

Classroom Activities
Math 1113 - Precalculus

University of Georgia
Department of Mathematics

June 18, 2018

Copyright (C) 2016-2018 Kelly Black University of Georgia Department of Mathematics

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".



Contents

1	Functions and Preliminaries	5
1.1	Coordinate Systems	9
1.2	Graphs of Equations	19
1.3	Functions	29
1.4	Linear Equations	39
1.5	Modeling With Linear Functions	49
1.6	Graphs of Functions	57
1.7	Piece-wise Defined Functions	67
1.8	Operations on Functions	77
1.9	Quadratic Functions	87
1.10	Inverse Functions	95
2	Exponential and Logarithmic Functions	103
2.1	Introduction to Exponential Functions	107
2.2	The Natural Exponential	117
2.3	Introduction to Logarithms	129
2.4	The Natural Logarithm	139
2.5	Exponential and Logarithmic Equations	149
2.6	Exponential and Logarithmic Models	159
3	Angle Measurement	165
3.1	Angle Measurement	169
3.2	Motion Around a Circle	179
4	Trigonometric Functions	191
4.1	Basic Trigonometry	195
4.2	Reference Angles	203
4.3	Trigonometric Functions	211
4.4	Graphs of Trigonometric Functions	221
4.5	Word Problems	229
4.6	Trigonometric Identities	235
4.7	Introduction to Inverse Trigonometric Functions	243
4.8	Inverse Trigonometric Functions	251
5	GNU Free Documentation License	259



Figure 1: Broad overview of the topics for the full course.

Chapter 1

Functions and Preliminaries



Figure 1.1: Topics for the first section of the course.

1. Make a sketch of a number line with zero at the center. Mark the locations of -2, -2.5, 1.1, and 2.3 on your number line. The relative distances between the points should be consistent.

Label the number line by putting an "x" to the right and an arrow indicating the positive direction.

2. Make a sketch of a number line with zero at the center. Mark the locations of -2 and 2.15 on your number line. What is the distance between the two points? (The relative distances between the points should be consistent.)

3. Make a sketch of a number line with zero at the center. Mark the locations of -1.54 and 2.07 on your number line. What is the distance between the two points? (The relative distances between the points should be consistent.)

8 Name:

Preclass Work - Finish Before Class Begins

1. A set of points is given below as a table. Each point is given on a line. The left side of the table has the x -coordinate of each point, and the right side has the y coordinate. Plot each point on the axes given below. Next to each point indicate which quadrant the point is located.

x	y	Quadrant
2.0	1.5	
2.2	-4.5	
-3.4	2.8	
-2.8	-1.3	
-4.5	2.5	



2. Mark the points $P_1(-2.1, -4.4)$ and $P_2(4.5, 1.2)$ on the coordinate plane below. Determine the distance between the two points. Include a sketch of a right triangle whose hypotenuse represents the distance between the two points.



3. Mark the point $P_3(1.3, -2.4)$ on the coordinate plane below. Determine the points on the x -axis that are a distance of 3 units from P_3 .



- (a) Place points on the graphs near to where you think the points **may** be located.
- (b) Label one of the points, (x, y) .
- (c) Write out the general distance formula.
- (d) What do you know about the point? Can you simplify the formula?
- (e) Solve for the unknown variable.

4. Mark the point $P_4(1, -2)$ on the coordinate plane below. Mark **all** of the points that are a distance of 2 units from P_4 .



What kind of figure did you draw?

5. Suppose a point, $P(x, y)$ is a distance of 2 units from the point $P_4(1, -2)$.
- (a) Use the distance formula to express the distance relationship between P and P_4 .

 - (b) Square both sides of the previous equation.
6. Suppose a point, $P(x, y)$ is a distance of R units from the point $P_4(1, -2)$.
- (a) Use the distance formula to express the distance relationship between P and P_4 .

 - (b) Square both sides of the previous equation.

 - (c) How does this formula relate to the figure you drew in question 4.

1. Briefly state two ideas from today's class.
 -
 -
2. For each equation below determine the values of x that satisfy the equation. Determine the exact answer and also determine an approximations to the answer using two decimal places.
 - (a) $2x^2 + 5x - 3 = 0$
 - (b) $5x - 1 = 8x + 7$
 - (c) $3x - 1 = 2x^2 + 2x + 6$
 - (d) $x^3 = 2$
3. Draw a coordinate axis, and properly label the axes. Use the axes to make a sketch of the graph of the relationship $y + x = 2$.
4. Draw a coordinate axis, and properly label the axes. Use the axes to make a sketch of the graph of the relationship $y^2 + x = 2$.
5. Make a sketch of a number line with zero at the center. Indicate the set of numbers that satisfy $x^2 > 2$.
6. Make a sketch of a number line with zero at the center. Indicate the set of numbers that satisfy $x > 2.2$ and $x < 5.4$.
7. Make a sketch of a number line with zero at the center. Indicate the set of numbers that satisfy $|x| > 1.5$.

1. Expand each of the functions below by FOILing the expression. The first one is done as an example. Recall what it means to FOIL an expression.

$$(a + b) \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$$

(a) $(x - 3)^2$

$$\begin{aligned} (x - 3)^2 &= (x - 3) \cdot (x - 3), \\ &= x \cdot x - 3 \cdot x - 3 \cdot x + (-3) \cdot (-3), \\ &= x^2 - 3x - 3x + 9, \\ &= x^2 - 6x + 9. \end{aligned}$$

(b) $(x + 4)^2$

(c) $(y - 5)^2$

(d) $(y + a)^2$ where a is a constant.

2. if $(x - a)^2 = x^2 + 10x + 25$ what is the value of a ?

18 Name:

Preclass Work - Finish Before Class Begins

1. Sketch the set of all points that are a distance of two from the point $Q(1, -3)$. What kind of figure do the points represent?



2. Suppose a point, $P(x, y)$ is a distance of R units from the point $C(x_0, y_0)$.
 - (a) Use the distance formula to express the distance relationship between P and C .

The values, x_0 and y_0 , are constants. The notation is awkward, but this is a convention that we have to adapt to.
 - (b) Square both sides of the previous equation.

3. Make a sketch of the circle with a radius of two centered at the point $P(-2, 3)$. Determine a formula for the circle.



4. Sketch a graph of the relationship given by

$$x^2 + 2x + y^2 - 8y = 8.$$

Determine the center and the radius of the circle. Make a sketch of the circle. (Include the axes and label the axes.)

In mathematics the idea of proportionality has a specific definition. The idea is that when two things are proportional then any changes in one yield a similar change in the other. For example, suppose we have two quantities. The first we call x , and the other we call y . If y is proportional to x , then if we double x then y will double. Likewise, if we triple x then y will triple.

We express this mathematically by noting that if y is proportional to x then the ratio of y to x must be a constant,

$$\frac{y}{x} = \text{constant}.$$

If we multiply both sides by x then

$$y = x \cdot \text{constant}.$$

As an example, it is estimated that the length of a person's femur is proportional to the person's total height. This implies that

$$\frac{\text{height}}{\text{femur length}} = \text{constant}.$$

In a paper by Obialor *et al*¹, it is estimated that in a specific area in Nigeria the mean height of women is 161.90 cm and the mean femur length of women is 40.82 cm. If a woman's femur has a length of 42.00 cm, what is her expected height?

First, we have to estimate the value of the constant. Assuming that the means are consistent then

$$\frac{161.90}{40.82} = \text{constant}.$$

Now we look at the expression for the unidentified woman,

$$\frac{\text{height}}{42.00} = \frac{161.90}{40.82}.$$

Solving for the height we get

$$\text{height} = 42.00 \cdot \frac{161.90}{40.82} \text{cm}.$$

¹Ambrose Obialor, Churchill Ihentuge and Frank Akpuaka, **Determination of Height Using Femur Length in Adult Population of Oguta Local Government Area of Imo State Nigeria**, The FASEB Journal, April 2015, vol. 29 no. 1 Supplement LB19

5. Windows are constructed, and their width is proportional to their height. One window is measured, and its width is 100cm, and its height is 200cm.

(a) Another window has a width of 75cm. What is its height?

- (b) Make a sketch of the relationship of the height of a window given its width. First, express the relationship as a mathematical formula. Second, sketch the graph of the relationship. Briefly discuss the relationship. How does the height change as the width changes?

*Annotate
your plot
and label
your axes!*

6. The surface area of a sparrow's wing is proportional to the square of the length of its wing. A sparrow is measured, and it has a wing length of 9cm and an area of 45cm^2 .
- (a) Another sparrow is measured, and the length of its wing is 8cm. What is the area of its wing?

- (b) Make a sketch of the relationship of the area of a sparrow's wing given the length. Briefly discuss the relationship. How does the area change as the length changes?

*Annotate
your plot
and label
your axes!*

1. Briefly state two ideas from today's class.

-
-

2. A square is circumscribed within a circle of radius R so it just touches the circle on each of the four corners of the square.



- (a) Sketch the radius of the circle at one of the points where the square touches the circle.
 - (b) Find a convenient right triangle in the new diagram using the radius that you drew, and use the triangle to determine the length of one of the sides of the square. (You may have to double the length of the triangle to get the length of the side.)
 - (c) Determine the area of the square.
3. A triangle with three equal sides is circumscribed within a circle of radius R so it just touches the circle on each of the three corners of the triangle.
 - (a) Make a sketch of the situation. (Label your axes and label the length of the sides of the triangle.)

- (b) Sketch the radius of the circle at one of the points where the triangle touches the circle.
- (c) Find a convenient right triangle in the new diagram, and use the triangle to determine the length of one of the sides of the triangle. (You may have to double the length of the triangle to get the length of the side.)

1. A biologist grows four different colonies of bacteria. The number of bacteria in the colonies is estimated to be 10,000, 20,000, 30,000, and 40,000. The mass for each colony is measured and is estimated to be 2.61×10^{-6} , 5.20×10^{-6} , 7.85×10^{-6} and 1.043×10^{-5} grams respectively.

Organize the information above into a table so that the mass can be more easily determined given the number of bacteria in the colony. Also, graph each point as a coordinate where the number of bacteria is on the horizontal axis, and the mass is on the vertical axis.

*Label your
axes and
properly
annotate
your plot.*

2. What is the change in mass when the number of bacteria increases by 10,000?
3. Make rough estimate for a relationship that will provide a prediction for the mass of a colony given the number of bacteria within it.

28 Name:

Preclass Work - Finish Before Class Begins

1. A balloon has a tether that is attached to the ground, and the tether can be extended or retracted as the balloon is raised or lowered. One end of the tether is attached to the ground 20m away from a point directly below the balloon, and the balloon moves straight up and down. If the length of the tether is x meters what is the altitude of the balloon?

- (a) Sketch a diagram of the physical situation. Label the known and unknown quantities. Make up names for the quantities that you think you **may** need to use later.

*Assume
that the
balloon only
moves up
and down
with no
lateral
motion.*

- (b) Determine the important relationships between the known and unknown quantities.

- (c) How can you use this relationship to determine the variable of interest?

- (d) Determine the height of the balloon given the length of the tether.

- (e) Determine the domain and range of the function that gives the height given the length of the tether.

2. A park has two distinct areas separated by a river, and each area has its own population of mice. The population East of the river is estimated to have 10,000 individuals at the beginning of the year, and each week it grows by a constant 200 individuals. The population West of the river is estimated to have 8,000 individuals at the beginning of the year, and each week it grows by a constant 250 individuals.

- (a) Make a rough sketch of the number of mice in the two populations on the same graph. The horizontal axis should be the time from the beginning of the year in weeks.

*Label your
axes and
properly
annotate
your plot.*

- (b) Describe what is happening to the two populations. Is there a time when the two populations are equal? If so when is it?

- (c) Determine a formula for the total number of mice in the park at any week after the beginning of the year.

3. The mass of a sparrow is proportional to the cube of the length of its wing.
 - (a) A sparrow is measured, and it has a wing length of 9cm and a mass of 30 grams. Determine the mass of a sparrow whose wing length is 10cm.
 - (b) It is estimated that the mass of a sparrow is 27 grams. Determine an estimate of its wing length.
 - (c) Determine the general function that returns the mass of a sparrow given the length of its wing.

4. A common task is to convert units. For each statement below determine the function that returns a quantity in the second unit given a quantity in the first units.

(a) One kilometer is approximately 0.62 miles.

(b) One meter is 100 centimeters.

(c) One US dollar is approximately 1.35 Canadian dollars.

(d) How do these questions relate to the idea of proportionality?

1. Briefly state two ideas from today's class.
 -
 -
2. The surface area of a sphere of radius r is $4\pi r^2$, and the volume is $\frac{4}{3}\pi r^3$. Determine the equation for the surface area of a sphere given its volume.
3. In the Star Trek television series a ship's velocity is given in terms of its warp factor, w . According to wikipedia², the actual speed is the warp factor cubed multiplied by the speed of light which is approximately 3.0×10^8 m/s.
 - (a) Determine the speed of a ship that is moving at warp factor 0.2.
 - (b) Determine the speed of a ship that is moving at warp factor 2.5.
 - (c) Determine the speed of a ship that is moving at warp factor 3.0.
 - (d) A ship is moving at warp factor 3.1. What warp factor would be required to double the ship's speed?
 - (e) A ship is moving at warp factor 4.1. What warp factor would be required to double the ship's speed?
 - (f) What is the general formula to determine the new warp factor required to double the speed given the current warp factor.
4. You watch a video from your favourite conspiracy theorist. He says that scientists are suppressing evidence about prehistoric sparrows. He says that giant sparrows once existed whose wing length was 10 meters. Use the results from exercises 3a and 3b to determine if this makes sense. Based on your result write out the comment that you will post in the comments section in response to the video.

²https://en.wikipedia.org/wiki/Warp_drive accessed June 2016

1. A tortoise and a hare move in a straight line, and the both start at $x = 0$. The tortoise's position is given by

$$x_T = \frac{1}{2}t,$$

where t is in minutes and x is in meters. The hare's position is given by

$$x_H = 2t,$$

where t is in minutes and x is in meters.

- (a) Determine the positions of the tortoise at $t = 0$, $t = 1$, and $t = 2$.

- (b) Determine the positions of the hare at $t = 0$, $t = 1$, and $t = 2$.

- (c) For each time, plot the coordinate of the relative positions on the set of axes below. Use the tortoise's position for the x -coordinate, and use the hare's position for the y -coordinate. For example, if the tortoise's position is 1m, and the hare's position is 4m, then the coordinate would be $P(1, 4)$.



2. A tortoise and a hare move in a straight line, and the both start at $x = 0$. The tortoise's position is given by

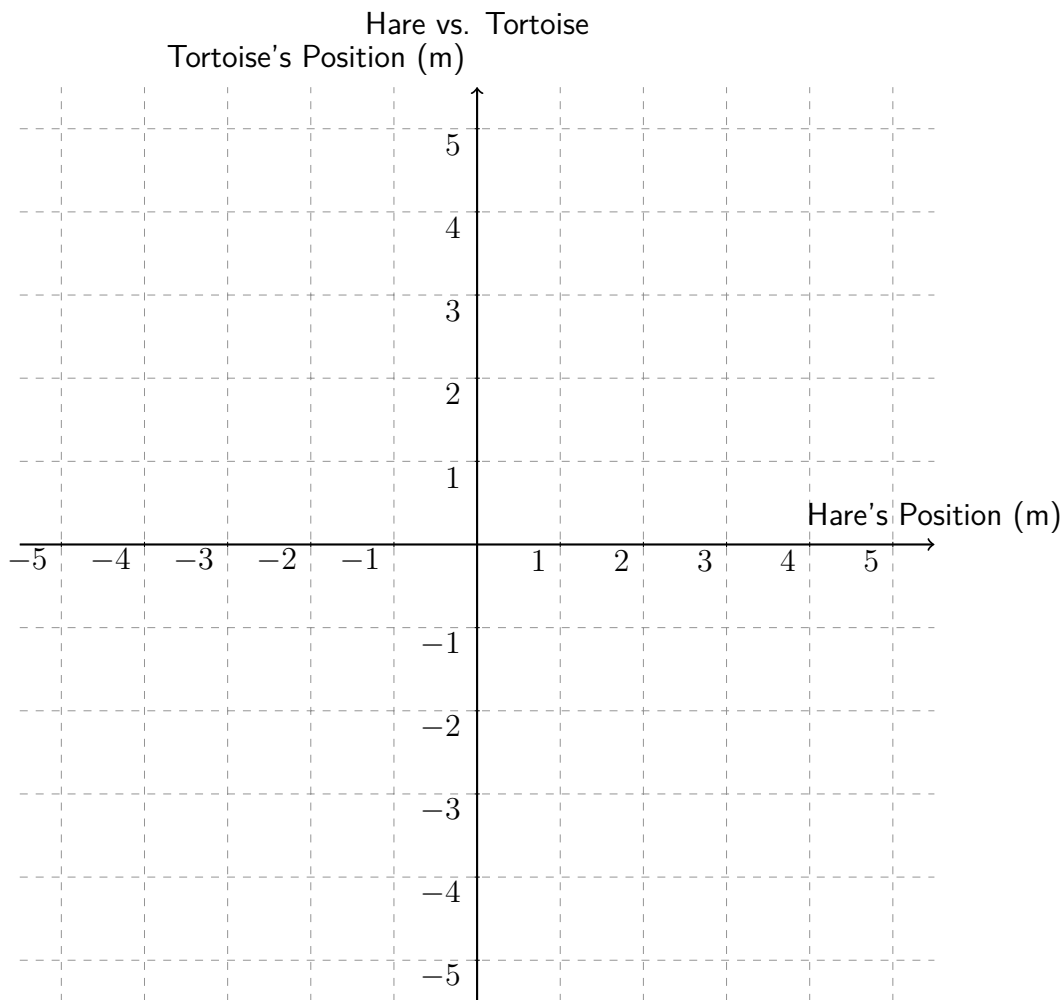
$$x_T = \frac{1}{2}t, \quad (1.1)$$

where t is in minutes and x is in meters. The hare's position is given by

$$x_H = 2t, \quad (1.2)$$

where t is in minutes and x is in meters.

Determine the relationship between the hare's and the tortoise's position. That is, given the hare's position determine the tortoise's position. Make a sketch of the graph of the relationship using the axes below.



What is the tortoise's position when the hare's position is 15 meters? (Mark the associated coordinate on the plot above.)

In equation 1.1 and 1.2 how many variables are there? How many equations? Which variables are changing?

38 Name:

Preclass Work - Finish Before Class Begins

1. In each case below determine the formulas for the lines that satisfy the given requirements. In each case make a rough sketch of the line.

(a) Goes through the point $P(-2, 5)$ and has a slope of -3 .

(b) Goes through the points $P_1(-3, -4)$ and $P_2(4, 1)$.

2. Two test plots are used to study the spread of an invasive plant. In the first test plot the conditions are dryer than in the second test plot. In the first test plot the invasive plant begins with a coverage of 10 square meters, and each day the area covered by the plant increases by 2 square meters. In the second test plot the invasive plant begins with a coverage of 15 square meters, and each day the area increases by 1 square meters.
- (a) Will there be a time when the area covered by the invasive test plant will be the same in the two test plots. Explain your reasoning.
- (b) Determine the area covered by the invasive plant in each test plot for a given time.
- (c) Determine the time that the area covered will be the same.

3. A group of researchers studies birds near a park. The birds tend to use cigarette butts in their nests, and it is believed to help reduce the number of parasitic insects. It is estimated that the number of cigarette butts used for nesting materials varies linearly with the distance from the nest to a nearby open air theater. A nest that is a distance of 30 meters appears to have 10 cigarette butts, and a nest that is a distance of 40 meters appears to have 8 cigarette butts.

- (a) Determine the relationship that will predict the number of cigarette butts in a nest given its distance from the theater. Use the relationship to predict the number of cigarette butts in a nest 50 meters from the theater. Also, make a sketch of the relationship.

