Curved Yang-Mills gauge theories

based on my preprint arXiv:2210.02924

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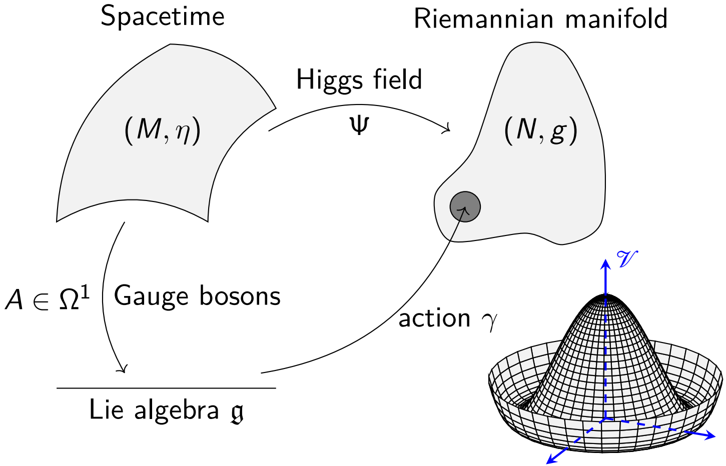
Infinitesimal curved Yang-Mills-Higgs gauge theory



Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0}A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

We will only focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathscr{g}$
Integrated	Lie group G	LGB[1] $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & M \end{tikzcd}$$

[1] LGB = Lie group bundle

Definition (LGB actions, simplified)

$$\begin{array}{ccc} & & \mathscr{G} \\ & & \downarrow \\ \mathscr{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathscr{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of** $\mathscr{G}$ **on** $\mathscr{P}$ is a smooth map $\mathscr{P} * \mathscr{G} := \pi^* \mathscr{G} = \mathscr{P} \times_M \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

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$\Rightarrow$ Think of the "classical" theory as coming from a trivial LGB

Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

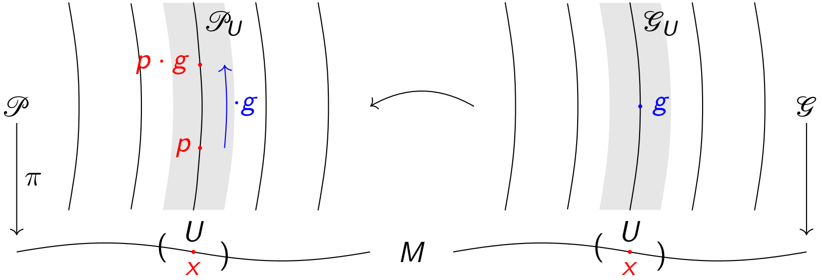
but with $\mathscr{G}$-action

$$\begin{array}{c} \cancel{\mathscr{P} \times G} \\ \mathscr{P} * \mathscr{G} \end{array} \rightarrow \mathscr{P}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids.
Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003

Connection on $\mathscr{P}$: Idea



But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$$\Rightarrow \qquad \mathrm{D}_p r_g \text{ only defined on vertical structure}$$

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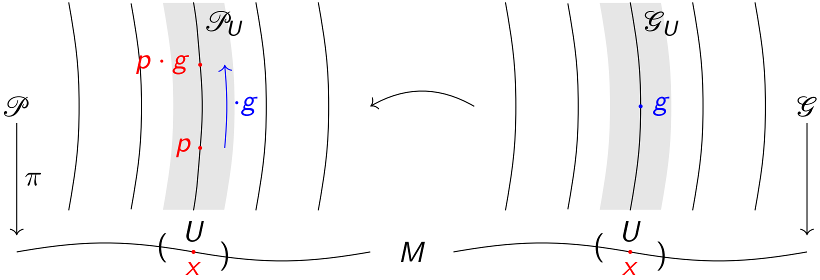

