

RECURSION, DICTIONARIES

(download slides and .py files and follow along!)

6.0001 LECTURE 6

QUIZ PREP

- a paper and an online component
- open book/notes
- not open Internet, not open computer
- start printing out whatever you may want to bring

LAST TIME

- tuples - immutable
- lists - mutable
- aliasing, cloning
- mutability side effects

TODAY

- recursion – divide/decrease and conquer
- dictionaries – another mutable object type

RECURSION

Recursion is the process of repeating items in a self-similar way.

WHAT IS RECURSION?

- Algorithmically: a way to design solutions to problems by **divide-and-conquer** or **decrease-and-conquer**
 - reduce a problem to simpler versions of the same problem
- Semantically: a programming technique where a **function calls itself**
 - in programming, goal is to NOT have infinite recursion
 - must have **1 or more base cases** that are easy to solve
 - must solve the same problem on **some other input** with the goal of simplifying the larger problem input

ITERATIVE ALGORITHMS SO FAR

- looping constructs (`while` and `for` loops) lead to **iterative** algorithms
- can capture computation in a set of **state variables** that update on each iteration through loop

MULTIPLICATION – ITERATIVE SOLUTION

- “multiply $a * b$ ” is equivalent to “add a to itself b times”

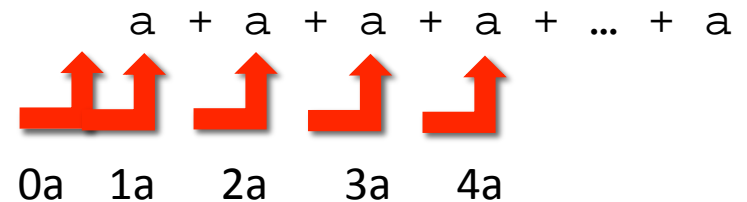
- capture **state** by

- an **iteration** number (i) starts at b

$i \leftarrow i - 1$ and stop when 0

- a current **value of computation** (result)

$\text{result} \leftarrow \text{result} + a$



```
def mult_iter(a, b):
```

```
    result = 0
```

```
    while b > 0:
```

```
        result += a
```

```
        b -= 1
```

```
    return result
```

iteration
current value of computation,
a running sum
current value of iteration variable

MULTIPLICATION – RECURSIVE SOLUTION

■ recursive step

- think how to reduce problem to a **simpler/smaller version** of same problem

$$\begin{aligned} a * b &= \underbrace{a + a + a + a + \dots + a}_{b \text{ times}} \\ &= a + \underbrace{a + a + a + \dots + a}_{b-1 \text{ times}} \\ &= a + \boxed{a * (b-1)} \end{aligned}$$

recursive reduction

■ base case

- keep reducing problem until reach a simple case that can be **solved directly**
- when $b = 1$, $a * b = a$

```
def mult(a, b):  
    if b == 1:  
        return a  
    else:  
        return a + mult(a, b-1)
```

base case

recursive step

FACTORIAL

$$n! = n * (n-1) * (n-2) * (n-3) * \dots * 1$$

- for what n do we know the factorial?

```
n = 1      →      if n == 1:
                        return 1
```

base case

- how to reduce problem? Rewrite in terms of something simpler to reach base case

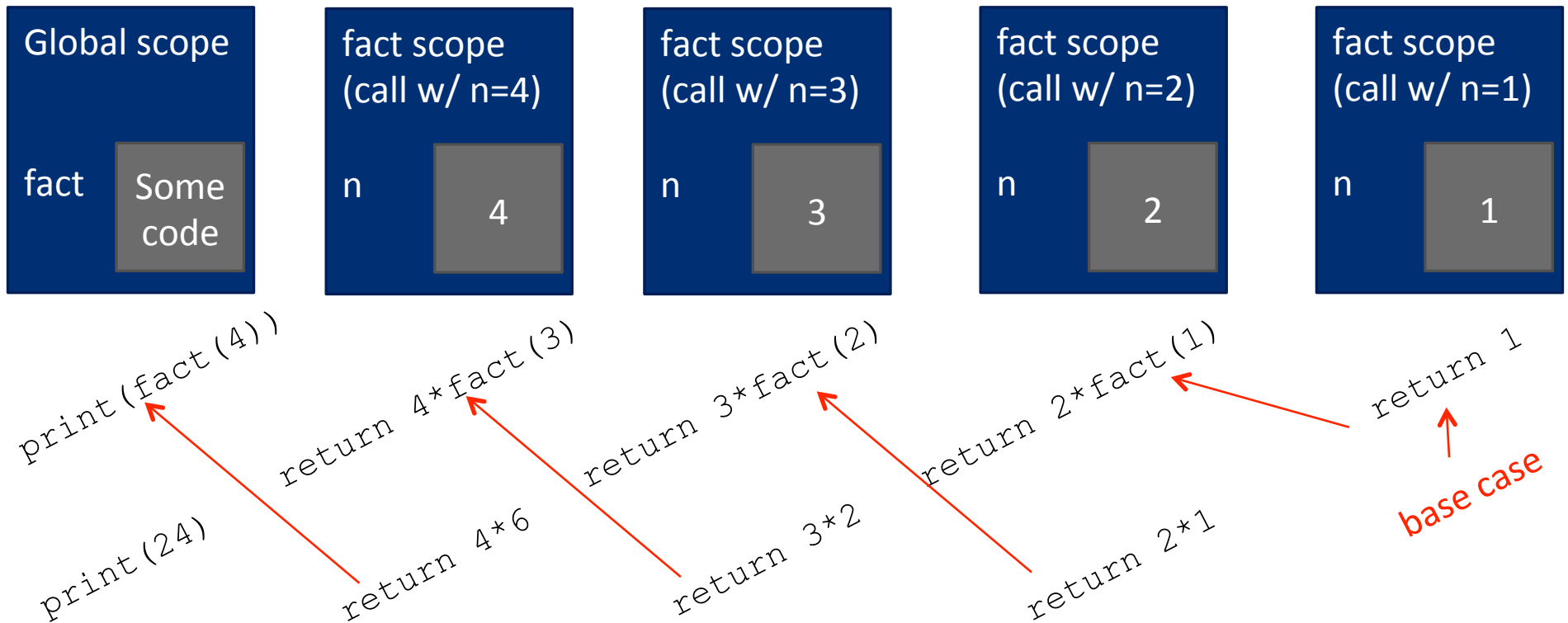
```
n*(n-1)!      →      else:
                        return n*factorial(n-1)
```

n-1)

recursive step

RECURSIVE FUNCTION SCOPE EXAMPLE

```
def fact(n):  
    if n == 1:  
        return 1  
    else:  
        return n*fact(n-1)  
  
print(fact(4))
```



