Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux
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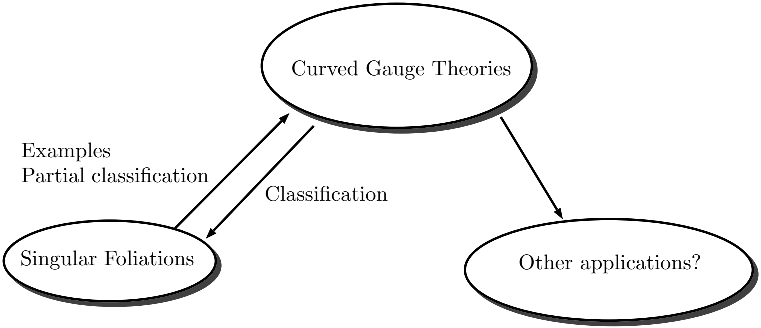
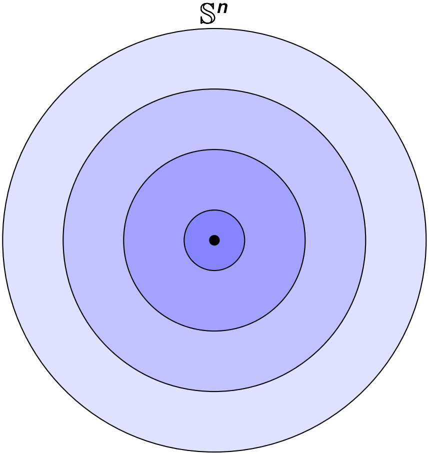
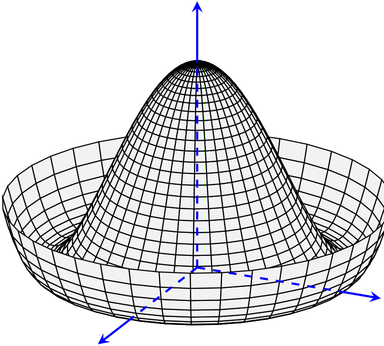


Remarks (Based on the following works)

1. S.-R. F. and Camille Laurent-Gengoux, *Classification of neighborhoods around leaves of singular foliations*, arXiv:2401.05966, (2024).
2. S.-R. F., Integrating curved Yang–Mills gauge theories, arXiv:2210.02924, (2022).

Singular Foliations





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

First idea

Definition (Partitionifolds)

Let M be a smooth manifold. A **partitionifold of** M is a partition of immersed connected submanifolds, which we call *leaves*.

Remarks

We will denote a partitionifold by $L_\bullet$, $p \mapsto L_p$, where L_p is the leaf through $p \in M$.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

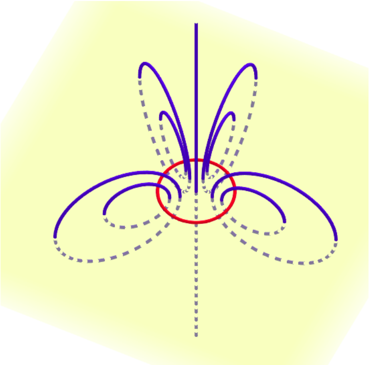


Figure: The magnetic partition

Remarks

A partitionifold with:

- All leaves are of the same dimension.
- **But:** It lacks regularity!

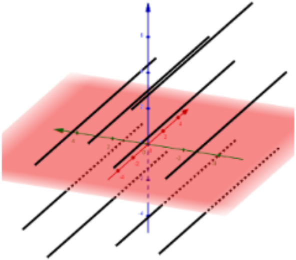


Figure: Isolated lasagna in a spaghetti dish

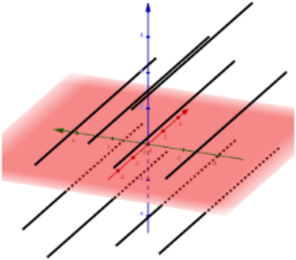
Remarks

A partitionifold with:

- Dimension is now different.
- **But:** Also no regularity!

Remarks

Isolated spaghetti in a lasagna dish: Regularity!



