020 at the beginning of class.

Theorem 3.3.
Let A be a square matrix.

(a) If a multiple of one row of A is added to another row to produce B, then $\det B = \det A$.

(b) If two rows of A are interchanged to produce B, then $\det B = -\det A$.

(c) If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

Exercise 1. Find the determinant of $A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{bmatrix}$ by row reducing to a matrix of row echelon form.

Theorem 3.4.
Let A be a square matrix. A is invertible if and only if $\det A \neq 0$.

Exercise 2. Does the matrix, A, from the previous exercise have an inverse? Why or why not?

Theorem 3.5.
If A is a square matrix, then $\det A = \det A^T$.

Theorem 3.6.
If A and B are both $n \times n$ matrices, then $\det AB = \det A \det B$.

Exercise 3. Suppose A and B are both 3×3 matrices where $\det A = 2$ and $\det B = -3$. Use the theorems from both sides to determine each of the following:

(a) $\det AB$

(b) $\det A^T$

(c) $\det 5A$

(d) $\det B^{-1}$

(e) $\det A^3$

5.3 Cramer's Rule and Volume

Name: ____________________

Theorem 3.7: Cramer's Rule.
Let A be an $n \times n$ invertible matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, ..., n.$$

Exercise 1. Consider the following system:

$$\begin{aligned} 5x_1 + 7x_2 &= 3 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$$

(a) Rewrite the system in the form $A\mathbf{x} = \mathbf{b}$.

(b) Verify that $\det A \neq 0$, so that Cramer's Rule will apply.

(c) Solve the system using Cramer's Rule.

Theorem 3.9.
If A is a 2×2 matrix, the area of the parallelogram determined by by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Theorem 3.10.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation determined by 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

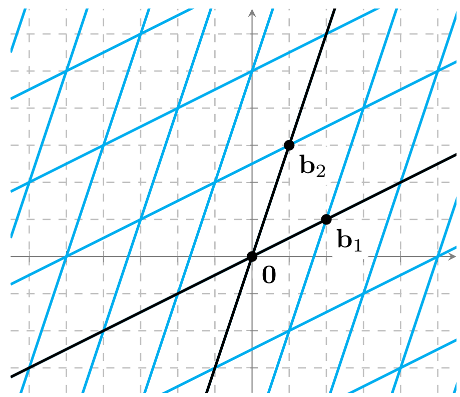
$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

Alternatively, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation determined by 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

Exercise 2. Let S be the parallelogram determined by the vectors $\mathbf{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{b_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(a) Shade in S in the picture below. Then, use a determinant to find the area of S.



(b) Find the area of the image of S under the mapping $\mathbf{x} \to A\mathbf{x}$ where $A = \begin{bmatrix} 5 & -1 \\ 2 & 5 \end{bmatrix}$.

6 Vector Spaces

8 Eigenvalues and Eigenvectors

9.1 Eigenvectors and Eigenvalues

Name: ______________________

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvector** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an **eigenvector corresponding to $\boldsymbol{\lambda}$**.

Exercise 1. Is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 3 & 3 & -1 \\ 2 & 8 & -2 \\ -2 & -6 & 4 \end{bmatrix}$? If so, find the corresonding eigenvalue.
Hint: Check to see if $A\mathbf{v} = \lambda\mathbf{v}$ for some λ.

Given an eigenvalue λ of a matrix A, an eigenvector $\mathbf{x}$ must be a nonzero vector satisfying $A\mathbf{x} = \lambda\mathbf{x}$. In order to solve for such an $\mathbf{x}$, we first rewrite $A\mathbf{x} = \lambda\mathbf{x}$ as follows:

$$\begin{aligned} A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

If $\mathbf{x}$ is a nonzero solution to this homogeneous equation, $\mathbf{x}$ is an eigenvector of A corresponding to λ.

Exercise 2.

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \qquad \lambda = 1, 5$$

(a) Compute $(A - \lambda I)$ for $\lambda = 1$.

$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$

(b) Find an eigenvector of A with eigenvalue 1.

