Curved Yang-Mills gauge theories and their recent applications

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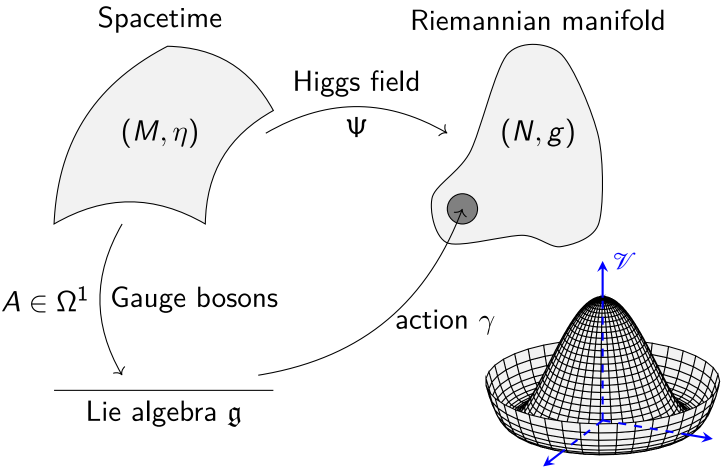

Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Consider a semisimple Lie group G and a principal G-bundle $P \to M$:

$$\mathrm{Ad}(P) \longhookrightarrow \mathrm{At}(P) \longrightarrow\!\!\!\!\rightarrow \mathrm{T}M$$

where $\mathrm{Ad}(P)$ and $\mathrm{At}(P)$ the adjoint and Atiyah bundle of a principal G-bundle P, respectively.

Gedankenexperiment

1. Adjoint connection $\leftrightarrow$ Ehresmann connection on P.
2. As parallel transport along a curve γ:

$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^P(p), v\right]$$

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Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection

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1. Adjoint connection $\leftrightarrow$ Ehresmann connection on P.
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$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^P(p) \cdot \kappa_\gamma, \kappa_\gamma^{-1} \cdot \mathrm{PT}_\gamma^0(v)\right],$$

Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection, κ_γ values in G & "suitable"

Theorem (Field Redefinitions S.-R. F.)

This leads to an equivalence relation of gauge theories, preserving dynamics and kinematics.
But: In the curved sense! Curvature terms appear.

Motivation (S.-R. F.)

1. How to formulate gauge theory such that it is invariant under field redefinitions?
2. Are there curved theories which are **not** equivalent to classical ones?

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Definition (LGB actions)

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\phi"] & L \end{tikzcd}$$

A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\phi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(t, g) \mapsto t \cdot g$, satisfying the following properties:

$$\phi(t \cdot g) = \phi(t), \tag{1}$$

$$(t \cdot g) \cdot h = t \cdot (gh), \tag{2}$$

$$t \cdot e_{\phi(t)} = p \tag{3}$$

for all $t \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\phi(t)}$, where $e_{\phi(t)}$ is the neutral element of $\mathscr{G}_{\phi(t)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Principal bundle)

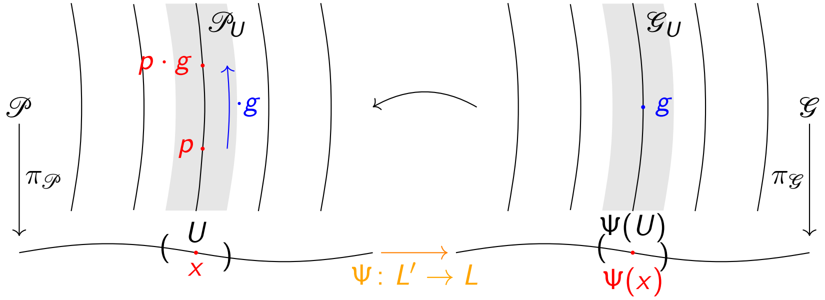
$$\begin{tikzcd}
\mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\
L' \arrow[r, "\Psi"] & L
\end{tikzcd}$$

A surjective submersion $\pi_{\mathscr{P}} \colon \mathscr{P} \to L'$, with $\mathscr{G}$-action

$$\cancel{\mathscr{P} \times G} \to \mathscr{P}$$
$$\mathscr{P} * \mathscr{G}$$

simply transitive on $\pi_{\mathscr{P}}$-fibres of $\mathscr{P}$, and "suitable" atlas.