Be sure to label your axes and annotate your plot.

- (b) What is the domain for the relationship?

- (c) A nest is found that has 4 cigarette butts. What is the prediction for the distance the nest is from the theater.

- (d) If the conjecture for the reason why birds use cigarette butts in their nests is true what would you expect is the general relationship between the fledgling success rate for birds and the location of their nests?

4. For the lines in the following plot sort the slopes for the lines in increasing order. (Write out the slopes, m_1 , m_2 , etc., in order from lowest to highest and do not try to estimate their values.)



5. The function, Larry(x), is defined to be

$$\text{Larry}(x) = \frac{1}{2}(x - 1)^2 + 2.$$

- (a) Make a sketch of Larry.



- (b) Determine the average rate of change of Larry from $x = -2$ to $x = 3$. Add a sketch of the secant line for these points on your sketch above.
- (c) Determine a value of x_0 where the average rate of change from $x = 2$ to $x = x_0$ is zero. Add the a sketch of the resulting secant line for these points on your sketch above.
- (d) Is there any value of $x = a$ where you cannot find another point so that the resulting average rate of change is zero? Explain your reasoning.

1. Briefly state two ideas from today's class.
 -
 -
2. The growth rate for a population is the change in the number of individuals per unit time. The per-capita growth rate is the growth rate divided by the total number of individuals in the population. Suppose that the per-capita growth rate for a particular species is approximated as a linear function. It is estimated that when the population is near zero the per-capita growth rate is highest due to a lack of competition and approaches 0.5 (the time units are hours). When the population approaches 1,000 the per-capita growth rate is estimated to be zero the death rate and birth rate are balanced, and the total growth rate is zero.
 - (a) What are the units for the per capita growth rate?
 - (b) Is the slope of the per-capita growth rate positive or negative? Explain why your answer makes sense given the physical situation.
 - (c) Determine the relationship that gives the per-capita growth rate as a function of the population, p .
 - (d) Make a sketch of the graph of the per-capita growth rate. (Make sure to annotate your graph and label your axes.)
 - (e) What happens to the per-capita growth rate as the population increases? Why might this happen?
 - (f) Determine the values where the per-capita growth rate is negative. Why would the per-capita growth rate be negative?

1. A chemical reaction has a single reactant that breaks down, and the resulting reaction produces two different products. It is estimated that for each gram of the reactant that $\frac{1}{3}$ g of the first product is produced and $\frac{1}{6}$ g of the original reactant remains. Everything else that remains is the second product.
 - (a) Write down all of the relevant information that is given in the statement above.
 - (b) If you start with four grams of reactant how many grams of the reactant will you get in the end?
 - (c) If you start with four grams of reactant how much of each product do you get? (Keep in mind that mass is conserved so you have to have the same total mass that you started with.)
 - (d) Determine the number of grams of the reactants will result when you start with x grams of reactant.
 - (e) Determine the number of grams of the products will result when you start with x grams of reactant.

48 Name:

Preclass Work - Finish Before Class Begins

1. The time required to bake a ceramic depends on the mass of the ceramic. It is estimated to be a linear relationship between the time and mass. **The goal is to determine the time required to bake a ceramic sample given its mass.**

- (a) Define your variables for the quantities described above. Then make a rough sketch of the relationships described above that will give an idea of the general relationship between the variables.

*Label your
axes and
properly
annotate
your
sketch.*

- (b) Should there be a positive or negative slope for the relationship? Briefly justify your answer.

- (c) With respect to the cost for each ceramic, would you prefer a larger or smaller slope for the relationship? Briefly justify your answer.

- (d) A set of samples will be baked, and it is estimated that the baking time for a ceramic whose mass is 2,000g is five hours. It is also estimated that the baking time for a ceramic whose mass is 3,000g is five and a half hours. Determine the baking time given the mass.

- (e) A sample ceramic will be tested, and its mass is 4,500g. How long would you expect it to take to bake the ceramic?

2. Alice was born on the same day as her father, Bob. This year Bob's age is three times Alice's age. In fifteen years, Bob will be twice his daughter's age. What are their ages this year?
- (a) Define the variables that you will be using for Alice's and Bob's ages.
 - (b) Write the algebraic expression that indicates that Bob's age is three times Alice's age.
 - (c) If Alice's age is now A what will her age be in fifteen years?
 - (d) If Bob's age is now B what will his age be in fifteen years?
 - (e) Use the two previous expressions, and write out the algebraic expression that indicates that Bob's future age will be twice Alice's future age.
 - (f) Use the two expressions from parts **2b** and **2e** to draw a sketch of the two linear relationships. Assume that Bob's age is a function of Alice's age, and the horizontal axis will Alice's age. Will the system have a solution that makes sense?
 - (g) Use the two expressions from parts **2b** and **2e** to determine Alice's and Bob's age.
 - (h) Next to each of the previous steps, write a note in the left margin. Your note should indicate the relationship of the step and what part of the original statement it corresponds to.

3. Trucks are unloaded at a warehouse, and during the summer it is estimated that it takes longer to unload a truck if the weather is warmer. When the temperature is 22°C it is estimated that it takes sixty minutes to unload a truck. For each increase of one degree Celsius it is estimated to take four additional minutes to unload a truck.

- (a) Define the variables that you will use.
- (b) How long will it take to unload a truck if the temperature is 23°C ?
- (c) How long will it take to unload a truck if the temperature is 24°C ?
- (d) How long will it take to unload a truck given a temperature of T degrees Celsius?

4. In the previous problem it was assumed that the truck was being unloaded during the summer. In the winter the relationship between unloading time and the temperature is different. When the temperature is 5°C it is estimated that it takes fifty minutes to unload a truck, and for each decrease of one degree Celsius it is estimated to take three additional minutes to unload the truck.
- (a) Define the variables that you will use.
 - (b) Determine how long it will take to unload the truck for any temperature less than 5°C .
 - (c) At what temperature should you switch from using the formula on the previous problem to using the formula above?
 - (d) Write out the formula to determine the time required to unload the truck given **any** temperature in Celsius.
 - (e) What is the shortest time to unload a time and what is the best temperature to unload a truck? (Explain your reasoning.)

1. Briefly state two ideas from today's class.
 -
 -
2. Determine the formula for a line that goes through the points $P(2, 50)$ and $Q(5, 80)$. What is the height of the graph of the line when $x = 3.5$?
3. The number of items sold at a given time of day is a linear function of the number of hours the office has been open. After the office has been open for two hours it is estimated that fifty items have been sold. After the office has been open for five hours it is estimated that eighty items have been sold. What is the estimate for the number of items sold after the office has been open for three and a half hours?
4. What is the relationship between the previous two questions?

1. Two populations of different species of bacteria interact. The number of bacteria (in tens of thousands) in the first population is given by

$$B(t) = 10 + t^2,$$

where t is the time in days since the beginning of the year. The number of bacteria (in tens of thousands) in the second population is given by

$$C(t) = 10 + (t - 2)^2,$$

where t is the time in days since the beginning of the year.

- (a) Make a sketch of the two functions below.

*Label your
axes and
properly
annotate
your plot.*

- (b) For what values of t does it make sense to use these functions?

- (c) A researcher decides to alter the situation and adds 50,000 bacteria to the first population given by $B(t)$. Determine a formula for the altered population. Make a sketch of the original and altered populations below.

56 Name:

Preclass Work - Finish Before Class Begins

1. The height, in meters, of a certain tree changes by the relationship

$$h(t) = \sqrt{\frac{t}{3}},$$

where t is the time in years from when the seed was germinated.

- (a) Make a sketch of the height of a tree as a function of time.

*Label your
axes and
properly
annotate
your plot.*

- (b) Two seeds are planted, and the first seed germinates immediately. The second seed germinates one year after the first is germinated, and then begins to grow.

- (i) Make a sketch of the graphs of the heights of the two trees.

- (ii) Determine the formulas for the height of the two trees with respect to the time that they were planted.

- (c) A new strain of the tree is developed that grows to the same height in half the time.
- (i) Make a sketch of the graph of the height of the new strain and the previous strain.
- (ii) Determine the formula for the height of the new strain with respect to the time that it is planted.

2. A function is defined as

$$f(x) = |x|.$$

- (a) Make a sketch of the function on the axes below.
- (b) Make a sketch of the following new functions on the same graph as well.
(Clearly annotation your plot.)

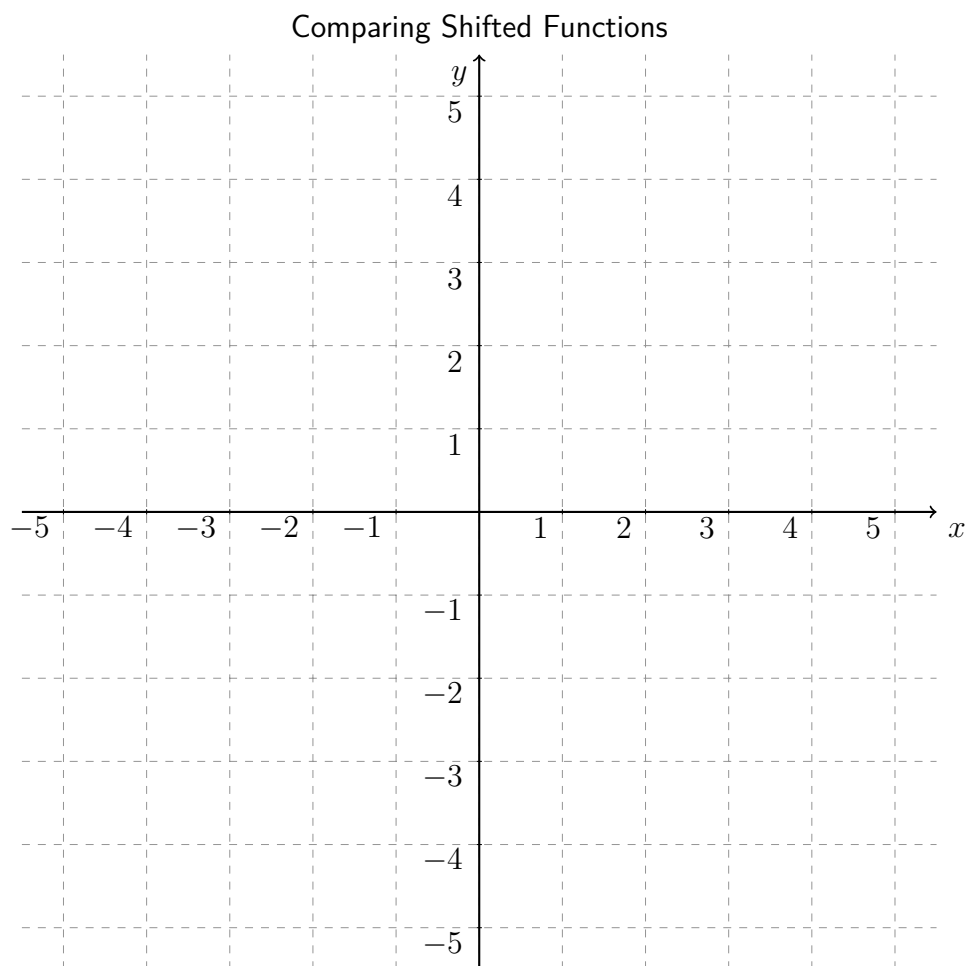
$$g(x) = f(3x),$$

$$h(x) = f(x) + 2,$$

$$p(x) = f(x + 2),$$

$$q(x) = 3f(x),$$

$$r(x) = -f(x) - 2.$$

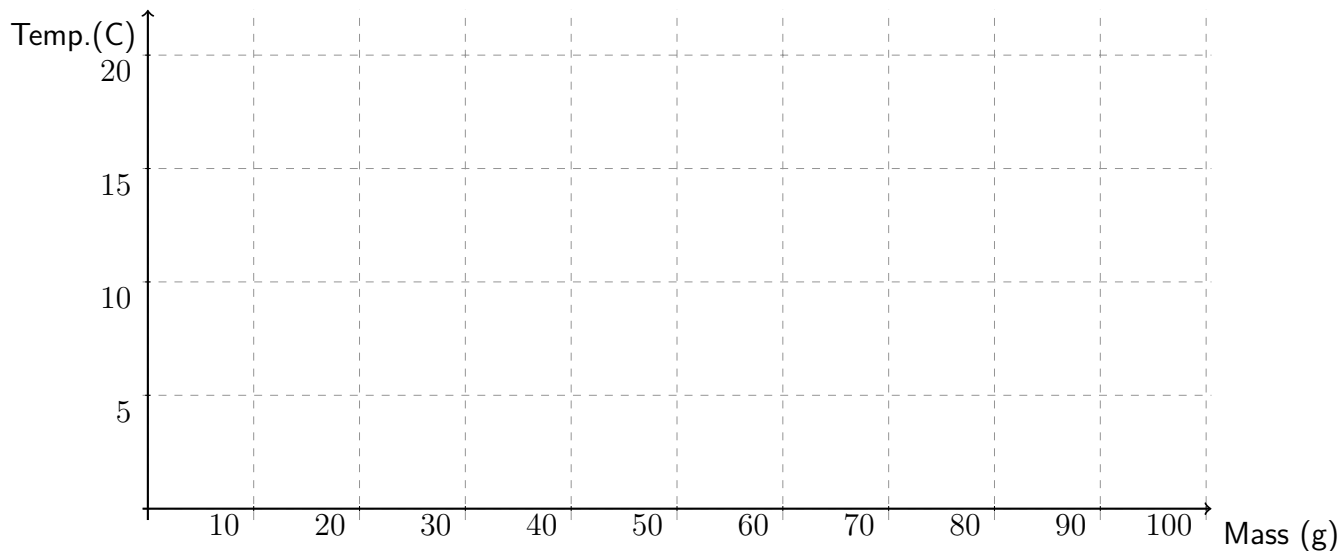


3. The temperature of a snake can change based on its activity level. Suppose that the temperature of a snake is taken at a fixed time after it consumes a rodent, and the snake's temperature depends on the mass of the rodent, denoted m ,

$$\text{Temperature}(m) = 12 + 0.04m,$$

where the temperature is in Celsius, and m is in grams.

- (a) Make a sketch of the relationship on the axes below.



- (b) To convert Celsius to Kelvin, you add 273.15K. What will happen to the graph if the temperature is converted to Kelvin? Can you draw the new plot on the existing axes?

- (c) To convert Celsius (C) to Fahrenheit (F), you use the function

$$\text{Fahrenheit}(C) = \frac{9}{5}C + 32.$$

If you convert the temperature to Fahrenheit how will the graph change?

- (d) If you convert grams to kilograms how will the graph change? (1kg=1,000g)

4. The water levels for the ocean near the Tybee lighthouse and the St. Simon's lighthouse are shown in the plot below. The solid line is the water level at Tybee, and the dotted line is at St. Simon's. According to the tidal charts for a given day it is estimated that the low tide at Tybee is at 3pm, and the low tide at St. Simon's is at 3:24pm. Determine the expression that will give the water level at St. Simon's in terms of the water level at Tybee.



1. Briefly state two ideas from today's class.
 -
 -
2. An enzyme in a solution decays, and the concentration (mg/liter) as a function of time in hours is

$$C(t) = \frac{3.5}{5.0 + t}.$$

- (a) Make a sketch of the concentration as a function of time. Assume that the time is positive. Annotate your plot and label your axes.
- (b) How long will it take for the enzyme to be reduced to half its original concentration?
- (c) Another enzyme is present, and its concentration is linked to the first. Its concentration is half the first enzyme's concentration 30 minutes in the past. Determine the formula for the second enzyme's concentration. (This is referred to as a delay relationship.)
- (d) How long will it take for the second enzyme to be reduced to half its original concentration?
- (e) Make a sketch of the concentration of both enzymes as a function of time. Assume that the time is positive. Annotate your plot and label your axes.

1. The graph of a function, g , is shown shown below:

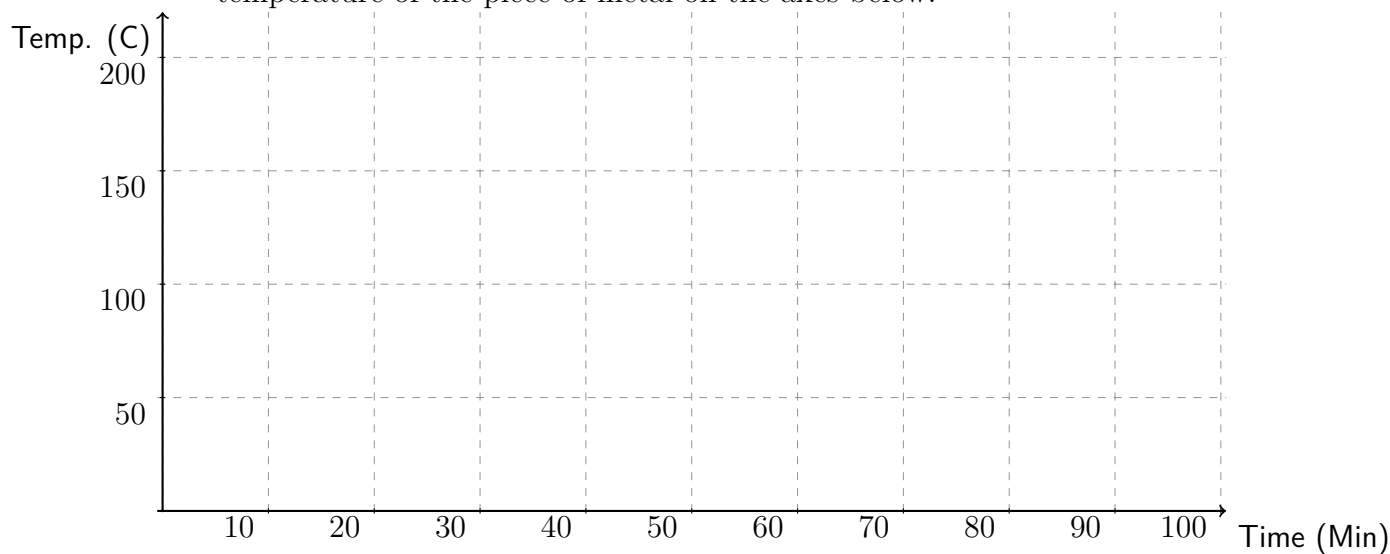


- (a) Determine the domain and range of the function.
- (b) Determine the formula for the function if $x \geq -2$ and $x < 0$.
- (c) Determine the formula for the function if $x > 0$ and $x \leq 4$.
- (d) For what values of x is the function increasing?
- (e) For what values of x is the function decreasing?

66 Name:

Preclass Work - Finish Before Class Begins

1. The temperature of a piece of metal is initially 20C. It is placed in a hot furnace for twenty-five minutes, and its temperature is raised to 200C. It is then removed from the oven and placed in cold water and cooled to room temperature.. After five minutes it is placed back into the furnace, and the process is repeated twice for a total number of three cycles. Make a sketch of the graph of the temperature of the piece of metal on the axes below.



- (a) What is the range and domain of the function?
- (b) Determine the values of the time where the function is increasing.
- (c) Determine the values of the time where the function is decreasing.

2. The questions below refer to the function whose graph is given below.



- (a) Determine the domain and range of the function.
- (b) Determine the values of x where the function is increasing.
- (c) Determine the values of x where the function is decreasing.

3. Make sketch of the graph of the function

$$\text{Harold}(x) = \begin{cases} \frac{1}{2}(x+2) & -2 \leq x < 0, \\ 4-x^2 & 0 < x < 2. \end{cases}$$

Determine the domain and range of the function. Determine where the function is increasing. Also determine where it is decreasing.



4. Part of the graph of a function is given below. The function is even. Sketch the rest of the function. Determine the domain and range of the function.



5. Part of the graph of a function is given below. The function is odd. Sketch the rest of the function. Determine the domain and range of the function.



