# Fintech and Financial Engineering

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# Week 4

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# Two example portfolios

Consider a one-period market with $n$ securities with identical expected returns and variances:

- $\mathbb{E}[R_i] = \mu$ and $\mathrm{Var}(R_i) = \sigma^2$ for $i = 1, ..., n$.
- $Cov(R_i, R_j) = 0$ for all $i \neq j$.
- Let $x_i$ denote the fraction of wealth invested in the $i$-th security at time $t = 0$.
- Note that we must have $\sum_{i=1}^n x_i = 1$ for any portfolio.

Consider now two portfolios:

- Portfolio A: All funds invested in security 1, that is, $x_1 = 1$ and $x_i = 0$ for $i > 1$.
- Portfolio B: Equally-weighted portfolio: $x_i = \frac{1}{n}$ for $i = 1, \ldots, n$.

Denote by $R_A$ and $R_B$ the returns of the portfolio. Then:

- $\mathbb{E}[R_A] = \mu = \mathbb{E}[R_B]$
- $\mathrm{Var}(R_A) = \sigma^2$, $\mathrm{Var}(R_B) = \sigma^2/n$.

# Markowitz Portfolio Optimization

**Economic perspective:** Two effects need to be clarified in the context of investment in capital markets:

- How does portfolio formation affect "value" and "risk"?
- How to optimally choose portfolio?

**Main characteristics and informal description of the Markowitz model:**

- Static, one-period model with $n$ assets
- Financial assets and portfolios are assessed using the standard deviation of returns as a measure of risk and the expected return as a measure of value.
- If two portfolios have the same risk, the investors prefer the portfolio with higher expected return.
- If two portfolios have the same expected return, then investors prefer the portfolio with lower risk.

**Setting:**

### Definition

A portfolio with return $R$ **dominates** a portfolio with return $\bar{R}$, if one of the following conditions hold:

- $\mathrm{Var}(R) < \mathrm{Var}(\bar{R})$ and $\mathbb{E}[R] \geq \mathbb{E}[\bar{R}]$
- $\mathbb{E}[R] > \mathbb{E}[\bar{R}]$ and $\mathrm{Var}(R) \leq \mathrm{Var}(\bar{R})$

A portfolio is **efficient**, if it is not dominated by any other portfolio.

- $R = (R_1, \ldots, R_n)$ is an $n$-dimensional random variable; $R_i$ is the return of asset $i$ "tomorrow"
- Let $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}$ the proportion of wealth invested in asset $i$.
- The return of the portfolio with weights $x$ is thus

$$R_p = R_p(x) = \sum_{i=1}^{n} x_i R_i$$

- **Notation**:
  - $\mu_i = \mathbb{E}[R_i]$
  - $\sigma_i = \sigma(R_i)$
  - $\rho_{ij} = \mathrm{Corr}(R_i, R_j)$
  - $\mu = (\mu_1 \ldots, \mu_n)$
  - $C = (\mathrm{Cov}(R_i, R_j))_{i,j=1,\ldots,n}$
  - $\mu_p = \mu_p(x) = \mathbb{E}[R_p(x)]$
  - $\sigma_p = \sigma_p(x) = \sigma(R_p(x)) = \sqrt{\mathrm{Var}(R_p(x))}$

- $e = (1, \ldots, 1)$, $\underline{0} = (0, \ldots, 0)$.
- Denote by $D$ the set of admissible weights and

$$M = \{(\sigma(R_p(x)), \mathbb{E}[R_p(x)]) : x \in D\}$$

- Restrictions on investment (we only consider two situations here):
  1. "Short sales are allowed":

$$D = D_1 := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1 \right\} = \{x \in \mathbb{R}^n : e \cdot x = 1\}$$

