

Classroom Activities
Math 1113 - Precalculus

University of Georgia
Department of Mathematics

July 9, 2016

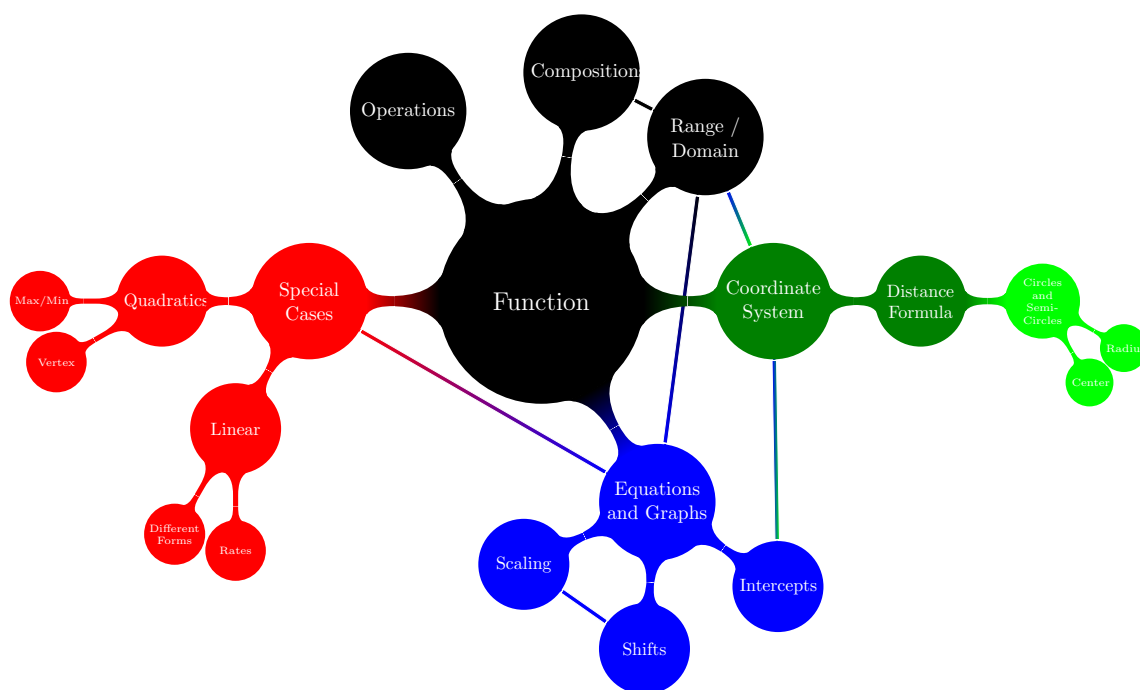
Copyright (C) 2016 Kelly Black University of Georgia Department of Mathematics
Permission is granted to copy, distribute and/or modify this document under the
terms of the GNU Free Documentation License, Version 1.3 or any later version
published by the Free Software Foundation; with no Invariant Sections, no Front-
Cover Texts, and no Back-Cover Texts. A copy of the license is included in the
section entitled "GNU Free Documentation License".

Contents

1	Functions and Preliminaries	5
1.1	Coordinate Systems	9
1.2	Graphs of Equations	17
1.3	Linear Equations	23
1.4	Functions	29
1.5	Graphs of Functions	35
1.6	Quadratic Functions	41
1.7	Operations on Functions	47
1.8	Inverse Functions	55
2	Exponential and Logarithmic Functions	61
2.1	Introduction to Exponential Functions	65
2.2	The Natural Exponential	75
2.3	Introduction to Logarithms	83
2.4	The Natural Logarithm	91
2.5	Exponential and Logarithmic Equations	101
3	Angle Measurement	109
3.1	Angle Measurement	113
3.2	Motion Around a Circle	121
4	Trigonometric Functions	129
4.1	Basic Trigonometry	133
4.2	Trigonometric Functions	141
4.3	Graphs of Trigonometric Functions	149
4.4	Word Problems	157
4.5	Trigonometric Identities	163
4.6	Introduction to Inverse Trigonometric Functions	171
4.7	Inverse Trigonometric Functions	179
5	GNU Free Documentation License	185

Chapter 1

Functions and Preliminaries

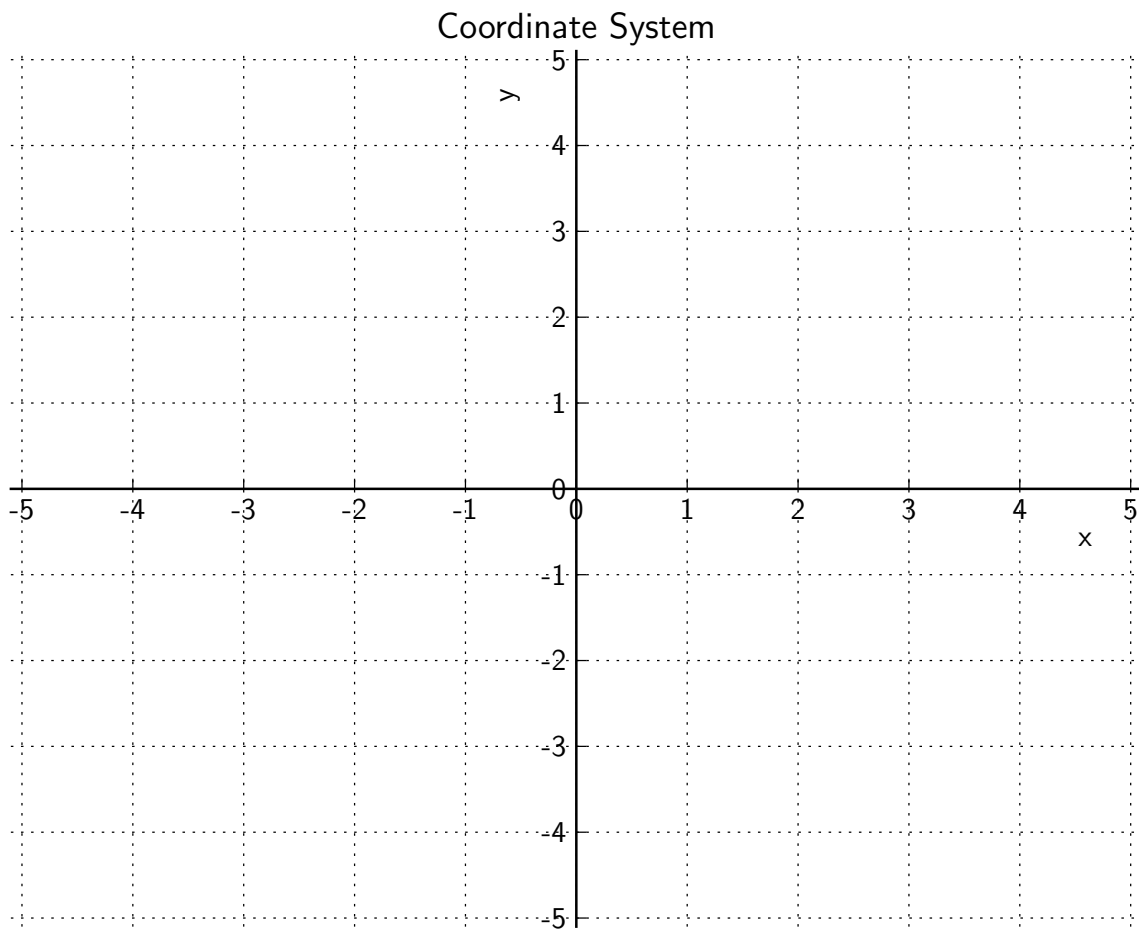


- Precalculus - July 9, 2016

8 Name:

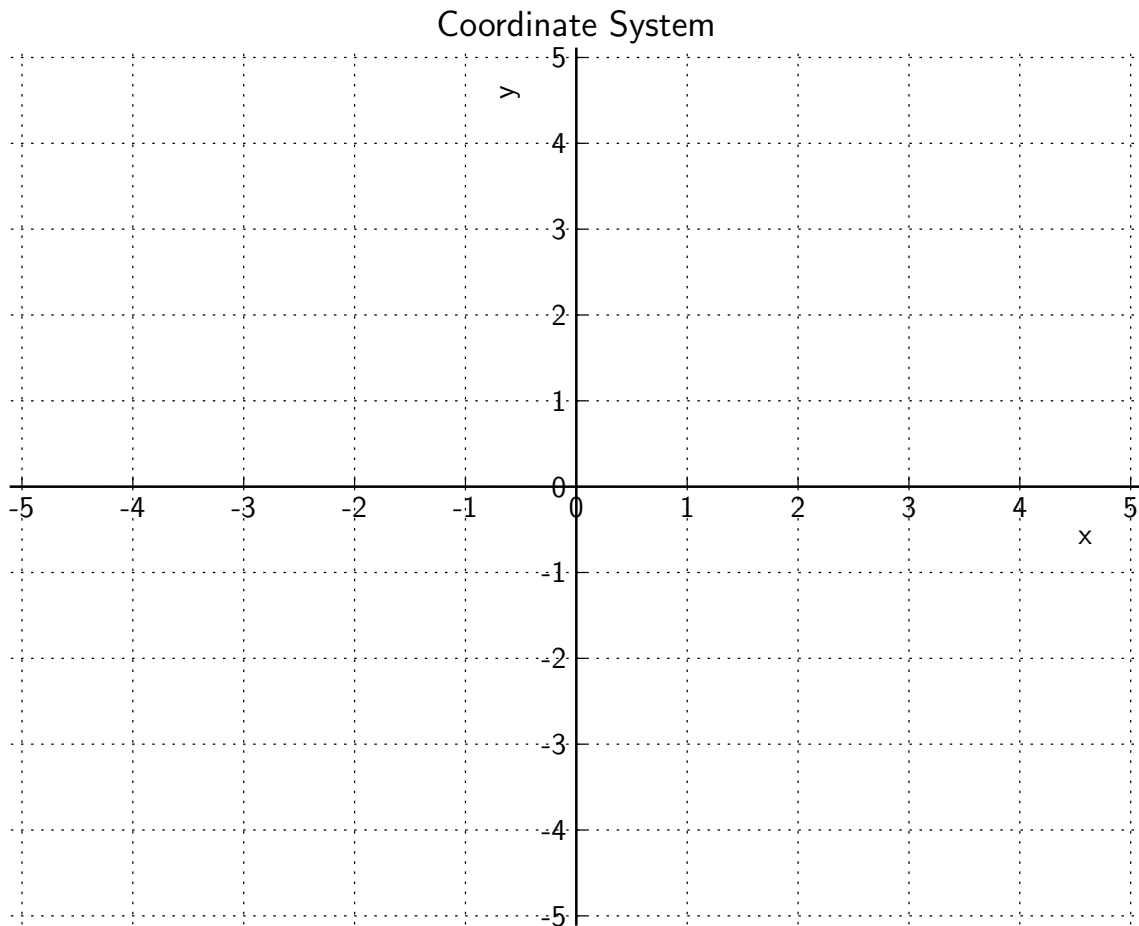
Preclass Work - Finish Before Class Begins

1. Mark the points $P_1(-2.1, -4.4)$ and $P_2(4.5, 1.2)$ on the axes below. Determine the distance between the two points. Include a sketch of a right triangle whose hypotenuse represents the distance between the two points.

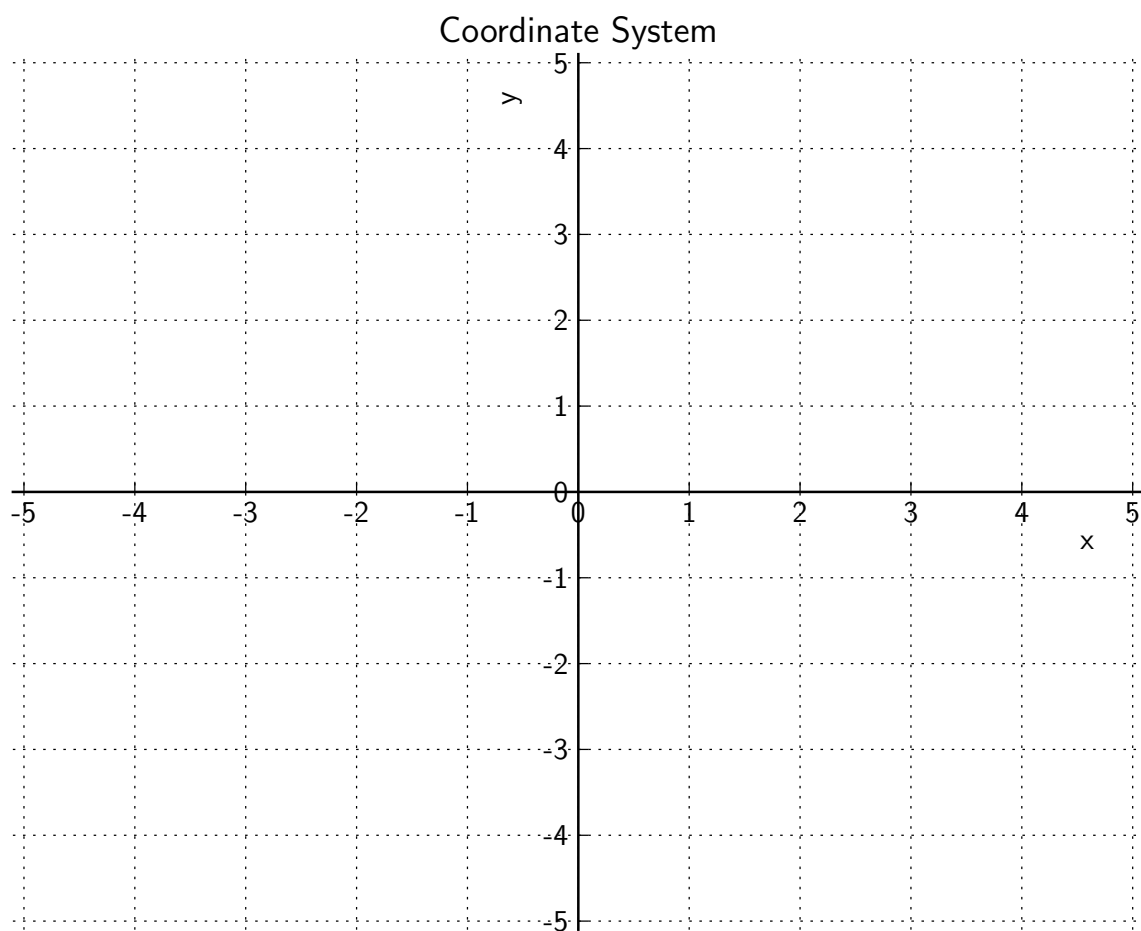


Write out
the distance
formula for
the
on the
x-axis
solve
the
unknown
variable

2. Mark the point $P_3(1.3, -2.4)$ on the axes below. Determine the points on the x -axis that are a distance of 3 units from P_3 . Mark the points on the axes below.



3. Mark the point $P_4(1, -2)$ on the axes below. Mark **all** of the points that are a distance of 2 units from P_4 .



4. Suppose a point, $P(x, y)$ is a distance of 2 units from the point $P_4(1, -2)$.
- (a) Use the distance formula to express the distance relationship between P and P_4 .
- (b) Square both sides of the previous equation.
5. Suppose a point, $P(x, y)$ is a distance of R units from the point $P_4(1, -2)$.
- (a) Use the distance formula to express the distance relationship between P and P_4 .
- (b) Square both sides of the previous equation.

1. Briefly state two ideas from today's class.

-
-

2. For each equation below determine the values of x that satisfy the equation. Express any approximations to at least two decimal places.

(a) $2x^2 + 5x - 3 = 0$

(b) $5x - 1 = 8x + 7$

(c) $x^2 + 3x - 1 = 2x^2 + 2x + 6$

(d) $x^3 = 2$

3. Make a sketch of a number line with zero at the center. Indicate the set of numbers that satisfy $x^2 > 2$.
4. Make a sketch of a number line with zero at the center. Indicate the set of numbers that satisfy $x > 2.2$ and $x < 5.4$.
5. Make a sketch of a number line with zero at the center. Indicate the set of numbers that satisfy $|x| > 1.5$.

1. A tortoise and a hare move in a straight line, and the both start at $x = 0$. The tortoise's position is given by

$$x_T = \frac{1}{2}t,$$

where t is in minutes and x is in meters. The hare's position is given by

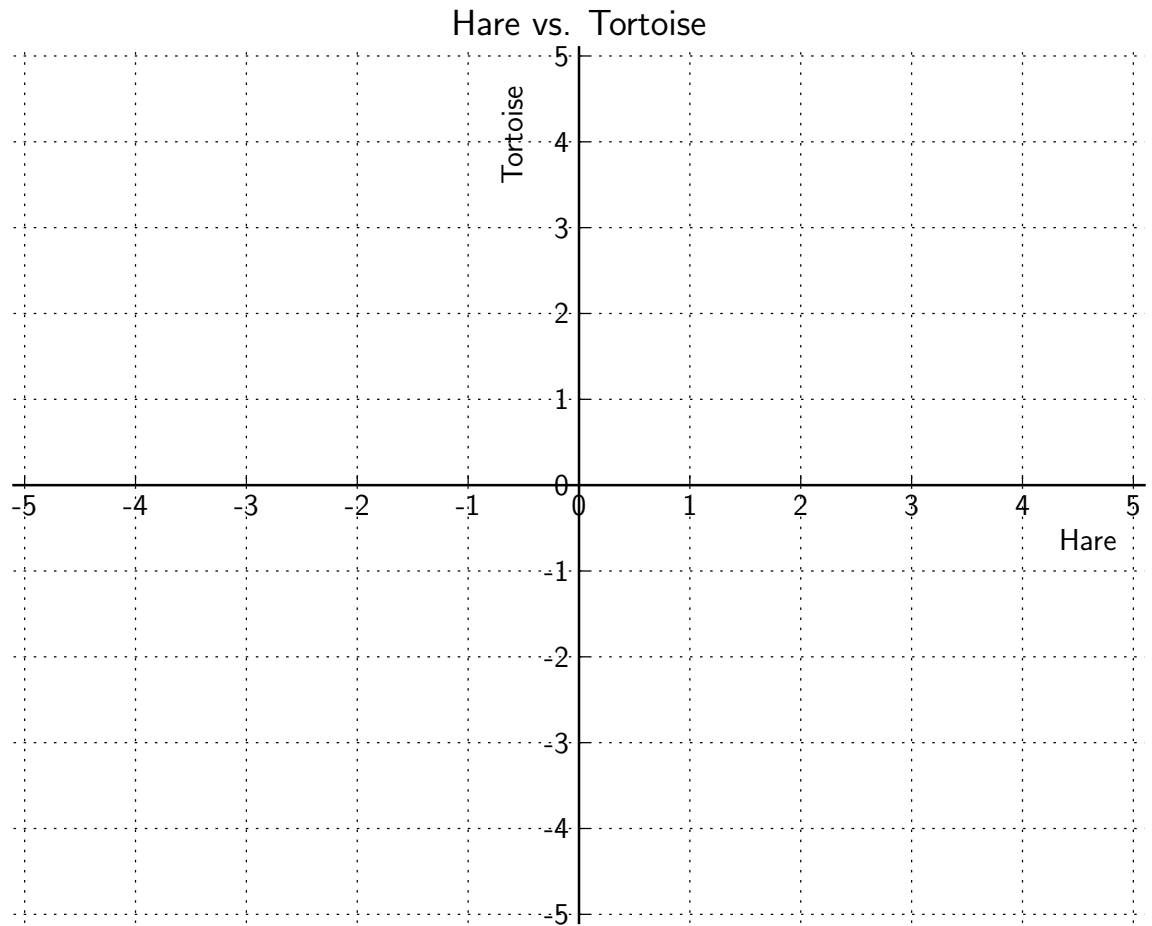
$$x_H = 2t,$$

where t is in minutes and x is in meters.