SOME OBSERVATIONS

- each recursive call to a function creates its **own scope/environment**
- **bindings of variables** in a scope are not changed by recursive call
- flow of control passes back to **previous scope** once function call returns value

using the same variable names but they are different objects in separate scopes

ITERATION vs. RECURSION

```
def factorial_iter(n):  
    prod = 1  
    for i in range(1,n+1):  
        prod *= i  
    return prod
```

```
def factorial(n):  
    if n == 1:  
        return 1  
    else:  
        return n*factorial(n-1)
```

- recursion may be simpler, more intuitive
- recursion may be efficient from programmer POV
- recursion may not be efficient from computer POV

INDUCTIVE REASONING

- How do we know that our recursive code will work?
- `mult_iter` terminates because `b` is initially positive, and decreases by 1 each time around loop; thus must eventually become less than 1
- `mult` called with `b = 1` has no recursive call and stops
- `mult` called with `b > 1` makes a recursive call with a smaller version of `b`; must eventually reach call with `b = 1`

```
def mult_iter(a, b):  
    result = 0  
    while b > 0:  
        result += a  
        b -= 1  
    return result
```

```
def mult(a, b):  
    if b == 1:  
        return a  
    else:  
        return a + mult(a, b-1)
```

MATHEMATICAL INDUCTION

- To prove a statement indexed on integers is true for all values of n :
 - Prove it is true when n is smallest value (e.g. $n = 0$ or $n = 1$)
 - Then prove that if it is true for an arbitrary value of n , one can show that it must be true for $n+1$

EXAMPLE OF INDUCTION

- $0 + 1 + 2 + 3 + \dots + n = (n(n+1))/2$
- Proof:
 - If $n = 0$, then LHS is 0 and RHS is $0*1/2 = 0$, so true
 - Assume true for some k , then need to show that
$$0 + 1 + 2 + \dots + k + (k+1) = ((k+1)(k+2))/2$$
 - LHS is $k(k+1)/2 + (k+1)$ by assumption that property holds for problem of size k
 - This becomes, by algebra, $((k+1)(k+2))/2$
 - Hence expression holds for all $n \geq 0$

RELEVANCE TO CODE?

- Same logic applies

```
def mult(a, b):  
    if b == 1:  
        return a  
    else:  
        return a + mult(a, b-1)
```

- Base case, we can show that `mult` must return correct answer
- For recursive case, we can assume that `mult` correctly returns an answer for problems of size smaller than `b`, then by the addition step, it must also return a correct answer for problem of size `b`
- Thus by induction, code correctly returns answer

TOWERS OF HANOI

- The story:
 - 3 tall spikes
 - Stack of 64 different sized discs – start on one spike
 - Need to move stack to second spike (at which point universe ends)
 - Can only move one disc at a time, and a larger disc can never cover up a small disc

TOWERS OF HANOI

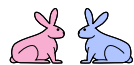
- Having seen a set of examples of different sized stacks, how would you write a program to print out the right set of moves?
- **Think recursively!**
 - Solve a smaller problem
 - Solve a basic problem
 - Solve a smaller problem

```
def printMove(fr, to):  
    print('move from ' + str(fr) + ' to ' + str(to))  
  
def Towers(n, fr, to, spare):  
    if n == 1:  
        printMove(fr, to)  
    else:  
        Towers(n-1, fr, spare, to)  
        Towers(1, fr, to, spare)  
        Towers(n-1, spare, to, fr)
```

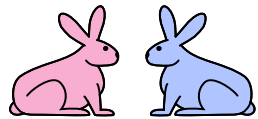
RECURSION WITH MULTIPLE BASE CASES

- Fibonacci numbers

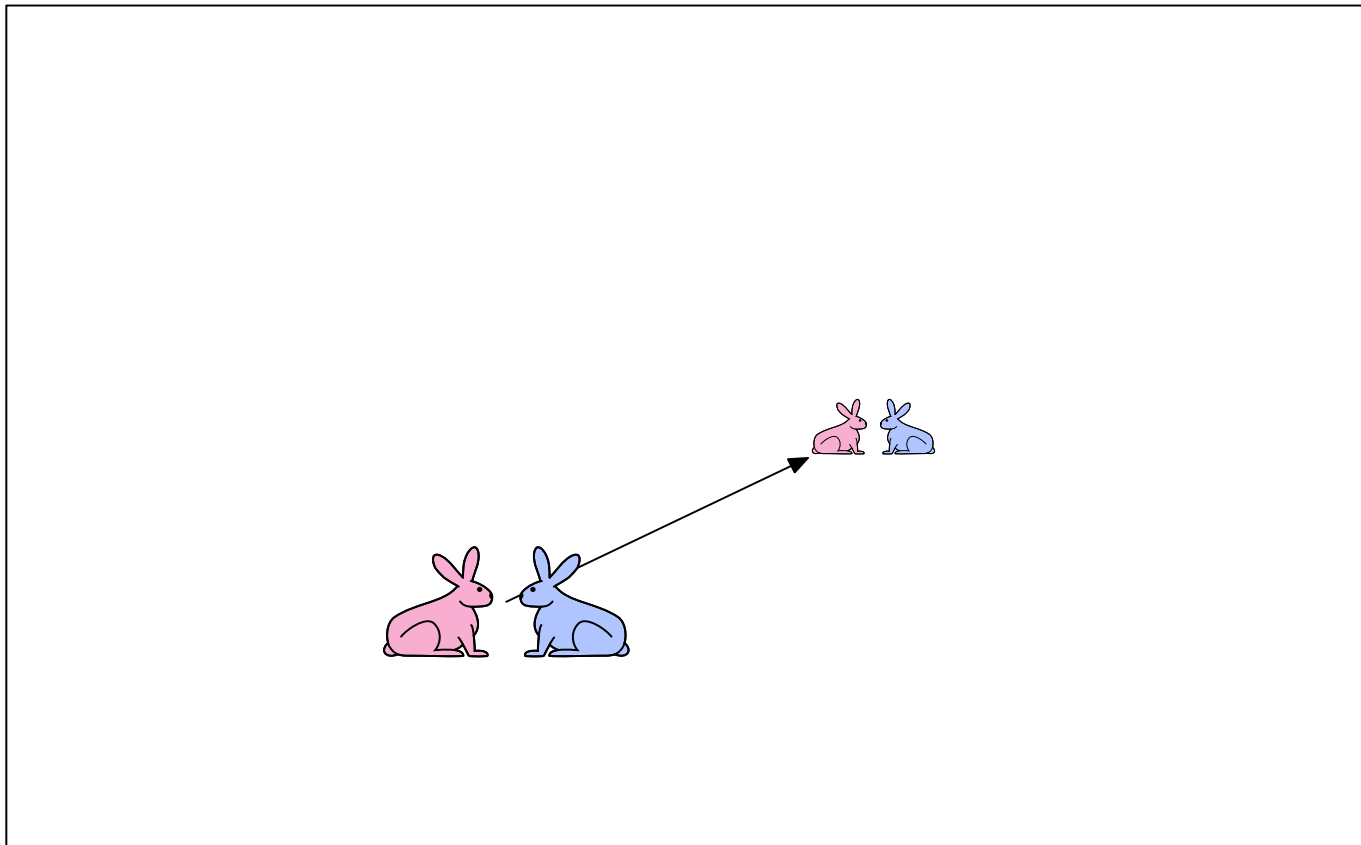
- Leonardo of Pisa (aka Fibonacci) modeled the following challenge
 - Newborn pair of rabbits (one female, one male) are put in a pen
 - Rabbits mate at age of one month
 - Rabbits have a one month gestation period
 - Assume rabbits never die, that female always produces one new pair (one male, one female) every month from its second month on.
 - How many female rabbits are there at the end of one year?