(c) Compute $(A - \lambda I)$ for $\lambda = 5$.

(d) Find an eigenvector of A with eigenvalue 5.

Theorem 5.1.
The eigenvalues of a triangular matrix are the entries on its main diagonal.

Note: If 0 is an eigenvalue of A, then we have some eigenvector $\mathbf{x}$ that satisfies $A\mathbf{x} = 0\mathbf{x}$. In other words, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. According to the invertible matrix theorem, this happens precisely when A is **not invertible**. So 0 is an eigenvalue of A if, and only if, A is not invertible!

Exercise 3. Use Theorem 5.1 or the note above along with the Invertible Matrix Theorem to answer the following questions.

(a) Find the eigenvalues of the matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

(b) Find one eigenvalue of the matrix without calculations. Explain your answer.

$$B = \begin{bmatrix} -1 & 3 & 7 \\ -1 & 3 & 7 \\ -1 & 3 & 7 \end{bmatrix}$$

For an $n \times n$ matrix A, the set of all the eigenvectors corresponding to an eigenvalue λ along with the zero vector forms a subspace of $\mathbb{R}^n$. This subspace, the set of all solutions to $(A - I\lambda)\mathbf{x} = \mathbf{0}$ (or the null space of $(A - \lambda I)$), is called the **eigenspace** of A corresponding to λ.

Exercise 4. Find a basis for the eigenspace corresponding to the given eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \qquad \lambda = 2$$

Hint: Find a basis for the null space of $(A - \lambda I)$.

9.2 The Characteristic Equation

Name: ______________________

Recall that eigenvalues of an $n \times n$ matrix A are scalars, λ, for which $A\mathbf{x} = \lambda\mathbf{x}$ has nonzero solutions. From this definition and what we have learned so far, we get the following equivalent notions:

(i) $A\mathbf{x} = \lambda\mathbf{x}$ has a nonzero solution. (Definition of eigenvalue)

(ii) $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial (nonzero) solution. (Matrix algebra)

(iii) $(A - \lambda I)$ is not invertible. (Invertible Matrix Theorem)

(iv) $\det(A - \lambda I) = 0$. (Invertible Matrix Theorem)

A scalar λ is an eigenvalue of an $n \times n$ matrix A if, and only if, λ satisfies the **characteristic equation**:

$$\det(A - \lambda I) = 0.$$

The expression $\det(A - \lambda I)$ is called the **characteristic polynomial**—it is a polynomial of degree n in the variable λ.

Exercise 1. Find the characteristic polynomial and the eigenvalues of the matrix.

(a) $A = \begin{bmatrix} 8 & 6 \\ 6 & 8 \end{bmatrix}$

(b) $B = \begin{bmatrix} -5 & 2 \\ 1 & -1 \end{bmatrix}$

Each eigenvalue of a matrix A has a (algebraic) **multiplicity** which is the same as its multiplicity as a root of the characteristic equation.

Exercise 2. For the matrices given, list the eigenvalues, repeated according to their multiplicity, i,e., if the multiplicity is 3, list the eigenvalue 3 times. Hint: Note that the matrices are both triangular.

(a) $A = \begin{bmatrix} 6 & -3 & 0 & 1 \\ 0 & 2 & 5 & 5 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

(b) $B = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 5 & 7 & 0 & 0 & 0 \\ 8 & 6 & 0 & 0 & 0 \\ 5 & 2 & -4 & 0 & 0 \\ 5 & 6 & -7 & -9 & 7 \end{bmatrix}$

Exercise 3. Find the characteristic polynomial for the given matrix. You should compute the determinant using either a cofactor expansion or the special formula for 3×3 determinants (Rule of Sarrus).

$A = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 4 & -4 \\ 0 & 5 & 0 \end{bmatrix}$

Exercise 4. Find bases for the eigenspaces corresponding to both eigenvalues of A. (Begin by finding the characteristic polynomial, and then using it to find the eigenvalues.)