- (a) Determine the positions of the tortoise at $t = 0$, $t = 1$, and $t = 2$.

- (b) Determine the positions of the hare at $t = 0$, $t = 1$, and $t = 2$.

- (c) For each time, plot the coordinate of the relative positions on the set of axes below. Use the tortoise's position for the x -coordinate, and use the hare's position for the y -coordinate. For example, if the tortoise's position is 1m, and the hare's position is 4m, then the coordinate would be $P(1, 4)$.



1. Sketch a graph of the relationship given by

$$x^2 + 2x + y^2 - 8y = 8.$$

Determine the center and the radius of the circle.

2. Windows are constructed, and their width is proportional to their height. One window is measured, and its width is 100cm, and its height is 200cm. Make a sketch of the relationship of the height of a window given its width. Briefly discuss the relationship. How does the height change as the width changes?

*Annotate
your plot
and label
your axes!*

3. The surface area of a sparrow's wing is proportional to the square of the length of the wing. A sparrow is measured, and it has a wing length of 9cm and an area of 45cm^2 . Make a sketch of the relationship of the area of a sparrow's wing given the length. Briefly discuss the relationship. How does the area change as the length changes?

*Annotate
your plot
and label
your axes!*

1. Briefly state two ideas from today's class.
 -
 -
2. A circle is circumscribed within a square so it just touches on each of the four edges of the square.
 - (a) Make a sketch of the square and circle. Mark the length of the square and the radius of the circle.
 - (b) Determine the relationships between the length of the edge, the radius of the circle, the area of the square, and the area of the circle.
 - (c) Determine the area of the square not covered by the circle as a function of the length of one side of the square.
 - (d) Make a sketch of a graph of the area of the square not covered by the circle. The horizontal axis should be the length of one side of the square, and the vertical axis should be the area. Label your axes.
 - (e) How does the area change as the length of one side of the square increases? (Does it change linearly, does the rate of change increase or decrease?)
 - (f) Determine the proportion of the area not covered by the circle with respect to the area of the square. What percentage of the area of the square is not covered? How does this change as the length of one side of the square changes?

1. A tortoise and a hare move in a straight line, and the both start at $x = 0$. The tortoise's position is given by

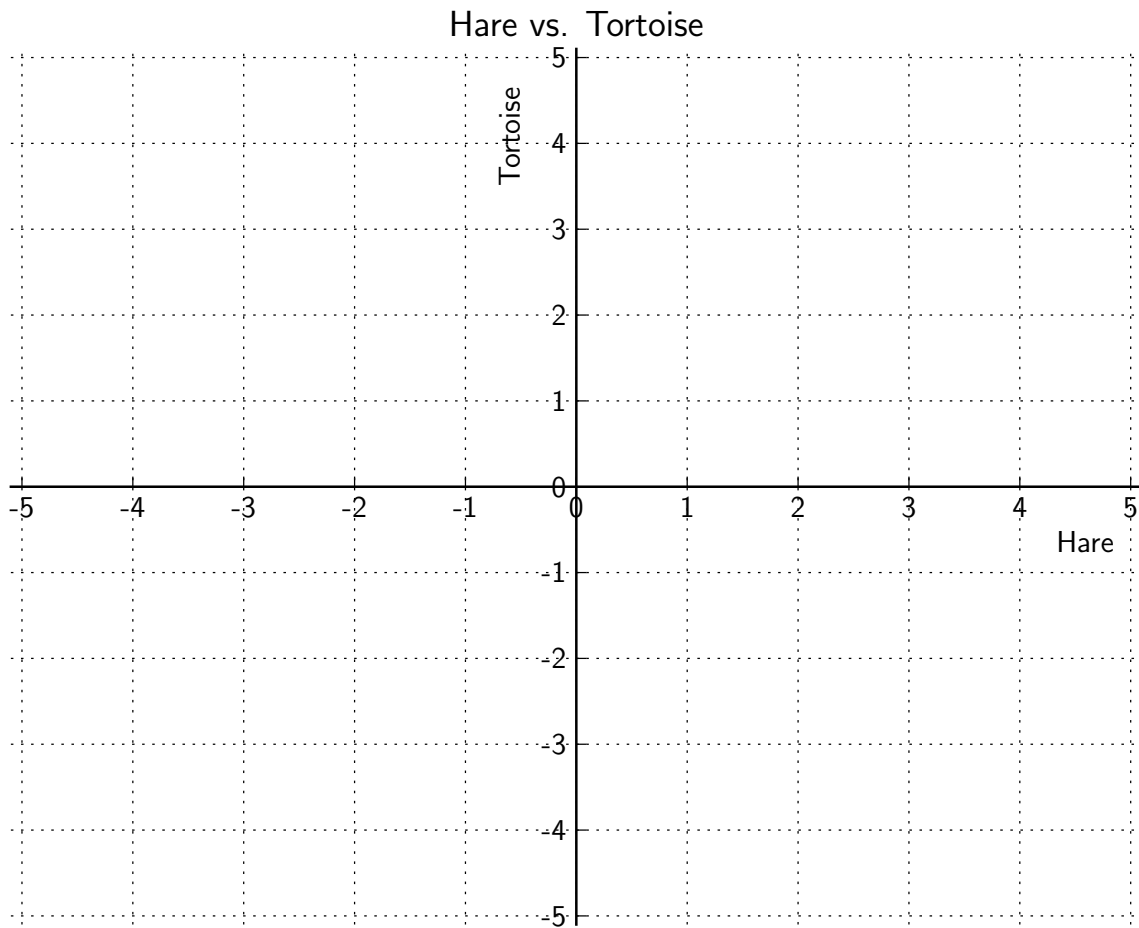
$$x_T = \frac{1}{2}t,$$

where t is in minutes and x is in meters. The hare's position is given by

$$x_H = 2t,$$

where t is in minutes and x is in meters.

Determine the relationship between the hare's and the tortoise's position. That is, given the hare's position determine the tortoise's position. Make a sketch of the plot of the relationship using the axes below.



What is the tortoise's position when the hare's position is 15 meters? (Mark the associated coordinate on the plot above.)

22 Name:

Preclass Work - Finish Before Class Begins

1. In each case below determine the formulas for the lines that satisfy the given requirements. In each case make a rough sketch of the line.

(a) Goes through the point $P(-2, 5)$ and has a slope of -3.

(b) Goes through the points $P_1(-3, -4)$ and $P_2(4, 1)$.

2. Birds near a park are studied by a group of researchers. The birds tend to use cigarette butts in their nests, and it is believed to help reduce the number of parasitic insects. It is estimated that the number of cigarette butts used for nesting materials varies linearly with the distance from the nest to a nearby open air theater. A nest that is a distance of 30 meters appears to have 10 cigarette butts, and a nest that is a distance of 40 meters appears to have 8 cigarette butts.
- (a) Determine the relationship that will predict the number of cigarette butts in a nest given its distance from the theater. Use it to predict the number of cigarette butts in a nest 50 meters from the theater. Also, make a sketch of the relationship.
- (b) What is the domain for the relationship?
- (c) A nest is found that has 4 cigarette butts. What is the prediction for the distance the nest is from the theater.
- (d) If the conjecture for the reason why birds use cigarette butts in their nests is true what would you expect is the general relationship between the fledgling success rate for birds and the location of their nests?

1. Briefly state two ideas from today's class.
 -
 -
2. The per-capita growth rate for a species is approximated as a linear function. It is estimated that when the population is near zero the per-capita growth rate is highest due to a lack of competition and approaches 0.5 (units are 1/hour). When the population approaches 1,000 the per-capita growth rate is estimated to be zero.
 - (a) Determine the relationship that gives the per-capita growth rate as a function of the population, p .
 - (b) Make a sketch of the graph of the per-capita growth rate. (Make sure to annotate your graph and label your axes.)
 - (c) What happens to the per-capita growth rate as the population increases? Why might this happen?
 - (d) Determine the values where the per-capita growth rate is negative. Why would the per-capita growth rate be negative?

1. A biologist grows four different colonies of bacteria. The number of bacteria in the colonies is estimated to be 10,000, 20,000, 30,000, and 40,000. The mass for each colony is measured and is estimated to be 2.61×10^{-6} , 5.20×10^{-6} , 7.90×10^{-6} and 1.04×10^{-5} grams respectively.

Organize the information above into a table so that the mass can be more easily determined given the number of bacteria in the colony. Also, graph each point as a coordinate where the number of bacteria is on the horizontal axis, and the mass is on the vertical axis.

2. Each time the number of bacteria increase by 10,000 what is the change in the mass?
3. Make rough estimate for a relationship that will provide a prediction for the mass of a colony given the number of bacteria within it.

28 Name:

Preclass Work - Finish Before Class Begins

1. A balloon has a tether to the ground, and it can be extended or retracted as the balloon is raised or lowered. One end of the tether is attached to the ground 20m away from a point directly below the balloon. If the balloon is x meters high in the air what is the length of the tether?

- (a) Sketch a diagram of the situation. Label the known and unknown quantities.

Assume that the balloon only moves up and down with no lateral motion.

- (b) Determine the important relationships between the known and unknown quantities.

- (c) Determine the length of the tether given the height.

- (d) Determine the domain and range of the function.

2. A park has two distinct areas separated by a river, and each area has its own population of mice. The population East of the river is estimated to have 10,000 individuals at the beginning of the year, and each week it grows by a constant 200 individuals. The population West of the river is estimated to have 8,000 individuals at the beginning of the year, and each week it grows by a constant 250 individuals.
- (a) Make a rough sketch of the number of mice in the two populations on the same graph. The horizontal axis should be the time from the beginning of the year in weeks.
- (b) Describe what is happening to the two populations. Is there a time when the two populations are equal? If so when is it?
- (c) Determine a formula for the total number of mice in the park at any week after the beginning of the year.

1. Briefly state two ideas from today's class.
 -
 -
2. The surface area of a sphere of radius r is $4\pi r^2$, and the volume is $\frac{4}{3}\pi r^3$. Determine the equation for the surface area of a sphere given its volume.
3. In the Star Trek television series a ship's velocity is given in terms of its warp factor, w . According to wikipedia, https://en.wikipedia.org/wiki/Warp_drive accessed June 2016, the actual speed is the warp factor cubed multiplied by the speed of light which is approximately 3.0×10^8 m/s.
 - (a) Determine the speed of a ship that is moving at warp factor 2.5.
 - (b) Determine the speed of a ship that is moving at warp factor 3.0.
 - (c) A ship is moving at warp factor 3.1. What warp factor would be required to double the ship's speed?
 - (d) A ship is moving at warp factor 4.1. What warp factor would be required to double the ship's speed?

1. Two populations of different species of bacteria interact. The number of bacteria (in millions) in the first population is given by

$$B(t) = 10 + t^2,$$

where t is the time in days since the beginning of the year. The number of bacteria (in millions) in the second population is given by

$$C(t) = 10 + (t + 2)^2,$$

where t is the time in days since the beginning of the year.

- (a) Make a sketch of the two functions below.

- (b) A researcher decides to alter the situation and adds 5 million bacteria to the first population given by $B(t)$. Determine a formula for the altered population. Make a sketch of the original and altered populations below.

34 Name:

Preclass Work - Finish Before Class Begins

1. The height, in meters, of a certain tree changes by the relationship

$$h(t) = \sqrt{\frac{t}{3}},$$

where t is the time in years from when the seed was germinated.

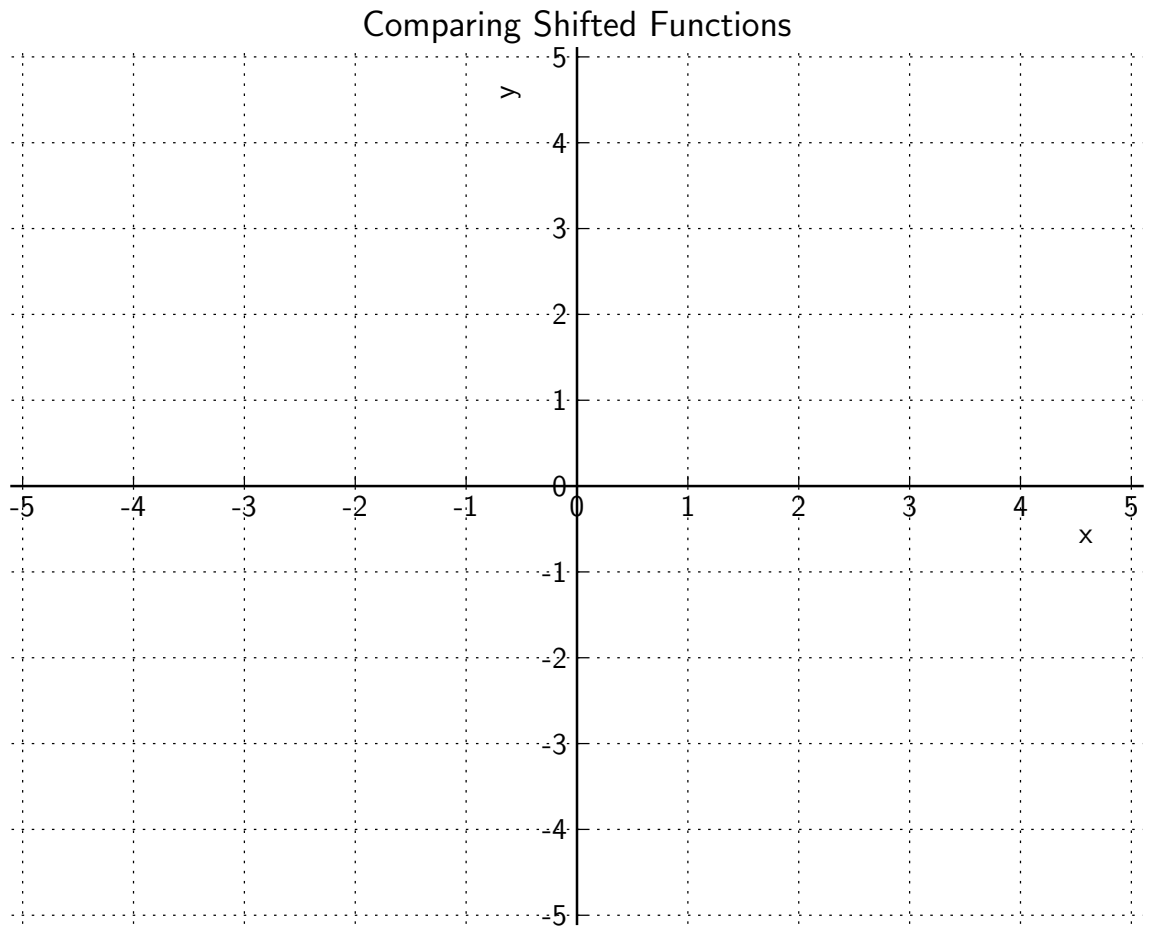
- (a) Make a sketch of the height of a tree as a function of time.
- (b) Two seeds are planted, and the first seed germinates immediately. The second seed germinates one year after the first is germinated, and then begins to grow. Determine the formulas for the height of the two trees with respect to the time that the first seed is germinated. Make a sketch of the two functions on the same graph.
- (c) A new strain of the tree is developed that grows at a rate twice as fast as the original. Determine the formula that will give the height of the new strain. Make a sketch comparing the height of the original and the new strains.

2. A function is defined to be

$$f(x) = |x|.$$

- (a) Make a sketch of the function on the axes below.
- (b) Make a sketch of the following new functions on the graph as well with clear annotations:

$$\begin{aligned}g(x) &= f(3x), \\h(x) &= f(x) + 2, \\p(x) &= f(x + 2), \\q(x) &= 3f(x).\end{aligned}$$



1. Briefly state two ideas from today's class.

-
-

2. An enzyme in a solution decays, and the concentration (mg/liter) as a function of time in hours is

$$C(t) = \frac{3.5}{5.0 + t}.$$

- (a) Make a sketch of the concentration as a function of time. Assume that the time is positive. Annotate your plot and label your axes.
- (b) How long will it take for the enzyme to be reduced to half its original concentration?
- (c) Another enzyme is present, and its concentration is linked to the first. Its concentration is half the first enzyme's concentration 30 minutes in the past. Determine the formula for the second enzyme's concentration. (This is referred to as a delay relationship.)
- (d) How long will it take for the second enzyme to be reduced to half its original concentration?
- (e) Make a sketch of the concentration of both enzymes as a function of time. Assume that the time is positive. Annotate your plot and label your axes.

