Curved Yang-Mills gauge theories and their recent applications

Simon-Raphael Fischer

國家理論科學研究中心
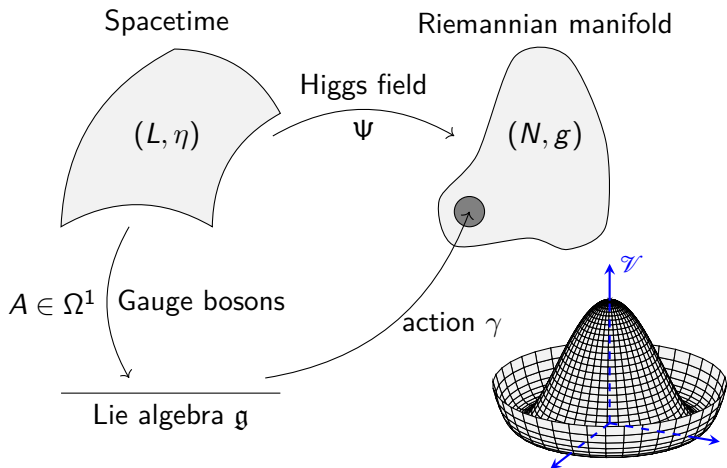National Center for Theoretical Sciences (National Taiwan University)

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Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $L \times \mathfrak{g}$	Lie algebroid $E \rightarrow N$
$\mathfrak{g}$-action γ	Anchor ρ of E
	& E-connections
Canonical flat connection ∇^0 on $L \times \mathfrak{g}$	General connection ∇ on E

Guide: Infinitesimal curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Sombrero hat potential made by Mark J.D. Hamilton.

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Foliations
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Future Prospects
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Field Redefinition

Consider a semisimple Lie group G and a principal G-bundle $P \to L$:

$$\mathrm{Ad}(P) \lhook\joinrel\longrightarrow \mathrm{At}(P) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{T}L$$

where $\mathrm{Ad}(P)$ and $\mathrm{At}(P)$ the adjoint and Atiyah bundle of a principal G-bundle P, respectively.

Gedankenexperiment

1. Adjoint connection $\leftrightarrow$ Ehresmann connection on P.
2. As parallel transport along a curve γ:

$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^P(p), v\right]$$

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Field Redefinition

Consider a semisimple Lie group G and a principal G-bundle $P \to L$:

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① Adjoint connection $\leftrightarrow$ Ehresmann connection on P.

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$$\text{PT}_\gamma^{\text{Ad}(P)}([p, v]) = \left[\text{PT}_\gamma^P(p), \text{PT}_\gamma^0(v)\right],$$

Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection

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$$\mathrm{Ad}(P) \lhook\joinrel\longrightarrow \mathrm{At}(P) \longrightarrow\!\!\!\!\twoheadrightarrow \mathrm{T}L$$

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Gedankenexperiment

① Adjoint connection $\leftrightarrow$ Ehresmann connection on P.

② As parallel transport:

$$\mathrm{PT}_\gamma^{\mathrm{Ad}(P)}([p, v]) = \left[\mathrm{PT}_\gamma^P(p) \cdot \kappa_\gamma, \kappa_\gamma^{-1} \cdot \mathrm{PT}_\gamma^0(v) \right],$$

Lie algebra $\mathfrak{g}$ as trivial bundle w/ canonical flat connection, κ_γ values in G & "suitable"

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Field Redefinition

Theorem (Field Redefinitions S.-R. F.)

This leads to an equivalence relation of gauge theories, preserving dynamics and kinematics.
But: In the curved sense! Curvature terms appear.

Motivation (S.-R. F.)

1. How to formulate gauge theory such that it is invariant under field redefinitions?

2. Are there curved theories which are **not** equivalent to classical ones?

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Motivation (S.-R. F.)

1. How to formulate gauge theory such that it is invariant under field redefinitions?
2. Are there curved theories which are **not** equivalent to classical ones?

