Bayesian Methods for Dynamic Multivariate Models

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I. Introduction

If multivariate dynamic models are to be used to guide decision-making, it is important that it be possible to provide probability assessments of their results, for example to give error bands around forecasts or policy projections. In Sims and Zha [1995] we showed how to compute Bayesian error bands for impulse responses estimated from reduced form vector autoregressions (VAR's) and from identified VAR's. We also explained there the conceptual and practical difficulties surrounding attempts to produce classical confidence bands for impulse responses. However in that paper we considered only various types of "flat" prior.

But if we are to take seriously the results from such models, we are forced either to make artificially strong assumptions to reduce the number of parameters, or to follow Litterman [1986] in introducing Bayesian prior information. In this paper we show that it is possible to introduce prior information in natural ways, without introducing substantial new computational burdens. Our framework is developed for what are known as "identified VAR" models, but it includes as a special case the reduced form VAR model with the Litterman prior on its coefficients.

The developments we describe here are important for at least two reasons. Until a recent paper by Kadiyala and Karlsson [1997], reduced-form Bayesian models under an informative prior (such as Litterman's) have not commonly been presented with error bands on forecasts or impulse responses based on the posterior distribution. The widely used method of constructing posterior distributions for VAR models that is packaged with the RATS computer program is justified only for the case of priors that have the same form in every equation of the system. Usually it has been applied to models estimated "without" a prior – i.e. with a flat prior, which is trivially symmetric across equations. Litterman's prior, however, differs across equations because it treats "own lags" as different from other coefficients.

The second reason, which motivates this paper, is that the identified VAR literature has been limited for the most part to working with models of 6 to 8 variables, probably because sampling error makes results erratic in larger models under a flat prior. Identified Bayesian VAR models under an informative prior have not at all, as far as we know, been handled in an internally consistent way. Moreover, when a model is not exactly identified, generating error bands on forecasts or impulse responses becomes a difficult problem both conceptually and numerically. A procedure that is implemented in the existing literature follows the convenient RATS programs to first draw the reduced-form innovation covariance matrix from an inverted Wishart distribution and then for each draw to estimate the structural parameters (e.g., Canova [1991], Gordon and Leeper [1994]). Such a procedure, appealing though it might be, lacks a theoretical foundation in finite samples. With our approach, identified VAR analysis under an informative prior becomes feasible even for large systems.

We consider linear multivariate models of the general form

$$A(L)\,y(t) + C = \varepsilon(t)\,, \tag{1}$$

where $y(t)$ is an $m \times 1$ vector of observations, $A(L)$ is an $m \times m$ matrix polynomial of lag operator L with lag length p and non-negative powers, and C is a constant vector. We assume

$$\varepsilon(t)|\, y(s), s<t \;\sim\; N(0,\; \underset{m\times m}{I}\;)\;. \tag{2}$$

Though we work with this model, in which the only exogenous component is the constant vector, much of our discussion generalizes to more complicated sets of exogenous regressors. We assume $A(0)$ is non-singular so that (1) and (2) provide a complete description of the p.d.f. for the data $y(1), y(2),..., y(T)$ conditional on the initial observations $y(-p+1),..., y(0)$.

II. General Bayesian Framework: Identified Approach

The recent identified VAR models that aim at identifying monetary policy effects (e.g., Sims [1986], Gordon and Leeper [1994], Cushman and Zha [1995], Bernanke and Mihov [1996]) give model coefficients economic interpretations that imply behavioral interpretations of estimation results. Such models work directly with the parameters in $A(L)$ from (1). The likelihood function is then

$$L\big(y(t), t=1,...,T\big|A(L)\big) \propto \big|A(0)\big|^T \exp\left[-\frac{1}{2}\sum_t \big(A(L)y(t)+C\big)'\big(A(L)y(t)+C\big)\right]. \tag{3}$$

Rewrite model (1) in matrix form:

$$\mathbf{Y}\mathbf{A}_0 - \mathbf{X}\mathbf{A}_+ = \mathbf{E}, \tag{4}$$

where $\mathbf{Y}$ is $T\times m$, $\mathbf{A}_0$ is $m\times m$, $\mathbf{X}$ is $T\times k$, $\mathbf{A}_+$ is $k\times m$, and $\mathbf{E}$ is $T\times m$. Note that $\mathbf{X}$ contains the lagged $\mathbf{Y}$'s and a column of 1's corresponding to the constant, T is the number of observations, m is the number of equations, and $k = mp+1$ is the number of coefficients corresponding to $\mathbf{X}$. Note that the arrangement of the elements in $\mathbf{A}_0$ is such that the columns in $\mathbf{A}_0$ correspond to the equations. That is to say, $\mathbf{A}_0 = A(0)'$.

Let

$$\mathbf{Z} = \big[\mathbf{Y} \quad -\mathbf{X}\big], \text{ and } \mathbf{A} = \begin{bmatrix}\mathbf{A}_0 \\ \mathbf{A}_+\end{bmatrix}. \tag{5}$$

We now introduce $\mathbf{a}$ as notation for $\mathbf{A}$ vectorized, i.e. the $m\cdot(k+m)\times 1$ vector formed by stacking the columns of $\mathbf{A}$, first column on top, and $\mathbf{a}_0$ and $\mathbf{a}_+$ correspondingly as notation for vectorized $\mathbf{A}_0$ and $\mathbf{A}_+$ respectively. Note that $\mathbf{a}_0$ and $\mathbf{a}_+$, though made up of elements of $\mathbf{a}$, do not arise from a simple partition of $\mathbf{a}$. The conditional likelihood function (3) can now be expressed in compact form:

$$L(\mathbf{Y}|\mathbf{A}) \propto \big|\mathbf{A}_0\big|^T \exp\Big[-0.5\,trace\big(\mathbf{Z}\mathbf{A}\big)'\big(\mathbf{Z}\mathbf{A}\big)\Big]$$

$$\propto \big|\mathbf{A}_0\big|^T \exp\big[-0.5\mathbf{a}'\big(I\otimes\mathbf{Z}'\mathbf{Z}\big)\mathbf{a}\big] \tag{6}$$

Let us assume $\mathbf{a}$ has prior p.d.f.

$$\pi(\mathbf{a}) = \pi_0(\mathbf{a}_0)\varphi\big(\mathbf{a}_+ - \mu(\mathbf{a}_0); H(\mathbf{a}_0)\big), \tag{7}$$

where $\pi_0(\ )$ is a marginal distribution of $\mathbf{a}_0$ and $\varphi(\cdot\,;\Sigma)$ is the standard normal p.d.f. with covariance matrix Σ. The marginal distribution π_0 can be very general, singularities generated by zero restrictions. Of course one special case of (7) occurs when π (or π_0) is itself a normal p.d.f. in the full $\mathbf{a}$ vector. Combining (6) and (7), we arrive at the posterior density function of $\mathbf{a}$:

$$q(\mathbf{a}) \propto \pi_0(\mathbf{a}_0)\left|A(0)\right|^T \left|H(\mathbf{a}_0)\right|^{-1/2} \exp\Big[-0.5\big(\mathbf{a}_0'\big(I\otimes\mathbf{Y}'\mathbf{Y}\big)\mathbf{a}_0 - 2\mathbf{a}_+'\big(I\otimes\mathbf{X}'\mathbf{Y}\big)\mathbf{a}_0 + \mathbf{a}_+'\big(I\otimes\mathbf{X}'\mathbf{X}\big)\mathbf{a}_+ + (\mathbf{a}_+ - \mu(\mathbf{a}_0))'H(\mathbf{a}_0)^{-1}(\mathbf{a}_+ - \mu(\mathbf{a}_0))\Big] \quad . \tag{8}$$