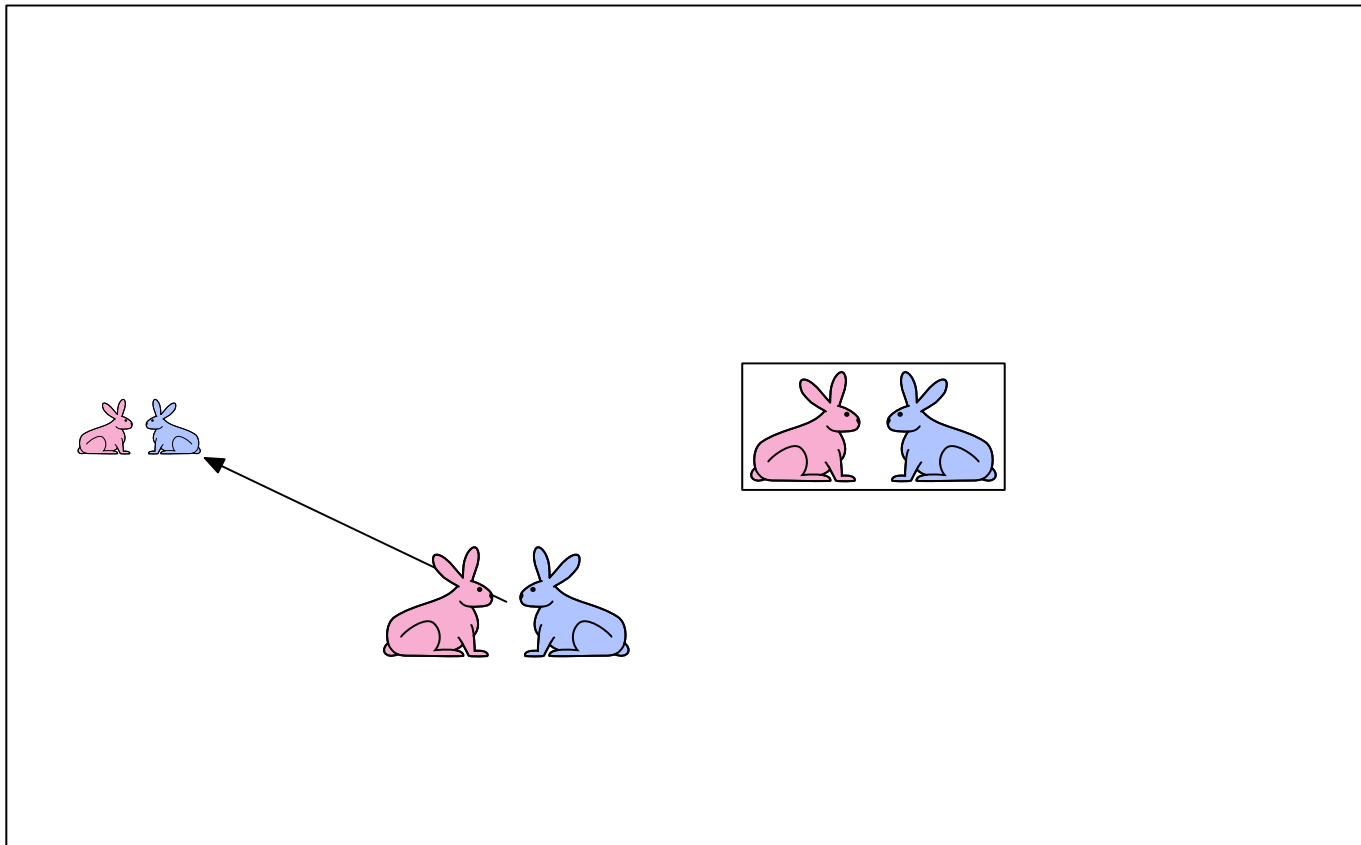
Demo courtesy of Prof. Denny Freeman and Adam Hartz