Connection on $\mathscr{P}$: Idea



But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

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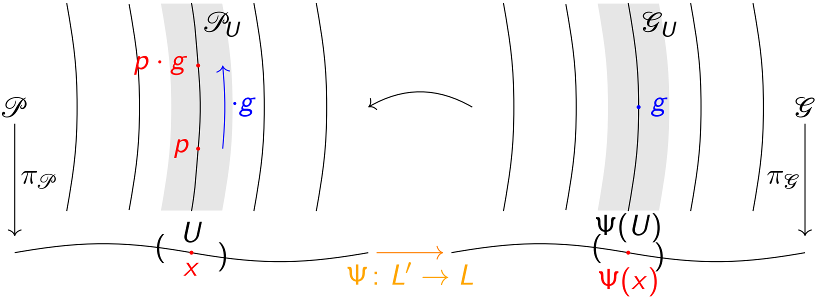

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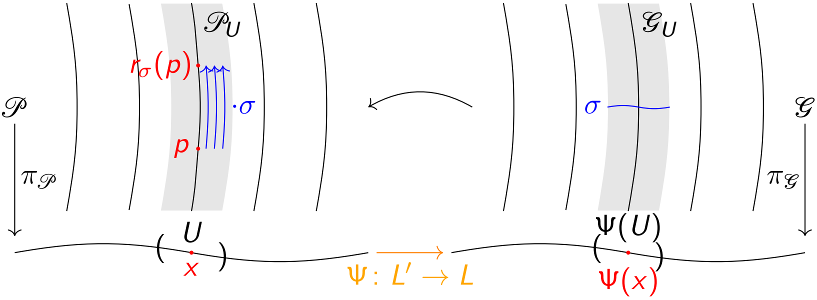
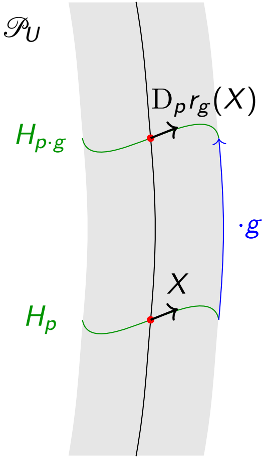


Use $\sigma \in \Gamma(\mathscr{G})$: $r_\sigma(p) := p \cdot \sigma_{\Psi(x)}$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



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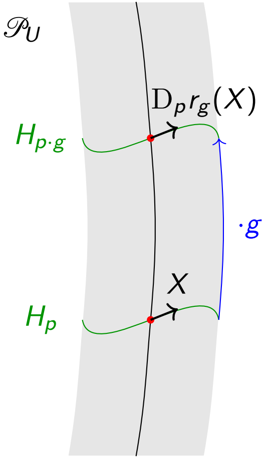

Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{P}}_{\gamma}$ in $\mathscr{P}$:

$$\mathrm{PT}^{\mathscr{P}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\gamma}(p) \cdot g$$

where $\gamma : I \to L'$ is a base path

Connection on $\mathscr{P}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{P}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi \circ \gamma}(g).$$

Back to the roots

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L'$ so that one has a commuting diagram

$$\begin{tikzcd} \mathscr{T} \arrow[d, "\pi_{\mathscr{T}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ L' \arrow[r, "\Psi"] & L \end{tikzcd}$$

1. **Ehresmann connection:** $\mathscr{G}$ preserving $\pi_{\mathscr{T}}$ and

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(t \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(t) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi \circ \gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = t \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}_{\phi(t)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
$$\mathrm{PT}^{\mathscr{G}}_{\gamma_0}(q) = g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

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Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

Remarks

There is a simplicial differential δ on $\mathcal{G} \overset{\pi_{\mathcal{G}}}{\rightarrow} L$ with Lie algebra bundle $\mathscr{g}$

$$\delta : \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathscr{g}) \to \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathscr{g})$$

such that the definition of the multiplicative Yang-Mills connection is equivalent to the **compatibility conditions**

- Connection closed
- Curvature exact ([S.-R. F.])

