

Classification of neighbourhoods of leaves of singular foliations

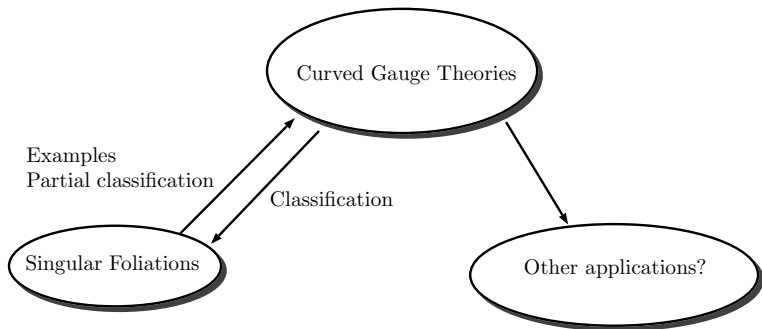
joint work with Camille Laurent-Gengoux
(Université de Lorraine)

Simon-Raphael Fischer



國家理論科學研究中心

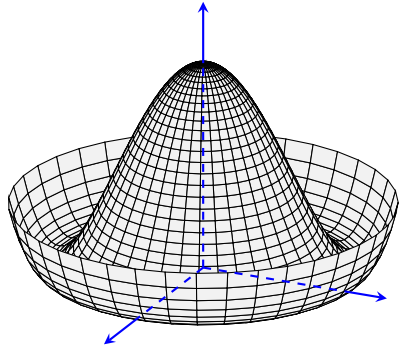
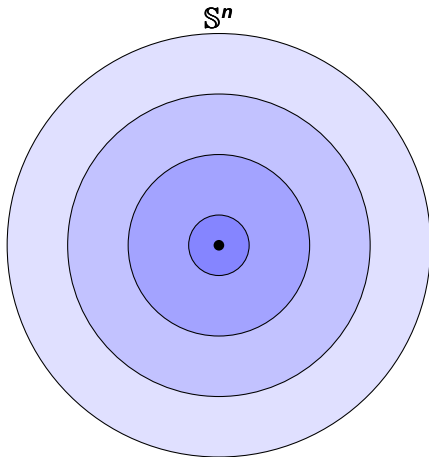
National Center for Theoretical Sciences (National Taiwan University)



Remarks (Based on the following work)

S.-R. F. and Camille Laurent-Gengoux, *Classification of neighborhoods around leaves of singular foliations*, arXiv:2401.05966, (2024).

Singular Foliations



Singular Foliations:

- Gauge Theory
- Poisson Geometry
(Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

First idea

Definition (Partitionifolds)

Let M be a smooth manifold. A **partitionifold of M** is a partition of immersed connected submanifolds, which we call *leaves*.

Remarks

We will denote a partitionifold by L_\bullet , $p \mapsto L_p$, where L_p is the leaf through $p \in M$.

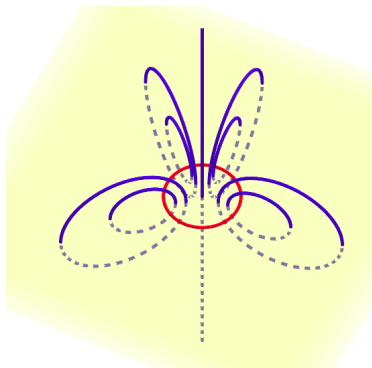


Figure: The magnetic partition

Remarks

A partitionifold with:

- All leaves are of the same dimension.
- **But:** It lacks regularity!

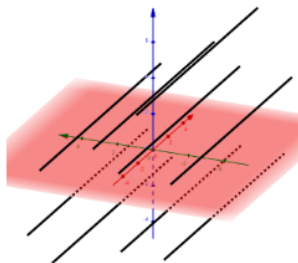


Figure: Isolated lasagna in a spaghetti dish

Remarks

A partitionifold with:

- Dimension is now different.
- **But:** Also no regularity!

Remarks

Isolated spaghetti in a lasagna dish: Regularity!

Definition

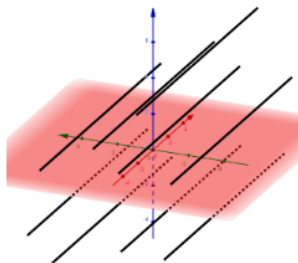


Figure: Isolated lasagna in a spaghetti dish

Remarks

A partitionifold with:

- Dimension is now different.
- **But:** Also no regularity!

Remarks

Isolated spaghetti in a lasagna dish: Regularity!

Definition (Smooth partitionifold)

A smooth partitionifold L_\bullet is smooth, if there is for all $p \in M$ and every vector $u \in T_p L_p$ a vector field X tangent to L_\bullet with

$$X_p = u.$$

Remarks

This definition is okay, but not widely used: It still has a problem...

Definition (Smooth partitionifold)

A smooth partitionifold L_\bullet is smooth, if there is for all $p \in M$ and every vector $u \in T_p L_p$ a vector field X tangent to L_\bullet with

$$X_p = u.$$

Remarks

This definition is okay, but not widely used: It still has a problem...

Example (Vector fields not necessarily finitely generated)

Consider the following smooth partition of \mathbb{R} :

- $M = \mathbb{R}$;
- 0-dimensional leaves:

$$\{1\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \dots, \left\{\frac{1}{n}\right\}, \dots, \{0\};$$

- 1-dimensional leaves: Remaining open intervals.