1. A function is defined to be

$$f(x) = x^2.$$

- (a) Make a sketch of the function on the axes below.
- (b) Make a sketch of the following new functions on the graph as well with clear annotations:

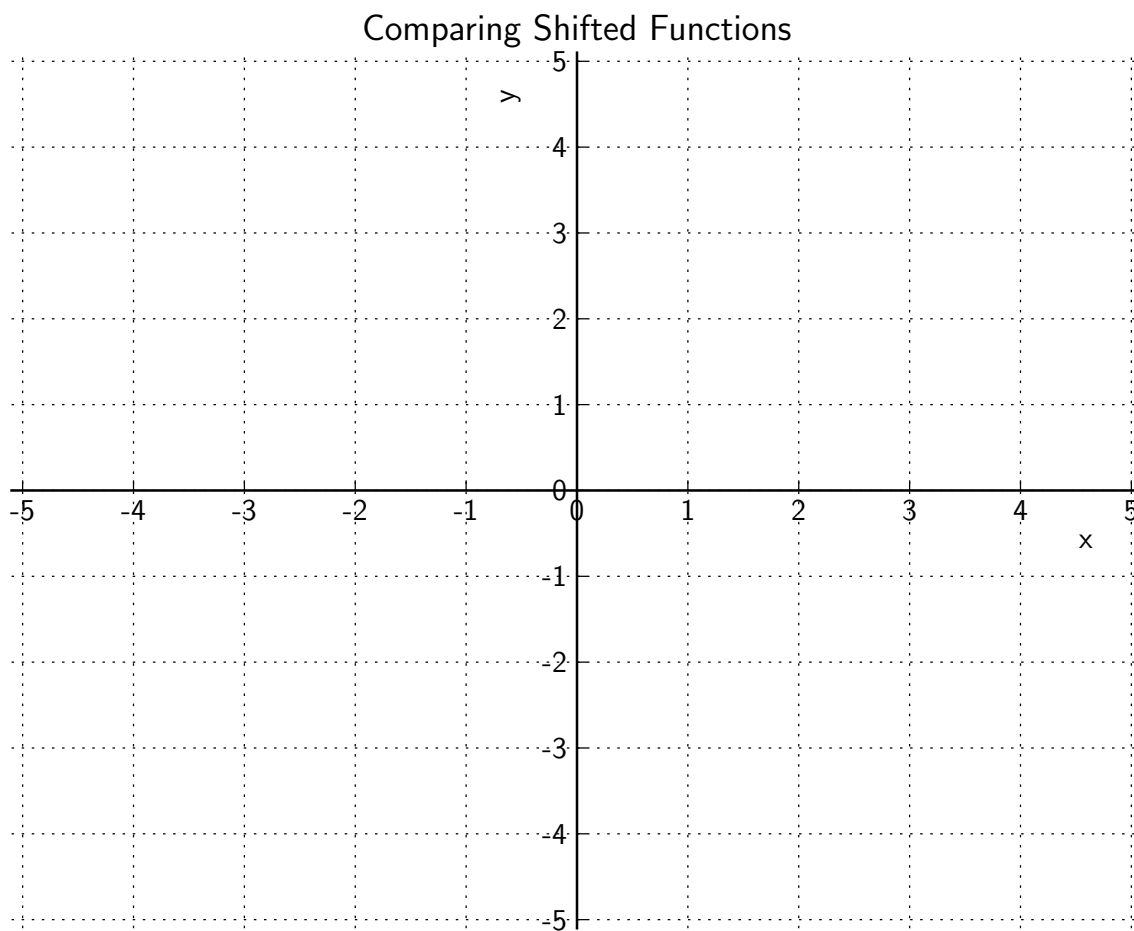
$$g(x) = f(3x),$$

$$h(x) = f(x) + 2,$$

$$p(x) = f(x + 2),$$

$$q(x) = -f(x),$$

$$r(x) = -f(x - 2).$$



40 Name:

Preclass Work - Finish Before Class Begins

1. A developer has a square plot of land that is 100 meters by 100 meters that is next to a river. The land will be divided into two parts by making a rectangle in one corner of the large plot with its width along the river. The height of the rectangle will be ten meters shorter than its width. It is estimated that the rectangular plot of land will sell for \$2.00 per square meter plus 100\$/meter times the width, and the remaining land will sell for \$4.00 per square meter. Determine the dimensions of the rectangular plot that will result in the highest profit.

(a) Make a sketch of the situation.

(b) Within the sketch above, identify and label the variables that you will use.

(c) Determine any relevant relationships between your variables and the total cost of the land.

(d) How will you solve this problem?

(e) Determine the dimensions of the rectangular plot that will result in the highest profit.

Check your work and make sure that you maximized the profit and did not minimize it.

2. A traffic engineer observes the movement of traffic on a busy street. She estimates that when the traffic density is 0 cars/meter that the rate of flow is 0 cars/minute. She notices that when the density is high, roughly 0.1 cars/meter or more, there is gridlock, and the rate of flow is 0 cars/minute. Finally, she notices that when there is 0.05 cars per meter the rate of flow is at a maximum of 20 cars per minute.
- (a) Assuming that the rate of flow is 0 if the density is greater than 0.1 make a rough sketch of the rate of flow as a function of the density below.
- (b) Assuming that the rate of flow is a quadratic function of the traffic density when the density is between 0 and 0.1 cars/meter inclusive, determine the formula for the traffic density.

1. Briefly state two ideas from today's class.

-
-

2. Determine the vertex of the following parabolas.

(a) $y = -x^2 + 10x - 5$

(b) $y = 2x^2 - 10x + 18$

(c) $y = 3x^2 + 4 + 1$

(d) $y = 5x^2 + 2$

(e) $y = 5x^2 + x + 2$

3. Two super hero capes will be sewn as part of a demonstration. One cape will be in the shape of a square. The other cape will be in the shape of a triangle, and the height is the same length as the base. They will be displayed together, and the sum of the two lengths must be 6 feet. What are the dimensions of the capes that will minimize the total area of the capes?

1. A restaurant would like to test a new menu item. They estimate that the cost for producing x servings in market A is 12\$ per serving. They estimate that the cost for producing y servings in market B is \$15 per serving. They will allocate a total of \$36,000 for producing the total number of servings. Write out an expression that relates the cost to the number of servings produced in market A and market B.
2. The number of mosquitoes per acre in an area is estimated to be 600 times the area of open water measured in acres. The area of open water in a location is declining over time and is $A(t) = 50 - \frac{1}{3}t$, where t is the number of years since January 1 of the current year. Determine the number of mosquitoes per acre in terms of t .
3. The delay time required for a neuron to recharge is estimate to be a function of the calcium concentration,

$$\text{Recharge}([\text{Ca}]) = 0.05 - [\text{Ca}]^2.$$

The concentration of calcium in an experiment is changed over time and is estimated to be

$$[\text{Ca}](t) = 0.01 + \frac{1}{1+t}.$$

Determine the formula used to estimate the recharge delay as a function of time, t . (*Do not simplify the expression.*)

46 Name:

Preclass Work - Finish Before Class Begins

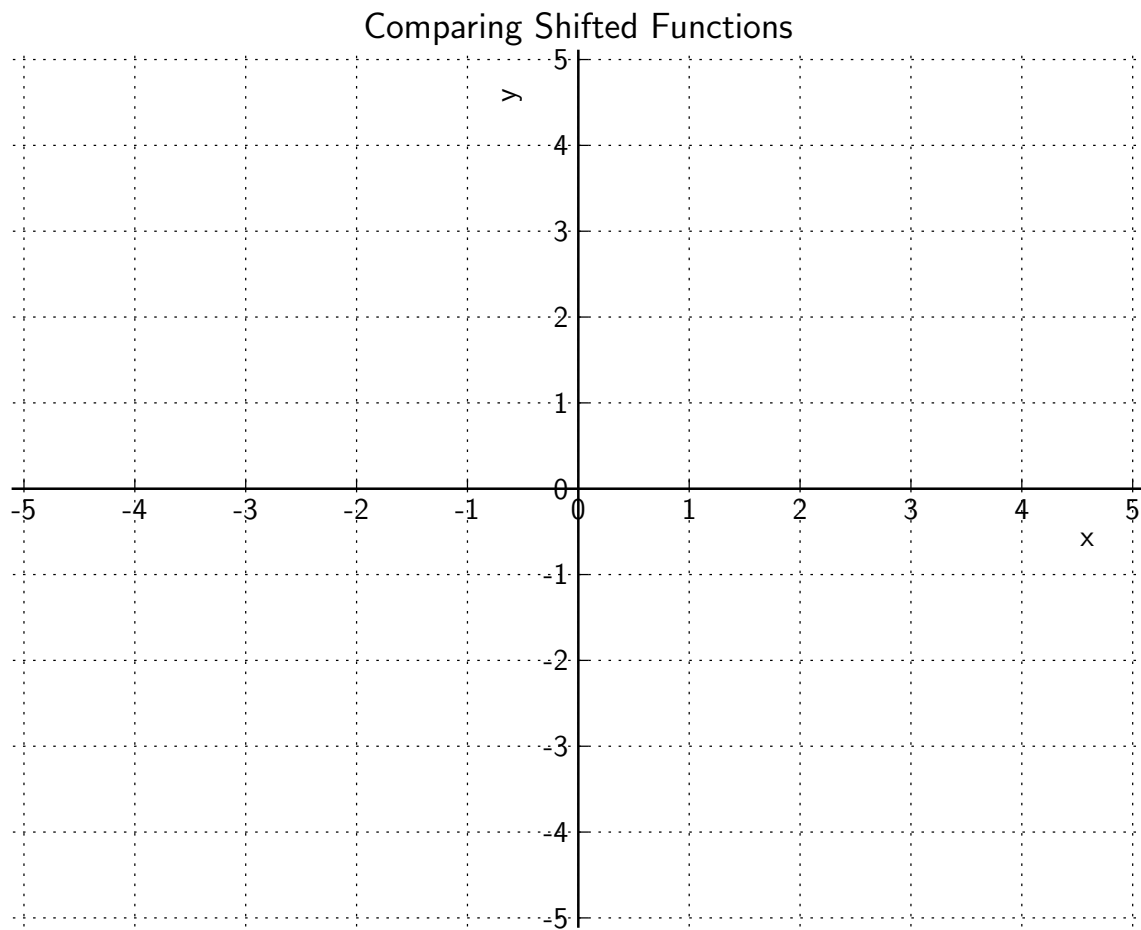
1. A function is defined to be

$$f(x) = x^2.$$

Determine the value of a and b so that the function

$$g(x) = f(x - a) + b$$

is the original function that is shifted up two units and left 3 units. Plot $f(x)$ and $g(x)$ on the plot below.



2. Two functions are given in the tables below.

x	0	1	2	3	4
$f(x)$	a	m	k	a	h

x	a	c	h	j	m
$g(x)$	♠	♦	□	♥	♦

(a) Determine the range and domain of f .

(b) Determine the range and domain of g .

(c) Determine the values of each of the following expressions:

$$f(2) =$$

$$f(4) =$$

$$g(f(2)) =$$

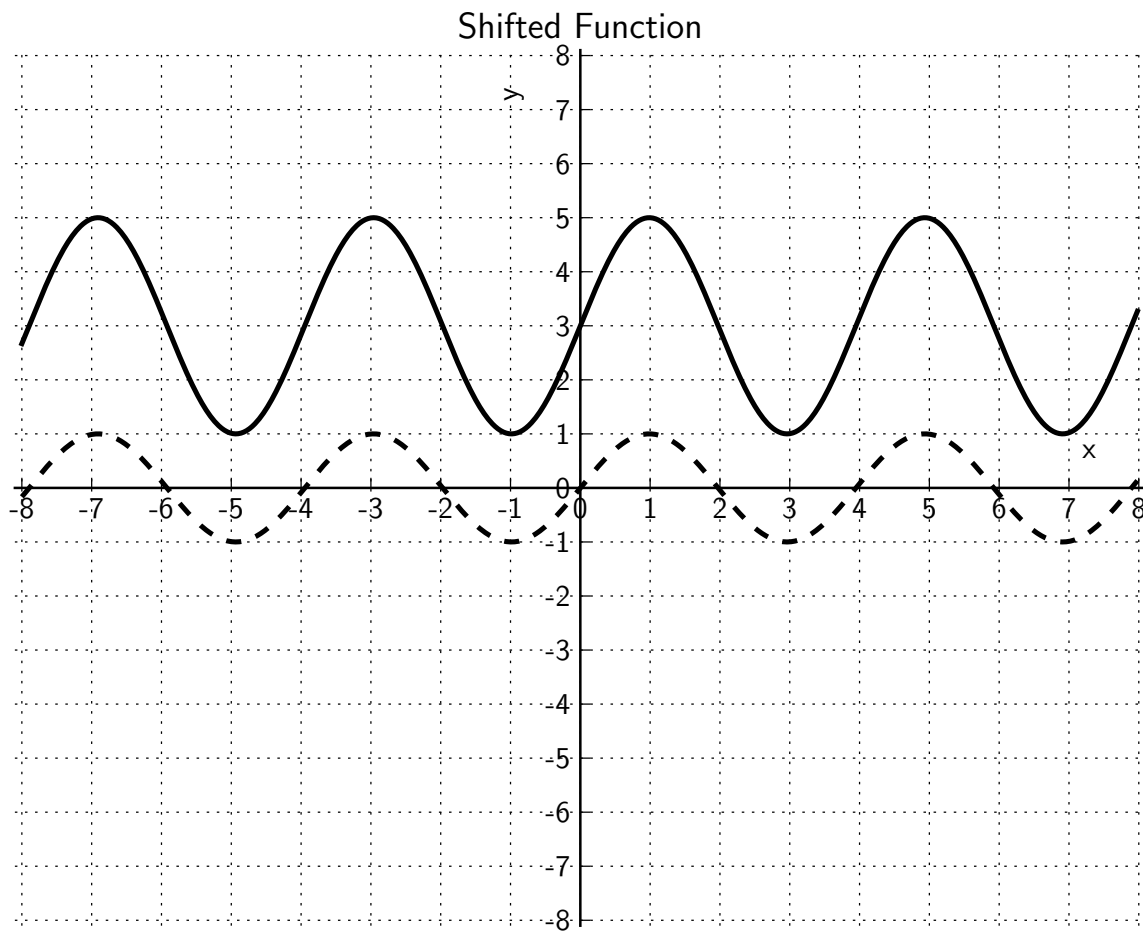
$$g(f(1)) =$$

$$g(f(0)) + g(f(3)) =$$

(d) If $f(x) = h$ what is the value of x ?

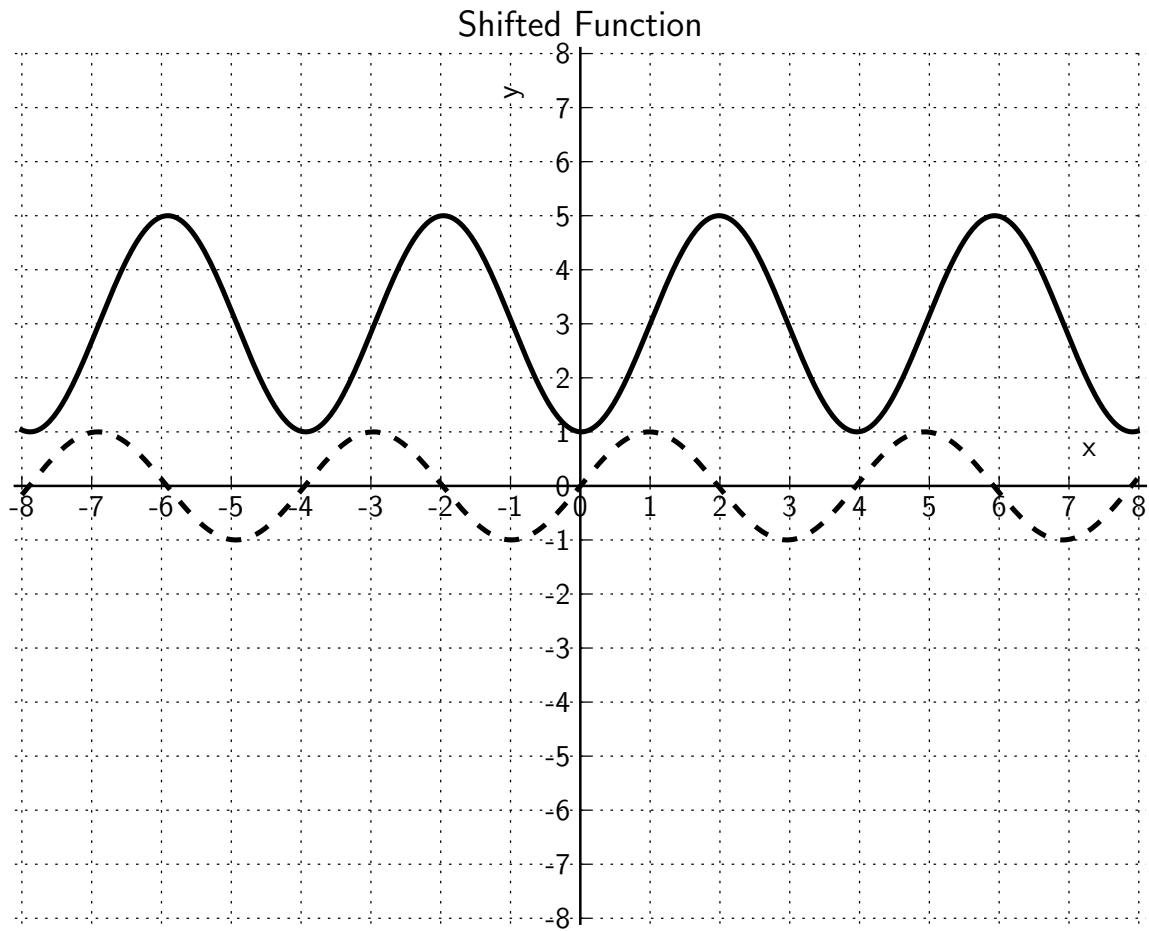
3. Two functions are shown in the figure below. The function plotted with the dotted line is $f(x)$, and the function plotted with the solid line is $g(x)$. Express $g(x)$ in terms of $f(x)$,

$$g(x) =$$



4. Two functions are shown in the figure below. The function plotted with the dotted line is $f(x)$, and the function plotted with the solid line is $g(x)$. Express $g(x)$ in terms of $f(x)$.

$$g(x) =$$



1. Briefly state two ideas from today's class.

-
-

2. Two functions are given in the tables below.

x	0	1	2	3	4
$f(x)$	a	m	k	a	h

x	a	c	h	j	m
$g(x)$	‡	◇	□	♡	◇

- (a) If $g(x) = \diamond$ what are the possible values of x ? Is this reverse procedure a function?
- (b) If $g(f(x)) = \ddagger$ what are the possible values of x ? Is this reverse procedure a function?
- (c) Determine the range and domain of $g(f(x))$.
- (d) Express the function $g(f(x))$ as a table.

1. The height, in meters, of a certain tree changes by the relationship

$$h(t) = \sqrt{\frac{t}{3}},$$

where t is the time in years from when the seed was germinated.

