

# Curved Yang-Mills gauge theories

based on my preprint [arXiv:2210.02924](https://arxiv.org/abs/2210.02924)

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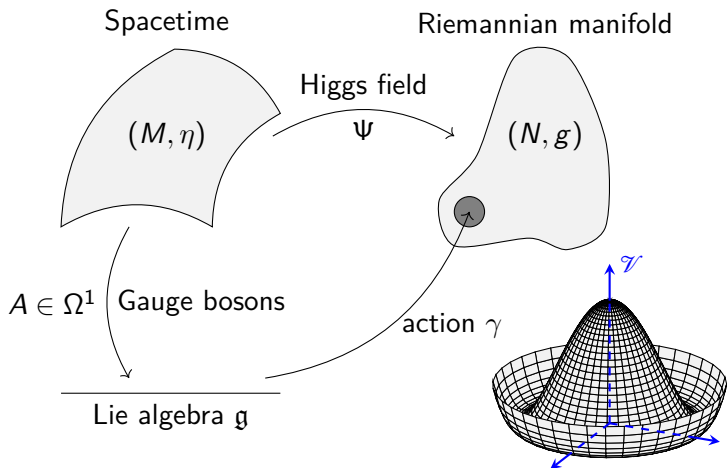
國家理論科學研究中心

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# Infinitesimal curved Yang-Mills-Higgs gauge theory



# Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \rightarrow N$
$\mathfrak{g}$ -action $\gamma$	Anchor $\rho$ of $E$ & $E$ -connections
Canonical flat connection $\nabla^0$ on $M \times \mathfrak{g}$	General connection $\nabla$ on $E$

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Classical formalism	CYMH GT
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## Remarks (Why a "curved theory"?)

Usually, the field strength  $F$  is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

$\rightsquigarrow$  We will use a general connection  $\nabla$  instead of  $\nabla^0$ , and  $\nabla$  may not be flat.

We will only focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB <sup>1</sup> $\mathcal{G}$
Integrated	Lie group $G$	LGB <sup>2</sup> $\mathcal{G}$

$$\begin{array}{ccc}
 G & \longrightarrow & \mathcal{G} \\
 & & \downarrow \\
 & & M
 \end{array}$$

---

<sup>1</sup>LAB = Lie algebra bundle

<sup>2</sup>LGB = Lie group bundle

## Definition (LGB actions, simplified)

$$\begin{array}{ccc} & \mathcal{G} & \\ & \downarrow & \\ \mathcal{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathcal{P} \xrightarrow{\pi} M$  a fibre bundle. A **right-action of  $\mathcal{G}$  on  $\mathcal{P}$**  is a smooth map  $\mathcal{P} * \mathcal{G} := \pi^* \mathcal{G} = \mathcal{P} \times_M \mathcal{G} \rightarrow \mathcal{P}$ ,  $(p, g) \mapsto p \cdot g$ , satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all  $p \in \mathcal{P}$  and  $g, h \in \mathcal{G}_{\pi(p)}$ , where  $e_{\pi(p)}$  is the neutral element of  $\mathcal{G}_{\pi(p)}$ .

# Examples

## Example

$\mathcal{G}$  acts canonically on itself:

$$\begin{aligned}\mathcal{G} * \mathcal{G} &\rightarrow \mathcal{G}, \\ (q, h) &\mapsto qh.\end{aligned}$$

## Example (Recovering Lie group action)

- Either by  $M = \{*\}$ .
- Or by  $\mathcal{G} \cong M \times G$ , then also  $\mathcal{P} * \mathcal{G} \cong \mathcal{P} \times G$ , and we can define

$$\begin{aligned}\mathcal{P} \times G &\rightarrow \mathcal{P}, \\ (p, g) &\mapsto p \cdot g := p \cdot (\pi(p), g),\end{aligned}$$

which is equivalent to  $\mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$ .



