Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux (Université de Lorraine)

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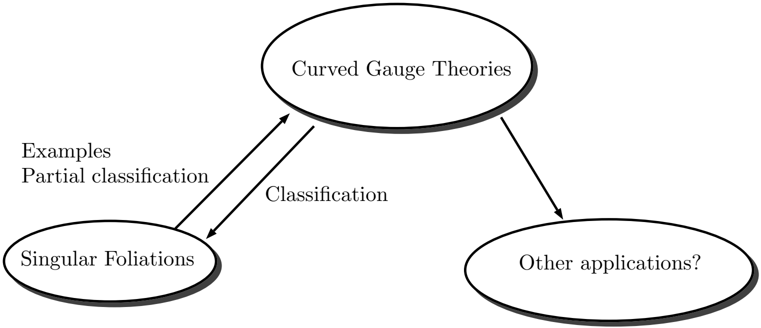
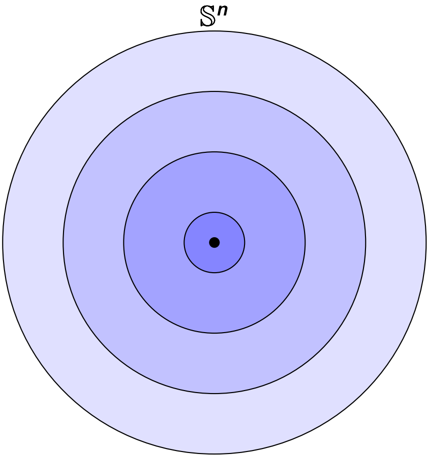
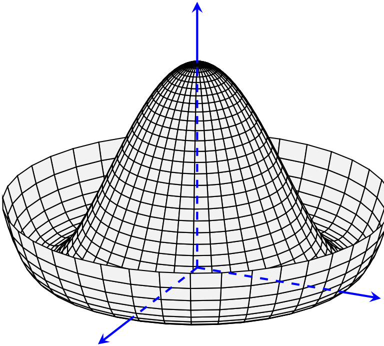


Curved Gauge Theories
Examples
Partial classification
Classification
Singular Foliations
Other applications?

Singular Foliations





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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- it is **locally finitely generated**, *i.e. around each* $p \in M$ *there is an open neighbourhood* U *and a finite family* $(X^i)_i^r$ $(X^i \in \mathscr{F})$ *such that for all* $X \in \mathscr{F}$ *there are* $f_i \in C^\infty(M)$ *satisfying on* U.

$$X = \sum_i f_i X^i.$$

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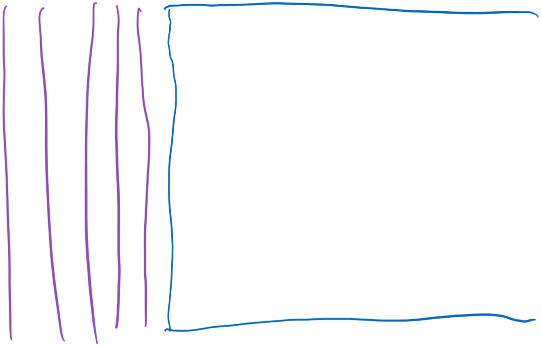
Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



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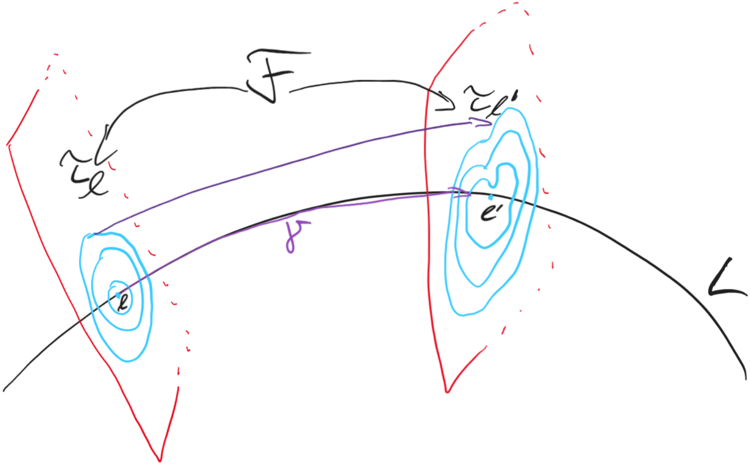
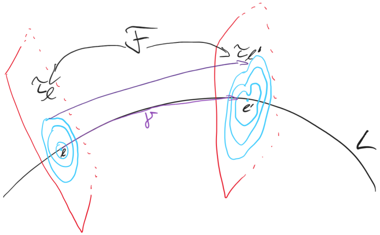
Idea: Relation to gauge theory



Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.



Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in* $\mathscr{F}$.
- *Parallel transport* PT_γ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop* γ_0 *at* l: PT_{γ_0} *values in* $\mathrm{Inner}(\tau_l)$.

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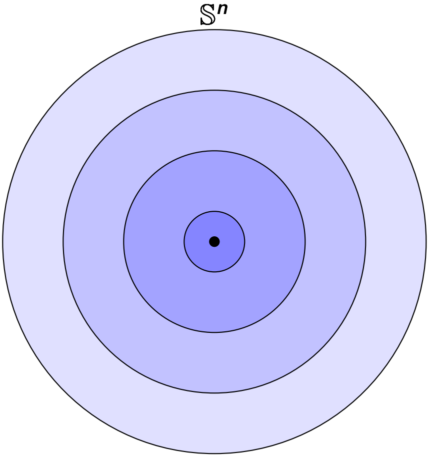
Example of a transverse foliation τ:



Remarks

- $\mathrm{Inner}(\tau_I)$ maps each circle to itself
- $\mathrm{Sym}(\tau_I)$ allows to exchange circles
- Both preserve τ_I and fix the origin

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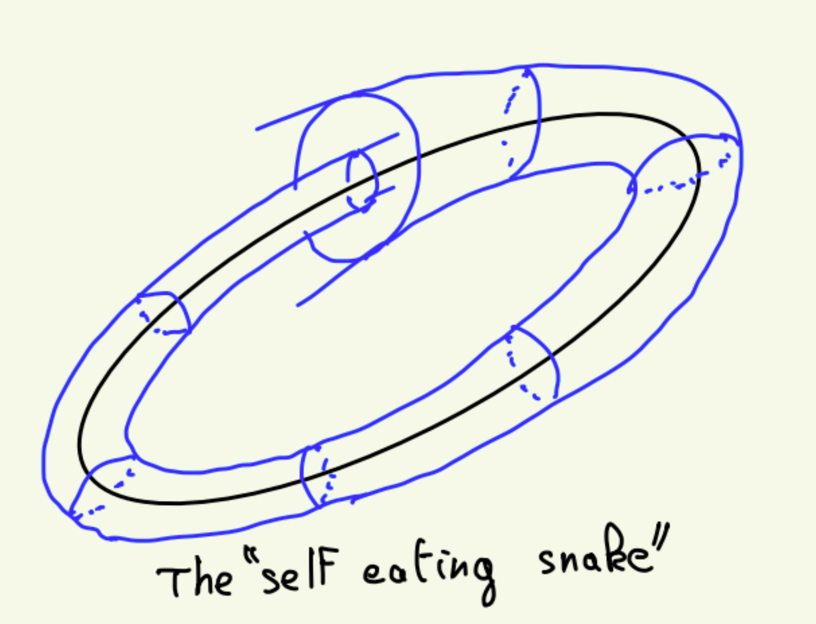
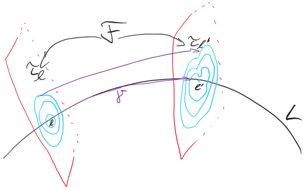


Diagram made by Camille Laurent-Gengoux.



Remarks ($\mathscr{F}$-connection)

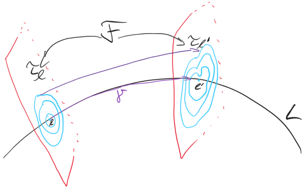
For $\phi \in \mathrm{Sym}(\tau_l)$ we have an induced parallel transport

$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) \coloneqq \mathrm{PT}_{\gamma} \circ \phi \circ \mathrm{PT}_{\gamma}^{-1}.$$

Then, on the normal bundle $\pi \colon \mathscr{T} \to L$,

$$\mathrm{PT}_{\gamma}(\phi \cdot p) = \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) \cdot \mathrm{PT}_{\gamma}(p)$$
$$\mathrm{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all $p \in \mathscr{T}_l$, $\phi \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.



