

# Classification of neighbourhoods of leaves of singular foliations

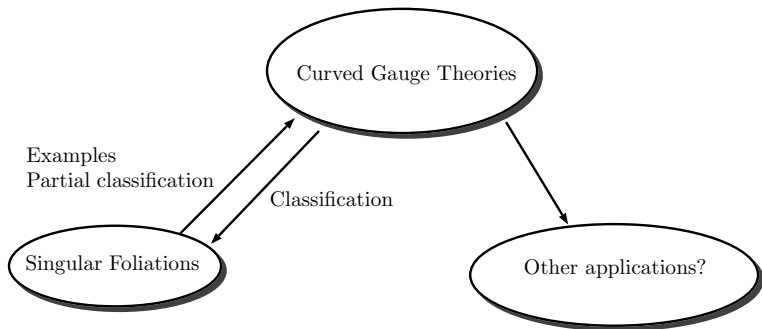
joint work with Camille Laurent-Gengoux  
(Université de Lorraine)

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National Center for Theoretical Sciences (National Taiwan University)



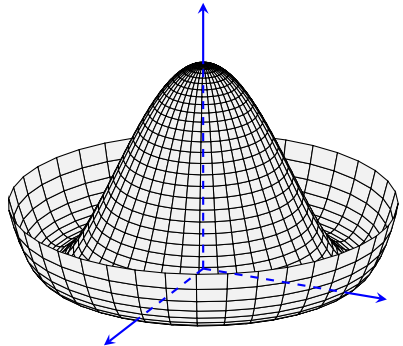
Remarks (Based on the following work)

S.-R. F. and Camille Laurent-Gengoux, *Classification of neighborhoods around leaves of singular foliations*, arXiv:2401.05966, (2024).

## Remarks (Also based on the following previous works)

- ① Camille Laurent-Gengoux and Leonid Ryvkin, *The neighborhood of a singular leaf*, Journal de l'École Polytechnique, (2021).
- ② Camille Laurent-Gengoux et Leonid Ryvkin, *The holonomy of a singular leaf*, Selecta Mathematica, (2022).
- ③ S.-R. F., Integrating curved Yang–Mills gauge theories, arXiv:2210.02924, (2022).

# **Singular Foliations**



## Singular Foliations:

- Gauge Theory
- Poisson Geometry  
(Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

# First idea

## Definition (Partitionifolds)

Let  $M$  be a smooth manifold. A **partitionifold of  $M$**  is a partition of immersed connected submanifolds, which we call *leaves*.

## Remarks

We will denote a partitionifold by  $L_\bullet$ ,  $p \mapsto L_p$ , where  $L_p$  is the leaf through  $p \in M$ .

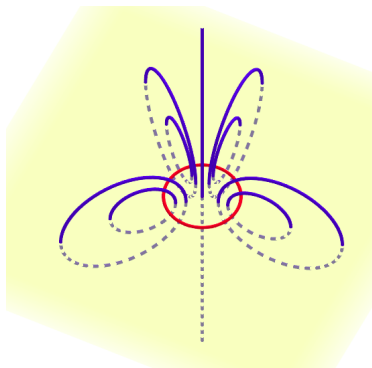


Figure: The magnetic partition

### Remarks

A partitionifold with:

- All leaves are of the same dimension.
- **But:** It lacks regularity!



## Definition

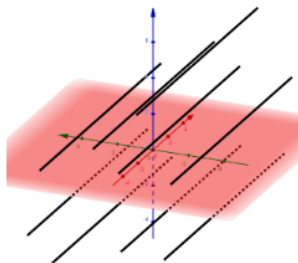


Figure: Isolated lasagna in a spaghetti dish

## Remarks

A partitionifold with:

- Dimension is now different.
- **But:** Also no regularity!

## Remarks

Isolated spaghetti in a lasagna dish: Regularity!