Remarks

One has a sort of “infinitesimal leaf” next to $\{0\}$.

Technically: Tangent vectors of L_\bullet are locally not finitely generated around 0.

Example (Vector fields not necessarily finitely generated)

Consider the following smooth partitionifold:

- $M = \mathbb{R}$;
- 0-dimensional leaves:

$$\{1\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \dots, \left\{\frac{1}{n}\right\}, \dots, \{0\};$$

- 1-dimensional leaves: Remaining open intervals.

Remarks

One has a sort of “infinitesimal leaf” next to $\{0\}$.

Technically: Tangent vectors of L_\bullet are locally not finitely generated around 0.

Recall:

Theorem (Frobenius Theorem)

Every integrable subbundle E of TM corresponds to a regular foliation in M .

Remarks ($\Gamma(E)$ is involutive)

Integrable:

$$[X, Y] \in \Gamma(E)$$

for all $X, Y \in \Gamma(E)$.

Remarks

Alternatively: An involutive submodule of $\mathfrak{X}(M)$, or equivalently of $\mathfrak{X}_c(M)$.

Recall:

Theorem (Frobenius Theorem)

Every integrable subbundle E of TM corresponds to a regular foliation in M .

Remarks ($\Gamma(E)$ is involutive)

Integrable:

$$[X, Y] \in \Gamma(E)$$

for all $X, Y \in \Gamma(E)$.

Remarks

Alternatively: An involutive submodule of $\mathfrak{X}(M)$, or equivalently of $\mathfrak{X}_c(M)$.

Definition (Smooth singular foliation)

A **smooth singular foliation** \mathcal{F} on a smooth manifold M is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$ -**multiplication**,
- it is **locally finitely generated**.

Definition (Smooth singular foliation)

A **smooth singular foliation** \mathcal{F} on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**, i.e. $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$,
- it is **stable under** $C^\infty(M)$ -**multiplication**,
- it is **locally finitely generated**.

Definition (Smooth singular foliation)

A **smooth singular foliation** \mathcal{F} on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**, i.e. $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$,
- it is **stable under $C^\infty(M)$ -multiplication**, i.e. $fX \in \mathcal{F}$ for all $f \in C^\infty(M)$ and $X \in \mathcal{F}$,
- it is **locally finitely generated**.

Definition (Smooth singular foliation)

A **smooth singular foliation** \mathcal{F} on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**, i.e. $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$,
- it is **stable under $C^\infty(M)$ -multiplication**, i.e. $fX \in \mathcal{F}$ for all $f \in C^\infty(M)$ and $X \in \mathcal{F}$,
- it is **locally finitely generated**, i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ ($X^i \in \mathcal{F}$) such that for all $X \in \mathcal{F}$ there are $f_i \in C^\infty(M)$ satisfying on U .

$$X = \sum_i f_i X^i.$$

Remarks (Leaves)

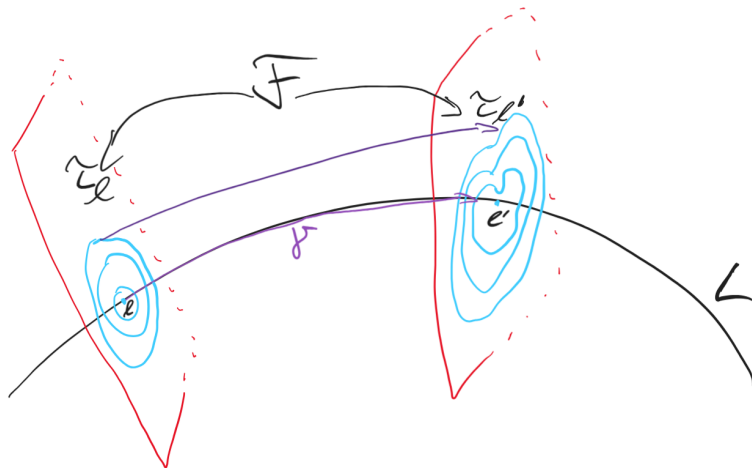
We have an induced smooth partitionifold L_\bullet ,

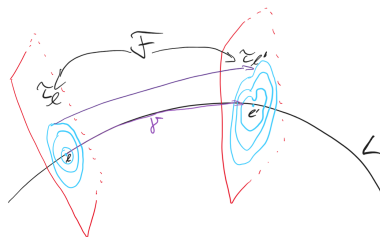
$$\mathcal{F} \Rightarrow L_\bullet,$$

but \mathcal{F} also encodes the information about the generators,

$$\begin{array}{ccc} \mathcal{F}_1 & \searrow & \\ & L_\bullet & \\ \mathcal{F}_2 & \nearrow & \end{array}$$

First step towards classification



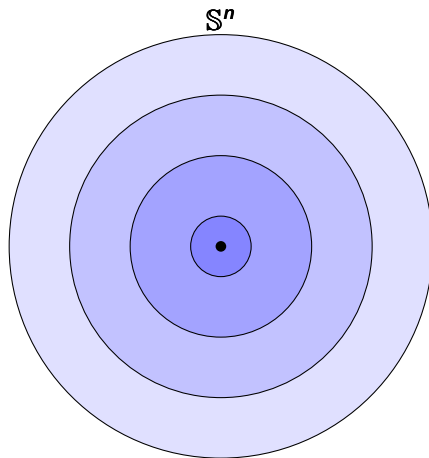


Theorem (\mathcal{F} -connections)

