Geometry of curved Yang-Mills-Higgs gauge theories

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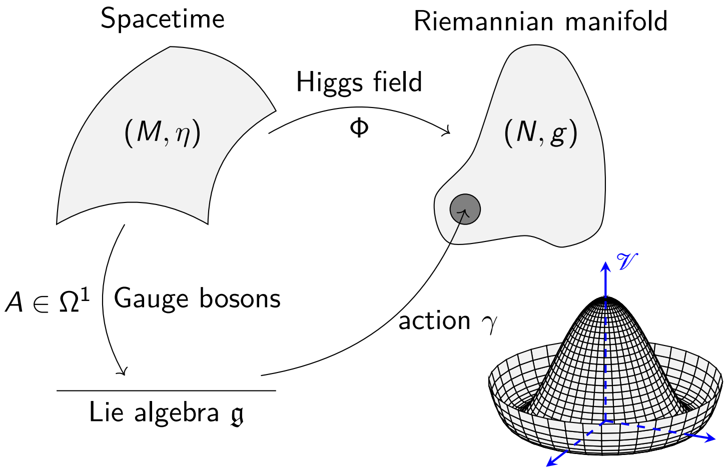
Infinitesimal gauge theory



Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathscr{g}$
Integrated	Lie group G	LGB[1] $\mathscr{G}$?

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

[1]LGB = Lie group bundle

Definition (LGB actions, simplified)

$$\begin{array}{ccc} & & \mathscr{G} \\ & & \downarrow \\ \mathscr{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathscr{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of $\mathscr{G}$ on $\mathscr{P}$** is a smooth map $\mathscr{P} * \mathscr{G} := \pi^* \mathscr{G} = \mathscr{P} \times_M \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

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Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

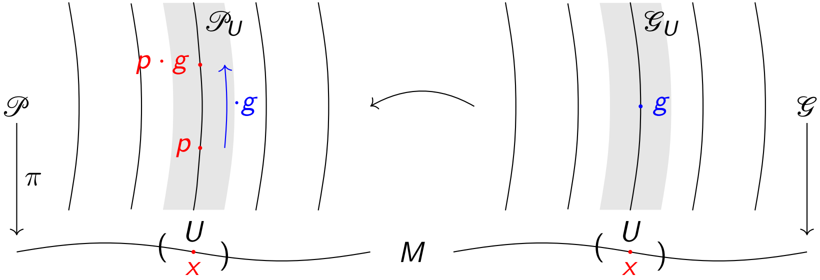
but with $\mathscr{G}$-action

$$\underset{\mathscr{P} * \mathscr{G}}{\cancel{\mathscr{P} \times G}} \to \mathscr{P}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.

Sources: Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids. *Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge*, 2003

Connection on $\mathscr{P}$: Idea



But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

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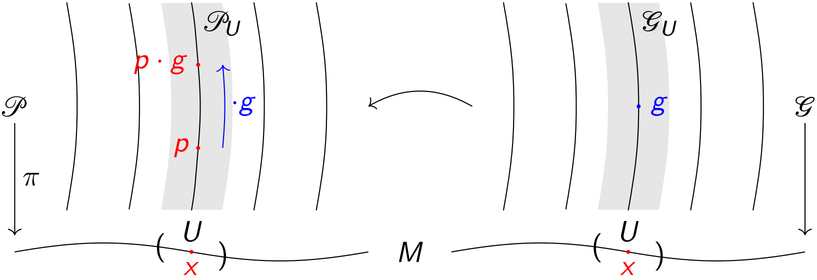

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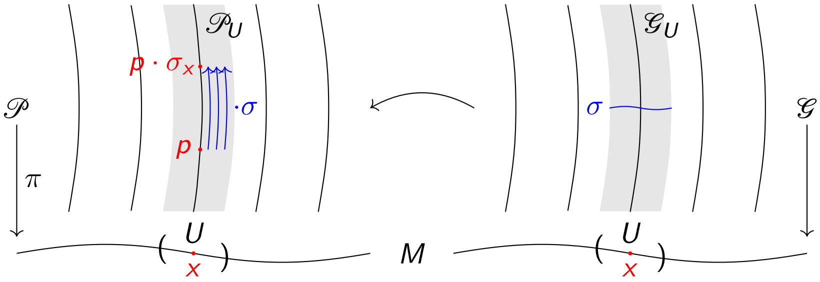
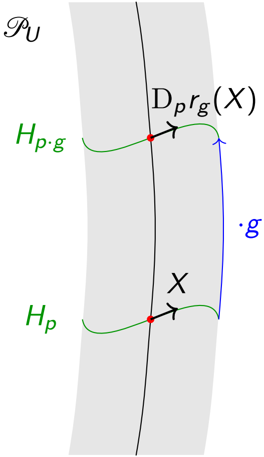


Use $\sigma \in \Gamma(\mathscr{G}) : r_\sigma(p) := p \cdot \sigma_{\pi(p)}$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



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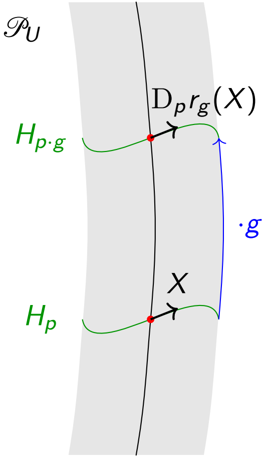


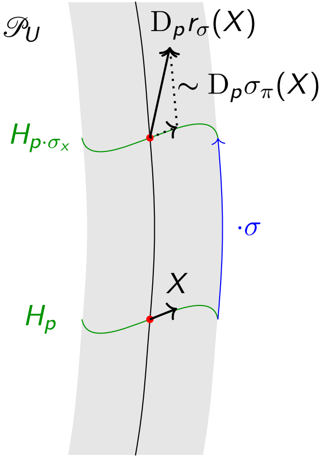
Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{P}}_{\alpha}$ in $\mathscr{P}$:

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot g$$

where $\alpha : I \to M$ is a base path

Connection on $\mathscr{P}$: General case

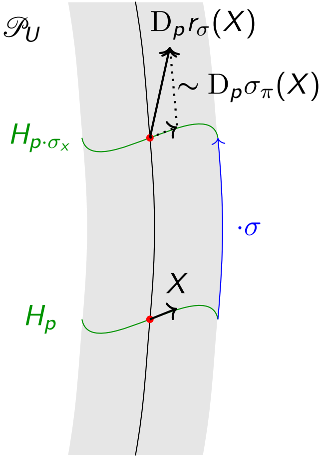


Remarks (Integrated case)

Ansatz:

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\alpha}(g)$$

Connection on $\mathscr{P}$: General case



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Ansatz:

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\alpha}(g)$$

Remarks (General situation)

Introduce connection on $\mathscr{G}$
$\Rightarrow \nabla$ on the LAB $\mathscr{g}$ of $\mathscr{G}$

Summary

	Locally	Globally
Curved Yang-Mills	Pre-classical	$\mathrm{ad}(\mathbb{S}^7 \to \mathbb{S}^4)$ curved

Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial
$\rightsquigarrow$ LGB action locally equivalent to a Lie group action

Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle $\mathscr{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathfrak{G}$?

$$\begin{tikzcd} & \mathfrak{G} \arrow[dl] \arrow[dr] & \\ M \arrow[rr, "\Phi"] & & N \end{tikzcd}$$

Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!