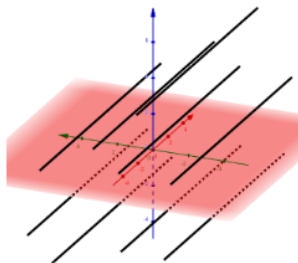


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### Definition (Smooth partitionifold)

A smooth partitionifold  $L_\bullet$  is smooth, if there is for all  $p \in M$  and every vector  $u \in T_p L_p$  a vector field  $X$  tangent to  $L_\bullet$  with

$$X_p = u.$$

### Remarks

This definition is okay, but not widely used: It still has a problem...

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### Example (Vector fields not necessarily finitely generated)

Consider the following smooth partitionifold:

- $M = \mathbb{R}$ ;
- 0-dimensional leaves:

$$\{1\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \dots, \left\{\frac{1}{n}\right\}, \dots, \{0\};$$

- 1-dimensional leaves: Remaining open intervals.

### Remarks

One has a sort of “infinitesimal leaf” next to  $\{0\}$ .

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Recall:

### Theorem (Frobenius Theorem)

*Every integrable subbundle  $E$  of  $TM$  corresponds to a regular foliation in  $M$ .*

Remarks ( $\Gamma(E)$  is involutive)

Integrable:

$$[X, Y] \in \Gamma(E)$$

for all  $X, Y \in \Gamma(E)$ .

Remarks

Alternatively: An involutive submodule of  $\mathfrak{X}(M)$ , or equivalently of  $\mathfrak{X}_c(M)$ .

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A **smooth singular foliation**  $\mathcal{F}$  on a smooth manifold  $M$  is a subspace of  $\mathfrak{X}_c(M)$  so that

- it is **involutive**,
- it is **stable under**  $C^\infty(M)$ -**multiplication**,
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- it is **locally finitely generated**, i.e. around each  $p \in M$  there is an open neighbourhood  $U$  and a finite family  $(X^i)_i^r$  ( $X^i \in \mathcal{F}$ ) such that for all  $X \in \mathcal{F}$  there are  $f_i \in C^\infty(M)$  satisfying on  $U$ .

$$X = \sum_i f_i X^i.$$

## Remarks (Leaves)

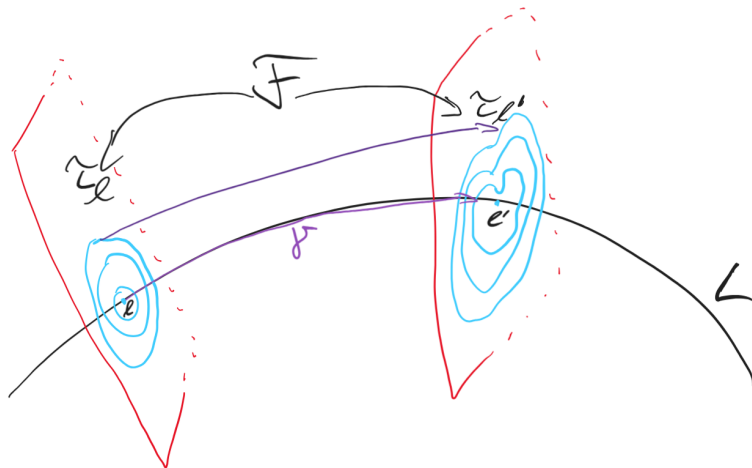
We have an induced smooth partitionifold  $L_\bullet$ ,

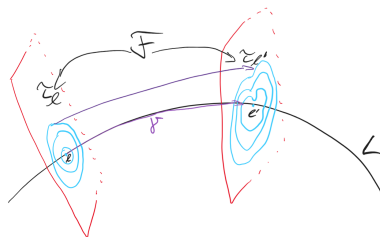
$$\mathcal{F} \Rightarrow L_\bullet,$$

but  $\mathcal{F}$  also encodes the information about the generators,

$$\begin{array}{ccc} \mathcal{F}_1 & \searrow & \\ & L_\bullet & \\ \mathcal{F}_2 & \nearrow & \end{array}$$