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## Guide to recover the classical theory (Part 1 of 3)

- 1  $\mathcal{G} \cong M \times G$

## Definition (Principal bundle)

Still a fibre bundle

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{P} \\ & & \downarrow \pi \\ & & M \end{array}$$

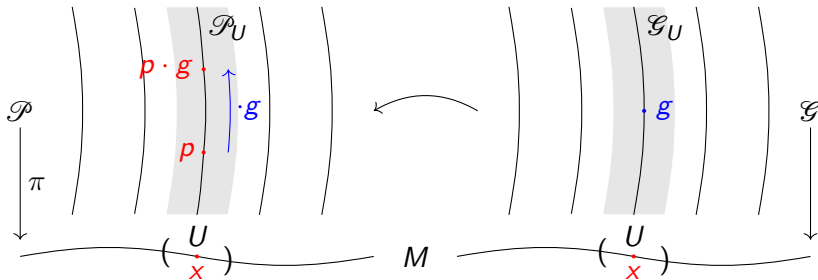
but with  $\mathcal{G}$ -action

$$\begin{array}{ccc} \cancel{\mathcal{P} * G} & \longrightarrow & \mathcal{P} \\ \mathcal{P} * \mathcal{G} & & \end{array}$$

simply transitive on fibres of  $\mathcal{P}$ , and "suitable" atlas.

Connections as parallel transport

# Connection on $\mathcal{P}$ : Idea



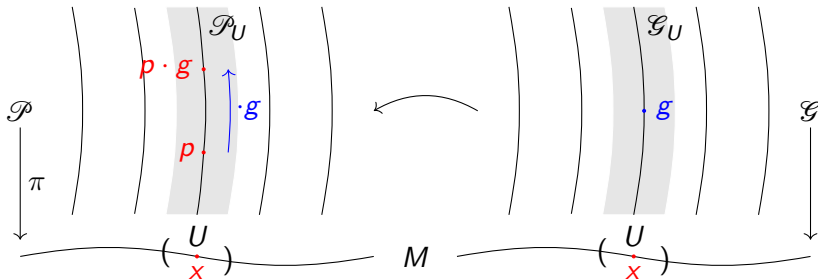
But:

$$r_g : \mathcal{P}_x \rightarrow \mathcal{P}_x$$

$\Rightarrow$

$D_p r_g$  only defined on vertical structure

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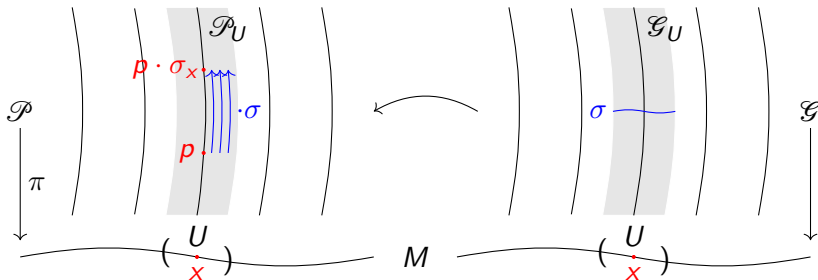


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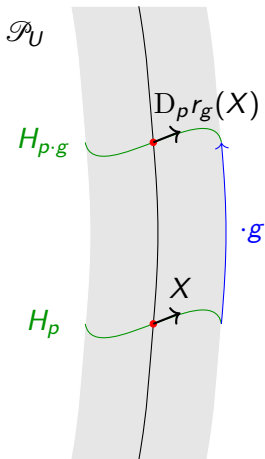
# Connection on $\mathcal{P}$ : Idea



Use  $\sigma \in \Gamma(\mathcal{E}) : r_\sigma(p) := p \cdot \sigma_x$

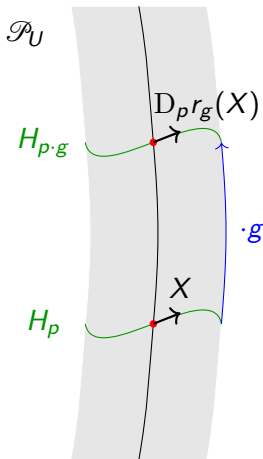
# Connection on $\mathcal{P}$ : Revisiting the classical setup

If  $\mathcal{P}$  a typical principal bundle  
 ( $\mathcal{G}$  trivial,  $\sigma \equiv g$  constant),  
 and  $H$  a connection:



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## Remarks (Integrated case)

