Curved Yang-Mills gauge theories and their recent applications

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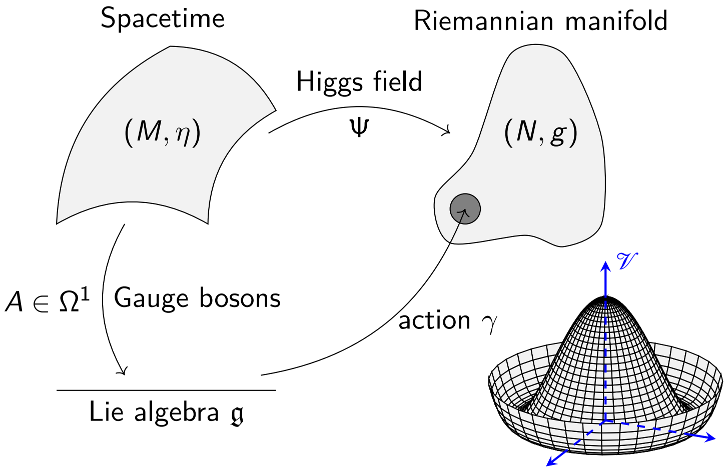
Infinitesimal Version

Infinitesimal curved Yang-Mills-Higgs gauge theory



Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Consider a semisimple Lie group G and a principal G-bundle $P \to M$:

$$\mathrm{Ad}(P) \hookrightarrow \mathrm{At}(P) \twoheadrightarrow \mathrm{T}M$$

where $\mathrm{Ad}(P)$ and $\mathrm{At}(P)$ the adjoint and Atiyah bundle of a principal G-bundle P, respectively.

Gedankenexperiment

1. Adjoint connection $\leftrightarrow$ Ehresmann connection on P.
2. As parallel transport along a curve γ:

$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^{P}(p), v\right]$$

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Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection

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$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^P(p)\cdot\kappa_\gamma, \kappa_\gamma^{-1}\cdot\mathrm{PT}_\gamma^0(v)\right],$$

Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection, κ_γ values in G

Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

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Definition (Field redefinition, [S.-R. F.])

Let $\lambda \in \Omega^1(N; E)$ such that $\Lambda := \mathbb{1}_E - \lambda \circ \rho$ is an automorphism of E. We then define the **field redefinitions** by

$$\widetilde{A}^\lambda := (\Phi^* \Lambda)(A) + \Phi^! \lambda, \tag{1}$$

$$\widetilde{\nabla}^\lambda := \nabla + \left(\Lambda \circ \mathrm{d}^{\nabla^{\mathrm{bas}}} \circ \Lambda^{-1}\right)\lambda, \tag{2}$$

$$\widetilde{\kappa}^\lambda := \kappa \circ \left(\Lambda^{-1}, \Lambda^{-1}\right), \tag{3}$$

$$\widetilde{g}^\lambda := g \circ \left(\widehat{\Lambda}^{-1}, \widehat{\Lambda}^{-1}\right), \tag{4}$$

where $\widehat{\Lambda} := \mathbb{1}_{\mathrm{T}N} - \rho \circ \lambda$, and for all $X, Y \in \mathfrak{X}(N)$ we also define

$$\widetilde{\zeta}^\lambda\left(\widehat{\Lambda}(X), \widehat{\Lambda}(Y)\right) := \Lambda(\zeta(X, Y)) - \left(\mathrm{d}^{\widetilde{\nabla}^\lambda} \lambda\right)(X, Y) + t_{\widetilde{\nabla}^\lambda_\rho}(\lambda(X), \lambda(Y)). \tag{5}$$

Proposition ([S.-R. F.])

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\widetilde{\mathfrak{L}}^{\lambda}_{\mathrm{CYMH}} = \mathfrak{L}_{\mathrm{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

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What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathcal{g}$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot, \cdot]_{\mathcal{g}} \in \Gamma\left(\bigwedge^2 \mathcal{g}^* \otimes \mathcal{g}\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

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- We need

$$\nabla_Y\left([\mu,\nu]_{\mathcal{g}}\right) = [\nabla_Y \mu, \nu]_{\mathcal{g}} + [\mu, \nabla_Y \nu]_{\mathcal{g}}, \tag{6}$$

$$R_\nabla(Y,Z)\mu = [\zeta(Y,Z),\mu]_{\mathcal{g}} \tag{7}$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu,\nu \in \Gamma(\mathcal{g})$.

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Theorem (Invariant for LABs, [S.-R. F.])

We have

$$\mathrm{d}^{\widetilde{\nabla}^\lambda}\widetilde{\zeta}^\lambda = \mathrm{d}^\nabla \zeta, \tag{8}$$

and $\mathrm{d}^\nabla \zeta$ *has values in the centre of* $\mathscr{g}$*.*

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^\nabla\zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^\lambda = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^\nabla\zeta \neq 0$.

