

Curved Yang-Mills gauge theories

based on my preprint [arXiv:2210.02924](https://arxiv.org/abs/2210.02924)

Simon-Raphael Fischer

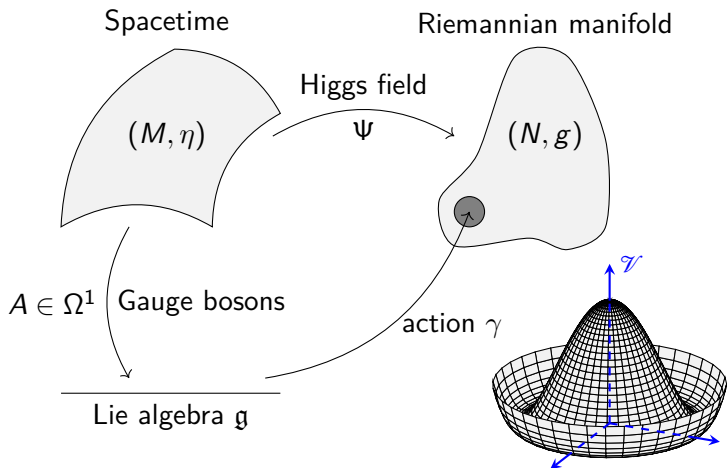
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Infinitesimal curved Yang-Mills-Higgs gauge theory



Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra \mathfrak{g} as $M \times \mathfrak{g}$	Lie algebroid $E \rightarrow N$
\mathfrak{g} -action γ	Anchor ρ of E & E -connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

\rightsquigarrow We will use a general connection ∇ instead of ∇^0 , and ∇ may not be flat.

We will only focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra \mathfrak{g}	LAB ¹ \mathcal{G}
Integrated	Lie group G	LGB ² \mathcal{G}

$$\begin{array}{ccc}
 G & \longrightarrow & \mathcal{G} \\
 & & \downarrow \\
 & & M
 \end{array}$$

¹LAB = Lie algebra bundle

²LGB = Lie group bundle

Definition (LGB actions, simplified)

$$\begin{array}{ccc} & \mathcal{G} & \\ & \downarrow & \\ \mathcal{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathcal{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of \mathcal{G} on \mathcal{P}** is a smooth map $\mathcal{P} * \mathcal{G} := \pi^* \mathcal{G} = \mathcal{P} \times_M \mathcal{G} \rightarrow \mathcal{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathcal{P}$ and $g, h \in \mathcal{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathcal{G}_{\pi(p)}$.

Examples

Example

\mathcal{G} acts canonically on itself:

$$\begin{aligned}\mathcal{G} * \mathcal{G} &\rightarrow \mathcal{G}, \\ (q, h) &\mapsto qh.\end{aligned}$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathcal{G} \cong M \times G$, then also $\mathcal{P} * \mathcal{G} \cong \mathcal{P} \times G$, and we can define

$$\begin{aligned}\mathcal{P} \times G &\rightarrow \mathcal{P}, \\ (p, g) &\mapsto p \cdot g := p \cdot (\pi(p), g),\end{aligned}$$

which is equivalent to $\mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$.

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Guide to recover the classical theory (Part 1 of 3)

- 1 $\mathcal{G} \cong M \times G$

Definition (Principal bundle)

Still a fibre bundle

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{P} \\ & & \downarrow \pi \\ & & M \end{array}$$

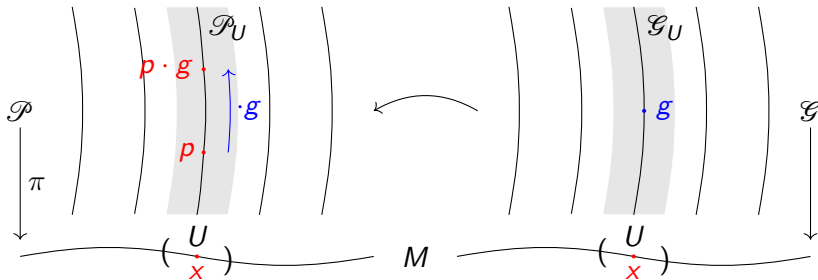
but with \mathcal{G} -action

$$\begin{array}{ccc} \cancel{\mathcal{P}} \times \cancel{G} & \longrightarrow & \mathcal{P} \\ \mathcal{P} * \mathcal{G} & & \end{array}$$

simply transitive on fibres of \mathcal{P} , and "suitable" atlas.