  2. "No short sales":

$$D = D_2 := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1 \right\}$$

- **Notice:**
  - $\mathbb{E}\left[R_p(x)\right] = \sum_{i=1}^{n} x_i \mu_i = \mu \cdot x$
  - $$\begin{aligned}
\sigma_p^2(x) &= \mathrm{Var}\left(R_p(x)\right) = \mathrm{Cov}\left(R_p(x), R_p(x)\right) = \mathrm{Cov}\left(\sum_{i=1}^{n} x_i R_i, \sum_{j=1}^{n} x_j R_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \underbrace{\mathrm{Cov}\left(R_i, R_j\right)}_{=C_{i,j}} \\
&= x^\top C x
\end{aligned}$$

**Goal:** Characterize the set $D^*$ of weights associated to the set $M^*$ of risk-return pairs associated to efficient portfolios, i.e.,

$$M^* = \{(\sigma_p(x^*), \mu_p(x^*)) : \ x^* \in D^*\}$$

To achieve this, we consider the following optimization problem: fix $t \geq 0$ and solve:

$$\text{maximize } Z(x) = t\mathbb{E}\left[R_p(x)\right] - \frac{1}{2}\operatorname{Var}\left(R_p(x)\right) = t\mu \cdot x - \frac{1}{2}x^\top Cx \qquad (\star)$$

over $x \in D$.

**Solution methods:**

- **Case 1:** $D = D_1$: Explicit solution by Lagrange-approach (see below)
- **Case 2:** $D = D_2$: Numerical solution (quadratic solution)

We now consider **Case 1** in more detail: Let

$$D^* = \{x^* \in \mathbb{R}^n : x^* \text{ is efficient}\}$$

.

### Theorem

*Let $D = D_1$ and assume $C$ is invertible. Then the set of efficient portfolios is given as*

$$D^* = \left\{x_t \in \mathbb{R}^n : x_t = t\left(C^{-1}\mu - C^{-1}e\alpha_0\right) + C^{-1}e\gamma_0, t \geq 0\right\},$$

*with* $\alpha_0 = \frac{e^\top C^{-1}\mu}{e^T C^{-1} e}$ *and* $\gamma_0 = \frac{1}{e^T C^{-1} e}$.

# Proof

We first solve $(\star)$; we show that

$$x_t = t\left(C^{-1}\mu - C^{-1}e\alpha_0\right) + C^{-1}e\gamma_0$$

is the unique solution to $(\star)$. Then we check that each $x_t$ corresponds to an efficient portfolio and vice versa. Set $L(x,\lambda) = t\mu \cdot x - \frac{1}{2}x^\top Cx - \lambda(e \cdot x - 1)$. Then

$$\begin{aligned} 0 \overset{!}{=} \partial_{x_k} L(x,\lambda) &= t\mu_k - \frac{1}{2}\left(2x_k C_{kk} + \sum_{\substack{i=1\\ i\neq k}}^{n} x_i C_{ik} + \sum_{\substack{j=1\\ j\neq k}}^{n} x_j C_{kj}\right) - \lambda \\ &= t\mu_k - (Cx)_k - \lambda \\ 0 \overset{!}{=} \partial_\lambda L(x,\lambda) &= -e\cdot x + 1. \end{aligned}$$

This yields $0 = t\mu - Cx - \lambda e$ and so $x = C^{-1}(t\mu - \lambda e)$.

Consequently

$$
\begin{aligned}
1 = e \cdot x = e \cdot \left(C^{-1}(t\mu - \lambda e)\right) \\
= \left(e^\top C^{-1}\mu\right) t - \lambda e^\top C^{-1} e
\end{aligned}
$$

which implies that

$$
\lambda = \frac{\left(e^\top C^{-1}\mu\right) t - 1}{e^\top C^{-1} e} = \alpha_0 t - \gamma_0
$$

and

$$
\begin{aligned}
x &= tC^{-1}\mu - C^{-1}e\left(\alpha_0 t - \gamma_0\right) \\
&= t\left(C^{-1}\mu - \alpha_0 C^{-1}e\right) + \gamma_0 C^{-1}e
\end{aligned}
$$

This identifies, for any $t \geq 0$, $x_t$ as the optimizer of $\max_{x \in D}$.