There is a connection on the normal bundle of a leaf L :

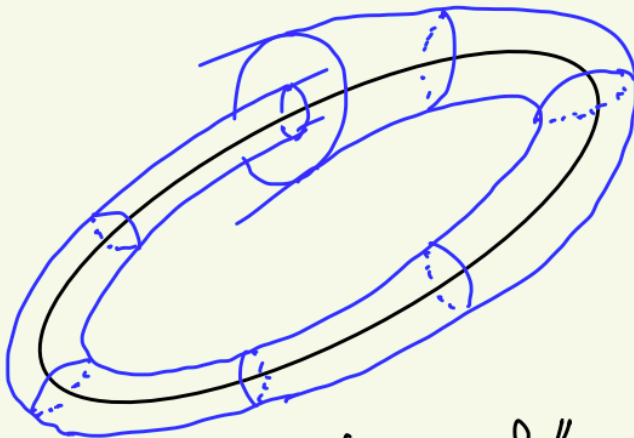
- *Horizontal vector fields are in \mathcal{F} .*
- *Parallel transport PT_γ has values in $\text{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l : PT_{γ_0} values in $\text{Inner}(\tau_l)$.*

Example of a transverse foliation τ in \mathbb{R}^d :



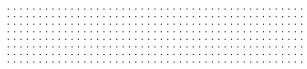
Remarks

- $\text{Inner}(\tau_I)$ maps each circle to itself
- $\text{Sym}(\tau_I)$ allows to exchange circles
- Both preserve τ_I and fix the origin



The "self eating snake"

Other example: Regular foliation



Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts

Remarks

$\text{Inner}(\tau_I)$: Trivial.

$\text{Sym}(\tau_I)$: We essentially need the image of a group morphism

$$\pi_1(L) \rightarrow \text{Diff}(\mathbb{R}^d, 0).$$

Idea

We guess:

$$\mathcal{F} = \left\{ \begin{array}{l} \text{Some map } \pi_1(L) \rightarrow \text{Diff}(\mathbb{R}^d, 0) \text{ (at } l) \\ \text{Bundle structure by } \tau_l, \text{Inner}(\tau_l), \text{Sym}(\tau_l), \dots \end{array} \right\}$$

Thus, we want to classify \mathcal{F} with given L and τ_l (for a fixed $l \in L$).

Danger

But $\text{Sym}(\tau_l)$ and $\text{Inner}(\tau_l)$ are in general infinite-dimensional, so that we have to deal with infinite-dimensional geometry!

Idea

We guess:

$$\mathcal{F} = \left\{ \begin{array}{l} \text{Some map } \pi_1(L) \rightarrow \text{Diff}(\mathbb{R}^d, 0) \text{ (at } l) \\ \text{Bundle structure by } \tau_l, \text{Inner}(\tau_l), \text{Sym}(\tau_l), \dots \end{array} \right\}$$

Thus, we want to classify \mathcal{F} with given L and τ_l (for a fixed $l \in L$).

Danger

But $\text{Sym}(\tau_l)$ and $\text{Inner}(\tau_l)$ are in general infinite-dimensional, so that we have to deal with infinite-dimensional geometry!

Definition (Formal foliation)

$X \in \mathcal{F}$ induces a derivation on $\hat{C} := C^\infty(M)/C_0^\infty(M)$, where $C_0^\infty(M)$ is the ideal of functions vanishing with all their derivatives along L . The image of \mathcal{F} under this is the **formal singular foliation**.

Remarks

$f \in \hat{C}$ a formal power series, w.r.t. (x_1, \dots, x_d) as “normal coordinates”:

$$f = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}$$

with $f_{i_1, \dots, i_d} \in C^\infty(L)$.

Remarks

$f \in \hat{C}$ a formal power series, w.r.t. (x_1, \dots, x_d) as “normal coordinates”:

$$f = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}$$

with $f_{i_1, \dots, i_d} \in C^\infty(L)$.

Example (Canonical example of a formal foliation)

For embedded submanifolds L :

- Normal bundle $\mathcal{T}: TM|_L/TL$.
- Formal vector fields via tubular neighbourhood embedding, a Lie algebra morphism $\mathfrak{X}(M) \rightarrow \mathfrak{X}^{\text{formal}}$.

Remarks

$f \in \hat{C}$ a formal power series, w.r.t. (x_1, \dots, x_d) as “normal coordinates”:

$$f = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}$$

with $f_{i_1, \dots, i_d} \in C^\infty(L)$.

Example (Canonical example of a formal foliation)

For embedded submanifolds L :

- Normal bundle $\mathcal{T}: TM|_L/TL$.
- Formal vector fields via tubular neighbourhood embedding, a Lie algebra morphism $\mathfrak{X}(M) \rightarrow \mathfrak{X}^{\text{formal}}$.
- Formal version of \mathcal{F} via this map.

Remarks

$f \in \hat{\mathcal{C}}$ a formal power series, w.r.t. (x_1, \dots, x_d) as “normal coordinates”:

$$f = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}$$

with $f_{i_1, \dots, i_d} \in C^\infty(L)$.