The posterior density (8) is non-standard in general, and the dimension of the parameter vector $\mathbf{a}$ is large even in relatively small systems of equations. A direct approach to analysis of the likelihood may therefore not be computationally feasible. However, the exponent in (8) is quadratic in $\mathbf{a}_+$ for fixed $\mathbf{a}_0$, meaning that the conditional distribution of $\mathbf{a}_+$ given $\mathbf{a}_0$ is Gaussian, making possible easy Monte Carlo sampling and analytic maximization or integration along the $\mathbf{a}_+$ dimension. Specifically, the conditional distribution of $\mathbf{a}_+$ given $\mathbf{a}_0$ and the marginal distribution of $\mathbf{a}_0$ can be derived as

$$q(\mathbf{a}_+|\mathbf{a}_0) = \varphi\Big(\mathbf{a}_0^*;\big(I\otimes\mathbf{X}'\mathbf{X} + H(\mathbf{a}_0)^{-1}\big)^{-1}\Big) \ , \tag{9}$$

$$q(\mathbf{a}_0) \propto \pi_0(\mathbf{a}_0)\left|A(0)\right|^T \left|(I\otimes\mathbf{X}'\mathbf{X})H(\mathbf{a}_0) + I\right|^{-1/2} \exp\Big[-0.5\big(\mathbf{a}_0'\big(I\otimes\mathbf{Y}'\mathbf{Y}\big)\mathbf{a}_0 + \mu(\mathbf{a}_0)'H(\mathbf{a}_0)^{-1}\mu(\mathbf{a}_0) - \mathbf{a}_0^{*\prime}\big(I\otimes\mathbf{X}'\mathbf{X} + H(\mathbf{a}_0)^{-1}\big)\mathbf{a}_0^*\Big] \ , \tag{10}$$

where

$$\mathbf{a}_0^* = \big(I\otimes\mathbf{X}'\mathbf{X} + H(\mathbf{a}_0)^{-1}\big)^{-1}\big(\big(I\otimes\mathbf{X}'\mathbf{Y}\big)\mathbf{a}_0 + H(\mathbf{a}_0)^{-1}\mu(\mathbf{a}_0)\big) \ .$$

III. Symmetry

It is clear from (9) that maximization or integration with respect to $\mathbf{a}_+$ conditional on a fixed value of $\mathbf{a}_0$ can easily involve computations that become heavy because of their high dimensionality. The $\mathbf{a}_+$ vector is of order $m\cdot(mp+1)$, so finding its conditional posterior mean will require, at each value of $\mathbf{a}_0$, a least-squares calculation of that order. The calculation has the same form as that for a seemingly-unrelated-regressions (SUR) model. When there is no special structure, such computations are manageable for models with, say, $m=6$ and $p=6$, making the order of $\mathbf{a}_+$ 222, as might be realistic for a small quarterly model. But we have applied these methods to models with 6 lags on as many as 20 variables, and even tested them on models with 13 lags on 20 variables (see, for example, Leeper, Sims and Zha [1996]). For a 6-lag, 20-variable model $\mathbf{a}_+$ is of order 2420. With 13 lags the order is 5220. Repeatedly solving least-squares problems of this order, using general algorithms, over many hundreds of iterations, is impractical on widely available workstations.

On the other hand, because the calculation is of a SUR type, it has the usual property that it breaks into m separate least-squares calculations, each of dimension only $mp+1$, when the matrix of "regressors" is common across equations. It was this observation that led Litterman [1986] (and subsequent followers of his approach) to use single-equation methods on his reduced form model, even though the prior he proposed satisfies the conditions needed to give conditional likelihood the common-regressors form at best approximately. Highfield [1987] and recently Kadiyala and Karlsson [1997] pointed out that by modifying Litterman's prior to make it symmetric across equations in the appropriate sense, one could make the full system posterior p.d.f. tractable. They were considering only a reduced form model, a special case of ours. In our more general framework we can also give the prior a form that makes the conditional posterior of $\mathbf{a}_+$ tractable, under conditions that are in some ways less restrictive than those Highfield or Kadiyala and Karlsson required.

The demanding part of the calculations required for integrating or maximizing (8) with respect to $\mathbf{a}_+$ is a matrix decomposition of the coefficient on the term quadratic in $\mathbf{a}_+$. This coefficient can be read off directly from (9) as

$$\left(I \otimes \mathbf{X}'\mathbf{X}\right) + H(\mathbf{a}_0)^{-1} \ . \tag{11}$$

Clearly to preserve the Kronecker-product structure of the first term in this expression, we will require that

$$H(\mathbf{a}_0) = B \otimes G \ , \tag{12}$$

where B and G have the same order as the I and $\mathbf{X}'\mathbf{X}$ in (11). Further, either B must be a scalar multiple of I, or G must be a scalar multiple of $\mathbf{X}'\mathbf{X}$, because otherwise the Kronecker-product structure will be lost after the summation. Because $\mathbf{X}'\mathbf{X}$ depends on random variables generated by the model, it does not make sense to have our prior distribution's form depend on $\mathbf{X}'\mathbf{X}$. Thus preserving Kronecker-product structure requires that B be scalar, i.e. that beliefs about coefficients on lagged variables in structural equations have precision that is independent across equations. Since there is just a single G matrix, (12) requires also that the precision of beliefs about coefficients be the same in every equation.

Once B has been restricted to be scalar, though, the computational advantages of the strict Kronecker-product structure are no longer decisive. Suppose we have a distinct covariance matrix G_i for the prior on each equation's component of $\mathbf{a}_+$, but maintain independence across equations. Then (11) becomes

$$\left(I \otimes \mathbf{X}'\mathbf{X}\right) + diag\left(G_1,...,G_m\right) = diag(G_1 + \mathbf{X}'\mathbf{X},...,G_m + \mathbf{X}'\mathbf{X}) \,, \tag{13}$$

where we have introduced the notation

$$diag\left(G_1,...,G_m\right) = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & G_m \end{bmatrix} . \tag{14}$$

While a matrix decomposition of the right-hand side of (13) is not as easy as a decomposition of a Kronecker product, it is still far easier than a decomposition of a general $mk \times mk$ matrix, because it can be done one block at a time for the diagonal blocks. In our example of a 20-variable, six-lag system, we are replacing a 2420×2420 decomposition with twenty 121×121 decompositions. Since these decompositions generally require computation time that is of cubic order in the size of the matrix, we have reduced the computations by a factor of 400.

It is interesting to contrast the situation in this model with what emerges from Highfield or Kadiyala and Karlsson's consideration of SUR symmetry restrictions for Bayesian system estimation of reduced form VAR's. The straightforward approach to the reduced form leaves the covariance matrix of disturbances free, so that in place of the $I \otimes \mathbf{X}'\mathbf{X}$ first term in (11), we have a $\Sigma \otimes \mathbf{X}'\mathbf{X}$ term, where Σ is the covariance matrix of the equation disturbances. This means that, to preserve a convenient system structure, prior beliefs must be treated as correlated across equations of the reduced form in the same pattern as Σ, a restriction that seems prima facie unappealing but, as we shall see in Section A, becomes more plausible if we begin in a simultaneous equations framework. The requirement that the covariance matrix of the prior be the same in every equation is also restrictive. In Litterman's approach, for example, the variances of coefficients on lags of the dependent variable in a reduced-form equation are larger than the variances of coefficients on other variables. This contradicts the requirement that the covariance matrix have the same structure in every equation. In this context, where the prior is directly on the reduced form, relaxing this requirement does greatly increase computational problems. We show that with our version of a prior on the simultaneous equations form, we can avoid most of these computational problems.

IV. Formulating a Prior Distribution

With the prior formulated as in (7), with a marginal p.d.f. on $\mathbf{a}_0$ multiplying a conditional p.d.f. for $\mathbf{a}_+ | \mathbf{a}_0$, our setup to this point has placed no restrictions on the conditional mean of $\mathbf{a}_+$. It restricts beliefs about $\mathbf{a}_+$ to be Gaussian and uncorrelated across equations conditional on $\mathbf{a}_0$, but allows them to be correlated in different ways in different equations. This leaves many degrees of freedom in specifying a prior, making the use of substantive economic knowledge to form a multivariate prior in these models an important task. In this section we suggest some approaches to the task.

