# Curved gauge theories and their applications

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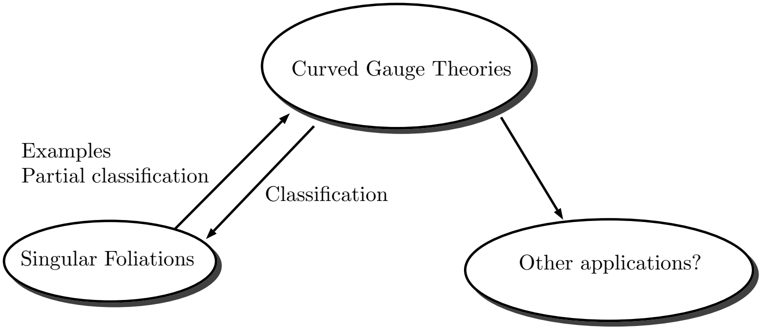
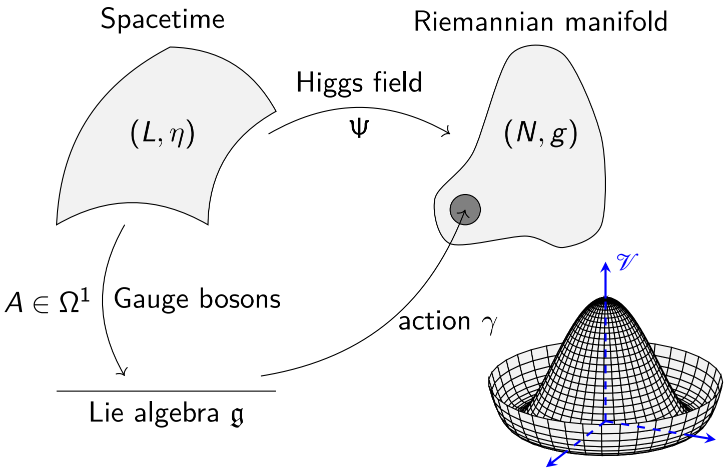

Curved Gauge Theories
Examples
Partial classification
Classification
Singular Foliations
Other applications?

# Curved gauge theory

# **Infinitesimal** curved Yang-Mills-Higgs gauge theory





Sombrero hat potential made by Mark J.D. Hamilton.

# Motivation 1 by Thomas Strobl and Alexei Kotov

| Classical formalism | CYMH GT |
|---|---|
| Lie algebra $\mathfrak{g}$ as $L \times \mathfrak{g}$ | Lie algebroid $E \to N$ |
| $\mathfrak{g}$-action $\gamma$ | Anchor $\rho$ of $E$ & $E$-connections |
| Canonical flat connection $\nabla^0$ on $L \times \mathfrak{g}$ | General connection $\nabla$ on $E$ |

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**Remarks (Why a "curved theory"?)**

Usually, the field strength $F$ is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection $\nabla$ instead of $\nabla^0$, and $\nabla$ may not be flat.

# Motivation 2 by S.-R. F.

Consider a semisimple Lie group $G$ and a principal $G$-bundle $P \to L$:

$$\begin{tikzcd} (P \times \mathfrak{g})/G \arrow[r, hook] & \mathrm{T}P/G \arrow[r, two heads] & \mathrm{T}L \arrow[l, bend right, "A"'] \end{tikzcd}$$

## Gedankenexperiment

1. Adjoint connection $\leftrightarrow$ Ehresmann connection on $P$.
2. Adjoint connection:

$$\nabla_X \nu := [A(X), \nu]_{\mathrm{T}P/G}$$

for all $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma((P \times \mathfrak{g})/G)$.

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2. As parallel transport:

$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^P(p), v\right]$$

for all $[p, v] \in (P \times \mathfrak{g})/G$.

