

Curved Yang-Mills gauge theories

based on my preprint [arXiv:2210.02924](https://arxiv.org/abs/2210.02924)

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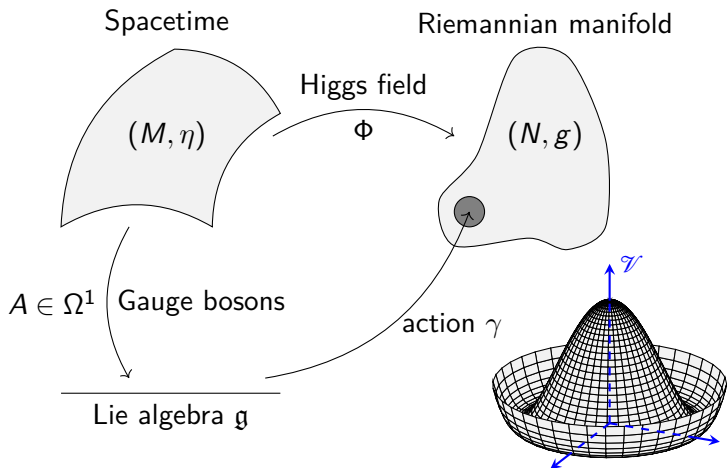
National Center for Theoretical Sciences

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Table of contents

- 1 Infinitesimal curved Yang-Mills-Higgs gauge theories
 - Motivation and short introduction
- 2 Integration: Ansatz
 - Principal bundles based on Lie group bundle actions
 - Connections as parallel transport
- 3 Connection
 - Basic notions
 - ... as horizontal distribution
 - ... as connection 1-form
- 4 Curvature
- 5 Conclusion

Infinitesimal gauge theory



Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra \mathfrak{g} as $M \times \mathfrak{g}$	Lie algebroid $E \rightarrow N$
\mathfrak{g} -action γ	Anchor ρ of E & E -connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

\rightsquigarrow We will use a general connection ∇ instead of ∇^0 , and ∇ may not be flat.

	Classical	Curved
Infinitesimal	Lie algebra \mathfrak{g}	LAB \mathcal{G}
Integrated	Lie group G	LGB ¹ \mathcal{G}

$$G \longrightarrow \mathcal{G}$$

$$M$$

¹LGB = Lie group bundle

Definition (LGB actions, simplified)

$$\begin{array}{ccc} & \mathcal{G} & \\ & \downarrow & \\ \mathcal{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathcal{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of \mathcal{G} on \mathcal{P}** is a smooth map $\mathcal{P} * \mathcal{G} := \pi^* \mathcal{G} = \mathcal{P} \times_M \mathcal{G} \rightarrow \mathcal{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \quad (1)$$

$$(p \cdot g) \cdot h = p \cdot (gh), \quad (2)$$

$$p \cdot e_{\pi(p)} = p \quad (3)$$

for all $p \in \mathcal{P}$ and $g, h \in \mathcal{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathcal{G}_{\pi(p)}$.

Examples

Example

\mathcal{G} acts canonically on itself:

$$\begin{aligned}\mathcal{G} * \mathcal{G} &\rightarrow \mathcal{G}, \\ (q, h) &\mapsto qh.\end{aligned}$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathcal{G} \cong M \times G$, then also $\mathcal{P} * \mathcal{G} \cong \mathcal{P} \times G$, and we can define

$$\begin{aligned}\mathcal{P} \times G &\rightarrow \mathcal{P}, \\ (p, g) &\mapsto p \cdot g := p \cdot (\pi(p), g),\end{aligned}$$

which is equivalent to $\mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$.