6. A small company builds solar panels. The amount of electricity produced by a panel is proportional to the intensity of sun light. When the sunlight is bright, 100,000 lux, the system produces 4,000 watts. On one day of operation there is a good deal of cloud cover, and the amount of sunlight varies linearly from 6am to noon from 0 lux to 50,000 lux. After noon it varies linearly to 0 lux at 6pm. On the second day the cycle repeats, but the maximum amount of light is 100,000 lux. Determine the amount of power produced by the panel at any time during the two days. (Include night time!)
- (a) Read the question again. Organize the information given to you. Underline the important parts, and determine the underlying assumption. (Hint: what happens when the light intensity is zero.)
 - (b) Make a rough sketch of what the relationship of interest might look like. (Label your axes.)
 - (c) Define the variables that you will use.
 - (d) List the relevant relationships.
 - (e) How will you solve this problem?
 - (f) Determine the relationship.
 - (g) Go back and check your work. Is your answer consistent with your prediction? Does it make sense?

1. Briefly state two ideas from today's class.

-
-

2. A train moves along a track. From $t = 0$ to $t = 12$ hours it moves from its initial position at a constant velocity to a new location twenty-five miles away. From $t = 12$ to $t = 24$ hours the train moves with a constant velocity to its start position. Determine a formula for the train's position. Use proper notation for piecewise defined functions.
3. A car moves along a straight track. The graph of the car's position is shown in the plot below. Write out a brief story describing the car's position but also how its position is changing in time.

Position (m)



1. A restaurant would like to test a new menu item. They estimate that the cost for producing x servings in market A is 12\$ per serving. They estimate that the cost for producing y servings in market B is \$15 per serving. They will allocate a total of \$36,000 for the test. Write out an expression that relates x , y , and the total cost of the test.
2. The number of mosquitoes per acre in an area is estimated to be 600 times the area of open water measured in acres. The area of open water in a location is declining over time and is $A(t) = 50 - \frac{1}{3}t$, where t is the number of years since January 1 of the current year. Determine the number of mosquitoes per acre in given the time, make t .
3. The delay time required for a neuron to recharge is estimate to be a function of the calcium concentration,

$$\text{Recharge}([\text{Ca}]) = 0.05 - [\text{Ca}]^2.$$

The concentration of calcium in an experiment is changed over time and is estimated to be

$$[\text{Ca}](t) = 0.01 + \frac{1}{1+t}.$$

Determine the formula used to estimate the recharge delay as a function of time, t . (*Do not simplify the expression.*)

76 Name:

Preclass Work - Finish Before Class Begins

1. A function is defined to be

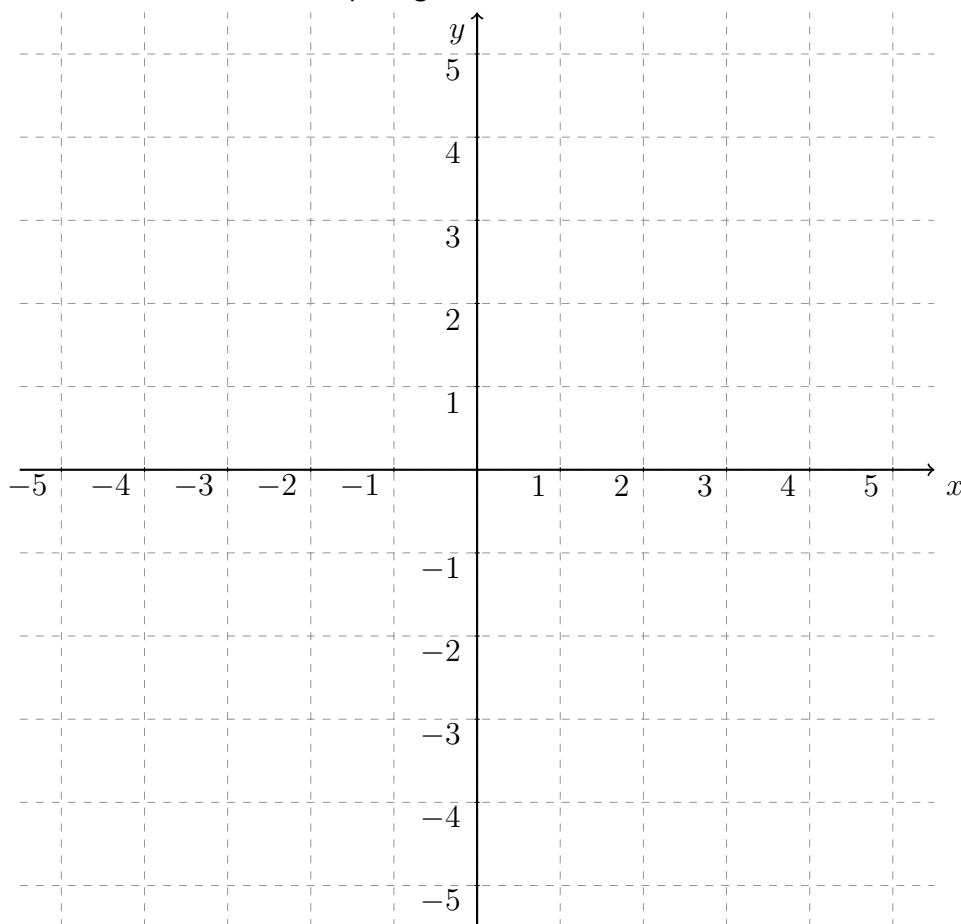
$$f(x) = x^2.$$

Determine the value of a and b so that the function

$$g(x) = f(x - a) + b$$

is the original function that is shifted up two units and left three units. Plot the graphs of $f(x)$ and $g(x)$ on the coordinate plane below.

Comparing Shifted Functions



2. Two functions are given in the tables below.

x	0	1	2	3	4
$f(x)$	a	m	k	a	h
x	a	c	h	j	m
$g(x)$	‡	◇	□	♥	◇

(a) Determine the range and domain of f .

(b) Determine the range and domain of g .

(c) Determine the values of each of the following expressions:

$$f(2) =$$

$$f(4) =$$

$$g(f(1)) =$$

$$g(f(2)) =$$

$$g(f(0)) + g(f(3)) =$$

(d) If $f(x) = h$ what is the value of x ?

3. The expected amount for insurance claims, I , for a small, college town depends on the number, N , of outside visitors to the town, and the graph of the relationship is shown in the plot below, on the left. The number of visitors to the town changes during a week, and the number of visitors based on the day of the week is shown in the graph on the right. (Zero corresponds to Sunday.)



- (a) What is the expected amount for insurance claims on Monday?
- (b) As the day of the week increases from Sunday to Monday what is the change in the expected amount for insurance claims?
- (c) As the day of the week increases from Wednesday to Thursday what is the change in the expected insurance claims?
- (d) Explain what meaning the expression $I(N(d))$ has where d is the day of the week.
- (e) How can the function $I(N(d))$ increase from Sunday to Monday when both functions are decreasing?

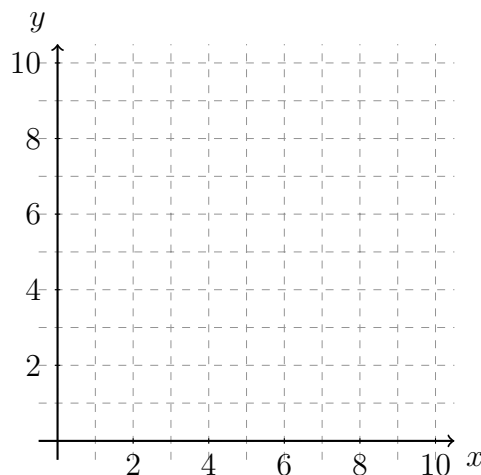
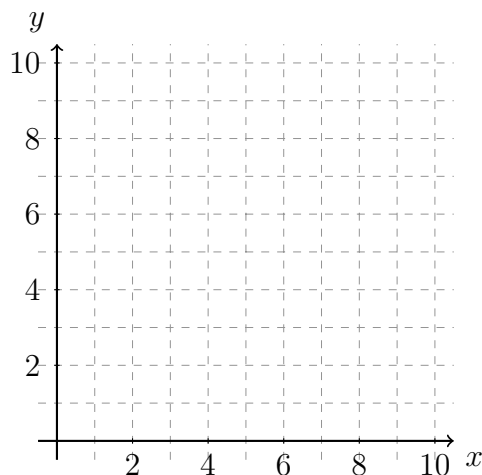
A negative number indicates a decrease, and a positive number indicates an increase.

4. The owner of a shop will be selling two similar items and hopes that customers will choose to buy either one or the other of the items. The cost to the owner depends on how many are purchased,

$$\begin{aligned}C_1(x) &= 100 - (x - 10)^2, \\C_2(y) &= 100 - (y - 10)^2,\end{aligned}$$

where x is the number of the first item purchased, and y is the number of the second item purchased. The owner plans on allocating **at least** 100\$ for purchasing the items. The owner has **at most** 5 m² of total space for the two sets of items, and each individual item takes up the same amount of space, 0.5 m².

- Determine the relationship between x , y , and the total costs.
- Determine the relationship between x , y , and the total space required.
- Make a sketch of the graphs of the two relationships. Use the left axes for the costs, and use the right axes for the space.



- If the owner chooses to purchase 4 of the first item, determine the possible numbers of the second item that could be purchased. (It is a range of values.)
- If the owner chooses to purchase 8 of the first item, determine the the possible numbers of the second item that could be purchased. (It is a range of values.)

1. Briefly state two ideas from today's class.

•

•

2. Two functions are given in the tables below.

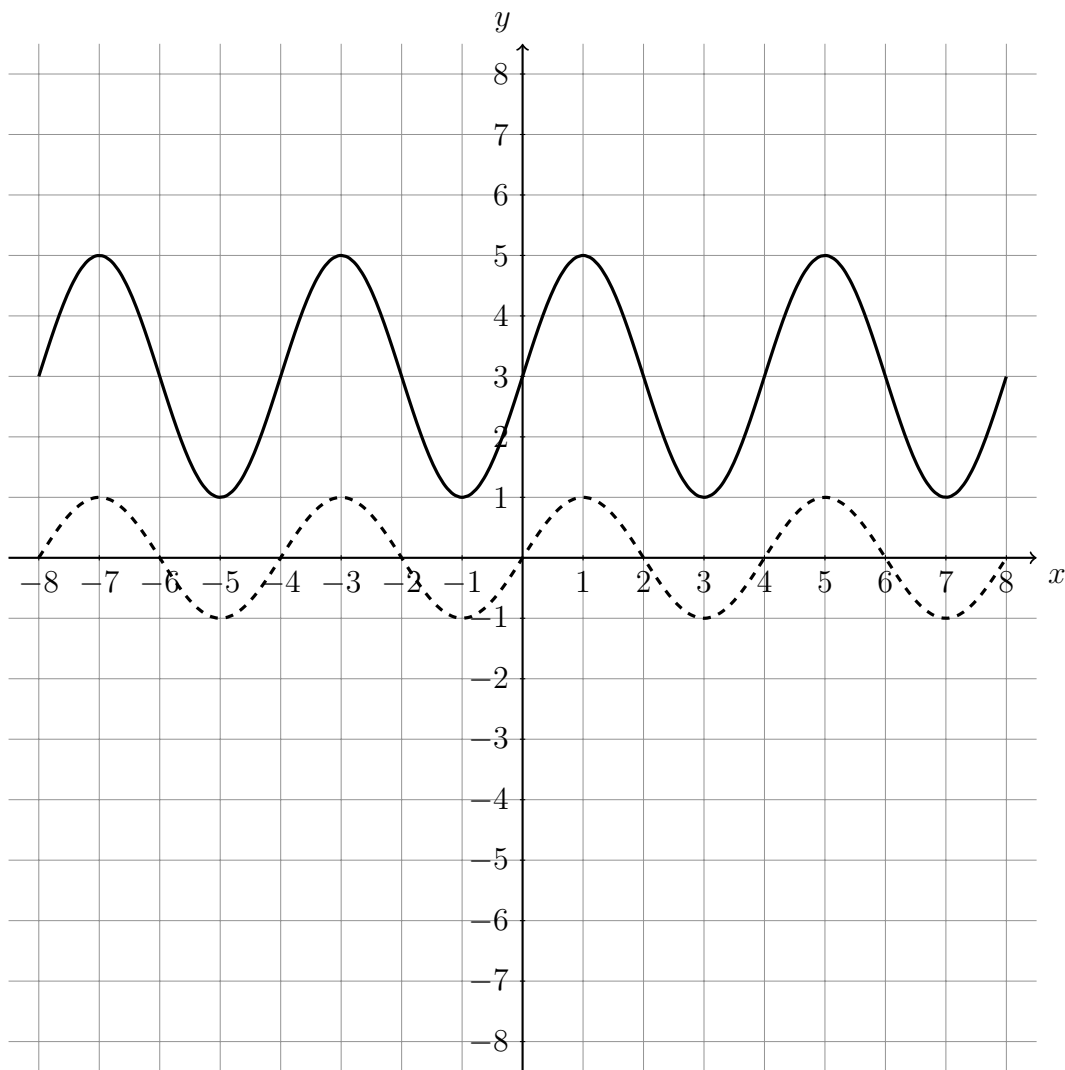
x	0	1	2	3	4
$f(x)$	a	m	k	a	h
x	a	c	h	j	m
$g(x)$	‡	◇	□	♥	◇

- (a) If $g(f(x)) = ‡$ what are the possible values of x ? Is this reverse procedure a function?
- (b) If $g(x) = ◇$ what are the possible values of x ? Is this reverse procedure a function?
- (c) Express the function $g(f(x))$ as a table.
- (d) Determine the range and domain of $g(f(x))$.

3. Two functions are shown in the figure below. The function plotted with the dotted line is $f(x)$, and the function plotted with the solid line is $g(x)$. Express $g(x)$ in terms of $f(x)$,

$$g(x) =$$

Shifted Function



4. Two functions are shown in the figure below. The function plotted with the dotted line is $f(x)$, and the function plotted with the solid line is $g(x)$. Express $g(x)$ in terms of $f(x)$.

$$g(x) =$$

Shifted Function



Add a sketch of the graph of the function $h(x) = 3f(x + 2) - 5$ to the plot above.

Can you find a different formula whose graph looks exactly the same as $g(x)$?

1. A function is defined to be

$$f(x) = x^2.$$

- (a) Make a sketch of the function on the axes below.
- (b) Make a sketch of the following new functions on the graph as well with clear annotations:

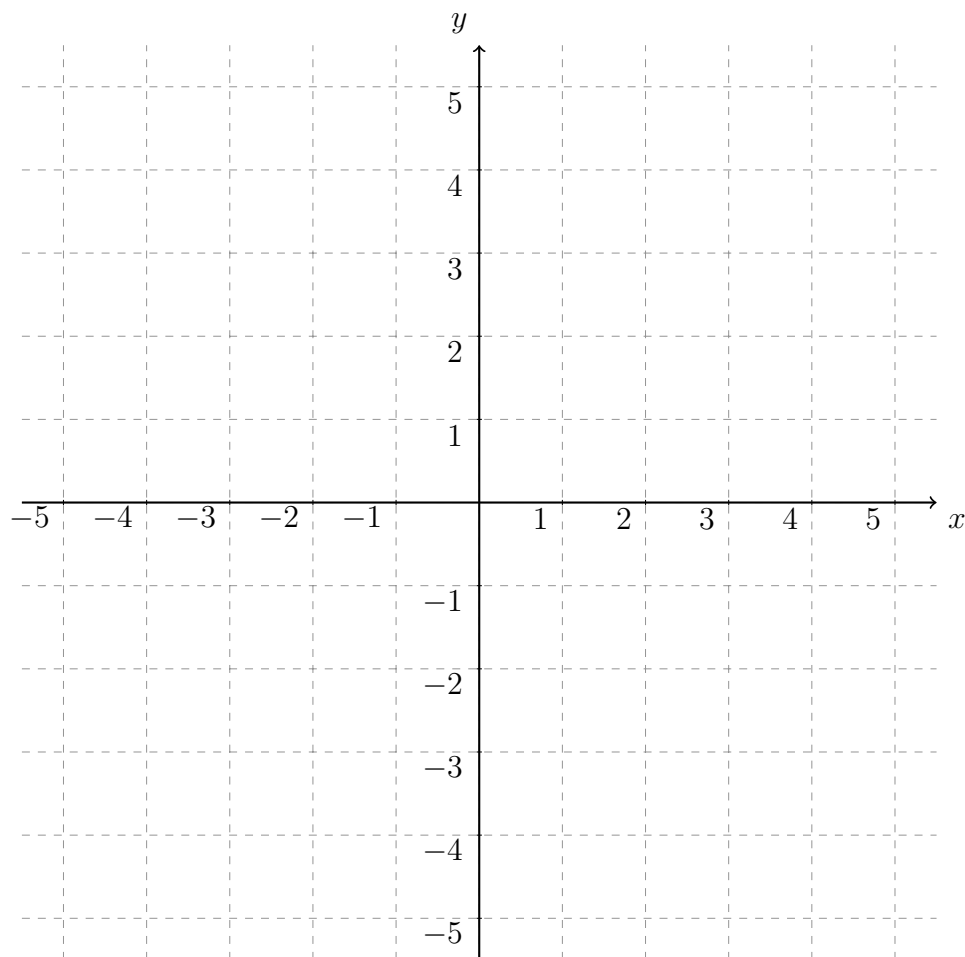
$$g(x) = f(3x),$$

$$h(x) = f(x) + 2,$$

$$p(x) = f(x + 2),$$

$$q(x) = -f(x).$$

Comparing Shifted Functions



86 Name:

Preclass Work - Finish Before Class Begins

1. A traffic engineer observes the movement of traffic on a busy street. She estimates that when the density of cars is low the rate of flow is also low, and as the density increases the rate of flow increases. However, once the density of cars reaches a certain point the traffic slows, and the rate of flow begins to decrease.

Some tests are run, and it is estimated that when the traffic density is 0 cars/meter that the rate of flow is 0 cars/minute. When the density is high, roughly 0.1 cars/meter or more, there is gridlock, and the rate of flow is 0 cars/minute. Finally, when there is 0.05 cars per meter the rate of flow is at a maximum of 20 cars per minute.

- (a) Assuming that the rate of flow is 0 if the density is greater than 0.1 make a rough sketch of the rate of flow as a function of the density of cars.

*Label your
axes and
properly
annotate
your graph.*

- (b) Assuming that the rate of flow is a quadratic function of the traffic density when the density is between 0 and 0.1 cars/meter inclusive, determine the formula for the traffic density.

2. Maximize the product of two positive numbers whose sum is five.

- (a) The product of two numbers can be thought of as the area of a rectangle.
Make a rough sketch of a rectangle.

*The product
of two
numbers is
the area of
a rectangle!*

- (b) Within the sketch above, identify and label the variables that you will use.

- (c) Determine the formula for the function to be maximized.

*This is
called the
"Cost
Function."
It should
have two
variables.*

- (d) Determine any other relevant relationships between your variables that must be true in all circumstances.

*This is
called the
"Con-
straint." It
should have
two
variables.*

- (e) How will you solve this problem?

- (f) Determine the values of the two numbers.

*Check your
work and
make sure
that you
maximized
the profit
and did not
minimize
it*

3. A developer has a square plot of land that is 100 meters by 100 meters, and the land is next to a river. The land will be divided into two parts. One part will be formed by cutting out a rectangle in one corner of the large plot, and its width will be along the river. The height of the rectangle will be ten meters shorter than its width. It is estimated that this new rectangular plot of land will sell for \$2.00 per square meter plus 100\$/meter times the width. The remaining land will be sold for \$4.00 per square meter. Determine the dimensions of the rectangular plot that will result in the highest profit.

- (a) Make a sketch of the land with a corner designated for one of the plots.

*Label
important
aspects of
the sketch.*

- (b) Within the sketch above, identify and label the variables that you will use.

- (c) Determine the total selling price for the land in terms of both variables.