We will only focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB[1] $\mathscr{g}$
Integrated	Lie group G	LGB[2] $\mathscr{G}$

$$G \longrightarrow \mathscr{G}$$
$$\downarrow$$
$$M$$

---

[1]LAB = Lie algebra bundle
[2]LGB = Lie group bundle

Definition (LGB actions, simplified)

$$
\begin{array}{c}
\mathscr{G} \\
\downarrow \\
\mathscr{P} \xrightarrow{\ \pi\ } L
\end{array}
$$

$\mathscr{P} \xrightarrow{\pi} L$ a fibre bundle. A **right-action of $\mathscr{G}$ on $\mathscr{P}$** is a smooth map $\mathscr{P} * \mathscr{G} := \pi^* \mathscr{G} = \mathscr{P} \times_L \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$
\pi(p \cdot g) = \pi(p), \tag{1}
$$

$$
(p \cdot g) \cdot h = p \cdot (gh), \tag{2}
$$

$$
p \cdot e_{\pi(p)} = p \tag{3}
$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

Sources   K. Mackenzie. General Theory of Lie Groupoids and Algebroids.
                       London Mathematical Society Lecture Note Series, 213, 2005.

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Field Redefinition

Definition (Principal bundle)

Still a fibre bundle

$$G \longrightarrow \mathscr{P}$$
$$\downarrow \pi$$
$$L$$

but with $\mathscr{G}$-action

$$\begin{array}{c} \mathscr{P} \times G \\ \mathscr{P} * \mathscr{G} \end{array} \rightarrow \mathscr{P}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.

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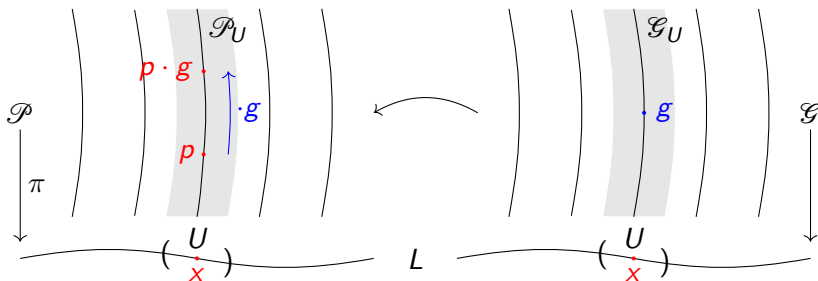
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Connections as parallel transport

Connection on $\mathscr{P}$: Idea

But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $D_p r_g$ only defined on vertical structure

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Connections as parallel transport
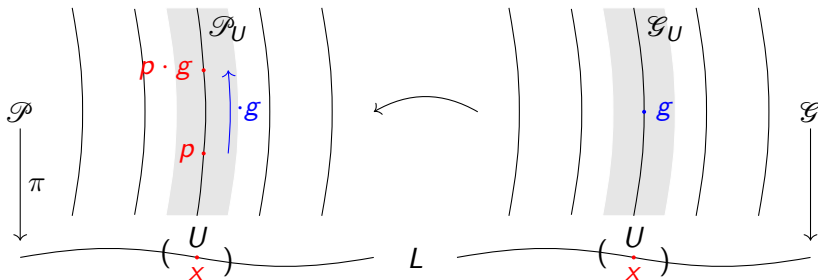
Connection on $\mathcal{P}$: Idea
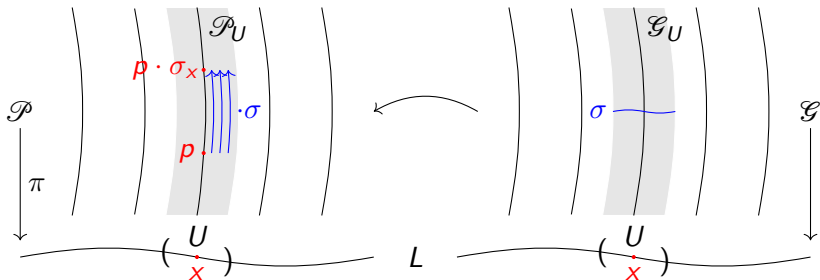
$$\text{Use } \sigma \in \Gamma(\mathcal{G}) : r_\sigma(p) \coloneqq p \cdot \sigma_x$$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle
($\mathscr{G}$ trivial, $\sigma \equiv g$ constant),
and H a connection:

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Future Prospects
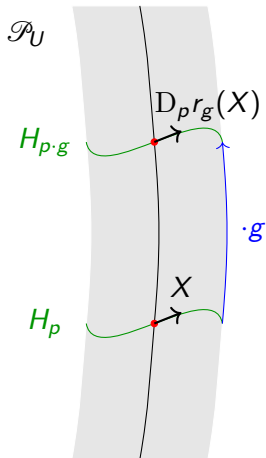○○