**First step towards classification**





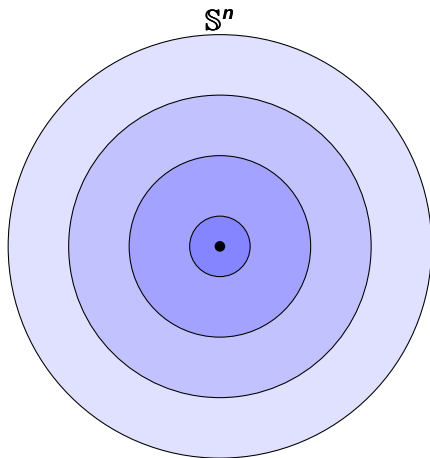
### Theorem ( $\mathcal{F}$ -connections)

*There is a connection on the normal bundle of a leaf  $L$ :*

- *Horizontal vector fields are in  $\mathcal{F}$ .*
- *Parallel transport  $PT_\gamma$  has values in  $\text{Sym}(\tau_l, \tau_{l'})$ .*
- *For a contractible loop  $\gamma_0$  at  $l$ :  $PT_{\gamma_0}$  values in  $\text{Inner}(\tau_l)$ .*

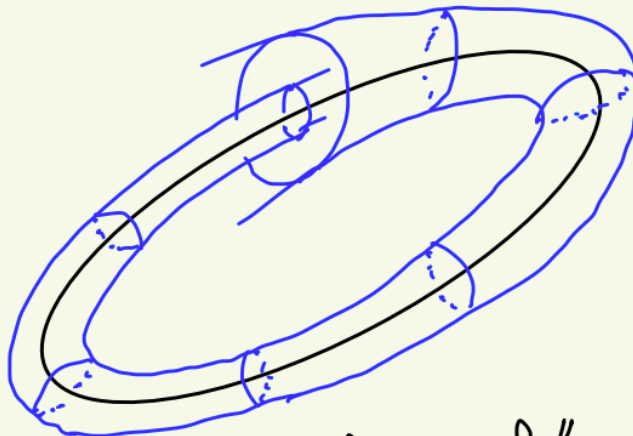


# Example of a transverse foliation $\tau$ in $\mathbb{R}^d$ :



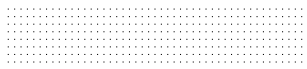
## Remarks

- $\text{Inner}(\tau_I)$  maps each circle to itself
- $\text{Sym}(\tau_I)$  allows to exchange circles
- Both preserve  $\tau_I$  and fix the origin



The "self eating snake"

# Other example: Regular foliation



## Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts

## Remarks

$\text{Inner}(\tau_I)$ : Trivial.

$\text{Sym}(\tau_I)$ : We essentially need the image of a group morphism

$$\pi_1(L) \rightarrow \text{Diff}(\mathbb{R}^d, 0).$$

## Idea

We guess:

$$\mathcal{F} = \left\{ \begin{array}{l} \text{Some map } \pi_1(L) \rightarrow \text{Diff}(\mathbb{R}^d, 0) \text{ (at } l) \\ \text{Bundle structure by } \tau_l, \text{Inner}(\tau_l), \text{Sym}(\tau_l), \dots \end{array} \right\}$$

Thus, we want to classify  $\mathcal{F}$  with given  $L$  and  $\tau_l$  (for a fixed  $l \in L$ ).

## Danger

But  $\text{Sym}(\tau_l)$  and  $\text{Inner}(\tau_l)$  are in general infinite-dimensional, so that we have to deal with infinite-dimensional geometry!

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## Definition (Formal foliation)

$X \in \mathcal{F}$  induces a derivation on  $\hat{C} := C^\infty(M)/C_0^\infty(M)$ , where  $C_0^\infty(M)$  is the ideal of functions vanishing with all their derivatives along  $L$ . The image of  $\mathcal{F}$  under this is the **formal singular foliation**.