- (a) Make a sketch of the height of a tree as a function of time.
- (b) A tree is measured, and it is estimated that its height is 5 meters. How long ago did its seed germinate?
- (c) Make a sketch of the time since germination as a function of the height of the tree. Make sure to properly annotate your graph and label the axes.

54 Name:

Preclass Work - Finish Before Class Begins

1. Two functions are defined in the following tables. Determine the values of each expression below. If a value does not exist write "NA."

x	0	1	2	3	4
$f(x)$	2	6	5	1	4

x	1	2	4	5	6
$g(x)$	3	3	2	9	4

(a) $f(g(4))$

(b) $g(f(4))$

(c) $f^{-1}(g(4))$

(d) $f(g^{-1}(4))$

(e) $f^{-1}(g^{-1}(3))$

(f) $f^{-1}(g^{-1}(9))$

(g) $g^{-1}(f^{-1}(5))$

2. The height, in meters, of a certain tree changes by the relationship

$$h(t) = \sqrt{\frac{t}{3}},$$

where t is the time in years from when the seed was germinated.

- (a) Determine the range and the domain of $h(t)$.
- (b) Determine the inverse of $h(t)$. How can it be interpreted?
- (c) Determine the range and the domain of the inverse of $h(t)$.

3. For each function below determine if it is one-to-one. For each function that is one-to-one determine the inverse of the function. For each function that is not one-to-one determine a subset of the domain for which the function is one-to-one on that subset.

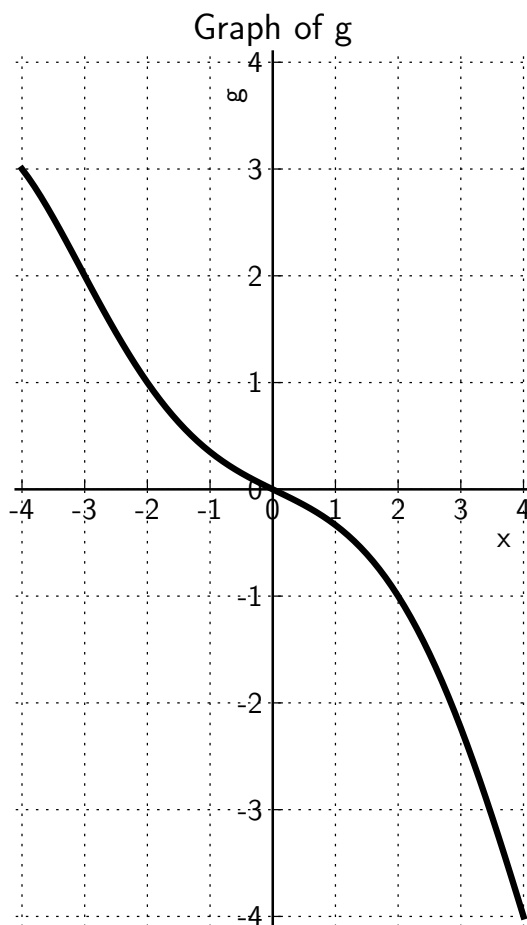
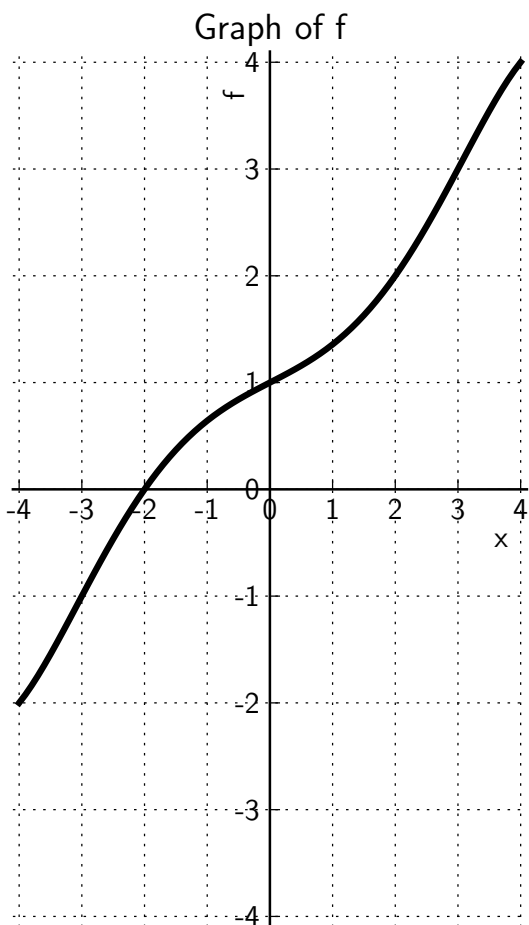
(a) $f(x) = 3x + 1$

(b) $f(x) = \frac{1}{1+x}$

(c) $f(x) = x^2$

(d) $f(x) = \sqrt{3x + 1}$

4. Given the graphs of the two functions below determine the values of the expressions below the graph.



(a) $f(g(3)) =$

(b) $f^{-1}(0) =$

(c) $f^{-1}(-2) =$

(d) $g(f(2)) =$

(e) $g^{-1}(f(0)) =$

(f) $g^{-1}(f^{-1}(1)) =$

1. Briefly state two ideas from today's class.

-
-

2. We examine the function

$$f(x) = x^2.$$

- (a) Determine if the function is 1-1.
- (b) Determine a restriction on the domain of f so that the function is 1-1 on the restricted domain.
- (c) Determine the inverse on the restricted domain.

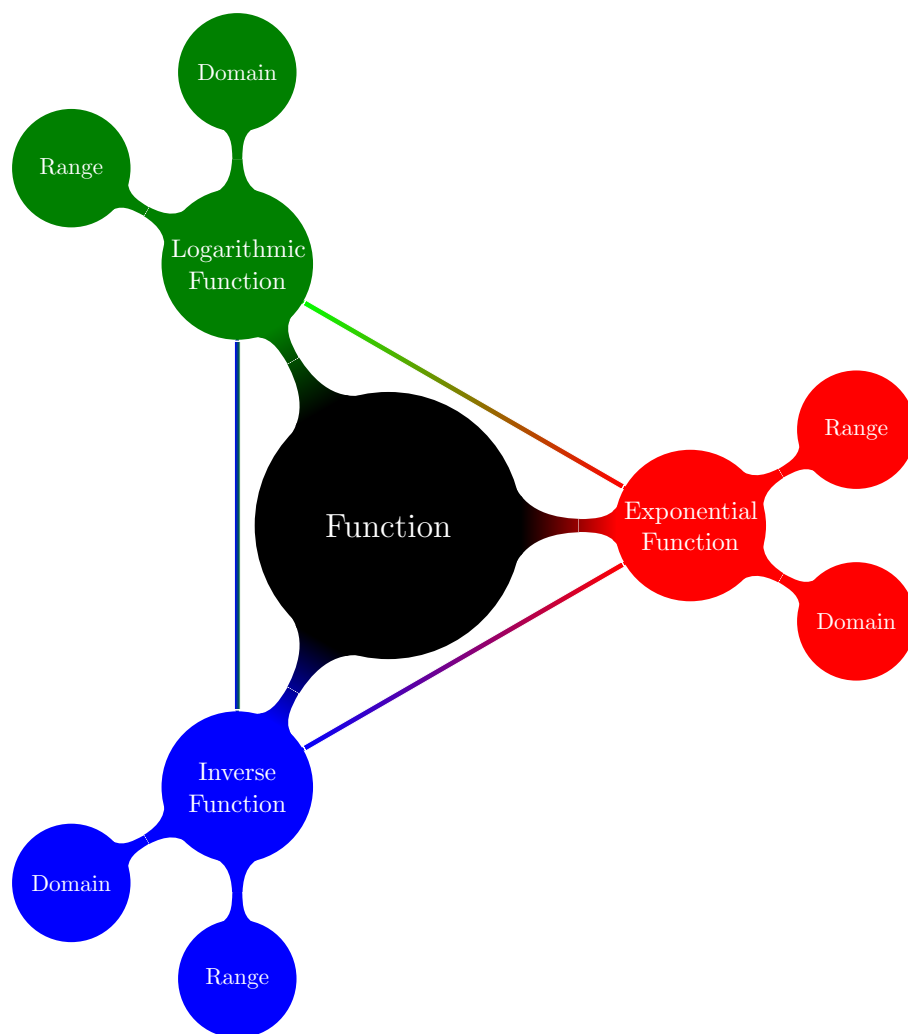
3. We examine the function

$$f(x) = x^2 - 4x.$$

- (a) Determine if the function is 1-1.
- (b) Determine a restriction on the domain of f so that the function is 1-1 on the restricted domain.
- (c) Determine the inverse on the restricted domain.

Chapter 2

Exponential and Logarithmic Functions



1. Carbon-15 has a half life of about 2.5 seconds. If an object has 2 grams of carbon-15 in it now, then in 2.5 seconds it will only have 1 gram due to its decay. An object has 8.0×10^{-6} grams of carbon-15, and it is placed in a sealed container. Determine how much carbon-15 is contained in the object at the following times:

(a) After 2.5 seconds.

(b) After 5.0 seconds.

(c) After 7.5 seconds.

(d) After 10.0 seconds.

64 Name:

Preclass Work - Finish Before Class Begins

1. A species of bacteria is able to divide every three hours. That is every three hours each bacteria splits into two new individuals. A colony starts with 10,000 individuals. Determine the number of bacteria at the following times"

(a) After 3 hours.

(b) After 6 hours.

(c) After 9 hours.

(d) After $n \times 3$ hours where n is an integer greater than zero.

(e) How many bacteria were in the colony 3 hours before the start of the experiment?

2. A species of bacteria is able to divide every three hours. That is every three hours each bacteria splits into two new individuals. After each division, only 75% of the remaining bacteria survive. A colony starts with 10,000 individuals. Determine the number of bacteria at the following times:

(a) After 3 hours.

(b) After 6 hours.

(c) After 9 hours.

(d) After $n \times 3$ hours where n is an integer greater than zero.

(e) How many bacteria were in the colony 3 hours before the start of the experiment?

3. A bank offers a savings account in which the interest is compounded 1.5% annually, and the interest is accrued each month. If a person places \$1,000 in an account how much money is in the account after n months? *Determine the amount of money in the account after the first, second, and third months. Do not simplify your results, and try to find the pattern.*

4. Simplify each expression below.

(a) $\frac{3^5 \cdot 3^2}{3^4}$

(b) $\frac{2^8}{2^5}$

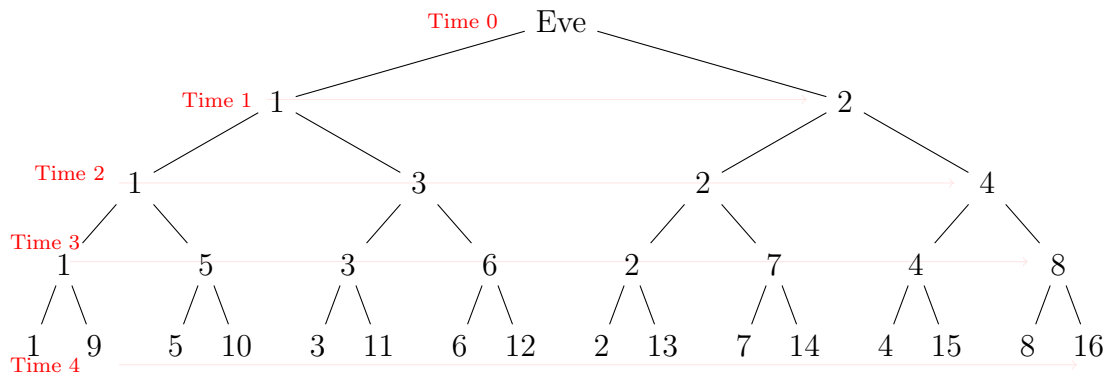
(c) $\frac{4^2 \cdot 2^2}{4^3}$

(d) $5^9 \cdot 5^2 \cdot 5^{-7}$

(e) $\left(\frac{1}{2}\right)^5 \cdot 2^9 \cdot 2^{-3}$

1. Briefly state two ideas from today's class.
 -
 -
2. A bank offers 1.5% annual interest compounded weekly (assume 52 weeks in a year). You will deposit \$5,000 into the account. How much money will be in the account at any time?
3. A bank offers 1.5% annual interest compounded monthly. You will deposit some money into an account and wish to have \$25,000 after two years. How much money should you deposit?
4. A compound is created that decays over time. It takes four years until half of the compound decays in a sample. You wish to store the compound for 5 years. How much should you store if you wish to get 4 kg after the storage period?

Exponential functions are used whenever some quantity has a proportional increase over fixed time spans. The simplest example is a bacteria population that increases by 100% every six hours. That means that every six hours the population doubles. In the diagram below, a single bacteria starts in a sample. After the first time period, six hours, there are two bacteria. After another six hours, for a total of 12 hours, each bacteria splits, and there are four bacteria. In each subsequent six hour time period the population doubles.



Exponential functions satisfy the algebraic properties given below. In each example it is assumed that a , b , and c are constants, and $a > 0$.

$$a^b \cdot a^c = a^{b+c} \quad (2.1)$$

$$\frac{a^b}{a^c} = a^{b-c} \quad (2.2)$$

$$(a^b)^c = a^{b \cdot c} \quad (2.3)$$

Also, e is a constant number, and we define the number e to be

$$e \approx 2.718.$$

It is common to use the number e as the base for exponentials. The number e plays the same role as the constant a in the equations above:

$$e^b \cdot e^c = e^{b+c} \quad (2.4)$$

$$\frac{e^b}{e^c} = e^{b-c} \quad (2.5)$$

$$(e^b)^c = e^{b \cdot c} \quad (2.6)$$

1. Determine an approximation for the value of each expression below. Your approximation should be to the nearest 0.01.

(a) 2^3

(b) 2.5^2

(c) $2.7^{1.5}$

(d) $2.718^{2.1}$

(e) $2.7183^{2.44}$

74 Name:

Preclass Work - Finish Before Class Begins

1. A bank manager is considering the impact of different terms for an account that offers compounded interest. She assumes that the interest rate is a constant annual one percent rate and then checks to see what happens for different lengths of time between compounding. Assume that one dollar is initially deposited.
 - (a) Determine the amount of money in the account after one hundred years, if the interest is compounded yearly.
 - (b) Determine the amount of money in the account after 100 years, if the interest is compounded once every six months.
 - (c) Determine the amount of money in the account after 100 years, if the interest is compounded once a month.
 - (d) Determine the amount of money in the account after 100 years, if the interest is compounded once a day.
 - (e) Determine the amount of money in the account after 100 years, if the interest is compounded twice a day.
 - (f) What is happening to the balance as the number of terms increases?

2. Generalize the value found on the previous page.
- (a) Determine a formula for the balance for 1% annual interest after 100 years if the interest is compounded n times per year.

- (b) Make a substitution, $u = 100n$, in the previous expression. Write out the expression for the balance as a function of u .

- (c) Determine the values of the balance for the following values of u .

u	balance
1	
2	
12	
365	
1000	

- (d) What is the value approaching as u gets large? This is a number that occurs in many situations, and we do not want to write it out every time we use it, so we use the symbol e as a form of short hand notation.

$$e \approx$$

3. When interest is compounded continuously, the balance is determined using the function

$$\text{Balance}(t) = Pe^{rt},$$

where P is the initial balance, r is the annual interest rate, and t is the time in years.

Sketch a plot of the balance over time if 1\$ is deposited with a rate of 1%.

4. As radioactive isotopes decay, the amount of isotope in a sample decreases. If the decay rate of an isotope is r then the amount of an isotope in a sample is expressed using the function

$$\text{Amount}(t) = Ae^{-rt},$$

where t is measured in years.

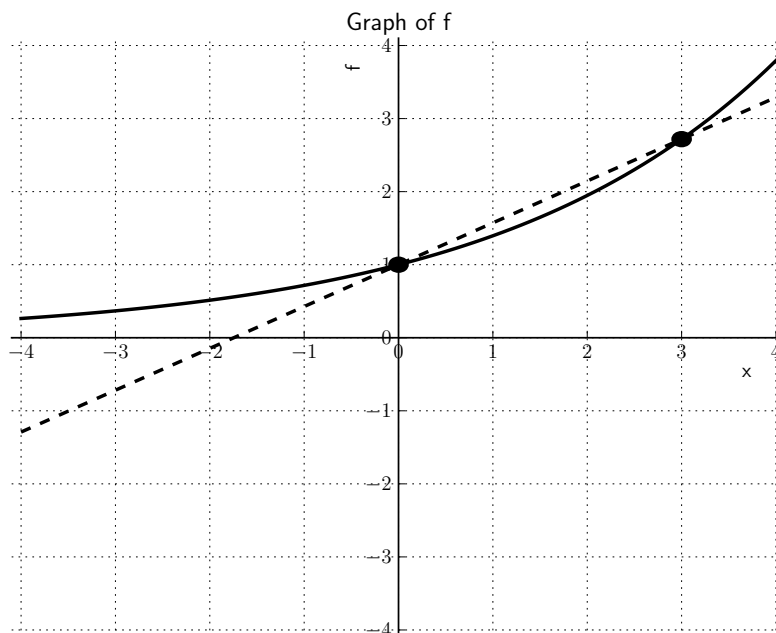
- (a) What is the physical interpretation of the constant A ?
- (b) If a sample of a radioactive substance initially contains 3 grams, and the radioactive decay is 0.00004, sketch a plot of the amount of the substance in the sample over time.

1. Briefly state two ideas from today's class.

-
-

2. The average rate of change for a function from $x = a$ to $x = b$ is defined to be

$$\text{Avg. Rate of Change} = \frac{f(b) - f(a)}{b - a}.$$



Visually, it can be thought of as the slope of the line that goes through the graph of the function at two points, $(a, f(a))$ and $(b, f(b))$.

- (a) The population of a colony of bacteria is two times its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (b) The population of a colony of bacteria is three times its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (c) The population of a colony of bacteria is 2.5 its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (d) The population of a colony of bacteria is 2.7 its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (e) Determine a value so that if a population multiplies its population by that number every hour the average rate of change approaches 1 from $a = 0$ and b gets close to zero.

1. A computer virus is constructed that will infect two new computer systems each day and then wipe the disk drive of its current computer clean. The virus is installed on one computer on 1 January.
 - (a) How many new computers will it infect on 2 January?
 - (b) How many new computers will it infect on 3 January?
 - (c) How many new computers will it infect on 4 January?
2. A computer virus is constructed that will infect two new computer systems each day and then wipe the disk drive of its current computer clean. The virus is installed on a computer but it is not clear when it was first installed.
 - (a) It is estimated that 128 computers were infected on a given day. How many days beforehand was it installed on the first computer?
 - (b) It is estimated that 1024 computers were infected on a given day. How many days beforehand was it installed on the first computer?
 - (c) It is estimated that 32768 computers were infected on a given day. How many days beforehand was it installed on the first computer?