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

9.3 Diagonalization

Name: ______________________

A matrix A is **diagonalizable** if A is similar to a diagonal matrix D. That is, for some invertible matrix P and diagonal matrix D, we have $A = PDP^{-1}$.

Powers of a diagonalizable matrix are easy to compute: $A^k = PD^kP^{-1}$. The matrix D^k is simply D with all diagonal entries raised to the k power.

Exercise 1. Let $A = PDP^{-1}$. Compute A^4 using the diagonalization of A.
(You may want to use a calculator to double check your matrix multiplication calculations.)

$$P = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 5.5. The Diagonalization Theorem
An $n \times n$ matrix A is diagonalizable if, and only if, A has n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D a diagonal matrix, if, and only if, the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Exercise 2. The matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$A = \begin{bmatrix} 4 & 0 & -4 \\ 4 & 6 & 8 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

Theorem 5.6.
An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Although n *distinct eigenvalues* ensure that an $n \times n$ matrix is definitely diagonalizable, some matrices with repeated eigenvalues are diagonalizable, too. If A has n *linearly independent eigenvectors*, one may construct an **eigenbasis** for $\mathbb{R}^n$—a basis comprised of just eigenvectors. The existence of an eigenbasis corresponding to a matrix A is enough to ensure diagonalizablility.

Exercise 3. Find an eigenbasis for the matrix A below. The eigenvalues are given.
Hint: Recall that the eigenspace corresponding to an eigenvalue λ is $\text{Nul}(A - \lambda I)$, the set of all solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Find a basis for each eigenspace and combine these vectors to form an eigenbasis.

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 4 & -4 \\ 1 & -2 & 6 \end{bmatrix} \qquad \lambda = 2, 7$$

Exercise 4. Find matrices P and D such that $A = PDP^{-1}$. Use your work from the previous exercise.

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 4 & -4 \\ 1 & -2 & 6 \end{bmatrix}$$

9.5 Complex Eigenvalues

Name: ______________________

We define $i = \sqrt{-1}$. Then, the complex numbers, $\mathbb{C}$, is the set of all numbers of the form $a + bi$ for $a, b \in \mathbb{R}$.

The Fundamental Theorem of Algebra
Let $p(x)$ be a polynomial with complex coefficients of degree $n > 0$. Then, with multiplicities, $p(x)$ has n zeroes.

Exercise 1. Consider $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a basis for each eigenspace of A. (Hint: Find a basis for one of the eigenvectors, λ. Then, the basis corresponding to $\overline{\lambda}$ is the complex conjugate of the basis for λ.)

Theorem 5.9.
Let A be a real 2×2 matrix with complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then,

$$A = PCP^{-1}, \text{ where } P = [\operatorname{Re}\mathbf{v} \;\; \operatorname{Im}\mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Exercise 2. Consider $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$. (Same matrix as previous page.)

(a) Find matrices P and C such that A, P, and C satisfy the conditions of the theorem above.

(b) Verify your answer in (a) is correct by verifying that $A = PCP^{-1}$. (Hint: You can do this by just verifying that $AP = PC$.)

(c) Factor the matrix C into a scaling matrix, and a rotation matrix. That is, let $r = |\lambda| = \sqrt{a^2 + b^2}$, where $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Then factor to get $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix}$.

10 Orthogonality and Least Squares

11.1 Inner Product, Length, & Orthogonality

Name: ____________________

The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product** (or **dot product**) of $\mathbf{u}$ and $\mathbf{v}$. It is written as $\mathbf{u}\cdot\mathbf{v}$.

Exercise 1. Let $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Compute the following.

(a) $\mathbf{w}\cdot\mathbf{w}$

(b) $\mathbf{x}\cdot\mathbf{w}$

(c) $\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$

The **length** (or **norm**) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

A vector with length 1 is called a **unit vector**. To create a unit vector $\mathbf{u}$ from $\mathbf{v}$, compute $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$. You can think of this as "dividing $\mathbf{v}$" by its length, $\|\mathbf{v}\|$. This is called **normalizing $\mathbf{v}$**. We say the new unit vector $\mathbf{u}$ is **in the same direction** as $\mathbf{v}$.