 For differential: Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Mathematische Zeitschrift*, 279(3):939–979, 2015.

Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection $\nabla^{\mathscr{G}}$ with

$$\nabla^{\mathscr{G}}\left([\mu,\nu]_{\mathscr{g}}\right) = \left[\nabla^{\mathscr{G}}\mu,\nu\right]_{\mathscr{g}} + \left[\mu,\nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}},$$

$$R_{\nabla^{\mathscr{G}}} = \mathrm{ad}\circ\zeta. \qquad \text{([S.-R. F.])}$$

Example

Given a short exact sequence of algebroids

$$\mathscr{g} \hookrightarrow E \twoheadrightarrow \mathrm{T}L$$

with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla^{\mathscr{G}}_X\nu = [\chi(X),\nu]_E,$$
$$\zeta(X,X') = [\chi(X),\chi(X')]_E - \chi([X,X']).$$

Sources: For the first equation: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

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Given a multiplicative Yang-Mills connection on $\mathscr{G}$:

Definition (Generalized curvature/field strength F of A, [S.-R. F.])

We define

$$F := \mathrm{d}^{\pi_{\mathscr{P}}^{*}\nabla} A + \frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi_{\mathscr{P}}^{*}\mathscr{g}} + \pi_{\mathscr{P}}^{!}\zeta.$$

Theorem (Lagrangian, [S.-R. F.])

- κ *be an* Ad*-invariant fibre metric on* $\mathscr{g}$,
- M *a spacetime, and* $*$ *its Hodge star operator,*
- $(U_i)_i$ *open covering of* M *with subordinate gauges* $s_i \in \Gamma(\mathscr{P}|_{U_i})$.

Then the Lagrangian $\mathfrak{L}_{\mathrm{CYM}}[A]$, *defined locally by*

$$\left(\mathfrak{L}_{\mathrm{CYM}}[A]\right)\big|_{U_i} := -\frac{1}{2}\kappa(F_{s_i} \overset{\wedge}{,} *F_{s_i}),$$

is well-defined, and

$$\mathfrak{L}_{\mathrm{CYM}}\left[K^! A\right] = \mathfrak{L}_{\mathrm{CYM}}[A]$$

for all principal bundle automorphisms K.

Back to the roots

1. $\mathscr{G} \cong M \times G$
2. Equip $\mathscr{G}$ with canonical flat connection
3. $\zeta \equiv 0$

Example (Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^4$, [S.-R. F.])

Let P be the Hopf bundle

$$\begin{array}{ccc} \mathrm{SU}(2) \cong \mathbb{S}^3 & \longrightarrow & \mathbb{S}^7 \\ & & \Big\downarrow \\ & & \mathbb{S}^4 \end{array}$$

Define $\mathscr{P} := \mathscr{G}$ as the inner group bundle of P,

$$\mathscr{G} := c_{\mathrm{SU}(2)}(P) := \left(P \times \mathrm{SU}(2)\right) \Big/ \mathrm{SU}(2).$$

This principal $c_{\mathrm{SU}(2)}(P)$-bundle admits the structure as curved Yang-Mills gauge theory; there is no description as classical gauge theory.

An attempt of classification

Definition (Field redefinition, [S.-R. F.])