Example (Canonical example of a formal foliation)

For embedded submanifolds L :

- Normal bundle $\mathcal{T}: TM|_L/TL$.
- Formal vector fields via tubular neighbourhood embedding, a Lie algebra morphism $\mathfrak{X}(M) \rightarrow \mathfrak{X}^{\text{formal}}$.
- Formal version of \mathcal{F} via this map.
- In particular: Formal version of $X \in \mathcal{F}$ can be evaluated at $l \in L$, coinciding with X_l .

Remarks

$f \in \hat{C}$ a formal power series, w.r.t. (x_1, \dots, x_d) as “normal coordinates”:

$$f = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}$$

with $f_{i_1, \dots, i_d} \in C^\infty(L)$.

Example (Canonical example of a formal foliation)

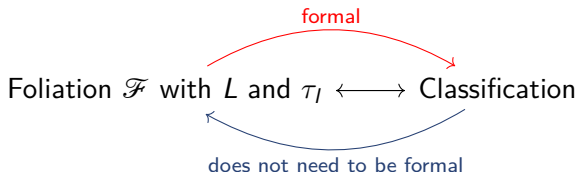
For embedded submanifolds L :

- Normal bundle $\mathcal{T}: TM|_L/TL$.
- Formal vector fields via tubular neighbourhood embedding, a Lie algebra morphism $\mathfrak{X}(M) \rightarrow \mathfrak{X}^{\text{formal}}$.
- Formal version of \mathcal{F} via this map.
- In particular: Formal version of $X \in \mathcal{F}$ can be evaluated at $l \in L$, coinciding with X_l .

Our aim

Remarks (Our assumptions)

- τ_I a formal singular foliation.
- L a manifold (connected immersed submanifold of M).



Remarks (Avoiding formal setting)

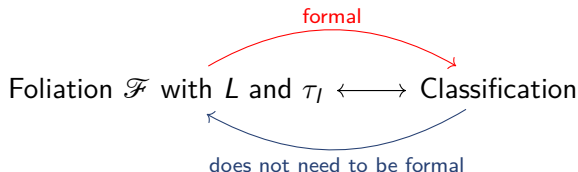
Either

- add real-analyticity conditions to the classification,
- or also
- assume embedded L and real-analytic \mathcal{F} .

Our aim

Remarks (Our assumptions)

- τ_I a formal singular foliation.
- L a manifold (connected immersed submanifold of M).

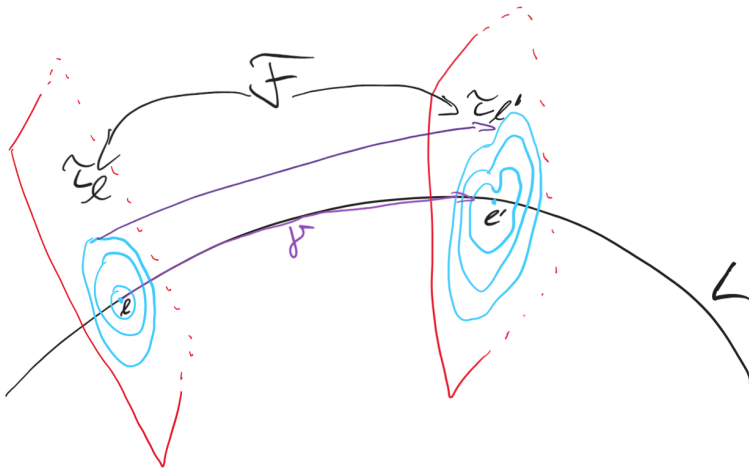


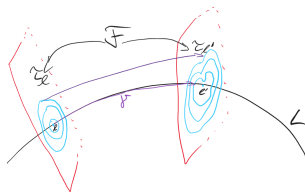
Remarks (Avoiding formal setting)

Either

- add real-analyticity conditions to the classification,
- or also
- assume embedded L and real-analytic \mathcal{F} .

Multiplicative Yang-Mills connections





Remarks (\mathcal{F} -connection)

For $\phi \in \text{Sym}(\tau_I)$ we have an induced parallel transport

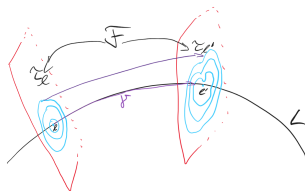
$$\text{PT}_\gamma^{\text{Sym}}(\phi) := \text{PT}_\gamma \circ \phi \circ \text{PT}_\gamma^{-1}.$$

Then, on the normal bundle $\pi: \mathcal{T} \rightarrow L$,

$$\text{PT}_\gamma(\phi \cdot p) = \text{PT}_\gamma^{\text{Sym}}(\phi) \cdot \text{PT}_\gamma(p)$$

$$\text{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all $p \in \mathcal{T}_I$, $\phi \in \text{Sym}(\tau_I)$, and for some $\varphi \in \text{Inner}(\tau_I)$.



Remarks (Sym-connection)

For $\phi \in \text{Sym}(\tau_I)$ we have an induced parallel transport

$$\text{PT}_\gamma^{\text{Sym}}(\phi) := \text{PT}_\gamma \circ \phi \circ \text{PT}_\gamma^{-1}.$$

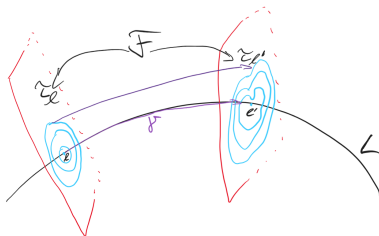
Then

$$\text{PT}_\gamma^{\text{Sym}}(\phi \circ \phi') = \text{PT}_\gamma^{\text{Sym}}(\phi) \circ \text{PT}_\gamma^{\text{Sym}}(\phi')$$

$$\text{PT}_{\gamma_0}^{\text{Sym}}(\phi) = \varphi \circ \phi \circ \varphi^{-1}$$

for all $\phi, \phi' \in \text{Sym}(\tau_I)$, and for some $\varphi \in \text{Inner}(\tau_I)$.