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Consider a semisimple Lie group $G$ and a principal $G$-bundle $P \to L$:

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Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection

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Consider a semisimple Lie group $G$ and a principal $G$-bundle $P \to L$:

$$\big(P \times_L \underline{\mathfrak{g}}\big)/G \hookrightarrow \mathrm{T}P/G \twoheadrightarrow \mathrm{T}L \quad (A: \mathrm{T}L \to \mathrm{T}P/G)$$

## Gedankenexperiment

1. Adjoint connection $\leftrightarrow$ Ehresmann connection on $P$.
2. As parallel transport:

$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \Big[\mathrm{PT}_\gamma^P(p)\cdot\kappa_\gamma, \kappa_\gamma^{-1}\cdot\mathrm{PT}_\gamma^0(v)\Big],$$

Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection, $\kappa_\gamma$ values in $G$ & "suitable"

**Theorem (Field Redefinitions S.-R. F.)**

*This leads to an equivalence relation of gauge theories, preserving dynamics and kinematics.*
*But: In the curved sense! Curvature terms appear.*

**Motivation (S.-R. F.)**

1. How to formulate gauge theory such that it is invariant under field redefinitions?
2. Are there curved theories which are **not** equivalent to classical ones?

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2. Are there curved theories which are **not** equivalent to classical ones?

**We will only focus on Yang-Mills theories:**

| | Classical | Curved |
|---|---|---|
| Infinitesimal | Lie algebra $\mathfrak{g}$ | LAB[1] $\mathscr{g}$ |
| Integrated | Lie group $G$ | LGB[2] $\mathscr{G}$ |

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{G} \\ & & \downarrow \\ & & M \end{array}$$

[1] LAB = Lie algebra bundle
[2] LGB = Lie group bundle

## Definition (LGB actions, simplified)

$$\begin{tikzcd} & \mathscr{G} \arrow[ld, bend right] \arrow[d] \\ \mathscr{P} \arrow[r, "\pi"] & L \end{tikzcd}$$

$\mathscr{P} \xrightarrow{\pi} L$ a fibre bundle. A **right-action of $\mathscr{G}$ on $\mathscr{P}$** is a smooth map $\mathscr{P} * \mathscr{G} := \pi^*\mathscr{G} = \mathscr{P} \times_L \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$

$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.


K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

## Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi"] \\ & L \end{tikzcd}$$

but with $\mathscr{G}$-action

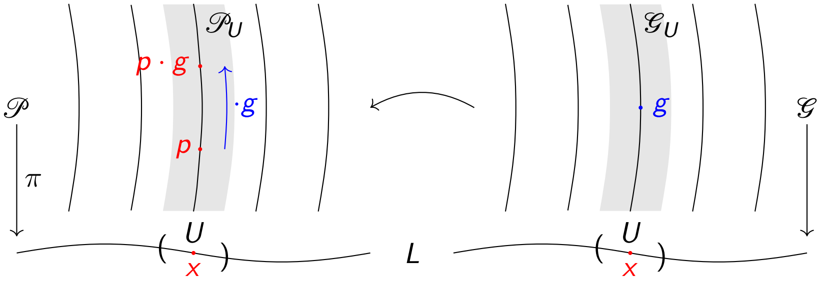
$$\cancel{\mathscr{P} \times G} \to \mathscr{P}$$

$$\mathscr{P} * \mathscr{G}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.


Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids. *Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge*, 2003

# Connection on $\mathscr{P}$: Idea




But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$$\Rightarrow \qquad \mathrm{D}_p r_g \text{ only defined on vertical structure}$$

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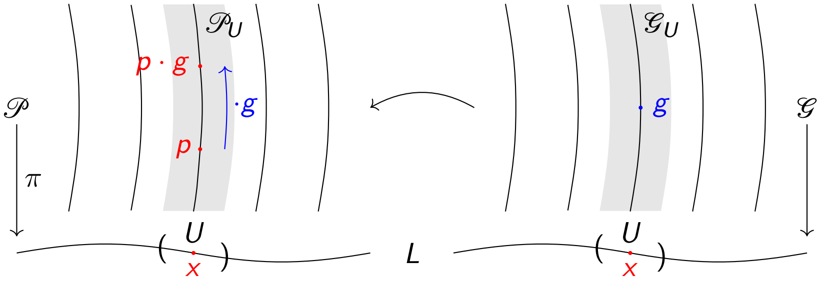


