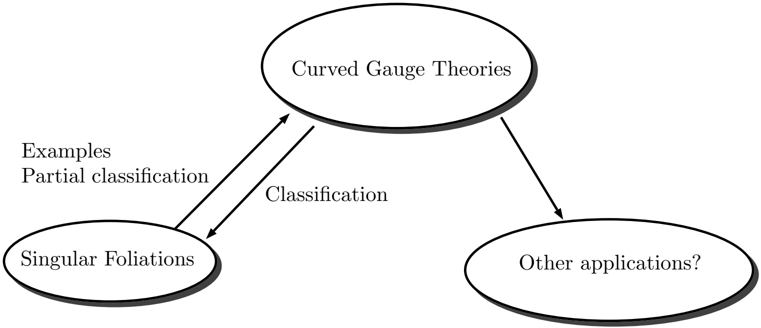
Classification of neighbourhoods of leaves of singular foliations

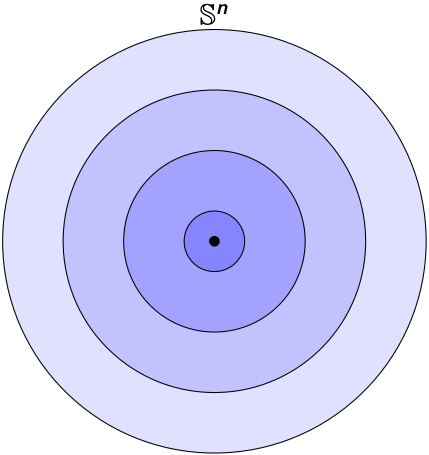
joint work with Camille Laurent-Gengoux (Université de Lorraine)

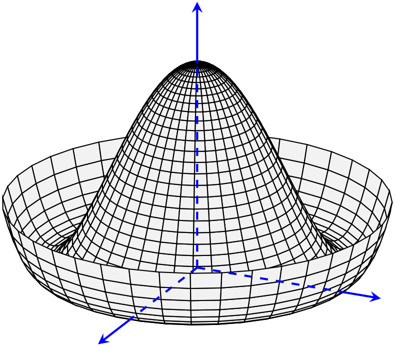
Simon-Raphael Fischer

國家理論科學研究中心
National Center for Theoretical Sciences (National Taiwan University)


Curved Gauge Theories
Examples
Partial classification
Classification
Singular Foliations
Other applications?

Singular Foliations





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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- it is **stable under** $C^\infty(M)$**-multiplication**, *i.e.* $fX \in \mathscr{F}$ *for all* $f \in C^\infty(M)$ *and* $X \in \mathscr{F}$,
- it is **locally finitely generated**, *i.e. around each* $p \in M$ *there is an open neighbourhood* U *and a finite family* $(X^i)_i^r$ $(X^i \in \mathscr{F})$ *such that for all* $X \in \mathscr{F}$ *there are* $f_i \in C^\infty(M)$ *satisfying on* U.

$$X = \sum_i f_i X^i.$$

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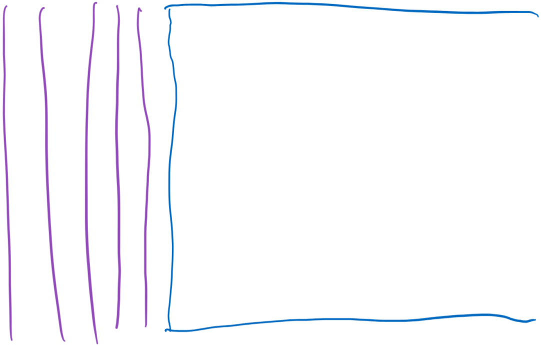
Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



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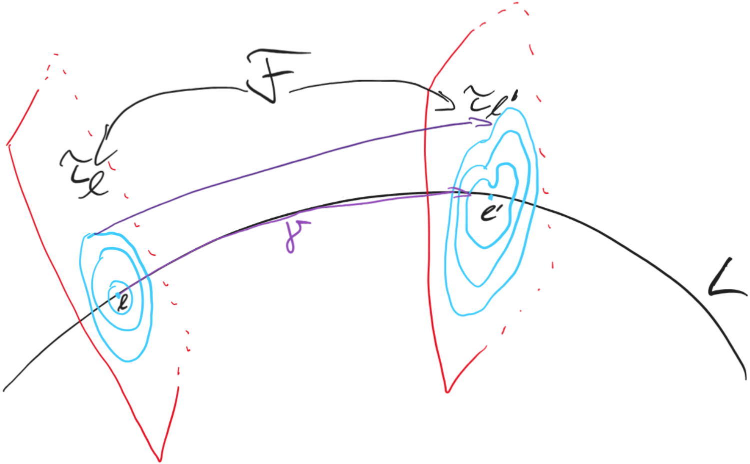
Idea: Relation to gauge theory