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Turning to the field redefinition of ∇:

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ restricts to the centre of $\mathscr{g}$ and induces a differential d^{Ξ} on centre-valued forms. Moreover, d^{Ξ} is independent of the field redefinitions.

Sketch of proof.

Recall

$$\nabla_Y\left([\mu,\nu]_{\mathscr{g}}\right) = [\nabla_Y\mu,\nu]_{\mathscr{g}} + [\mu,\nabla_Y\nu]_{\mathscr{g}},$$
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Theorem (Closedness of $\mathrm{d}^{\nabla}\zeta$, [S.-R. F.])

We have

$$\mathrm{d}^{\Xi}\mathrm{d}^{\nabla}\zeta = 0. \tag{9}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^{\nabla}\zeta\right]_{\mathrm{d}^{\Xi}}. \tag{10}$$

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If $\mathrm{Obs}(\Xi) \neq 0$, then there is no field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

Theorem (Locally always pre-classical)

If N is contractible, then there is a field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

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Second theorem follows as a result by K. Mackenzie (General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005). Mackenzie derived $\mathrm{Obs}(\Xi)$ in the context of extending Lie algebroids by LABs.

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Example (Zero obstruction class not necessarily pre-classical)

Let P be the Hopf fibration

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{S}^4 \end{array}$$

Then for the adjoint bundle

$$\mathcal{g} := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left(\mathbb{S}^7 \times \mathfrak{su}(2)\right) \Big/ \mathrm{SU}(2)$$

we have a non-flat ∇ satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but $\mathrm{Obs}(\Xi) = 0$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical.
In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $\mathbb{S}^7 \to \mathbb{S}^4$?
$\rightsquigarrow$ Integration

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical. In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $\mathbb{S}^7 \to \mathbb{S}^4$?
$\rightsquigarrow$ Integration

Integrated Version

We will only focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB[1] $\mathcal{g}$
Integrated	Lie group G	LGB[2] $\mathcal{G}$

$$\begin{tikzcd} G \arrow[r] & \mathcal{G} \arrow[d] \\ & M \end{tikzcd}$$

[1] LAB = Lie algebra bundle
[2] LGB = Lie group bundle

Definition (LGB actions, simplified)

$$
\begin{array}{ccc}
 & & \mathscr{G} \\
 & & \downarrow \\
\mathscr{P} & \xrightarrow{\pi} & M
\end{array}
$$

$\mathscr{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of $\mathscr{G}$ on $\mathscr{P}$** is a smooth map $\mathscr{P} * \mathscr{G} := \pi^*\mathscr{G} = \mathscr{P} \times_M \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{11}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{12}$$

$$p \cdot e_{\pi(p)} = p \tag{13}$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

Sources
K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

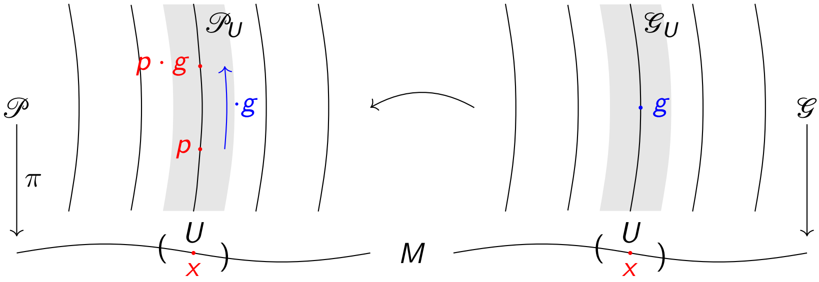
but with $\mathscr{G}$-action

$$\underset{\mathscr{P} * \mathscr{G}}{\cancel{\mathscr{P} \times G}} \to \mathscr{P}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids. *Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge*, 2003

Connection on $\mathscr{P}$: Idea



But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$$\Rightarrow \qquad \mathrm{D}_p r_g \text{ only defined on vertical structure}$$

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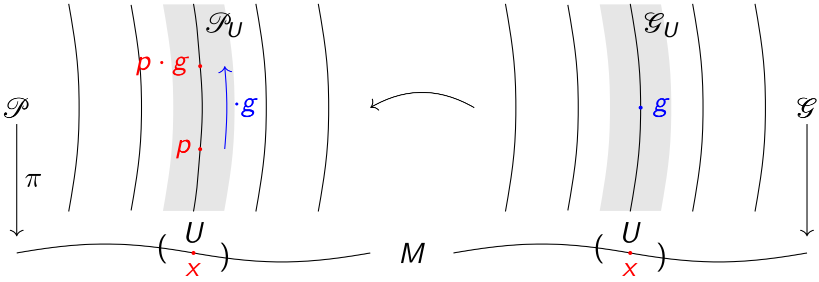