A. A Base: The Random Walk Prior

The Litterman prior for a reduced form model expresses a belief that a random-walk model for each variable in the system is a reasonable "center" for beliefs about the behavior of the variables. Since this idea concerns behavior of the reduced form, it does not in itself restrict $\mathbf{A}_0$. It suggests that beliefs about the reduced form coefficient matrix

$$\mathbf{B} = \mathbf{A}_+ \mathbf{A}_0^{-1} \tag{15}$$

should be centered on an identity matrix for the top m rows and zeros for the remaining rows. We make this notion concrete by making the conditional distribution for $\mathbf{A}_+$ Gaussian with mean of $\mathbf{A}_0$ in the first m rows and 0 in the remaining rows, or

$$E\left[\mathbf{A}_+ \middle| \mathbf{A}_0\right] = \begin{bmatrix} \mathbf{A}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{16}$$

As a starting point, we assume the prior conditional covariance matrix of the coefficients in $\mathbf{A}_+$ follows the same pattern that Litterman gave to the prior covariance matrix on reduced form coefficients. That is, we make the conditional prior independent across elements of $\mathbf{A}_+$ and with the conditional standard deviation of the coefficient on lag ℓ of variable j in equation i given by

$$\frac{\lambda_0 \lambda_1}{\sigma_j \ell^{\lambda_3}} . \tag{17}$$

The hyperparameter λ_0 controls the tightness of beliefs on $\mathbf{A}_0$, λ_1 controls what Litterman called overall tightness of beliefs around the random walk prior, and λ_3 controls the rate at which prior variance shrinks with increasing lag length. The vector of parameters $\sigma_1,...,\sigma_m$ are scale factors, allowing for the fact that the units of measurement or scale of variation may not be uniform across variables. While in principle these should be chosen on the basis of a priori reasoning or knowledge, we have in practice followed Litterman in choosing these as the sample standard deviations of residuals from univariate autoregressive models fit to the individual series in the sample.

This specification differs from Litterman's in a few respects. There is no distinction here between the prior conditional variances on "own lags" versus "others" as there is in Litterman's framework. Because our model is a simultaneous equations model, there is no dependent variable in an equation, other than what might be set by an arbitrary normalization, so the "own" versus "other" distinction among variables is not possible. But note also that the unconditional prior for the top m rows in $\mathbf{A}_+$ will be affected by the prior on $\mathbf{A}_0$. In fact the unconditional prior variance of an element of the first m rows of $\mathbf{A}_+$ will be, because of (16), the sum of the prior variance of the corresponding element of $\mathbf{A}_0$ and the conditional variance specified in (17). Thus if our prior on $\mathbf{A}_0$ puts high probability on large coefficients on some particular variable j in structural equation i, then the prior probability on large coefficients on the corresponding variable j at the first lag is high as well.

Litterman's specification also has the scale factors entering as the ratio σ_i/σ_j, rather than only in the denominator as in (17). This reflects the fact that our specification normalizes the variances of disturbances in the structural equations to one.

The last row of $\mathbf{A}_+$ corresponds to the constant term. We give it a conditional prior mean of zero and a standard deviation controlled by $\lambda_0\lambda_4$ where λ_4 is a separate hyperparameter.

However it is not a good idea in practice to work with a prior in which beliefs about the constant term are uncorrelated with beliefs about the coefficients on lagged *y*'s. Some of our suggestions in Section B for modifying the prior via dummy observations are aimed at correcting this deficiency in the base setup.

The fact that this prior has a structure similar to Litterman's and can be similarly motivated should not obscure the fact that, because it is a prior on the conditional distribution of $\mathbf{A}_+|\mathbf{A_0}$ rather than on $\mathbf{B}|\mathbf{A}_0$, it entails different beliefs about the behavior of the data. In particular, as can be seen from (15), the prior described here makes beliefs about **B** correlated across equations in a way dependent on beliefs about $\mathbf{A}_0$, or equivalently about the covariance matrix of reduced form disturbances. Indeed in the special case where the prior covariance matrices G_i are the same across equations, the prior conditional distribution for $\mathbf{B}|\mathbf{A}_0$ is Gaussian with covariance matrix

$$\Sigma \otimes G \ , \tag{18}$$

which is exactly the form assumed by Highfield [1987] and the Normal-Wishart form preferred by Kadiyala and Karlsson [1997]. Recall, though, that unlike Highfield or Kadiyala and Karlsson our framework is more general and allows us to handle the case of differing G_i's without additional computational burden, where there is no Kronecker product structure like (18) for the prior conditional covariance matrix of **B**.

B. Dummy Observations, Unruly Trends

Litterman's work exploited the insight of Theil mixed estimation, that prior information in a regression model can be introduced in the form of extra "dummy" observations in the data matrix. A similar idea applies to the simultaneous equations framework we are considering here. For example, suppose we want to follow Litterman in starting with a prior centered on a reduced form implying the data series all follow random walks, correlated only through the correlation of innovations. Litterman, following an equation-by-equation approach to estimation, could implement his prior by adding to the data matrix used for estimating the *i*'th equation a set of $k-1$ dummy observations indexed by $j=1,...,m$, $\ell=1,...,p$, with data taking the values specified in Table 1. (Litterman also used a prior on the constant term, but practice has varied on the form of the prior on the constant, so we omit it here.)

Table 1

$y_i(r,j)\,;\ \ r=1,..k-1;\ j=1,...,m$	$x_i(r,s)\,;\ \ r=1,...,k-1;\ s=1,...,k-1$
$\begin{cases} \mu_0\mu_1\mu_2^{\delta(i,j)}\sigma_r/\ell^{\mu_3} & r=j, r\le m \\ 0 & otherwise \end{cases}$	$\begin{cases} \mu_0\mu_1\mu_2^{\delta(i,j)}\sigma_r/\ell^{\mu_3} & r=s \\ 0 & otherwise \end{cases}$

Here we have defined

$$\delta(i,j)=\begin{cases}0, i\neq j\\ 1, i=j\end{cases} \tag{19}$$

Note that both r and s are indexed by $j=1,...,m$, $\ell=1,...,p$. We are also introducing a convention that scale factors for variances in the prior covariance matrix are λ's, while scale factors on dummy observations are μ's. When the same distribution can be formulated either directly with a prior covariance matrix or indirectly via dummy observations, the similarly numbered λ's and μ's correspond, with $\lambda_i = 1/\mu_i$. In Litterman's framework, μ_2 controls tightness of a belief on the coefficients of variables relative to the independent variable in a given equation. As pointed out in the previous section, μ_2 should be set to 1 in our simultaneous equations framework. The *i*'th equation's dummy observations can be written as

$$\underset{(k-1)\times m}{\mathbf{Y}_{id}}\ \underset{m\times 1}{\mathbf{A}_{0\bullet i}} = \underset{(k-1)\times(k-1)}{\mathbf{X}_{id}}\ \underset{(k-1)\times 1}{\mathbf{A}_{+\bullet i}} + \underset{(k-1)\times 1}{\mathbf{E}_{\bullet i}}\ , \tag{20}$$

where a "$\bullet i$" subscript on a matrix refers to the *i*'th column of the matrix and the matrices $\mathbf{Y}_{id}$ and $\mathbf{X}_{id}$ are formed directly from Table 1 so that $\mathbf{Y}_{id}=\{y_i(r,j)\}$ and $\mathbf{X}_{id}=\{x_i(r,j)\}$.

