

# Classification of neighbourhoods of leaves of singular foliations

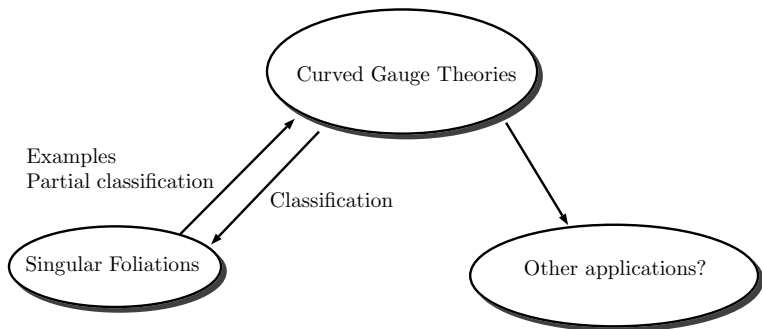
joint work with Camille Laurent-Gengoux  
(Université de Lorraine)

Simon-Raphael Fischer



國家理論科學研究中心

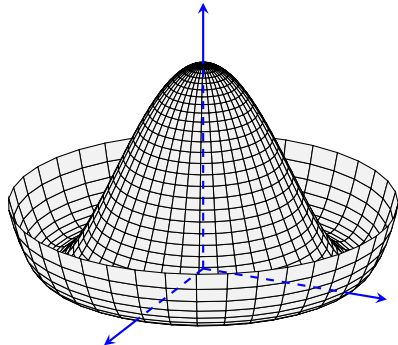
National Center for Theoretical Sciences (National Taiwan University)



### Remarks (Based on the following works)

- 1 S.-R. F. and Camille Laurent-Gengoux, *Classification of neighborhoods around leaves of singular foliations*, arXiv:2401.05966, (2024).
- 2 S.-R. F., *Integrating curved Yang–Mills gauge theories*, arXiv:2210.02924, (2022).

# **Singular Foliations**



## Singular Foliations:

- Gauge Theory
- Poisson Geometry  
(Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

# First idea

## Definition (Partitionifolds)

Let  $M$  be a smooth manifold. A **partitionifold of  $M$**  is a partition of immersed connected submanifolds, which we call *leaves*.

## Remarks

We will denote a partitionifold by  $L_\bullet$ ,  $p \mapsto L_p$ , where  $L_p$  is the leaf through  $p \in M$ .

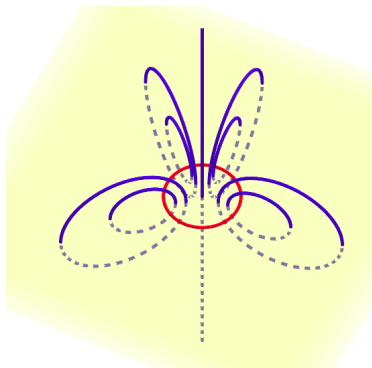


Figure: The magnetic partition

### Remarks

A partitionifold with:

- All leaves are of the same dimension.
- **But:** It lacks regularity!

## Definition

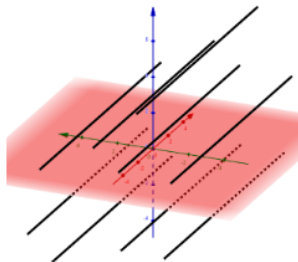


Figure: Isolated lasagna in a spaghetti dish

## Remarks

A partitionifold with:

- Dimension is now different.
- **But:** Also no regularity!



### Definition (Smooth singular foliation)

A **smooth singular foliation**  $\mathcal{F}$  on a smooth manifold  $M$  is a subspace of  $\mathfrak{X}_c(M)$  so that

- it is **involutive**,
- it is **stable under**  $C^\infty(M)$ -**multiplication**,
- it is **locally finitely generated**.