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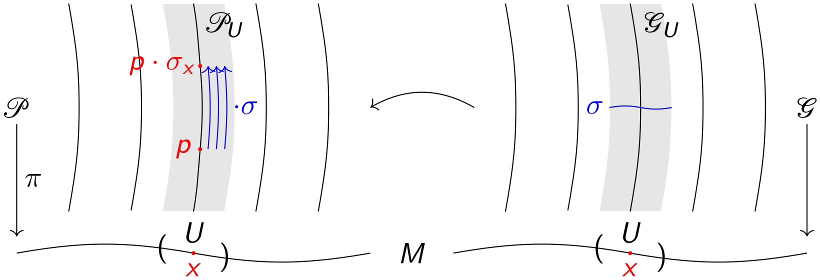
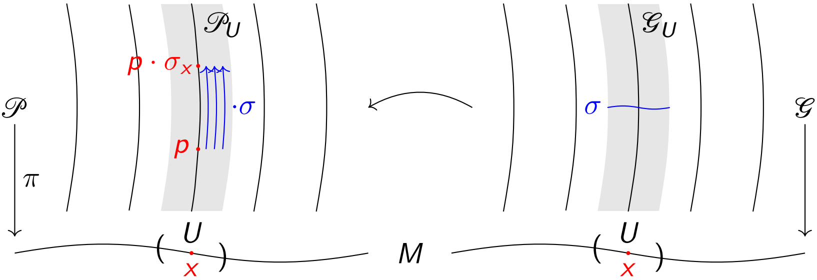


$$\text{Use } \sigma \in \Gamma(\mathscr{G}) : r_\sigma(p) := p \cdot \sigma_x$$

Remarks (Problem!)

Ambiguity in the choice of $\sigma \Rightarrow$ Fix a horizontal distribution

Connection on $\mathscr{P}$: Idea



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Ambiguity in the choice of $\sigma \Rightarrow$ Fix a horizontal distribution

Definition (Fundamental vector fields)

Fundamental vector fields defined by

$$\overline{\nu}_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(p \cdot \mathrm{e}^{t\nu_x}\right)$$

for all $\nu \in \Gamma(\mathscr{g})$ and $p \in \mathscr{P}_x$, where $\mathscr{g}$ is the LAB of $\mathscr{G}$.

 Straightforward generalization of classical definition as in: Mark JD Hamilton. Mathematical Gauge Theory. *Springer*, 2017.

Definition (Darboux derivative)

For $\sigma \in \Gamma(\mathscr{G})$ we define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathscr{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathscr{G}},$$

where $\mu_{\mathscr{G}}$ is given by

$$(\mu_{\mathscr{G}})_g := \mathrm{D}_g L_{g^{-1}} \circ \pi^v,$$

π^v the projection onto the vertical bundle.

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π^v the projection onto the vertical bundle.

Remarks

If $\mathscr{G}$ a trivial LGB with canonical flat connection and if Lie group additionally a matrix group, then

$$\Delta\sigma = \sigma^{-1}\mathrm{d}\sigma.$$

Trivial generalization of classical definition as in: K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Modified right-pushforward, [S.-R. F.])

Define

$$\mathscr{r}_{g*}(X) := \mathrm{D}_p r_\sigma(X) - \left.\overline{(\pi^! \Delta\sigma)|_p(X)}\right|_{p\cdot g}$$

for all $(p, g) \in \mathscr{P}_x \times \mathscr{G}_x$ and $X \in \mathrm{T}_p\mathscr{P}$, where σ is any section of $\mathscr{G}$ with $\sigma_x = g$.

Proposition (Well-defined isomorphism, [S.-R. F.])

We have that

$$\begin{aligned} \mathrm{T}\mathscr{P}|_{\mathscr{P}_x} &\to \mathrm{T}\mathscr{P}|_{\mathscr{P}_x}, \\ X &\mapsto \mathscr{r}_{g*}(X), \end{aligned}$$

is a well-defined automorphism over r_g.

Definition (Ehresmann connection, [S.-R. F.])

H a horizontal distribution of $\mathrm{T}\mathscr{P}$ with

$$\mathscr{r}_{g*}(H_p) = H_{p \cdot g}$$

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Definition (Equivalently: Connection 1-form, [S.-R. F.])

$A \in \Omega^1(\mathscr{P}; \pi^* \mathscr{g})$ with

$$A(\overline{\nu}) = \pi^* \nu,$$
$$\mathscr{r}_\sigma^! A = \mathrm{Ad}_{\sigma^{-1}} \circ A$$

for all $\sigma \in \Gamma(\mathscr{G})$ and $\nu \in \Gamma(\mathscr{g})$.