82 Name:

Preclass Work - Finish Before Class Begins

The approximate average lengths of various items are given in the table below. Use this information to answer each of the questions below.

Item	Size
Person	1.75 m
Finger	9.2×10^{-2} m
DNA	5.0×10^{-2} m
Hair (Width)	9.0×10^{-5} m
Bacteria	0.8×10^{-7} m

- (a) Make a sketch of a number line with the lengths for each item marked on the number line.

- (b) For each item in the table above what is the approximate power of ten for the length? For example, if a length is 1×10^{-4} m it is a power of -4. Sketch a number line and indicate the powers of ten on the number line.

2. In one cycle a type of mayfly lays eggs and on average three females survive to lay more eggs. One of the mayfly are introduced to a stream for the first time and lays eggs.
 - (a) Draw a tree diagram of the individuals from each cycle that survive starting with the original mother. Include three cycles in your diagram.
 - (b) On the top of the tree diagram above indicate the total number of mayflies that survive each cycle.
 - (c) On the bottom of the tree diagram indicate the cycle with the first cycle labeled as cycle 0 (zero).
 - (d) It is estimated that there are 2187 mayfly in the river. How many cycles have there been?
 - (e) It is estimated that there are 19683 mayfly in the river. How many cycles have there been?

3. Mosquitoes lay between 50 to 200 eggs each cycle. Assume that for a given species in a particular area roughly 75 eggs hatch and survive to be adults and lay eggs.
- (a) Assume that in the spring there is one surviving female mosquito starting with cycle 0. Make a table to indicate how many mosquitoes there will be from cycle 0 to cycle 4.
- (b) Using your table, if it is estimated that there are 5625 mosquitoes what cycle in the season is it?
- (c) Using your table, if it is estimated that there are 421875 mosquitoes what cycle in the season is it?

1. Briefly state two ideas from today's class.

-
-

2. Determine the value of x in each expression below. Any approximations should be to at least two decimal places.

(a) $\log_5(2x + 1) = 9$

(b) $\log_7(x - 1) + \log_7(x + 1) = 2$

(c) $3^{\log_3(x+1)} = 4$

(d) $\log_8(8^{5-2x}) = 9$

(e) $\log_{10}(2x + 1) = \log_3(4.1)$

3. Use the natural logarithm to determine the value of x in each expression below. Any approximations should be to at least two decimal places.

(a) $3.5^{x+1} = 2^{x-1}$

(b) $10 = e^{-2x}$

(c) $1 = 2^{3x} \cdot 4^{8x-1}$

(d) $5 = e^{3x-1}$

(e) $e^{9x} = 2^{9x}$

1. Determine an approximation for the numerical value of each number below. Also determine the natural logarithm of each of the following values. Express the numbers to two decimal places.

(a) e

(b) e^2

(c) $e^3 \cdot e^{-4}$

(d) $e^{-2} \cdot e^{-3}$

90 Name:

Preclass Work - Finish Before Class Begins

1. Determine the natural logarithm of each value below, and use the properties of logarithms to express the value as a sum or differences of logarithms.

(a) $a \cdot b \cdot c$

(b) $a \cdot b^2 \cdot c \cdot d^3$

(c) $\frac{(x-1) \cdot (x+3)}{(x-2)}$

(d) $\frac{(x-4) \cdot (x+2)^3 \cdot (x+4)}{(x-9)^2}$

2. A computer virus is constructed that will infect two new computer systems each day and then wipe the disk drive of its current computer clean. The virus is installed on one computer on day 0 (zero).

- (a) Determine the formula that gives the number of computers infected on the number of days since the first computer is infected.

$$\text{Number}(t) =$$

- (b) Make a rough sketch of the number of new computers infected as a function of time.

- (c) Solve the equation above for t . That is determine a formula for t given the number of newly infected computers.

- (d) It is determined that 524288 new computers were infected on a given day. How long ago was the first virus installed?

- (e) It is determined that 67108864 new computers were infected on a given day. How long ago was the first virus installed?

3. Radon 222 has a half life of 3.8 days. It is estimated that there is 0.2g of radon 222 in a basement. Assume that no more radon enters the basement after a treatment is applied, and the area is not ventilated.

- (a) Determine a formula for the amount of radon in the basement at a given time, in days, from when the treatment is applied.

$$\text{Amount}(t) =$$

- (b) Make a rough sketch of the amount of Radon 222 in the basement as a function of time.

- (c) Solve the equation for t . That is determine a formula for t given the amount of radon in the basement.

- (d) How long will it take before the amount of radon is down to one tenth the original amount?

4. A patient has a growth, and a treatment is applied that is estimated to reduce the size of the growth by ten percent each week.

- (a) Determine a formula for the size of the growth at a given time, in days, from when the treatment is applied.

$$\text{Size}(t) =$$

- (b) Make a rough sketch of the size of the growth as a function of time.

- (c) Solve the equation for t . That is determine a formula for t given the size of the growth.

- (d) How long will it take before the growth is down to one tenth the original size?

1. Briefly state two ideas from today's class.

-
-

2. Determine the value of x in each expression below:

(a) $e^{2x} - 2e^x - 3 = 0$.

(b) $e^{2x} - 4e^x + 4 = 0$.

(c) $e^{2x} - e^x + 6 = 0$.

(d) $e^{2x} - e^x - 12 = 0$.

(e) $e^{2x} - e^x + 4 = 0$.

3. A bank offers 1.5% annual interest compounded weekly (assume 52 weeks in a year). How long will it take for the balance to double?
4. A bank offers 1.5% annual interest compounded monthly. How long will it take for the balance to double?
5. A compound is created that decays over time. It takes four years until half of the compound decays in a sample. How long will it take for 80% of the compound to decay?

Logarithmic functions satisfy the algebraic properties given below. In each example it is assumed that a , b , and r are constants.

$$\ln(a \cdot b) = \ln(a) + \ln(b) \quad (2.7)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) \quad (2.8)$$

$$\ln(a^r) = r \ln(a) \quad (2.9)$$

$$\log_{10}(a \cdot b) = \log_{10}(a) + \log_{10}(b) \quad (2.10)$$

$$\log_{10}\left(\frac{a}{b}\right) = \log_{10}(a) - \log_{10}(b) \quad (2.11)$$

$$\log_{10}(a^r) = r \log_{10}(a) \quad (2.12)$$

98Name:

Properties of logarithms (Properties: 3)

1. Solve for the variable x in each expression below.

(a) $e^{4x-1} = 8$

(b) $\ln(2x + 1) = -4$

(c) $3^{2x+1} = e^{4x}$

(d) $8^{3x+2} = 6e^{4x}$

100 Name:

Preclass Work - Finish Before Class Begins

1. A population of bacteria doubles every four hours. How long does it take for the population to triple?
2. Carbon 14 has a half life of 5,730 years. How long will it take for a sample to decay to two thirds of its original value?

3. Determine the value of x in each of the following expressions.

(a) $10 = 20(1 - e^{-3t})$

(b) $\frac{1}{20} = \frac{1}{3\sqrt{2\pi}}e^{-x^2/6}$

(c) $xe^{-x} + e^{-x} = 0$

(d) $e^x + 2 - e^{-x} = 0$

(e) $e^{-x^2} = 4^{2x-1}$

4. The number of animals in a population of a small mammals follows a logistic function,

$$P(t) = \frac{10,000}{1 + e^{-\frac{1}{2}t}}.$$

- (a) What is the initial population?

- (b) How many animals will the population approach after a very long time?

- (c) How long will it take for the number of animals to reach 75% of its long time value?

- (d) How long will it take for the number of animals to reach 80% of its long time value?

5. A spill occurs and a chemical is introduced into the lake. It is estimated that the amount of the chemical in the lake decays like

$$\text{Amount}(t) = Ae^{rt}.$$

- (a) Should the value of r be positive or negative? (Briefly explain how you arrive at your conclusion.)
- (b) At some time after the spill occurs it is estimated that there are 4,000 kg of the chemical in the lake. A month later it is estimated that there is 3,500 kg in the lake. What is the half life of the chemical?
- (c) How long will it take for the amount of the chemical to be reduced to 1% of the original amount that was spilled?

1. Briefly state two ideas from today's class.

-
-

2. Determine the value of x in each expression below.

(a) $3^x = 4^{x+1}$

(b) $3^x = 9 \cdot 4^{x+1}$

(c) $2^x = 9^{5x-1} \cdot 4^{x+1}$

(d) $14^{3x+4} = 10^{9x/2} \cdot 20^{6x-1}$

(e) $3^{x^2} = 3 \cdot 8^x$

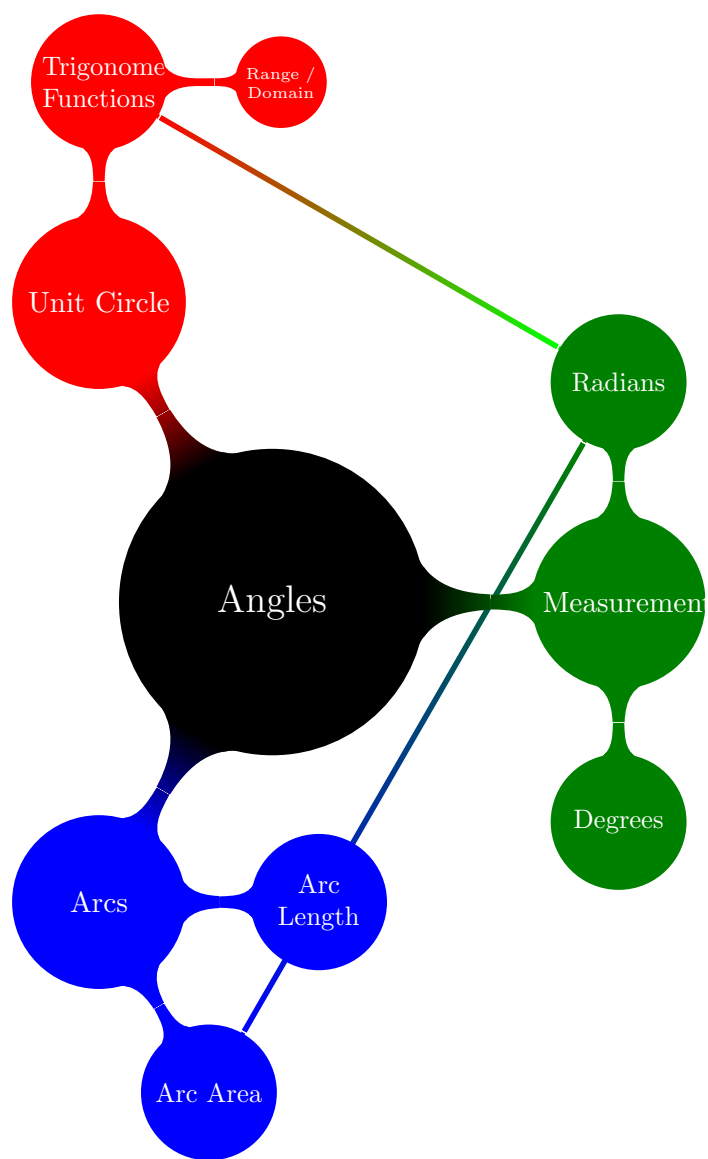
3. The number of animals in a population is approximated using a logistic function,

$$P(t) = \frac{80,000}{1 + 10e^{-\frac{1}{10}t}}.$$

- (a) Make a sketch of the function for t going from 0 to 50.
- (b) How long will it take for the population to double with respect to its original population?
- (c) How many animals does the population approach in the long run? (i.e. what happens to the population after a very long time?)

Chapter 3

Angle Measurement



1. Make a sketch of a horizontal and vertical axes with labels for the x and y axes. Make a sketch of a circle centered on the origin of your axes with a radius r . (Just mark where r and $-r$ occurs on the axes.) Label the radius and the circumference. Finally, answer each of the questions below.

- (a) If the radius is 1m what is the circumference?
- (b) If the radius is 2m what is the circumference?
- (c) What is the general formula that relates the radius and the circumference?
- (d) How many degrees are there in the angle that represents one complete turn around a circle?
- (e) On your circle above draw a ray from the origin along the line that forms a 45 degree angle with the positive horizontal axis. Indicate and label the angle.

112 Name:

Preclass Work - Finish Before Class Begins

1. Make a sketch of a horizontal and vertical axes with labels for the x and y axes. Make a sketch of a circle of centered on the origin of your axes. Label the radius and the circumference. Finally, answer each of the questions below.
 - (a) What is the length of the top half of the circle assuming its radius is 1?
 - (b) What is the length of the top half of the circle assuming its radius is r ?
 - (c) What is the length of the part of the circle in the first quadrant assuming its radius is r ?
 - (d) What is the length of the sector that forms part of a circle with an angle of 180 degrees assuming its radius is 1?
 - (e) What is the length of the sector that forms part of a circle with an angle of 180 degrees assuming its radius is r ?
 - (f) What is the length of the sector that forms part of a circle with an angle of 90 degrees assuming its radius is r ?

2. Make a sketch of a circle of radius r , and mark the radius and circumference of the circle.

(a) What is the general relationship between the radius and the circumference?

(b) Mark a sector on your circle above whose angle is one half of the angle needed to make one complete turn around the circle. What is the length of the sector of the circle?

*This should
be a
function of
 r .*

(c) Mark a sector on your circle above whose angle is one third of the angle needed to make one complete turn around the circle. What is the length of the sector?

*This should
be a
function of
 r .*

(d) If the sector has an angle is $p\%$ of one whole turn around the circle, what is the length of the sector?

*This should
be a
function of
 r and p .*

3. From the previous problem you should have a general formula that relates the length of the sector with radius r given the percentage, $p\%$, that its angle is of one complete turn around a circle.

(a) Rewrite your expression, and label the distance along the sector as s .

- (b) Divide both sides of your formula by the radius, and you should have an expression for

$$\frac{s}{r} =$$

- (c) The value on the right side of your expression is the definition of radian measure for an angle. In one sentence explain the meaning of the ratio on the left side of the expression.

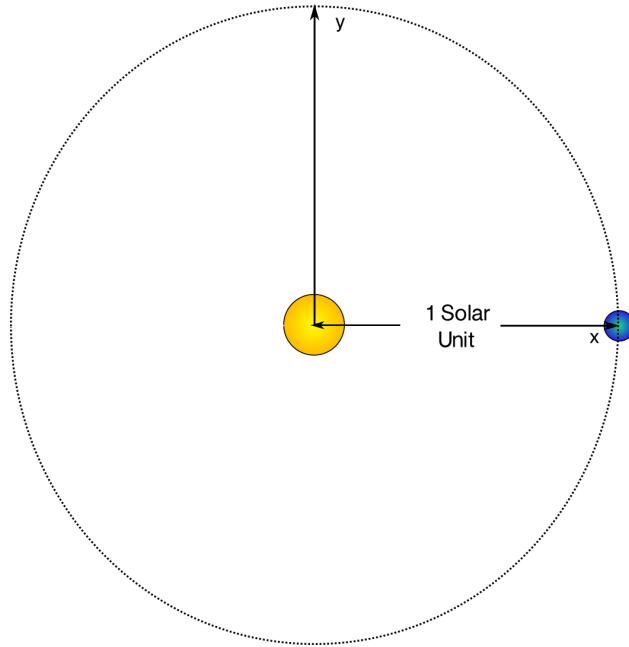
4. The following questions refer to the measure of an angle in radians.
- (a) How many radians are there in one complete turn around a circle?
 - (b) How many radians are there in one half of one complete turn around a circle?
 - (c) How many radians are there in one fourth of one complete turn around a circle?
 - (d) How many radians are there in one third of one complete turn around a circle?
 - (e) If an angle is measured as being 45 degrees, how many radians is it?
 - (f) If an angle is measured as being 120 degrees, how many radians is it?

1. Briefly state two ideas from today's class.

-
-

2. (a)

1. When viewed above the north pole of the Sun, the earth appears to move around the sun in a counter-clockwise direction. The path can be roughly approximated as a circle. It takes one year to make one revolution, and assume that the distance from the center of the sun to the earth is one solar unit.



- (a) What distance does the earth traverse in one year?
- (b) What distance does the earth traverse in two years?
- (c) What distance does the earth traverse in ten years?
- (d) Assume that the eastern direction is to the right, and the northern direction is up. What is the furthest East that the earth moves? What is the furthest West, North, and South that the earth moves?

120 Name:

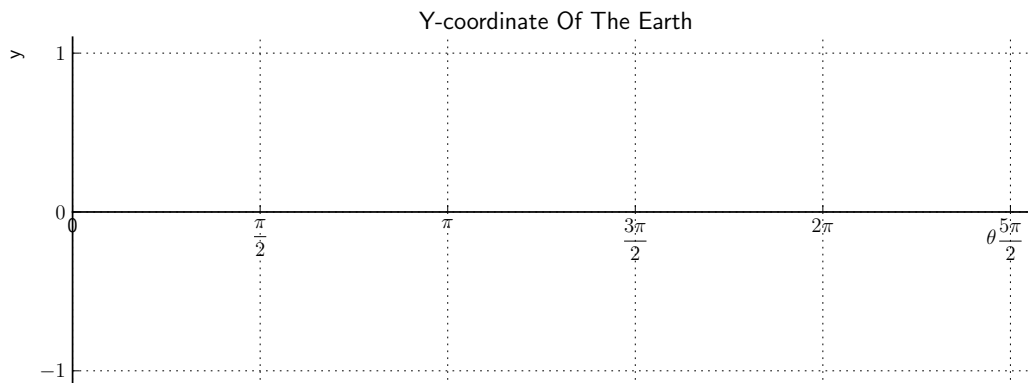
Preclass Work - Finish Before Class Begins

1. When viewed above the north pole of the Sun, the earth appears to move around the sun in a counter-clockwise direction. The path can be roughly approximated as a circle. It takes one year to make one revolution, and assume that the distance from the center of the sun to the earth is one solar unit. (See the image in the preclass activity.)