To see that $x_t \in D^*$ is (the weights of) an efficient portfolio, let $\bar{x} \in D$ be any other portfolio. Then, as $x_t$ maximizes $Z$ over $D$,

$$Z(\bar{x}) = t\mu_p(\bar{x}) - \frac{1}{2}\sigma_p^2(\bar{x}) \leq t\mu_p(x_t) - \frac{1}{2}\sigma_p^2(x_t)$$

and so:

- If $\sigma_p^2(\bar{x}) < \sigma_p^2(x_t)$, then $\mu_p(\bar{x}) < \mu p(x_t)$
- If $\mu_p(\bar{x}) > \mu_p(x_t)$, then $\sigma_p^2(\bar{x}) > \sigma_p^2(x_t)$

and so $R_p(\bar{x})$ does not dominate $R_p(x_t)$. Hence $x_t$ is efficient.

Conversely, if $x^*$ is any efficient portfolio, then we solve $\mu_p\left(x^n\right) \stackrel{!}{=} \alpha_0 + \alpha_1 t$. Let $t = \frac{\mu_p(x^*)-\lambda_0}{\lambda_1}$. Then $x^* = x_t$ (where $x_t$ is the unique maximizer of $\max_{x \in D}\left(t\mu_p(x) - \frac{1}{2}\sigma_p^2(x)\right)$), because otherwise

$$
\begin{aligned}
\mu_p(x^*) = \lambda_0 + \lambda_1 t &= \mu_p(x_t) \\
\sigma_p^2(x_t) &\leq \sigma_p^2(x^*)
\end{aligned}
$$

and so $x_t$ would dominate $x^*$ ($\rightarrow x^*$ is not efficient!). This proves that

$$\left\{x^* : \mathbb{R}_p\left(x^*\right) \text{ is efficient } \right\} = \left\{x_t \in \mathbb{R}^n : t \geq 0\right\}.$$

This result can be used to describe the set of optimal risk-return pairs, and, for each fixed level of risk, describe the maximal expected return that can be achieved by an efficient portfolio. To do this, introduce the set of risk-return pairs associated to efficient portfolios:
$M^* = \{(\sigma_p(x^*), \mu_p(x^*)) : x^* \text{ is efficient}\}$. Then:

**Corollary**

$$M^* = \left\{(\sigma_t, \mu_t) \in (0,\infty) \times \mathbb{R} : \mu = \alpha_0 + \alpha_1 t,\ \sigma_t^2 = \gamma_0 + \alpha_1 t^2,\ t \geq 0\right\}$$
$$= \left\{(\sigma, \mu(\sigma)) \in (0,\infty) \times \mathbb{R} : \mu(\sigma) = \alpha_0 + \sqrt{\alpha_1(\sigma^2 - \gamma_0)}\right\},$$

*where* $\alpha_1 = \mu^\top C^{-1}\mu - \frac{(e^\top C^{-1}\mu)^2}{e^\top C^{-1} e}$.

# Proof

Firstly, note for any $t \geq 0$ :

$$
\begin{aligned}
\mu p\left(x_t\right) &= \mu^\top x_t \\
&= t\left(\mu^\top C^{-1}\mu - \mu^\top C^{-1} e\alpha_0\right) + \mu^\top C^{-1} e\gamma_0 \\
&= t\alpha_1 + \alpha_0 \\
\sigma_p^2\left(x_t\right) &= x_t^\top C x_t = \left[t\left(\mu^\top C^{-1} - e^\top C^{-1}\alpha_0\right) + e^\top C^{-1}\gamma_0\right]\left[t_\mu - te\alpha_0 + e\gamma_0\right] \\
&= t^2\left(\mu^\top C^{-1} - e^\top C^{-1}\alpha_0\right)\left(\mu - e\alpha_0\right) + t\left(\mu^\top C^{-1} - e^\top C^{-1}\alpha_0\right) e\gamma_0 \\
&\quad + e^\top C^{-1}\gamma_0\left[t\mu - te\alpha_0\right] + \left(e^\top C^{-1} e\right)\gamma_0^2 \\
&= t^2\left(\mu^\top C^{-1}\mu - 2\mu^\top C{-1}e\alpha_0 + e^\top C^{-1} e\left(\alpha_0^2\right)\right) \\
&\quad + 2t\left[\mu^\top C^{-1} e - e^\top C^{-1} e\alpha_0\right]\gamma_0 + \gamma_0 \\
&= t^2\left(\mu^\top C^{-1}\mu - \frac{\left(e^\top C^{-1}\mu\right)^2}{e^\top C^{-1} e}\right) + \gamma_0 \\
&= \alpha_1 t^2 + \gamma_0.
\end{aligned}
$$