Remarks

$$\left(\mathscr{r}_\sigma^! A\right)_p (X) = A_{p\sigma_x}\left(\mathscr{r}_{\sigma_x *}(X)\right).$$

Proposition (Connection on $\mathscr{G}$, [S.-R. F.])

We have an induced vector bundle connection on $\mathscr{G}$ given by

$$\nabla^{\mathscr{G}} \nu := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta \mathrm{e}^{t\nu}.$$

 Similar construction as in: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

Remarks

Recall, $\mathscr{G}$ a principal $\mathcal{G}$-bundle.

Definition (Compatibility conditions, [S.-R. F.])

$\mu_{\mathscr{G}}$ a **Yang-Mills connection (w.r.t. $\zeta \in \Omega^2(M; \mathscr{g})$)** if it satisfies the **compatibility conditions**:

1. $\mu_{\mathscr{G}}$ a connection 1-form on $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$;
2. $\mu_{\mathscr{G}}$ satisfies the **generalised Maurer-Cartan equation**

$$\left(\mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}} + \frac{1}{2}[\mu_{\mathscr{G}} \overset{\wedge}{,} \mu_{\mathscr{G}}]_{\pi_{\mathscr{G}}^* \mathscr{g}}\right)\Big|_g = \mathrm{Ad}_{g^{-1}} \circ \pi_{\mathscr{G}}^! \zeta\Big|_g - \pi_{\mathscr{G}}^! \zeta\Big|_g$$

Proposition ($\nabla^{\mathscr{G}}$ a Lie bracket derivation)

Let $\mu_{\mathscr{G}}$ be a connection 1-form on $\mathscr{G}$, then

$$\nabla^{\mathscr{G}}\left([\mu,\nu]_{\mathscr{g}}\right) = \left[\nabla^{\mathscr{G}}\mu,\nu\right]_{\mathscr{g}} + \left[\mu,\nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}}.$$

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Theorem (Curvature of LAB connection exact, [S.-R. F.])

$\mu_{\mathscr{G}}$ satisfies the generalized Maurer-Cartan equation w.r.t. ζ if and only if

$$R_{\nabla^{\mathscr{G}}} = \mathrm{ad}\circ\zeta.$$

For the first statement: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

Remarks

Both compatibility conditions are related to a cohomology, that is:

1. μ_g closed
2. Curvature of μ_g exact with primitive ζ

For differential: Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Mathematische Zeitschrift*, 279(3):939–979, 2015.

Given a Yang-Mills connection on $\mathscr{G}$:

Definition (Generalized curvature/field strength F of A, [S.-R. F.])

We define

$$F := \mathrm{d}^{\pi^*\nabla^{\mathscr{G}}} A + \frac{1}{2}[A \overset{\wedge}{,} A]_{\pi^*\mathscr{g}} + \pi^! \zeta.$$

Proposition (Properties of F, [S.-R. F.])

- $F(X, \cdot) = 0$, *if* X *vertical,*
- $\mathscr{r}^!_\sigma F = \mathrm{Ad}_{\sigma^{-1}} \circ F$.

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Theorem (Gauge transformation, [S.-R. F.])

Let s_i, s_j be two sections of $\mathscr{P}$ over U_i and U_j, respectively, which are open subsets of M with $U_i \cap U_j \neq \emptyset$. Then over $U_i \cap U_j$

$$F_{s_i} = \mathrm{Ad}_{\sigma_{ji}^{-1}} \circ F_{s_j},$$

where $F_{s_i} := s_i^! F$ and σ_{ji} a section of $\mathscr{G}$ with $s_i = s_j \cdot \sigma_{ji}$.

Theorem (Lagrangian, [S.-R. F.])

- *κ be an Ad-invariant fibre metric on $\mathscr{g}$,*
- *M a spacetime, and $*$ its Hodge star operator,*
- *$(U_i)_i$ open covering of M with subordinate gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$.*

Then the Lagrangian $\mathfrak{L}_{\mathrm{CYM}}[A]$, defined locally by

$$\left(\mathfrak{L}_{\mathrm{CYM}}[A]\right)\big|_{U_i} := -\frac{1}{2}\kappa(F_{s_i} \overset{\wedge}{,} *F_{s_i}),$$

is well-defined, and

$$\mathfrak{L}_{\mathrm{CYM}}\left[L^! A\right] = \mathfrak{L}_{\mathrm{CYM}}[A]$$

for all principal bundle automorphisms L.

Back to the roots

1. $\mathscr{G} \cong M \times G$
2. Equip $\mathscr{G}$ with canonical flat connection
3. $\zeta \equiv 0$

Example (Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^4$, [S.-R. F.])

