Curved Yang-Mills gauge theories

based on my preprint arXiv:2210.02924

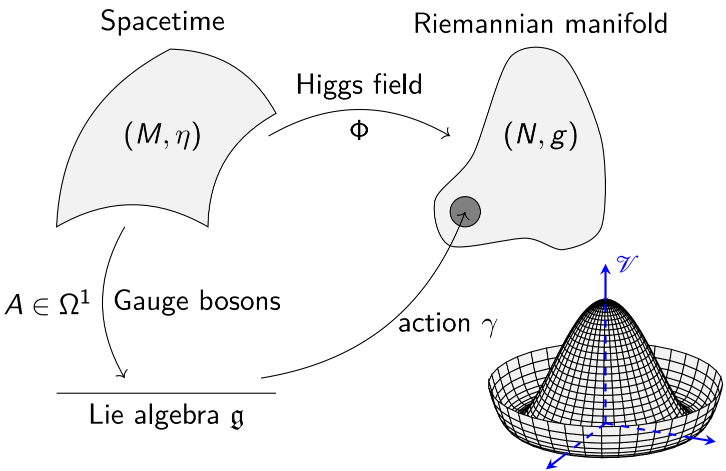
Simon-Raphael Fischer

National Center for Theoretical Sciences

30 December 2022

Table of contents

Infinitesimal gauge theory



Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathscr{g}$
Integrated	Lie group G	LGB[1] $\mathscr{G}$

$$G \longrightarrow \mathscr{G}$$

$$M$$

[1] LGB = Lie group bundle

Definition (LGB actions, simplified)

$$\begin{array}{ccc} & & \mathscr{G} \\ & & \downarrow \\ \mathscr{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathscr{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of $\mathscr{G}$ on $\mathscr{P}$** is a smooth map $\mathscr{P} * \mathscr{G} := \pi^* \mathscr{G} = \mathscr{P} \times_M \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

$\Rightarrow$ Think of the "classical" theory as coming from a trivial LGB

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

$\Rightarrow$ Think of the "classical" theory as coming from a trivial LGB

Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

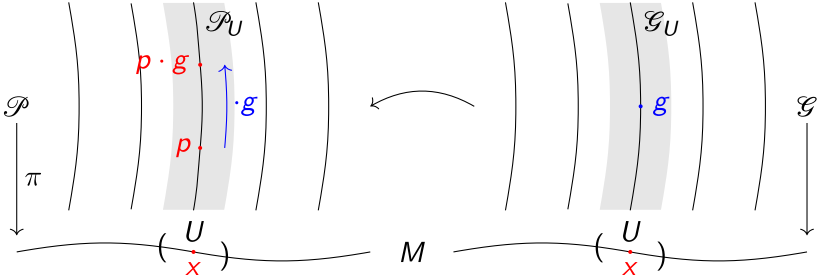
but with $\mathscr{G}$-action

$$\cancel{\mathscr{P} \times G} \to \mathscr{P}$$
$$\mathscr{P} * \mathscr{G}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids.
Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003

Connection on $\mathscr{P}$: Idea

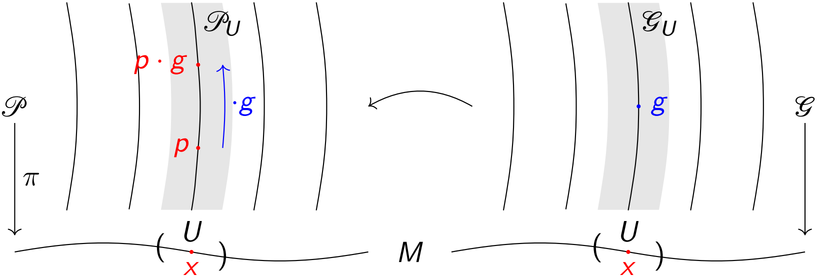


But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

Connection on $\mathscr{P}$: Idea

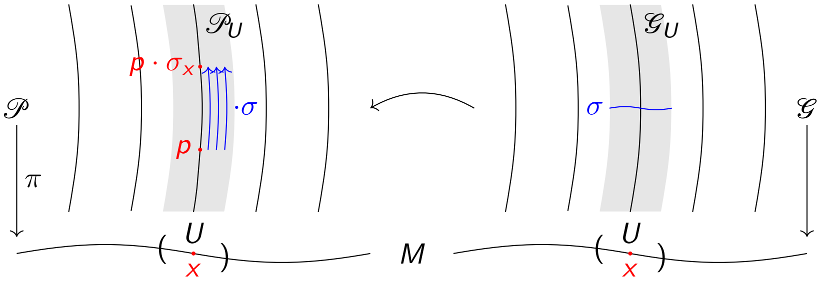


But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$$\Rightarrow \qquad \mathrm{D}_p r_g \text{ only defined on vertical structure}$$