Let $\lambda \in \Omega^1(L; \mathscr{g})$, then we define the **field redefinitions** by

$$\begin{aligned}
\widetilde{A}^\lambda &:= A - \pi^!_{\mathscr{P}} \lambda, \\
\widetilde{\nabla}^\lambda &:= \nabla + \mathrm{ad} \circ \lambda, \\
\widetilde{\zeta}^\lambda &:= \zeta + \mathrm{d}^{\widetilde{\nabla}^\lambda} \lambda + \frac{1}{2} [\lambda \stackrel{\wedge}{,} \lambda]_{\mathscr{g}},
\end{aligned}$$

where $A \in \Omega^1(\mathscr{P}; \pi^*_{\mathscr{P}} \mathscr{g})$ is the connection 1-form on $\mathscr{P}$.

Proposition ([S.-R. F.])

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\widetilde{\mathfrak{L}}^{\lambda}_{\mathrm{CYMH}} = \mathfrak{L}_{\mathrm{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

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Theorem (Invariant for LABs, [S.-R. F.])

We have

$$\mathrm{d}^{\widetilde{\nabla}^\lambda}\widetilde{\zeta}^\lambda = \mathrm{d}^\nabla \zeta,$$

and $\mathrm{d}^\nabla\zeta$ *has values in the centre of* $\mathcal{g}$.

Proof.

Bianchi identity given by $\mathrm{d}^\nabla\zeta$.
Compatibilities:

- κ needs to be ad-invariant
- We need

$$\nabla_Y\left([\mu,\nu]_{\mathcal{g}}\right) = [\nabla_Y\mu,\nu]_{\mathcal{g}} + [\mu,\nabla_Y\nu]_{\mathcal{g}}, \tag{4}$$

$$R_\nabla(Y,Z)\mu = [\zeta(Y,Z),\mu]_{\mathcal{g}} \tag{5}$$

for all $Y,Z \in \mathfrak{X}(N)$ and $\mu,\nu \in \Gamma(\mathcal{g})$.

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Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^\nabla \zeta \neq 0$*, then there is no field redefinition such that* $\widetilde{\zeta}^\lambda = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^\nabla \zeta \neq 0$.

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However, by $R_\nabla = \mathrm{ad}_g \circ \zeta$ it may still be that ∇ becomes flat.

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However, by $R_\nabla = \mathrm{ad}_{\mathscr{g}} \circ \zeta$ it may still be that ∇ becomes flat.

Turning to the field redefinition of ∇:

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ restricts to the centre of $\mathscr{g}$ and induces a differential d^{Ξ} on centre-valued forms. Moreover, d^{Ξ} is independent of the field redefinitions.

Sketch of proof.

Recall

$$\nabla_Y\Big([\mu,\nu]_{\mathscr{g}}\Big) = [\nabla_Y\mu,\nu]_{\mathscr{g}} + [\mu,\nabla_Y\nu]_{\mathscr{g}},$$
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Theorem (Closedness of $\mathrm{d}^{\nabla}\zeta$, [S.-R. F.])

We have

$$\mathrm{d}^{\Xi}\mathrm{d}^{\nabla}\zeta = 0.$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^{\nabla}\zeta\right]_{\mathrm{d}^{\Xi}}.$$

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Theorem (Obstruction for non-pre-classical theories, [S.-R. F.])

If $\mathrm{Obs}(\Xi) \neq 0$, then there is no field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

Theorem (Locally always pre-classical)

If L is contractible, then there is a field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

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Example (Zero obstruction class not necessarily pre-classical)

Let P be the Hopf fibration

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathbb{S}^7 \\ & & \big\downarrow \\ & & \mathbb{S}^4 \end{array}$$