In our approach, where all equations are estimated jointly, the fact that these "dummy observations" are equation-specific means that they are not algebraically equivalent to adding rows to the data matrix. One might nonetheless introduce them into the exponent in the posterior p.d.f. (8) as

$$\mathbf{a}_0'\cdot diag\left(\{\mathbf{Y}_{id}'\mathbf{Y}_{id}\}_{i=1}^m\right)\cdot\mathbf{a}_0 - 2\mathbf{a}_+'\cdot diag\left(\{\mathbf{X}_{id}'\mathbf{Y}_{id}\}_{i=1}^m\right)\cdot\mathbf{a}_0 + \mathbf{a}_+'\cdot diag\left(\{\mathbf{X}_{id}'\mathbf{X}_{id}\}_{i=1}^m\right)\cdot\mathbf{a}_+\ . \tag{21}$$

These terms do not introduce any complications in the numerical analysis, because they preserve the block diagonal structure of the coefficient matrix for the term quadratic in $\mathbf{a}_+$. In fact, terms of this form can be used to implement exactly the conditional prior for $\mathbf{A}_+|\mathbf{A}_0$ given by (16) and (17). To implement that prior we would want to set $\mu_0 = 1/\lambda_0$, $\mu_1 = 1/\lambda_1$, and $\mu_2 = 1$ in the formulas in Table 1. One can easily verity that (21) is exactly the last term in the exponent (8), where

$$diag\left(\{\mathbf{X}_{id}'\mathbf{X}_{id}\}_{i=1}^m\right) = H(\mathbf{a}_0)^{-1},\ diag\left(\{\mathbf{X}_{id}'\mathbf{Y}_{id}\}_{i=1}^m\right) = H(\mathbf{a}_0)^{-1}\mu(\mathbf{a}_0).$$

In work following Litterman's, modifications of his prior have been introduced that improve forecasting performance and take the form of true, system-wide dummy observations. The "sums of coefficients" component of a prior, introduced in Doan, Litterman, and Sims [1984], expresses a belief that when the average of lagged values of a variable is at some level $\bar{y}_i$, that same value $\bar{y}_i$ is likely to be a good forecast of $y_i(t)$. It also implies that knowing the average of lagged values of variable *j* does not help in predicting a variable $i\neq j$. In a system of *m* equations it introduces *m* observations, indexed by i, of the form

Table 2

$y(i,j)\,;\quad i=1,...,m;\ j=1,...,m$	$x(i,s)\,;\quad i=1,...,m;\ s=1,...,k$
$\begin{cases}\mu_5\bar{y}_{0i} & i=j\\ 0 & otherwise\end{cases}$	$\begin{cases}\mu_5\bar{y}_{0i} & i=j,\ \text{all } l\\ 0 & otherwise\end{cases}$

Here s is indexed by $j=1,...m, l=1,...,p$, and the constant term (so that $k=pm+1$), and $\bar{y}_{0i}$ is the average of initial values of variable i. Clearly, the dummy observations for the whole system can be written as

$$\mathbf{Y}_d\mathbf{A}_0=\mathbf{X}_d\mathbf{A}_+ +\mathbf{E}\ , \tag{22}$$

where $\mathbf{Y}_d=\{y(i,j)\}$ and $\mathbf{X}_d=\{x(i,s)\}$. In these dummy observations, the last column of the data matrix $\mathbf{X}_d$, corresponding to the constant term, is set to zero. These dummy observations introduce correlation among coefficients on a given variable in a given equation. When $\mu_5\to\infty$, the model tends to a form that can be expressed entirely in terms of differenced data. In such a limiting form, there are as many unit roots as variables and there is no cointegration. Since the constant term is not included (the dummy observations have zeros in place of the usual ones in the constant vector), this type of prior information allows, even in the limit as $\mu_5\to\infty$, nonzero constant terms, and thus linearly trending drift.

The "dummy initial observation" component of a prior, introduced by Sims [1993], introduces a single dummy observation in which, up to a scaling factor, all values of all variables are set equal to the corresponding averages of initial conditions (i.e., $\bar{y}_{0i}$), and the last column of the data matrix is set at its usual value of 1. We designate the scale factor for this single dummy observation as μ_6 and the corresponding equation (22) takes the form

Table 3

$y(j)\,;\quad j=1,...,m$	$x(s)\,;\quad s=1,...,k$
$\mu_6\bar{y}_{0j}$	$\begin{cases}\mu_6\bar{y}_{0j} & s\le k-1\\ \mu_6 & s=k\end{cases}$

This type of dummy observation reflects a belief that when lagged values of y_i have averaged $\bar{y}_i$, that same value $\bar{y}_i$ should be a good forecast, but without any implication that there are no cross effects among variables or that the constant term is small. This kind of dummy observation introduces correlations in prior beliefs about all coefficients (including the constant term) in a given equation. As $\mu_6\to\infty$, the model tends to a form in which either all variables are stationary with means equal to the sample averages of the initial conditions, or there are unit root components without drift (linear trend) terms. To see this, note that as $\mu_6\to\infty$ the prior imposes the constraint (in the notation of (1))

$$\big(I-B(1)\big)\bar{y}_0+A(0)^{-1}C=0\ . \tag{23}$$

where $B(L) = A(0)^{-1} A(L)$. This implies the existence of one unit root in the system if $C = 0$, but not otherwise. When the data imply near-unit roots, this type of dummy observation suggests C will be small. Since as $\mu_6 \to \infty$ this type of dummy observation implies only a single unit root even with $C = 0$, cointegrated models are not ruled out in this limit.

Unlike our base prior (21), both of these types of dummy observations are symmetric across equations, so that they can be introduced as extra rows in the data matrix, making them easy to handle efficiently in computation (see (22), Table 2, and Table 3).

These latter two types of true dummy observations, taken together, favor unit roots and cointegration, which fits the beliefs reflected in the practices of many applied macroeconomists. More importantly, they have been found to improve forecasts in a variety of contexts with economic time series.

Some insight into why this should be so is provided in Sims [1992], which shows that without such elements in the prior, fitted multivariate time series models tend to imply that an unreasonably large share of the sample period variation in the data is accounted for by deterministic components. That is, if we construct from the estimated coefficients, treating them as non-random, the vector of time series $E\left[y(t) \middle| y(s), s \le 0 \right]$, t=1,…,T, we find that they show substantial variation, while remaining close to $y(t)$ itself, even for large values of t. Figure 1 displays such an example. The forecast monthly time series of M1 and the unemployment rate as of 1959:12 are constructed from the 18-variable VAR model of Leeper, Sims and Zha [1996] but (unlike the estimates used in that paper) without any prior. Clearly the estimates imply that it was possible to forecast three decades into the future with unbelievable accuracy. This bias toward attributing unreasonable amounts of variation to deterministic components is the other side of the well-known bias toward stationarity of least-squares estimates of dynamic autoregressions. The problem is in fact prevalent even for small-sized systems.

We do not have a clear theoretical explanation for why estimated multivariate time series models should show stronger bias of this type than do lower dimensional models, but it seems that they do. One can certainly see how it is possible. A reduced-form system in m variables with p lags and constant terms can generally fit without error an arbitrary collection of time series that are all polynomials in t of order mp. Suppose the data are in fact all mp'th order polynomials in t. The $mp+1$ vectors of length T formed by the constant vector and the first p lags of the data will in general all be linearly independent and all will by construction lie in the space spanned by the 0'th through mp'th power of t. They therefore form a basis for the space of mp'th order polynomials in t. Thus some linear combination of them exactly matches the dependent variable, which in each equation of the reduced form system is itself by assumption such a polynomial.

This result means that a least-squares algorithm, attempting to fit, say, 6 time series with a 5'th order VAR, always has the option of taking a form in which $E\left[y(t) \middle| y(s), s \le 0 \right]$, t=1,…,T is a freely chosen set of 30'th order polynomials in t. Such high order polynomials are likely to be able to fit many economic time series quite well (as demonstrated in Figure 1), while still being implausible for out-of-sample projections. Of course, this argument is only heuristic, because it has not shown that VAR coefficients that exactly fit high-order polynomial approximations to the

data series will provide a good fit to the data series themselves. Nonetheless, there seems to be cause for concern about overfitting of low-frequency deterministic components both from a theoretical point of view and, as shown in Sims [1992], from an applied perspective.