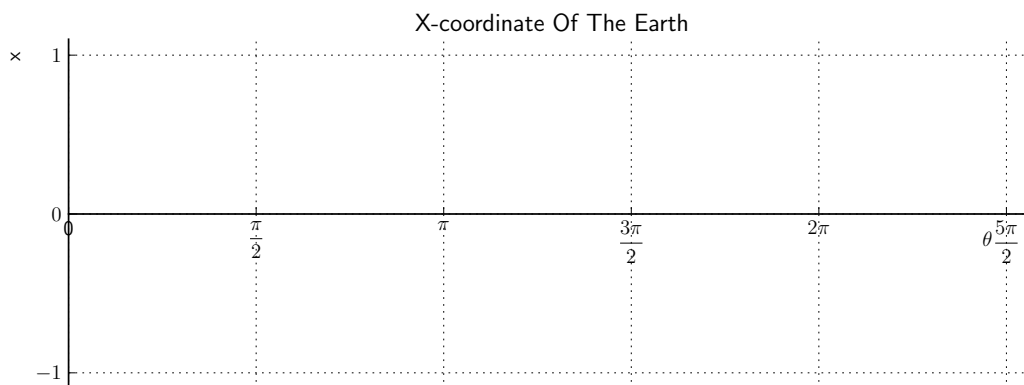
(a) What angle (in radians) does the earth make from the x -axis after 3 months?

(b) What angle (in radians) does the earth make from the x -axis after 6 months?

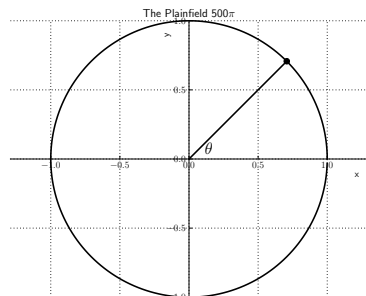
(c) Make a rough sketch of the earth's y position as a function of the angle.



(d) Make a rough sketch of the earth's x position as a function of the angle.



2. At the annual Plainfield 500π race a car will make 250 laps around a circular track. The track has a radius of 1 km. A single car will make a trial run by going around the track at 1 km per hour. (It is not a very fast race.) The car starts on the point furthest East and is initially moving to the North.



- (a) How far will the car travel in all? Determine the distance traveled as a function of time, and then determine the angle, θ , at any time.

- (b) Assuming that the origin is the center of the track, sketch a plot of the car's y position as a function of time for the first two laps.

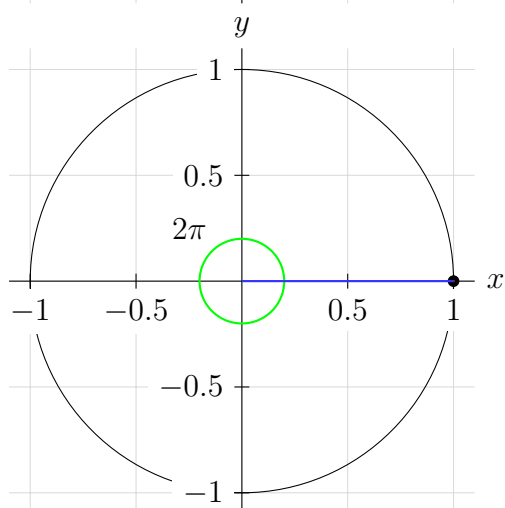
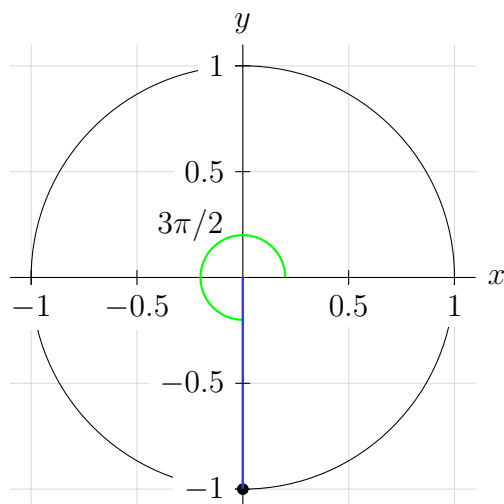
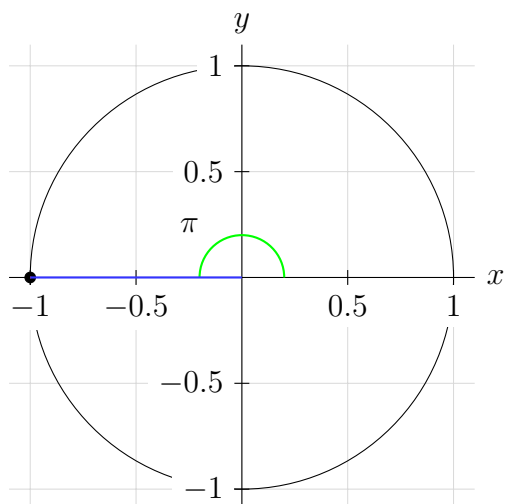
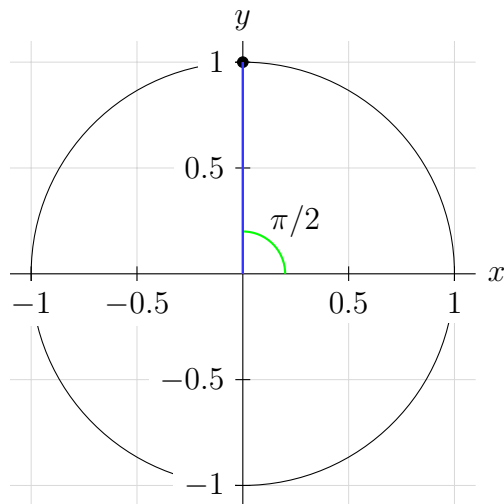
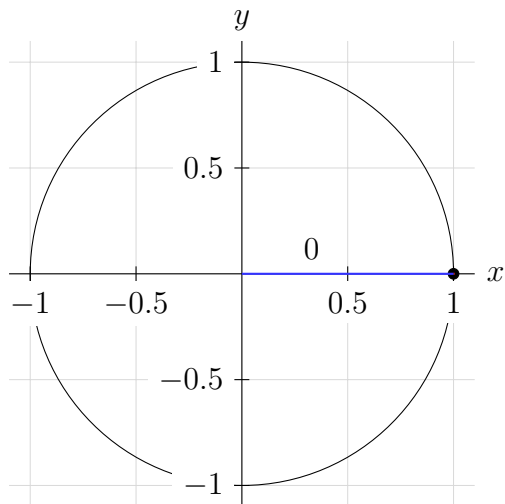
Label the axes and annotate the intercepts.

- (c) Assuming that the origin is the center of the track, sketch a plot of the car's x position as a function of time for the first two laps.

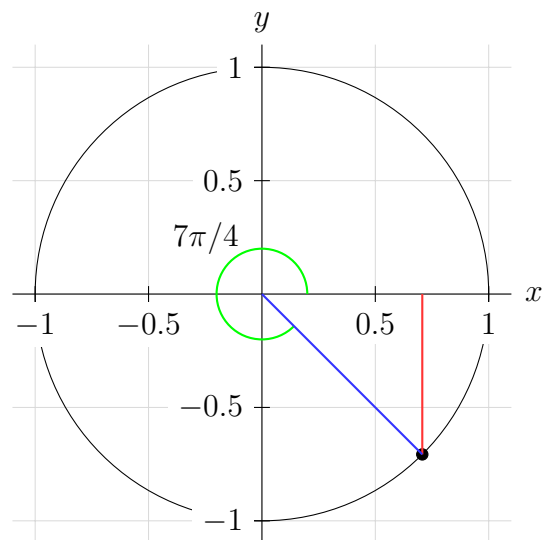
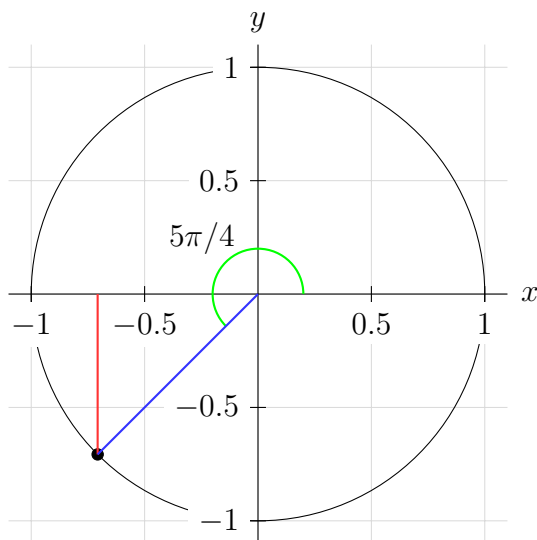
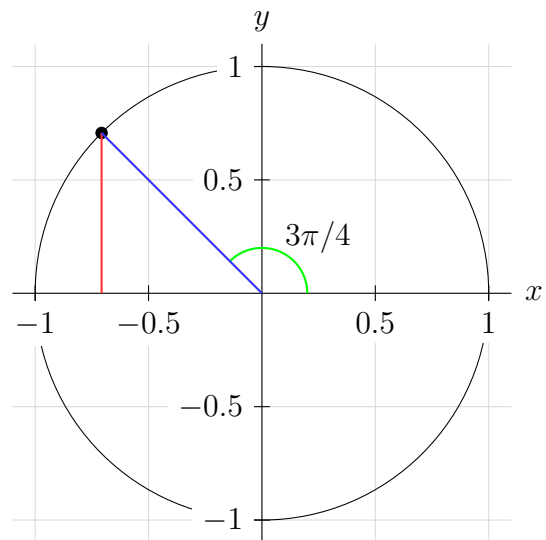
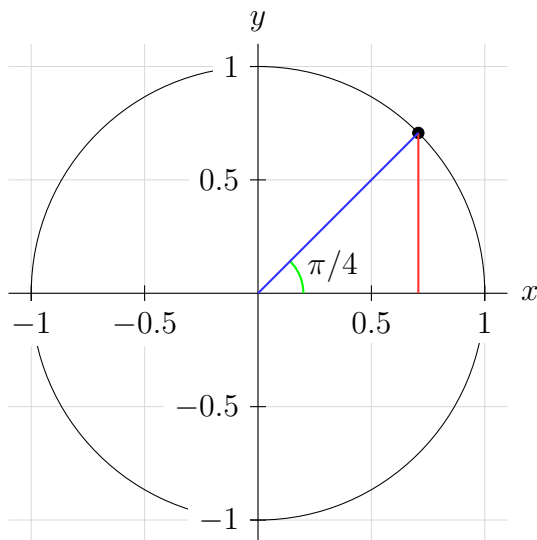
Label the axes and annotate the intercepts.

1. Briefly state two ideas from today's class.
 -
 -
2. Make a sketch of the sine function. Annotate the graph to show the amplitude and period of the function.
3. Make a sketch of the cosine function. Annotate the graph to show the amplitude and period of the function.
4. Make a sketch of the function $f(x) = \sin(\pi x)$. Annotate the graph to show the amplitude and period of the function.
5. Make a sketch of the function $f(x) = \cos(\pi x)$. Annotate the graph to show the amplitude and period of the function.
6. Make a sketch of the function $f(x) = \sin(x) + 2$. Annotate the graph to show the amplitude and period of the function.
7. Make a sketch of the function $f(x) = \cos(x) + 2$. Annotate the graph to show the amplitude and period of the function.
8. Make a sketch of the function $f(x) = 3\sin(x) + 1$. Annotate the graph to show the amplitude and period of the function.
9. Make a sketch of the function $f(x) = 3\cos(x) + 1$. Annotate the graph to show the amplitude and period of the function.

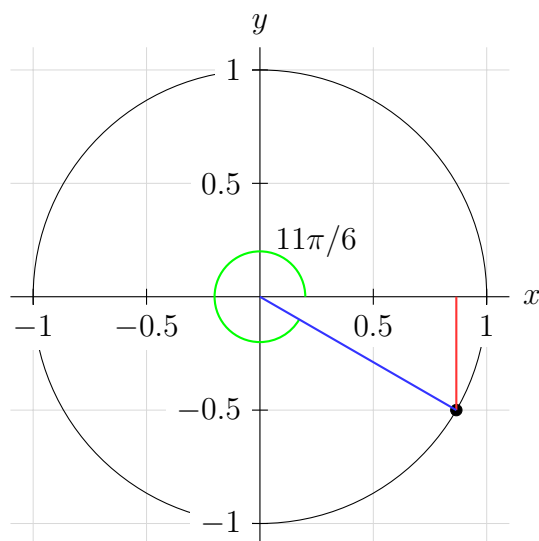
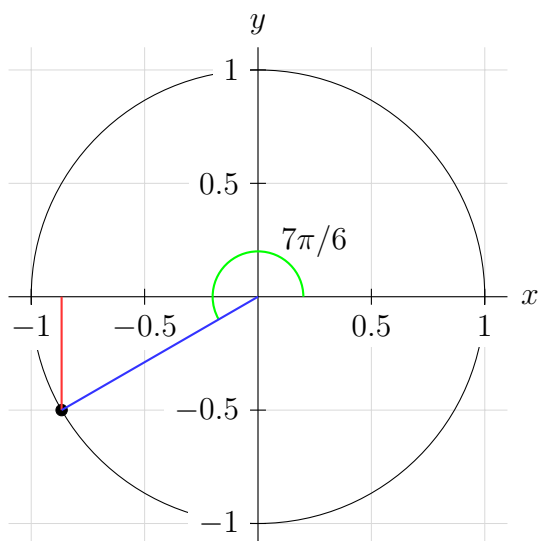
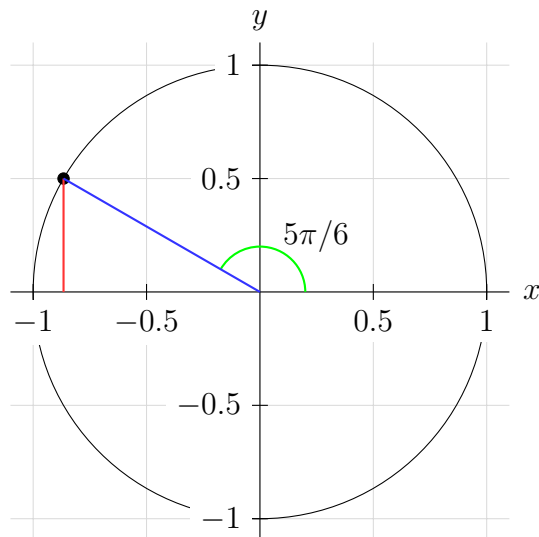
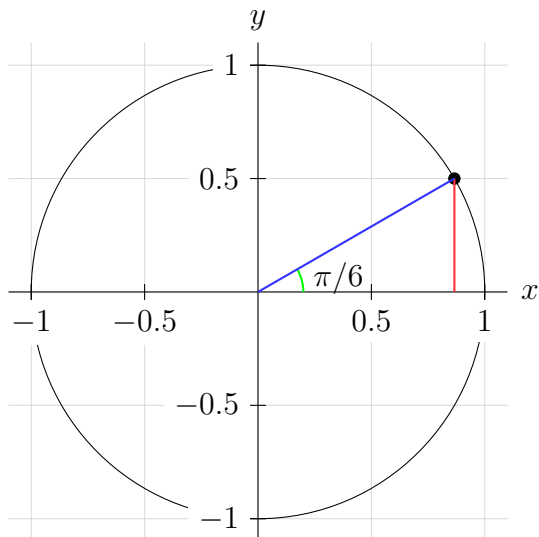
Angles that are a multiple of $\frac{\pi}{2}$ are aligned with the x and y axis. For each plot below determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



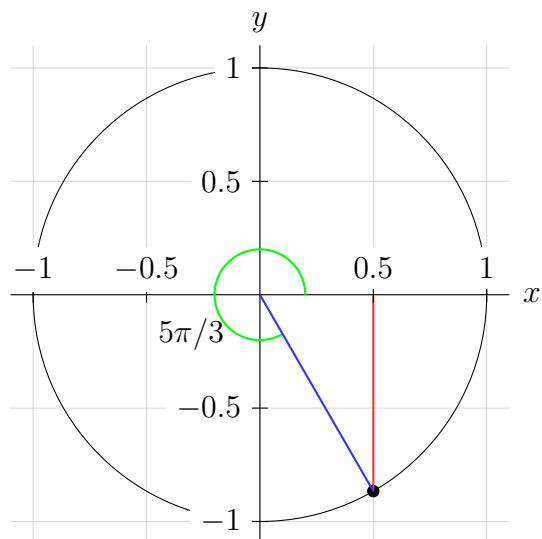
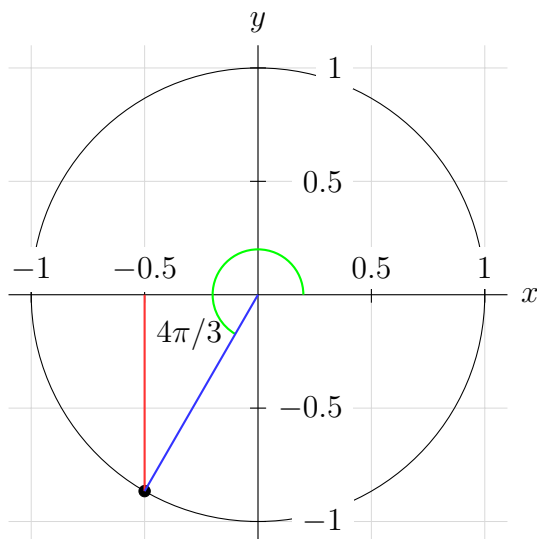
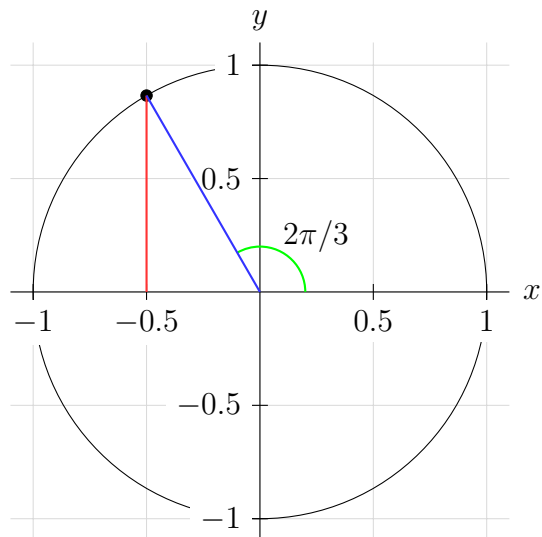
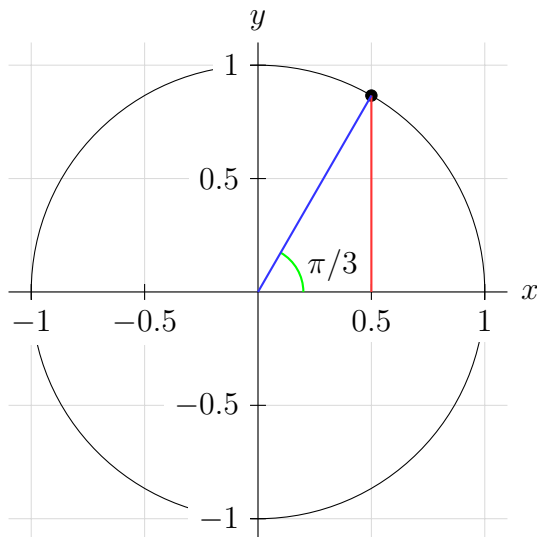
Angles whose reference angles are $\frac{\pi}{4}$ are aligned with the diagonal lines from the origin. In each plot below label and define the reference angle. Also, determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



Angles whose reference angles are $\frac{\pi}{6}$ have y values that are $\pm\frac{1}{2}$. In each plot below label and define the reference angle. Also, determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



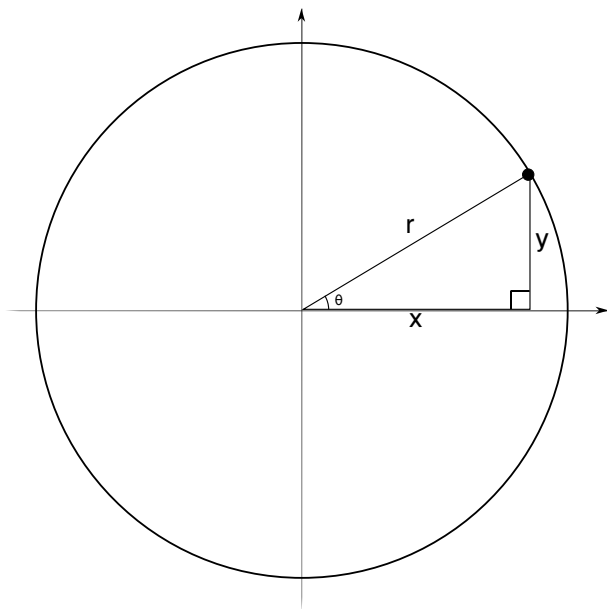
Angles whose reference angles are $\frac{\pi}{3}$ have x values that are $\pm\frac{1}{2}$. In each plot below label and define the reference angle. Also, determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



Chapter 4

Trigonometric Functions

1. A circle of radius r is centered at the origin, and a ray originates from the origin at a given angle, θ . The ray passes through the circle at the coordinate (x, y) . A function of θ can be defined in terms of the coordinate.