Remarks (Sym-connection)

For $\phi \in \mathrm{Sym}(\tau_l)$ we have an induced parallel transport

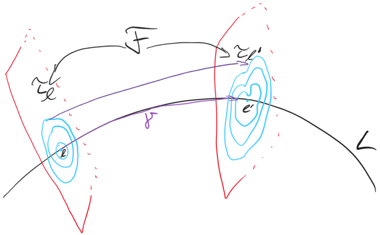
$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) := \mathrm{PT}_{\gamma} \circ \phi \circ \mathrm{PT}_{\gamma}^{-1}.$$

Then

$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi \circ \phi') = \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) \circ \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi')$$
$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma_0}(\phi) = \varphi \circ \phi \circ \varphi^{-1}$$

for all $\phi, \phi' \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.

Idea

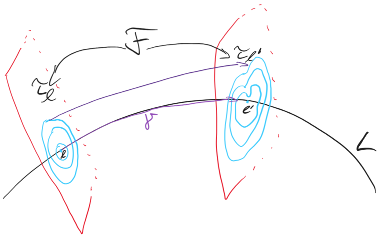


Idea

Generators of $\mathscr{F}$ given by:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.



Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$
\begin{aligned}
[\mathbb{H}(X)+\overline{\nu}, \mathbb{H}(X')+\overline{\mu}] &= \mathbb{H}([X, X']) + \dots \\
&= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{ curvature}} \\
&\qquad + \underbrace{[\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu, \mu]}
\end{aligned}
$$

Summary

$$
\begin{tikzcd}
 & \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
 & \{\text{Connection on } \mathrm{Sym}(\tau)\} &
\end{tikzcd}
$$

Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given $\mathscr{F}$!

Multiplicative Yang-Mills connections

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

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Definition (LGB actions)

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi"] & L \end{tikzcd}$$

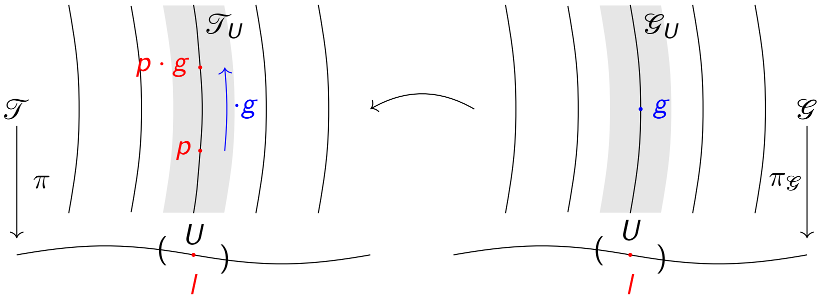
A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\pi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$
$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$
$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Connection on $\mathscr{T}$: Idea



But:

$$r_g : \mathscr{T}_l \to \mathscr{T}_l$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

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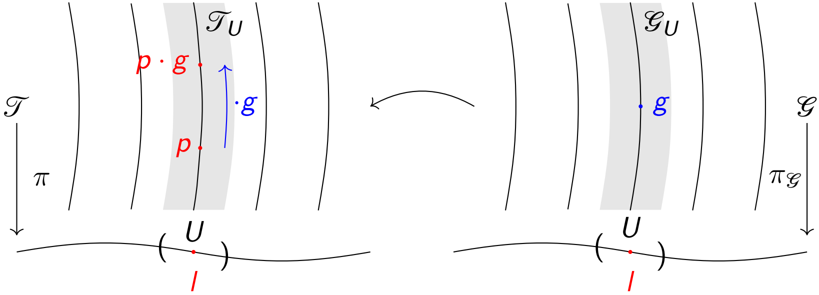