Exercise 2. For each problem below, find a unit vector in the direction of the given vector.
Hint: You may wish to scale the vector by a positive constant before normalizing. This will not affect the final answer, but it can simplify calculations.

(a) $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

(b) $\begin{bmatrix} 7/4 \\ 1/2 \\ -1/2 \end{bmatrix}$

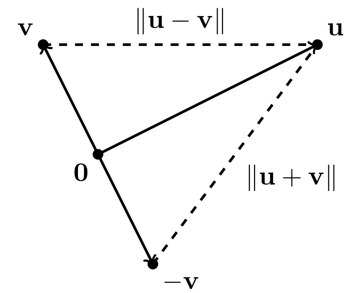
For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $(\mathbf{u} - \mathbf{v})$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector, $\mathbf{0}$, is orthogonal to every vector.

Exercise 3. The notion of orthogonality is a generalization of perpendicular lines. The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if the distance between $\mathbf{u}$ and $\mathbf{v}$ is the same as the distance between $\mathbf{u}$ and $-\mathbf{v}$ (see the figure). Given $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$, answer the following.

(a) Determine if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal by comparing $\|\mathbf{u} - \mathbf{v}\|$ and $\|\mathbf{u} + \mathbf{v}\|$.



(b) Determine if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal by computing $\mathbf{u} \cdot \mathbf{v}$.

If $\mathbf{z}$ is orthogonal to every vector in a subspace W, then we say $\mathbf{z}$ is **orthogonal** to W. The set of all $\mathbf{z}$ orthogonal to W is called the **orthogonal complement** of W, denoted $W^\perp$. A vector $\mathbf{z}$ is in $W^\perp$ if, and only if, $\mathbf{z}$ is orthogonal to every vector in a spanning set for W.

Exercise 4. Suppose W is spanned by the set $\left\{ \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -1 \\ 4 \end{bmatrix} \right\}$. Is $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ in $W^\perp$?

11.2 Orthogonal Sets

Name: ______________________

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.

Exercise 1. Determine if the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ -4 \\ 7 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 4 \\ 7 \\ 6 \\ 0 \end{bmatrix}$$

Theorem 6.4.
If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Theorem 6.5.
Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

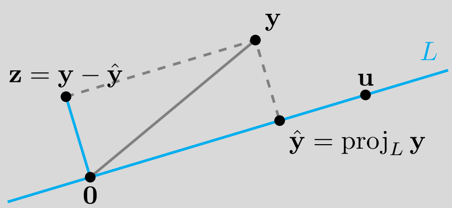
are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad (\text{for } j = 1, \dots, p).$$

Exercise 2. The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal. By Theorem 6.4 (and Theorem 2.15 or 4.12, The Basis Theorem), the set is a basis for $\mathbb{R}^3$. Write $\mathbf{x}$ as a linear combination of the basis vectors: $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

Suppose $\mathbf{u}$ and $\mathbf{y}$ are given. It will be useful to write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is some scalar multiple of $\mathbf{u}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$. The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of y onto u**, and the vector $\mathbf{z}$ is called the **component of y orthogonal to u**.



If we let $L = \text{Span}\{\mathbf{u}\}$, then we write

$$\hat{\mathbf{y}} = \text{proj}_L\,\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

Exercise 3. (a) Compute the orthogonal projection of $\mathbf{y} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ onto the line through $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and the origin.

(b) Using part (a), how could you compute the shortest distance from $\mathbf{y}$ to $L = \text{Span}\,\mathbf{u}$?

A set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an **orthonormal** if it is an orthogonal set of unit vectors. If $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then the set is an **orthonormal basis** for W.

Exercise 4. Determine if the set of vectors is orthonormal. If the set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$\mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix}$$

11.3 Orthogonal Projections

Name: ______________________________

Theorem 6.8. The Orthogonal Decomposition Theorem
Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \qquad \text{and} \qquad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto W**, and the vector $\mathbf{z}$ is called the **component of $\mathbf{y}$ orthogonal to W**.