Connections as parallel transport

Connection on \mathcal{P} : Idea



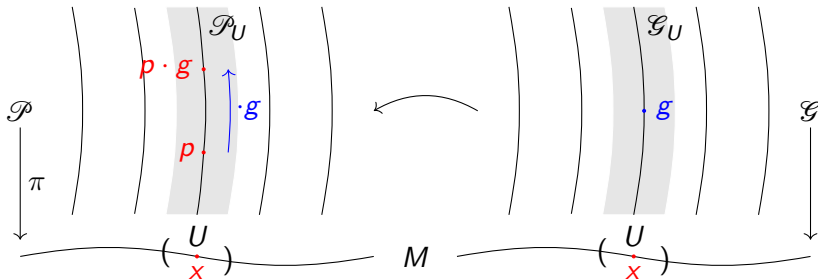
But:

$$r_g : \mathcal{P}_x \rightarrow \mathcal{P}_x$$

\Rightarrow

$D_p r_g$ only defined on vertical structure

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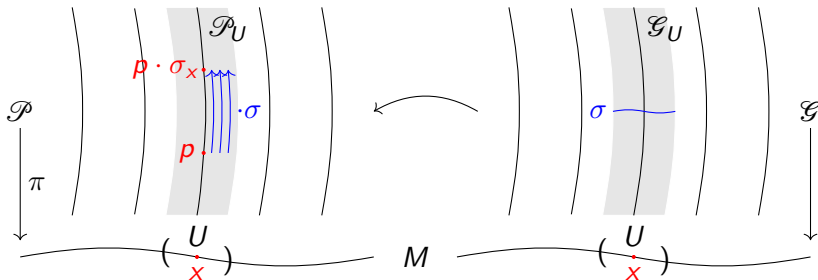
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Connections as parallel transport

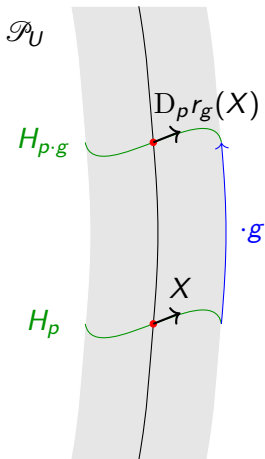
Connection on \mathcal{P} : Idea



Use $\sigma \in \Gamma(\mathcal{G}) : r_\sigma(p) := p \cdot \sigma_x$

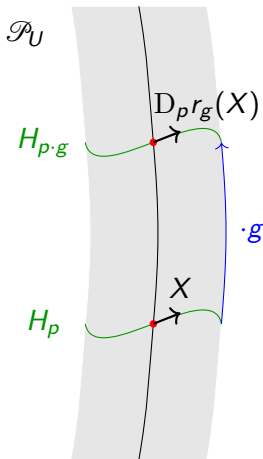
Connection on \mathcal{P} : Revisiting the classical setup

If \mathcal{P} a typical principal bundle
 (\mathcal{G} trivial, $\sigma \equiv g$ constant),
 and H a connection:



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Remarks (Integrated case)

Parallel transport $\text{PT}_\alpha^{\mathcal{P}}$ in \mathcal{P} :

$$\text{PT}_\alpha^{\mathcal{P}}(p \cdot g) = \text{PT}_\alpha^{\mathcal{P}}(p) \cdot g$$

where $\alpha : I \rightarrow M$ is a base path

Connection on \mathcal{P} : General case

Remarks (Integrated case)

Ansatz: Introduce connection on \mathcal{G} ,

$$\mathrm{PT}_{\alpha}^{\mathcal{P}}(p \cdot g) = \mathrm{PT}_{\alpha}^{\mathcal{P}}(p) \cdot \mathrm{PT}_{\alpha}^{\mathcal{G}}(g).$$

Guide to recover the classical theory (Part 2 of 3)

- ① $\mathcal{G} \cong M \times G$
- ② Equip \mathcal{G} with canonical flat connection

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Classical situation: Differential of Lie group action

Remarks (Lie group G situation with Lie algebra \mathfrak{g})

In the case of a right G -action on \mathcal{P} , $\Phi : \mathcal{P} \times G \rightarrow \mathcal{P}$, we have

$$D_{(p,g)}\Phi(X, Y) = D_p r_g(X) + \overline{(\mu_G)_g(Y)} \Big|_{p \cdot g}$$

for all $p \in \mathcal{P}$, $g \in G$, $X \in T_p \mathcal{P}$ and $Y \in T_g G$, where

- $\bar{\nu}$ denotes the fundamental vector field on \mathcal{P} of $\nu \in \mathfrak{g}$,
- μ_G is the Maurer-Cartan form of G .

Definition (Fundamental vector fields)

Fundamental vector fields defined by

$$\bar{\nu}_p := \left. \frac{d}{dt} \right|_{t=0} (p \cdot e^{t\nu_x})$$

for all $\nu \in \Gamma(\mathfrak{g})$ and $p \in \mathcal{P}_x$, where \mathfrak{g} is the LAB of \mathcal{G} .