Let P be the Hopf bundle

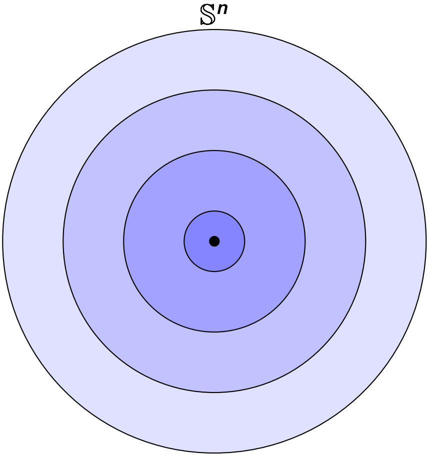
$$\begin{array}{ccc} \mathrm{SU}(2) \cong \mathbb{S}^3 & \longrightarrow & \mathbb{S}^7 \\ & & \Big\downarrow \\ & & \mathbb{S}^4 \end{array}$$

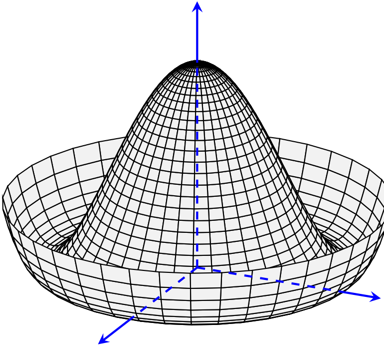
Define $\mathscr{P} := \mathscr{G}$ as the inner group bundle of P,

$$\mathscr{G} := c_{\mathrm{SU}(2)}(P) := \left(P \times \mathrm{SU}(2)\right) \Big/ \mathrm{SU}(2).$$

This principal $c_{\mathrm{SU}(2)}(P)$-bundle admits the structure as curved Yang-Mills gauge theory; there is no description as classical gauge theory.

Applications: Singular foliations
(joint work w/ Camille Laurent-Gengoux)





 Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under $C^\infty(M)$-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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- it is **locally finitely generated**, *i.e. around each* $p \in M$ *there is an open neighbourhood* U *and a finite family* $(X^i)_i^r$ $(X^i \in \mathscr{F})$ *such that for all* $X \in \mathscr{F}$ *there are* $f_i \in C^\infty(M)$ *satisfying on* U.

$$X = \sum_i f_i X^i.$$

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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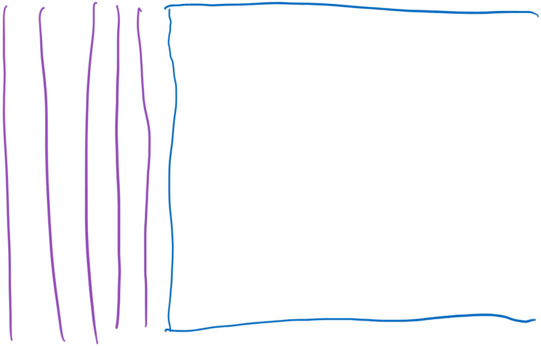
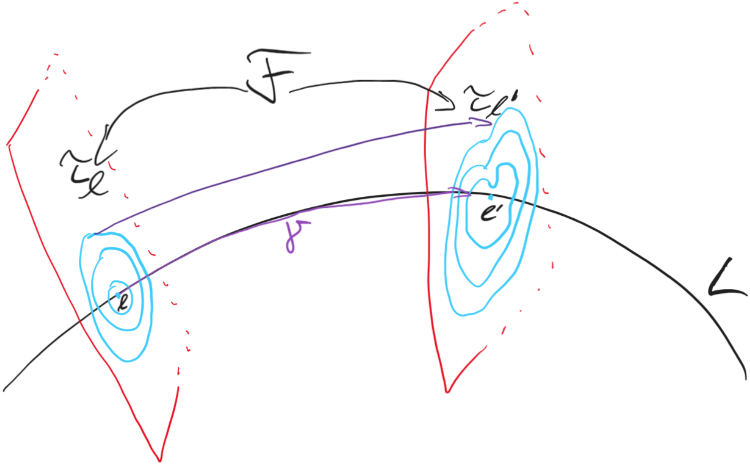
Remarks (Leaves)

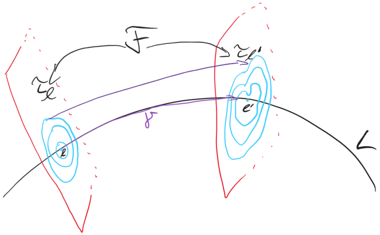
Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



Idea: Relation to gauge theory



Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.



Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in* $\mathscr{F}$.
- *Parallel transport* PT_γ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop* γ_0 *at* l*:* PT_{γ_0} *values in* $\mathrm{Inner}(\tau_l)$.