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\Rightarrow Think of the "classical" theory as coming from a trivial LGB

Definition (Principal bundle)

Still a fibre bundle

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{P} \\ & & \downarrow \pi \\ & & M \end{array}$$

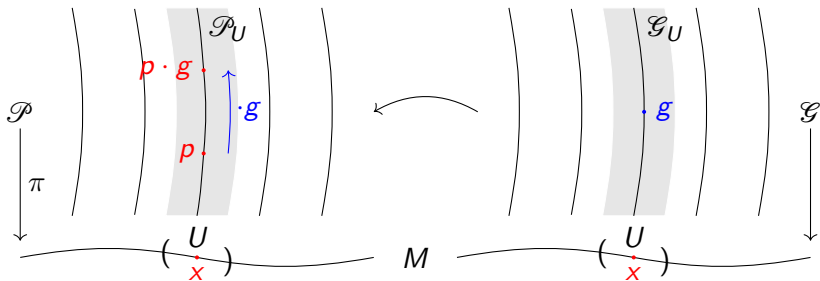
but with \mathcal{G} -action

$$\begin{array}{ccc} \cancel{\mathcal{P} * G} & \longrightarrow & \mathcal{P} \\ \mathcal{P} * \mathcal{G} & & \end{array}$$

simply transitive on fibres of \mathcal{P} , and "suitable" atlas.

Connections as parallel transport

Connection on \mathcal{P} : Idea



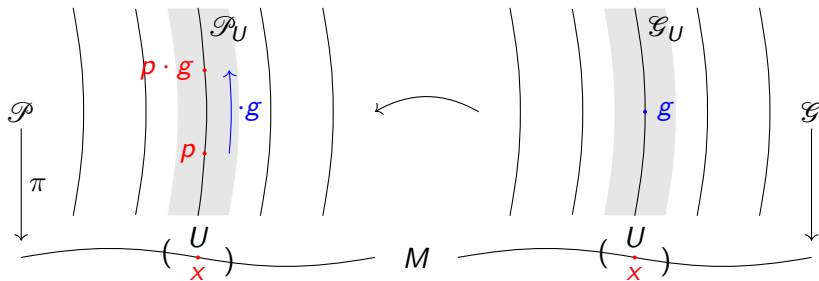
But:

$$r_g : \mathcal{P}_x \rightarrow \mathcal{P}_x$$

\Rightarrow

$D_p r_g$ only defined on vertical structure

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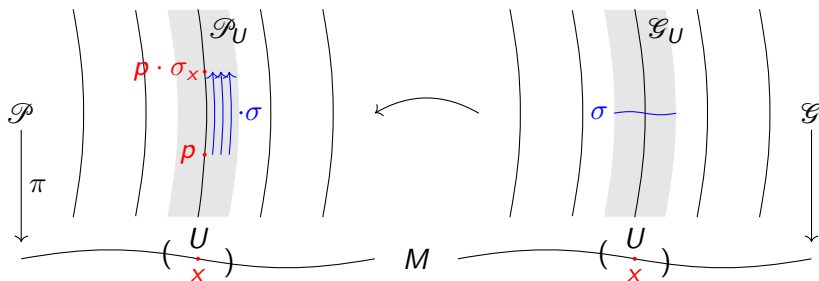


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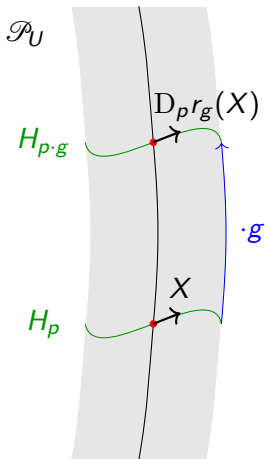
Connection on \mathcal{P} : Idea



Use $\sigma \in \Gamma(\mathcal{G}) : r_\sigma(p) := p \cdot \sigma_{\pi(p)}$

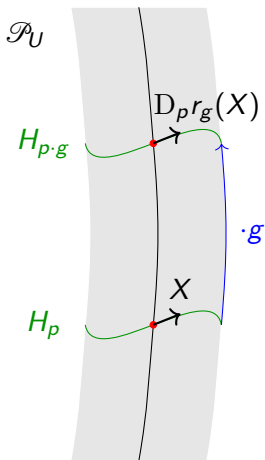
Connection on \mathcal{P} : Revisiting the classical setup