## Remarks

$f \in \hat{C}$  a formal power series, w.r.t.  $(x_1, \dots, x_d)$  as “normal coordinates”:

$$f = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}$$

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## Example (Canonical example of a formal foliation)

For embedded submanifolds  $L$ :

- Normal bundle  $\mathcal{T}: TM|_L/TL$ .
- Formal vector fields via tubular neighbourhood embedding, a Lie algebra morphism  $\mathfrak{X}(M) \rightarrow \mathfrak{X}^{\text{formal}}$ .

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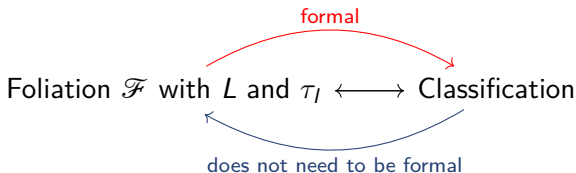
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# Our aim

## Remarks (Our assumptions)

- $\tau_I$  a formal singular foliation.
- $L$  a manifold (connected immersed submanifold of  $M$ ).



## Remarks (Avoiding formal setting)

Either

- add real-analyticity conditions to the classification,

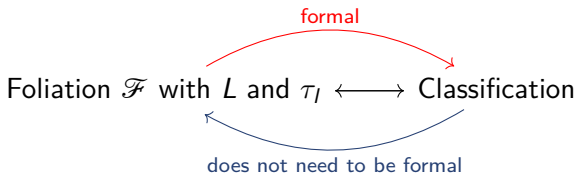
or

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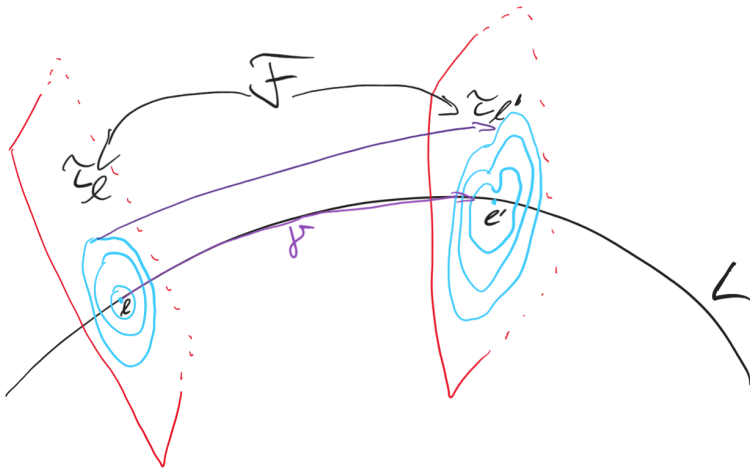
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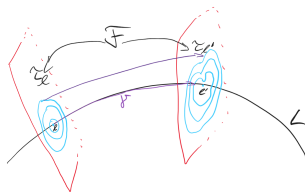
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## **Multiplicative Yang-Mills connections**





### Remarks ( $\mathcal{F}$ -connection)

For  $\phi \in \text{Sym}(\tau_I)$  we have an induced parallel transport

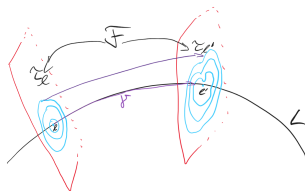
$$\text{PT}_\gamma^{\text{Sym}}(\phi) := \text{PT}_\gamma \circ \phi \circ \text{PT}_\gamma^{-1}.$$

Then, on the normal bundle  $\pi: \mathcal{T} \rightarrow L$ ,

$$\text{PT}_\gamma(\phi \cdot p) = \text{PT}_\gamma^{\text{Sym}}(\phi) \cdot \text{PT}_\gamma(p)$$

$$\text{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all  $p \in \mathcal{T}_I$ ,  $\phi \in \text{Sym}(\tau_I)$ , and for some  $\varphi \in \text{Inner}(\tau_I)$ .