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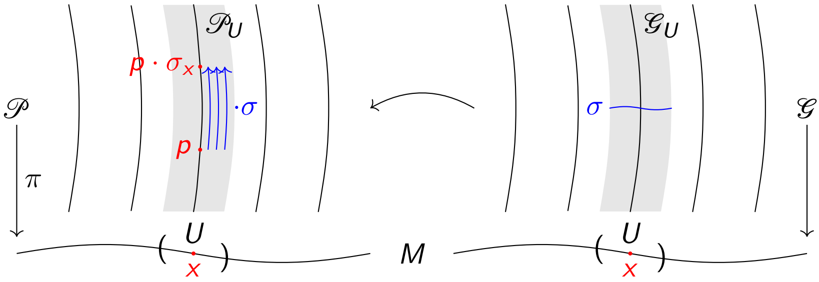
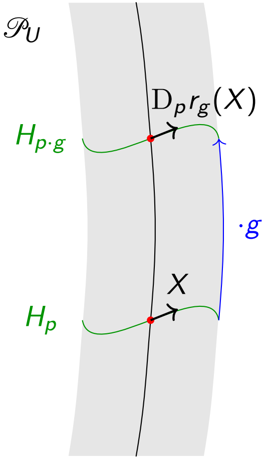


Use $\sigma \in \Gamma(\mathscr{G}) : r_\sigma(p) := p \cdot \sigma_x$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



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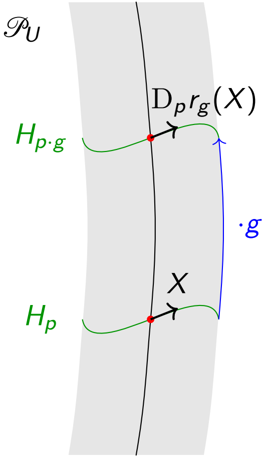


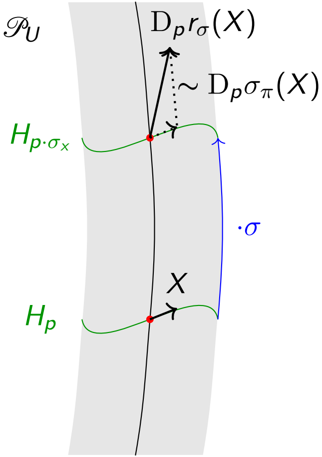
Remarks (Integrated case)

Parallel transport $\mathrm{PT}_\alpha^{\mathscr{P}}$ in $\mathscr{P}$:

$$\mathrm{PT}_\alpha^{\mathscr{P}}(p \cdot g) = \mathrm{PT}_\alpha^{\mathscr{P}}(p) \cdot g$$

where $\alpha : I \to M$ is a base path

Connection on $\mathscr{P}$: General case

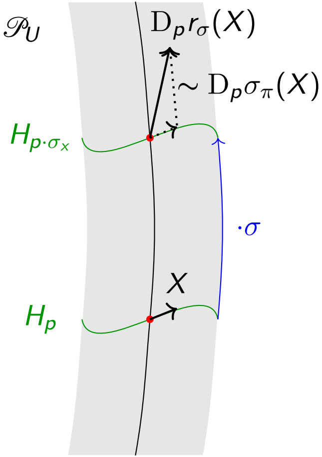


Remarks (Integrated case)

Ansatz:

$$\mathsf{PT}^{\mathscr{P}}_\alpha(p \cdot g) = \mathsf{PT}^{\mathscr{P}}_\alpha(p) \cdot \mathsf{PT}^{\mathscr{G}}_\alpha(g)$$

Connection on $\mathscr{P}$: General case



Remarks (Integrated case)

Ansatz:

$$\mathrm{PT}^{\mathscr{P}}_\alpha(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_\alpha(p) \cdot \mathrm{PT}^{\mathscr{G}}_\alpha(g)$$

$\Rightarrow$ Introduce connection on $\mathscr{G}$

Classical situation: Differential of Lie group action

Remarks (Lie group G situation with Lie algebra $\mathfrak{g}$)