Connections as parallel transport

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle
($\mathscr{G}$ trivial, $\sigma \equiv g$ constant),
and H a connection:

$\mathscr{P}_U$

$\mathrm{D}_p r_g(X)$

$H_{p \cdot g}$

$\cdot g$

X

H_p
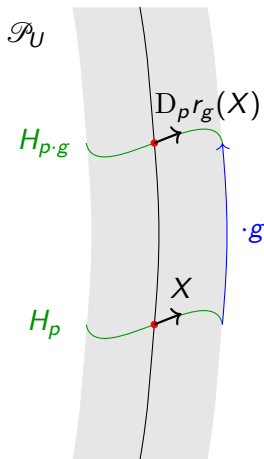
Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{P}}_\gamma$ in $\mathscr{P}$:

$$\mathrm{PT}^{\mathscr{P}}_\gamma(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_\gamma(p) \cdot g$$

where $\gamma : I \to L$ is a base path

Connections as parallel transport

Connection on $\mathscr{P}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}_\gamma^\mathscr{P}(p \cdot g) = \mathrm{PT}_\gamma^\mathscr{P}(p) \cdot \mathrm{PT}_\gamma^\mathscr{G}(g).$$

Back to the roots

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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Connections as parallel transport

Connection on $\mathscr{P}$: General case

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General notion of Ehresmann and Yang-Mills connections

Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L$ so that one has a commuting diagram

$$
\begin{array}{ccc}
 & & \mathscr{G} \\
 & \nearrow & \downarrow \pi_{\mathscr{G}} \\
\mathscr{T} & \xrightarrow{\pi_{\mathscr{T}}} & L
\end{array}
$$

1 **Ehresmann connection:**

$$
\mathrm{PT}_{\gamma}^{\mathscr{T}}(t \cdot g) = \mathrm{PT}_{\gamma}^{\mathscr{T}}(t) \cdot \mathrm{PT}_{\gamma}^{\mathscr{G}}(g)
$$

2 **Yang-Mills connection:** Additionally

$$
\mathrm{PT}_{\gamma_0}^{\mathscr{T}}(t) = t \cdot g_{\gamma_0}
$$

for some $g_{\gamma_0} \in \mathscr{G}_{\pi_{\mathscr{T}}(t)}$, where γ_0 is a contractible loop.

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General notion of Ehresmann and Yang-Mills connections

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}_\gamma^{\mathscr{G}}(q \cdot g) = \mathrm{PT}_\gamma^{\mathscr{G}}(q) \cdot \mathrm{PT}_\gamma^{\mathscr{G}}(g),$$
$$\mathrm{PT}_{\gamma_0}^{\mathscr{G}}(q) = g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
$$\mathrm{PT}^{\mathscr{G}}_{\gamma_0}(q) = g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

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Connections as parallel transport

Remarks

There is a simplicial differential δ on $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} L$ with Lie algebra bundle $\mathfrak{g}$

$$\delta : \Omega^{\bullet}(\underbrace{\mathscr{G} * \ldots * \mathscr{G}}_{k \text{ times}}; \pi_{\mathscr{G}}^* \mathfrak{g}) \to \Omega^{\bullet}(\underbrace{\mathscr{G} * \ldots * \mathscr{G}}_{k+1 \text{ times}}; \pi_{\mathscr{G}}^* \mathfrak{g})$$

such that the definition of the multiplicative Yang-Mills connection is equivalent to the **compatibility conditions**

- Connection closed
- Curvature exact ([S.-R. F.])

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Connections as parallel transport

Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection ∇ with

$$\nabla\Big([\mu,\nu]_{\mathscr{g}}\Big) = [\nabla\mu,\nu]_{\mathscr{g}} + [\mu,\nabla\nu]_{\mathscr{g}},$$

$$R_{\nabla} = \mathrm{ad} \circ \zeta.$$

Example

Given a short exact sequence of algebroids

$$\mathscr{g} \longhookrightarrow E \longrightarrow\!\!\!\!\!\rightarrow \mathrm{T}L$$

with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla_X \nu = [\chi(X), \nu]_E,$$

$$\zeta(X, X') = [\chi(X), \chi(X')]_E - \chi([X, X']).$$

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Example

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with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla_X\nu = [\chi(X),\nu]_E,$$
$$\zeta(X,X') = [\chi(X),\chi(X')]_E - \chi([X,X']).$$

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Field strength

Given a multiplicative Yang-Mills connection on $\mathscr{G}$:

Definition (Generalized curvature/field strength F of A, [S.-R. F.])