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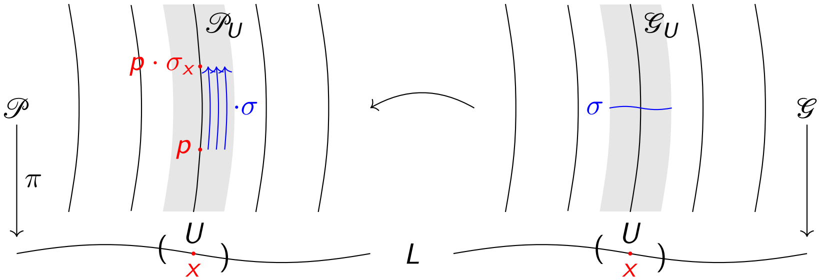
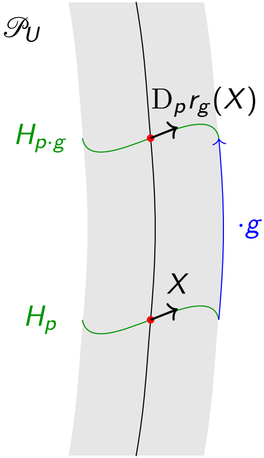



Use $\sigma \in \Gamma(\mathscr{G}) : r_\sigma(p) := p \cdot \sigma_x$

# Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and $H$ a connection:



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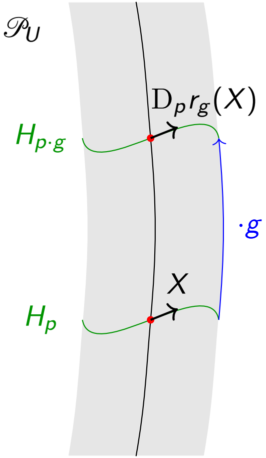


**Remarks (Integrated case)**

Parallel transport $\mathrm{PT}^{\mathscr{P}}_\gamma$ in $\mathscr{P}$:

$$\mathrm{PT}^{\mathscr{P}}_\gamma(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_\gamma(p) \cdot g$$

where $\gamma : I \to L$ is a base path

# Connection on $\mathscr{P}$: General case

## Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{P}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g).$$

## Back to the roots

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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## Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L$ so that one has a commuting diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi_{\mathscr{T}}"] & L \end{tikzcd}$$

1. **Ehresmann connection:**

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(t \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(t) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = t \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}_{\pi_{\mathscr{T}}(t)}$, where $\gamma_0$ is a contractible loop.

## Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
$$\mathrm{PT}^{\mathscr{G}}_{\gamma_0}(q) = g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1}$$

## Remarks

There is a simplicial differential $\delta$ on $\mathcal{G} \overset{\pi_{\mathcal{G}}}{\to} L$ with Lie algebra bundle $\mathscr{g}$

$$\delta : \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathscr{g}) \to \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathscr{g})$$

such that the definition of the multiplicative Yang-Mills connection is equivalent to the **compatibility conditions**

- Connection closed
- Curvature exact ([S.-R. F.])

Sources For differential: Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Mathematische Zeitschrift*, 279(3):939–979, 2015.

## Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection $\nabla$ with

$$\nabla\Big([\mu,\nu]_{\mathscr{g}}\Big) = [\nabla\mu,\nu]_{\mathscr{g}} + [\mu,\nabla\nu]_{\mathscr{g}},$$
$$R_\nabla = \mathrm{ad}\circ\zeta.$$

## Example

Given a short exact sequence of algebroids

$$\mathscr{g} \hookrightarrow E \twoheadrightarrow \mathrm{T}L$$

with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla_X\nu = [\chi(X),\nu]_E,$$
$$\zeta(X,X') = [\chi(X),\chi(X')]_E - \chi([X,X']).$$

Sources

For the first equation: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

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# Integrating Alexei Kotov's and Thomas Strobl's idea

**Definition (Principal bundle connection, [S.-R. F.])**

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

**Definition (Generalized curvature/field strength $F$ of $A$, [S.-R. F.])**

We define

$$F := \mathrm{d}^{\pi^*\nabla} A + \frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}} + \pi^! \zeta.$$

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## Theorem (Lagrangian, [S.-R. F.])

- *$\kappa$ be an $\mathrm{Ad}$-invariant fibre metric on $\mathscr{g}$,*
- *$L$ a spacetime, and $*$ its Hodge star operator,*
- *$(U_i)_i$ open covering of $L$ with subordinate gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$.*

*Then the Lagrangian $\mathfrak{L}_{\mathrm{CYM}}[A]$, defined locally by*

$$\left(\mathfrak{L}_{\mathrm{CYM}}[A]\right)\big|_{U_i} := -\frac{1}{2}\kappa\left(F_{s_i} \stackrel{\wedge}{,} *F_{s_i}\right),$$

*is well-defined, and*

$$\mathfrak{L}_{\mathrm{CYM}}\left[K^! A\right] = \mathfrak{L}_{\mathrm{CYM}}[A]$$

*for all principal bundle automorphisms $K$.*

## Back to the roots

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection
3. $\zeta \equiv 0$

## Example (Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^4$, [S.-R. F.])

Let $P$ be the Hopf bundle

$$\begin{array}{ccc} \mathrm{SU}(2) \cong \mathbb{S}^3 & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{S}^4 \end{array}$$

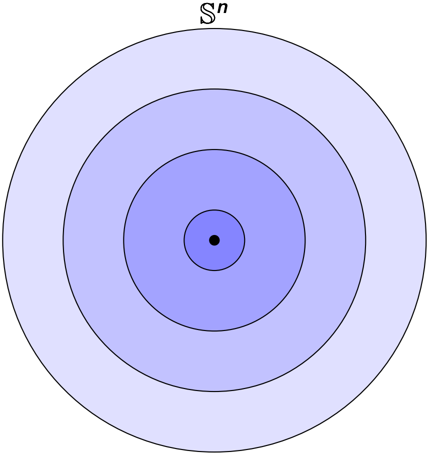
Define $\mathscr{P} := \mathscr{G}$ as the inner group bundle of $P$,

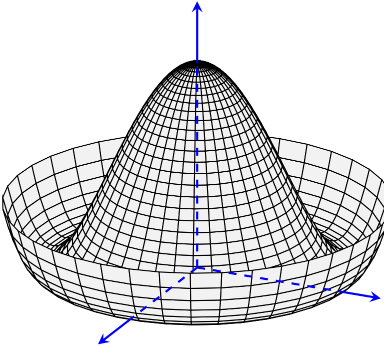
$$\mathscr{G} := c_{\mathrm{SU}(2)}(P) := \left(P \times \mathrm{SU}(2)\right) \Big/ \mathrm{SU}(2).$$

This principal $c_{\mathrm{SU}(2)}(P)$-bundle admits the structure as curved Yang-Mills gauge theory; there is no description as classical gauge theory.

# Applications: Classifying singular foliations (joint work w/ Camille Laurent-Gengoux)

## Why foliations?




Right diagram made by Mark J.D. Hamilton.