Idea

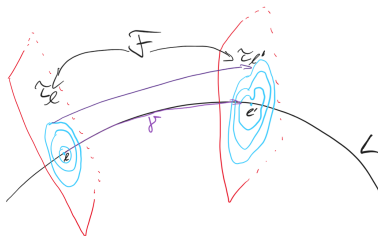


Idea

Generators of \mathcal{F} given by:

$$\mathbb{H}(X) + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift,
 $\nu \in \Gamma(\text{inner}(\tau))$ and $\bar{\nu}$ its fundamental vector field.



Idea

Fix $I \in L$, given τ and \mathbb{H} . Reconstruct \mathcal{F} .

$$\begin{aligned}
 [\mathbb{H}(X) + \bar{\nu}, \mathbb{H}(X') + \bar{\mu}] &= \mathbb{H}([X, X']) + \dots \\
 &= [\mathbb{H}(X), \mathbb{H}(X')] \\
 &\quad + [\mathbb{H}(X), \bar{\mu}] - [\mathbb{H}(X'), \bar{\nu}] + \overline{[\nu, \mu]}
 \end{aligned}$$

Idea

Fix I and given τ_I : Reconstruct \mathcal{F} .

$$\begin{aligned}
 [\mathbb{H}(X) + \bar{\nu}, \mathbb{H}(X') + \bar{\mu}] &= \mathbb{H}([X, X']) + \dots \\
 &= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{curvature}} \\
 &\quad + \underbrace{[\mathbb{H}(X), \bar{\mu}] - [\mathbb{H}(X'), \bar{\nu}]}_{\rightsquigarrow \text{connection}} + [\bar{\nu}, \bar{\mu}]
 \end{aligned}$$

Idea ($\dots \in \tau$)

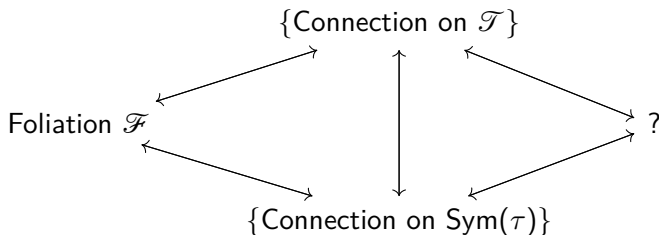
We need:

- ① Lie algebra bundle τ with structure τ_I
- ② A horizontal lift \mathbb{H} into \mathcal{F} satisfying

$$\text{Curvature:} \quad [\mathbb{H}(X), \mathbb{H}(X')] - \mathbb{H}([X, X']) \in \tau$$

$$\text{Connection:} \quad [\mathbb{H}(X), \bar{\mu}] \in \tau$$

Summary



Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given \mathcal{F} !

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle \mathcal{G}

$$\begin{array}{ccc}
 G & \longrightarrow & \mathcal{G} \\
 & & \downarrow \\
 & & L
 \end{array}$$

Motivation

What are Ehresmann connections, preserving \mathcal{G} -actions?

Definition (LGB actions)

$$\begin{array}{ccc} \mathcal{G} & & \\ \downarrow \pi_{\mathcal{G}} & \searrow & \\ L & \xleftarrow{\pi} & \mathcal{T} \end{array}$$

A **left-action of \mathcal{G} on \mathcal{T}** is a smooth map

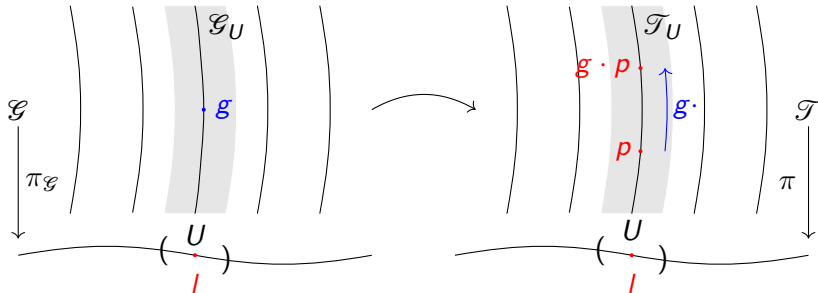
$\mathcal{G} * \mathcal{T} := \mathcal{G}_{\pi_{\mathcal{G}}} \times_{\pi} \mathcal{T} \rightarrow \mathcal{T}$, $(g, p) \mapsto g \cdot p$, satisfying the following properties:

$$\begin{aligned} \pi(g \cdot p) &= \pi(p), \\ h \cdot (g \cdot p) &= (hg) \cdot p, \\ e_{\pi(p)} \cdot p &= p \end{aligned}$$

for all $p \in \mathcal{T}$ and $g, h \in \mathcal{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathcal{G}_{\pi(p)}$.