Exercise 1. Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find $\hat{\mathbf{y}}$, the orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Hint: If you do the problem correctly, $\hat{\mathbf{y}}$ has all integer entries.

$$\mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} -5 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Exercise 2. Write $\mathbf{x}$ as the sum of two vectors, one in $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and the other in $\text{Span}\{\mathbf{u}_4\}$. You may assume $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$$

Hint: You could compute the orthogonal projections of $\mathbf{x}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\text{Span}\{\mathbf{u}_4\}$, but there is a much quicker method using Theorem 6.8. If we let $\hat{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ onto $W = \text{Span}\{\mathbf{u}_4\}$. Then $\mathbf{z} = \mathbf{x} - \hat{\mathbf{x}}$ is in $W^{\perp} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. So $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$ will be the sum you want.

Theorem 6.9. The Best Approximation Theorem
Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|$$

for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$. The vector $\hat{\mathbf{y}}$ is **the best approximation to y by elements of $\boldsymbol{W}$**.

Exercise 3. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

Theorem 6.10.
If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\operatorname{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

If $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$, then $\operatorname{proj}_W \mathbf{y} = UU^T\mathbf{y}$ for all $\mathbf{y}$ in $\mathbb{R}^n$.

Exercise 4. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, and $W = \operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. You may assume $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthonormal.

(a) Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$ and compute UU^T.

$$UU^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} =$$

(b) Compute $\operatorname{proj}_W \mathbf{y} = UU^T\mathbf{y}$.

11.4 The Gram-Schmidt Process

Name: ______________________________

Theorem 6.11. The Gram-Schmidt Process
Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\ \vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \ldots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W, and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $k \leq p$.

Exercise 1. The set $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W. Hint: Scaling vectors before you begin may simplify calculations.

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$$

Exercise 2. A matrix A with linearly independent columns is given below. Find an orthogonal basis for the column space of A. Note that columns 1 and 2 are already orthogonal. Hint: Recall that one possible basis for $\text{Col}\, A$ consists of the pivot columns of A.

$$A = \begin{bmatrix} -1 & 1 & 3 \\ 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Exercise 3. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for a subspace W. Find an orthonormal basis for W. Hint: Since scaling vectors does not affect orthogonality, you may wish to scale $\mathbf{v}_1$ and $\mathbf{v}_2$ before normalizing.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$$

Theorem 6.12. The $\boldsymbol{QR}$ Factorization
If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\operatorname{Col} A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

To produce Q, orthogonalize and normalize the columns of A, i.e., apply the Gram-Schmidt process (with normalization) to the columns of A. To produce R, use the following:

$$\begin{aligned} QR &= A \\ Q^T QR &= Q^T A \\ IR &= Q^T A && \text{(By Thm 6.6, since } Q \text{ has orthonormal columns, } Q^T Q = I) \\ R &= Q^T A \end{aligned}$$

Exercise 4. The columns of Q were obtained by applying the Gram-Schmidt process (with normalization) to the columns of A. Find an upper triangular matrix R such that $A = QR$.

$$A = \begin{bmatrix} -2 & -3 \\ 5 & 7 \\ -2 & -2 \\ -4 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} -2/7 & -1/\sqrt{14} \\ 5/7 & 2/\sqrt{14} \\ -2/7 & 0 \\ -4/7 & 3/\sqrt{14} \end{bmatrix}$$

11.5 Least-Squares Problems

Name: ______________________

Theorem 6.13.
The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.

Exercise 1. Describe all the least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 1 \end{bmatrix}$$

(a) Compute $A^T A$ and $A^T\mathbf{b}$.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 5 \\ 1 \end{bmatrix} =$$

(b) Solve $A^T A\mathbf{x} = A^T\mathbf{b}$.
(You may use a calculator or computer)

Theorem 6.14.
Let A be an $m \times n$ matrix. The following statements are logically equivalent:

(a) The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

(b) The columns of A are linearly independent.

(c) The matrix $A^T A$ is invertible.