We define

$$F := \mathrm{d}^{\pi_{\mathscr{P}}^* \nabla} A + \frac{1}{2} [A \stackrel{\wedge}{,} A]_{\pi_{\mathscr{P}}^* \mathscr{G}} + \pi_{\mathscr{P}}^! \zeta.$$

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Curved Yang-Mills gauge theory

Theorem (Lagrangian, [S.-R. F.])

- κ be an Ad-invariant fibre metric on $\mathcal{g}$,
- M a spacetime, and $*$ its Hodge star operator,
- $(U_i)_i$ open covering of M with subordinate gauges $s_i \in \Gamma(\mathcal{P}|_{U_i})$.

Then the Lagrangian $\mathfrak{L}_{\mathrm{CYM}}[A]$, defined locally by

$$\left(\mathfrak{L}_{\mathrm{CYM}}[A]\right)\big|_{U_i} := -\frac{1}{2}\kappa(F_{s_i} \overset{\wedge}{,} *F_{s_i}),$$

is well-defined, and

$$\mathfrak{L}_{\mathrm{CYM}}\Big[K^! A\Big] = \mathfrak{L}_{\mathrm{CYM}}[A]$$

for all principal bundle automorphisms K.

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Classical theory

Back to the roots

1. $\mathscr{G} \cong M \times G$

2. Equip $\mathscr{G}$ with canonical flat connection

3. $\zeta \equiv 0$

Example (Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^4$, [S.-R. F.])

Let P be the Hopf bundle

$$\mathrm{SU}(2) \cong \mathbb{S}^3 \longrightarrow \mathbb{S}^7$$
$$\downarrow$$
$$\mathbb{S}^4$$

Define $\mathscr{P} := \mathscr{G}$ as the inner group bundle of P,

$$\mathscr{G} := c_{\mathrm{SU}(2)}(P) := (P \times \mathrm{SU}(2)) \big/ \mathrm{SU}(2).$$

This principal $c_{\mathrm{SU}(2)}(P)$-bundle admits the structure as curved Yang-Mills gauge theory; there is no description as classical gauge theory.

Applications: Singular foliations
(joint work w/ Camille Laurent-Gengoux)

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Curved Yang-Mills gauge theory

$\mathbb{S}^n$

Right diagram made by Mark J.D. Hamilton.

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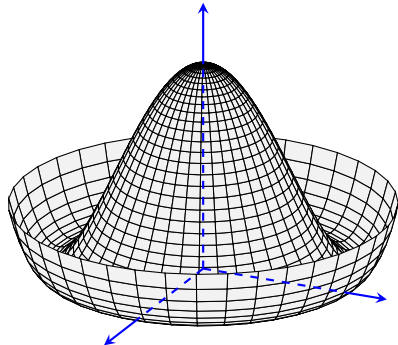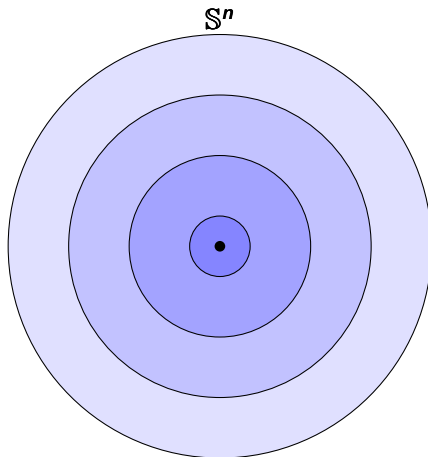
Singular Foliations:

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)

- Poisson Geometry
  (Singular foliation of symplectic leaves)

- Lie groupoids and algebroids

- Dirac structures

- Generalised complex manifolds

- Non-commutative geometry

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Definition

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under $C^\infty(M)$-multiplication**,
- it is **locally finitely generated**.