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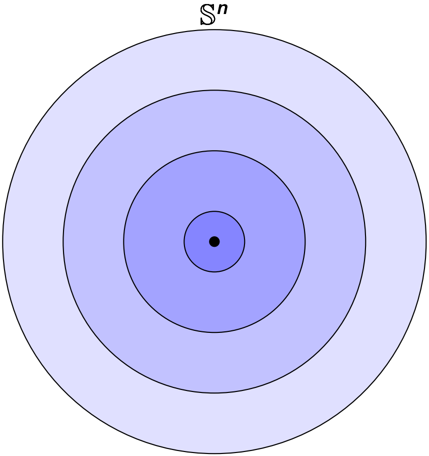
Example of a transverse foliation τ:

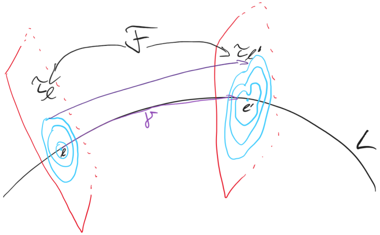


Remarks

- Inner(τ_I) maps each circle to itself
- Sym(τ_I) allows to exchange circles
- Both preserve τ_I and fix the origin

Sources: Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Idea



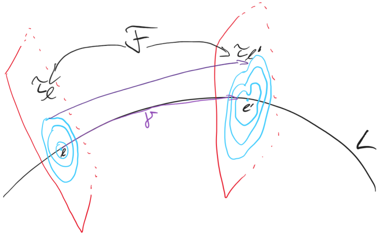
Idea

Generators of $\mathscr{F}$ given by $\mathscr{F}_{\text{projectable}}$:

$$X^\uparrow + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $X^\uparrow$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.

Idea



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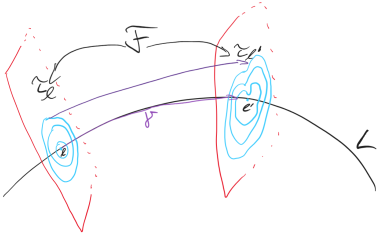

Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$\left[X^{\uparrow}+\overline{\nu}, X'^{\uparrow}+\overline{\mu}\right] = [X,X']^{\uparrow}+\dots$$

$$= \underbrace{\left[X^{\uparrow}, X'^{\uparrow}\right]}_{\rightsquigarrow\ \text{curvature}} + \underbrace{\left[X^{\uparrow}, \overline{\mu}\right] - \left[X'^{\uparrow}, \overline{\nu}\right]}_{\rightsquigarrow\ \text{connection}} + \overline{[\nu,\mu]}$$

Curved Yang-Mills gauge theory

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Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$
\begin{tikzcd}
G \arrow[r] & \mathscr{G} \arrow[d] \\
& L
\end{tikzcd}
$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

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$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Definition (LGB actions)

$$
\begin{tikzcd}
 & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\
\mathscr{T} \arrow[r, "\phi"] & L
\end{tikzcd}
$$

A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T}\ {}_{\phi}\!\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(t, g) \mapsto t \cdot g$, satisfying the following properties:

$$\phi(t \cdot g) = \phi(t), \tag{14}$$

$$(t \cdot g) \cdot h = t \cdot (gh), \tag{15}$$

$$t \cdot e_{\phi(t)} = p \tag{16}$$

for all $t \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\phi(t)}$, where $e_{\phi(t)}$ is the neutral element of $\mathscr{G}_{\phi(t)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Principal bundle)

$$\begin{tikzcd} \mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ L' \arrow[r, "\Psi"] & L \end{tikzcd}$$

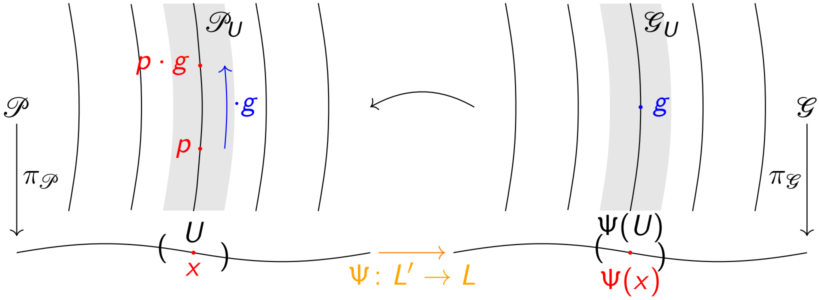
A surjective submersion $\pi_{\mathscr{P}} \colon \mathscr{P} \to L'$, with $\mathscr{G}$-action

$$\begin{array}{l} \cancel{\mathscr{P} \times G} \to \mathscr{P} \\ \mathscr{P} * \mathscr{G} \end{array}$$

simply transitive on $\pi_{\mathscr{P}}$-fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids. *Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge*, 2003