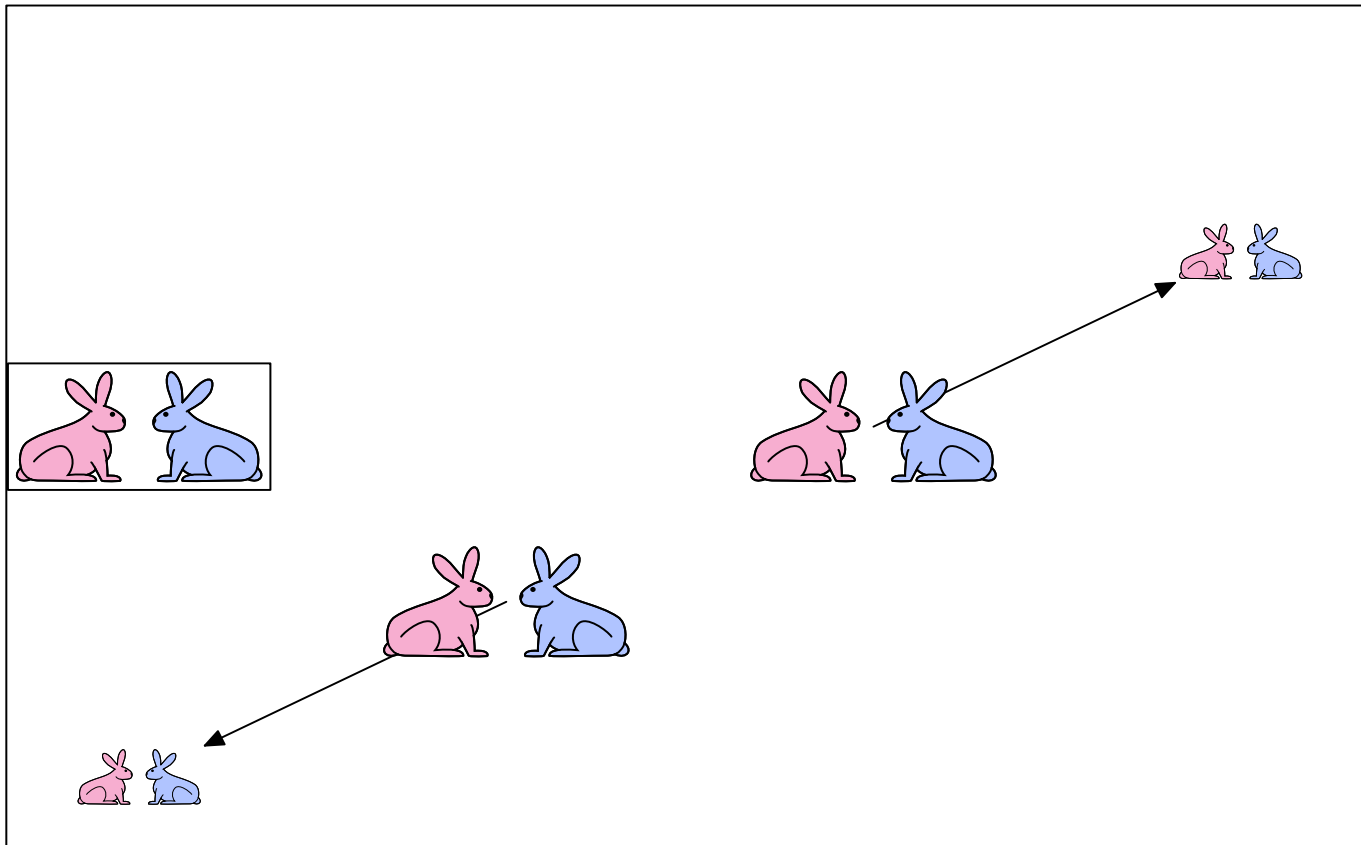
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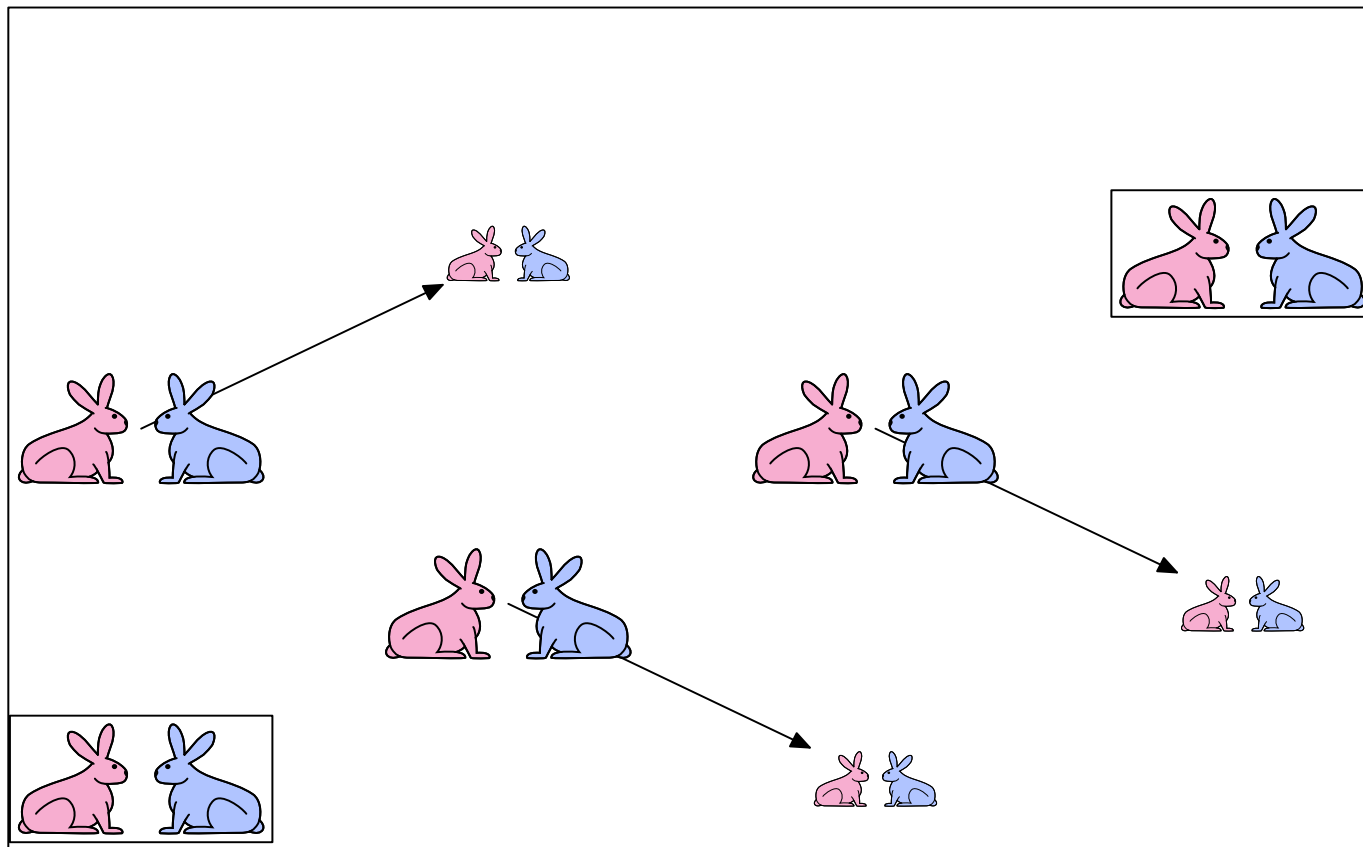
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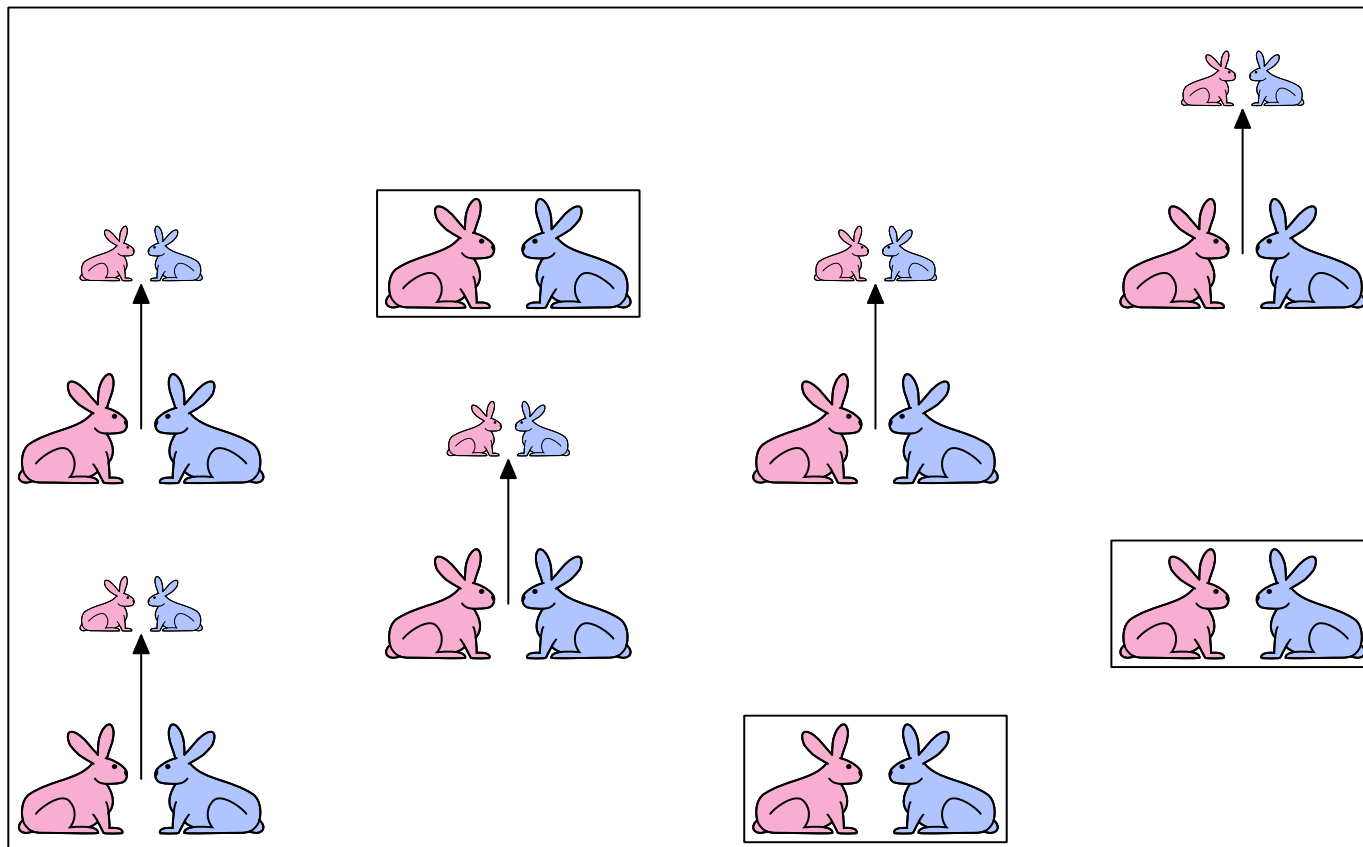