C. A Prior on $\mathbf{A}_0$

In many applications the prior on $\mathbf{A}_0$ will reflect the substantive knowledge that makes it possible to distinguish the equations as behavioral mechanisms – i.e., the identifying restrictions. Since this paper is concerned mainly with explaining the econometric technique, we do not here discuss how to come up with identifying restrictions.[1] We should note, though, that in this approach it is often desirable to aim for distinct behavioral interpretations only of blocks of equations, not the complete list of individual equations. Within a block of equations that are not separately identified, we can simultaneously make coefficients unique and the disturbance matrix diagonal by a triangular normalization – we impose zero restrictions on an otherwise unrestricted square block of coefficients within the block of equations so as to force it to take on a triangular form.[2]

A reduced form model can be estimated within the framework of this section by taking $\mathbf{A}_0$ to be triangular, as a *normalization*, and imposing no other prior restrictions on it. As we will discuss in Section A, the prior on $\mathbf{A}_0$ is then equivalent to a prior on the reduced form innovation covariance matrix Σ.

V. Examples

We display results for two cases: One matching the structure laid out in the preceding sections, with a simple exactly identified parameterization of $\mathbf{A}_0$ and a Gaussian prior on its elements; one matching the usual interpretation of Litterman's prior, in which it is independent across elements of $\mathbf{B}|\mathbf{A}_0$ rather than $\mathbf{A}_+|\mathbf{A}_0$, and using a non-Gaussian "flat" prior on $\mathbf{A}_0$. We show that the former results in much more convenient calculations and gives quite reasonable results. We discuss the different implications of the two types of prior, arguing that usually the more convenient form is also more appealing substantively. We also show that some apparently plausible numerical shortcuts for the latter example that have appeared in the literature, can have substantial effects on results.

The variables in both models are quarterly data on: the 3-month T-bill rate (R), money stock (M1), real GNP (y, \$1982), GNP deflator (P), the unemployment rate (U), and gross private domestic fixed investment (I). Money stock, real GNP, GNP deflator, and fixed investment are

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[1] The reader who is interested in the application of identified VAR models with informative priors can consult Leeper, Sims and Zha [1996].

[2] If we consider responses to a shock in an equation (call it the "policy equation") outside a block of equations subject only to normalizing restrictions, the responses of all variables in the system to policy shocks are invariant to the form of the normalizing restrictions within the block. This point is made precise in Zha [1996].

all in logarithms, and the sample period is 48:1-82:4. Also in both models we include 6 lags, i.e. set $p = 6$.

A. Prior Independence Across Structural Equations

We assume an upper triangular normalization of $\mathbf{A}_0$. In the calculations reported below, we assumed a joint normal prior on the non-zero elements of $\mathbf{A}_0$. The individual elements were assumed independent, with prior standard deviations of all elements in the *i*'th row set to $\lambda_0/\hat{\sigma}_i$, where λ_0 and $\hat{\sigma}_i$ were defined above, below (17). This prior implies a belief that the lower right diagonal elements of Σ, the reduced form innovation covariance matrix, are larger than those in the upper left. Yet if we had assumed a lower triangular normalization of $\mathbf{A}_0$ we would have obtained the opposite implication. It is an unreasonable aspect of our setup that it makes our beliefs about Σ depend on this arbitrary aspect of our normalization. A better procedure, which however would not have been very different in practice, would have been to derive our prior on $\mathbf{A}_0$ from a natural prior on Σ^{-1}, the Wishart, which has p.d.f.

$$\left|\Sigma^{-1}\right|^{(\nu-m-1)/2} \exp\left[-0.5 trace\left(\mathbf{S}\Sigma^{-1}\right)\right] .^{3} \tag{24}$$

In this expression ν is the degrees of freedom parameter. We could have derived a prior on $\mathbf{A}_0$ from (24), using the 1-1 mapping between Σ^{-1} and $\mathbf{A}_0$ given by the Choleski decomposition. With $\nu = m+1$ and $\mathbf{S}$ diagonal with diagonal elements $\left(\lambda_0/\hat{\sigma}_i\right)^2$, we would have arrived at the same p.d.f. for the elements of $\mathbf{A}_0$, except for a Jacobian term. The Jacobian is

$$\left|\frac{\partial \Sigma^{-1}}{\partial \mathbf{A}_0}\right| = 2^m \prod_{j=1}^{m} a_{jj}^{j} \ .^{4} \tag{25}$$

Since the likelihood itself contains a factor $\left|\mathbf{A}_0\right|^T = \prod_{i=1}^{m} a_{ii}^T$, ignoring the Jacobian term as we did will not give results very different from what would have been obtained from the better practice of including the Jacobian term, so long as *T* is considerably larger than *m*.

The prior on $\mathbf{A}_+ \left| \mathbf{A}_0 \right.$ is of the same form that we have described in Section A Base: The Random Walk Prior, including both types of dummy observations.

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[3] Note that this is the p.d.f. for the elements of Σ^{-1}, not Σ. If we were integrating with respect to the elements of Σ, the appropriate p.d.f. would be the inverse-Wishart, which has $\nu + m + 1$ as the exponent of the determinant in place of $\nu - m - 1$.

[4] Note that we have to take account here of the fact that Σ is symmetric and that not all elements of $\mathbf{A}_0$ vary in calculating the Jacobian. We are using the convention that $\mathbf{A}_0$ is chosen upper triangular.

We generated draws from the posterior distribution of the data for the period after 1982:4, the end of the sample period. In the prior, we set the weights μ_5 and μ_6 on the two types of dummy observations to 1, $\lambda_0 = 1$, $\lambda_1 = 0.2$, and $\lambda_2 = \lambda_3 = \lambda_4 = 1$.[5] Figure 2 displays the results, based on 5000 MC draws. The solid lines shown are the actual data series. The central dotted line, made up of alternating long and short dashes, is the posterior mean, and the two outer dotted lines represent 16th and 84th percentiles, so that the bands contain about the same amount of probability (68%) as one-standard-error bands. The bands are calculated pointwise, so that the posterior probability of the future data being in the band is 68% at each forecast horizon individually, not for the band as a whole.

This is a difficult period for VAR models to forecast, particularly for prices. The bands and forecasts nonetheless look reasonable.[6] The actual data lie in or close to the 68% posterior band except for prices. The price forecast predicts substantially more inflation than actually occurred, and gives very low probability to actual values as far from the forecast as actually occurred. The tendency of fixed-coefficient Bayesian VAR's to do badly in forecasting prices in the 80's was a main motivation for the extensions to the BVAR framework introduced in Sims [1993].

B. Prior Independence Across Reduced-Form Equations

We now consider a different type of prior on the same model, aiming to match the usual interpretation of Litterman's prior on the reduced form. We keep the prior conditional on $\mathbf{A}_0$ in the same form as in sections A Base: The Random Walk Prior and Dummy Observations, Unruly Trends, but interpret it now as applying to $\mathbf{B}|\mathbf{A}_0$ rather than to $\mathbf{A}_+|\mathbf{A}_0$. Also, to keep it in line with common usage of this sort of prior, we make $\lambda_2 = .3$, so that coefficients representing cross-variable effects are taken as a priori likely to be smaller, and to maintain comparability with Litterman's original work we drop the two kinds of true dummy observations, i.e. set

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[5] In principle, these hyperparameters can be estimated or integrated out in a hierarchical framework.