### Remarks (Sym-connection)

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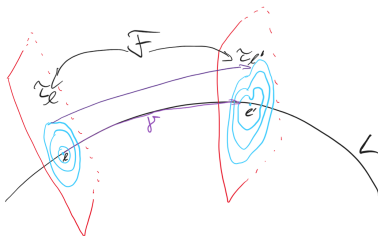
$$\text{PT}_{\gamma}^{\text{Sym}}(\phi \circ \phi') = \text{PT}_{\gamma}^{\text{Sym}}(\phi) \circ \text{PT}_{\gamma}^{\text{Sym}}(\phi')$$

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# Idea

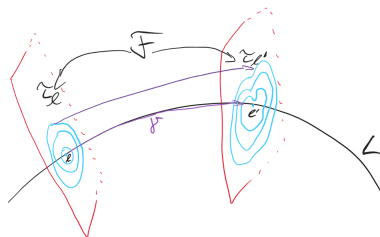


## Idea

Generators of  $\mathcal{F}$  given by:

$$\mathbb{H}(X) + \bar{\nu},$$

where  $X \in \mathfrak{X}(L)$ ,  $\mathbb{H}(X)$  its projectable horizontal lift,  
 $\nu \in \Gamma(\text{inner}(\tau))$  and  $\bar{\nu}$  its fundamental vector field.



## Idea

Fix  $I$  and given  $\tau_I$ : Reconstruct  $\mathcal{F}$ .

$$\begin{aligned}
 [\mathbb{H}(X) + \bar{\nu}, \mathbb{H}(X') + \bar{\mu}] &= \mathbb{H}([X, X']) + \dots \\
 &= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{curvature}} \\
 &\quad + \underbrace{[\mathbb{H}(X), \bar{\mu}] - [\mathbb{H}(X'), \bar{\nu}]}_{\rightsquigarrow \text{connection}} + [\bar{\nu}, \bar{\mu}]
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Idea ( $\dots \in \tau$ )

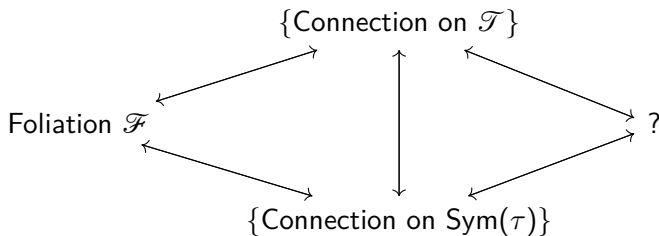
We need:

- ① Lie algebra bundle  $\tau$  with structure  $\tau_I$
- ② A horizontal lift  $\mathbb{H}$  into  $\mathcal{F}$  satisfying

$$\text{Curvature:} \quad [\mathbb{H}(X), \mathbb{H}(X')] - \mathbb{H}([X, X']) \in \tau$$

$$\text{Connection:} \quad [\mathbb{H}(X), \bar{\mu}] \in \tau$$

# Summary



## Remarks

### Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given  $\mathcal{F}$ !

## Curved Yang-Mills gauge theories:

Classical	Curved
Lie group $G$	Lie group bundle $\mathcal{G}$

$$\begin{array}{ccc}
 G & \longrightarrow & \mathcal{G} \\
 & & \downarrow \\
 & & L
 \end{array}$$

### Motivation

What are Ehresmann connections, preserving  $\mathcal{G}$ -actions?