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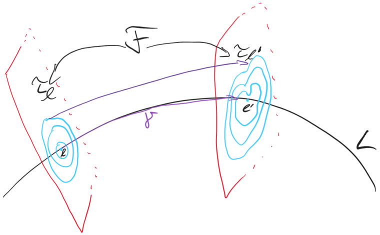

Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport PT_γ has values in $\mathrm{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l: PT_{γ_0} values in $\mathrm{Inner}(\tau_l)$.*

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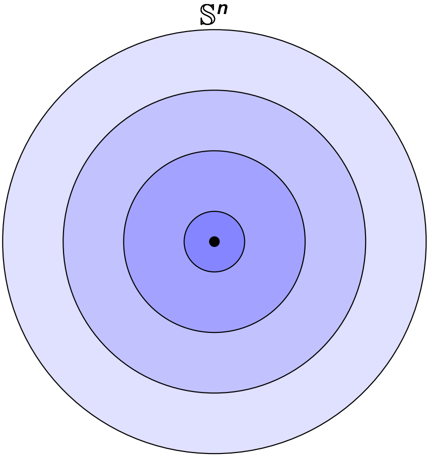
Example of a transverse foliation τ:



Remarks

- Inner(τ_I) maps each circle to itself
- Sym(τ_I) allows to exchange circles
- Both preserve τ_I and fix the origin

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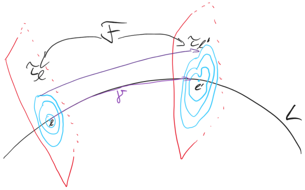


Remarks ($\mathscr{F}$-connection)

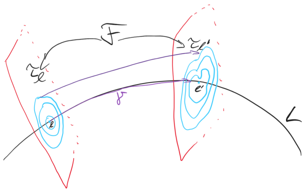
For $\phi \in \mathrm{Sym}(\tau_l)$ we have an induced parallel transport

$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) := \mathrm{PT}_{\gamma} \circ \phi \circ \mathrm{PT}_{\gamma}^{-1}.$$

Then, on the normal bundle $\pi \colon \mathscr{T} \to L$,

$$\mathrm{PT}_{\gamma}(\phi \cdot p) = \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) \cdot \mathrm{PT}_{\gamma}(p)$$
$$\mathrm{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all $p \in \mathscr{T}_l$, $\phi \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.



Remarks (Sym-connection)

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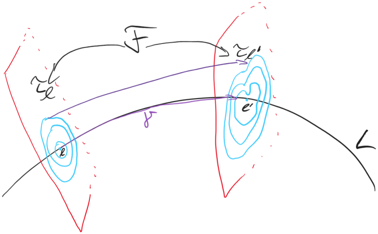
Then

$$\mathrm{PT}_{\gamma}(\phi \circ \phi') = \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) \circ \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi')$$
$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma_0}(\phi) = \varphi \circ \phi \circ \varphi^{-1}$$

for all $\phi, \phi' \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.

Idea

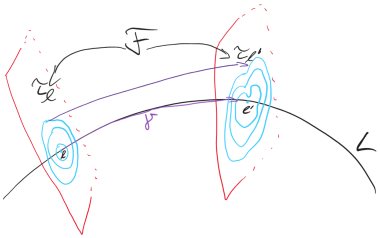


Idea

Generators of $\mathscr{F}$ given by:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.



Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$\begin{aligned}
\left[\mathbb{H}(X)+\overline{\nu},\mathbb{H}(X')+\overline{\mu}\right] &= \mathbb{H}([X,X'])+\dots \\
&= \underbrace{\left[\mathbb{H}(X),\mathbb{H}(X')\right]}_{\rightsquigarrow \text{ curvature}} \\
&\qquad + \underbrace{\left[\mathbb{H}(X),\overline{\mu}\right]-\left[\mathbb{H}(X'),\overline{\nu}\right]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu,\mu]}
\end{aligned}$$

Summary

$$
\begin{tikzcd}
& \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
& \{\text{Connection on Sym}(\tau)\} &
\end{tikzcd}
$$

Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given $\mathscr{F}$!

Multiplicative Yang-Mills connections

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0}A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

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Definition (LGB actions)

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi"] & L \end{tikzcd}$$

A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\pi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

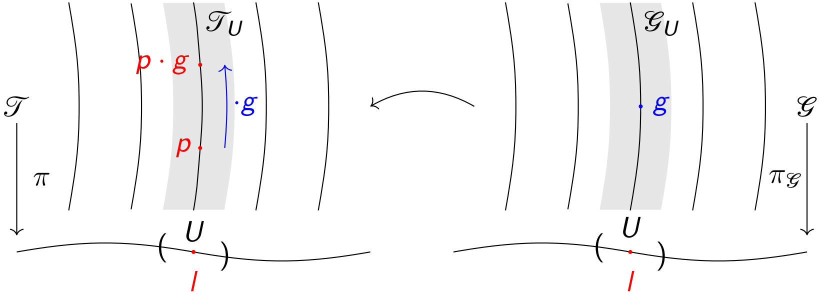
$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Connection on $\mathscr{T}$: Idea



But:

$$r_g : \mathscr{T}_x \to \mathscr{T}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