[6] There is nothing contradictory about assessing a prior based on the reasonableness of the results it produces in a context like this. Our prior is a computationally convenient distribution, meant to capture widely held beliefs about how economic time series behave. We are proposing it for wide, standardized use in scientific reporting of data analyses. Because the parameters it describes determine the data in such complicated ways, we cannot be sure in advance that the prior does not give credence to behavior in the data that we actually do not believe likely. Also, because there is a great deal of collective experience in forecasting economic time series, it may be easier to assess whether the prior succeeds in capturing widely held beliefs about how time series behave by examining what forecasts it leads to under various conditions than by directly examining its implied distributions of VAR coefficients. Of course if we were a single, isolated, Bayesian decision-maker whose prior represented an exhaustive assessment of her own beliefs, there would be no sense in judging the "reasonableness" of the results the prior implied. But in that case one could also argue that the distinction between stochastic model of the data and prior beliefs about the parameters makes no sense.

$\mu_5 = \mu_6 = 0$. The joint posterior on $\mathbf{A}_0$ and $\mathbf{A}_+$ is still in the form (8), but with $\mu(\mathbf{a}_0)$ given by (16) and with

$$H(\mathbf{a}_0) = (\mathbf{A}_0' \otimes I) \cdot diag(\{G_i\}) \cdot (\mathbf{A}_0 \otimes I) \ . \tag{26}$$

Note that in going back and forth between a prior using the $\mathbf{A}_0, \mathbf{B}$ parameterization and one using the $\mathbf{A}_0, \mathbf{A}_+$ parameterization, the Jacobian, $|\partial \mathbf{B} / \partial \mathbf{A}_+| = |\mathbf{A}_0|^{-k}$, must be taken into account. We in fact use $|\mathbf{A}_0|^k$ as an improper prior for $\mathbf{A}_0$ here when writing the prior in terms of $\mathbf{B}$, so that when transformed to the $(\mathbf{A}_0, \mathbf{A}_+)$ parameter space, the prior on $\mathbf{A}_0$ is flat. Because we express this prior, like that in the previous section, in terms of $\mathbf{A}_0$ rather than Σ, there is also a $|\partial \Sigma^{-1} / \partial \mathbf{A}_0|$ Jacobian, as expressed in (25), to consider. We have not included such a term, though it probably would have made sense to do so to avoid the implied arbitrary asymmetry in beliefs about Σ. If we had included the Jacobian term, the prior on $\mathbf{B}|\mathbf{A}_0$ specified here would have been equivalent to the Normal-Diffuse prior considered by Kadiyala and Karlsson [1997]. Here as in the preceding section, the Jacobian term is unlikely to affect results much because $T >> m$.

Clearly with the conditional covariance matrix in the form of (26), the outcome of the addition in (11) is not a matrix with any standard special structure we can exploit in a matrix decomposition algorithm. Nonetheless, the models we consider here are small enough, with $mp = 36$ and $k = m \cdot (mp + 1) = 222$, that direct manipulation of the 222×222 conditional covariance matrix is computationally feasible. We include this case in part to display the nature of the gains in computational convenience from using the formulation in the preceding section.

Since the marginal posterior p.d.f. $q(\mathbf{a}_0)$ in (10) is now non-standard, our numerical procedure was first to find the posterior mode of it over unrestricted elements in $\mathbf{a}_0$, then to use importance sampling.[7] That is to say, we drew values of $\mathbf{a}_0$ from the multivariate *t*-distribution with 9 degrees of freedom centered on the posterior mode and with covariance matrix given by minus the inverse of the Hessian of the log likelihood at this mode. To make the results reflect the true posterior distribution, we weighted $\mathbf{a}_0$ draws by the ratio of the true posterior p.d.f. to the p.d.f. of the multivariate *t* from which we were drawing. For each draw of $\mathbf{a}_0$ generated this way, we generated an associated $\mathbf{a}_+$ by drawing from the conditional normal posterior on $\mathbf{a}_+$. This of course involved solving a large least squares problem at each draw.

Figure 3 shows error bands computed with this prior. Besides the posterior mean and the 68% band shown in Figure 2, Figure 3 also shows (as a dashed line with equal-length long dashes) the forecast as Litterman originally constructed it, from single-equation estimates, ignoring randomness in the coefficients conditional on the data. It is apparent that the prior differs from that underlying Figure 2 and that this affects results. Particularly noticeable is the shift in the

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[7] Alternatively, one can work directly on $\mathbf{B}$ and Σ (the reduced-form innovation covariance matrix) and use a Gibbs sampler technique outlined by Kadiyala and Karlsson [1997].

location of the unemployment forecast and its error band, with this latter estimate showing less decline in unemployment and with the actual path of unemployment largely outside the 68% band. The Figure 2 forecast is also more optimistic for output. Note that this was a period in which Litterman's model outperformed commercial forecasts by making forecasts for output that were much more optimistic than those of most commercial forecasters. The Figure 2 forecasts, perhaps because they embody prior belief in correlation of coefficients in reduced form equations related to correlation of innovations, make the optimism on unemployment match the optimism on output, while this is less true of the Figure 3 forecasts.

The computing time for obtaining the peak of the marginal posterior on $\mathbf{a}_0$ for Figure 3, using our own code for optimization, written in Matlab,[8] is about 2.8 hours on a Pentium/120 machine, and additional computation for generating weighted Monte Carlo draws takes about 16 minutes per 1,000 draws. In contrast, 1000 Monte Carlo draws of $\mathbf{a}_0$ and $\mathbf{a}_+$ altogether for Figure 2 take 31 seconds. These ratios of times would become greater with larger systems. As noted above, we have used these methods with $\mathbf{A}_+|\mathbf{A}_0$ priors on overidentified VAR's, in which the $\mathbf{A}_0$ matrix is restricted, with up to 20 variables and 6 lags.[9] Handling models of this size with a $\mathbf{B}|\mathbf{A}_0$ prior would be prohibitively time-consuming.

Of course the computational convenience of our $\mathbf{A}_+|\mathbf{A}_0$ prior does reflect a restriction on the forms of allowable prior beliefs: it implies that our beliefs about the deviations of structural equation coefficients from their prior means are independent across equations, conditional on $\mathbf{A}_0$. We believe that in most instances this is a reasonable starting place for a prior. In our example forecasting model, it implies that once we know that reduced form forecast errors for unemployment and GDP are positively correlated, we are likely to believe that coefficients on lagged variables differ from the random walk prior in the same way in both the unemployment and GDP reduced form equations. We believe that this is reasonable, and probably accounts for the better unemployment forecast that emerged from the $\mathbf{A}_+|\mathbf{A}_0$ prior.[10]

Of course deviating from the assumption of independence across columns of $\mathbf{A}_+$ may sometimes be substantively appealing and worth its computational price. For example, if we were modeling a collection of stock price time series, we might believe all were likely to be approximately random walks, but with a few deviating because they were thinly traded.

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[8] The Matlab code used for the maximization is available via the web at http://www.econ.yale.edu/~sims or directly via ftp at http://ftp.econ.yale.edu/pub/sims. It is more robust to deterioration of numerical accuracy and to certain kinds of one-dimensional discontinuities than are most such programs.

[9] Leeper, Sims and Zha [1996] also experimented with asymmetric priors on $(\mathbf{A}_0, \mathbf{A}_+)$.

[10] Remember that here we are using "the $\mathbf{A}_+|\mathbf{A}_0$ prior" to refer to a prior that makes the columns of $\mathbf{A}_+$ independent conditional on $\mathbf{A}_0$. Our computations actually expressed the Litterman prior itself in the form of a prior on $\mathbf{A}_+|\mathbf{A}_0$, but with dependence across columns of $\mathbf{A}_+$.

Correlations among innovations in the series would probably be determined by whether the companies the series represented were in related industries, and might have little to do with the likelihood that the stock was thinly traded. In this case, a prior that forced correlations of beliefs about reduced form coefficients to match correlations of innovations would not make sense; the Litterman prior might be more appealing despite its computational costs.