In the case of a right G-action on $\mathscr{P}$, $\Phi : \mathscr{P} \times G \to \mathscr{P}$, we have

$$\mathrm{D}_{(p,g)}\Phi(X, Y) = \mathrm{D}_p r_g(X) + \left.\overline{(\mu_G)_g(Y)}\right|_{p \cdot g}$$

for all $p \in \mathscr{P}$, $g \in G$, $X \in \mathrm{T}_p\mathscr{P}$ and $Y \in \mathrm{T}_g G$, where

- $\overline{\nu}$ denotes the fundamental vector field on $\mathscr{P}$ of $\nu \in \mathfrak{g}$,
- μ_G is the Maurer-Cartan form of G.

Mark JD Hamilton. Mathematical Gauge Theory. *Springer*, 2017.

Definition (Fundamental vector fields)

Fundamental vector fields defined by

$$\overline{\nu}_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(p \cdot \mathrm{e}^{t\nu_x}\right)$$

for all $\nu \in \Gamma(\mathscr{g})$ and $p \in \mathscr{P}_x$, where $\mathscr{g}$ is the LAB[a] of $\mathscr{G}$.

[a] Lie algebra bundle

 Straightforward generalization of classical definition as in: Mark JD Hamilton. Mathematical Gauge Theory. *Springer*, 2017.

Definition (Darboux derivative)

For $\sigma \in \Gamma(\mathscr{G})$ we define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathscr{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathscr{G}},$$

where $\mu_{\mathscr{G}}$ is given by

$$(\mu_{\mathscr{G}})_g := \mathrm{D}_g L_{g^{-1}} \circ \pi^v,$$

π^v the projection onto the vertical bundle.

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Remarks

If $\mathscr{G}$ a trivial LGB with canonical flat connection and if Lie group additionally a matrix group, then

$$\Delta\sigma = \sigma^{-1}\mathrm{d}\sigma.$$

 Trivial generalization of classical definition as in: K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Proposition (Differential of LGB action Φ, [S.-R. F.])

We have

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_\sigma(X) - \left.\overline{\left.(\pi^! \Delta\sigma)\right|_p(X)}\right|_{p \cdot g} + \left.\overline{(\mu_{\mathscr{G}})_g(Y)}\right|_{p \cdot g}$$

for all $(p, g) \in \mathscr{P}_x \times \mathscr{G}_x$, $(X, Y) \in \mathrm{T}_{(p,g)}(\mathscr{P} * \mathscr{G})$, *where* σ *is any section of* $\mathscr{G}$ *with* $\sigma_x = g$.

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Definition (Modified right-pushforward, [S.-R. F.])

$$\begin{aligned}
\mathscr{r}_{g*}(X) &:= \mathrm{D}_p r_\sigma(X) - \overline{(\pi^!\Delta\sigma)|_p(X)}\Big|_{p\cdot g}, \\
\mathscr{r}_{\sigma*}(X) &:= \mathscr{r}_{\sigma_x *}(X).
\end{aligned}$$

Proposition (Well-defined isomorphism, [S.-R. F.])

We have that

$$\begin{aligned} \mathrm{T}\mathscr{P}|_{\mathscr{P}_x} &\to \mathrm{T}\mathscr{P}|_{\mathscr{P}_x}, \\ X &\mapsto \mathscr{r}_{g*}(X), \end{aligned}$$

is a well-defined automorphism over r_g. Similarly,

$$\begin{aligned} \mathrm{T}\mathscr{P} &\to \mathrm{T}\mathscr{P}, \\ X &\mapsto \mathscr{r}_{\sigma*}(X), \end{aligned}$$

is an automorphism over r_σ.

Definition (Ehresmann connection, [S.-R. F.])

H a horizontal distribution of $\mathrm{T}\mathscr{P}$ with

$$\mathscr{r}_{g*}(H_p) = H_{p \cdot g}$$

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$$r_{g*}(H_p) = H_{p \cdot g}$$

Definition (Connection 1-form, [S.-R. F.])