*This is
called the
"Cost
Function."
It should
have at
least two
variables in
it.*

This is called the "Constraint." It should have two variables in it.

- (d) Determine any other relevant relationships between your variables that must be true in all circumstances.

- (e) How will you solve this problem?

- (f) Determine the dimensions of the rectangular plot that will result in the highest profit.

Check your work and make sure that you maximized the profit and did not minimize it.

1. Briefly state two ideas from today's class.
 -
 -
2. Determine the vertex of the following parabolas.
 - (a) $y = -x^2 + 10x - 5$
 - (b) $y = 2x^2 - 10x + 18$
 - (c) $y = 3x^2 + 4 + 1$
 - (d) $y = 5x^2 + 2$
 - (e) $y = 5x^2 + x + 2$
3. Two super hero capes will be sewn as part of a demonstration. One cape will be in the shape of a square. The other cape will be in the shape of a triangle, and the height is the same length as its base. They will be displayed together, and the sum of the two lengths must be 6 feet. What are the dimensions of the capes that will minimize the total area of the capes?
 - (a) What are the steps you will take to solve this problem?
 - (b) Follow your steps and determine the dimensions of the two capes that will meet the given criteria.
4. Determine the numbers whose sum is 30 and their product is as large as possible. (Show your work and justify your answers mathematically.)
5. Two types of bacteria will be used to produce a medicinal compound. Bacteria A will be given x hours, and it produces 3g of the compound per hour. Bacteria B will be given y hours, and it produces 4g of the compound per hour. The total cost for the type A bacteria is x^2 dollars, and the total cost for the type B bacteria is $2y^2$ dollars. A total of 40g of product will be required. How many hours should be allocated to each bacteria to minimize the total cost
6. You have been asked to help design a channel taking water from a lake into a reservoir. The channel must have a rectangular cross section with a fixed depth and width throughout its entire length. The bigger the cross section, the more the channel will cost. In fact, the total cost of building the channel will be \$30,000 dollars per meter of depth and \$10,000 per meter of width of its cross section, and the government has budgeted a total of \$100,000 to construct the channel. The government wants you to use this money to build a channel which will maximize the rate at which water flows through the channel (which is proportional to the cross sectional area of the channel). What is the depth and width that will accomplish this?
7. Two chicken coops will be constructed using 120 feet of fencing. The coops will be in the shape of a rectangle with fencing across the middle dividing the

rectangle into two equal rectangles. What dimensions will give the largest total area for the two coops?

1. The height, in meters, of a certain tree changes by the relationship

$$h(t) = \frac{50 \cdot t}{25 + t},$$

where t is the time in years from when the seed was germinated.

- (a) Make a sketch of the height of a tree as a function of time.
- (b) A tree is measured, and it is estimated that its height is 5 meters. How long ago did its seed germinate?
- (c) A tree is measured, and it is estimated that its height is 10 meters. How long ago did its seed germinate?
- (d) Determine the function that takes the height of the tree and then determines its time since germination.

94 Name:

Preclass Work - Finish Before Class Begins

1. Two functions are defined in the following tables. Determine the values of each expression below. If a value does not exist write “NA.”

x	0	1	2	3	4
$f(x)$	2	6	5	1	4
x	1	2	4	5	6
$g(x)$	3	3	2	9	4

(a) $f(g(4))$

(b) $g(f(4))$

(c) $f^{-1}(g(4))$

(d) $f(g^{-1}(4))$

(e) $f^{-1}(g^{-1}(3))$

(f) $f^{-1}(g^{-1}(9))$

(g) $g^{-1}(f^{-1}(5))$

2. The height, in meters, of a certain tree changes by the relationship

$$h(t) = \frac{50 \cdot t}{25 + t},$$

where t is the time in years from when the seed was germinated.

- (a) Determine the range and the domain of $h(t)$.
- (b) Determine the inverse of $h(t)$. How can it be interpreted?
- (c) Determine the range and the domain of the inverse of $h(t)$.

3. For each function below determine if it is one-to-one. For each function that is one-to-one determine the inverse of the function. For each function that is not one-to-one determine a subset of the domain for which the function is one-to-one on that subset.

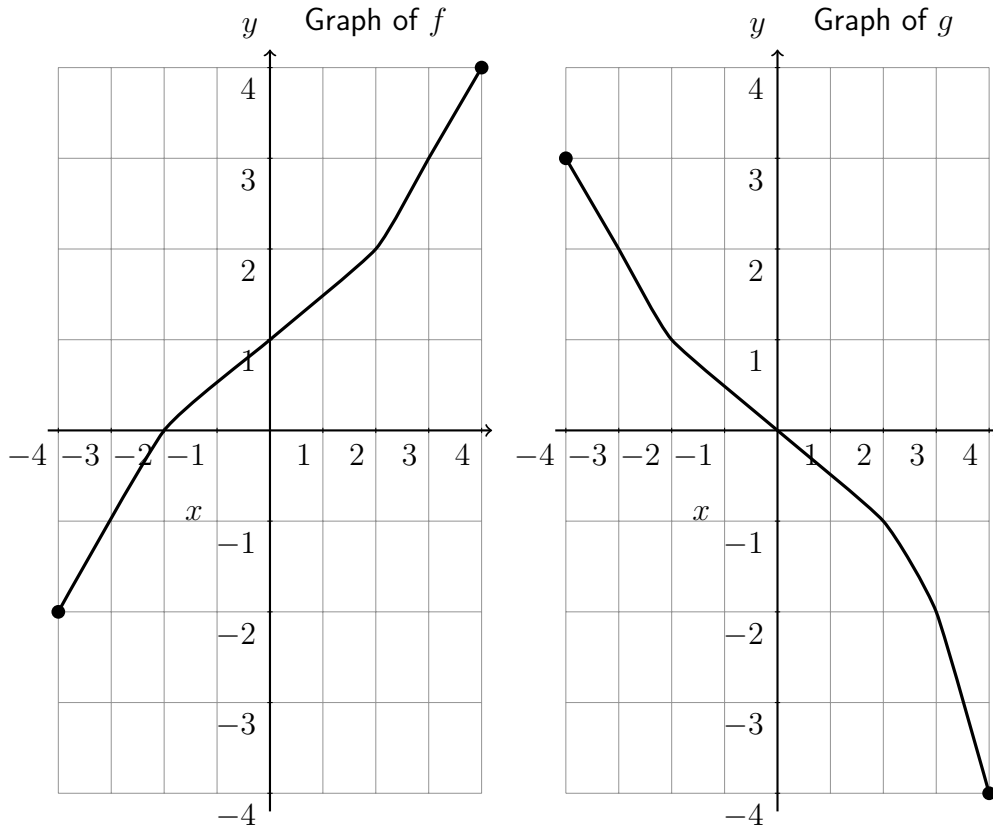
(a) $f(x) = 3x + 1$

(b) $f(x) = \frac{1}{1+x}$

(c) $f(x) = x^2$

(d) $f(x) = \sqrt{3x + 1}$

4. Given the graphs of the two functions below determine the values of the expressions below the graph.



(a) $f(g(3)) =$

(b) $f^{-1}(0) =$

(c) $f^{-1}(-2) =$

(d) $g(f(2)) =$

(e) $g^{-1}(f(0)) =$

(f) $g^{-1}(f^{-1}(1)) =$

5. The impact of sex determination in Pine Snakes was explored in a paper in **The American Naturalist**.³ It was found that if the temperature of the Pine Snake's eggs was kept constant then the sex ratio in the brood can be approximated using a linear function of the temperature. The data suggests the following estimate:

$$\text{Sex Ratio}(\text{temp.}) \approx 0.68 \cdot \text{temp.} - 0.95.$$

(The sex ratio is calculated by taking the number of male snakes and dividing by the number of female snakes in the brood.)

- (a) What is the largest possible domain that can be used for this function? Does this result make physical sense?

- (b) What is the meaning for the inverse function? What is the domain and range of the inverse?

- (c) Determine the inverse function.

- (d) Sketch a graph of the function and its inverse function on the following page. Label your axes and annotate your plots.

³*Effects of Incubation Temperature on Sex Ratios in Pine Snakes: Differential Vulnerability of Males and Females*, Joanna Burger and R. T. Zappalorti, **The American Naturalist** Vol. 132, No. 4 (Oct., 1988), pp. 492-505.

1. Briefly state two ideas from today's class.

-

-

2. We examine the function

$$f(x) = x^2.$$

- (a) Determine if the function is 1-1.
- (b) Determine a restriction on the domain of f so that the function is 1-1 on the restricted domain.
- (c) Determine the inverse on the restricted domain.

3. We examine the function

$$f(x) = x^2 - 4x.$$

- (a) Determine if the function is 1-1.
- (b) Determine a restriction on the domain of f so that the function is 1-1 on the restricted domain.
- (c) Determine the inverse on the restricted domain.

Chapter 2

Exponential and Logarithmic Functions



Figure 2.1: Topics for the second section of the course.

1. Carbon-15 has a half life of about 2.5 seconds. If an object has 2 grams of carbon-15 in it now, then in 2.5 seconds it will only have 1 gram due to its decay. After an additional 2.5 seconds there will only be about $\frac{1}{2}$ gram within the object.

Suppose that an object has 8.0×10^{-6} grams of carbon-15, and it is placed in a sealed container. Determine how much carbon-15 is contained in the object at the following times:

(a) After 2.5 seconds.

(b) After an additional 2.5 seconds for a total of 5.0 seconds.

(c) After an additional 2.5 seconds for a total of 7.5 seconds.

(d) After an additional 2.5 seconds for a total of 10.0 seconds.

106 Name:

Preclass Work - Finish Before Class Begins

1. A species of bacteria is able to divide every three hours. Every three hours each individual bacteria splits into two new individuals. Suppose that a colony starts with 10,000 individuals. For each time below determine the number of bacteria and also determine an expression for the total time in terms of the three hour time span. (For example, 6 hours = 2×3 hours.)

(a) At $t = 3$ hours. (Do not simplify the expression.)

(b) At $t = 6$ hours.

Your results should be kept in terms of products of terms and do not simplify. Look for a pattern.

(c) At $t = 9$ hours.

(d) At $t = n \times 3$ hours where n is an integer greater than or equal to zero.

(e) How many bacteria were in the colony 3 hours before the start of the experiment?

2. A species of bacteria is able to divide every five hours. That is every three hours each bacteria splits into two new individuals. After each division, only 75% of the remaining bacteria survive. A colony starts with 10,000 individuals. For each time below determine the number of bacteria and also determine an expression for the total time in terms of the three hour time span. (For example, 6 hours = 2×3 hours.)

*Keep the
fractions
and do not
simplify.
Look for a
pattern.*

(a) At $t = 5$ hours.

(b) At $t = 10$ hours.

(c) At $t = 15$ hours.

(d) At $t = n \times 5$ hours where n is an integer greater than zero.

(e) How many bacteria were in the colony 5 hours before the start of the experiment?

3. A bank offers a savings account in which the interest is compounded 1.5% annually, and the interest is accrued each month. If a person places \$1,000 in an account how much money is in the account after n months? From the resulting expression determine the money in the account for any time, t , where t is measured in years. *Determine the amount of money in the account after the first, second, and third months. Do not simplify your results, and try to determine the pattern.*

4. Simplify each expression below.

(a) $\frac{3^5 \cdot 3^2}{3^4}$

(b) $\frac{2^8}{2^5}$

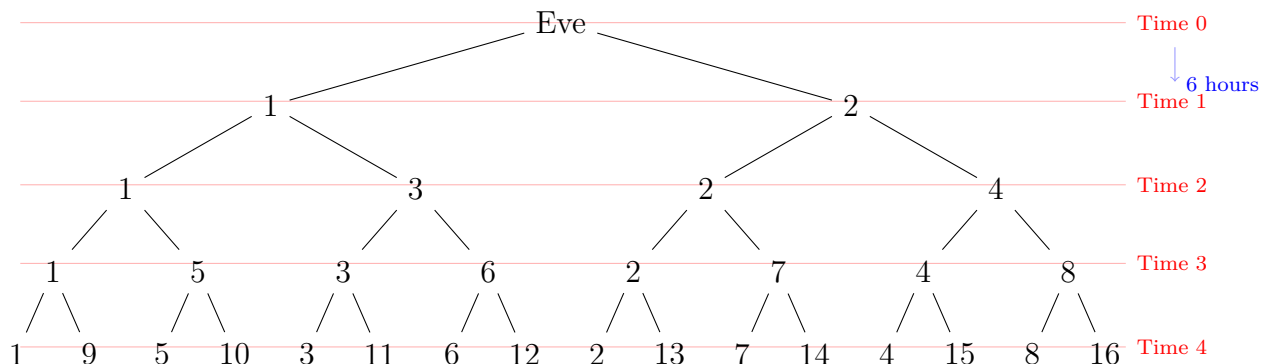
(c) $\frac{4^2 \cdot 2^2}{4^3}$

(d) $5^9 \cdot 5^2 \cdot 5^{-7}$

(e) $\left(\frac{1}{2}\right)^5 \cdot 2^9 \cdot 2^{-3}$

1. Briefly state two ideas from today's class.
 -
 -
2. A bank offers 1.5% annual interest compounded weekly (assume 52 weeks in a year). You will deposit \$5,000 into the account. How much money will be in the account at any time?
3. A bank offers 1.5% annual interest compounded monthly. You will deposit some money into an account and wish to have \$25,000 after two years. How much money should you deposit?
4. A compound is created that decays over time. It takes four years until half of the compound decays in a sample. You wish to store the compound for 5 years. How much should you store so that there will be 4 kg of material at the end of the time period?
5. A colony of bacteria starts with 10,000 individuals. Each individual divides into two new individuals every 4 hours. After each division only 80% of the total number of bacteria survive to divide later. Determine the number of bacteria present at any time.

Exponential functions are used whenever some quantity has a proportional increase over fixed time spans. An example is a bacteria population that increases by 100% every six hours. That means that every six hours the population doubles. In the diagram below, a single bacteria starts in a sample. After the first time period, six hours, there are two bacteria. After another six hours, a total of 12 hours, there are four bacteria. In each six hour time period that follows the population doubles.



Exponential functions satisfy the algebraic properties given below. In each example it is assumed that a , b , and c are constants, and $a > 0$.

$$a^b \cdot a^c = a^{b+c} \quad (2.1)$$

$$\frac{a^b}{a^c} = a^{b-c} \quad (2.2)$$

$$(a^b)^c = a^{b \cdot c} \quad (2.3)$$

Also, e is a constant number, and we define the number e to be

$$e \approx 2.718.$$

It is common to use the number e as the base for exponentials. The number e plays the same role as the constant a in the equations above:

$$e^b \cdot e^c = e^{b+c} \quad (2.4)$$

$$\frac{e^b}{e^c} = e^{b-c} \quad (2.5)$$

$$(e^b)^c = e^{b \cdot c} \quad (2.6)$$

1. Determine an approximation for the value of each expression below. Your approximation should be to the nearest 0.01.

(a) 2^3

(b) 2.5^2

(c) $2.7^{1.5}$

(d) $2.718^{2.1}$

(e) $2.7183^{2.44}$

116 Name:

Preclass Work - Finish Before Class Begins

1. A bank manager is considering the impact of different terms for an account that offers compounded interest. She assumes that the interest rate is a constant annual one percent rate and then checks to see what happens for different lengths of time between compounding. Assume that one dollar is initially deposited.
 - (a) Determine the amount of money in the account after one hundred years, if the interest is compounded yearly.
 - (b) Determine the amount of money in the account after 100 years, if the interest is compounded once every six months.
 - (c) Determine the amount of money in the account after 100 years, if the interest is compounded once a month.
 - (d) Determine the amount of money in the account after 100 years, if the interest is compounded once a day.
 - (e) Determine the amount of money in the account after 100 years, if the interest is compounded twice a day.
 - (f) What is happening to the balance as the number of terms increases?

2. Generalize the value found on the previous page.
- (a) Determine a formula for the balance for 1% annual interest after 100 years if the interest is compounded n times per year.

- (b) Make a substitution, $u = 100n$, in the previous expression. Write out the expression for the balance as a function of u .

- (c) Determine the values of the balance for the following values of u .

u	balance
1	
2	
12	
365	
1000	

- (d) What is the value approaching as u gets large? This is a number that occurs in many situations, and we do not want to write it out every time we use it, so we use the symbol e as a form of short hand notation.

$$e \approx$$

3. When interest is compounded continuously, the balance is determined using the function

$$\text{Balance}(t) = Pe^{rt},$$

where P is the initial balance, r is the annual interest rate, and t is the time in years.

- (a) Sketch a plot of the balance over time if 1\$ is deposited with a rate of 1%.

- (b) What happens to the balance as time increases?

- (c) What would happen to the graph if you make r bigger? What if r is smaller?

4. As radioactive isotopes decay, the amount of isotope in a sample decreases. If the decay rate of an isotope is r then the amount of an isotope in a sample is expressed using the function

$$\text{Amount}(t) = Ae^{-rt},$$

where t is measured in years.

- (a) What is the physical interpretation of the constant A ?
- (b) If a sample of a radioactive substance initially contains 3 grams, and the radioactive decay is 0.00004, sketch a plot of the amount of the substance in the sample over time.
- (c) What happens to the amount of the radioactive substance in the sample as time increases?
- (d) What would happen to the graph if you make r bigger? What if r is smaller?

5. Simplify each of the following expressions.

(a) $(e^{4.22})^3$

(b) $e^{3.2} \cdot e^{1.8}$

(c) $e^{9.33} \cdot e^t$

(d) $(e^{4.19} \cdot e^{2t})^2$

(e) $(e^{1.9})^t$

(f) $(e^r)^t = \left(\frac{1}{2}\right)^t$

6. We look at one important property of the logarithm. In particular we want to examine ways to write the expression $e^a \cdot e^b$. For each question below, solve the relationships as requested.

- (a) Use the properties of the exponential to represent the product $e^a \cdot e^b$ as a single exponential.

Write the
result as
 $e^a \cdot e^b =$
 $e^\#$

$$e^a \cdot e^b = \quad (2.7)$$

- (b) If $e^a = x$ substitute this value for e^a into the left hand side of equation 2.7.

- (c) If $e^b = y$ substitute this value for e^b into the left hand side of equation 2.7.

- (d) Take the logarithm of both sides of your current form of the expression and simplify the expression.

- (e) Solve $e^a = x$ for a and solve $e^b = y$ for b in the previous expression and substitute the result into the expression.

- (f) What does this imply about the expression

$$\ln(x \cdot y) = \quad .$$

7. We now examine another property with respect to raising a number to a power. In particular we look at the logarithm of x^r where r is a constant.

(a) We can use the algebraic rule $x^r = (x)^r$. Substitute the identity $x = e^{\ln(x)}$ to rewrite the left side of the expression below.

$$x^r = \quad (2.8)$$

(b) Expand and simplify the exponent in the right hand side of the expression.

(c) Take the logarithm of both sides of the expression. Do not simplify the left hand side of the expression, but simplify the right hand side of the expression.