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Remarks

Isolated spaghetti in a lasagna dish: Regularity!

Definition (Smooth partitionifold)

A smooth partitionifold $L_\bullet$ is smooth, if there is for all $p \in M$ and every vector $u \in \mathrm{T}_p L_p$ a vector field X tangent to $L_\bullet$ with

$$X_p = u.$$

Remarks

This definition is okay, but not widely used: It still has a problem...

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Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.

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Example (Vector fields not necessarily finitely generated)

Consider the following smooth partitionifold:

- $M = \mathbb{R}$;
- 0-dimensional leaves:

$$\{1\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \dots, \left\{\frac{1}{n}\right\}, \dots, \{0\};$$

- 1-dimensional leaves: Remaining open intervals.

Remarks

One has a sort of "infinitesimal leaf" next to $\{0\}$.
Technically: Tangent vectors of $L_\bullet$ are locally not finitely generated around 0.

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Recall:

Theorem (Frobenius Theorem)

Every integrable subbundle E of $\mathrm{T}M$ corresponds to a regular foliation in M.

Remarks ($\Gamma(E)$ is involutive)

Integrable:

$$[X, Y] \in \Gamma(E)$$

for all $X, Y \in \Gamma(E)$.

Remarks

Alternatively: An involutive submodule of $\mathfrak{X}(M)$, or equivalently of $\mathfrak{X}_c(M)$.

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A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold M is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

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- it is **stable under $C^\infty(M)$-multiplication**, *i.e.* $fX \in \mathscr{F}$ *for all* $f \in C^\infty(M)$ *and* $X \in \mathscr{F}$,
- it is **locally finitely generated**, *i.e. around each* $p \in M$ *there is an open neighbourhood* U *and a finite family* $(X^i)_i^r$ $(X^i \in \mathscr{F})$ *such that for all* $X \in \mathscr{F}$ *there are* $f_i \in C^\infty(M)$ *satisfying on* U.

$$X = \sum_i f_i X^i.$$

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Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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Remarks (Leaves)

We have an induced smooth partitionifold $L_\bullet$,

$$\mathscr{F} \Rightarrow L_\bullet,$$

but $\mathscr{F}$ also encodes the information about the generators,

$$\begin{tikzcd}
\mathscr{F}_1 \arrow[dr, Rightarrow] & \\
& L_\bullet \\
\mathscr{F}_2 \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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Why happens without being locally finitely generated?

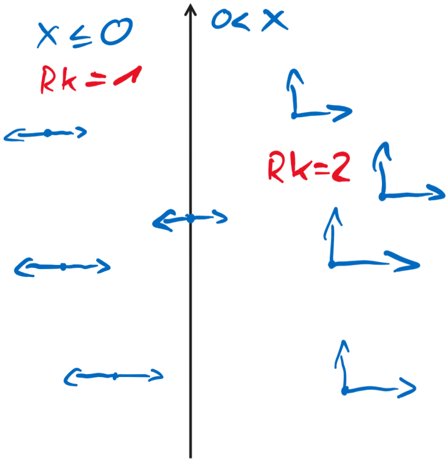
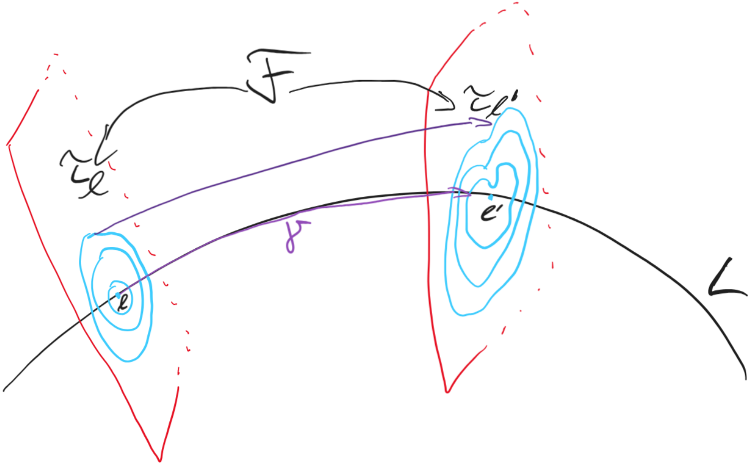