$A \in \Omega^1(\mathscr{P}; \pi^* g)$ with

$$\begin{aligned} r_\sigma^! A &= \mathrm{Ad}_{\sigma^{-1}} \circ A, \\ A(\overline{\nu}) &= \pi^* \nu \end{aligned}$$

for all $\sigma \in \Gamma(\mathscr{G})$ and $\nu \in \Gamma(g)$.

Remarks

$$\left(r_\sigma^! A\right)_p (X) = A_{p\sigma_x}\left(r_{\sigma*}(X)\right).$$

Theorem (Equivalence of both definitions, [S.-R. F.])

There is the usual 1:1 correspondence between both definitions:

- *Given H, define A by*

$$A_p(\overline{\nu}_p + X) \coloneqq (\pi^* \nu)_p,$$

  where $X \in H_p$.

- *Given A, define H by*

$$H_p \coloneqq \mathrm{Ker}(A_p).$$

Theorem (Gauge transformation, [S.-R. F.])

Let s_i, s_j be two sections of $\mathscr{P}$ over U_i and U_j, respectively, which are open subsets of M with $U_i \cap U_j \neq \emptyset$. Then over $U_i \cap U_j$

$$A_{s_i} = \mathrm{Ad}_{\sigma_{ji}^{-1}} \circ A_{s_j} + \Delta\sigma_{ji},$$

where $A_{s_i} := s_i^! A$ and σ_{ji} a section of $\mathscr{G}$ with $s_i = s_j \cdot \sigma_{ji}$.

Proposition (Connection on $\mathscr{G}$, [S.-R. F.])

We have an induced vector bundle connection on $\mathscr{G}$ given by

$$\nabla^{\mathscr{G}} \nu := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta \mathrm{e}^{t\nu}.$$

 Similar construction as in: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

Remarks

Recall, $\mathscr{G}$ a principal $\mathcal{G}$-bundle.

Definition (Compatibility conditions, [S.-R. F.])

$\mu_{\mathscr{G}}$ a **Yang-Mills connection (w.r.t. $\zeta \in \Omega^2(M; \mathscr{g})$)** if it satisfies the **compatibility conditions**:

1. $\mu_{\mathscr{G}}$ a connection 1-form on $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$;
2. $\mu_{\mathscr{G}}$ satisfies the **generalised Maurer-Cartan equation**

$$\left(\mathrm{d}^{\pi_{\mathscr{G}}^{*}\nabla^{\mathscr{G}}}\mu_{\mathscr{G}} + \frac{1}{2}[\mu_{\mathscr{G}} \stackrel{\wedge}{,} \mu_{\mathscr{G}}]_{\pi_{\mathscr{G}}^{*}\mathscr{g}}\right)\bigg|_{g} = \mathrm{Ad}_{g^{-1}} \circ \pi_{\mathscr{G}}^{!}\zeta\Big|_{g} - \pi_{\mathscr{G}}^{!}\zeta\Big|_{g}$$

Proposition ($\nabla^{\mathscr{G}}$ a Lie bracket derivation)

Let $\mu_{\mathscr{G}}$ be a connection 1-form on $\mathscr{G}$, then

$$\nabla^{\mathscr{G}}\left([\mu, \nu]_{\mathscr{g}}\right) = \left[\nabla^{\mathscr{G}}\mu, \nu\right]_{\mathscr{g}} + \left[\mu, \nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}}.$$

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Theorem (Curvature of LAB connection exact, [S.-R. F.])