(d) What does this imply about the expression

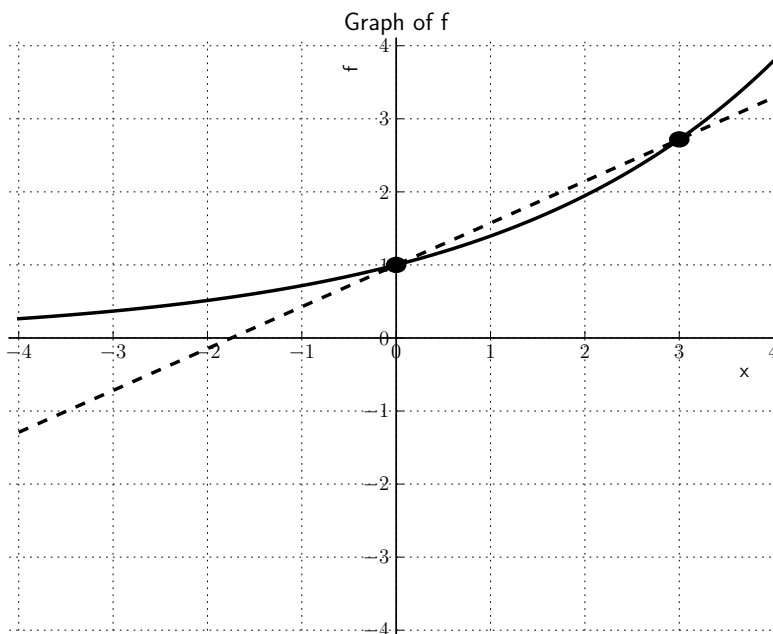
$$\ln(x^r) = \quad .$$

1. Briefly state two ideas from today's class.

-
-

2. The average rate of change for a function from $x = a$ to $x = b$ is defined to be

$$\text{Avg. Rate of Change} = \frac{f(b) - f(a)}{b - a}.$$



Visually, it can be thought of as the slope of the line that goes through the graph of the function at two points, $(a, f(a))$ and $(b, f(b))$.

- (a) The population of a colony of bacteria is two times its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (b) The population of a colony of bacteria is three times its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (c) The population of a colony of bacteria is 2.5 its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (d) The population of a colony of bacteria is 2.7 its previous population every one hour. Determine the average rate of change from $a = 0$ to the times given in the table below.

b	1	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$
Avg Rate Change						

What number is the average rate of change approaching as b gets close to zero?

- (e) Determine a value so that if a population multiplies its population by that number every hour the average rate of change approaches 1 from $a = 0$ and b gets close to zero.

1. A computer virus is constructed that will infect two new computer systems each day and then wipe the disk drive of its current computer clean. The virus is installed on one computer on 1 January.
 - (a) How many new computers will it infect on 2 January?

 - (b) How many new computers will it infect on 3 January?

 - (c) How many new computers will it infect on 4 January?

2. A computer virus is constructed that will infect two new computer systems each day and then wipe the disk drive of its current computer clean. The virus is installed on a computer but it is not clear when it was first installed.
 - (a) It is estimated that 32 computers were infected on a given day. How many days beforehand was it installed on the first computer?

 - (b) It is estimated that 64 computers were infected on a given day. How many days beforehand was it installed on the first computer?

 - (c) It is estimated that 512 computers were infected on a given day. How many days beforehand was it installed on the first computer?

3. Make a sketch of the number line. Mark zero on your number line. Indicate the locations of 1, 1.5, -0.5, and 9.5. Try to make the points consistent with the relative distances between the points.

128 Name:

Preclass Work - Finish Before Class Begins

1. The approximate distances in kilometers from the earth to various destinations is given in the table below. Use this information to answer each of the questions below.

Destination	Distance
The moon	3.84×10^5 km
Mars	5.46×10^7 km
Saturn	1.28×10^9 km
Proxima Centauri (Closest star)	4.01×10^{13} km
Center of the Milky Way	2.83×10^{17} km

- (a) Make a sketch of a number line with the distances for each destination marked on the number line.

*Try to keep
the relevant
distances
between
points
consistent.*

- (b) For each distance in the table above what is the approximate power of ten for the distance? For example, if a distance is 1×10^4 km it is a power of 4. Sketch a number line and indicate the powers of ten on the number line.

The approximate average lengths of various items are given in the table below. Use this information to answer each of the questions below.

Item	Size
Person	1.75 m
Finger	9.2×10^{-2} m
DNA	5.0×10^{-2} m
Hair (Width)	9.0×10^{-5} m
Bacteria	0.8×10^{-7} m

- (a) Make a sketch of a number line with the lengths for each item marked on the number line.

*Try to keep
the relevant
distances
between
points
consistent.*

- (b) For each item in the table above what is the approximate power of ten for the length? For example, if a length is 1×10^{-4} m it is a power of -4. Sketch a number line and indicate the powers of ten on the number line.

2. In one cycle a type of mayfly lays eggs and on average three females survive to lay more eggs. One of the mayfly are introduced to a stream for the first time and lays eggs.
- (a) Draw a tree diagram of the individuals from each cycle that survive starting with the original mother. Include three cycles in your diagram. The node for the first female mayfly should be at the top center of the space, and then the generations that follow should be below the previous generations in the diagram.
 - (b) On the right side of the tree diagram indicate the total number of female mayflies that survive each cycle.
 - (c) On the left side of the tree diagram indicate the corresponding cycle with the first cycle labeled as cycle 0 (zero), and the second cycle is 1 (one).
 - (d) It is estimated that there are 2187 female mayfly in the river. How many cycles have there been?
 - (e) It is estimated that there are 19683 female mayfly in the river. How many cycles have there been?
 - (f) If the number of female mayfly, f , is given by $f = 3^n$ where n is the number of cycles, solve the equation for n . (What is this equation called in relation to the original function?)

3. Mosquitoes lay between 50 to 200 eggs each cycle. Assume that for a given species in a particular area roughly 34 eggs hatch and survive to be female adults and lay eggs.
- (a) Assume that in the spring there is one surviving female mosquito starting with cycle 0. Make a table to indicate how many mosquitoes there will be from cycle 0 to cycle 4.
- (b) Using your table, if it is estimated that there are 39304 female mosquitoes what cycle in the season is it?
- (c) Using your table, if it is estimated that there are 1,336,336 female mosquitoes what cycle in the season is it?
- (d) If the number of female mosquitoes, f , is given by $f = 34^n$ where n is the number of cycles, solve the equation for n . (What is this equation called in relation to the original function?)

1. Briefly state two ideas from today's class.

•

•

2. Determine the value of x in each expression below. Any approximations should be to at least two decimal places.

(a) $\log_5(2x + 1) = 9$

(b) $\log_7(x - 1) + \log_7(x + 1) = 2$

(c) $3^{\log_3(x+1)} = 4$

(d) $\log_8(8^{5-2x}) = 9$

(e) $\log_{10}(2x + 1) = \log_3(4.1)$

3. Use the natural logarithm to determine the value of x in each expression below. Any approximations should be to at least two decimal places.

(a) $3.5^{x+1} = 2^{x-1}$

(b) $10 = e^{-2x}$

(c) $1 = 2^{3x} \cdot 4^{8x-1}$

(d) $5 = e^{3x-1}$

(e) $e^{9x} = 2^{9x}$

Name:

1. Determine an approximation for the numerical value of each number below. Also determine the natural logarithm of each of the following values. Express the numbers to two decimal places.

(a) e

(b) e^2

(c) $e^3 \cdot e^{-4}$

(d) $e^{-2} \cdot e^{-3}$

138 Name:

Preclass Work - Finish Before Class Begins

1. Determine the natural logarithm of each value below, and use the properties of logarithms to express the value as a sum or differences of logarithms.

(a) $a \cdot b \cdot c$

(b) $a \cdot b^2 \cdot c \cdot d^3$

(c) $\frac{(x-1) \cdot (x+3)}{(x-2)}$

(d) $\frac{(x-4) \cdot (x+2)^3 \cdot (x+4)}{(x-9)^2}$

2. A computer virus is constructed that will infect two new computer systems each day and then wipe the disk drive of its current computer clean. The virus is installed on one computer on day 0 (zero).

- (a) Determine the formula that gives the number of computers infected on the number of days since the first computer is infected.

$$\text{Number}(t) =$$

- (b) Make a rough sketch of the number of new computers infected as a function of time.

- (c) Solve the equation above for t . That is determine a formula for t given the number of newly infected computers.

- (d) It is determined that 524288 new computers were infected on a given day. How long ago was the first virus installed?

- (e) It is determined that 67108864 new computers were infected on a given day. How long ago was the first virus installed?

3. Radon 222 has a half life of 3.8 days. It is estimated that there is 0.2g of radon 222 in a basement. Assume that no more radon enters the basement after a treatment is applied, and the area is not ventilated.
- (a) Determine a formula for the amount of radon in the basement at a given time, in days, from when the treatment is applied.

$$\text{Amount}(t) =$$

- (b) Make a rough sketch of the amount of Radon 222 in the basement as a function of time.
- (c) Solve the equation for t . That is determine a formula for t given the amount of radon in the basement.
- (d) How long will it take before the amount of radon is down to one tenth the original amount?

4. A patient has a growth, and a treatment is applied that is estimated to reduce the size of the growth by ten percent each week.

(a) Determine a formula for the size of the growth at a given time, in days, from when the treatment is applied.

$$\text{Size}(t) =$$

(b) Make a rough sketch of the size of the growth as a function of time.

(c) Solve the equation for t . That is determine a formula for t given the size of the growth.

(d) How long will it take before the growth is down to one tenth the original size?

1. Briefly state two ideas from today's class.

-

-

2. Determine the value of x in each expression below:

(a) $e^{2x} - 2e^x - 3 = 0$.

(b) $e^{2x} - 4e^x + 4 = 0$.

(c) $e^{2x} - e^x + 6 = 0$.

(d) $e^{2x} - e^x - 12 = 0$.

(e) $e^{2x} - e^x + 4 = 0$.

3. A bank offers 1.5% annual interest compounded weekly (assume 52 weeks in a year). How long will it take for the balance to double?
4. A bank offers 1.5% annual interest compounded monthly. How long will it take for the balance to double?
5. A compound is created that decays over time. It takes four years until half of the compound decays in a sample. How long will it take for 80% of the compound to decay?
6. A species of plant produces two hundred seeds each year, but on average only 5% of the seeds germinate and grow in to plants that produce seeds. A survey is done in a large area and it is estimated that the area contains fifty plants. How many plants are expected to be present after 3 years? Determine a formula to estimate the number of plants at any year, t , in the future.

Logarithmic functions satisfy the algebraic properties given below. In each example it is assumed that a , b , and r are constants.

$$\ln(a \cdot b) = \ln(a) + \ln(b) \quad (2.9)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) \quad (2.10)$$

$$\ln(a^r) = r \ln(a) \quad (2.11)$$

$$\log_{10}(a \cdot b) = \log_{10}(a) + \log_{10}(b) \quad (2.12)$$

$$\log_{10}\left(\frac{a}{b}\right) = \log_{10}(a) - \log_{10}(b) \quad (2.13)$$

$$\log_{10}(a^r) = r \log_{10}(a) \quad (2.14)$$

Name:

Preclass Work - Finish Before Class Begins 147

1. Solve for the variable x in each expression below.

(a) $\ln(x - 1) = 5$

(b) $e^{3x+1} = 5$

(c) $\log_4(x + 1) = 2$

(d) $3^{-x+1} = 4$

148 Name:

Preclass Work - Finish Before Class Begins

1. Determine the value of x or t as appropriate in each of the following expressions.
(Justify the steps that you make.)

(a) $10 = 20(1 - e^{-3t})$

(b) $\frac{1}{10} = \frac{1}{\sqrt{12\pi}}e^{-x^2/6}$

(c) $xe^{-x} + e^{-x} = 0$

(d) $e^t + 2 - e^{-t} = 0$

(e) $e^{-x^2} = 3 \cdot 4^{2x-1}$

2. The goal is to determine the value of x that satisfies the equation

$$\log_4(x) = 3 + \log_8(x).$$

- (a) First focus on the left hand side of the equation.
 - (i) Define a new variable, u , by setting u equal to the left hand side of the equation.
 - (ii) Exponentiate both sides of the equation using a clever choice for the base so that there will not be any logarithms in the equation. Simplify the result.
 - (iii) Take the natural logarithm of both sides and solve for u .
- (b) Now focus on the right hand side of the equation.
 - (i) Define a new variable, v , by setting v equal to the logarithm in the right hand side of the equation.
 - (ii) Exponentiate both sides of the equation using a clever choice for the base so that there will not be any logarithms in the equation. Simplify the result.
 - (iii) Take the natural logarithm of both sides and solve for v .
- (c) Substitute your value for u into the left hand side and the value for v into the right hand side.
- (d) Solve the new equation for x .

3. A population of bacteria doubles every four hours. Determine a function that gives the number of individuals in the population for a given time.
- (a) Express the given information in function form. (Assume the population is given by $P(t)$.)

 - (b) The assumption is that the population is modeled as an exponential function. Determine the general form of the function. (Why would an exponential function be used in this instance?)

 - (c) Using the function above express the given information as a mathematical equation.

 - (d) Which variable can you solve for? Identify the variable and solve for it.

 - (e) What is the general form for the function that models the population?

4. Carbon 14 has a half life of 5,730 years. How long will it take for a sample to decay to two thirds of its original value?
- (a) Express the given information in function form. (Assume the amount is given by $C(t)$.)
- (b) The assumption is that the amount is modeled as an exponential function. Determine the general form of the function. (Why would an exponential function be used in this instance?)
- (c) Using the function above express the given information as a mathematical equation.
- (d) Which variable can you solve for? Identify the variable and solve for it.
- (e) What is the general form for the function that models the amount of material?

1. Briefly state two ideas from today's class.

-

-

2. Determine the value of x in each expression below.

(a) $3^x = 4^{x+1}$

(b) $3^x = 9 \cdot 4^{x+1}$

(c) $2^x = 9^{5x-1} \cdot 4^{x+1}$

(d) $14^{3x+4} = 10^{9x/2} \cdot 20^{6x-1}$

(e) $3^{x^2} = 3 \cdot 8^x$

Name:

Preclass Work - Finish Before Class Begins 157

1. Solve for the variable x in each expression below.

(a) $e^{4x-1} = 8$

(b) $\ln(2x + 1) = -4$

(c) $3^{2x+1} = e^{4x}$

(d) $8^{3x+2} = 6e^{4x}$

158 Name:

Preclass Work - Finish Before Class Begins

1. The number of animals in a population of a small mammals follows a logistic function,

$$P(t) = \frac{10,000}{1 + e^{-\frac{1}{2}t}},$$

where t is the number of years since the initial observation.

- (a) What is the initial population?

- (b) How many animals will the population approach after a very long time?

- (c) How long will it take for the number of animals to reach 75% of its long term value?

- (d) How long will it take for the number of animals to reach 80% of its long term value?

2. A spill occurs and a chemical is introduced into a lake. The amount of chemical decays over time.
- (a) In this case an exponential function should be used. Why?
 - (b) Should the rate, r , used in the exponential function be positive or negative? (Briefly explain how you arrive at your conclusion.)
 - (c) Write the general form for the equation.
 - (d) At some time after the spill occurs it is estimated that there are 4,000 kg of the chemical in the lake. A month later it is estimated that there is 3,500 kg in the lake. What is the half life of the chemical?
 - (i) Write out the equations that result from the given information.
 - (ii) Use the two equations to solve for the value of the decay rate.
 - (iii) Determine the half life of the chemical.

3. A population of bacteria doubles every four hours. How long does it take for the population to triple?
- (a) Is this decay or growth? Make a (very) rough sketch of the graph of the function you expect to get.
 - (b) Write out the information that is given to you.
 - (c) Write out a general form of a function that will model this situation.
 - (d) Use the given information and the function above to construct a system of equations.
 - (e) Solve the system of equations for the constants necessary to solve the problem.
 - (f) How long will it take for the population to triple?

*Use the
same steps
used in the
previous
question.*

4. Carbon 14 has a half life of 5,730 years. How long will it take for a sample to decay to two thirds of its original value?

1. Briefly state two ideas from today's class.

-

-

2. The number of animals in a population is approximated using a logistic function,

$$P(t) = \frac{80,000}{1 + 10e^{-\frac{1}{10}t}}.$$

- (a) Make a sketch of the function for t going from 0 to 50.
- (b) How long will it take for the population to double with respect to its original population?
- (c) How many animals does the population approach in the long run? (i.e. what happens to the population after a very long time?)

Chapter 3

Angle Measurement

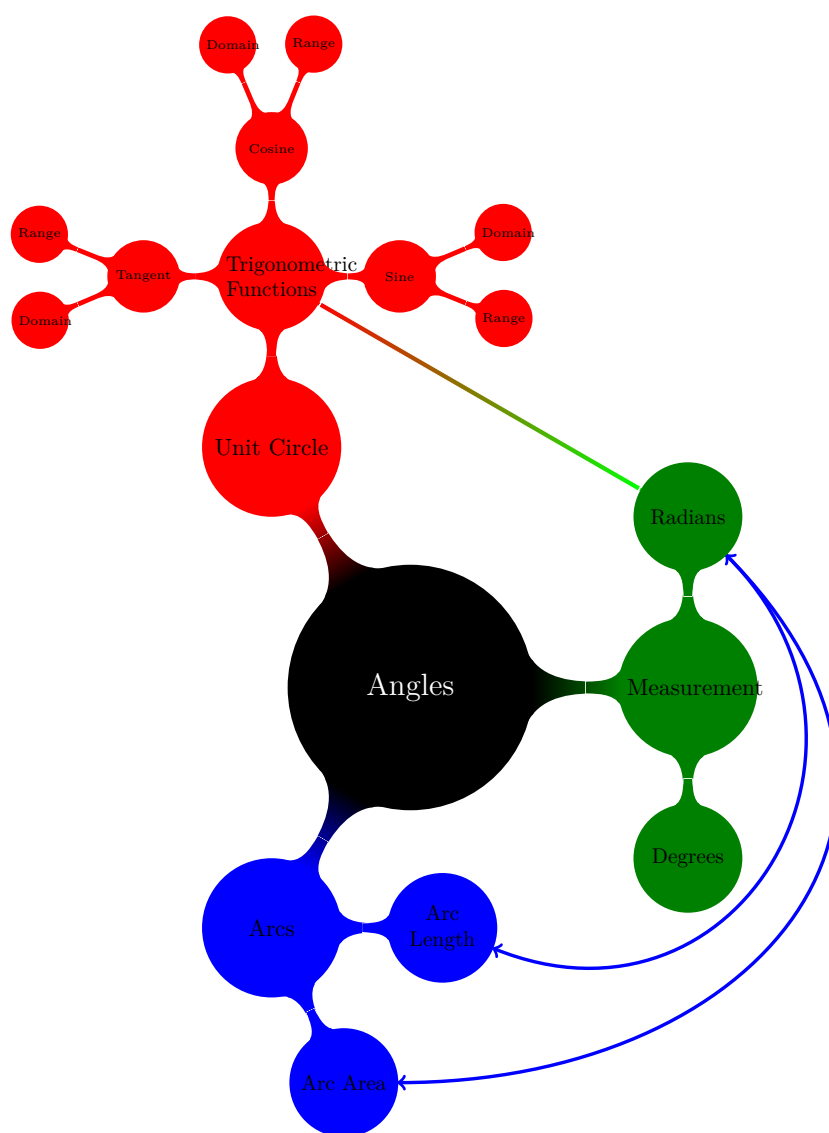


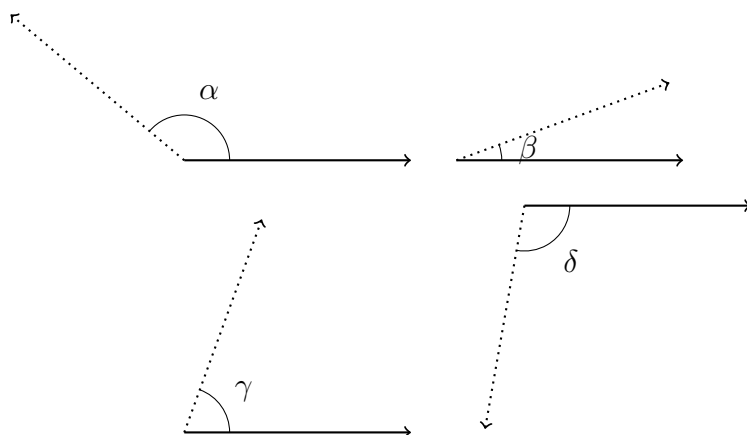
Figure 3.1: Topics for the section on angles.