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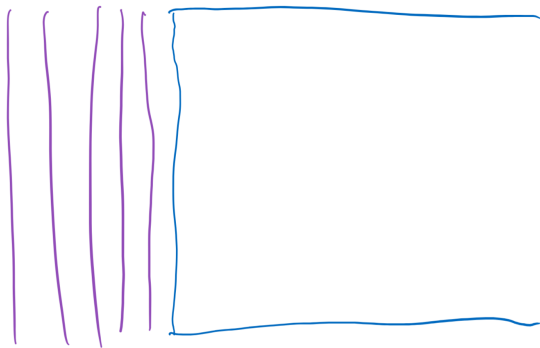
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- it is **stable under  $C^\infty(M)$ -multiplication**, i.e.  $fX \in \mathcal{F}$  for all  $f \in C^\infty(M)$  and  $X \in \mathcal{F}$ ,
- it is **locally finitely generated**, i.e. around each  $p \in M$  there is an open neighbourhood  $U$  and a finite family  $(X^i)_i^r$  ( $X^i \in \mathcal{F}$ ) such that for all  $X \in \mathcal{F}$  there are  $f_i \in C^\infty(M)$  satisfying on  $U$ .

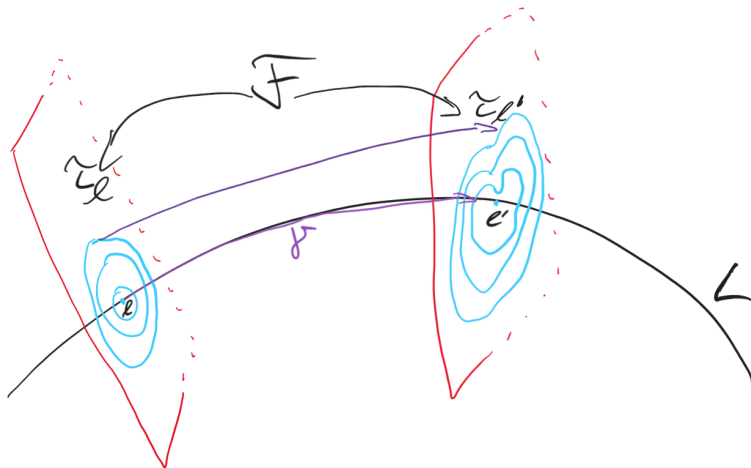
$$X = \sum_i f_i X^i.$$

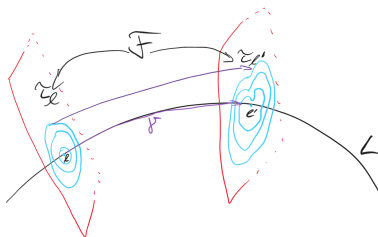
## Remarks (Leaves)

Following the flows in  $\mathcal{F}$ , this gives rise to a partition of connected immersed submanifolds in  $M$ .



# Why locally finitely generated?





### Theorem ( $\mathcal{F}$ -connections)

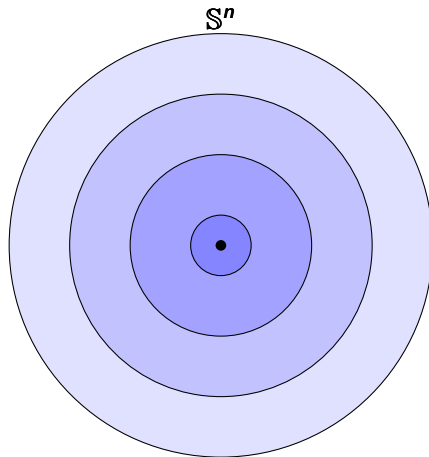
There is a connection on the normal bundle of a leaf  $L$ :

- Horizontal vector fields are in  $\mathcal{F}$ .
- Parallel transport  $PT_\gamma$  has values in  $\text{Sym}(\tau_l, \tau_{l'})$ .
- For a contractible loop  $\gamma_0$  at  $l$ :  $PT_{\gamma_0}$  values in  $\text{Inner}(\tau_l)$ .