If \mathcal{P} a typical principal bundle
 (\mathcal{G} trivial, $\sigma \equiv g$ constant),
 and H a connection:



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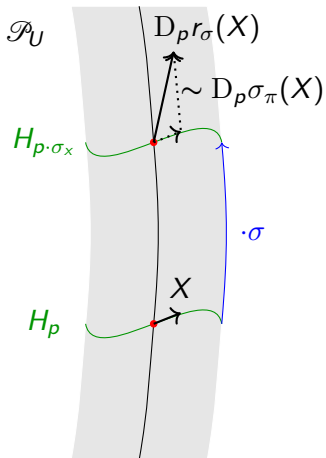
Remarks (Integrated case)

Parallel transport $\text{PT}_\alpha^{\mathcal{P}}$ in \mathcal{P} :

$$\text{PT}_\alpha^{\mathcal{P}}(p \cdot g) = \text{PT}_\alpha^{\mathcal{P}}(p) \cdot g$$

where $\alpha : I \rightarrow M$ is a base path

Connection on \mathcal{P} : General case



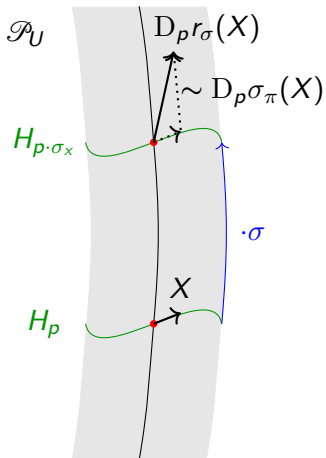
Remarks (Integrated case)

Ansatz:

$$\text{PT}_\alpha^{\mathcal{P}}(p \cdot g) = \text{PT}_\alpha^{\mathcal{P}}(p) \cdot \text{PT}_\alpha^{\mathcal{G}}(g)$$

Connections as parallel transport

Connection on \mathcal{P} : General case



Remarks (Integrated case)

Ansatz:

$$\text{PT}_\alpha^{\mathcal{P}}(p \cdot g) = \text{PT}_\alpha^{\mathcal{P}}(p) \cdot \text{PT}_\alpha^{\mathcal{G}}(g)$$

\Rightarrow Introduce connection on \mathcal{G}

Classical situation

Remarks (Lie group G situation with Lie algebra \mathfrak{g})

In the case of a right G -action on \mathcal{P} , $\Phi : \mathcal{P} \times G \rightarrow \mathcal{P}$, we have

$$D_{(p,g)}\Phi(X, Y) = D_p r_g(X) + \widetilde{(\mu_G)_g(Y)} \Big|_{p \cdot g}$$

for all $p \in \mathcal{P}$, $g \in G$, $X \in T_p \mathcal{P}$ and $Y \in T_g G$, where

- $\tilde{\nu}$ denotes the fundamental vector field on \mathcal{P} of $\nu \in \mathfrak{g}$
- μ_G is the Maurer-Cartan form of G

Definition (Fundamental vector fields)

Fundamental vector fields defined by

$$\tilde{\nu}_p := \left. \frac{d}{dt} \right|_{t=0} (p \cdot e^{t\nu_{\pi(p)}})$$

for all $\nu \in \Gamma(\mathfrak{g})$ and $p \in \mathcal{P}$, where \mathfrak{g} is the LAB^a of \mathcal{G} .

^aLie algebra bundle

Darboux derivative

... as horizontal distribution

Ehresmann connection

... as connection 1-form

Field of gauge bosons

Summary

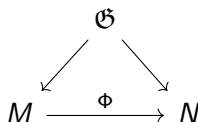
	Locally	Globally
Curved Yang-Mills	Pre-classical	$\text{ad}(\mathbb{S}^7 \rightarrow \mathbb{S}^4)$ curved

Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial
 \leadsto LGB action locally equivalent to a Lie group action

Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle \mathcal{G}
Curved Yang-Mills-Higgs	Lie groupoid \mathfrak{G} ?



Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!