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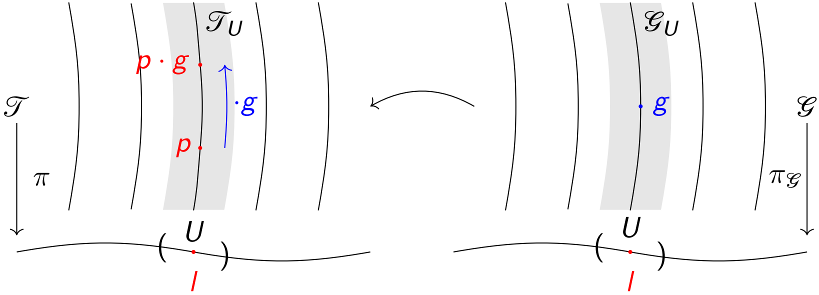

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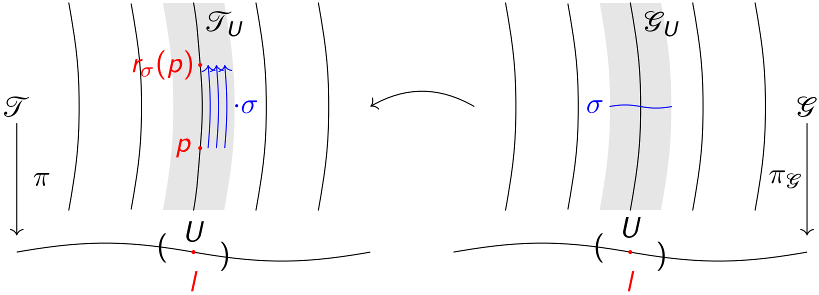
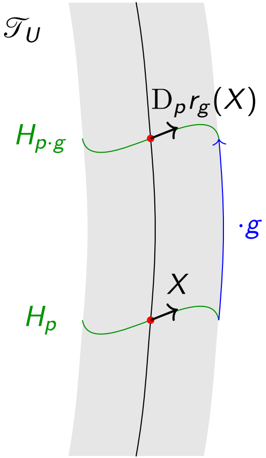


Use $\sigma \in \Gamma(\mathscr{G})$: $r_\sigma(p) := p \cdot \sigma_x$

Connection on $\mathcal{T}$: Revisiting the classical setup

If $\mathcal{G}$ is trivial, $\sigma \equiv g$ constant,
and H a connection:



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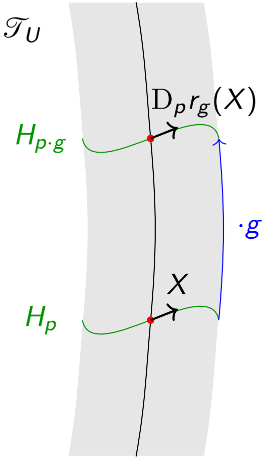

Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{T}}_{\gamma}$ in $\mathscr{T}$:

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(p) \cdot g$$

where $\gamma : I \to L$ is a base path

Connection on $\mathscr{T}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g).$$

Back to the roots

1. $\mathscr{G} \cong L \times G$
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1. $\mathscr{G} \cong L \times G$
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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi\colon \mathscr{T} \to L$ so that one has a commuting diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi"] & L \end{tikzcd}$$

1. **Ehresmann connection:**

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(p) = p \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}^0_{\pi(p)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
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Remarks

Compare this with the Maurer-Cartan form and its curvature equation!

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Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection ∇ with

$$\nabla\left([\mu,\nu]_{\mathscr{g}}\right) = [\nabla\mu,\nu]_{\mathscr{g}} + [\mu,\nabla\nu]_{\mathscr{g}},$$
$$R_\nabla = \mathrm{ad}\circ\zeta.$$

Example

Given a short exact sequence of algebroids

$$\begin{tikzcd} \mathscr{g} \arrow[r, hook] & E \arrow[r, two heads] & \mathrm{T}L \arrow[l, bend right, "\chi"'] \end{tikzcd}$$

with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla_X\nu = [\chi(X),\nu]_E,$$
$$\zeta(X,X') = [\chi(X),\chi(X')]_E - \chi([X,X']).$$

Sources: For the first equation: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

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Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla_X \nu},$$
$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathscr{g})$.

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

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- $\mathscr{G}$ acts on $\mathscr{T}$ (canonically from the left).

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

Proof.

- The adjoint bundle of P, $\mathrm{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of $\mathscr{G}$
- $\tau = \overline{\mathrm{Ad}(P)}$
- Difference of two connections on P has values in $\mathrm{Ad}(P)$

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Summary

Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf L and transverse model $\left(\mathbb{R}^d, \tau_l\right)$
- Principal $\mathrm{Inner}(\tau_l)$-bundles P over L

$$
\begin{tikzcd}
 & \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & P \arrow[dl, leftrightarrow] \\
 & \{\text{Connection on } \mathrm{Sym}(\tau)\} &
\end{tikzcd}
$$

Thank you!