1. A circle has a radius of r meters. Answer each of the following questions:
 - (a) Determine the lengths described below.
 - (i) What is the circumference of the whole circle?
 - (ii) What is the length around half of the circle?
 - (iii) What is the length around one-fourth of the circle?
 - (b) Determine the areas of the circle described below.
 - (i) What is the area of the whole circle?
 - (ii) What is the area of half of the circle?
 - (iii) What is the area of one-fourth of the circle?
 - (c) How many degrees are there in a full circle? Where did this number come from?
 - (d) Make a sketch of a circle centered at the origin. Include the x and y -axes. Add the ray from the origin that forms a 45 degree angle with the positive x axis.

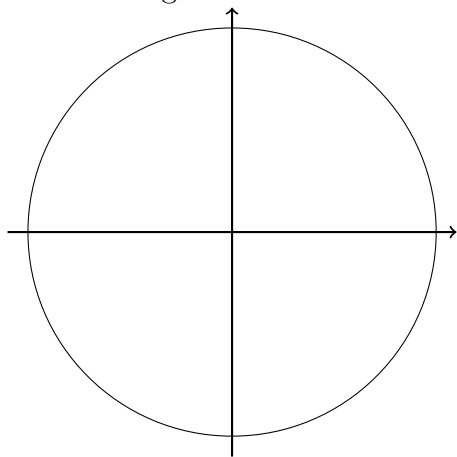
168 Name:

Preclass Work - Finish Before Class Begins

1. For each of the angles below, print the angles in order from smallest to largest. In each picture the solid line is the initial side, and the dotted line is the terminal side.



2. Use the diagram below to make the points indicated in the descriptions below.



- Mark and label a point, P , where the ray from the origin to the point form an angle of π radians.
- Mark and label a point, Q , where the ray from the origin to the point form an angle of 2π radians.
- Mark and label a point, R , where the ray from the origin to the point form an angle of $\frac{\pi}{2}$ radians.
- Mark and label a point, S , where the ray from the origin to the point form an angle of $\frac{\pi}{4}$ radians.
- Mark and label a point, T , where the ray from the origin to the point form an angle of 0 radians.
- Mark and label a point, U , where the ray from the origin to the point form

an angle of $\frac{3\pi}{4}$ radians.

3. Make a sketch of a circle of radius r , and mark the radius and circumference of the circle.

(a) What is the general relationship between the radius and the circumference?

- (b) Mark a sector on your circle above whose angle is one half of the angle needed to make one complete turn around the circle. What is the length of the sector of the circle?

*This should
be a
function of
 r .*

- (c) Mark a sector on your circle above whose angle is one third of the angle needed to make one complete turn around the circle. What is the length of the sector?

*This should
be a
function of
 r .*

- (d) If the angle of a sector is a fraction, p , of one whole turn around the circle, what is the length of the sector? (If $p = 0.5$ then it represents one half of a full turn around the circle.)

*This should
be a
function of
 r and p .*

4. From the previous problem you should have a general formula that relates the length of the sector with radius r given the fraction, p , that its angle is of one complete turn around a circle.

(a) Rewrite your expression, and label the distance along the sector as s .

- (b) Divide both sides of your formula by the radius, and you should have an expression for

$$\frac{s}{r} =$$

- (c) The value on the right side of your expression is the definition of radian measure for an angle. In one sentence explain the meaning of the value on the right side of the expression.

5. A hare is placed on a track that is a circle with radius 10m. The hare moves around the track in the counter-clockwise direction.

(a) Make a sketch of the track below.

(b) Determine the angle that the hare moves around after moving 1m.

*All angle
measures
should be in
radians.*

(c) Determine the angle that the hare moves around after moving 10m.

(d) Mark and annotate the sector formed after the hare moves 10m. What is the angle of the sector, and what is the area of the sector?

(e) Mark and annotate the sector formed after the hare moves 2m. What is the angle of the sector, and what is the area of the sector?

(f) Mark and annotate the sector formed after the hare moves 15m. What is the angle of the sector, and what is the area of the sector?

6. A turtle and a hare are placed at the same start point, and they move around a circle of radius 10m. The hare moves around the circle counter-clockwise at 1m per minute. The turtle moves around the circle counter-clockwise at 0.1m per minute.

(a) After one hour where is the hare on the circle?

(b) After one hour what angle has the hare moved around the circle?

(c) After one hour where is the turtle on the circle?

(d) After one hour what angle has the turtle moved around the circle?

(e) How long will it take until the hare and the turtle are at the start point at the same time?

1. Briefly state two ideas from today's class.

-
-

2. The following questions refer to the measure of an angle in radians.

- (a) How many radians are there in one complete turn around a circle?
- (b) How many radians are there in one half of one complete turn around a circle?
- (c) How many radians are there in one fourth of one complete turn around a circle?
- (d) How many radians are there in one third of one complete turn around a circle?
- (e) If an angle is measured as being 45 degrees, how many radians is it?
- (f) If an angle is measured as being 120 degrees, how many radians is it?

1. When viewed above the north pole of the Sun, the earth appears to move around the sun in a counter-clockwise direction. The path can be roughly approximated as a circle, and its coordinate at any time is (x, y) . It takes one year to make one revolution, and assume that the distance from the center of the sun to the earth is one solar unit.



- (a) What distance does the earth traverse in one year?
- (b) What distance does the earth traverse in two years?
- (c) What distance does the earth traverse in ten years?
- (d) What is the largest value that the x -coordinate can be? What is the smallest value that the x -coordinate can be? What is the largest value that the y -coordinate can be? What is the smallest value that the y -coordinate can be?

178 Name:

Preclass Work - Finish Before Class Begins

1. An angle is in the second quadrant, and the sine of the angle is 0.2. Determine the cosine and tangent of the angle.
 - (a) Make a sketch of the unit circle and place an angle somewhere in the second quadrant. Mark the point on the edge of the circle, (x, y) .
 - (b) Will the cosine be positive or negative? (Explain how you came to that conclusion.) Do the same for the tangent.
 - (c) Determine the cosine of the angle using the Pythagorean relationship between the sine and the cosine.
 - (d) Determine the tangent of the angle using the definition of the tangent.

2. A turtle moves around the edge of a circle of radius 1 meter.

- (a) After the turtle moves a distance of 2m what is its position? What quadrant is the point in?

Make a sketch of the unit circle first and estimate its location.

- (b) After the turtle moves a distance of 4m what is its position? What quadrant is the point in?

Make a sketch of the unit circle first and estimate its location.

- (c) After the turtle moves a distance of 6m what is its position? What quadrant is the point in?

Make a sketch of the unit circle first and estimate its location.

3. A turtle moves around the edge of a circle of radius 1 meter.

(a) After it moves a distance of $\frac{\pi}{2}$ meters where is it on the circle?

Make a rough sketch of the circle and indicate its position.

(b) After it moves a distance of π meters where is it on the circle?

(c) After it moves a distance of $\frac{3\pi}{2}$ meters where is it on the circle?

Make a rough sketch of the circle and indicate its position.

(d) After it moves a distance of 2π meters where is it on the circle?

(e) After it moves a distance of 5π meters where is it on the circle?

Make a rough sketch of the circle and indicate its position.

Precalculus - June 18, 2018

4. A turtle moves around the edge of a circle of radius 1 meter.

Make a rough sketch of the circle and indicate its position.

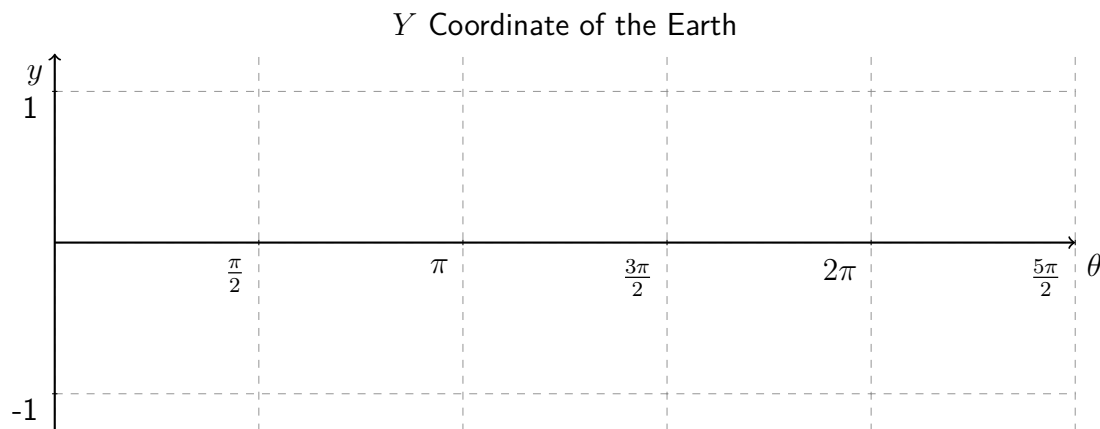
- (a) After it moves a distance of $\frac{\pi}{4}$ meters where is it on the circle?
- (b) As the turtle moves from a distance of 0 meters to $\frac{\pi}{2}$ meters what is happening to the value of its x coordinate. What about its y coordinate?
- (c) As the turtle moves from a distance of $\frac{\pi}{2}$ meters to π meters what is happening to the value of its x coordinate. What about its y coordinate?
- (d) As the turtle moves from a distance of π meters to $\frac{3\pi}{2}$ meters what is happening to the value of its x coordinate. What about its y coordinate?
- (e) As the turtle moves from a distance of $\frac{3\pi}{2}$ meters to 2π meters what is happening to the value of its x coordinate. What about its y coordinate?

5. When viewed above the north pole of the Sun, the earth appears to move around the sun in a counter-clockwise direction. The path can be roughly approximated as a circle. It takes one year to make one revolution, and assume that the distance from the center of the sun to the earth is one solar unit. (See the image in the preclass activity.)

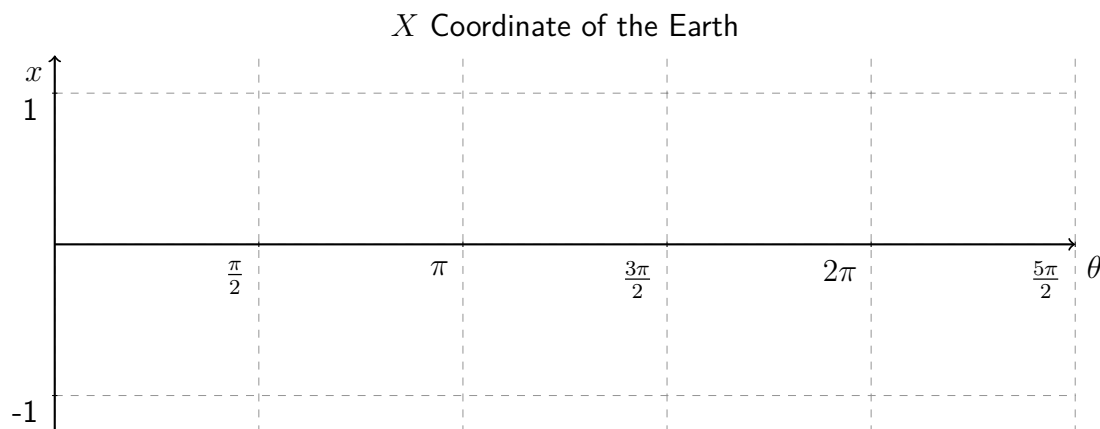
(a) What angle (in radians) does the earth make from the x -axis after 3 months?

(b) What angle (in radians) does the earth make from the x -axis after 6 months?

(c) Make a rough sketch of the earth's y position as a function of the angle.



(d) Make a rough sketch of the earth's x position as a function of the angle.



6. At the annual Plainfield 500π race a tractor will make 250 laps around a circular track. The track has a radius of 1 km. A single tractor will make a trial run by going around the track at 1 km per hour. (It is not a very fast race.) The car starts on the point furthest East and is initially moving to the North.

- (a) How far will the tractor travel in all? Determine the distance traveled as a function of time, and then determine the angle, θ , at any time.

- (b) What are the possible values of the angle, θ ?

- (c) Assuming that the origin is the center of the track, sketch a plot of the tractor's y position as a function of time for the first two laps.

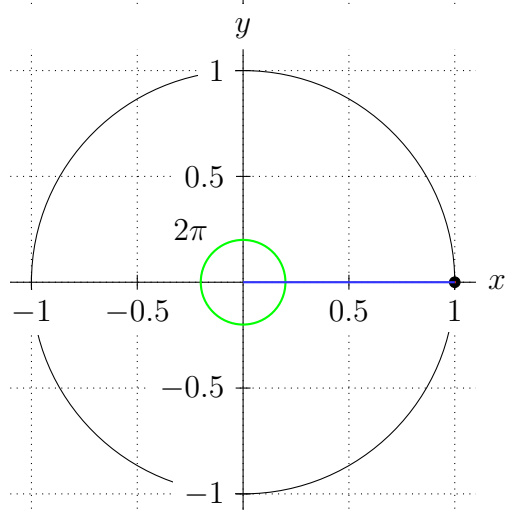
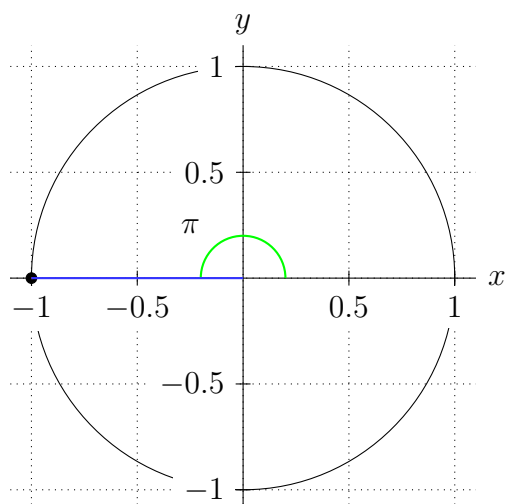
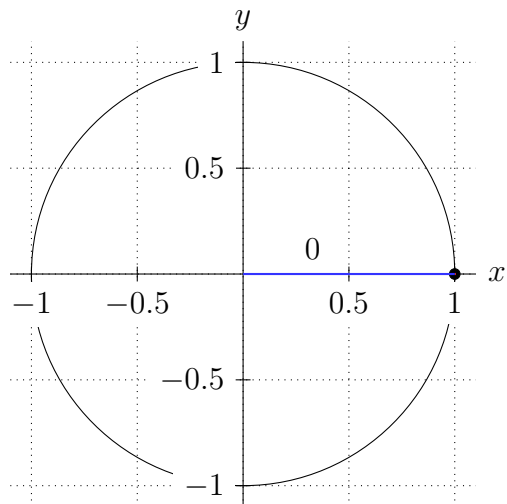
Label the axes and annotate the intercepts.

- (d) Assuming that the origin is the center of the track, sketch a plot of the tractor's x position as a function of time for the first two laps.

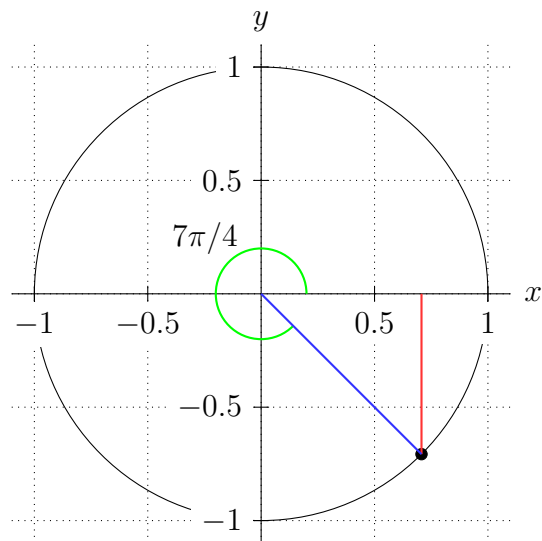
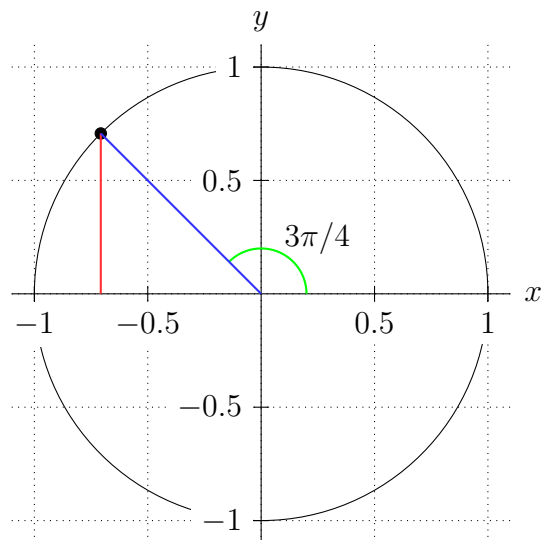
Label the axes and annotate the intercepts.

1. Briefly state two ideas from today's class.
 -
 -
2. Make a sketch of the sine function. Annotate the graph to show the amplitude and period of the function.
3. Make a sketch of the cosine function. Annotate the graph to show the amplitude and period of the function.
4. Make a sketch of the function $f(x) = \sin(\pi x)$. Annotate the graph to show the amplitude and period of the function.
5. Make a sketch of the function $f(x) = \cos(\pi x)$. Annotate the graph to show the amplitude and period of the function.
6. Make a sketch of the function $f(x) = \sin(x) + 2$. Annotate the graph to show the amplitude and period of the function.
7. Make a sketch of the function $f(x) = \cos(x) + 2$. Annotate the graph to show the amplitude and period of the function.
8. Make a sketch of the function $f(x) = 3 \sin(x) + 1$. Annotate the graph to show the amplitude and period of the function.
9. Make a sketch of the function $f(x) = 3 \cos(x) + 1$. Annotate the graph to show the amplitude and period of the function.