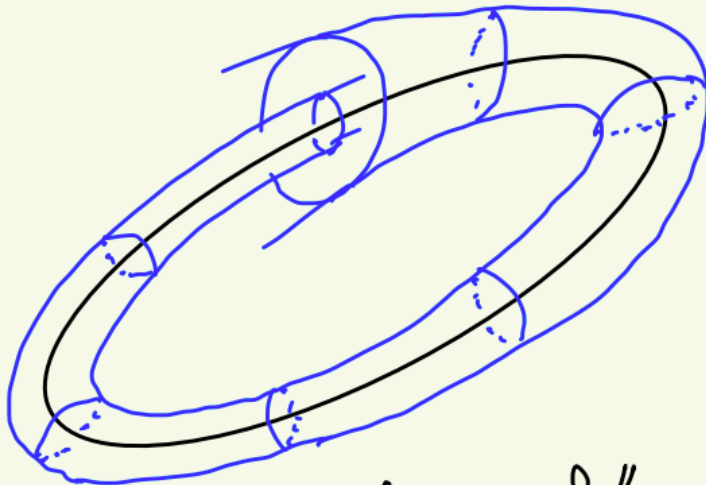
Idea: Relation to gauge theory

## Example of a transverse foliation $\tau$ :



### Remarks

- $\text{Inner}(\tau_I)$  maps each circle to itself
- $\text{Sym}(\tau_I)$  allows to exchange circles
- Both preserve  $\tau_I$  and fix the origin



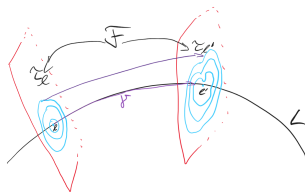
The "self eating snake"

# Other example: Regular foliation



## Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts



### Remarks ( $\mathcal{F}$ -connection)

For  $\phi \in \text{Sym}(\tau_I)$  we have an induced parallel transport

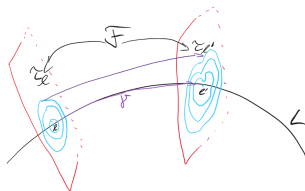
$$\text{PT}_\gamma^{\text{Sym}}(\phi) := \text{PT}_\gamma \circ \phi \circ \text{PT}_\gamma^{-1}.$$

Then, on the normal bundle  $\pi: \mathcal{T} \rightarrow L$ ,

$$\text{PT}_\gamma(\phi \cdot p) = \text{PT}_\gamma^{\text{Sym}}(\phi) \cdot \text{PT}_\gamma(p)$$

$$\text{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all  $p \in \mathcal{T}_I$ ,  $\phi \in \text{Sym}(\tau_I)$ , and for some  $\varphi \in \text{Inner}(\tau_I)$ .



### Remarks (Sym-connection)

For  $\phi \in \text{Sym}(\tau_l)$  we have an induced parallel transport

$$\text{PT}_\gamma^{\text{Sym}}(\phi) := \text{PT}_\gamma \circ \phi \circ \text{PT}_\gamma^{-1}.$$

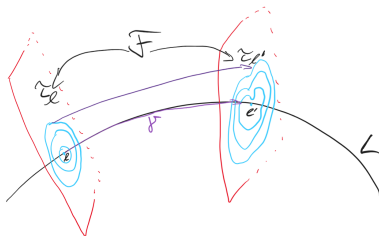
Then

$$\text{PT}_\gamma^{\text{Sym}}(\phi \circ \phi') = \text{PT}_\gamma^{\text{Sym}}(\phi) \circ \text{PT}_\gamma^{\text{Sym}}(\phi')$$

$$\text{PT}_{\gamma_0}^{\text{Sym}}(\phi) = \varphi \circ \phi \circ \varphi^{-1}$$

for all  $\phi, \phi' \in \text{Sym}(\tau_l)$ , and for some  $\varphi \in \text{Inner}(\tau_l)$ .

# Idea

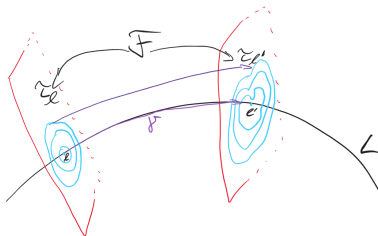


## Idea

Generators of  $\mathcal{F}$  given by:

$$\mathbb{H}(X) + \bar{\nu},$$

where  $X \in \mathfrak{X}(L)$ ,  $\mathbb{H}(X)$  its projectable horizontal lift,  
 $\nu \in \Gamma(\text{inner}(\tau))$  and  $\bar{\nu}$  its fundamental vector field.