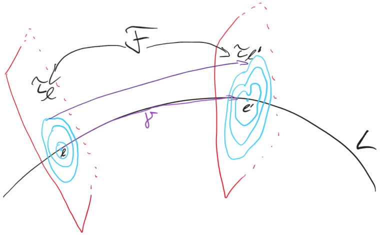
Figure: Infinite comb

First step towards classification

First steps



Sources Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.



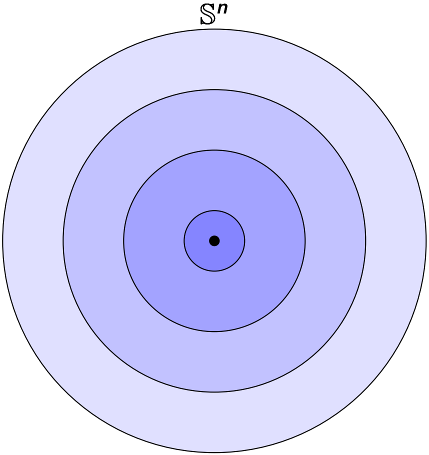
Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport PT_γ has values in $\mathrm{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l: PT_{γ_0} values in $\mathrm{Inner}(\tau_l)$.*

Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Example of a transverse foliation τ in $\mathbb{R}^d$:



Remarks

- Inner(τ_l) maps each circle to itself
- Sym(τ_l) allows to exchange circles
- Both preserve τ_l and fix the origin

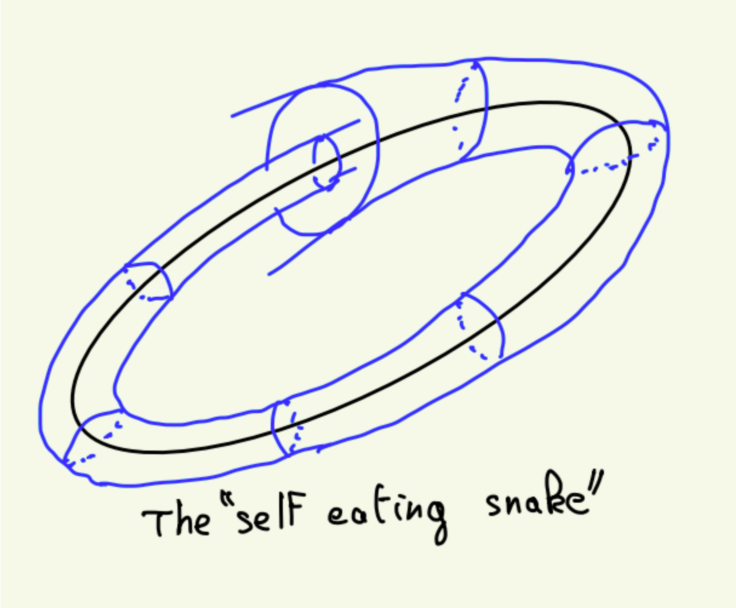
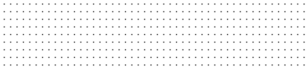


Diagram drawn by Camille Laurent-Gengoux.

Other example: Regular foliation



Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts

Remarks

Inner(τ_I): Trivial.

Sym(τ_I): The image of a group morphism $\pi_1(L) \to \mathrm{Diff}\left(\mathbb{R}^d, 0\right)$.

Idea

We guess:

$$\mathscr{F} = \left\{ \begin{array}{c} \text{Some map } \pi_1(L) \to \mathrm{Diff}\left(\mathbb{R}^d, 0\right) \ \text{(at l)} \\ \text{Bundle structure by } \tau_l, \mathrm{Inner}(\tau_l), \mathrm{Sym}(\tau_l), \ldots \end{array} \right\}$$

Thus, we want to classify $\mathscr{F}$ with given L and τ_l (for a fixed $l \in L$).