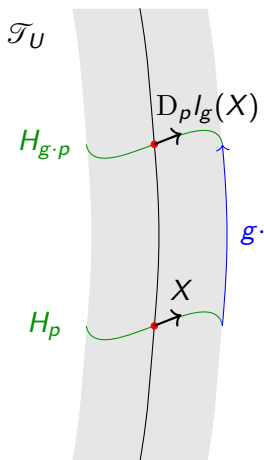
Connections as parallel transport

Connection on \mathcal{T} : Idea



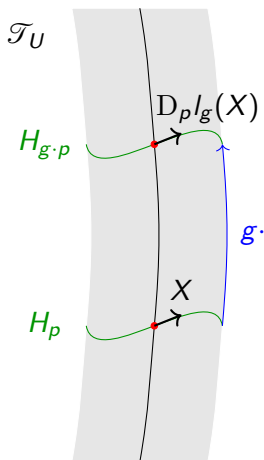
Connection on \mathcal{T} : Revisiting the classical setup

If \mathcal{G} is trivial, and H a connection:



Connection on \mathcal{T} : Revisiting the classical setup

If \mathcal{G} is trivial, and H a connection:



Remarks (Integrated case)

Parallel transport $\text{PT}_\gamma^{\mathcal{T}}$ in \mathcal{T} :

$$\text{PT}_\gamma^{\mathcal{T}}(g \cdot p) = g \cdot \text{PT}_\gamma^{\mathcal{T}}(p),$$

where $\gamma : I \rightarrow L$ is a base path

Connection on \mathcal{T} : General case

Remarks (Integrated case)

Ansatz: Introduce connection on \mathcal{G} ,

$$\mathrm{PT}_{\gamma}^{\mathcal{T}}(g \cdot p) = \mathrm{PT}_{\gamma}^{\mathcal{G}}(g) \cdot \mathrm{PT}_{\gamma}^{\mathcal{T}}(p).$$

Recovering the ordinary definition

- ① $\mathcal{G} \cong L \times G$
- ② Equip \mathcal{G} with canonical flat connection

Connection on \mathcal{T} : General case

Remarks (Integrated case)

Ansatz: Introduce connection on \mathcal{G} ,

$$\mathrm{PT}_{\gamma}^{\mathcal{T}}(g \cdot p) = \mathrm{PT}_{\gamma}^{\mathcal{G}}(g) \cdot \mathrm{PT}_{\gamma}^{\mathcal{T}}(p).$$

Recovering the ordinary definition

- ① $\mathcal{G} \cong L \times G$
- ② Equip \mathcal{G} with canonical flat connection

Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.]

A surjective submersion $\pi: \mathcal{T} \rightarrow L$ so that one has a commuting diagram

$$\begin{array}{ccc}
 \mathcal{G} & & \\
 \downarrow \pi_{\mathcal{G}} & \searrow & \\
 L & \xleftarrow{\pi} & \mathcal{T}
 \end{array}$$

1 Ehresmann connection:

$$\mathrm{PT}_{\gamma}^{\mathcal{T}}(g \cdot p) = \mathrm{PT}_{\gamma}^{\mathcal{G}}(g) \cdot \mathrm{PT}_{\gamma}^{\mathcal{T}}(p)$$

2 Yang-Mills connection: Additionally

$$\mathrm{PT}_{\gamma_0}^{\mathcal{T}}(p) = g_{\gamma_0} \cdot p$$

for some $g_{\gamma_0} \in \mathcal{G}_{\pi(p)}^0$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.]

On \mathcal{G} there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\begin{aligned} \text{PT}_{\gamma}^{\mathcal{G}}(q \cdot g) &= \text{PT}_{\gamma}^{\mathcal{G}}(q) \cdot \text{PT}_{\gamma}^{\mathcal{G}}(g), \\ \text{PT}_{\gamma_0}^{\mathcal{G}}(q) &= g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1} \end{aligned}$$

Remarks

Compare this with the Maurer-Cartan form and its curvature equation!

Definition (Multiplicative YM connection, [S.-R. F.]

On \mathcal{G} there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\begin{aligned} \text{PT}_{\gamma}^{\mathcal{G}}(q \cdot g) &= \text{PT}_{\gamma}^{\mathcal{G}}(q) \cdot \text{PT}_{\gamma}^{\mathcal{G}}(g), \\ \text{PT}_{\gamma_0}^{\mathcal{G}}(q) &= g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1} \end{aligned}$$

Remarks

Compare this with the Maurer-Cartan form and its curvature equation!

On the Lie algebra bundle \mathfrak{g} we have a connection ∇ with

$$\begin{aligned}\nabla([\mu, \nu]_g) &= [\nabla\mu, \nu]_g + [\mu, \nabla\nu]_g, \\ R_\nabla &= \text{ad} \circ \zeta.\end{aligned}$$

Consider the Atiyah sequence of a principal G -bundle P :

$$g := (P \times \mathfrak{g})/G \hookrightarrow TP/G \overset{\text{H}}{\rightrightarrows} TL$$

with splitting $\mathbb{H}: TL \rightarrow TP/G$, where \mathfrak{g} is the Lie algebra. Then

$$\begin{aligned}\nabla_X \nu &= [\mathbb{H}(X), \nu]_{TP/G}, \\ \zeta(X, X') &= [\mathbb{H}(X), \mathbb{H}(X')]_{TP/G} - \mathbb{H}([X, X']).\end{aligned}$$