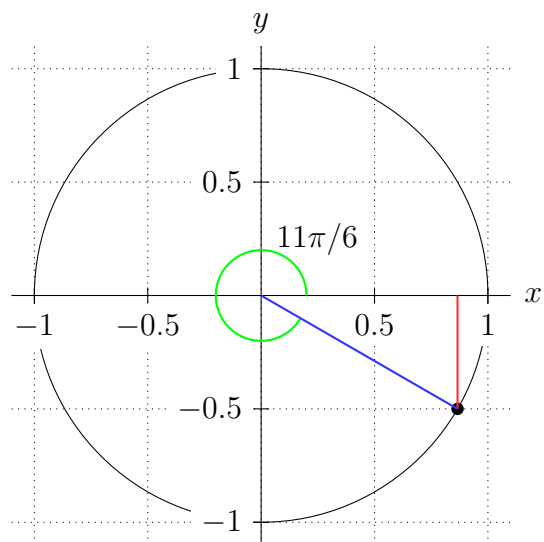
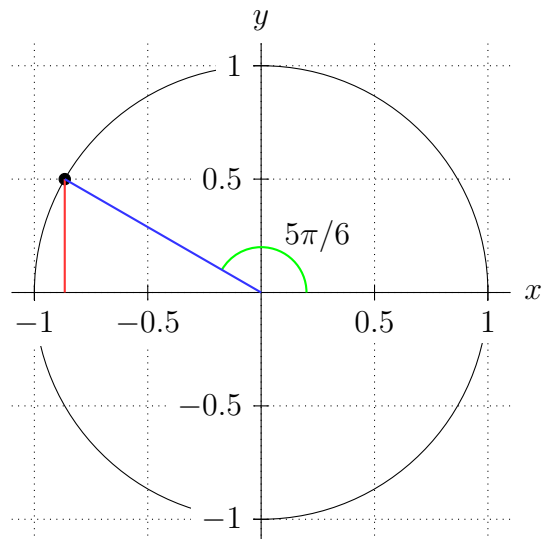
Angles that are a multiple of $\frac{\pi}{2}$ are aligned with the x and y axis. For each plot below determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



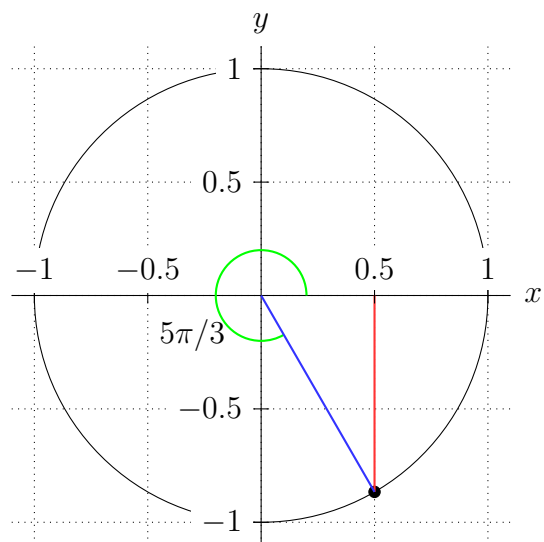
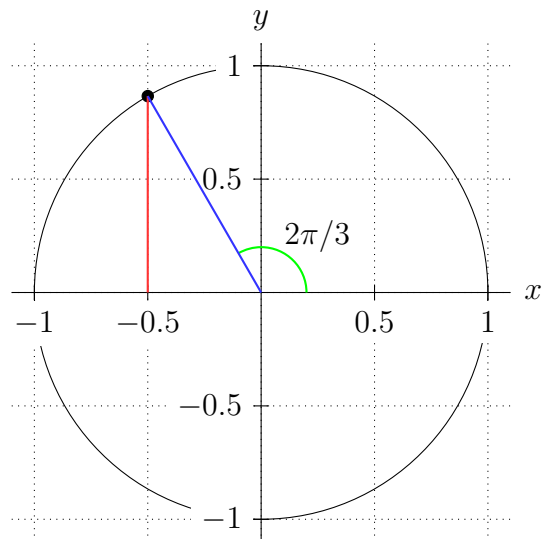
Angles whose reference angles are $\frac{\pi}{4}$ are aligned with the diagonal lines from the origin. In each plot below label and define the reference angle. Also, determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



Angles whose reference angles are $\frac{\pi}{6}$ have y values that are $\pm\frac{1}{2}$. In each plot below label and define the reference angle. Also, determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



Angles whose reference angles are $\frac{\pi}{3}$ have x values that are $\pm\frac{1}{2}$. In each plot below label and define the reference angle. Also, determine and label the sine and cosine of the angle based on the x and y position of the point on the unit circle.



Chapter 4

Trigonometric Functions



Figure 4.1: Topics for the section on trigonometric functions.

1. A circle of radius r is centered at the origin, and a ray originates from the origin at a given angle, θ . The ray passes through the circle at the coordinate (x, y) . A function of θ can be defined in terms of the coordinate.



- (a) A function, sine, is defined to be

$$\sin(\theta) = \frac{y}{r}.$$

Determine the formula for the value of y given r and θ in terms of the sine function.

Solve the equation above for y .

- (b) A function, cosine, is defined to be

$$\cos(\theta) = \frac{x}{r}.$$

Determine the formula for the value of x given r and θ in terms of the cosine function.

Solve the equation above for x .

194 Name:

Preclass Work - Finish Before Class Begins

1. Answer each of the following questions where the given point is $P(2, 4)$.

- (a) Make a sketch of the coordinate plane and include the point $P(2, 4)$. Draw the ray from the origin to the point.

*Label your
axes and
annotate
your plot.*

- (b) Add a circle to your sketch whose center is the origin and goes through the point. Label the angle θ as the angle between the ray and the x -axis.

- (c) What is the radius of the circle? (Add a label to your plot for the radius.)

- (d) Determine the values of the sine and cosine for the angle.

$$\sin(\theta) =$$

$$\cos(\theta) =$$

2. Different points are given below. For each point assume it is on the edge of a circle whose center is at the origin. Determine the radius of the circle as well as the value of the sine and cosine of the angle associated with each point on its circle.

(a) $P(1, 0)$

(b) $P(0, 1)$

(c) $P(-1, 0)$

(d) $P(0, -1)$

(e) $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3. In each item below some information is given about an angle. Use the information to determine the sine, cosine, and tangent of the angle. In each case determine if the sine, cosine, and tangent will increase or decrease if the angle is increased by a small amount.

(a) The angle is in the first quadrant and $\cos(\theta) = 0.3$.

(b) The angle is in the third quadrant and $\sin(\theta) = -0.25$.

(c) The angle is in the second quadrant and $\cos(\theta) = -0.8$.

4. An airplane is observed, and it is flying directly away from an observer in level flight. The observer measures the angle of elevation for the plane at two different times. You wish to determine the height of the plane.

(a) The plane is moving away from the observer. Will the angle increase or decrease?

(b) At the first time the angle of elevation is 20° . At the second time the angle of elevation is 13° . It is estimated that the plane flew a distance of 3 miles between observations. Draw a rough sketch of the situation.

*You should
have two
triangles.*

(c) Determine the number of unknowns and determine how many equations you need.

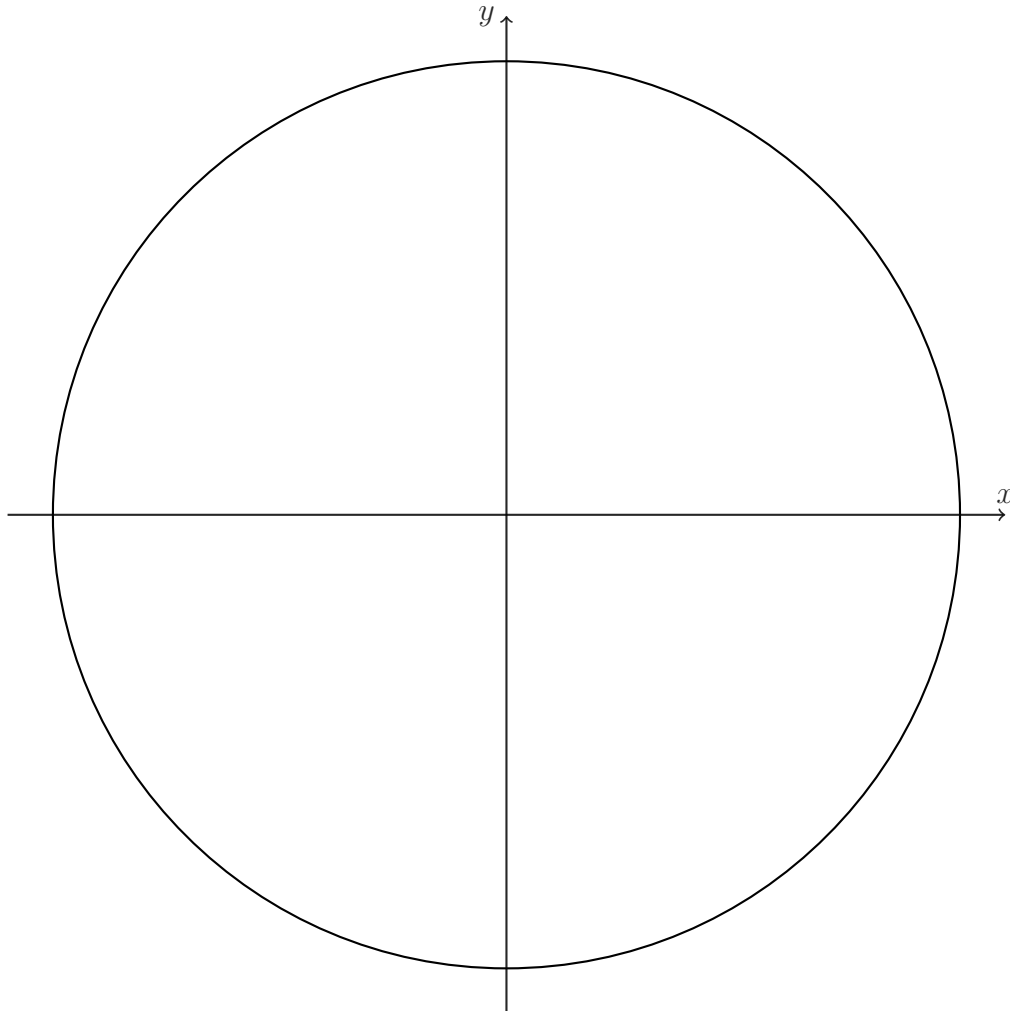
(d) Determine a relevant formula each of the two triangles plus any other equations you may need.

(e) Solve the system of equations to determine the height of the plane.

1. Briefly state two ideas from today's class.

-
-

2. The circle below is centered at the origin and has a radius of one.



- (a) Mark the locations on the circle whose associated angles are 0 , $\pi/4$, $\pi/2$, $3\pi/4$, π , $5\pi/4$, $3\pi/2$, and $7\pi/4$. Determine the coordinates for the points. (Label the points and annotate your plot.)
 - (b) Determine the (x, y) coordinates for each angle.
 - (c) Determine the cosine and sine of each angle.
3. An airplane is observed, and it is flying directly away from an observer in level flight. The observer measures the angle of elevation for the plane at three different times. You wish to determine the height of the plane.
 - (a) The plane is moving away from the observer. Will the angle increase or decrease?

- (b) At the first time the angle of elevation is 20° . At the second time the angle of elevation is 13° . At the third time the angle of elevation is 10° . It is estimated that the plane is flying at a constant speed. The time between each measurement is 20 seconds, and the distance the plane flies between each measurement is $20 \cdot v$ where v is measured in meters per second. Draw a rough sketch of the situation.

*You should
have three
triangles.*

- (c) Determine the number of unknowns and determine how many equations you need. other equations you may need.

- (d) Solve the system of equations to determine the height of the plane.

- Precalculus - June 18, 2018

202 Name:

Preclass Work - Finish Before Class Begins

1. Make a sketch of the unit circle. Indicate the point whose angle coincides with the angle $\theta = \frac{11\pi}{6}$.

(a) Determine the reference angle for θ .

- (b) According to some website, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Assuming this is correct determine the sine of the original angle, θ .

- (c) Determine the cosine of the original angle, θ .

*Do not just
use your
calculator!
Use
reference
angles and
justify your
answer.*

2. Make a sketch of the unit circle. Indicate the point whose angle coincides with the angle $\theta = \frac{13\pi}{12}$.

(a) Determine the reference angle for θ .

- (b) According to some website, $\cos\left(\frac{\pi}{12}\right) = \frac{1}{4}(\sqrt{6} + \sqrt{2})$. Assuming this is correct determine the cosine of the original angle, θ .

*Do not just
use your
calculator!
Use
reference
angles and
justify your
answer.*

- (c) Determine the sine of the original angle, θ .

3. The angle θ is in the second quadrant.

(a) How do you calculate the reference angle given θ ?

Make a sketch of the unit circle and label the angles.

(b) What happens to the sine of the angle if its reference angle is increased and the angle remains in the second quadrant?

(c) What happens to the cosine of the angle if its reference angle is increased and the angle remains in the second quadrant?

4. The angle θ is in the third quadrant.

Make a sketch of the unit circle and label the angles.

- (a) How do you calculate the reference angle given θ ?

- (b) What happens to the sine of the angle if its reference angle is increased and the angle remains in the third quadrant?

- (c) What happens to the cosine of the angle if its reference angle is increased and the angle remains in the third quadrant?

1. Briefly state two ideas from today's class.

•

•

2. (a)

1. An ant starts at the coordinate $P(1, 0)$, and it moves counter-clockwise around a circle of radius one centered at the origin. It moves at a constant 1 meter per minute.

(a) Sketch a plot of the ant's path.

*Label your
axes and
annotate
your plot.*

(b) Sketch a plot of the ant's x -coordinate as a function of time.

*Label your
axes and
annotate
your plot.*

(c) Sketch a plot of the ant's y -coordinate as a function of time.

*Label your
axes and
annotate
your plot.*

210 Name:

Preclass Work - Finish Before Class Begins

1. (a) Make a sketch of the sine function.



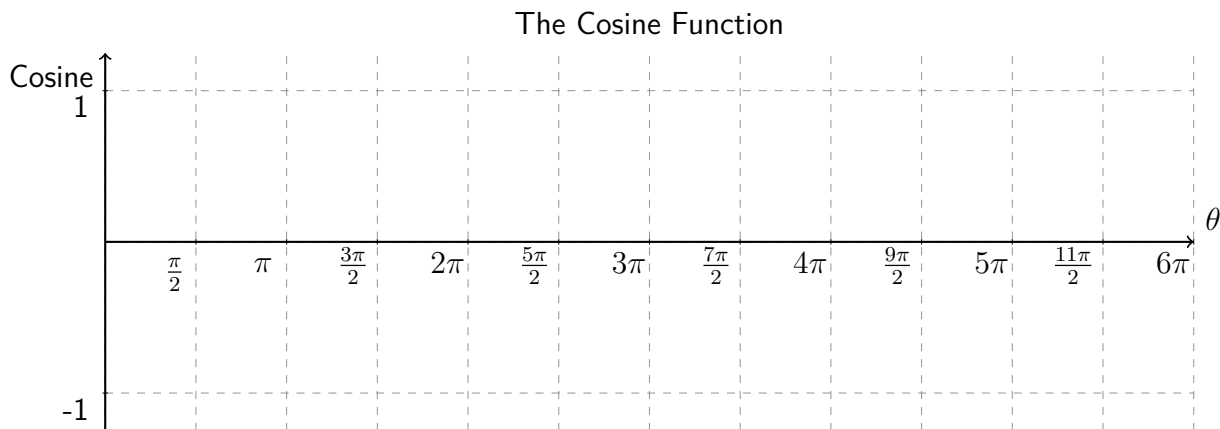
- (b) After what values of θ does the sine function start to repeat itself?

- (c) For what value of a is

$$\sin(\theta) = \sin(\theta - a)?$$

- (d) Is the sine function an invertible function?

- (e) Make a sketch of the cosine function.



- (f) For what values of θ does the cosine function start to repeat itself?

- (g) For what value of a is

$$\cos(\theta) = \cos(\theta - a)?$$

- (h) Is the cosine function an invertible function?

2. What is the relationship between the sine and cosine functions? For each relationship below determine the value and use the unit circle to explain why these relationships should be expected.

(a) Determine a value of a where

$$\cos(\theta) = \sin(\theta + a).$$

(b) Determine a value of a where

$$\cos(\theta) = -\sin(\theta + a).$$

3. Determine values of a , and b so that the function

$$f(\theta) = a \sin(\theta) + b$$

oscillates between 2 and 6.

4. Determine values of a , and b so that the function

$$g(\theta) = a \cos(\theta) + b$$

oscillates between -5 and -3.

5. Determine values of m , and c so that the function

$$f(\theta) = a \sin(mt + c) + b$$

has a minimum when $t = 1$ and has a period of 3.

6. Determine values of m , and c so that the function

$$g(\theta) = a \cos(mt + c) + b$$

has a minimum when $t = 3$ and has a period of 2.

7. Determine values of a , b , m , and c so that the function

$$f(\theta) = a \sin(\theta) + b$$

oscillates between 2 and 6 and has a minimum when $t = 1$ and has a period of 3.

8. Determine values of a , b , m , and c so that the function

$$f(\theta) = a \cos(\theta) + b$$

oscillates between -5 and -3 and has a minimum when $t = 3$ and has a period of 2.

1. Briefly state two ideas from today's class.

•

•

2. (a)

Name:

1. Make a sketch of the sine function.

The Sine Function



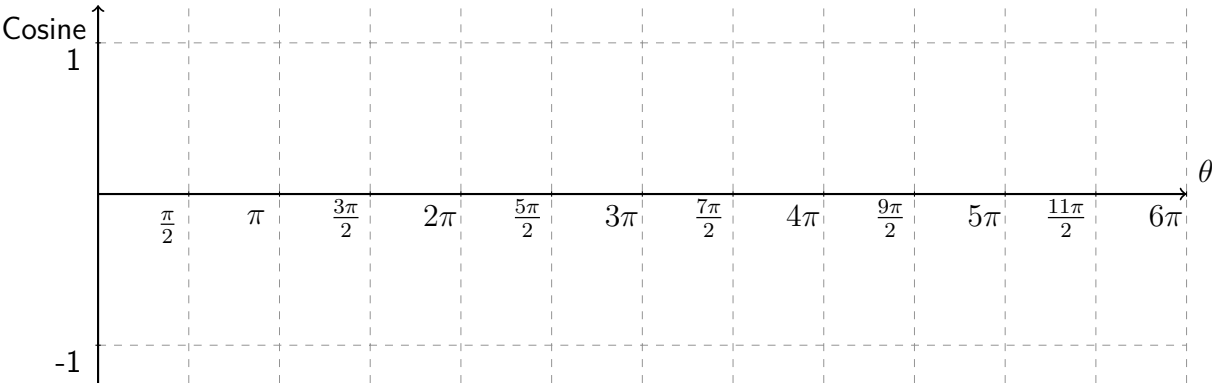
2. Make a sketch of the sine function shifted left $\frac{\pi}{2}$ units.

The Sine Function



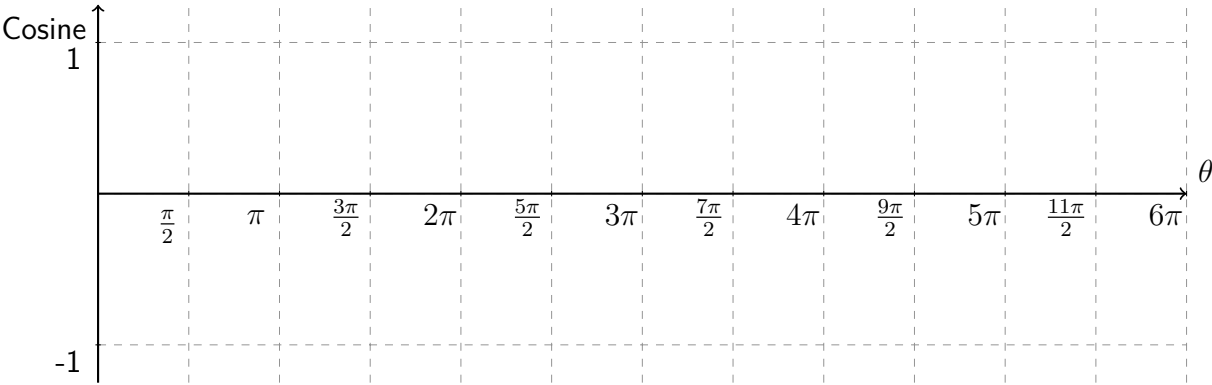
3. Make a sketch of the cosine function.

The Cosine Function



4. Make a sketch of the cosine function shifted left $\frac{\pi}{2}$ units.

The Cosine Function



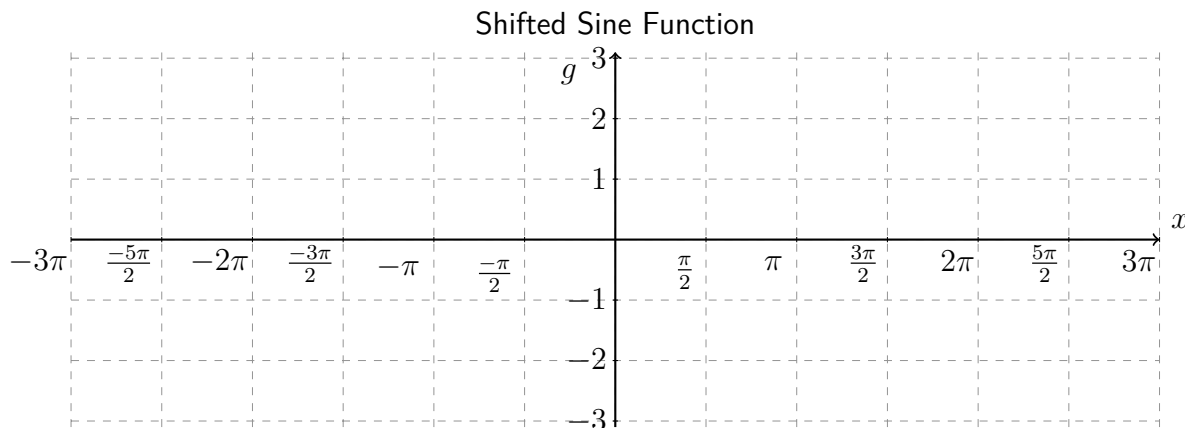
1. A sine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a sine function that is shifted left $\frac{\pi}{2}$ units, oscillates between 2 and -2, and has a period of 2π .