## Idea

Fix  $I$  and given  $\tau_I$ : Reconstruct  $\mathcal{F}$ .

$$\begin{aligned}
 [\mathbb{H}(X) + \bar{\nu}, \mathbb{H}(X') + \bar{\mu}] &= \mathbb{H}([X, X']) + \dots \\
 &= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{curvature}} \\
 &\quad + \underbrace{[\mathbb{H}(X), \bar{\mu}] - [\mathbb{H}(X'), \bar{\nu}]}_{\rightsquigarrow \text{connection}} + [\bar{\nu}, \bar{\mu}]
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Idea ( $\dots \in \tau$ )

We need:

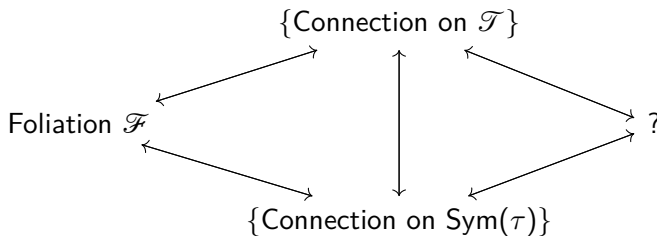
- ① Lie algebra bundle  $\tau$  with structure  $\tau_I$
- ② A horizontal lift  $\mathbb{H}$  into  $\mathcal{F}$  satisfying

$$\text{Curvature:} \quad [\mathbb{H}(X), \mathbb{H}(X')] - \mathbb{H}([X, X']) \in \tau$$

$$\text{Connection:} \quad [\mathbb{H}(X), \bar{\mu}] \in \tau$$



# Summary



## Remarks

### Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given  $\mathcal{F}$ !

## **Multiplicative Yang-Mills connections**

## Curved Yang-Mills gauge theories:

Classical                      Curved  
Lie group  $G$     Lie group bundle  $\mathcal{G}$

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{G} \\ & & \downarrow \\ & & L \end{array}$$

Remarks (Why a "curved theory"?)

Usually, the field strength  $F$  is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

$\rightsquigarrow$  We will use a general connection  $\nabla$  instead of  $\nabla^0$ , and  $\nabla$  may not be flat.

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## Definition (LGB actions)

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 \swarrow & & \downarrow \pi_{\mathcal{G}} \\
 \mathcal{T} & \xrightarrow{\pi} & L
 \end{array}$$

A **right-action** of  $\mathcal{G}$  on  $\mathcal{T}$  is a smooth map

$\mathcal{T} * \mathcal{G} := \mathcal{T} \times_{\pi \times \pi_{\mathcal{G}}} \mathcal{G} \rightarrow \mathcal{T}$ ,  $(p, g) \mapsto p \cdot g$ , satisfying the following properties:

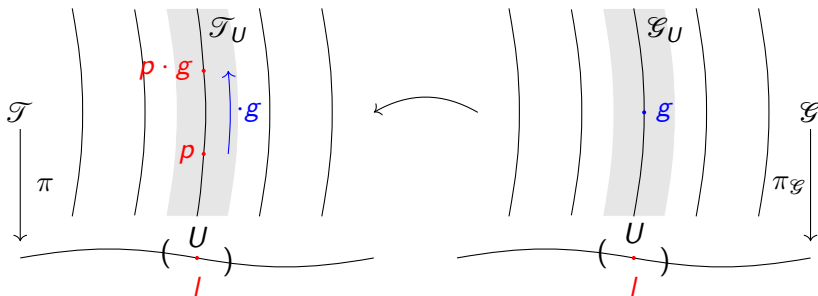
$$\pi(p \cdot g) = \pi(p), \quad (1)$$

$$(p \cdot g) \cdot h = p \cdot (gh), \quad (2)$$

$$p \cdot e_{\pi(p)} = p \quad (3)$$

for all  $p \in \mathcal{T}$  and  $g, h \in \mathcal{G}_{\pi(p)}$ , where  $e_{\pi(p)}$  is the neutral element of  $\mathcal{G}_{\pi(p)}$ .