Parallel transport  $\text{PT}_\alpha^{\mathcal{P}}$  in  $\mathcal{P}$ :

$$\text{PT}_\alpha^{\mathcal{P}}(p \cdot g) = \text{PT}_\alpha^{\mathcal{P}}(p) \cdot g$$

where  $\alpha : I \rightarrow M$  is a base path



# Connection on $\mathcal{P}$ : General case

## Remarks (Integrated case)

Ansatz: Introduce connection on  $\mathcal{G}$ ,

$$\text{PT}_{\alpha}^{\mathcal{P}}(p \cdot g) = \text{PT}_{\alpha}^{\mathcal{P}}(p) \cdot \text{PT}_{\alpha}^{\mathcal{G}}(g).$$

Guide to recover the classical theory (Part 2 of 3)

- ①  $\mathcal{G} \cong M \times G$
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# Classical situation: Differential of Lie group action

## Remarks (Lie group $G$ situation with Lie algebra $\mathfrak{g}$ )

In the case of a right  $G$ -action on  $\mathcal{P}$ ,  $\Phi : \mathcal{P} \times G \rightarrow \mathcal{P}$ , we have

$$D_{(p,g)}\Phi(X, Y) = D_p r_g(X) + \overline{(\mu_G)_g(Y)} \Big|_{p \cdot g}$$

for all  $p \in \mathcal{P}$ ,  $g \in G$ ,  $X \in T_p \mathcal{P}$  and  $Y \in T_g G$ , where

- $\bar{\nu}$  denotes the fundamental vector field on  $\mathcal{P}$  of  $\nu \in \mathfrak{g}$ ,
- $\mu_G$  is the Maurer-Cartan form of  $G$ .

## Definition (Fundamental vector fields)

**Fundamental vector fields** defined by

$$\bar{\nu}_p := \left. \frac{d}{dt} \right|_{t=0} (p \cdot e^{t\nu_x})$$

for all  $\nu \in \Gamma(\mathfrak{g})$  and  $p \in \mathcal{P}_x$ , where  $\mathfrak{g}$  is the LAB of  $\mathcal{G}$ .

## Definition (Darboux derivative)

For  $\sigma \in \Gamma(\mathcal{G})$  we define the **Darboux derivative**  $\Delta\sigma \in \Omega^1(M; \mathfrak{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathcal{G}},$$

where  $\mu_{\mathcal{G}}$  is given by

$$(\mu_{\mathcal{G}})_g := D_g L_{g^{-1}} \circ \pi^{\vee},$$

$\pi^{\vee}$  the projection onto the vertical bundle.

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$\pi^{\vee}$  the projection onto the vertical bundle.

## Remarks

If  $\mathcal{G}$  a trivial LGB with canonical flat connection and if Lie group additionally a matrix group, then

$$\Delta\sigma = \sigma^{-1} d\sigma.$$

# Proposition (Differential of LGB action $\Phi$ , [S.-R. F.])

We have

$$D_{(p,g)}\Phi(X, Y) = D_p r_\sigma(X) - \overline{(\pi^! \Delta \sigma)|_p(X)} \Big|_{p \cdot g} + \overline{(\mu_{\mathcal{G}})_g(Y)} \Big|_{p \cdot g}$$

for all  $(p, g) \in \mathcal{P}_x \times \mathcal{G}_x$ ,  $(X, Y) \in T_{(p,g)}(\mathcal{P} * \mathcal{G})$ , where  $\sigma$  is any section of  $\mathcal{G}$  with  $\sigma_x = g$ .

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# Definition (Modified right-pushforward, [S.-R. F.])

Define

$$r'_{g*}(X) := D_p r_\sigma(X) - \overline{(\pi^! \Delta \sigma)|_p(X)} \Big|_{p \cdot g}.$$



# Proposition (Well-defined isomorphism, [S.-R. F.])

*We have that*

$$\begin{aligned} \mathrm{T}\mathcal{P}|_{\mathcal{P}_x} &\rightarrow \mathrm{T}\mathcal{P}|_{\mathcal{P}_x}, \\ X &\mapsto r_{g*}(X), \end{aligned}$$

*is a well-defined automorphism over  $r_g$ .*

# Definition (Ehresmann connection, [S.-R. F.])

$H$  a horizontal distribution of  $T\mathcal{P}$  with

$$\tau_{g*}(H_p) = H_{p \cdot g}$$

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# Definition (Connection 1-form, [S.-R. F.])