By the previous theorem, we know

$$\begin{aligned} D^* &= \{x_t : t \geq 0\} \text{ and so} \\ M^* &= \{(\sigma_p(x^*), \mu p(x^*)) : x^* \text{ is efficient } \} \\ &\overset{\circ}{=} \{(\sigma_p(x_t), \mu_p(x_t)) : t \geq 0\} \\ &= \left\{(\sigma_t, \mu_t) : \sigma_t^2 = \alpha_1 t^2 + \gamma_0,\ \mu_t = t\alpha_1 + \alpha_0\right\}, \end{aligned}$$

where in $\circ$ we used the theorem above. To prove the last equality, we solve $\sigma_t^2 = \gamma_0 + \alpha_1 t^2$ for $t \geq 0$ to obtain $t = \sqrt{\frac{\sigma_t^2 - \gamma_0}{\alpha_1}}$ and so $\mu_t = \alpha_0 + \sqrt{\alpha_1 \left(\sigma_t^2 - \gamma_0\right)}$.

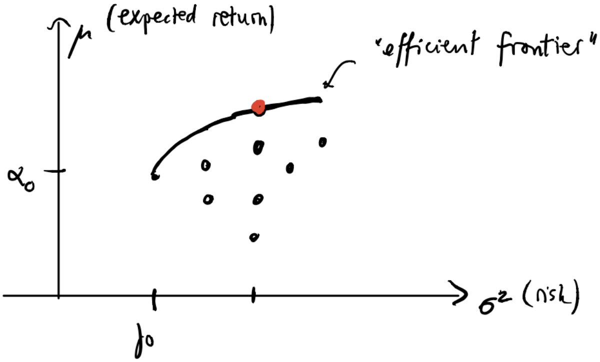


Figure: Efficient frontier (illustration)

## Portfolio selection

For Portfolio selection the investor faces a trade-off between risk and expected return. She selects a pair $(\sigma, \mu(\sigma)) \in M^*$. She determines it based on her risk preferences. There are several (equivalent) ways to do this:

- Fix $\sigma$ (level of risk she is willing to take) and select the portfolio associated to $(\sigma, \mu(\sigma))$ (see Corollary above).
- Fix $\mu_0$ (expected return she would like to achieve) and select the portfolio associated to $(\sigma, \mu(\sigma))$.
- Fix $t \geq 0$ and maximize $Z(x) = t\mathbb{E}\left[R_p(x)\right] - \frac{1}{2}\operatorname{Var}\left(R_p(x)\right)$ over $x \in D$. This leads to the portfolio $x_t \in D^*$.
- Fix a risk-aversion parameter $\beta > 0$ and maximize $\bar{Z}(x) = \mathbb{E}\left[R_p(x)\right] - \frac{\beta}{2}\operatorname{Var}\left(R_p(x)\right)$ over $x \in D$ ($\bar{Z}(x) = \beta Z(x)$ with $t = \frac{1}{b}$). This leads to the portfolio $x_{\frac{1}{\beta}} \in D^*$.

A challenge in practice is to appropriately select $t$ (or $\beta$).
In practice, one often considers an additional constraint which restricts the VaR of the selected portfolios.