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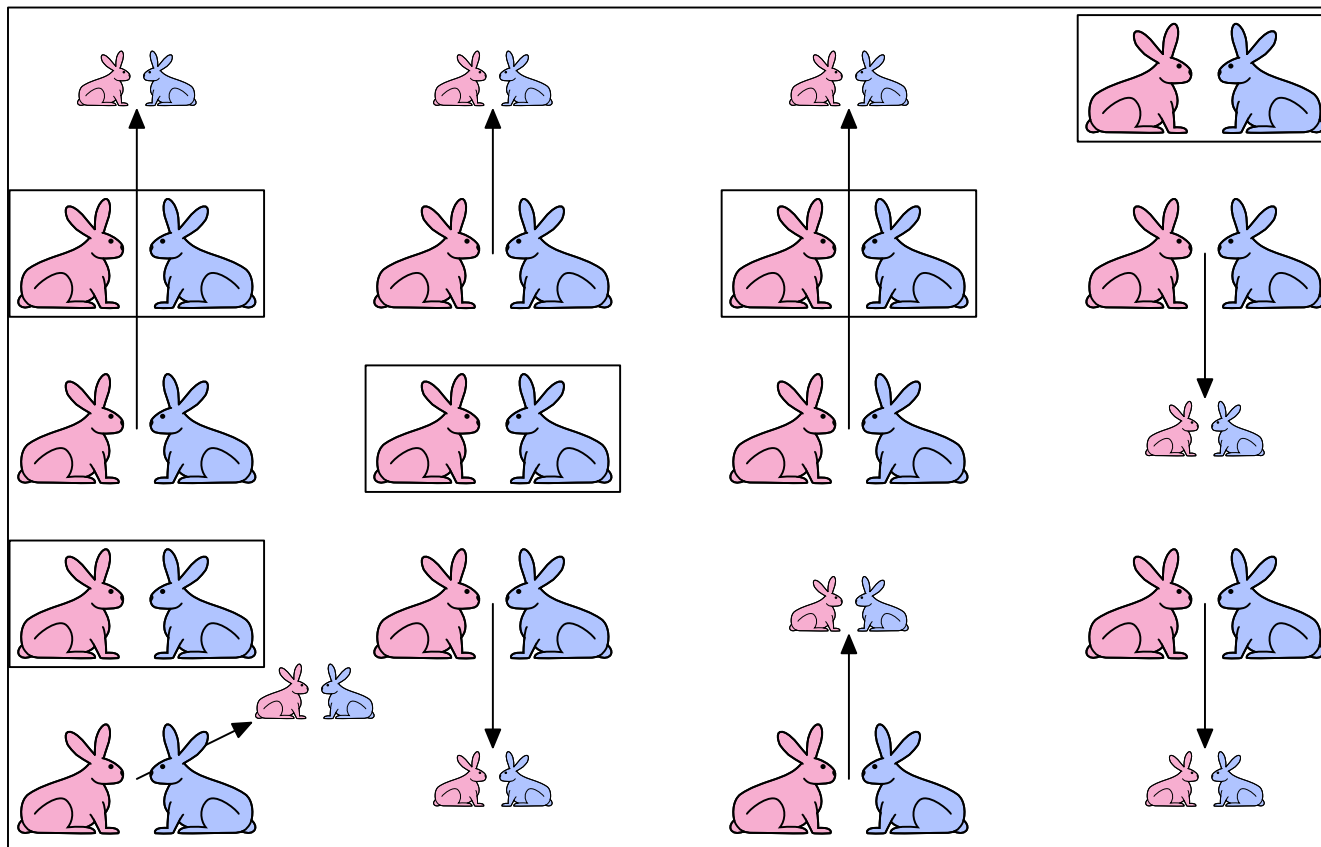
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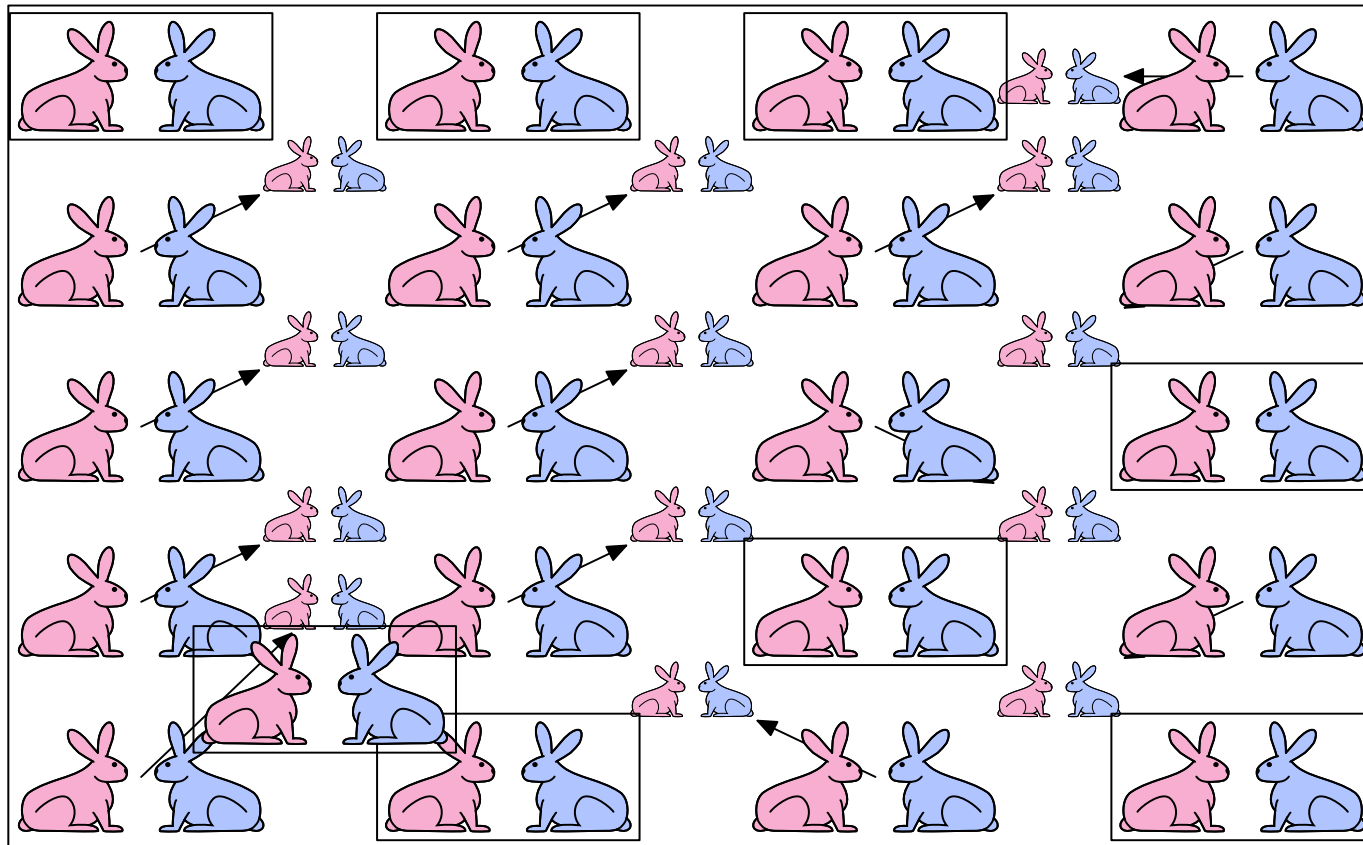