$A \in \Omega^1(\mathcal{P}; \pi^*\mathcal{G})$  with

$$A(\bar{\nu}) = \pi^*\nu,$$

$$\tau_\sigma^! A = \text{Ad}_{\sigma^{-1}} \circ A$$

for all  $\sigma \in \Gamma(\mathcal{G})$  and  $\nu \in \Gamma(\mathcal{G})$ .

# Remarks

$$(\tau_\sigma^! A)_p(X) = A_{p\sigma_x}(\tau_{\sigma*}(X)).$$

## Theorem (Equivalence of both definitions, [S.-R. F.]

*There is the usual 1:1 correspondence between both definitions:*

- *Given  $H$ , define  $A$  by*

$$A_p(\bar{\nu}_p + X) := (\pi^* \nu)_p,$$

*where  $X \in H_p$ .*

- *Given  $A$ , define  $H$  by*

$$H_p := \text{Ker}(A_p).$$

### Theorem (Gauge transformation, [S.-R. F.])

*Let  $s_i, s_j$  be two sections of  $\mathcal{P}$  over  $U_i$  and  $U_j$ , respectively, which are open subsets of  $M$  with  $U_i \cap U_j \neq \emptyset$ . Then over  $U_i \cap U_j$*

$$A_{s_i} = \text{Ad}_{\sigma_{ji}^{-1}} \circ A_{s_j} + \Delta\sigma_{ji},$$

*where  $A_{s_i} := s_i^! A$  and  $\sigma_{ji}$  a section of  $\mathcal{G}$  with  $s_i = s_j \cdot \sigma_{ji}$ .*

### Proposition (Connection on $\mathcal{G}$ , [S.-R. F.]

*We have an induced vector bundle connection on  $\mathcal{G}$  given by*

$$\nabla^{\mathcal{G}}_{\nu} := \left. \frac{d}{dt} \right|_{t=0} \Delta e^{t\nu}.$$

## Remarks

Recall,  $\mathcal{G}$  a principal  $G$ -bundle.

## Definition (Compatibility conditions, [S.-R. F.])

$\mu_{\mathcal{G}}$  a **Yang-Mills connection** (w.r.t.  $\zeta \in \Omega^2(M; \mathfrak{g})$ ) if it satisfies the **compatibility conditions**:

- ①  $\mu_{\mathcal{G}}$  a connection 1-form on  $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ ;
- ②  $\mu_{\mathcal{G}}$  satisfies the **generalised Maurer-Cartan equation**

$$\left( d\pi_{\mathcal{G}}^* \nabla^{\mu_{\mathcal{G}}} \mu_{\mathcal{G}} + \frac{1}{2} [\mu_{\mathcal{G}} \wedge \mu_{\mathcal{G}}]_{\pi_{\mathcal{G}}^* \mathfrak{g}} \right) \Big|_g = \text{Ad}_{g^{-1}} \circ \pi_{\mathcal{G}}^! \zeta \Big|_g - \pi_{\mathcal{G}}^! \zeta \Big|_g$$

## Proposition ( $\nabla^{\mathcal{G}}$ a Lie bracket derivation)

Let  $\mu_{\mathcal{G}}$  be a connection 1-form on  $\mathcal{G}$ , then

$$\nabla^{\mathcal{G}}([\mu, \nu]_{\mathcal{G}}) = [\nabla^{\mathcal{G}}\mu, \nu]_{\mathcal{G}} + [\mu, \nabla^{\mathcal{G}}\nu]_{\mathcal{G}}.$$



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## Theorem (Curvature of LAB connection exact, [S.-R. F.])

$\mu_{\mathcal{G}}$  satisfies the generalized Maurer-Cartan equation w.r.t.  $\zeta$  if and only if

$$R_{\nabla^{\mathcal{G}}} = \text{ad} \circ \zeta.$$

## Remarks

There is a simplicial differential  $\delta$  on  $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$

$$\delta : \Omega^{\bullet}(\underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{G}) \rightarrow \Omega^{\bullet}(\underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{G})$$

such that the compatibility conditions are equivalent to

$$\delta \mu_{\mathcal{G}} = 0 \quad \text{and} \quad d^{\pi_{\mathcal{G}}^* \nabla^{\mathcal{G}}} \mu_{\mathcal{G}} + \frac{1}{2} [\mu_{\mathcal{G}} \wedge \mu_{\mathcal{G}}]_{\pi_{\mathcal{G}}^* \mathcal{G}} = \delta \zeta.$$