- (b) Determine the formula for the new function

$$f(x) =$$

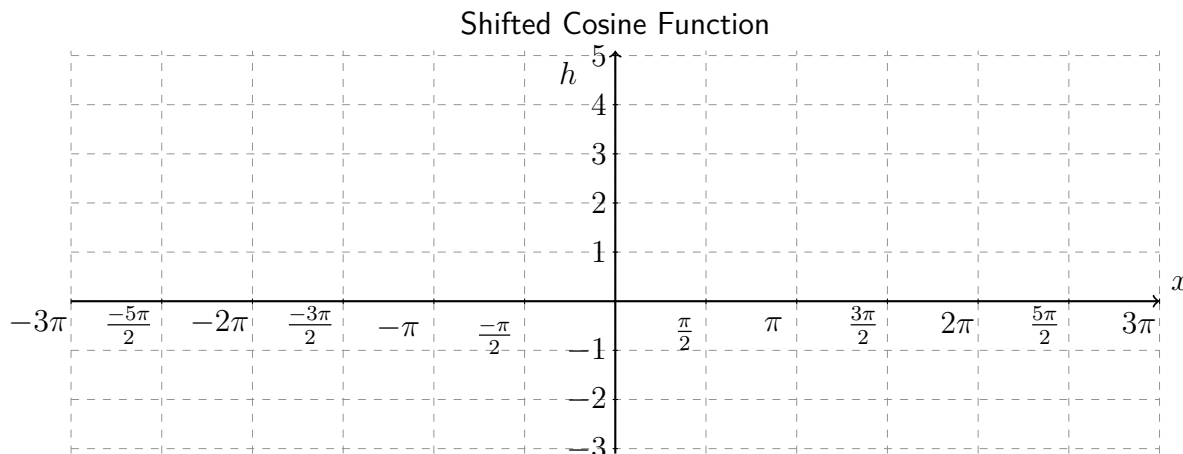
2. A sine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a sine function that is shifted left $\frac{\pi}{2}$ units, oscillates between 3 and -1, and has a period of 2π .



- (b) Determine the formula for the new function

$$g(x) =$$

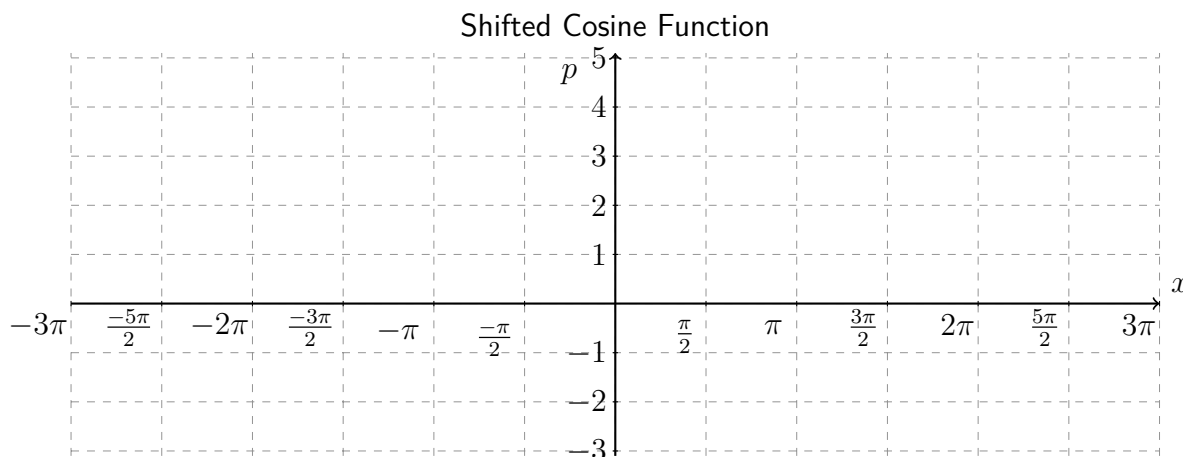
3. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a cosine function that is shifted right $\frac{\pi}{2}$ units, oscillates between 3 and -3, and has a period of 2π .



- (b) Determine the formula for the new function

$$h(x) =$$

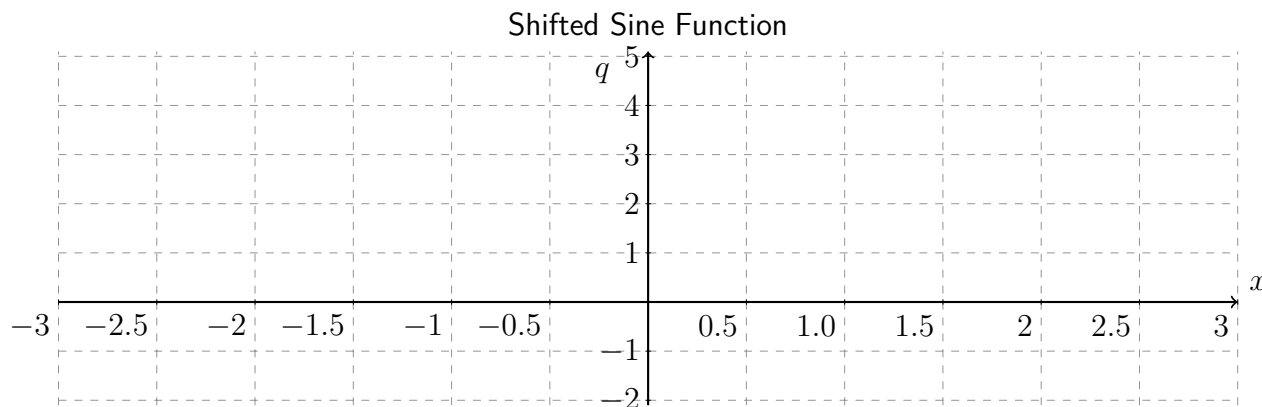
4. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a cosine function that is shifted right $\frac{\pi}{2}$ units, oscillates between 5 and -1, and has a period of 2π .



- (b) Determine the formula for the new function

$$p(x) =$$

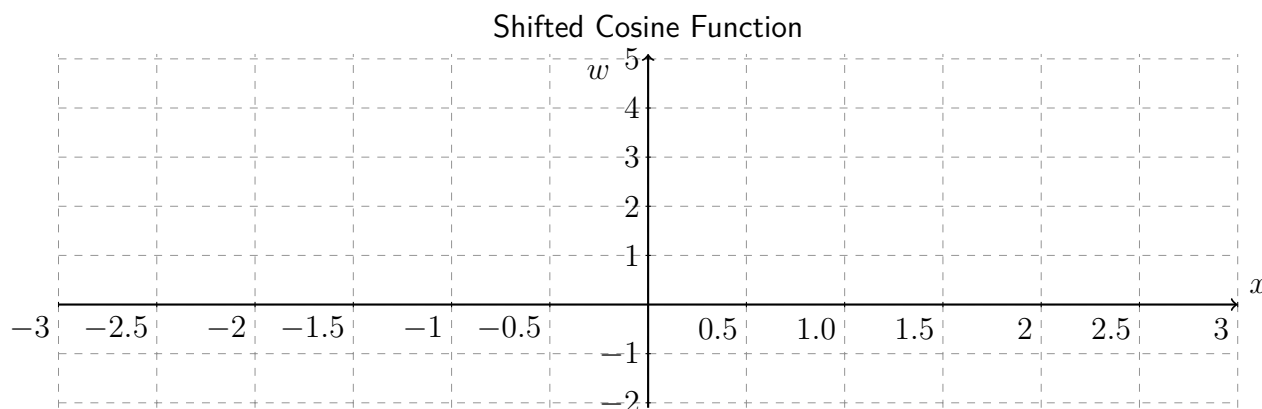
5. A sine function is scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a sine function that has a period of 1 and an amplitude of 1.



- (b) Determine the formula for the new function

$$q(x) =$$

6. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a cosine function that is shifted right 2 units, oscillates between 5 and -1, and has a period of 1.



- (b) Determine the formula for the new function

$$w(x) =$$

7. A cosine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a cosine function that is shifted right 1 units, oscillates between 2 and -1, and has a period of 2.



- (b) Determine the formula for the new function

$$D(x) =$$

8. A sine function is shifted and scaled as described below. Make a sketch of its graph first, and then determine the formula for the new function.
- (a) Make a sketch of a sine function that is shifted left 2 units, oscillates between -1 and -4, and has a period of $\frac{2}{3}$.



- (b) Determine the formula for the new function

$$L(x) =$$

1. Briefly state two ideas from today's class.

•

•

2. (a)

1. A triangle, ABC, has lengths $a = 5$, $b = 4$, and the angle directly across from c is $\gamma = \frac{\pi}{6}$. (The triangle is **not** a right triangle.)
 - (a) Draw a picture of the triangle.

(b) Label the sides and the angle in your picture.

(c) Determine the area of the triangle. (Hint: draw a vertical line that represents the height in your triangle and use the appropriate trigonometric functions to determine the height.)

228 Name:

Preclass Work - Finish Before Class Begins

1. A surveyor sets up a transit 2m above the surface of the ground. The transit is 80m away from the base of a building. The transit is pointing at the top of the building, and its angle of elevation is 35 degrees. How tall is the building?
 - (a) Make a sketch of the situation. (It may take a couple tries!)
 - (b) Indicate and label all of the information that is given and indicate any variables that are not known in your diagram above.
 - (c) Identify the relationships between the variables.
 - (d) How do you plan on solving the problem?
 - (e) Determine the height of the building.

2. Chris Hadfield is in the International Space Station and is 360km above the surface of the earth. He looks down toward the center of the earth and then to the horizon of the earth. He measures an angle of 71 degrees between the two directions. What is the radius of the earth?

(a) Make a sketch of the situation. (It may take a couple tries!)

(b) Indicate and label all of the information that is given and indicate any variables that are not known in your diagram above.

(c) Identify the relationships between the variables.

(d) How do you plan on solving the problem?

(e) Determine the radius of the earth.

1. Briefly state two ideas from today's class.
 -
 -
2. Captain Horatio McCallister is standing on the bridge of his ship. He spies a buoy through his telescope. He estimates that the straight line distance to the buoy is 180m, and the angle of depression is 4 degrees. How far must he sail to get to the buoy?
3. A regular polygon has seven equal sides. The distance from each vertex to the center of the polygon is 0.5m. What is the area of the polygon?
4. A regular polygon with 8 sides is inscribed within a circle of radius 2m. What is the area between the circle and the polygon?
5. A circle of radius 2m is inscribed within a regular polygon with six sides. What is the area between the polygon and the circle?
6. The base of a rectangular box has dimensions 20cm by 15cm, and the height is h cm. The angle between the bottom of the box and the diagonal is 21 degrees. What is the height of the box?

Name:

1. Use the unit circle and the definition of sine and cosine to provide a justification for the identity

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$



Choose a point on the circle, draw the associated triangle, and then use the appropriate definitions.

234 Name:

Preclass Work - Finish Before Class Begins

1. Is the equation

$$(\sec(\theta) - \tan(\theta)) \cdot (\csc(\theta) + 1) = \cot(\theta)$$

true for all θ ? (Fully justify your answer!)

2. Is the equation

$$\frac{1 + \csc(3\beta)}{\sec(3\beta)} - \cot(3\beta) = \cos(3\beta)$$

true for all β ? (Fully justify your answer!)

3. Is the equation

$$\cos^2(4\alpha) = 1 - \sin^2(4\alpha)$$

true for all α ? (Fully justify your answer!)

4. Is the equation

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

true for all θ ? (Fully justify your answer!)

1. Briefly state two ideas from today's class.

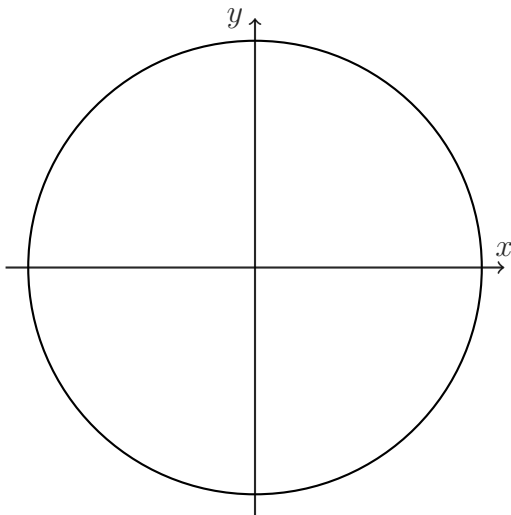
-

-

2. (a)

1. Determine all values of θ that satisfy

$$\sin(\theta) = \frac{\sqrt{2}}{2}.$$



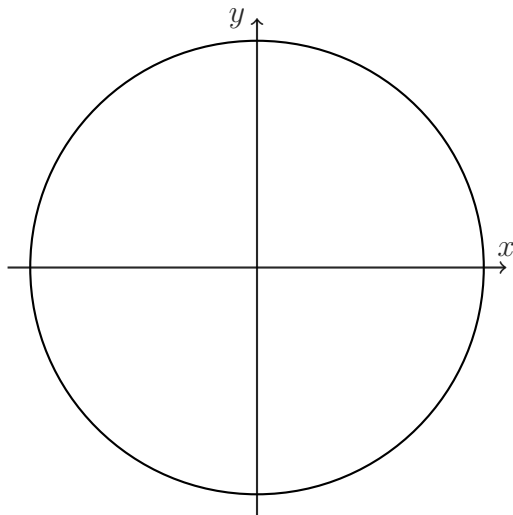
Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

242 Name:

Preclass Work - Finish Before Class Begins

1. Determine all values of θ that satisfy

$$\cos(\theta) = -\frac{\sqrt{2}}{2}.$$



Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

2. Determine all values of x that satisfy

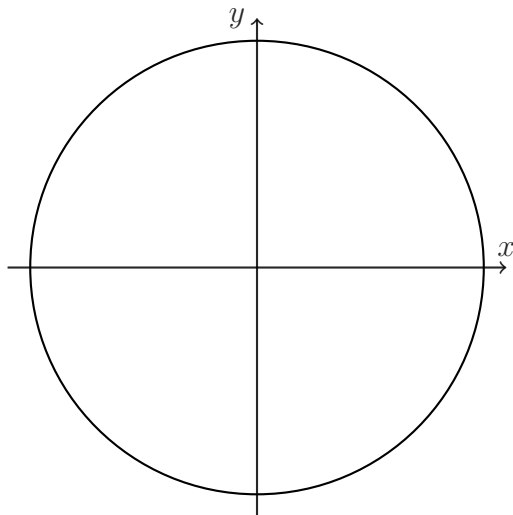
$$\cos(2x + 1) = -\frac{\sqrt{2}}{2}.$$



Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

3. Determine all values of x that satisfy

$$\tan(x^2 + 1) = \frac{\sqrt{3}}{3}.$$



Use the plot of the unit circle above, and sketch the angles that satisfy the equation above. Use the plot to determine the values of the angles in radians.

4. The voltage within a circuit element is given by

$$V(t) = 100 - 50 \cos \left(\frac{3\pi}{120}t + 3\pi \right).$$

What is the minimum voltage and when does it occur? (Make a sketch of the unit circle!)

1. Briefly state two ideas from today's class.

•

•

2. (a)

1. Use the axes below to make a sketch of the graph of $\sin(x)$.



- (a) Is the sine function one-to-one? (Justify your answer!)
- (b) What is the domain for the sine function?
- (c) Define a small part of the domain for which the sine function is one-to-one. (There is not a unique answer - just find one small part.) Make a sketch of the sine function restricted to your domain and explain why it is now one-to-one.

250 Name:

Preclass Work - Finish Before Class Begins

1. Use your calculator to evaluate the following expressions. Explain why it gives your results.

(a) $\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right)$

(b) $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

2. For each equation below determine all possible values of x that satisfy the equations.

(a) $\sin(3x - 1) = \frac{\sqrt{3}}{2}$

(b) $\ln(\cos(x + 1) + 2) = 0.6$

3. The commander of a radar station just received word that a super secret spy plane will fly directly over it. The plane is expected to have an elevation of two miles, and the distance between the station and a point on the ground directly below the plane is roughly ninety miles away from the station's antenna.
- (a) What should the angle of elevation of the antenna be to point directly at the plane.
- (b) Due to a mix up it is estimated that the orders are thirty minutes late. If the plane is traveling level flight at a constant speed of five-hundred miles per hour determine the angle of elevation.
- (c) Determine the necessary angle of elevation if the orders were actually thirty-five minutes late.

4. Determine the value of

$$\tan \left(\arcsin \left(\frac{1}{3} \right) \right)$$

(a) Let θ be the angle that satisfies $\sin(\theta) = \frac{1}{3}$. How does this definition relate to the expression above? Rewrite the expression in terms of the angle θ .

(b) Make a sketch of a right triangle. Mark the appropriate angle and sides given that $\sin(\theta) = \frac{1}{3}$.

(c) Determine the value of the expression above based on your diagram.

5. Determine the value of

$$\tan(\operatorname{arcsec}(x))$$

for any value of x without using any trigonometric functions.

(a) Make a sketch of a right triangle. Mark the appropriate angle and sides.

(b) Determine the value based on your diagram.

1. Briefly state two ideas from today's class.

•

•

2. (a)

Chapter 5

GNU Free Documentation License

Version 1.3, 3 November 2008

Copyright © 2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc.

<<http://fsf.org/>>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

Preamble

The purpose of this License is to make a manual, textbook, or other functional and useful document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The “**Document**”, below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as “**you**”. You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A “**Modified Version**” of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A “**Secondary Section**” is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The “**Invariant Sections**” are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document

is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The “**Cover Texts**” are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A “**Transparent**” copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not “Transparent” is called “**Opaque**”.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The “**Title Page**” means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

The “**publisher**” means any person or entity that distributes copies of the Document to the public.

A section “**Entitled XYZ**” means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “**Acknowledgements**”, “**Dedications**”, “**Endorsements**”, or “**History**”.) To “**Preserve the Title**” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

- A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
- B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
- C. State on the Title page the name of the publisher of the Modified Version, as the publisher.
- D. Preserve all the copyright notices of the Document.
- E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
- F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
- G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
- H. Include an unaltered copy of this License.
- I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.
- J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
- K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.
- L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

- M. Delete any section Entitled “Endorsements”. Such a section may not be included in the Modified Version.
- N. Do not retitle any existing section to be Entitled “Endorsements” or to conflict in title with any Invariant Section.
- O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version’s license notice. These titles must be distinct from any other section titles.

You may add a section Entitled “Endorsements”, provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled “History” in the various original documents, forming one section Entitled “History”; likewise combine any sections Entitled “Acknowledgements”, and any sections Entitled “Dedications”. You must delete all sections Entitled “Endorsements”.

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an “aggregate” if the copyright resulting from the compilation is not used to limit the legal rights of the compilation’s users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document’s Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled “Acknowledgements”, “Dedications”, or “History”, the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60

days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it.

10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License “or any later version” applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy’s public statement of acceptance of a version permanently authorizes you to choose that version for the Document.

11. RELICENSING

“Massive Multiauthor Collaboration Site” (or “MMC Site”) means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A “Massive Multiauthor Collaboration” (or “MMC”) contained in the site means any set of copyrightable works thus published on the MMC site.

“CC-BY-SA” means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.

“Incorporate” means to publish or republish a Document, in whole or in part, as part of another Document.

An MMC is “eligible for relicensing” if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.

The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the “with ... Texts.” line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.