Sources   Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

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- it is **involutive**, *i.e.* $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$,
- it is **stable under $C^\infty(M)$-multiplication**, *i.e.* $fX \in \mathscr{F}$ for all $f \in C^\infty(M)$ and $X \in \mathscr{F}$,
- it is **locally finitely generated**.

Sources   Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

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Future Prospects
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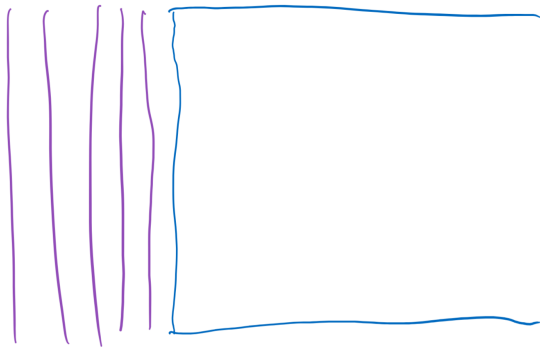
Definition

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**, *i.e.* $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$,
- it is **stable under** $C^\infty(M)$-**multiplication**, *i.e.* $fX \in \mathscr{F}$ for all $f \in C^\infty(M)$ and $X \in \mathscr{F}$,
- it is **locally finitely generated**, *i.e.* around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ $(X^i \in \mathscr{F})$ such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on U.
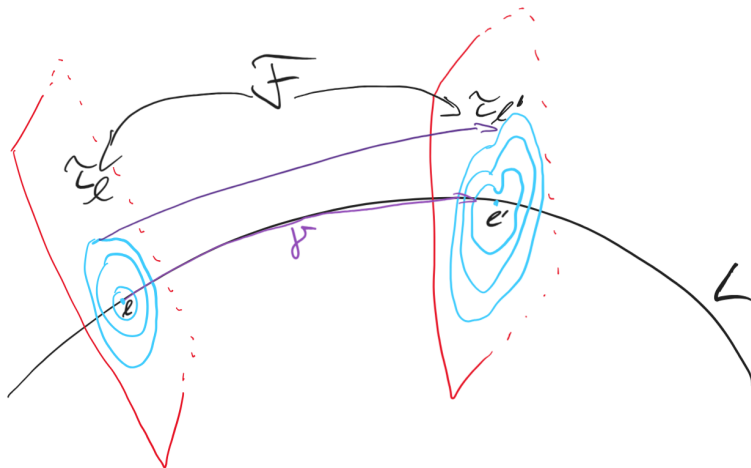
$$X = \sum_i f_i X^i.$$

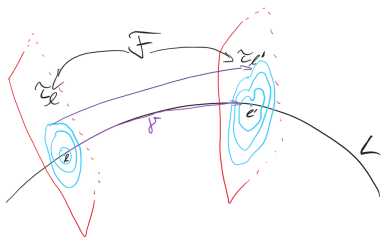Sources

Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.

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Definition

Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.

Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

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Idea: Relation to gauge theory

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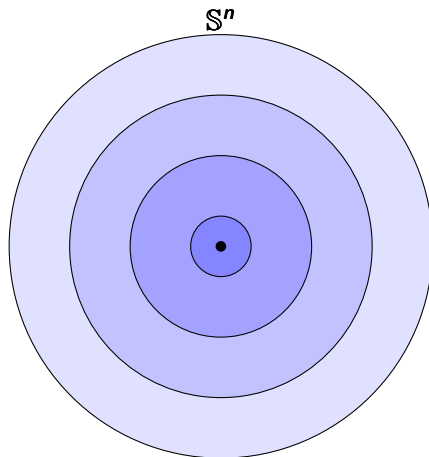○○

Idea: Relation to gauge theory

Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport PT_γ has values in $\mathrm{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l: PT_{γ_0} values in $\mathrm{Inner}(\tau_l)$.*

Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf,
Selecta Mathematica 28, no. 2, 45, 2022.

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Idea: Relation to gauge theory
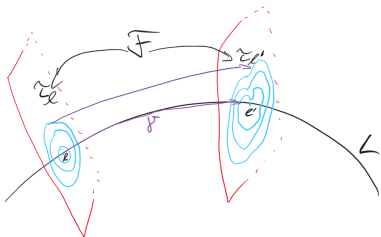
Example of a transverse foliation τ:

Remarks

- Inner(τ_l) maps each circle to itself
- Sym(τ_l) allows to exchange circles
- Both preserve τ_l and fix the origin

Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf,
Selecta Mathematica 28, no. 2, 45, 2022.