**Singular Foliations:**

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

## Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
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- it is **stable under** $C^\infty(M)$**-multiplication**, *i.e.* $fX \in \mathscr{F}$ for all $f \in C^\infty(M)$ and $X \in \mathscr{F}$,
- it is **locally finitely generated**, *i.e. around each $p \in M$ there is an open neighbourhood $U$ and a finite family $(X^i)_i^r$ ($X^i \in \mathscr{F}$) such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on $U$*.

$$X = \sum_i f_i X^i.$$

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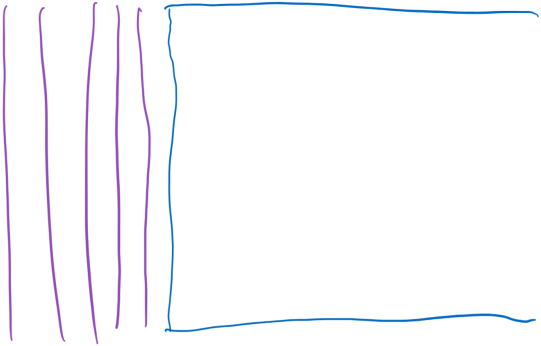
## Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in $M$.



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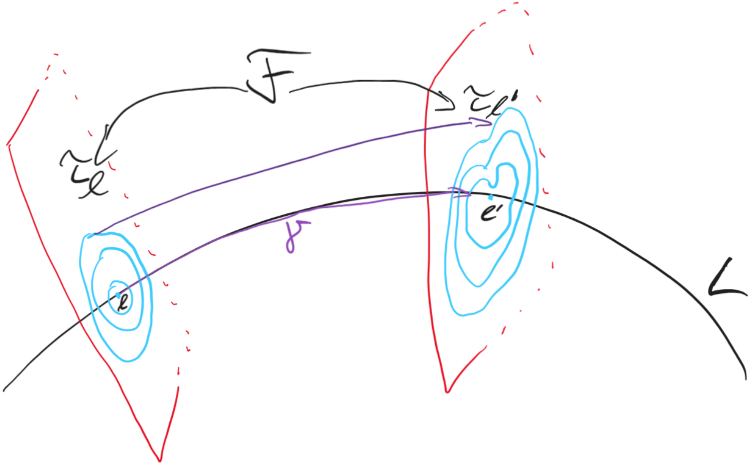
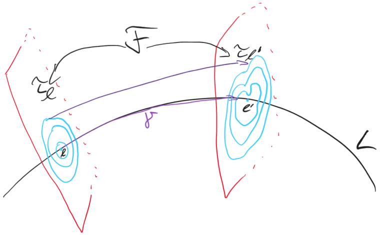
## Idea: Relation to gauge theory






Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.




### Theorem ($\mathscr{F}$-connections)

*There is a connection on the normal bundle of a leaf L:*

- *Horizontal vector fields are in* $\mathscr{F}$.
- *Parallel transport* $\mathrm{PT}_\gamma$ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop* $\gamma_0$ *at l:* $\mathrm{PT}_{\gamma_0}$ *values in* $\mathrm{Inner}(\tau_l)$.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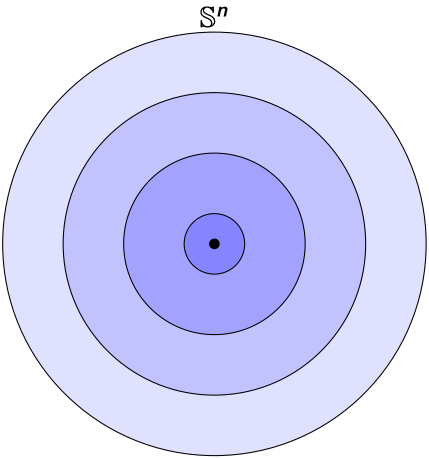
# Example of a transverse foliation $\tau$:

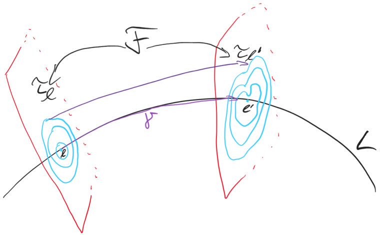



## Remarks

- Inner($\tau_I$) maps each circle to itself
- Sym($\tau_I$) allows to exchange circles
- Both preserve $\tau_I$ and fix the origin


Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

# Idea



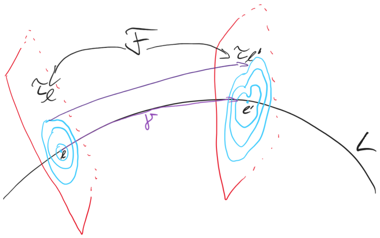

**Idea**

Generators of $\mathscr{F}$ given by $\mathscr{F}_{\text{projectable}}$:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.