Connection on $\mathscr{P}$: Idea



But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

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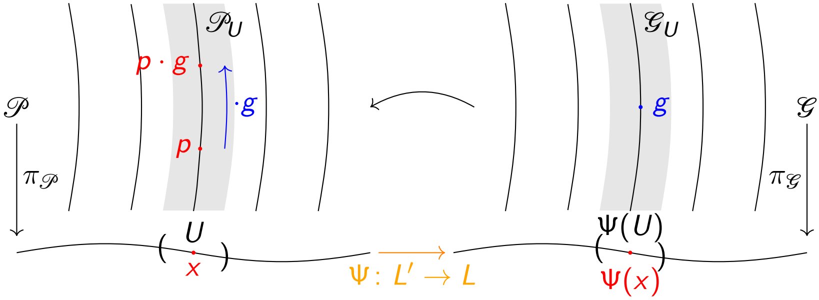

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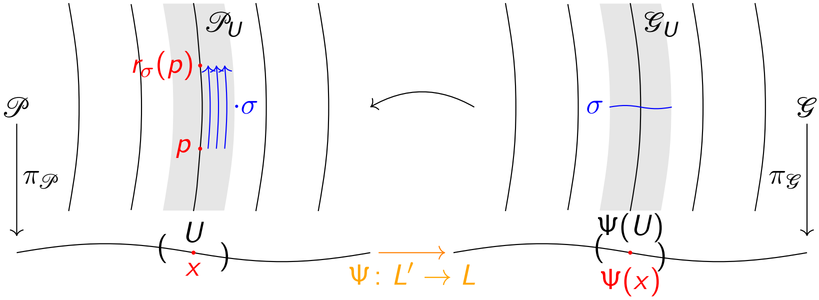
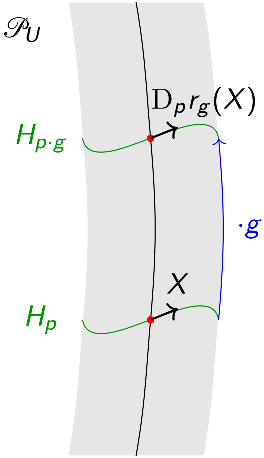


Use $\sigma \in \Gamma(\mathscr{G})$: $r_\sigma(p) := p \cdot \sigma_{\Psi(x)}$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



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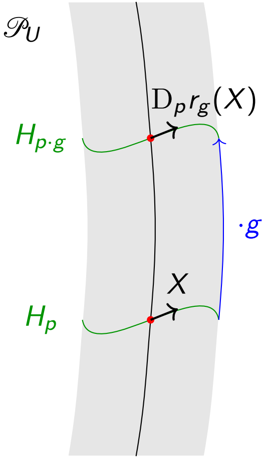

Remarks (Integrated case)

Parallel transport $\mathrm{PT}_\gamma^{\mathscr{P}}$ in $\mathscr{P}$:

$$\mathrm{PT}_\gamma^{\mathscr{P}}(p \cdot g) = \mathrm{PT}_\gamma^{\mathscr{P}}(p) \cdot g$$

where $\gamma : I \to L'$ is a base path

Connection on $\mathscr{P}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{P}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi \circ \gamma}(g).$$

Back to the roots

1. $\mathscr{G} \cong L \times G$
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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L'$ so that one has a commuting diagram

$$\begin{tikzcd}
\mathscr{T} \arrow[d, "\pi_{\mathscr{T}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\
L' \arrow[r, "\Psi"] & L
\end{tikzcd}$$

1. **Ehresmann connection:** $\mathscr{G}$ preserving $\pi_{\mathscr{T}}$ and
$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(t \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(t) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi\circ\gamma}(g)$$
2. **Yang-Mills connection:** Additionally
$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = t \cdot g_{\gamma_0}$$
for some $g_{\gamma_0} \in \mathscr{G}_{\phi(t)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
$$\mathrm{PT}^{\mathscr{G}}_{\gamma_0}(q) = g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

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Remarks

There is a simplicial differential δ on $\mathcal{G} \overset{\pi_{\mathcal{G}}}{\to} L$ with Lie algebra bundle $\mathcal{g}$

$$\delta : \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{g}) \to \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{g})$$

such that the definition of the multiplicative Yang-Mills connection is equivalent to the **compatibility conditions**

- Connection closed
- Curvature exact ([S.-R. F.])

For differential: Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Mathematische Zeitschrift*, 279(3):939–979, 2015.

Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection $\nabla^{\mathscr{G}}$ with

$$\nabla^{\mathscr{G}}\left([\mu,\nu]_{\mathscr{g}}\right) = \left[\nabla^{\mathscr{G}}\mu,\nu\right]_{\mathscr{g}} + \left[\mu,\nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}},$$

$$R_{\nabla^{\mathscr{G}}} = \mathrm{ad}\circ\zeta. \qquad \text{([S.-R. F.])}$$

Example

Given a short exact sequence of algebroids

$$\mathscr{g} \hookrightarrow E \twoheadrightarrow \mathrm{T}L$$

with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla^{\mathscr{G}}_X \nu = [\chi(X),\nu]_E,$$

$$\zeta(X,X') = [\chi(X),\chi(X')]_E - \chi([X,X']).$$

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For the first equation: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$X^{\uparrow} + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$\left[X^{\uparrow}, \overline{\nu}\right] = \overline{\nabla^{\mathscr{G}}_X \nu},$$
$$\left[X^{\uparrow}, X'^{\uparrow}\right] = [X, X']^{\uparrow} + \overline{\zeta(X, X')},$$

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. $G = \mathrm{Inn}(\tau_l)$
2. P a principal G-bundle, equipped with an ordinary connection

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

Lemma ([C. L.-G., S.-R. F.])

$\mathscr{P} := (P \times P)\big/ G$, the ***Atiyah groupoid****, is a principal $\mathscr{G}$-bundle*

$$\begin{tikzcd}
\mathscr{P} \arrow[d, "{(t,s)}"'] \arrow[dr, "s"] & \mathscr{G} \arrow[l, bend right] \arrow[d] \\
L \times L \arrow[r, "\mathrm{pr}_2"'] & L
\end{tikzcd}$$

where t and s are the target and source arrows, respectively. A connection on P induces an Ehresmann connection on $\mathscr{P}$.

Definition (Associated bundles, [C. L.-G., S.-R. F.])

$$\begin{tikzcd}
\mathscr{P} \arrow[d, "{(t,s)}"'] \arrow[dr, "s"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\
L' \arrow[r, "\mathrm{pr}_2"] & L & \mathscr{T} \arrow[l, "\pi_{\mathscr{T}}"]
\end{tikzcd}$$

Equivalence relation on $\mathscr{P} \,{}_s\!\times_{\pi_{\mathscr{T}}} \mathscr{T}$

$$(p, t) \sim \left(p \cdot g, g^{-1} \cdot t\right)$$

defines the **associated bundle** $\mathscr{P} \tilde{\times} \mathscr{T}$ over L'.

Theorem (Associated connection, [C. L.-G., S.-R. F.])

$$\mathrm{PT}_{\gamma}^{\mathscr{P}\tilde{\times}\mathscr{T}}[p,t] := \left[\mathrm{PT}_{\gamma}^{\mathscr{P}}(p), \mathrm{PT}_{\mathrm{pr}_2\circ\gamma}^{\mathscr{T}}(t)\right]$$

is a well-defined connection.

Remarks ([C. L.-G., S.-R. F.])

Associated connection independent of the choice of connection on P!

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Explicitly, one possible way:

Remarks

Corresponding to $\mathscr{P}$ there is an Atiyah sequence:

$$\mathsf{Ad}(P) \hookrightarrow \mathsf{At}(P) \twoheadrightarrow \mathrm{T}L$$

Via pullback to $\mathscr{T}$ we have a transitive algebroid over $\mathscr{T}$:

$$\pi_{\mathscr{T}}^{!}\mathsf{Ad}(P) \hookrightarrow \pi_{\mathscr{T}}^{!}\mathsf{At}(P) \twoheadrightarrow \mathrm{T}\mathscr{T}$$

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Observe

$$\pi_{\mathscr{T}}^{!}\mathsf{At}(P) \subset \mathrm{T}(\mathscr{P}\,{}_{s}\!\times_{\pi_{\mathscr{T}}}\mathscr{T})$$

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Ad(P) and At(P) the adjoint and Atiyah bundle of P, respectively:

$$\begin{tikzcd}
\Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi^!_{\mathscr{T}}\mathrm{Ad}(P)\right) \arrow[r, hook] \arrow[d] & \Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi^!_{\mathscr{T}}\mathrm{At}(P)\right) \arrow[r, two heads] \arrow[d] & \mathfrak{X}(L) \arrow[d, equal] \\
\tau \arrow[r, hook] & \mathscr{F}_{\text{projectable}} \arrow[r, two heads] & \mathfrak{X}(L)
\end{tikzcd}$$

Future Prospects

$$\begin{tikzcd} & \mathscr{F} \arrow[dl] \arrow[dr] & \\ \mathrm{At}(P) & & \text{Associated connection} \end{tikzcd}$$

Remarks (Why curved gauge theory?)

- Associated connection invariant under choice of $\mathscr{F}$-connection $\nabla^{\mathscr{F}}$
- Associated connection has the form

$$\nabla^{\mathscr{F}} + A\cdot$$

where A is the connection 1-form on $\mathscr{P}$

Thank you!