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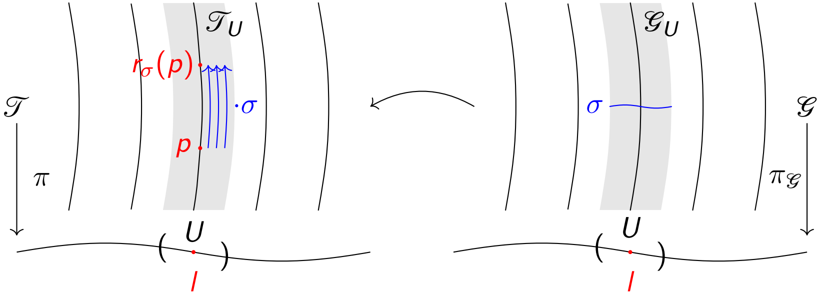
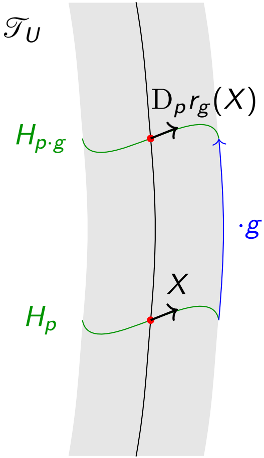


Use $\sigma \in \Gamma(\mathscr{G})$: $r_\sigma(p) := p \cdot \sigma_I$

Connection on $\mathscr{T}$: Revisiting the classical setup

If $\mathscr{G}$ is trivial, $\sigma \equiv g$ constant,
and H a connection:



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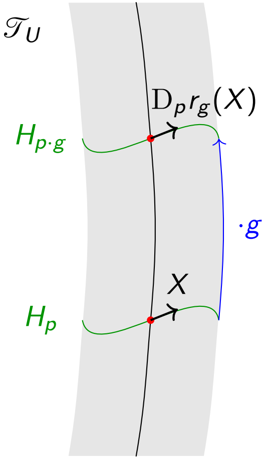

Remarks (Integrated case)

Parallel transport $\mathrm{PT}_\gamma^{\mathscr{T}}$ in $\mathscr{T}$:

$$\mathrm{PT}_\gamma^{\mathscr{T}}(p \cdot g) = \mathrm{PT}_\gamma^{\mathscr{T}}(p) \cdot g$$

where $\gamma : I \to L$ is a base path

Connection on $\mathscr{T}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g).$$

Back to the roots

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi\colon \mathscr{T} \to L$ so that one has a commuting diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi"] & L \end{tikzcd}$$

1. **Ehresmann connection:**

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(p) = p \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}^0_{\pi(p)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q\cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q)\cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
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Compare this with the Maurer-Cartan form and its curvature equation!

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Remarks

On the Lie algebra bundle $\mathcal{g}$ we have a connection ∇ with

$$\nabla\left([\mu,\nu]_{\mathcal{g}}\right) = [\nabla\mu,\nu]_{\mathcal{g}} + [\mu,\nabla\nu]_{\mathcal{g}},$$
$$R_\nabla = \mathrm{ad}\circ\zeta.$$

Example

Consider the Atiyah sequence of a principal G-bundle P:

$$\begin{tikzcd} (P\times\mathfrak{g})/G \arrow[r, hook] & \mathrm{T}P/G \arrow[r, two heads] & \mathrm{T}L \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

with splitting $\mathbb{H}\colon \mathrm{T}L\to E$, where $\mathfrak{g}$ is the Lie algebra. Then

$$\nabla_X\nu = [\mathbb{H}(X),\nu]_E,$$
$$\zeta(X,X') = [\mathbb{H}(X),\mathbb{H}(X')]_E - \mathbb{H}([X,X']).$$

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Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla_X \nu},$$
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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. $G = \mathrm{Inn}(\tau_l)$
2. P a principal G-bundle, equipped with an ordinary connection

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- $\mathscr{G}$ acts on $\mathscr{T}$ (canonically from the left).

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

Proof.

- The adjoint bundle of P, $\mathrm{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of $\mathscr{G}$
- $\tau = \overline{\mathrm{Ad}(P)}$
- Difference of two connections on P has values in $\mathrm{Ad}(P)$

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Summary

Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf L and transverse model $\left(\mathbb{R}^d, \tau_l\right)$
- Principal $\mathrm{Inner}(\tau_l)$-bundles P over L

$$\begin{tikzcd}
 & \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & P \arrow[dl, leftrightarrow] \\
 & \{\text{Connection on } \mathrm{Sym}(\tau)\} &
\end{tikzcd}$$

Thank you!