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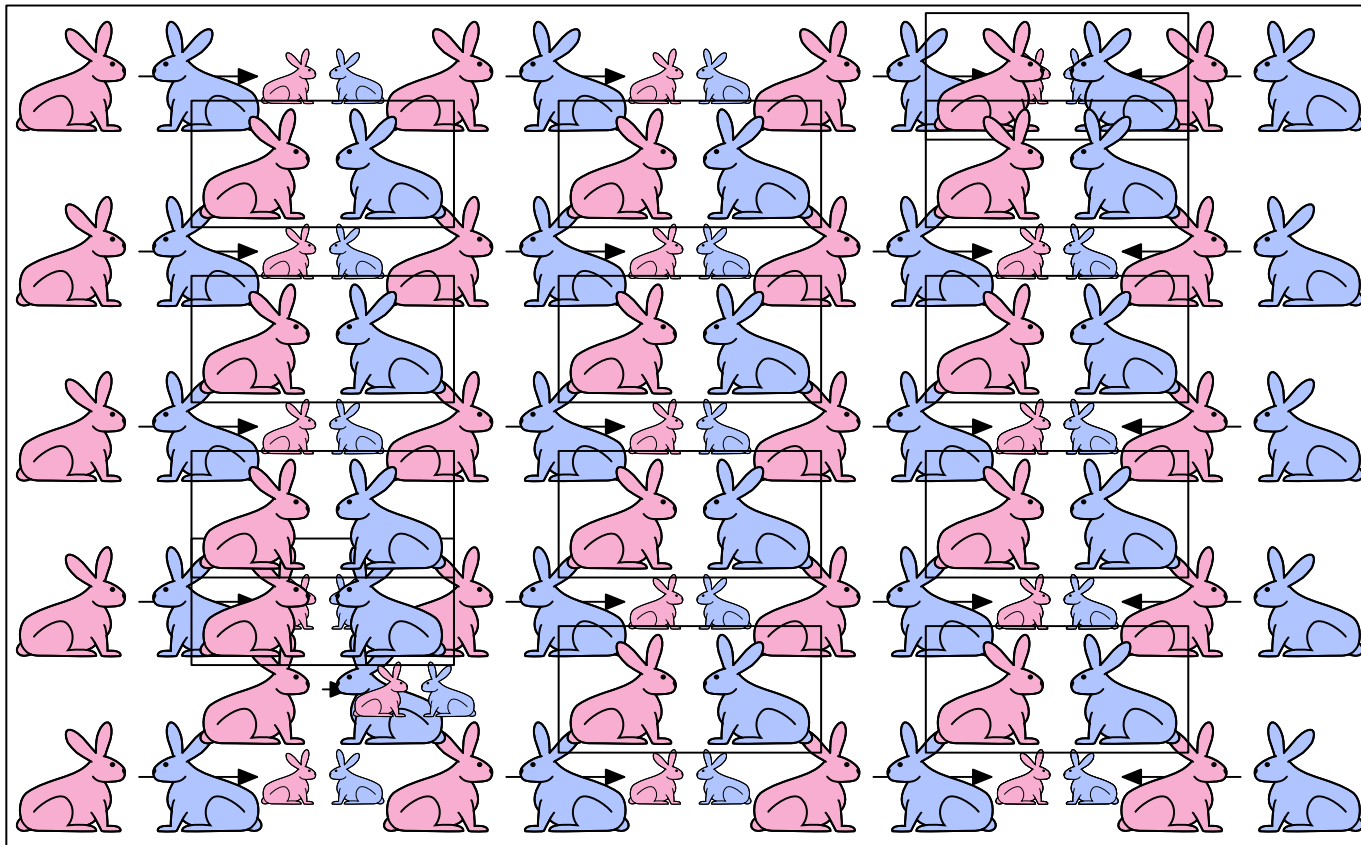