Definition (Darboux derivative)

For $\sigma \in \Gamma(\mathcal{G})$ we define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathfrak{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathcal{G}},$$

where $\mu_{\mathcal{G}}$ is given by

$$(\mu_{\mathcal{G}})_g := D_g L_{g^{-1}} \circ \pi^{\vee},$$

π^{\vee} the projection onto the vertical bundle.

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π^{\vee} the projection onto the vertical bundle.

Remarks

If \mathcal{G} a trivial LGB with canonical flat connection and if Lie group additionally a matrix group, then

$$\Delta\sigma = \sigma^{-1} d\sigma.$$

Proposition (Differential of LGB action Φ , [S.-R. F.])

We have

$$D_{(p,g)}\Phi(X, Y) = D_p r_\sigma(X) - \overline{(\pi^! \Delta \sigma)|_p(X)} \Big|_{p \cdot g} + \overline{(\mu_{\mathcal{G}})_g(Y)} \Big|_{p \cdot g}$$

for all $(p, g) \in \mathcal{P}_x \times \mathcal{G}_x$, $(X, Y) \in T_{(p,g)}(\mathcal{P} * \mathcal{G})$, where σ is any section of \mathcal{G} with $\sigma_x = g$.

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Definition (Modified right-pushforward, [S.-R. F.])

Define

$$r'_{g*}(X) := D_p r_\sigma(X) - \overline{(\pi^! \Delta \sigma)|_p(X)} \Big|_{p \cdot g}.$$

Proposition (Well-defined isomorphism, [S.-R. F.])

We have that

$$\begin{aligned} \mathrm{T}\mathcal{P}|_{\mathcal{P}_x} &\rightarrow \mathrm{T}\mathcal{P}|_{\mathcal{P}_x}, \\ X &\mapsto r_{g*}(X), \end{aligned}$$

is a well-defined automorphism over r_g .

Definition (Ehresmann connection, [S.-R. F.]

H a horizontal distribution of $T\mathcal{P}$ with

$$\tau_{g*}(H_p) = H_{p \cdot g}$$

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$$\mathcal{r}_{g*}(H_p) = H_{p \cdot g}$$

Definition (Connection 1-form, [S.-R. F.])

$A \in \Omega^1(\mathcal{P}; \pi^*\mathcal{G})$ with

$$A(\bar{\nu}) = \pi^*\nu,$$

$$\mathcal{r}_\sigma^! A = \text{Ad}_{\sigma^{-1}} \circ A$$

for all $\sigma \in \Gamma(\mathcal{G})$ and $\nu \in \Gamma(\mathcal{G})$.

Remarks

$$\left(\mathcal{r}_\sigma^! A\right)_p(X) = A_{p\sigma_x}(\mathcal{r}_{\sigma_x*}(X)).$$

Theorem (Equivalence of both definitions, [S.-R. F.]

There is the usual 1:1 correspondence between both definitions:

- *Given H , define A by*

$$A_p(\bar{\nu}_p + X) := (\pi^* \nu)_p,$$

where $X \in H_p$.

- *Given A , define H by*

$$H_p := \text{Ker}(A_p).$$

Theorem (Gauge transformation, [S.-R. F.])

Let s_i, s_j be two sections of \mathcal{P} over U_i and U_j , respectively, which are open subsets of M with $U_i \cap U_j \neq \emptyset$. Then over $U_i \cap U_j$

$$A_{s_i} = \text{Ad}_{\sigma_{ji}^{-1}} \circ A_{s_j} + \Delta\sigma_{ji},$$

where $A_{s_i} := s_i^! A$ and σ_{ji} a section of \mathcal{G} with $s_i = s_j \cdot \sigma_{ji}$.

Proposition (Connection on \mathcal{G} , [S.-R. F.]

We have an induced vector bundle connection on \mathcal{G} given by

$$\nabla^{\mathcal{G}}_{\nu} := \left. \frac{d}{dt} \right|_{t=0} \Delta e^{t\nu}.$$

Remarks

Recall, \mathcal{G} a principal G -bundle.

Definition (Compatibility conditions, [S.-R. F.])