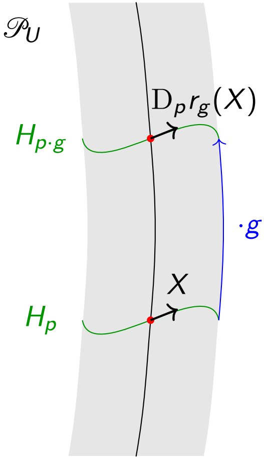
Connection on $\mathscr{P}$: Idea



Use $\sigma \in \Gamma(\mathscr{G}) : r_\sigma(p) := p \cdot \sigma_x$

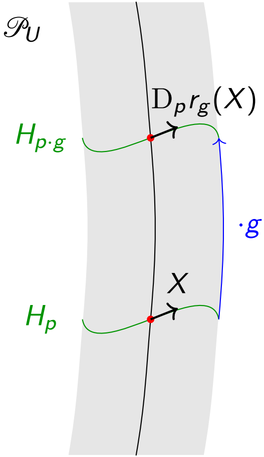
Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



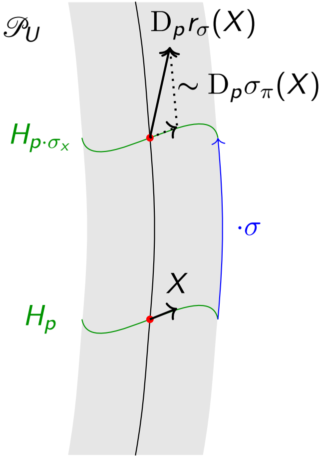
Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{P}}_{\alpha}$ in $\mathscr{P}$:

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p\cdot g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p)\cdot g$$

where $\alpha : I \to M$ is a base path

Connection on $\mathscr{P}$: General case

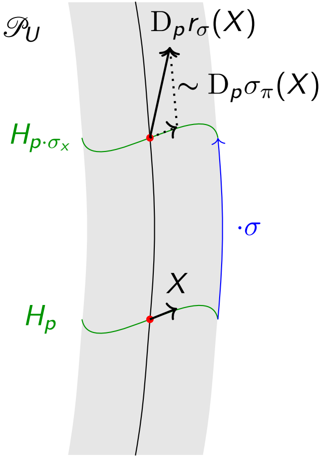


Remarks (Integrated case)

Ansatz:

$$\mathrm{PT}^{\mathscr{P}}_\alpha(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_\alpha(p) \cdot \mathrm{PT}^{\mathscr{G}}_\alpha(g)$$

Connection on $\mathscr{P}$: General case



Remarks (Integrated case)

Ansatz:

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\alpha}(g)$$

$\Rightarrow$ Introduce connection on $\mathscr{G}$

Classical situation: Differential of Lie group action

Remarks (Lie group G situation with Lie algebra $\mathfrak{g}$)

In the case of a right G-action on $\mathscr{P}$, $\Phi : \mathscr{P} \times G \to \mathscr{P}$, we have

$$\mathrm{D}_{(p,g)}\Phi(X, Y) = \mathrm{D}_p r_g(X) + \left.\overline{(\mu_G)_g(Y)}\right|_{p \cdot g}$$

for all $p \in \mathscr{P}$, $g \in G$, $X \in \mathrm{T}_p\mathscr{P}$ and $Y \in \mathrm{T}_g G$, where

- $\overline{\nu}$ denotes the fundamental vector field on $\mathscr{P}$ of $\nu \in \mathfrak{g}$,
- μ_G is the Maurer-Cartan form of G.

Mark JD Hamilton. Mathematical Gauge Theory. *Springer*, 2017.

Definition (Fundamental vector fields)

Fundamental vector fields defined by

$$\overline{\nu}_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(p \cdot \mathrm{e}^{t\nu_x}\right)$$

for all $\nu \in \Gamma(\mathscr{g})$ and $p \in \mathscr{P}_x$, where $\mathscr{g}$ is the LAB[a] of $\mathscr{G}$.

[a] Lie algebra bundle

Sources: Straightforward generalization of classical definition as in: Mark JD Hamilton. Mathematical Gauge Theory. *Springer*, 2017.

Definition (Darboux derivative)

For $\sigma \in \Gamma(\mathscr{G})$ we define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathscr{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathscr{G}},$$

where $\mu_{\mathscr{G}}$ is the connection 1-form on $\mathscr{G}$.