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FIBONACCI

After one month (call it 0) – 1 female

After second month – still 1 female (now pregnant)

After third month – two females, one pregnant, one not

In general, $\text{females}(n) = \text{females}(n-1) + \text{females}(n-2)$

- Every female alive at month $n-2$ will produce one female in month n ;
- These can be added those alive in month $n-1$ to get total alive in month n

Month	Females
0	1

FIBONACCI

- Base cases:
 - $\text{Females}(0) = 1$
 - $\text{Females}(1) = 1$
- Recursive case
 - $\text{Females}(n) = \text{Females}(n-1) + \text{Females}(n-2)$

FIBONACCI

```
def fib(x):  
    """assumes x an int >= 0  
        returns Fibonacci of x"""  
    if x == 0 or x == 1:  
        return 1  
    else:  
        return fib(x-1) + fib(x-2)
```

RECURSION ON NON-NUMERICS

- how to check if a string of characters is a palindrome, i.e., reads the same forwards and backwards
 - “Able was I, ere I saw Elba” – attributed to Napoleon
 - “Are we not drawn onward, we few, drawn onward to new era?” – attributed to Anne Michaels



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SOLVING RECURSIVELY?

- First, convert the string to just characters, by stripping out punctuation, and converting upper case to lower case
- Then
 - Base case: a string of length 0 or 1 is a palindrome
 - Recursive case:
 - If first character matches last character, then is a palindrome if middle section is a palindrome

EXAMPLE

- 'Able was I, ere I saw Elba' → 'ablewasiereisawleba'
- `isPalindrome('ablewasiereisawleba')`
is same as
 - `'a' == 'a'` and
`isPalindrome('blewasiereisawleb')`

```
def isPalindrome(s):  
  
    def toChars(s):  
        s = s.lower()  
        ans = ''  
        for c in s:  
            if c in 'abcdefghijklmnopqrstuvwxyz':  
                ans = ans + c  
        return ans  
  
    def isPal(s):  
        if len(s) <= 1:  
            return True  
        else:  
            return s[0] == s[-1] and isPal(s[1:-1])  
  
    return isPal(toChars(s))
```

DIVIDE AND CONQUER

- an example of a “divide and conquer” algorithm
- solve a hard problem by breaking it into a set of sub-problems such that:
 - sub-problems are easier to solve than the original
 - solutions of the sub-problems can be combined to solve the original

DICTIONARIES

HOW TO STORE STUDENT INFO

- so far, can store using separate lists for every info

```
names = ['Ana', 'John', 'Denise', 'Katy']
```

```
grade = ['B', 'A+', 'A', 'A']
```

```
course = [2.00, 6.0001, 20.002, 9.01]
```

- a **separate list** for each item
- each list must have the **same length**
- info stored across lists at **same index**, each index refers to info for a different person

HOW TO UPDATE/RETRIEVE STUDENT INFO

```
def get_grade(student, name_list, grade_list, course_list):  
    i = name_list.index(student)  
    grade = grade_list[i]  
    course = course_list[i]  
    return (course, grade)
```

- **messy** if have a lot of different info to keep track of
- must maintain **many lists** and pass them as arguments
- must **always index** using integers
- must remember to change multiple lists

A BETTER AND CLEANER WAY – A DICTIONARY

- nice to **index item of interest directly** (not always int)
- nice to use **one data structure**, no separate lists

A list

0	Elem 1
1	Elem 2
2	Elem 3
3	Elem 4
...	...

index

element

A dictionary

Key 1	Val 1
Key 2	Val 2
Key 3	Val 3
Key 4	Val 4
...	...

custom
index by
label

element

A PYTHON DICTIONARY

- store pairs of data
 - key
 - value

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'

custom
index by
label

element

my_dict = { } *empty dictionary*

grades = { 'Ana': 'B', 'John': 'A+', 'Denise': 'A', 'Katy': 'A' }

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
key1 val1 key2 val2 key3 val3 key4 val4

DICTIONARY LOOKUP

- similar to indexing into a list
- **looks up** the **key**
- **returns** the **value** associated with the key
- if key isn't found, get an error

'Ana'	'B'
'Denise'	'A'
'John'	'A+ '
'Katy'	'A'

```
grades = { 'Ana': 'B', 'John': 'A+', 'Denise': 'A', 'Katy': 'A' }
```

```
grades[ 'John' ]      → evaluates to 'A+ '
```

```
grades[ 'Sylvan' ]    → gives a KeyError
```

DICTIONARY OPERATIONS

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'
'Sylvan'	'A'

```
grades = { 'Ana': 'B', 'John': 'A+', 'Denise': 'A', 'Katy': 'A' }
```

- **add** an entry

```
grades[ 'Sylvan' ] = 'A'
```

- **test** if key in dictionary

```
'John' in grades      → returns True  
'Daniel' in grades   → returns False
```

- **delete** entry

```
del(grades[ 'Ana' ])
```

DICTIONARY OPERATIONS

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'

```
grades = { 'Ana': 'B', 'John': 'A+', 'Denise': 'A', 'Katy': 'A' }
```

- get an **iterable that acts like a tuple of all keys** *no guaranteed order*
`grades.keys()` → returns ['Denise', 'Katy', 'John', 'Ana']
- get an **iterable that acts like a tuple of all values**
`grades.values()` → returns ['A', 'A', 'A+', 'B'] *no guaranteed order*

DICTIONARY KEYS and VALUES

■ values

- any type (**immutable and mutable**)
- can be **duplicates**
- dictionary values can be lists, even other dictionaries!