VI. Conclusion

We have shown that there is a form for a prior on the coefficients of a multiple-equation linear model that is both substantively appealing and computationally tractable. It should make it possible to extend Bayesian methods to larger models and to models with overidentifying restrictions. We hope that this will allow the transparency and reproducibility of Bayesian methods to be more widely available for tasks of forecasting and policy analysis.

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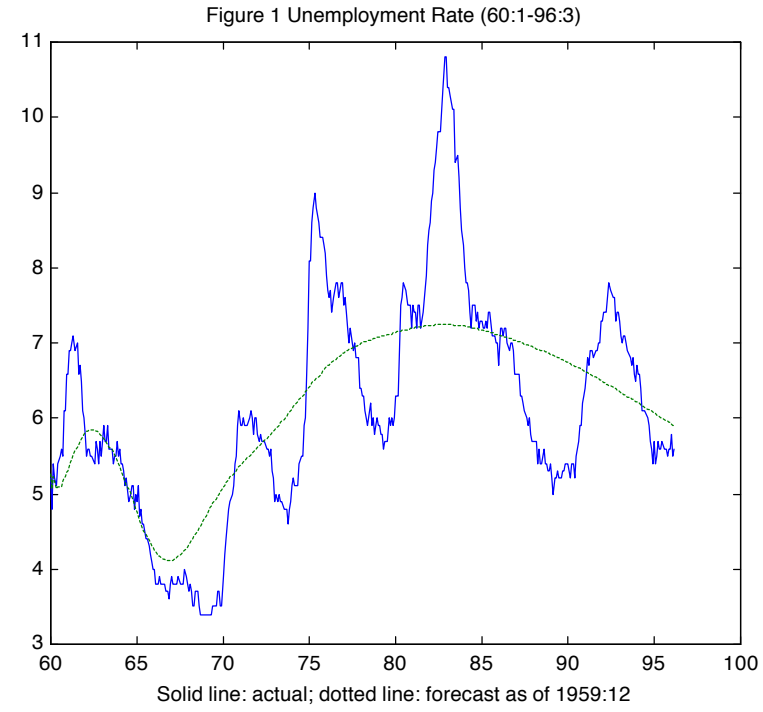
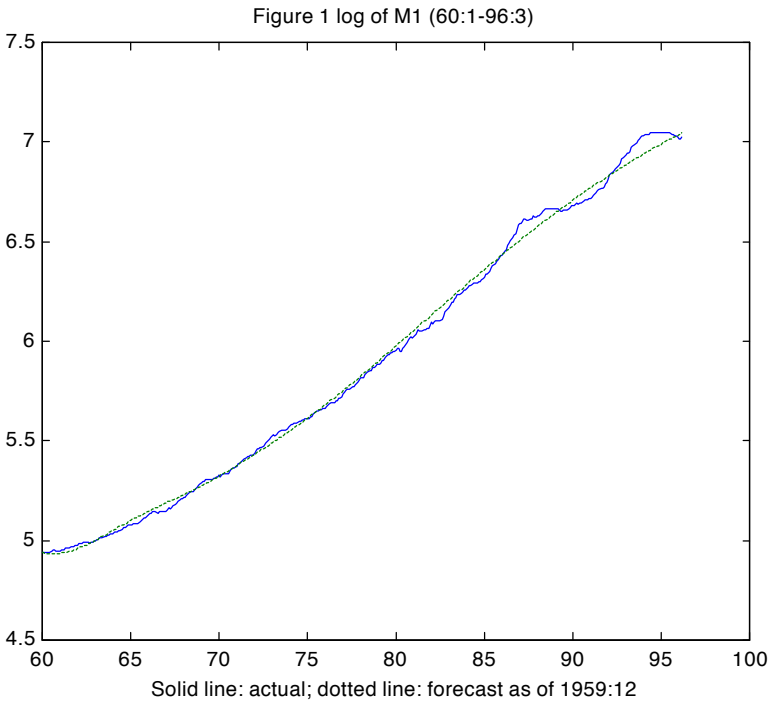
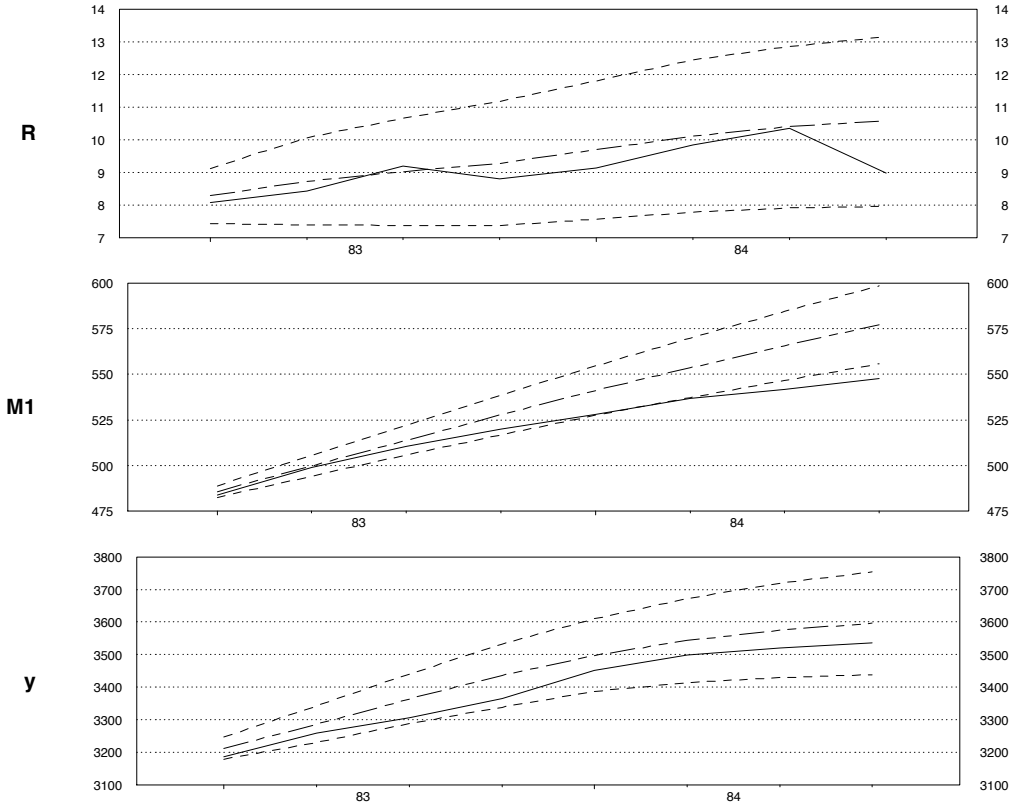
Figure 1 Actual and forecast values of M1 and the unemployment rate: 1960:1-1996:3





(Source: adapted from Leeper, Sims and Zha [1996])