# Problematic aspects of Markowitz theory

- Standard deviation as a measure of risk does not take into account asymmetry and (only marginally) extreme losses.
- (Markowitz)-optimal portfolios often have (unrealistic) extreme positions, for instance: if short sales are allowed, then very high short-sale positions are taken...
- **Robustness:** The optimal portfolio is very sensitive to the input data (estimated expected values); varying these slightly may give structurally completely different portfolios. Thus, the input data (in particular estimated expected values) play a major role for the quality of portfolio optimization.
- **Parameter estimation:** For $n$ assets one needs to estimate $\frac{n(n+1)}{2}$ entries of the covariance matrix. For $n = 100$ this corresponds to around 5000 entries; for $n = 250$ already around 30000 covariances. Due to this high-dimensionality in practice one doesn't estimate for single titles, but first one applies a dimension reduction to a multi-factor model.

# Alternative Methods for Portfolio Optimization

In the Markowitz model specific choices for measuring risk and value are made (expected value/standard deviation) Various alternative approaches exist:

1. Risk is measured by a different risk measure, "Value" is still measured by the expected value.
2. Also "value" is measured differently
3. ...

We now look at an example for 1. A first attempt could be to measure risk by $\mathrm{VaR}_\alpha$ instead of $\sqrt{\mathrm{Var}(\cdot)}$. However, $x \mapsto \mathrm{VaR}_\alpha(R_p(x))$ is not convex $\rightsquigarrow$ optimization can be problematic.

On the other hand, if $\rho$ is convex, then $x \mapsto \rho(R_p(x))$ is convex (Example: $\rho = \mathrm{ES}_\alpha$). So, an alternative to the optimization problem $(\star)$:

$$\text{maximize } Z(x) = t\mathbb{E}\left[R_p(x)\right] - \frac{1}{2}\rho\left(R_p(x)\right] \text{ over } x \in D.$$

Or, we turn to portfolio selection directly: fix a desired/target expected return $r$, solve:

$$\text{minimize } \rho(\underbrace{R_p(x)}_{=R\cdot x}) \text{ over } x \in D \text{ subject to } \mathbb{E}[\underbrace{R_p(x)}_{=R\cdot x}] = r \qquad (\otimes)$$

In general the solution of this problem depends on further properties of/assumptions on the distribution of $R$.

The following sample-based variant is commonly used in the case $\rho = \mathrm{ES}_\alpha$. Firstly, one can represent expected shortfall as follows:

$$\mathrm{ES}_\alpha(X) = \inf_{l \in \mathbb{R}} \left\{ l + \frac{1}{\alpha} \mathbb{E}[(-X - l)^+] \right\}, \text{ so}$$

$$\rho\left(R_p(x)\right) = \inf_{l \in \mathbb{R}} \left\{ l + \frac{1}{\alpha} \mathbb{E}\left[(-R \cdot x - l)^+\right] \right\}.$$

Then, with $r_1, \ldots, r_M$ a sample of $R$ we now replace "$\mathbb{E}[\cdot]$" by its sample average. Then the optimization problem ($\otimes$) turns into:

$$\text{minimize } \inf_{l \in \mathbb{R}} \left\{ l + \frac{1}{\alpha} \left( \frac{1}{M} \sum_{i=1}^{M} (-r_i \cdot x - l) \right)^+ \right\} \text{over } x \in D \text{ subject to } \frac{1}{M} \sum_{i=1}^{M} r_i \cdot x = r$$

which is equivalent to

$$\text{minimize } l + \frac{1}{\alpha M} \sum_{i=1}^{M} z_i$$

subject to

- $z_i \geq 0, \ i = 1, \ldots, M$
- $z_i + r_i \cdot x + l \geq 0, \ i = 1, \ldots, M$
- $x \cdot \left( \frac{1}{M} \sum_{i=1}^{M} r_i \right) = r, \ x \cdot e = 1$

which corresponds to a **Linear programming problem**.