Danger

But $\mathrm{Sym}(\tau_l)$ and $\mathrm{Inner}(\tau_l)$ are in general infinite-dimensional, so that we have to deal with infinite-dimensional geometry!

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Definition (Formal foliation)

$X \in \mathscr{F}$ induces a derivation on $\hat{C} := C^\infty(M)/C_0^\infty(M)$, where $C_0^\infty(M)$ is the ideal of functions vanishing with all their derivatives along L. The image of $\mathscr{F}$ under this is the **formal singular foliation**.

Remarks

$f \in \hat{C}$ a formal power series, w.r.t. $(x_1, \ldots, x_d)$ as "normal coordinates":

$$f = \sum_{i_1, \ldots, i_d \geq 0} f_{i_1, \ldots, i_d} x_1^{i_1} \ldots x_d^{i_d}$$

with $f_{i_1, \ldots, i_d} \in C^\infty(L)$.

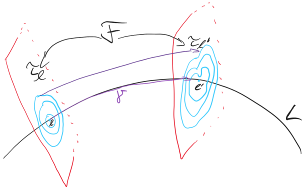
Camille Laurent-Gengoux and Leonid Ryvkin, The neighborhood of a singular leaf, *Journal de l'École polytechnique*, Math. 8 (2021), 1037–1064.

Our aim

Remarks (Our assumptions)

- τ_I a formal singular foliation.
- L a manifold (connected immersed submanifold of M).

$$\begin{tikzcd}
\text{Foliation } \mathscr{F} \arrow[r, leftrightarrow] \arrow[r, bend left, "\text{formal}"] & \text{Classification} \arrow[l, bend left, "\text{does not need to be formal}"]
\end{tikzcd}$$



Remarks ($\mathscr{F}$-connection)

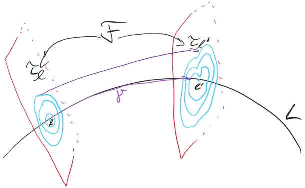
For $\phi \in \mathrm{Sym}(\tau_l)$ we have an induced parallel transport

$$\mathrm{PT}^{\mathrm{Sym}}_\gamma(\phi) := \mathrm{PT}_\gamma \circ \phi \circ \mathrm{PT}_\gamma^{-1}.$$

Then, on the normal bundle $\pi\colon \mathscr{T} \to L$,

$$\begin{aligned}\mathrm{PT}_\gamma(\phi \cdot p) &= \mathrm{PT}^{\mathrm{Sym}}_\gamma(\phi) \cdot \mathrm{PT}_\gamma(p)\\ \mathrm{PT}_{\gamma_0}(p) &= \varphi \cdot p\end{aligned}$$

for all $p \in \mathscr{T}_l$, $\phi \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.



Remarks (Sym-connection)

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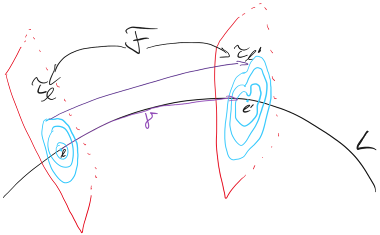
$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) := \mathrm{PT}_{\gamma} \circ \phi \circ \mathrm{PT}_{\gamma}^{-1}.$$

Then

$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi \circ \phi') = \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi) \circ \mathrm{PT}^{\mathrm{Sym}}_{\gamma}(\phi')$$
$$\mathrm{PT}^{\mathrm{Sym}}_{\gamma_0}(\phi) = \varphi \circ \phi \circ \varphi^{-1}$$

for all $\phi, \phi' \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.

Idea

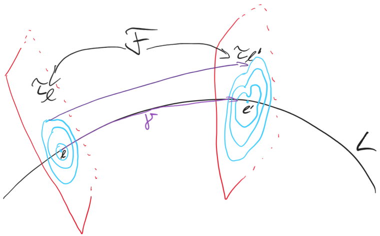


Idea

Generators of $\mathscr{F}$ given by:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.



Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$\begin{aligned}
[\mathbb{H}(X)+\overline{\nu}, \mathbb{H}(X')+\overline{\mu}] &= \mathbb{H}([X,X'])+\dots \\
&= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{ curvature}} \\
&\qquad + \underbrace{[\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu,\mu]}
\end{aligned}$$

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\end{aligned}
$$

Idea $(\dots \in \tau)$

We need:

1. Lie algebra bundle τ with structure τ_I
2. A horizontal lift $\mathbb{H}$ into $\mathscr{T}$ satisfying

$$
\begin{aligned}
\text{Curvature:} &\qquad [\mathbb{H}(X), \mathbb{H}(X')] - \mathbb{H}([X,X']) \in \tau \\
\text{Connection:} &\qquad [\mathbb{H}(X), \overline{\mu}] \in \tau
\end{aligned}
$$

Summary

$$
\begin{tikzcd}
 & \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
 & \{\text{Connection on Sym}(\tau)\} &
\end{tikzcd}
$$

Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given $\mathscr{F}$!

Multiplicative Yang-Mills connections

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{G} \\ & & \downarrow \\ & & L \end{array}$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

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Definition (LGB actions)

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi"] & L \end{tikzcd}$$

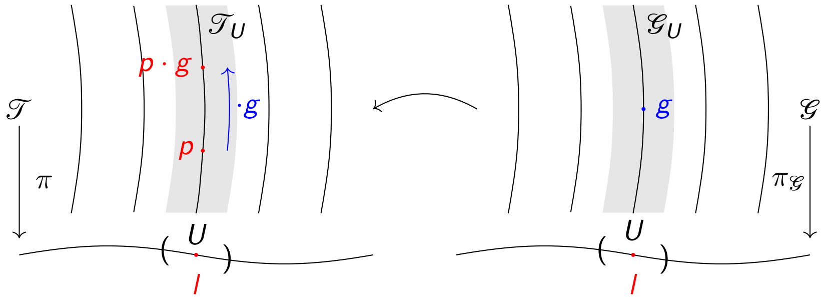
A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\pi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{1}$$
$$(p \cdot g) \cdot h = p \cdot (gh), \tag{2}$$
$$p \cdot e_{\pi(p)} = p \tag{3}$$

for all $p \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Connection on $\mathscr{T}$: Idea

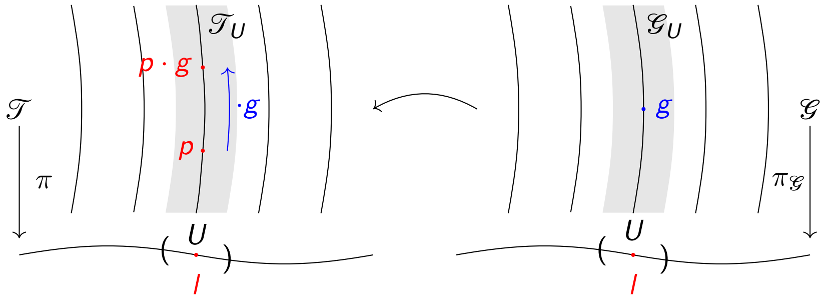


But:

$$r_g : \mathscr{T}_l \to \mathscr{T}_l$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

Connection on $\mathscr{T}$: Idea



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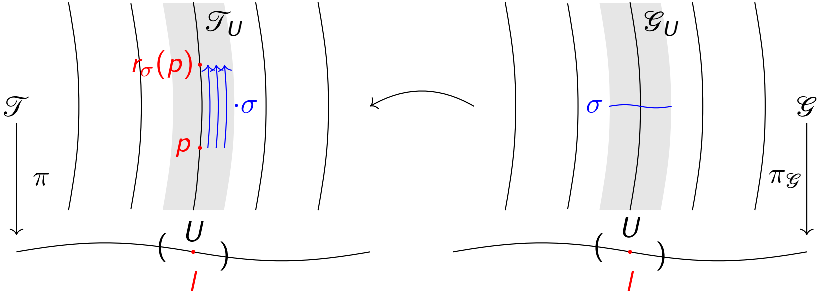
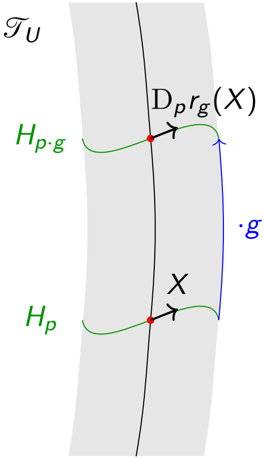
Connection on $\mathscr{T}$: Idea