- (a) A function, sine, is defined to be

$$\sin(\theta) = \frac{y}{r}.$$

Determine the formula for the value of y given r and θ in terms of the sine function.

- (b) A function, cosine, is defined to be

$$\cos(\theta) = \frac{x}{r}.$$

Determine the formula for the value of x given r and θ in terms of the cosine function.

132 Name:

Preclass Work - Finish Before Class Begins

1. Answer each of the following questions where the given point is $P(2, 4)$.
 - (a) Make a sketch of the coordinate plane and include the point $P(2, 4)$. Draw the ray from the origin to the point. (Label your axes!)
 - (b) Add a circle to your sketch show center is the origin and goes through the point. Label the angle θ as the angle between the ray and the x -axis.
 - (c) What is the radius of the circle?
 - (d) Determine the values of the sine and cosine for the angle.

$$\sin(\theta) =$$

$$\cos(\theta) =$$

2. For each point below determine the radius and the value of the sine and cosine of the angle associated with each point.

(a) $P(1, 0)$

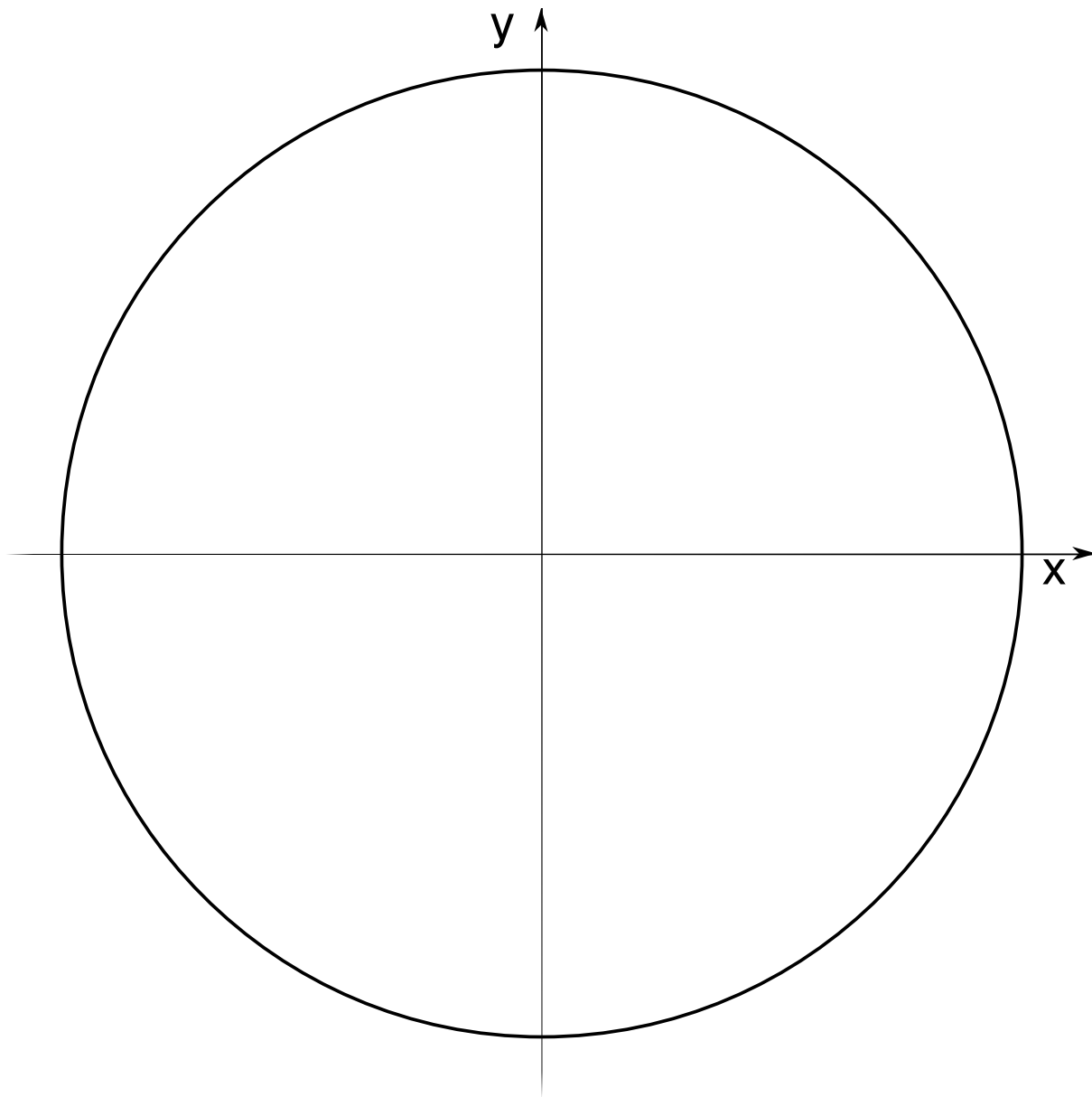
(b) $P(0, 1)$

(c) $P(-1, 0)$

(d) $P(0, -1)$

(e) $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3. The circle below is centered at the origin and has a radius of one.



- (a) Mark the locations on the circle whose associated angles are 0 , $\pi/4$, $\pi/2$, $3\pi/4$, π , $5\pi/4$, $3\pi/2$, and $7\pi/4$.

(b) Determine the (x, y) coordinates for each angle.

(c) Determine the cosine and sine of each angle.

1. Briefly state two ideas from today's class.

-
-

2. (a)

1. An ant starts at the coordinate $P(1, 0)$, and it moves counter-clockwise around a circle of radius one centered at the origin. It moves at a constant 1 meter per minute.

(a) Sketch a plot of the ant's path.

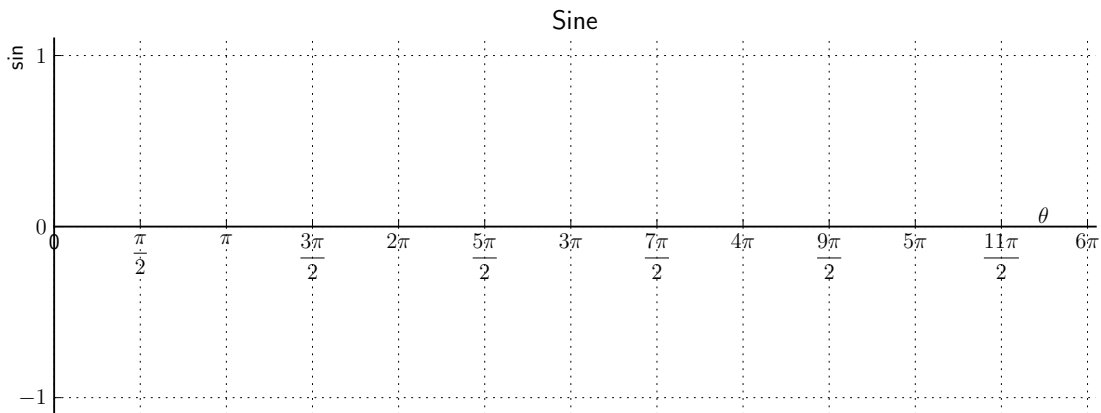
(b) Sketch a plot of the ant's x -coordinate as a function of time.

(c) Sketch a plot of the ant's y -coordinate as a function of time.

140 Name:

Preclass Work - Finish Before Class Begins

1. (a) Make a sketch of the sine function.



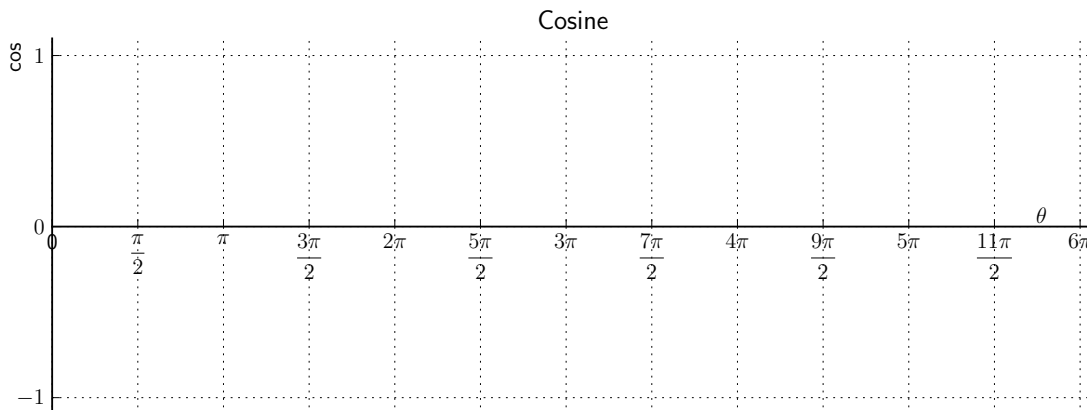
- (b) Over what values of θ does the sine function repeat itself?

- (c) For what value of a is

$$\sin(\theta) = \sin(\theta - a)?$$

- (d) Is the sine function an invertible function?

(e) Make a sketch of the cosine function.



(f) Over what values of θ does the cosine function repeat itself?

(g) For what value of a is

$$\cos(\theta) = \cos(\theta - a)?$$

(h) Is the cosine function an invertible function?

2. What is the relationship between the sine and cosine functions? For each relationship below determine the value and use the unit circle to explain why these relationships should be expected.

(a) Determine a value of a where

$$\cos(\theta) = \sin(\theta + a).$$

(b) Determine a value of a where

$$\cos(\theta) = -\sin(\theta + a).$$

- (c) Determine values of a , and b so that the function

$$f(\theta) = a \sin(\theta) + b$$

oscillates between 2 and 6.

- (d) Determine values of a , and b so that the function

$$f(\theta) = a \sin(\theta) + b$$

oscillates between -5 and -3.

- (e) Determine values of a , and b so that the function

$$f(\theta) = a \sin(\theta) + b$$

oscillates between -1 and 4.

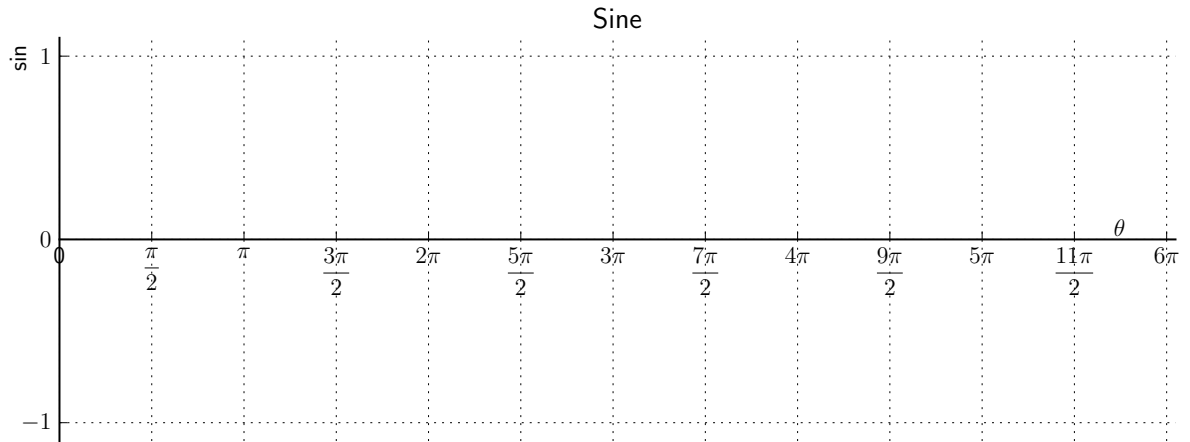
1. Briefly state two ideas from today's class.

-
-

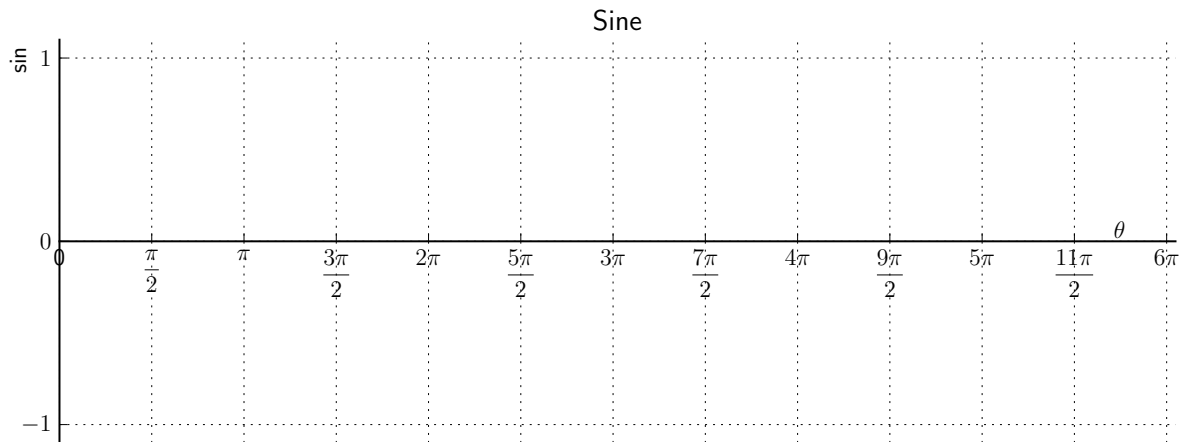
2. (a)

Name:

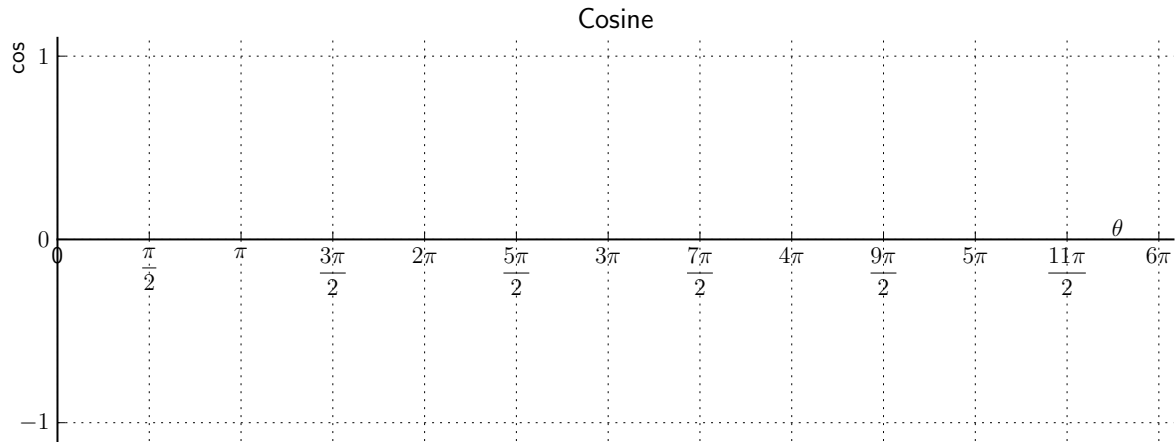
1. Make a sketch of the sine function.



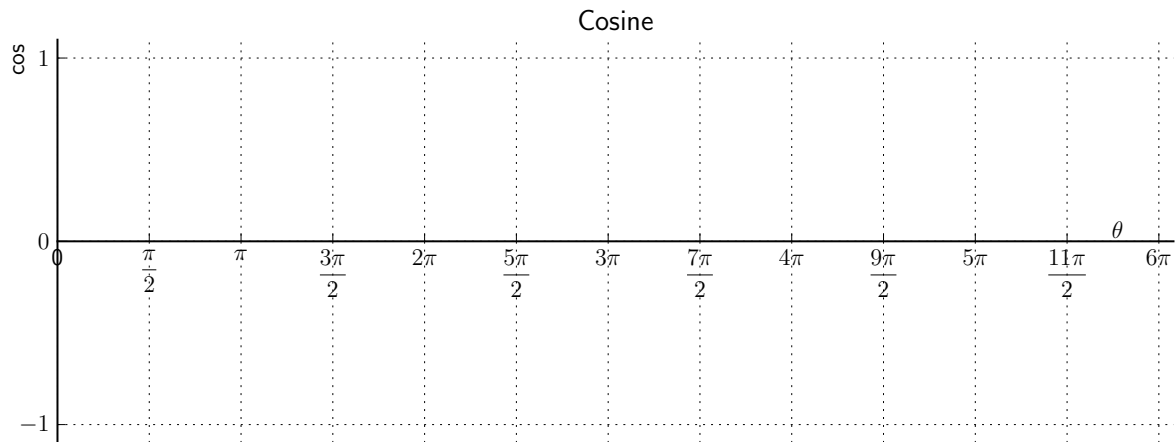
2. Make a sketch of the sine function shifted left $\frac{\pi}{2}$ units.