# Connection on $\mathcal{T}$ : Idea



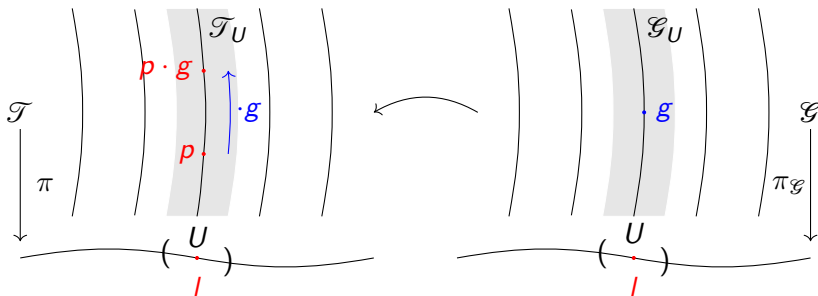
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$D_p r_g$  only defined on vertical structure

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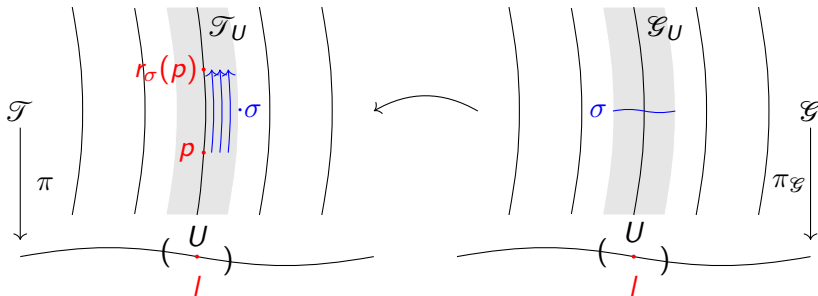


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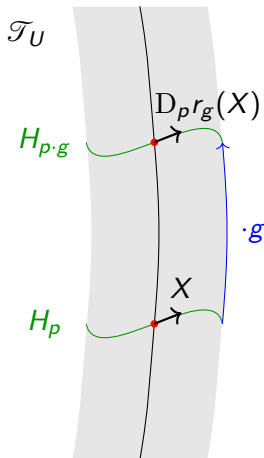
Connection on  $\mathcal{T}$ : Idea

Use  $\sigma \in \Gamma(\mathcal{G}): r_\sigma(p) := p \cdot \sigma_I$



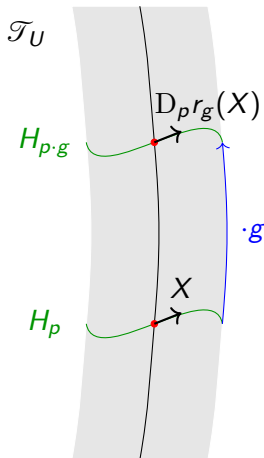
# Connection on $\mathcal{T}$ : Revisiting the classical setup

If  $\mathcal{G}$  is trivial,  $\sigma \equiv g$  constant,  
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## Remarks (Integrated case)

Parallel transport  $\text{PT}_\gamma^{\mathcal{T}}$  in  $\mathcal{T}$ :

$$\text{PT}_\gamma^{\mathcal{T}}(p \cdot g) = \text{PT}_\gamma^{\mathcal{T}}(p) \cdot g$$

where  $\gamma : I \rightarrow L$  is a base path

# Connection on $\mathcal{T}$ : General case

## Remarks (Integrated case)

Ansatz: Introduce connection on  $\mathcal{G}$ ,

$$\mathrm{PT}_{\gamma}^{\mathcal{T}}(p \cdot g) = \mathrm{PT}_{\gamma}^{\mathcal{T}}(p) \cdot \mathrm{PT}_{\gamma}^{\mathcal{G}}(g).$$

## Recovering the ordinary definition

- ①  $\mathcal{G} \cong L \times G$
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## Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.]