Then for the adjoint bundle

$$\mathcal{g} := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left(\mathbb{S}^7 \times \mathfrak{su}(2)\right) \Big/ \mathrm{SU}(2)$$

we have a non-flat ∇ satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but $\mathrm{Obs}(\Xi) = 0$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical.
In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $\mathbb{S}^7 \to \mathbb{S}^4$?
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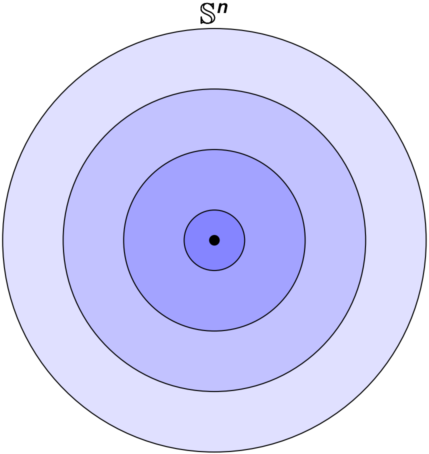
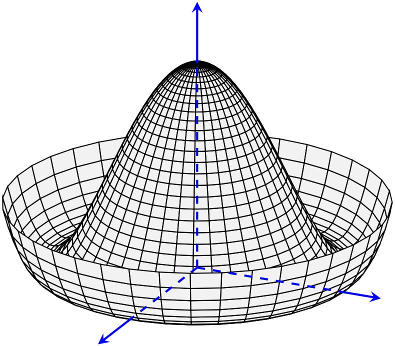
$\rightsquigarrow$ Integration

Applications: Singular foliations (joint work w/ Camille Laurent-Gengoux)





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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- it is **locally finitely generated**, *i.e. around each* $p \in M$ *there is an open neighbourhood* U *and a finite family* $(X^i)_i^r$ $(X^i \in \mathscr{F})$ *such that for all* $X \in \mathscr{F}$ *there are* $f_i \in C^\infty(M)$ *satisfying on* U.

$$X = \sum_i f_i X^i.$$

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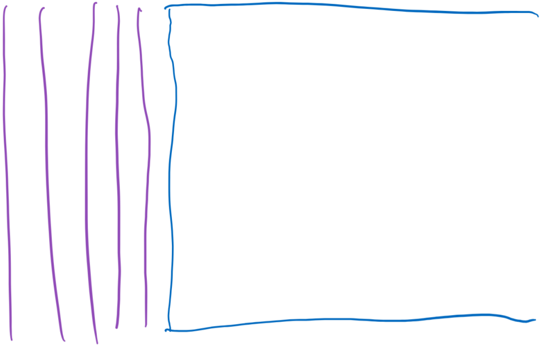
Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



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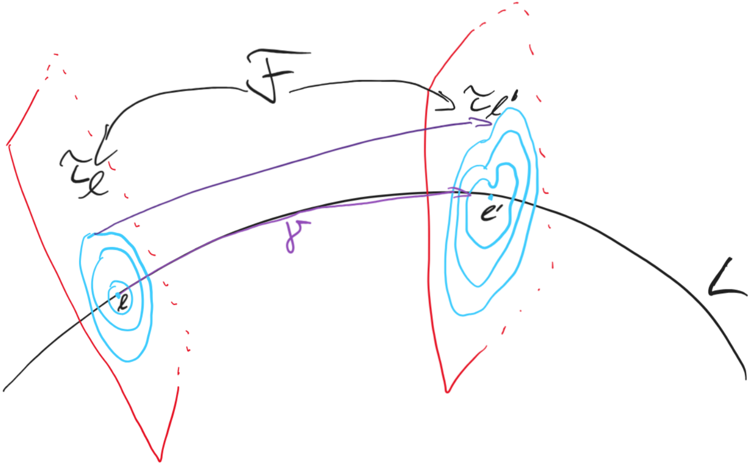
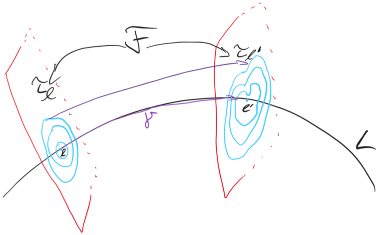
Idea: Relation to gauge theory



Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.



Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport PT_γ has values in $\mathrm{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l: PT_{γ_0} values in $\mathrm{Inner}(\tau_l)$.*

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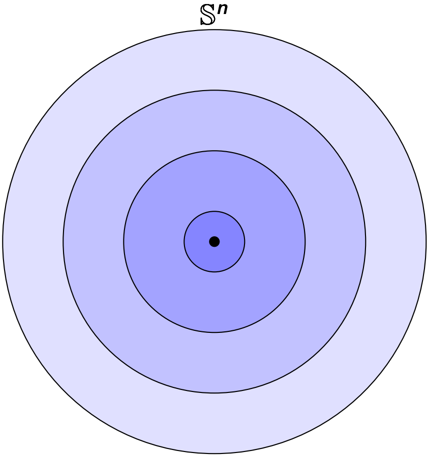
Example of a transverse foliation τ:



Remarks

- Inner(τ_I) maps each circle to itself
- Sym(τ_I) allows to exchange circles
- Both preserve τ_I and fix the origin

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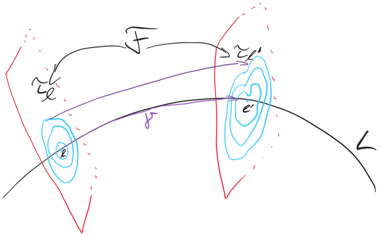
Idea

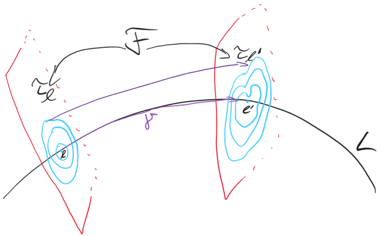


Idea

Generators of $\mathscr{F}$ given by $\mathscr{F}_{\text{projectable}}$:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.



Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$
\begin{aligned}
[\mathbb{H}(X)+\overline{\nu},\mathbb{H}(X')+\overline{\mu}] &= \mathbb{H}([X,X'])+\dots \\
&= \underbrace{[\mathbb{H}(X),\mathbb{H}(X')]}_{\rightsquigarrow\ \text{curvature}} \\
&\qquad + \underbrace{[\mathbb{H}(X),\overline{\mu}]-[\mathbb{H}(X'),\overline{\nu}]}_{\rightsquigarrow\ \text{connection}} + \overline{[\nu,\mu]}
\end{aligned}
$$

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla^{\mathscr{G}}_X \nu},$$
$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathscr{g})$.

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. $G = \mathrm{Inn}(\tau_l)$
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Remarks

- Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.
- $\mathscr{G}$ acts on $\mathscr{T}$ (canonically from the left).

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

$$\begin{tikzcd}
\Gamma(\mathrm{Ad}(P)) \arrow[rrrr, "\nu\mapsto\overline{\nu}"] \arrow[dr, hook] & & & & \tau \arrow[dl, hook'] \\
& \Gamma(\mathrm{At}(P)) \arrow[rr, "\xi\mapsto\overline{\xi}"] \arrow[dr, two heads] & & \mathscr{F}_\pi \arrow[dl, two heads] & \\
& & \mathfrak{X}(L) \arrow[ul, bend left, "A"] \arrow[ur, bend right, "\mathbb{H}"'] & &
\end{tikzcd}$$

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Summary

Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf L and transverse model τ_l
- Principal $\mathrm{Inner}(\tau_l)$-bundles over L

Remarks (Classification of curved Yang-Mills gauge theories)

If $\mathscr{G}$ acts faithfully on $\mathscr{T}$, then a curved Yang-Mills gauge theory can be flattened if and only if P is flat.

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Future Prospects

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- Total space and Atiyah bundles of groupoid-based principal bundles
- Further applications of curved gauge theories (existence of Cartan connections,...)
- Field redefinitions as natural symmetry in associated applications

Thank you!