$\mu_{\mathcal{G}}$ a **Yang-Mills connection** (w.r.t. $\zeta \in \Omega^2(M; \mathfrak{g})$) if it satisfies the **compatibility conditions**:

- ① $\mu_{\mathcal{G}}$ a connection 1-form on $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$;
- ② $\mu_{\mathcal{G}}$ satisfies the **generalised Maurer-Cartan equation**

$$\left(d\pi_{\mathcal{G}}^* \nabla^{\mu_{\mathcal{G}}} \mu_{\mathcal{G}} + \frac{1}{2} [\mu_{\mathcal{G}} \wedge \mu_{\mathcal{G}}]_{\pi_{\mathcal{G}}^* \mathfrak{g}} \right) \Big|_g = \text{Ad}_{g^{-1}} \circ \pi_{\mathcal{G}}^! \zeta \Big|_g - \pi_{\mathcal{G}}^! \zeta \Big|_g$$

Proposition ($\nabla^{\mathcal{G}}$ a Lie bracket derivation)

Let $\mu_{\mathcal{G}}$ be a connection 1-form on \mathcal{G} , then

$$\nabla^{\mathcal{G}}([\mu, \nu]_{\mathcal{G}}) = [\nabla^{\mathcal{G}}\mu, \nu]_{\mathcal{G}} + [\mu, \nabla^{\mathcal{G}}\nu]_{\mathcal{G}}.$$

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Theorem (Curvature of LAB connection exact, [S.-R. F.])

$\mu_{\mathcal{G}}$ satisfies the generalized Maurer-Cartan equation w.r.t. ζ if and only if

$$R_{\nabla^{\mathcal{G}}} = \text{ad} \circ \zeta.$$

Remarks

There is a simplicial differential δ on $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$

$$\delta : \Omega^{\bullet}(\underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{G}) \rightarrow \Omega^{\bullet}(\underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{G})$$

such that the compatibility conditions are equivalent to

$$\delta \mu_{\mathcal{G}} = 0 \quad \text{and} \quad d^{\pi_{\mathcal{G}}^* \nabla^{\mathcal{G}}} \mu_{\mathcal{G}} + \frac{1}{2} [\mu_{\mathcal{G}} \wedge \mu_{\mathcal{G}}]_{\pi_{\mathcal{G}}^* \mathcal{G}} = \delta \zeta.$$

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Given a Yang-Mills connection on \mathcal{G} :

Definition (Generalized curvature/field strength F of A , [S.-R. F.]

π^H denotes the projection onto $H \subset T\mathcal{P}$, then we define

$$F := d^{\pi^* \nabla^{\mathcal{G}}} A \circ (\pi^H, \pi^H) + \pi^! \zeta.$$

Theorem (Structure equation, [S.-R. F.]

$$F = d^{\pi^* \nabla^{\mathcal{G}}} A + \frac{1}{2} [A \wedge A]_{\pi^* \mathcal{G}} + \pi^! \zeta.$$

Proposition (Properties of F , [S.-R. F.]

- $F(X, \cdot) = 0$, if X vertical,
- $\pi_\sigma^! F = \text{Ad}_{\sigma^{-1}} \circ F$.

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- $F(X, \cdot) = 0$, if X vertical,
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Theorem (Lagrangian, [S.-R. F.])

- κ be an Ad-invariant fibre metric on \mathfrak{g} ,
- M a spacetime, and $*$ its Hodge star operator,
- $(U_i)_i$ open covering of M with subordinate gauges $s_i \in \Gamma(\mathcal{P}|_{U_i})$.

Then the Lagrangian $\mathfrak{L}_{\text{CYM}}[A]$, defined locally by

$$(\mathfrak{L}_{\text{CYM}}[A])|_{U_i} := -\frac{1}{2}\kappa(F_{s_i} \wedge *F_{s_i}),$$

is well-defined, and

$$\mathfrak{L}_{\text{CYM}}[L^!A] = \mathfrak{L}_{\text{CYM}}[A]$$

for all principal bundle automorphisms L .

Example (Hopf fibration $S^7 \rightarrow S^4$, [S.-R. F.])

Let P be the Hopf bundle

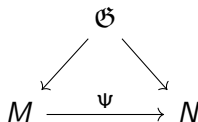
$$\begin{array}{ccc} \mathrm{SU}(2) \cong S^3 & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

Define $\mathcal{P} := \mathcal{G}$ as the inner group bundle of P ,

$$\mathcal{G} := c_{\mathrm{SU}(2)}(P) := (P \times \mathrm{SU}(2)) / \mathrm{SU}(2).$$

This principal $c_{\mathrm{SU}(2)}(P)$ -bundle admits the structure as curved Yang-Mills gauge theory; there is no description as flat theory.

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle \mathcal{G}
Curved Yang-Mills-Higgs	Lie groupoid $\mathcal{G}?$



Remarks

- Richer set of principal bundles, containing Lie groupoids equipped with "non-flat Maurer-Cartan forms".
- Principal bundle for the whole of Yang-Mills-Higgs theory
- Even if \mathcal{G} is trivial, what happens if its connection is not flat?
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!