Remarks

On the Lie algebra bundle \mathfrak{g} we have a connection ∇ with

$$\begin{aligned}\nabla([\mu, \nu]_{\mathfrak{g}}) &= [\nabla\mu, \nu]_{\mathfrak{g}} + [\mu, \nabla\nu]_{\mathfrak{g}}, \\ R_{\nabla} &= \text{ad} \circ \zeta.\end{aligned}$$

Example

Consider the Atiyah sequence of a principal G -bundle P :

$$\mathfrak{g} := (P \times \mathfrak{g})/G \hookrightarrow TP/G \overset{\mathbb{H}}{\longleftarrow} TL$$

with splitting $\mathbb{H}: TL \rightarrow TP/G$, where \mathfrak{g} is the Lie algebra. Then

$$\begin{aligned}\nabla_X \nu &= [\mathbb{H}(X), \nu]_{TP/G}, \\ \zeta(X, X') &= [\mathbb{H}(X), \mathbb{H}(X')]_{TP/G} - \mathbb{H}([X, X']).\end{aligned}$$

Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on \mathcal{G} and a Yang-Mills connection \mathbb{H} on \mathcal{T} , then there is a natural foliation on \mathcal{T} generated by

$$\mathbb{H}(X) + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathfrak{g})$.

Proof.

We have

$$[\mathbb{H}(X), \bar{\nu}] = \overline{\nabla_X \nu},$$

$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathfrak{g})$.

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on \mathcal{G} and a Yang-Mills connection \mathbb{H} on \mathcal{T} , then there is a natural foliation on \mathcal{T} generated by

$$\mathbb{H}(X) + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathfrak{g})$.

Proof.

We have

$$[\mathbb{H}(X), \bar{\nu}] = \overline{\nabla_X \nu},$$

$$[\mathbb{H}(X), \mathbb{H}(X')] = \mathbb{H}([X, X']) + \overline{\zeta(X, X')},$$

where $\zeta \in \Omega^2(L; \mathfrak{g})$.

Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

- 1 $G = \text{Inn}(\tau_l)$
- 2 P a principal G -bundle, equipped with an ordinary connection

Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

- ① $G = \text{Inn}(\tau_l)$
- ② P a principal G -bundle, equipped with an ordinary connection
- ③ $\mathcal{G} := (P \times G) / G$, the **inner group bundle**

Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

- ① $G = \text{Inn}(\tau_l)$
- ② P a principal G -bundle, equipped with an ordinary connection
- ③ $\mathcal{G} := (P \times G) / G$, the **inner group bundle**
- ④ $\mathcal{T} := (P \times \mathbb{R}^d) / G$, the **normal bundle**

Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

- ① $G = \text{Inn}(\tau_l)$
- ② P a principal G -bundle, equipped with an ordinary connection
- ③ $\mathcal{G} := (P \times G) / G$, the **inner group bundle**
- ④ $\mathcal{T} := (P \times \mathbb{R}^d) / G$, the **normal bundle**

Remarks

- Think of the induced connection on \mathcal{T} as the \mathcal{F} -connection.
- \mathcal{G} acts on \mathcal{T} (canonically from the left).

Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

- ① $G = \text{Inn}(\tau_l)$
- ② P a principal G -bundle, equipped with an ordinary connection
- ③ $\mathcal{G} := (P \times G) / G$, the **inner group bundle**
- ④ $\mathcal{T} := (P \times \mathbb{R}^d) / G$, the **normal bundle**

Remarks

- Think of the induced connection on \mathcal{T} as the \mathcal{F} -connection.
- \mathcal{G} acts on \mathcal{T} (canonically from the left).

The associated connection on \mathcal{G} is a multiplicative Yang-Mills connection and the one on \mathcal{T} is a corresponding Yang-Mills connection.

Proof.

Recall

$$[p, g] \cdot [p, v] = [p, g \cdot v]$$

for all $[p, g] \in \mathcal{G}$ and $[p, v] \in \mathcal{T}$, and

$$\text{PT}_{\gamma}^{\mathcal{T}}([p, v]) = [\text{PT}_{\gamma}^P(p), v].$$

Remarks

Thus, we have a singular foliation on \mathcal{T} , which, by construction, admits L as a leaf and τ_L as transverse data.

Proposition ([C. L.-G., S.-R. F.])

The associated connection on \mathcal{G} is a multiplicative Yang-Mills connection and the one on \mathcal{T} is a corresponding Yang-Mills connection.

Proof.

Recall

$$[p, g] \cdot [p, v] = [p, g \cdot v]$$

for all $[p, g] \in \mathcal{G}$ and $[p, v] \in \mathcal{T}$, and

$$\text{PT}_{\gamma}^{\mathcal{T}}([p, v]) = [\text{PT}_{\gamma}^P(p), v].$$

Remarks

Thus, we have a singular foliation on \mathcal{T} , which, by construction, admits L as a leaf and τ_l as transverse data.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P .

Proof.

- The adjoint bundle of P , $\text{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of \mathcal{G}
- $\tau = \overline{\text{Ad}(P)}$
- Difference of two connections on P has values in $\text{Ad}(P)$

Generators take this already into account; recall:

$$\mathbb{H}(X) + \bar{\nu}$$

with $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\text{Ad}(P))$.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P .

Proof.