3. Make a sketch of the cosine function.



4. Make a sketch of the cosine function shifted left $\frac{\pi}{2}$ units.



1. A sine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a sine function that is shifted left $\frac{\pi}{2}$ units, oscillates between 2 and -2, and has a period of 2π .



- (b) Determine the formula for the new function

$$f(x) =$$

2. A sine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a sine function that is shifted left $\frac{\pi}{2}$ units, oscillates between 3 and -1, and has a period of 2π .

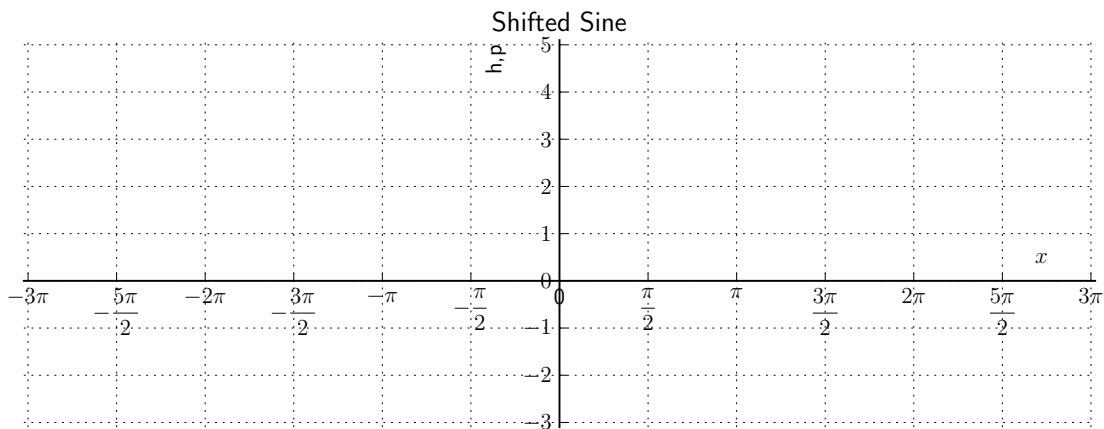


- (b) Determine the formula for the new function

$$g(x) =$$

3. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a cosine function that is shifted right $\frac{\pi}{2}$ units, oscillates between 3 and -3, and has a period of 2π .

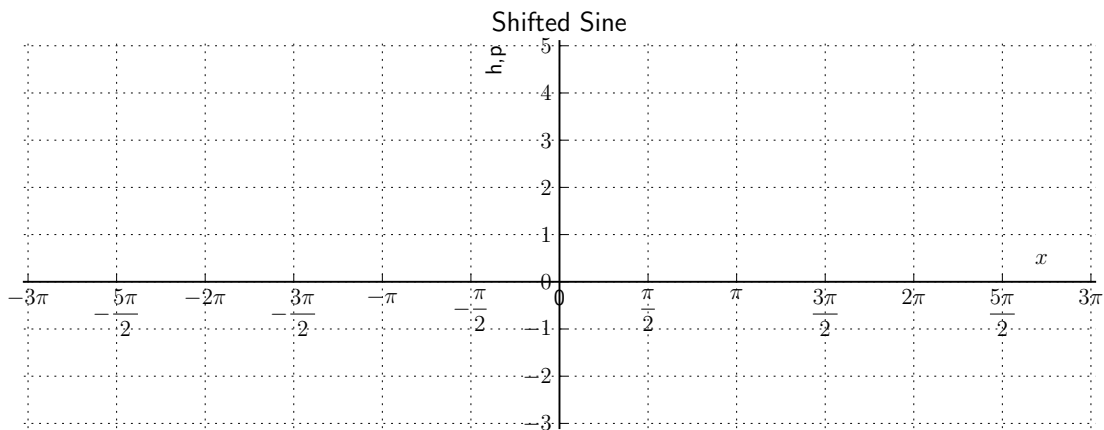


- (b) Determine the formula for the new function

$$h(x) =$$

4. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a cosine function that is shifted right $\frac{\pi}{2}$ units, oscillates between 5 and -1, and has a period of 2π .

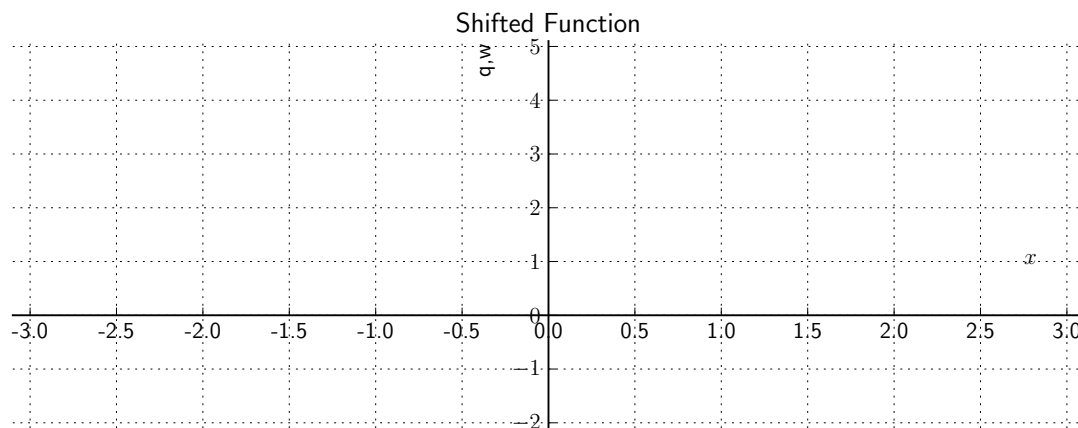


- (b) Determine the formula for the new function

$$p(x) =$$

5. A sine function is scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a sine function that has a period of 1 and an amplitude of 1.

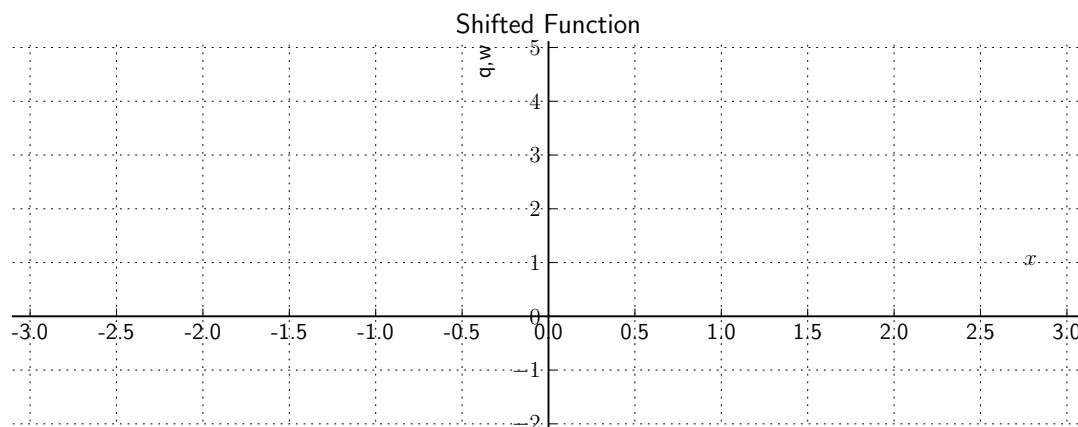


- (b) Determine the formula for the new function

$$q(x) =$$

6. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a cosine function that is shifted right 2 units, oscillates between 5 and -1, and has a period of 1.



- (b) Determine the formula for the new function

$$w(x) =$$

7. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a cosine function that is shifted right 1 units, oscillates between 2 and -1, and has a period of 2.



- (b) Determine the formula for the new function

$$D(x) =$$

8. A sine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.

- (a) Make a sketch of a sine function that is shifted left 2 units, oscillates between -1 and -4, and has a period of $\frac{2}{3}$.



- (b) Determine the formula for the new function

$$L(x) =$$

1. Briefly state two ideas from today's class.

-
-

2. (a)

1. A triangle, ABC, has lengths $a = 5$, $b = 4$, and the angle directly across from c is $\gamma = \frac{\pi}{6}$.

(a) Draw a picture of the triangle.

(b) Label the sides and the angle in your picture.

(c) Determine the area of the triangle. (Hint: draw a vertical line that represents the height in your triangle and use the appropriate trigonometric functions to determine the height.)

156 Name:

Preclass Work - Finish Before Class Begins

1. A surveyor sets up a transit 2m above the surface of the ground. It is 80m away from a building. The transit is pointing at the top of the building, and its angle of elevation is 35 degrees. How tall is the building?

- (a) Make a sketch of the situation. (It may take a couple tries!)
- (b) Indicate and label all of the information that is given and indicate any variables that are not known in your diagram above.
- (c) Identify the relationships between the variables.
- (d) How do you plan on solving the problem?
- (e) Determine the height of the building.

2. Chris Hadfield is in the International Space Station and is 360km above the surface of the earth. He looks down toward the center of the earth and then to the horizon of the earth. He measures an angle of 71 degrees between the two directions. What is the radius of the earth?

- (a) Make a sketch of the situation. (It may take a couple tries!)

- (b) Indicate and label all of the information that is given and indicate any variables that are not known in your diagram above.

- (c) Identify the relationships between the variables.

- (d) How do you plan on solving the problem?

- (e) Determine the radius of the earth.

1. Briefly state two ideas from today's class.

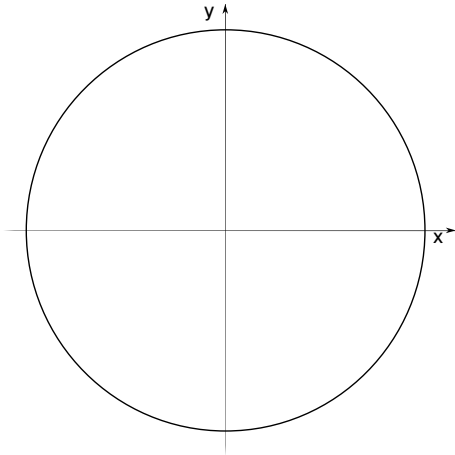
-
-

2. (a)

Name:

1. Use the unit circle and the definition of sine and cosine to provide a justification for the identity

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$



Choose a point on the circle, draw the associated triangle, and then use the appropriate definitions.

162 Name:

Preclass Work - Finish Before Class Begins

1. Is the equation

$$(\sec(\theta) - \tan(\theta)) \cdot (\csc(\theta) + 1) = \cot(\theta)$$

true for all θ ? (Fully justify your answer!)

2. Is the equation

$$\frac{1 + \csc(3\beta)}{\sec(3\beta)} - \cot(3\beta) = \cos(3\beta)$$

true for all β ? (Fully justify your answer!)

3. Is the equation

$$\cos^2(4\alpha) = 1 - \sin^2(4\alpha)$$

true for all α ? (Fully justify your answer!)

4. Is the equation

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

true for all θ ? (Fully justify your answer!)

1. Briefly state two ideas from today's class.

-
-

2. (a)

Name:

1. Determine all values of θ that satisfy

$$\sin(\theta) = \frac{\sqrt{2}}{2}.$$



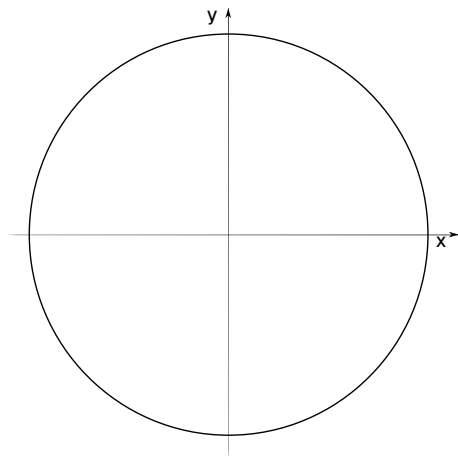
Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

170 Name:

Preclass Work - Finish Before Class Begins

1. Determine all values of θ that satisfy

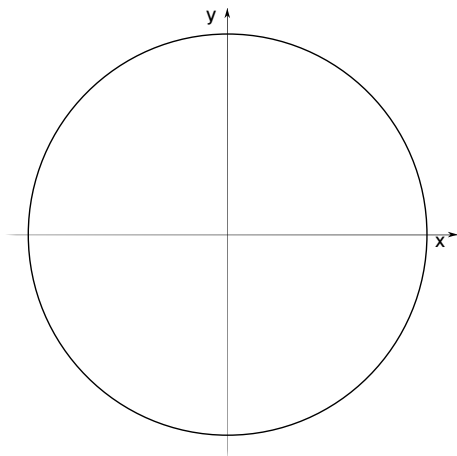
$$\cos(\theta) = -\frac{\sqrt{2}}{2}.$$



Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

2. Determine all values of x that satisfy

$$\cos(2x + 1) = -\frac{\sqrt{2}}{2}.$$



Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

3. Determine all values of x that satisfy

$$\tan(x^2 + 1) = \frac{\sqrt{3}}{3}.$$



Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

4. The voltage within a circuit element is given by

$$V(t) = 100 - 50 \cos \left(\frac{3\pi}{120} t + 3\pi \right).$$

What is the minimum voltage and when does it occur? (Make a sketch of the unit circle!)

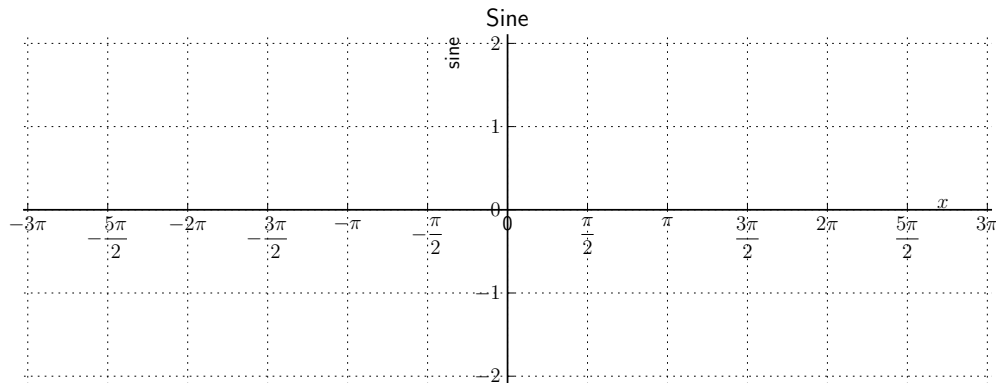
1. Briefly state two ideas from today's class.

-
-

2. (a)

Name:

1. Use the axes below to make a sketch of $\sin(x)$.



- (a) Is the sine function one-to-one? (Justify your answer!)
- (b) What is the domain for the sine function?
- (c) Define a small part of the domain for which the sine function is one-to-one. (There is not a unique answer - just find one small part.) Make a sketch of the sine function restricted to your domain and explain why it is now one-to-one.

178 Name:

Preclass Work - Finish Before Class Begins

1. Use your calculator to evaluate the following expressions. Explain why it gives your results.

(a) $\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right)$

(b) $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

2. A plane is a straight line distance of 10,000 meters away from a radar station. Its altitude is 3,000 meters above the ground. It is flying west away from a radar station. What is the distance between the radar station and a point on the ground directly below the plane?

3. Determine the value of

$$\tan \left(\arcsin \left(\frac{1}{3} \right) \right)$$

(a) Make a sketch of a right triangle. Mark the appropriate angle and sides.

(b) Determine the value based on your diagram.

4. Determine the value of

$$\tan \left(\operatorname{arcsec} \left(\frac{4}{3} \right) \right)$$

without using a calculator.

- (a) Make a sketch of a right triangle. Mark the appropriate angle and sides.

- (b) Determine the value based on your diagram.

1. Briefly state two ideas from today's class.

-
-

2. (a)

Chapter 5

GNU Free Documentation License

Version 1.3, 3 November 2008

Copyright © 2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc.

<<http://fsf.org/>>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

Preamble

The purpose of this License is to make a manual, textbook, or other functional and useful document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The “**Document**”, below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as “**you**”. You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A “**Modified Version**” of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A “**Secondary Section**” is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The “**Invariant Sections**” are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document

is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The “**Cover Texts**” are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A “**Transparent**” copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not “Transparent” is called “**Opaque**”.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The “**Title Page**” means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

The “**publisher**” means any person or entity that distributes copies of the Document to the public.

A section “**Entitled XYZ**” means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “**Acknowledgements**”, “**Dedications**”, “**Endorsements**”, or “**History**”.) To “**Preserve the Title**” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

- A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
- B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
- C. State on the Title page the name of the publisher of the Modified Version, as the publisher.
- D. Preserve all the copyright notices of the Document.
- E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
- F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
- G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
- H. Include an unaltered copy of this License.
- I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.
- J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
- K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.
- L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

- M. Delete any section Entitled “Endorsements”. Such a section may not be included in the Modified Version.
- N. Do not retitle any existing section to be Entitled “Endorsements” or to conflict in title with any Invariant Section.
- O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version’s license notice. These titles must be distinct from any other section titles.

You may add a section Entitled “Endorsements”, provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled “History” in the various original documents, forming one section Entitled “History”; likewise combine any sections Entitled “Acknowledgements”, and any sections Entitled “Dedications”. You must delete all sections Entitled “Endorsements”.

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an “aggregate” if the copyright resulting from the compilation is not used to limit the legal rights of the compilation’s users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document’s Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled “Acknowledgements”, “Dedications”, or “History”, the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60

days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it.

10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License “or any later version” applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy’s public statement of acceptance of a version permanently authorizes you to choose that version for the Document.

11. RELICENSING

“Massive Multiauthor Collaboration Site” (or “MMC Site”) means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A “Massive Multiauthor Collaboration” (or “MMC”) contained in the site means any set of copyrightable works thus published on the MMC site.

“CC-BY-SA” means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.

“Incorporate” means to publish or republish a Document, in whole or in part, as part of another Document.

An MMC is “eligible for relicensing” if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.

The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the “with ... Texts.” line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.