## Definition (LGB actions)

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 \swarrow & & \downarrow \pi_{\mathcal{G}} \\
 \mathcal{T} & \xrightarrow{\pi} & L
 \end{array}$$

A **right-action** of  $\mathcal{G}$  on  $\mathcal{T}$  is a smooth map

$\mathcal{T} * \mathcal{G} := \mathcal{T} \times_{\pi \times \pi_{\mathcal{G}}} \mathcal{G} \rightarrow \mathcal{T}$ ,  $(p, g) \mapsto p \cdot g$ , satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \quad (1)$$

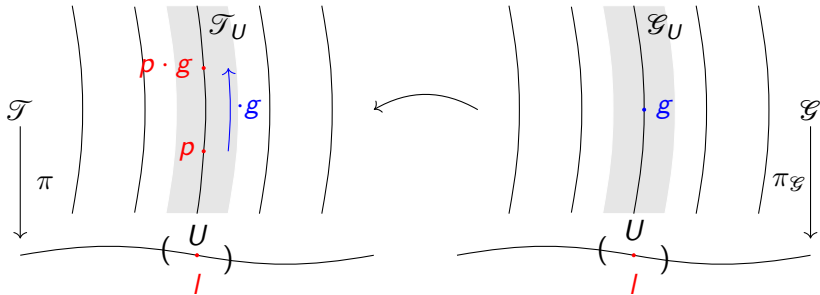
$$(p \cdot g) \cdot h = p \cdot (gh), \quad (2)$$

$$p \cdot e_{\pi(p)} = p \quad (3)$$

for all  $p \in \mathcal{T}$  and  $g, h \in \mathcal{G}_{\pi(p)}$ , where  $e_{\pi(p)}$  is the neutral element of  $\mathcal{G}_{\pi(p)}$ .

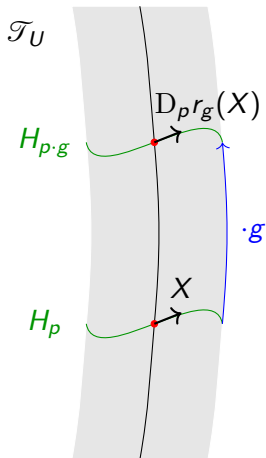
Connections as parallel transport

# Connection on $\mathcal{T}$ : Idea



# Connection on $\mathcal{T}$ : Revisiting the classical setup

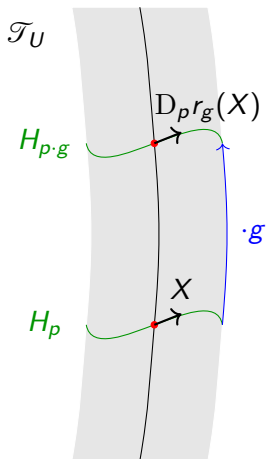
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and  $H$  a connection:



## Remarks (Integrated case)

Parallel transport  $\text{PT}_\gamma^{\mathcal{T}}$  in  $\mathcal{T}$ :

$$\text{PT}_\gamma^{\mathcal{T}}(p \cdot g) = \text{PT}_\gamma^{\mathcal{T}}(p) \cdot g$$

where  $\gamma : I \rightarrow L$  is a base path

# Connection on $\mathcal{T}$ : General case

## Remarks (Integrated case)

Ansatz: Introduce connection on  $\mathcal{G}$ ,

$$\mathrm{PT}_{\gamma}^{\mathcal{T}}(p \cdot g) = \mathrm{PT}_{\gamma}^{\mathcal{T}}(p) \cdot \mathrm{PT}_{\gamma}^{\mathcal{G}}(g).$$

## Recovering the ordinary definition

- ①  $\mathcal{G} \cong L \times G$
- ② Equip  $\mathcal{G}$  with canonical flat connection

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## Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.]

A surjective submersion  $\pi: \mathcal{T} \rightarrow L$  so that one has a commuting diagram

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 \swarrow & \downarrow \pi_{\mathcal{G}} & \\
 \mathcal{T} & \xrightarrow{\pi} & L
 \end{array}$$

### 1 Ehresmann connection:

$$\text{PT}_{\gamma}^{\mathcal{T}}(p \cdot g) = \text{PT}_{\gamma}^{\mathcal{T}}(p) \cdot \text{PT}_{\gamma}^{\mathcal{G}}(g)$$

### 2 Yang-Mills connection: Additionally

$$\text{PT}_{\gamma_0}^{\mathcal{T}}(p) = p \cdot g_{\gamma_0}$$

for some  $g_{\gamma_0} \in \mathcal{G}_{\pi(p)}^0$ , where  $\gamma_0$  is a contractible loop.