- The adjoint bundle of P , $\text{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of \mathcal{G}
- $\tau = \overline{\text{Ad}(P)}$
- Difference of two connections on P has values in $\text{Ad}(P)$

Generators take this already into account; recall:

$$\mathbb{H}(X) + \bar{\nu}$$

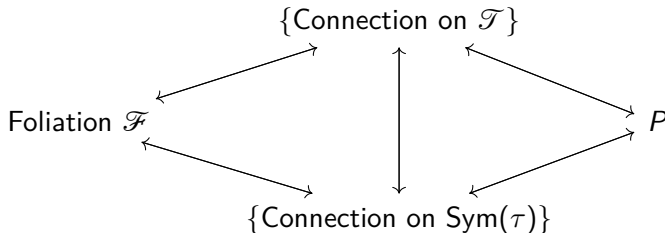
with $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\text{Ad}(P))$.

Main Theorems

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_l)*
- *Principal $\text{Inner}(\tau_l)$ -bundles P over L*



Main Theorems

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_L)*
- *Principal $\text{Inner}(\tau_L)/\text{Inner}(\tau_L)_{\geq 2}$ -bundles P over L*

Remarks

P is now finite-dimensional!

Main Theorems

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_L)*
- *Principal $\text{Inner}(\tau_L)/\text{Inner}(\tau_L)_{\geq 2}$ -bundles P over L*

Remarks

P is now finite-dimensional!

Danger

Formal setting is here very important! Especially here is where one has to think about real-analytic conditions for the general case.

Main Theorems

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_L)*
- *Principal $\text{Inner}(\tau_L)/\text{Inner}(\tau_L)_{\geq 2}$ -bundles P over L*

Remarks

P is now finite-dimensional!

Danger

Formal setting is here very important! Especially here is where one has to think about real-analytic conditions for the general case.

Examples

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_l)*
- *Principal $\text{Inner}(\tau_l)/\text{Inner}(\tau_l)_{\geq 2}$ -bundles P over L*

Corollary

L simply connected and τ_l is made of vector fields vanishing quadratically at 0. Then the unique singular foliation is the trivial one, i.e. the trivial product of $(L, \mathfrak{X}(L))$ and (\mathbb{R}^d, τ_l) .

Examples

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_l)*
- *Principal $\text{Inner}(\tau_l)/\text{Inner}(\tau_l)_{\geq 2}$ -bundles P over L*

Corollary ([C. L.-G., S.-R. F.])

L contractible. Then the unique singular foliation is the trivial one.

Examples

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

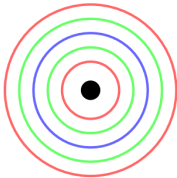
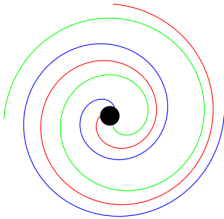
- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_l)*
- *Principal $\text{Inner}(\tau_l)/\text{Inner}(\tau_l)_{\geq 2}$ -bundles P over L*

Corollary ([C. L.-G., S.-R. F.])

L simply connected. Then \mathcal{F} is the trivial foliation if and only if it admits a flat \mathcal{F} -connection.

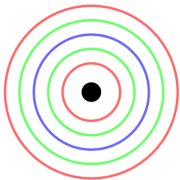
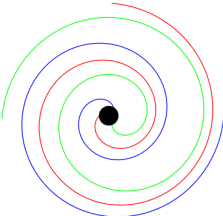
Examples [C. L.-G., S.-R. F.]

$L = \mathbb{S}^2$, $M = \text{TS}^2$. Let us consider two possible τ_l :

Name	Concentric circles	Spirals
Generator	$x\partial_y - y\partial_x$	$x\partial_y - y\partial_x + (x^2 + y^2)(x\partial_x + y\partial_y)$
Picture of the leaves		
$\text{Inner}(\tau_l)/\text{Inner}(\tau_l)_{\geq 2}$	\mathbb{S}^1	\mathbb{R}

Examples [C. L.-G., S.-R. F.]

$L = \mathbb{S}^2$, $M = T\mathbb{S}^2$. Let us consider two possible τ_l :

Name	Concentric circles	Spirals
Generator	$x\partial_y - y\partial_x$	$x\partial_y - y\partial_x + (x^2 + y^2)(x\partial_x + y\partial_y)$
Picture of the leaves	 A diagram showing several concentric circles in red and blue, centered around a black dot, representing the leaves of a foliation.	 A diagram showing several spirals in red and blue, winding outwards from a black dot, representing the leaves of a foliation.
$\text{Inner}(\tau_l)/\text{Inner}(\tau_l)_{\geq 2}$	\mathbb{S}^1	\mathbb{R}
Foliation	☺	☹

Thank you! 😊

Total classification

Remarks

$\text{Inner}(\tau_I)$ is a normal subgroup of $\text{Sym}(\tau_I)$, thus we have a quotient:

$$\text{Inner}(\tau_I) \hookrightarrow \text{Sym}(\tau_I) \twoheadrightarrow \text{Out}(\tau_I)$$

Theorem ([C. L.-G., S.-R. F.])

The following are equivalent:

- *Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_I) ,*
- $\textcircled{1}$ *a group morphism $\Xi: \pi_1(L) \rightarrow \text{Out}(\tau_I)$, and*
- $\textcircled{2}$ *a finite-dimensional principal $H/\text{Inner}(\tau_I)_{\geq 2}$ -bundle, with H a subgroup of $\text{Sym}(\tau_I)$ containing $\text{Inner}(\tau_I)$.*