## Idea: Relation to gauge theory




### Idea

Fix $l$ and given $\tau_l$: Reconstruct $\mathscr{F}$.

$$
\begin{aligned}
[\mathbb{H}(X)+\overline{\nu}, \mathbb{H}(X')+\overline{\mu}] &= \mathbb{H}([X,X']) + \overline{\ldots} \\
&= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{ curvature}} \\
&\qquad + \underbrace{[\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu, \mu]}
\end{aligned}
$$

### Theorem ([C. L.-G., S.-R. F.])

*Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by*

$$\mathbb{H}(X) + \overline{\nu},$$

*where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.*

### Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla_X \nu},$$
$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathscr{g})$.

## Theorem ([C. L.-G., S.-R. F.])

*Given a multiplicative Yang-Mills connection on* $\mathscr{G}$ *and a Yang-Mills connection* $\mathbb{H}$ *on* $\mathscr{T}$*, then there is a natural foliation on* $\mathscr{T}$ *generated by*

$$\mathbb{H}(X) + \overline{\nu},$$

*where* $X \in \mathfrak{X}(L)$ *and* $\nu \in \Gamma(\mathscr{g})$*.*

## Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla_X \nu},$$
$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathscr{g})$.

## Idea (Leaf $L$ simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

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## Remarks

- Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.
- $\mathscr{G}$ acts on $\mathscr{T}$ (canonically from the left).

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### Proposition ([C. L.-G., S.-R. F.])

*The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.*

### Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits $L$ as a leaf and $\tau_l$ as transverse data.

## Proposition ([C. L.-G., S.-R. F.])

*The reconstructed foliation is independent of the choice of connection on $P$.*

$$
\begin{tikzcd}
\Gamma(\mathrm{Ad}(P)) \arrow[rrrr, "\nu\mapsto\overline{\nu}"] \arrow[dr, hook] & & & & \mathcal{T} \arrow[dl, hook'] \\
& \Gamma(\mathrm{At}(P)) \arrow[rr, "\xi\mapsto\overline{\xi}"] \arrow[dr, two heads] & & \mathscr{F}_{\text{projectable}} \arrow[dl, two heads] & \\
& & \mathfrak{X}(L) \arrow[ul, bend left, "A"] \arrow[ur, bend right, "\mathbb{H}"'] & &
\end{tikzcd}
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& \Gamma(\mathrm{At}(P)) \arrow[rr, "\xi\mapsto\overline{\xi}"] \arrow[dr, two heads] & & \mathscr{F}_{\text{projectable}} \arrow[dl, two heads] & \\
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\end{tikzcd}
$$

# Summary

**Remarks ([C. L.-G., S.-R. F.])**

In the simply connected case, the following are equivalent:

- Singular foliations with leaf $L$ and transverse model $\left(\mathbb{R}^d, \tau_l\right)$
- Principal $\mathrm{Inner}(\tau_l)$-bundles over $L$

**Remarks (Classification of curved Yang-Mills gauge theories)**

If $\mathscr{G}$ acts faithfully on $\mathscr{T}$, preserving $L$, then a curved Yang-Mills gauge theory can be flattened if and only if $P$ is flat.

| Curved YM Gauge Theory | Singular Foliations $\mathscr{F}$ |
|---|---|
| Multiplicative Yang-Mills connection | $\mathscr{F}$-connection |
| Flat gauge theory | Flat singular foliation |
| Field redefinition of connection on $\mathscr{G}$ | Different choice of $\mathscr{F}$-connection |

**Thank you!**