# Markowitz Theory with a Riskless Asset

Consider the Markowitz model with an additional risk-free asset at the safe interest rate $r_0$. Assume any amount can be borrowed/invested at this rate. Suppose you have invested a "proportion" $\lambda \in [0, \infty)$ in a "risky" portfolio $R_p (= R_p(x)$ for some $x \in D^*)$ and invested (respectively borrowed) $1 - \lambda \in (-\infty, 1]$ in the riskless asset. Then the overall return of this portfolio is

$$R_{\tilde{p}} = \lambda R_p + (1 - \lambda) r_0.$$

The expected return and return variance are

$$\begin{aligned}
\mu_{\tilde{p}} &= \mathbb{E}\left[R_{\tilde{p}}\right] = \lambda \mu_p + (1 - \lambda)\, r_0 = r_0 + \lambda \left(\mu_p - r_0\right) \\
(\sigma \tilde{p})^2 &= \operatorname{Var}\left(\lambda R_p + (1 - \lambda) r_0\right) = \lambda^2 \operatorname{Var}\left(R_p\right) = \lambda^2 \sigma_p^2 \\
&\Rightarrow \mu_{\tilde{p}} = r_0 + \frac{\mu_p - r_0}{\sigma_p} \sigma_{\tilde{p}}
\end{aligned}$$

Thus, the set of risk-return pairs that can he realized with such extended portfolios is given as

$$\tilde{M} = \left\{ (\tilde{\sigma}, \tilde{\mu}(\tilde{\sigma})) : \tilde{\mu}(\tilde{\sigma}) = r_0 + \frac{\mu_p - r_0}{\sigma} \tilde{\sigma},\ (\sigma_p, \mu_p) \in M \right\}.$$

If we fix a portfolio $p$ (i.e. $\mu_p$ and $\sigma_p$ are fired) and only vary $\lambda$, this corresponds to risk-return pairs

$$(\tilde{\sigma}, \tilde{\mu}(\tilde{\sigma})) = \left( \tilde{\sigma}, r_0 + \frac{\mu_p - r_0}{\sigma_p} \tilde{\sigma} \right) \qquad \text{for varying } \tilde{\sigma}$$

$\rightsquigarrow$ risk-return pairs lie on a line passing through $(0, r_0)$ and with slope

$$\mathrm{SR}(R_p) = \frac{\mathbb{E}[R_p] - r_0}{\sqrt{\mathrm{Var(R_p)}}} = \frac{\mu_p - r_0}{\sigma_p},$$

the so-called **Sharpe ratio**.

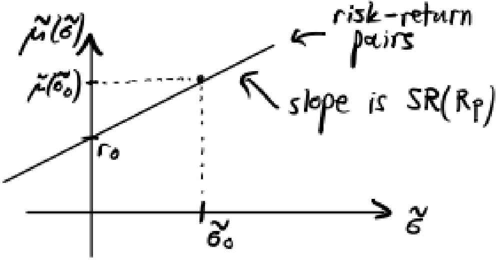


Figure: Sharpe ratio (illustration)

How about the risk-return pairs associated to efficient portfolios?

$$\tilde{M}^* = \{(\tilde{\sigma}, \tilde{\mu}(\tilde{\sigma})) \in \tilde{M} : (\tilde{\sigma}, \tilde{\mu}(\tilde{\sigma})) \text{ efficient } \}$$

It turns out (see below) that

$$\tilde{M}^* = \left\{(\sigma, \mu(\sigma)) : \mu(\sigma) = r_0 + \frac{\mu_T - r_0}{\sigma_T}\sigma\right\}$$

for some portfolio $T$, called "tangential portfolio", i.e.,

$\mu_T = \mathbb{E}\left[R_p\left(x_T\right)\right], \sigma_T = \sigma\left(R_p\left(x_T\right)\right)$ for some fixed $x_T$ (denoted $\omega_T$ below),

which is the same for all optimal portfolios - these only differ by the choice of $\lambda$. The sharpe ratio of $T$ is the maximal sharpe ratio.