■ keys

- must be **unique**
- **immutable** type (`int`, `float`, `string`, `tuple`, `bool`)
 - actually need an object that is **hashable**, but think of as immutable as all immutable types are hashable
- careful with `float` type as a key

■ **no order** to keys or values!

```
d = {4:{1:0}, (1,3):"twelve", 'const':[3.14,2.7,8.44]}
```


list vs dict

- **ordered** sequence of elements
- look up elements by an integer index
- indices have an **order**
- index is an **integer**

- **matches** “keys” to “values”
- look up one item by another item
- **no order** is guaranteed
- key can be any **immutable** type

EXAMPLE: 3 FUNCTIONS TO ANALYZE SONG LYRICS

- 1) create a **frequency dictionary** mapping `str: int`
- 2) find **word that occurs the most** and how many times
 - use a list, in case there is more than one word
 - return a tuple `(list, int)` for `(words_list, highest_freq)`
- 3) find the **words that occur at least X times**
 - let user choose “at least X times”, so allow as parameter
 - return a list of tuples, each tuple is a `(list, int)` containing the list of words ordered by their frequency
 - IDEA: From song dictionary, find most frequent word. Delete most common word. Repeat. It works because you are mutating the song dictionary.

CREATING A DICTIONARY

```
def lyrics_to_frequencies(lyrics):  
    myDict = {}  
    for word in lyrics:  
        if word in myDict:  
            myDict[word] += 1  
        else:  
            myDict[word] = 1  
    return myDict
```

can iterate over list
can iterate over keys
in dictionary
update value
associated with key

USING THE DICTIONARY

```
def most_common_words(freqs):  
    values = freqs.values()  
    best = max(values)  
    words = []  
    for k in freqs:  
        if freqs[k] == best:  
            words.append(k)  
    return (words, best)
```

*this is an iterable, so can
apply built-in function*

*can iterate over keys
in dictionary*

LEVERAGING DICTIONARY PROPERTIES

```
def words_often(freqs, minTimes):  
    result = []  
    done = False  
    while not done:  
        temp = most_common_words(freqs)  
        if temp[1] >= minTimes:  
            result.append(temp)  
            for w in temp[0]:  
                del(freqs[w])  
        else:  
            done = True  
    return result
```

*can directly mutate
dictionary; makes it
easier to iterate*

```
print(words_often(beatles, 5))
```

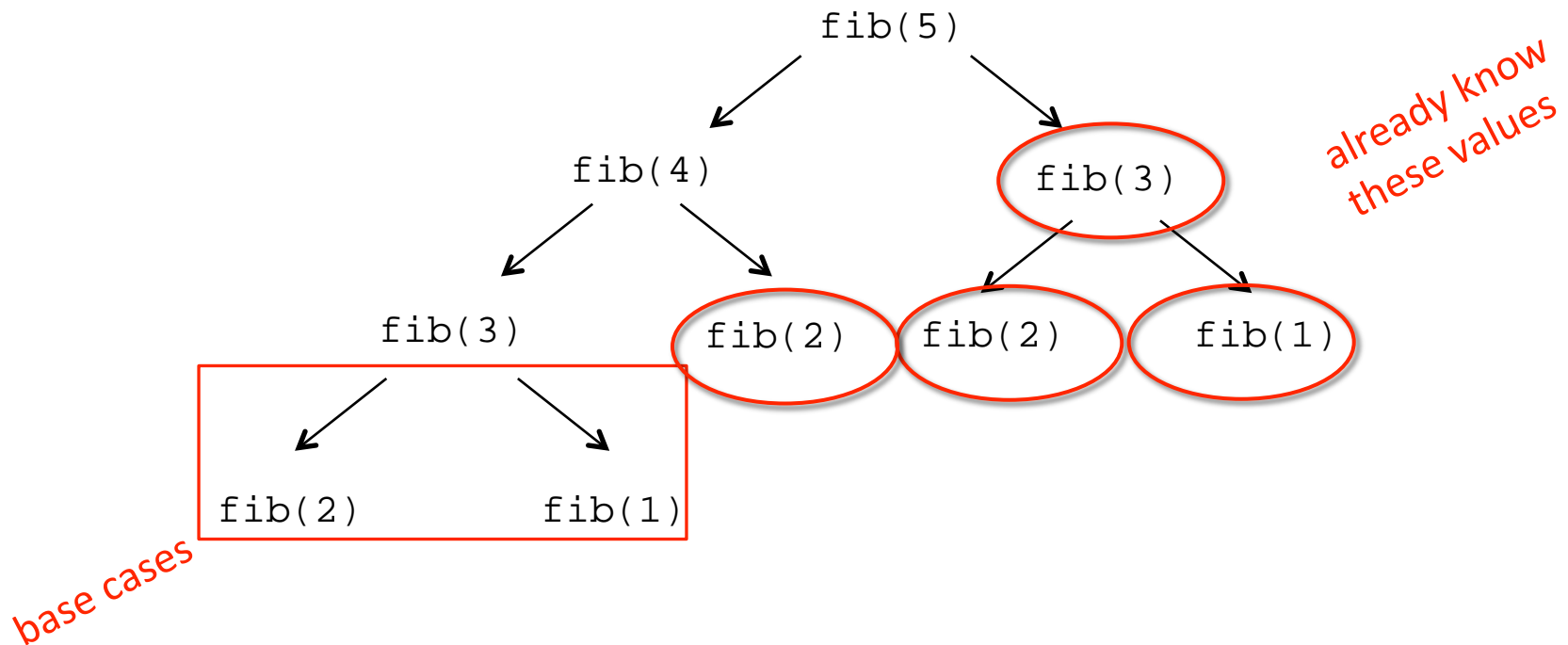
FIBONACCI RECURSIVE CODE

```
def fib(n):  
    if n == 1:  
        return 1  
    elif n == 2:  
        return 2  
    else:  
        return fib(n-1) + fib(n-2)
```

- two base cases
- calls itself twice
- this code is inefficient

INEFFICIENT FIBONACCI

$$\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2)$$



- **recalculating** the same values many times!
- could keep **track** of already calculated values

FIBONACCI WITH A DICTIONARY

```
def fib_efficient(n, d):  
    if n in d:  
        return d[n]  
    else:  
        ans = fib_efficient(n-1, d) + fib_efficient(n-2, d)  
        d[n] = ans  
        return ans  
  
d = {1:1, 2:2}  
print(fib_efficient(6, d))
```

Method sometimes
called "memoization"

Initialize dictionary
with base cases

- do a **lookup first** in case already calculated the value
- **modify dictionary** as progress through function calls

EFFICIENCY GAINS

- Calling `fib(34)` results in 11,405,773 recursive calls to the procedure
- Calling `fib_efficient(34)` results in 65 recursive calls to the procedure
- Using dictionaries to capture intermediate results can be very efficient
- But note that this only works for procedures without side effects (i.e., the procedure will always produce the same result for a specific argument independent of any other computations between calls)

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