Figure 2 Forecast as of 82:4: Sims-Zha proposed framework



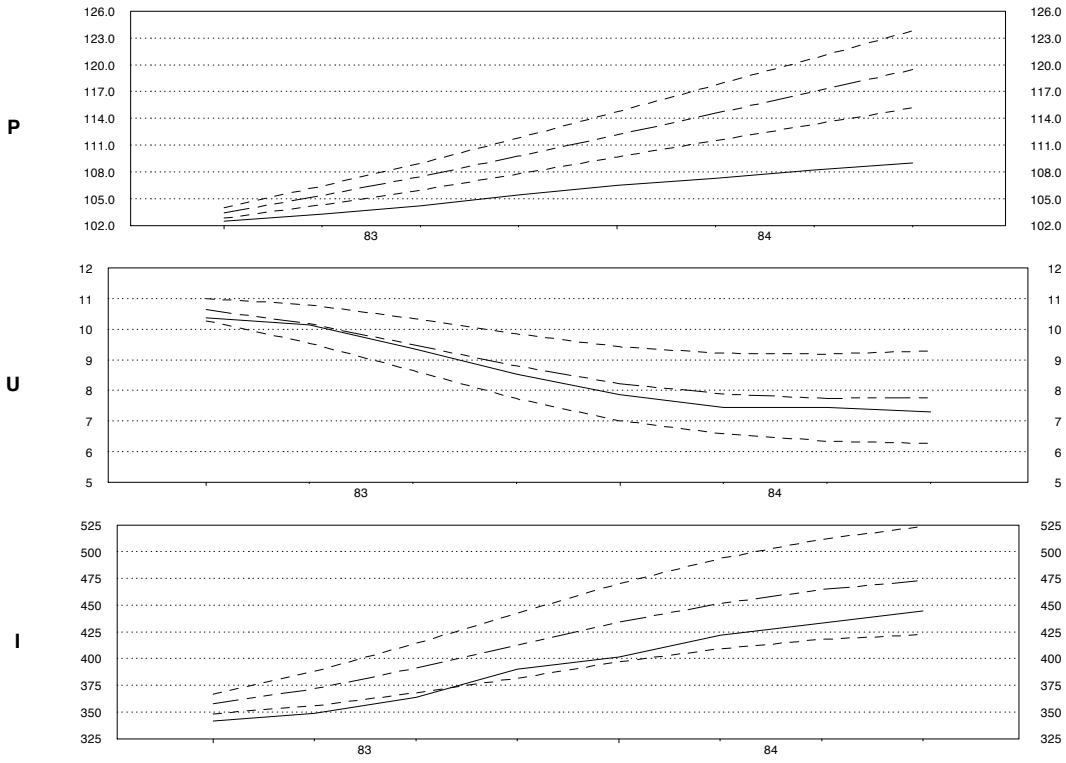
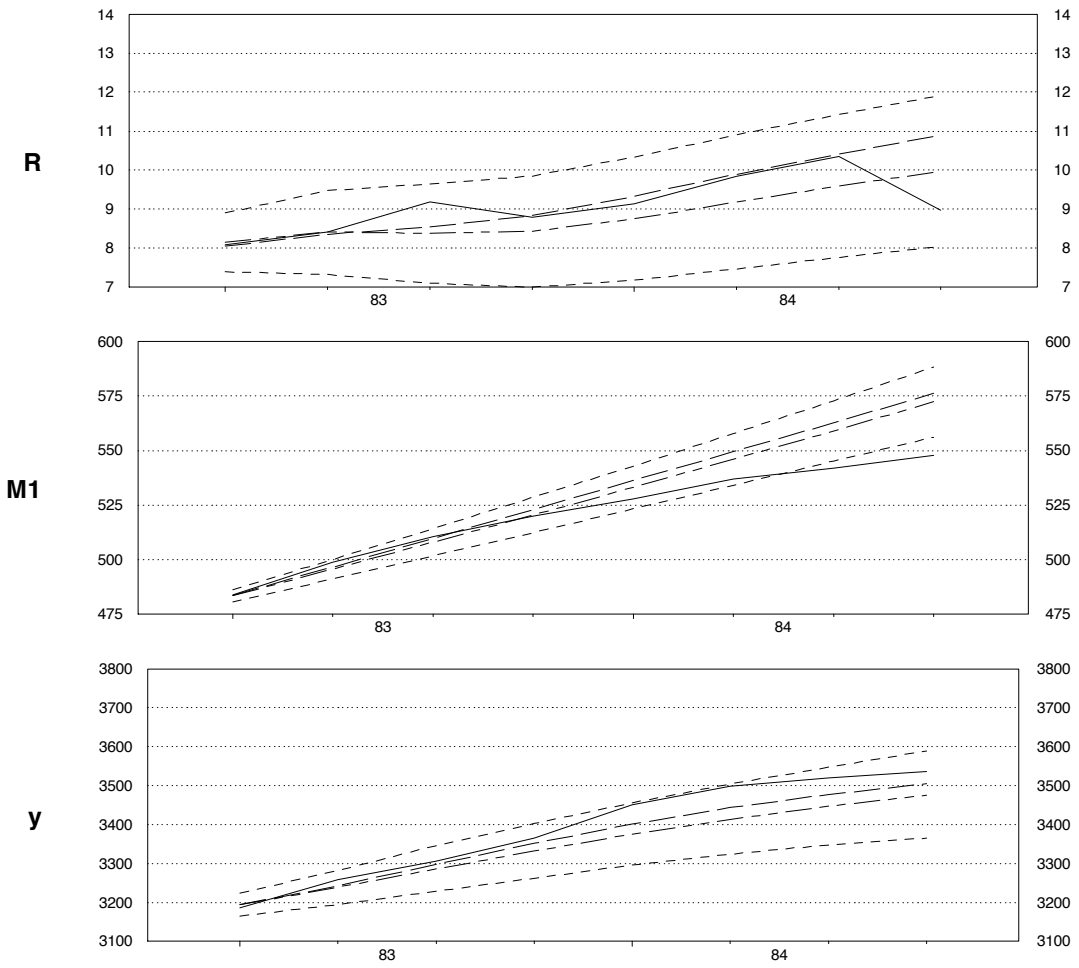


Figure 3 Forecast as of 82:4: Litterman's framework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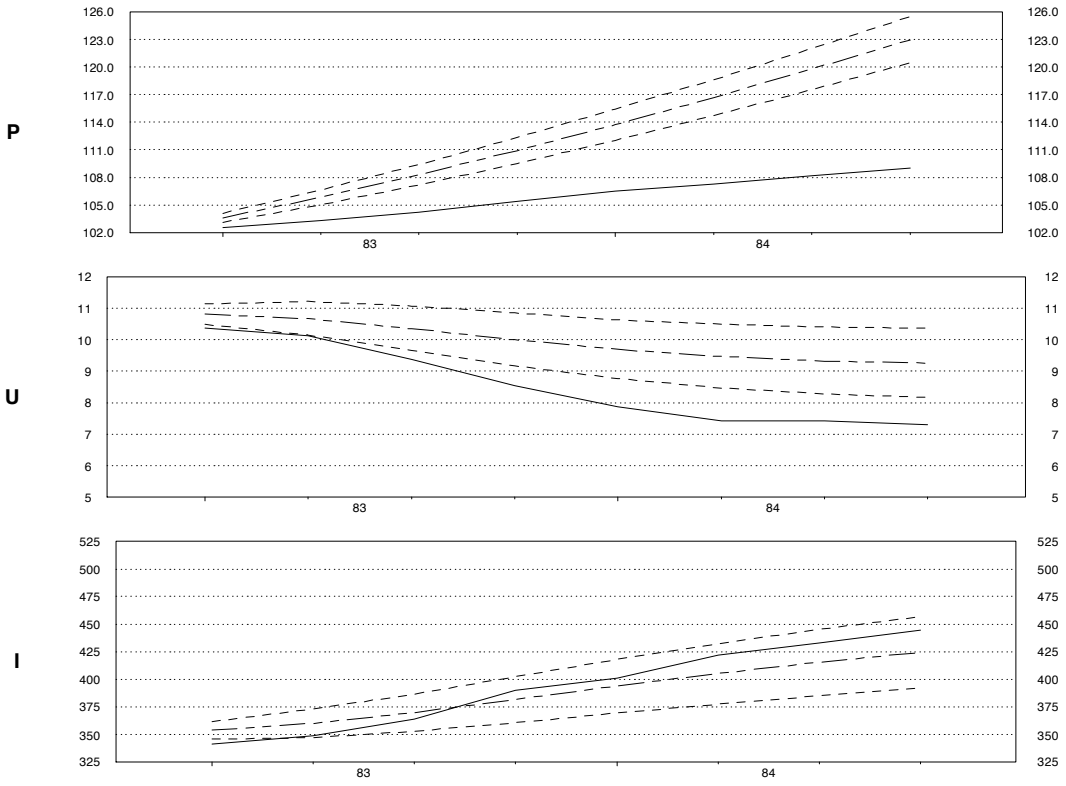
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