## In more detail/more precisely

Consider $n+1$ assets ($i = 0, \ldots, n$), where asset 0 corresponds to a riskless asset at rate $r_0$. Denote the portfolio weights by $\omega_0, w_1, \ldots, w_n$ and write $\omega = (\omega_1, \ldots, \omega_n)$. Now the constraint is

$$w_0 = 1 - \sum_{i=1}^{n} w_i = 1 - w \cdot e$$

$w \cdot e$ is not restricted anymore, as money can be borrowed from/invested in the risk-free asset. The remaining notation/assumptions are as in the standard Markowitz model above. In addition, we denote by

$$r = \begin{pmatrix} \mu_1 - r_0 \\ \mu_2 - r_0 \\ \vdots \\ \mu_n - r_0 \end{pmatrix} = \mu - r_0 e$$

the vector of **expected excess returns**.

The return is now given as

$$R_p = R_p(\omega_0, \omega) = \omega_0 r_0 + \omega \cdot R = (1 - \omega \cdot e) r_0 + \omega \cdot R$$

and

$$\mu_p = \mu_p(\omega_0, \omega) = \mathbb{E}[R_p(\omega_0, \omega)] = \omega_0 r_0 + \omega \cdot \mu = (1 - \omega \cdot e) r_0 + \omega \cdot \mu$$
$$\sigma_p^2 = \sigma_p^2(\omega_0, \omega) = \mathrm{Var}(R_p(\omega_0, \omega)) = \mathrm{Var}(\omega_0 r_0 + \omega \cdot R) = \mathrm{Var}(\omega \cdot R) = \omega^\top C \omega$$

Thus,

$$r_p := r_p(\omega_0, \omega) = \mu_p(\omega_0, \omega) - r_0 = -\omega \cdot e r_0 + \omega \cdot \mu = \omega \cdot r.$$

The optimization problem is now (see $(\star)$):

$$\begin{aligned} \text{maximize } \tilde{Z}(\omega) &= t\mathbb{E}[Rp(\omega)] - \frac{1}{2}\mathrm{Var}(R_p(\omega)] \\ &= t[(1 - \omega \cdot e) r_0 + \omega \cdot \mu] - \frac{1}{2}\omega^\top C \omega \end{aligned}$$

which is equivalent to

$$\Leftrightarrow \text{ maximize } Z(\omega) = t\left(\mathbb{E}\left[R_p(\omega)\right] - r_0\right) - \frac{1}{2}\operatorname{Var}\left(R_p(\omega)\right)$$
$$= t\omega^\top r - \frac{1}{2}\omega^\top C\omega$$

### Proposition

*Assume $C$ is invertible. Then the set of efficient portfolios is given as*

$$\tilde{D}^* = \left\{(\omega_0, \omega_t) \in \mathbb{R} \times \mathbb{R}^n : \omega_t = tC^{-1}r,\ \omega_0 = 1 - w_t \cdot e,\ t \geq 0\right\},$$

*the set of risk-return pairs associated to efficient portfolios is given as*

$$\begin{aligned}\tilde{M}^* &= \left\{(\sigma_t, \mu_t) : \mu_t = r_0 + Bt,\ \sigma_t^2 = Bt^2,\ t \geq 0\right\} \\ &= \left\{(0, \mu(\sigma)) : \mu(\sigma) = r_0 + \sqrt{B}\sigma\right\}\end{aligned}$$

*where $A = e^\top Ce$, $B = r^\top C^{-1}r$ and $\sqrt{B}$ is the sharpe ratio of the (tangential) portfolio $\omega_T = \frac{1}{A}C^{-1}r$.*

We skip the proof here.

- One can show: the tangential portfolio $\omega_T$ is an element of the "purely risky" efficient frontier, i.e. $\omega_T \in D^*$.
- $(\sigma_p(\omega_T), \mu_p(\omega_T))$, i.e., the risk-return associated to the tangential portfolio is the tangential point of the tangent (through $(0, r_0)$) at the efficient frontier $M^*$.
- "Two-fund theorem": For each efficient portfolio $\omega_t$ we have

$$(w_0, w_t) = \lambda(0, \omega_T) + (1-\lambda)(1, 0)$$

  for $\lambda = tA$ $(t \geq 0)$ the relative investment in the tangential portfolio.
- Natural connection to Capital Asset Pricing Model (CAPM)!