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Idea: Relation to gauge theory
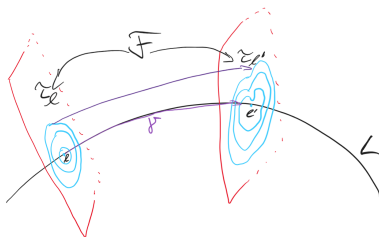
Idea

Idea

Generators of $\mathscr{F}$ given by $\mathscr{F}_{\text{projectable}}$:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift,
$\nu \in \Gamma(\text{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.

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Future Prospects
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Idea: Relation to gauge theory

Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$\begin{aligned}
[\mathbb{H}(X) + \overline{\nu}, \mathbb{H}(X') + \overline{\mu}] &= \mathbb{H}([X, X']) + \overline{\cdots} \\
&= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{ curvature}} \\
&\quad + \underbrace{[\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu, \mu]}
\end{aligned}$$

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla^{\mathscr{G}}_X \nu},$$
$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathscr{g})$.

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. $G = \mathrm{Inn}(\tau_l)$
2. P a principal G-bundle, equipped with an ordinary connection

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Future Prospects
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Reconstructing Foliations

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1. $G = \mathrm{Inn}(\tau_l)$
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3. $\mathscr{G} := (P \times G)\big/ G$, the **inner group bundle**

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3. $\mathscr{G} \coloneqq (P \times G)\big/ G$, the **inner group bundle**
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Remarks

- Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.
- $\mathscr{G}$ acts on $\mathscr{T}$ (canonically from the left).

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Reconstructing Foliations

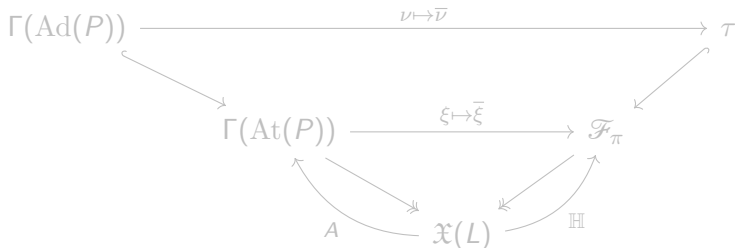## Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathcal{G}$ is a multiplicative Yang-Mills connection and the one on $\mathcal{T}$ is a corresponding Yang-Mills connection.

Remarks

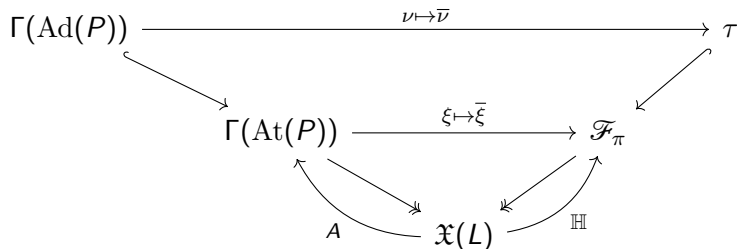Thus, we have a singular foliation on $\mathcal{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

$$
\begin{array}{ccc}
\Gamma(\mathrm{Ad}(P)) & \xrightarrow{\quad \nu \mapsto \overline{\nu} \quad} & \tau \\
\downarrow & & \\
\Gamma(\mathrm{At}(P)) & \xrightarrow{\quad \xi \mapsto \overline{\xi} \quad} & \mathscr{F}_\pi \\
A \quad \nwarrow & \mathfrak{X}(L) & \nearrow \mathbb{H}
\end{array}
$$

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Future Prospects
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Independency of choice of connection

Summary

Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf L and transverse model $\left(\mathbb{R}^d, \tau_l\right)$
- Principal $\mathrm{Inner}(\tau_l)$-bundles over L

Remarks (Classification of curved Yang-Mills gauge theories)

If $\mathscr{G}$ acts faithfully on $\mathscr{T}$, then a curved Yang-Mills gauge theory can be flattened if and only if P is flat.

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Future Prospects
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Future Prospects

> **Remarks (Future prospects)**
>
> - Further applications of curved gauge theories (existence of Cartan connections,...)
> - Total space and Atiyah bundles of groupoid-based principal bundles
> - Field redefinitions as natural symmetry in associated applications

Thank you!