## Guide to recover the classical theory (Part 3 of 3)

- ①  $\mathcal{G} \cong M \times G$
- ② Equip  $\mathcal{G}$  with canonical flat connection
- ③  $\zeta \equiv 0$

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- ①  $\mathcal{G} \cong M \times G$
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- ③  $\zeta \equiv 0$

Given a Yang-Mills connection on  $\mathcal{G}$ :

**Definition (Generalized curvature/field strength  $F$  of  $A$ , [S.-R. F.]**

$\pi^H$  denotes the projection onto  $H \subset T\mathcal{P}$ , then we define

$$F := d^{\pi^* \nabla^{\mathcal{G}}} A \circ (\pi^H, \pi^H) + \pi^! \zeta.$$

**Theorem (Structure equation, [S.-R. F.]**

$$F = d^{\pi^* \nabla^{\mathcal{G}}} A + \frac{1}{2} [A \wedge A]_{\pi^* \mathcal{G}} + \pi^! \zeta.$$

## Proposition (Properties of $F$ , [S.-R. F.])

- $F(X, \cdot) = 0$ , if  $X$  vertical,
- $\pi_\sigma^! F = \text{Ad}_{\sigma^{-1}} \circ F$ .

## Proposition (Properties of $F$ , [S.-R. F.])

- $F(X, \cdot) = 0$ , if  $X$  vertical,
- $\kappa_\sigma^! F = \text{Ad}_{\sigma^{-1}} \circ F$ .

## Theorem (Gauge transformation, [S.-R. F.])

Let  $s_i, s_j$  be two sections of  $\mathcal{P}$  over  $U_i$  and  $U_j$ , respectively, which are open subsets of  $M$  with  $U_i \cap U_j \neq \emptyset$ . Then over  $U_i \cap U_j$

$$F_{s_i} = \text{Ad}_{\sigma_{ji}^{-1}} \circ F_{s_j},$$

where  $F_{s_i} := s_i^! F$  and  $\sigma_{ji}$  a section of  $\mathcal{G}$  with  $s_i = s_j \cdot \sigma_{ji}$ .

## Theorem (Lagrangian, [S.-R. F.])

- $\kappa$  be an Ad-invariant fibre metric on  $\mathfrak{g}$ ,
- $M$  a spacetime, and  $*$  its Hodge star operator,
- $(U_i)_i$  open covering of  $M$  with subordinate gauges  $s_i \in \Gamma(\mathcal{P}|_{U_i})$ .

Then the Lagrangian  $\mathfrak{L}_{\text{CYM}}[A]$ , defined locally by

$$(\mathfrak{L}_{\text{CYM}}[A])|_{U_i} := -\frac{1}{2}\kappa(F_{s_i} \wedge *F_{s_i}),$$

is well-defined, and

$$\mathfrak{L}_{\text{CYM}}[L^!A] = \mathfrak{L}_{\text{CYM}}[A]$$

for all principal bundle automorphisms  $L$ .

## Example (Hopf fibration $S^7 \rightarrow S^4$ , [S.-R. F.] )

Let  $P$  be the Hopf bundle

$$\begin{array}{ccc} \mathrm{SU}(2) \cong S^3 & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

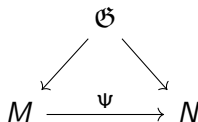
Define  $\mathcal{P} := \mathcal{G}$  as the inner group bundle of  $P$ ,

$$\mathcal{G} := c_{\mathrm{SU}(2)}(P) := (P \times \mathrm{SU}(2)) / \mathrm{SU}(2).$$

This principal  $c_{\mathrm{SU}(2)}(P)$ -bundle admits the structure as curved Yang-Mills gauge theory; there is no description as flat theory.



Gauge theory	Structure
Yang-Mills	Lie group $G$
Curved Yang-Mills	Lie group bundle $\mathcal{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathcal{G}?$



## Remarks

- Richer set of principal bundles, containing Lie groupoids equipped with "non-flat Maurer-Cartan forms".
- Principal bundle for the whole of Yang-Mills-Higgs theory
- Even if  $\mathcal{G}$  is trivial, what happens if its connection is not flat?
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

**Thank you!**