A surjective submersion  $\pi: \mathcal{T} \rightarrow L$  so that one has a commuting diagram

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 \swarrow & \downarrow \pi_{\mathcal{G}} & \\
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 \end{array}$$

### 1 Ehresmann connection:

$$\text{PT}_{\gamma}^{\mathcal{T}}(p \cdot g) = \text{PT}_{\gamma}^{\mathcal{T}}(p) \cdot \text{PT}_{\gamma}^{\mathcal{G}}(g)$$

### 2 Yang-Mills connection: Additionally

$$\text{PT}_{\gamma_0}^{\mathcal{T}}(p) = p \cdot g_{\gamma_0}$$

for some  $g_{\gamma_0} \in \mathcal{G}_{\pi(p)}^0$ , where  $\gamma_0$  is a contractible loop.

### Definition (Multiplicative YM connection, [S.-R. F.]

On  $\mathcal{G}$  there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\begin{aligned} \mathrm{PT}_{\gamma}^{\mathcal{G}}(q \cdot g) &= \mathrm{PT}_{\gamma}^{\mathcal{G}}(q) \cdot \mathrm{PT}_{\gamma}^{\mathcal{G}}(g), \\ \mathrm{PT}_{\gamma_0}^{\mathcal{G}}(q) &= g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1} \end{aligned}$$

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Compare this with the Maurer-Cartan form and its curvature equation!

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On the Lie algebra bundle  $\mathfrak{g}$  we have a connection  $\nabla$  with

$$\begin{aligned}\nabla([\mu, \nu]_{\mathfrak{g}}) &= [\nabla\mu, \nu]_{\mathfrak{g}} + [\mu, \nabla\nu]_{\mathfrak{g}}, \\ R_{\nabla} &= \text{ad} \circ \zeta.\end{aligned}$$

## Example

Consider the Atiyah sequence of a principal  $G$ -bundle  $P$ :

$$(P \times \mathfrak{g})/G \hookrightarrow TP/G \overset{\mathbb{H}}{\rightrightarrows} TL$$

with splitting  $\mathbb{H}: TL \rightarrow E$ , where  $\mathfrak{g}$  is the Lie algebra. Then

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**Going back to foliations**

**Theorem ([C. L.-G., S.-R. F.])**

*Given a multiplicative Yang-Mills connection on  $\mathcal{G}$  and a Yang-Mills connection  $\mathbb{H}$  on  $\mathcal{T}$ , then there is a natural foliation on  $\mathcal{T}$  generated by*

$$\mathbb{H}(X) + \bar{\nu},$$

*where  $X \in \mathfrak{X}(L)$  and  $\nu \in \Gamma(\mathfrak{g})$ .*

**Proof.**

We have

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### Idea (Leaf $L$ simply connected)

Fix a point  $l \in L$  with transverse model  $(\mathbb{R}^d, \tau_l)$ :

- ①  $G = \text{Inn}(\tau_l)$
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### Proposition ([C. L.-G., S.-R. F.])

*The associated connection on  $\mathcal{G}$  is a multiplicative Yang-Mills connection and the one on  $\mathcal{T}$  is a corresponding Yang-Mills connection.*

### Remarks

Thus, we have a singular foliation on  $\mathcal{T}$ , which, by construction, admits  $L$  as a leaf and  $\tau_l$  as transverse data.

### Proposition ([C. L.-G., S.-R. F.])

*The reconstructed foliation is independent of the choice of connection on  $P$ .*

### Proof.

- The adjoint bundle of  $P$ ,  $\text{Ad}(P) := (P \times \mathfrak{g})/G$ , is the Lie algebra bundle of  $\mathcal{G}$
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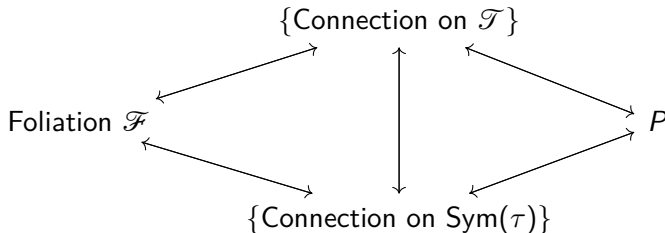
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# Summary

## Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf  $L$  and transverse model  $(\mathbb{R}^d, \tau_l)$
- Principal  $\text{Inner}(\tau_l)$ -bundles  $P$  over  $L$



# Examples

**Thank you!**