Sources: Trivial generalization of classical definition as in: K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Darboux derivative)

For $\sigma \in \Gamma(\mathscr{G})$ we define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathscr{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathscr{G}},$$

where $\mu_{\mathscr{G}}$ is the connection 1-form on $\mathscr{G}$.

Remarks

If $\mathscr{G}$ a trivial LGB with canonical flat connection, then $\mu_{\mathscr{G}}$ is the Maurer-Cartan form.
If Lie group additionally a matrix group, then

$$\Delta\sigma = \sigma^{-1}\mathrm{d}\sigma.$$

Sources: Trivial generalization of classical definition as in: K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Proposition (Differential of LGB action Φ, [S.-R. F.])

We have

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_\sigma(X) - \left.\overline{\left.(\pi^!\Delta\sigma)\right|_p(X)}\right|_{p\cdot g} + \left.\overline{(\mu_{\mathscr{G}})_g(Y)}\right|_{p\cdot g}$$

for all $(p,g) \in \mathscr{P}_x \times \mathscr{G}_x$, $(X,Y) \in \mathrm{T}_{(p,g)}(\mathscr{P} * \mathscr{G})$, *where* σ *is any section of* $\mathscr{G}$ *with* $\sigma_x = g$*.*

Proposition (Differential of LGB action Φ, [S.-R. F.])

We have

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_\sigma(X) - \left.\overline{\left.(\pi^!\Delta\sigma)\right|_p(X)}\right|_{p\cdot g} + \left.\overline{\left(\mu_{\mathscr{G}}\right)_g(Y)}\right|_{p\cdot g}$$

for all $(p,g) \in \mathscr{P}_x \times \mathscr{G}_x$, $(X,Y) \in \mathrm{T}_{(p,g)}(\mathscr{P} * \mathscr{G})$, *where* σ *is any section of* $\mathscr{G}$ *with* $\sigma_x = g$.

Definition (Modified right-pushforward, [S.-R. F.])

$$\begin{aligned} \mathscr{r}_{g*}(X) &:= \mathrm{D}_p r_\sigma(X) - \left.\overline{\left.(\pi^!\Delta\sigma)\right|_p(X)}\right|_{p\cdot g}, \\ \mathscr{r}_{\sigma*}(X) &:= \mathscr{r}_{\sigma_x*}(X). \end{aligned}$$

Proposition (Well-defined isomorphism, [S.-R. F.])

We have that

$$\begin{aligned} \mathrm{T}\mathscr{P}|_{\mathscr{P}_x} &\to \mathrm{T}\mathscr{P}|_{\mathscr{P}_x}, \\ X &\mapsto \mathscr{r}_{g*}(X), \end{aligned}$$

is a well-defined automorphism over r_g. *Similarly,*

$$\begin{aligned} \mathrm{T}\mathscr{P} &\to \mathrm{T}\mathscr{P}, \\ X &\mapsto \mathscr{r}_{\sigma*}(X), \end{aligned}$$

is an automorphism over r_σ.

Definition (Ehresmann connection, [S.-R. F.])

H a horizontal distribution of $\mathrm{T}\mathscr{P}$ with

$$\mathscr{r}_{g*}(H_p) = H_{p \cdot g}$$

Definition (Ehresmann connection, [S.-R. F.])

H a horizontal distribution of $\mathrm{T}\mathscr{P}$ with

$$r_{g*}(H_p) = H_{p\cdot g}$$

Definition (Connection 1-form, [S.-R. F.])

$A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ with

$$r_\sigma^! A = \mathrm{Ad}_{\sigma^{-1}} \circ A,$$
$$A(\overline{\nu}) = \pi^*\nu$$

for all $\sigma \in \Gamma(\mathscr{G})$ and $\nu \in \Gamma(\mathscr{g})$.

Remarks

$$\left(r_\sigma^! A\right)_p (X) = A_{p\sigma_x}\left(r_{\sigma*}(X)\right).$$

Theorem (Equivalence of both definitions, [S.-R. F.])

There is the usual 1:1 correspondence between both definitions:

- *Given H, define A by*

$$A_p(\overline{\nu}_p + X) := (\pi^* \nu)_p,$$

where $X \in H_p$.

- *Given A, define H by*

$$H_p := \mathrm{Ker}(A_p).$$

Gauge transformation

Summary

	Locally	Globally
Curved Yang-Mills	Pre-classical	$\mathrm{ad}(\mathbb{S}^7 \to \mathbb{S}^4)$ curved

Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial
$\rightsquigarrow$ LGB action locally equivalent to a Lie group action

Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle $\mathscr{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathfrak{G}$?

$$\begin{tikzcd} & \mathfrak{G} \arrow[dl] \arrow[dr] & \\ M \arrow[rr, "\Phi"] & & N \end{tikzcd}$$

Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!