$\mu_{\mathscr{G}}$ satisfies the generalized Maurer-Cartan equation w.r.t. ζ if and only if

$$R_{\nabla^{\mathscr{G}}} = \mathrm{ad}\circ\zeta.$$

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Remarks

There is a simplicial differential on $\mathscr{G} \overset{\pi_{\mathscr{G}}}{\to} M$

$$\delta : \Omega^\bullet(\underbrace{\mathscr{G} * \ldots * \mathscr{G}}_{k \text{ times}}; \pi_{\mathscr{G}}^* \mathscr{g}) \to \Omega^\bullet(\underbrace{\mathscr{G} * \ldots * \mathscr{G}}_{k+1 \text{ times}}; \pi_{\mathscr{G}}^* \mathscr{g})$$

such that the compatibility conditions are equivalent to

$$\delta \mu_{\mathscr{G}} = 0,$$

$$\mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}} + \frac{1}{2} [\mu_{\mathscr{G}} \overset{\wedge}{,} \mu_{\mathscr{G}}]_{\pi_{\mathscr{G}}^* \mathscr{g}} = \delta \zeta.$$

Sources For differential: Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Mathematische Zeitschrift*, 279(3):939–979, 2015.

Definition (Generalized curvature/field strength F of A, [S.-R. F.])

π^H denotes the projection onto $H \subset \mathrm{T}\mathscr{P}$, then we define

$$F := \mathrm{d}^{\pi^*\nabla^{\mathscr{G}}} A \circ \left(\pi^{\mathrm{H}}, \pi^{\mathrm{H}}\right) + \pi^! \zeta.$$

Theorem (Structure equation, [S.-R.])

$$F = \mathrm{d}^{\pi^*\nabla^{\mathscr{G}}} A + \frac{1}{2}[A \overset{\wedge}{,} A]_{\pi^*\mathscr{g}} + \pi^! \zeta.$$

Proposition (Properties of F, [S.-R. F.])

- $r_\sigma^! F = \mathrm{Ad}_{\sigma^{-1}} \circ F$,
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where $F_{s_i} := s_i^! F$ *and* σ_{ji} *a section of* $\mathscr{G}$ *with* $s_i = s_j \cdot \sigma_{ji}$.

Theorem (Lagrangian, [S.-R. F.])

- *κ be an Ad-invariant fibre metric on $\mathscr{g}$,*
- *M a spacetime, and $*$ its Hodge star operator,*
- *$(U_i)_i$ open covering of M with subordinate gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$.*

Then the Lagrangian $\mathfrak{L}_{\mathrm{CYM}}[A]$, defined locally by

$$\left(\mathfrak{L}_{\mathrm{CYM}}[A]\right)\big|_{U_i} := -\frac{1}{2}\kappa\left(F_{s_i} \overset{\wedge}{,} *F_{s_i}\right),$$

is well-defined, and

$$\mathfrak{L}_{\mathrm{CYM}}\left[L^! A\right] = \mathfrak{L}_{\mathrm{CYM}}[A]$$

for all principal bundle automorphisms L.

Example (Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^4$, [S.-R. F.])

Let P be the Hopf bundle

$$\begin{tikzcd} \mathrm{SU}(2) \cong \mathbb{S}^3 \arrow[r] & \mathbb{S}^7 \arrow[d] \\ & \mathbb{S}^4 \end{tikzcd}$$

Define $\mathscr{P} := \mathscr{G}$ as the inner group bundle of P,

$$\mathscr{G} := c_{\mathrm{SU}(2)}(P) := (P \times G) \Big/ G.$$

This principal $c_{\mathrm{SU}(2)}(P)$-bundle admits the structure as curved Yang-Mills gauge theory; there is no description as flat theory.

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle $\mathscr{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathfrak{G}$?

$$\begin{tikzcd} & \mathfrak{G} \arrow[dl] \arrow[dr] & \\ M \arrow[rr, "\Psi"] & & N \end{tikzcd}$$

Remarks

- Richer set of principal bundles, containing Lie groupoids equipped with "non-flat Maurer-Cartan forms".
- Principal bundle for the whole of Yang-Mills-Higgs theory
- Even if $\mathscr{G}$ is trivial, what happens if its connection is not flat?
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!