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On  $\mathcal{G}$  there is also the notion of **multiplicative Yang-Mills connections**, that is,

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Consider the Atiyah sequence of a principal  $G$ -bundle  $P$ :

$$(P \times \mathfrak{g})/G \hookrightarrow TP/G \overset{\mathbb{H}}{\rightrightarrows} TL$$

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**Going back to foliations**

## Theorem ([C. L.-G., S.-R. F.])

*Given a multiplicative Yang-Mills connection on  $\mathcal{G}$  and a Yang-Mills connection  $\mathbb{H}$  on  $\mathcal{T}$ , then there is a natural foliation on  $\mathcal{T}$  generated by*

$$\mathbb{H}(X) + \bar{\nu},$$

*where  $X \in \mathfrak{X}(L)$  and  $\nu \in \Gamma(\mathfrak{g})$ .*

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We have

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Fix a point  $l \in L$  with transverse model  $(\mathbb{R}^d, \tau_l)$ :

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Recall

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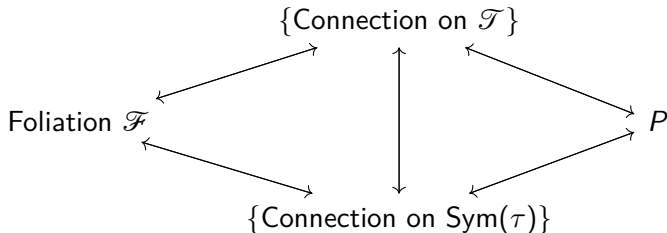
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# Main Theorems

## Theorem ([C. L.-G., S.-R. F.])

*In the simply connected case, the following are equivalent:*

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## Corollary

*$L$  simply connected and  $\tau_l$  is made of vector fields vanishing quadratically at 0. Then the unique singular foliation is the trivial one, i.e. the trivial product of  $(L, \mathfrak{X}(L))$  and  $(\mathbb{R}^d, \tau_l)$ .*

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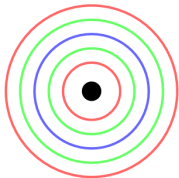
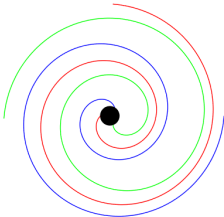
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## Corollary ([C. L.-G., S.-R. F.])

*$L$  simply connected. Then  $\mathcal{F}$  is the trivial foliation if and only if it admits a flat  $\mathcal{F}$ -connection.*

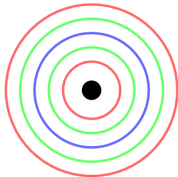
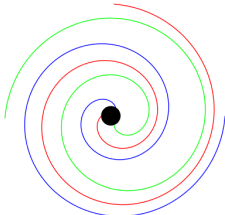
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$L = \mathbb{S}^2$ ,  $M = T\mathbb{S}^2$ . Let us consider two possible  $\tau_l$ :

Name	Concentric circles	Spirals
Generator	$x\partial_y - y\partial_x$	$x\partial_y - y\partial_x + (x^2 + y^2)(x\partial_x + y\partial_y)$
Picture of the leaves	 A diagram showing five concentric circles centered at a black dot. The circles are colored from innermost to outermost: red, blue, green, blue, and red.	 A diagram showing three spiral curves (red, blue, and green) that wind outwards from a central black dot. The spirals are interleaved, with the red spiral being the outermost and the blue spiral being the innermost.
$\text{Inner}(\tau_l)/\text{Inner}(\tau_l)_{\geq 2}$	$\mathbb{S}^1$	$\mathbb{R}$

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**Thank you!**