Use $\sigma \in \Gamma(\mathscr{G})$: $r_\sigma(p) := p \cdot \sigma_I$

Connection on $\mathscr{T}$: Revisiting the classical setup

If $\mathscr{G}$ is trivial, $\sigma \equiv g$ constant,
and H a connection:



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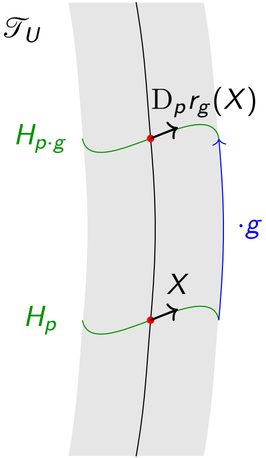

Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{T}}_\gamma$ in $\mathscr{T}$:

$$\mathrm{PT}^{\mathscr{T}}_\gamma(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_\gamma(p) \cdot g$$

where $\gamma : I \to L$ is a base path

Connection on $\mathscr{T}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g).$$

Recovering the ordinary definition

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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2. Equip $\mathscr{G}$ with canonical flat connection

Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi\colon \mathscr{T} \to L$ so that one has a commuting diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\pi"] & L \end{tikzcd}$$

1. **Ehresmann connection:**

$$\mathrm{PT}_\gamma^{\mathscr{T}}(p \cdot g) = \mathrm{PT}_\gamma^{\mathscr{T}}(p) \cdot \mathrm{PT}_\gamma^{\mathscr{G}}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}_{\gamma_0}^{\mathscr{T}}(p) = p \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}^0_{\pi(p)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
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Compare this with the Maurer-Cartan form and its curvature equation!

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$$\nabla\left([\mu,\nu]_{\mathscr{g}}\right) = [\nabla\mu,\nu]_{\mathscr{g}} + [\mu,\nabla\nu]_{\mathscr{g}},$$
$$R_\nabla = \mathrm{ad}\circ\zeta.$$

Example

Consider the Atiyah sequence of a principal G-bundle P:

$$\begin{tikzcd} (P\times\mathfrak{g})/G \arrow[r, hook] & \mathrm{T}P/G \arrow[r, two heads] & \mathrm{T}L \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

with splitting $\mathbb{H}\colon \mathrm{T}L\to\mathrm{T}P/G$, where $\mathfrak{g}$ is the Lie algebra. Then

$$\nabla_X\nu = [\mathbb{H}(X),\nu]_{\mathrm{T}P/G},$$
$$\zeta(X,X') = [\mathbb{H}(X),\mathbb{H}(X')]_{\mathrm{T}P/G} - \mathbb{H}([X,X']).$$

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Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla_X \nu},$$
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Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Proof.

Recall

$$[p, g] \cdot [p, v] = [p, g \cdot v]$$

for all $[p, g] \in \mathscr{G}$ and $[p, v] \in \mathscr{T}$, and

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}([p, v]) = \left[\mathrm{PT}^{P}_{\gamma}(p), v\right].$$

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

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Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

Proof.

- The adjoint bundle of P, $\mathrm{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of $\mathscr{G}$
- $\tau = \overline{\mathrm{Ad}(P)}$
- Difference of two connections on P has values in $\mathrm{Ad}(P)$

Generators take this already into account:

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Summary

Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf L and transverse model $\left(\mathbb{R}^d, \tau_l\right)$
- Principal $\mathrm{Inner}(\tau_l)$-bundles P over L

$$\begin{tikzcd}
& \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & P \\
& \{\text{Connection on } \mathrm{Sym}(\tau)\} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

Examples

Thank you!