When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by $\hat{\mathbf{x}} = \left(A^T A\right)^{-1} A^T\mathbf{b}$.

Exercise 2. Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$. Use Theorem 6.14 if applicable.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 4 \\ 8 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 16 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -16 \\ 44 \end{bmatrix}$$

Exercise 3. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$.

(a) Find the orthogonal projection of $\mathbf{b}$ onto $\operatorname{Col} A$. Note that the columns of A are linearly independent and orthogonal, so you already have an orthogonal basis for $\operatorname{Col} A$.

(b) Use your calculations in part (a) to find a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

Hint: Think of the entries of $\mathbf{x}$ as weights for a linear combination of the columns of A.

Theorem 6.15.
Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorization of A as in Theorem 6.12. Then, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, given by

$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. (Numerical Note: it is usually much quicker to solve $R\hat{\mathbf{x}} = Q^T\mathbf{b}$.)

Exercise 4. Use the factorization $A = QR$ given below to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$$

(a) Compute $Q^T\mathbf{b}$.

$$Q^T\mathbf{b} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} =$$

(b) Solve $R\hat{\mathbf{x}} = Q^T\mathbf{b}$ via back-substitution.

12 Symmetric Matrices and Quadratic Forms

13.1 Diagonalization of Symmetric Matrices

Name: ____________________

A **symmetric matrix** is a matrix A such that $A^T = A$.

Theorem 7.1.
If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Exercise 1. (a) Fill in the following matrix so that it is symmetric:

$$A = \begin{bmatrix} 2 & & 8 \\ -4 & 8 & 4 \\ & & 2 \end{bmatrix}$$

(b) The vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ are eigenvectors of the symmetric matrix A above. The eigenvalue for $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\lambda = 10$, and the eigenvalue for $\mathbf{v}_3$ is $\lambda = -8$. What does Theorem 7.1 tell you about $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_3$, and $\mathbf{v}_1 \cdot \mathbf{v}_3$, if anything?

An **orthogonal matrix** (section 6.2) is a square matrix A with orthonormal columns (so $A^{-1} = A^T$). A matrix A is **orthogonally diagonalizable** if there are an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1} = PDP^T$.

Theorem 7.2.
An $n \times n$ matrix A is orthogonally diagonalizable if, and only if, A is a symmetric matrix.

Exercise 2. (a) Is B an orthogonal matrix? Why or why not?

$$B = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

(b) Give an example of a 3×3 matrix that is orthogonally diagonalizable.

Steps to Orthogonally Diagonalize an $n \times n$ Symmetric Matrix

(i) Find the eigenvalues of A

(ii) Find n linearly independent eigenvectors for A, orthogonalize the set (if necessary), and normalize

(iii) Construct P from the orthonormalized eigenvectors you obtain in step (ii)

(iv) Construct D by placing the corresponding eigenvectors along the diagonal of D

Exercise 3. A matrix and two eigenvectors are given below. Orthogonally diagonalize the matrix (you only need to determine P and D). Steps (i) and (ii) should go quickly since you already have 2 eigenvectors.
$A = \begin{bmatrix} 13 & 4 \\ 4 & 7 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The set of eigenvalues of a matrix A is called the **spectrum** of A. A **spectral decomposition** for A can be obtained from the columns of P and eigenvalues from D in the orthogonal diagonalization of A:

$$A = PDP^T = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \cdots & \lambda_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

This last expression can be rewritten as a spectral decomposition: $A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$. This is a sum of matrices each of which relies on just one column of P and its corresponding eigenvalue.

Exercise 4. Find a spectral decomposition for the matrix A using the given orthogonal diagonalization, $A = PDP^T$.

$$A = \begin{bmatrix} 2 & 9 \\ 9 & 2 \end{bmatrix} \qquad PDP^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -7 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(a) Compute $\mathbf{u}_1 \mathbf{u}_1^T$ and $\mathbf{u}_2 \mathbf{u}_2^T$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the first and second columns of P, respectively.

(b) Write a spectral decomposition for A.