

Classification of neighbourhoods of leaves of singular foliations

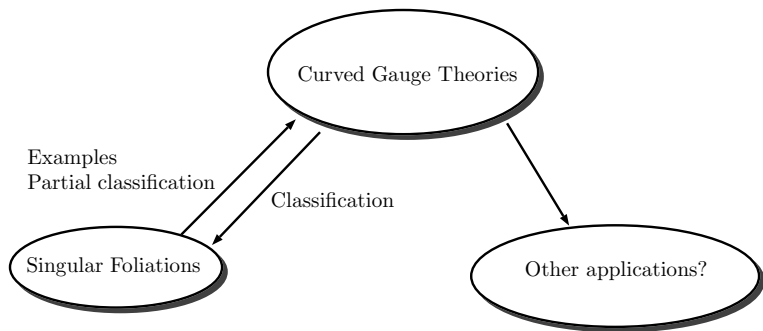
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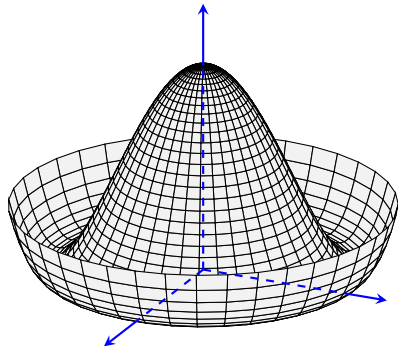
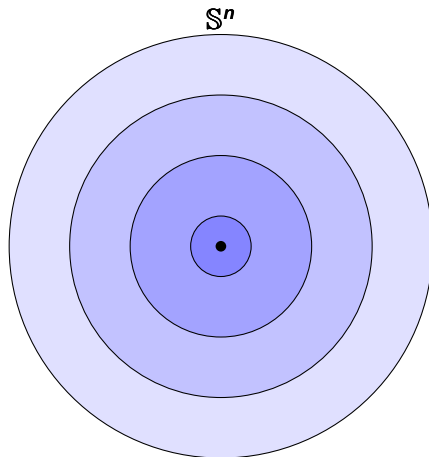


國家理論科學研究中心

National Center for Theoretical Sciences (National Taiwan University)



Singular Foliations



Singular Foliations:

- Gauge Theory
- Poisson Geometry
(Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** \mathcal{F} on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$ -**multiplication**,
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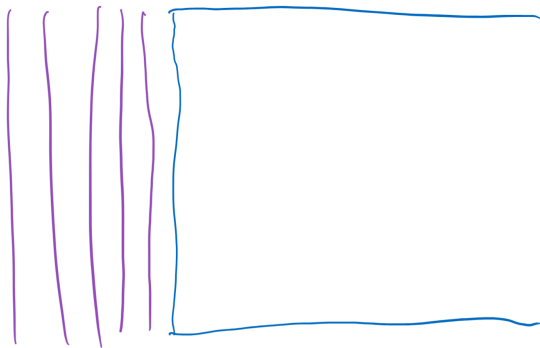
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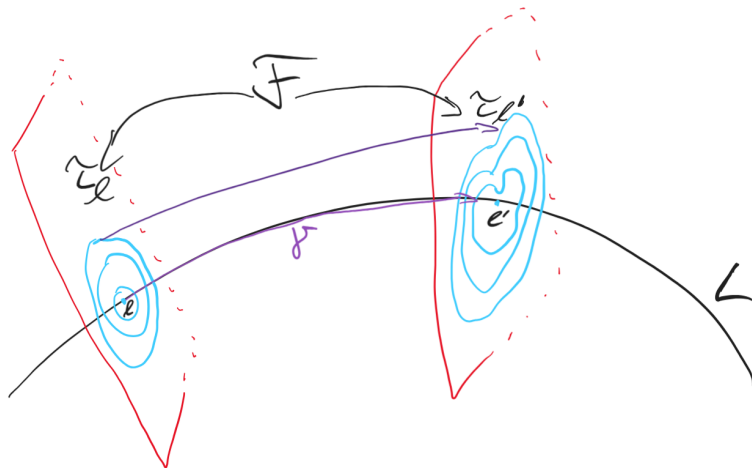
- it is **involutive**, i.e. $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$,
- it is **stable under $C^\infty(M)$ -multiplication**, i.e. $fX \in \mathcal{F}$ for all $f \in C^\infty(M)$ and $X \in \mathcal{F}$,
- it is **locally finitely generated**, i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ ($X^i \in \mathcal{F}$) such that for all $X \in \mathcal{F}$ there are $f_i \in C^\infty(M)$ satisfying on U .

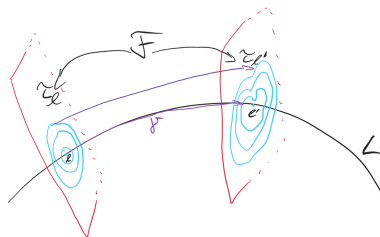
$$X = \sum_i f_i X^i.$$

Remarks (Leaves)

Following the flows in \mathcal{F} , this gives rise to a partition of connected immersed submanifolds in M .







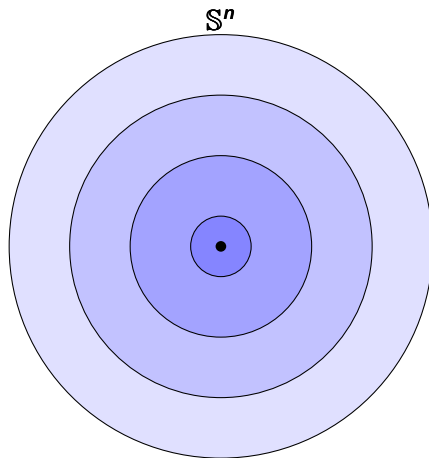
Theorem (\mathcal{F} -connections)

There is a connection on the normal bundle of a leaf L :

- *Horizontal vector fields are in \mathcal{F} .*
- *Parallel transport PT_γ has values in $\text{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l : PT_{γ_0} values in $\text{Inner}(\tau_l)$.*

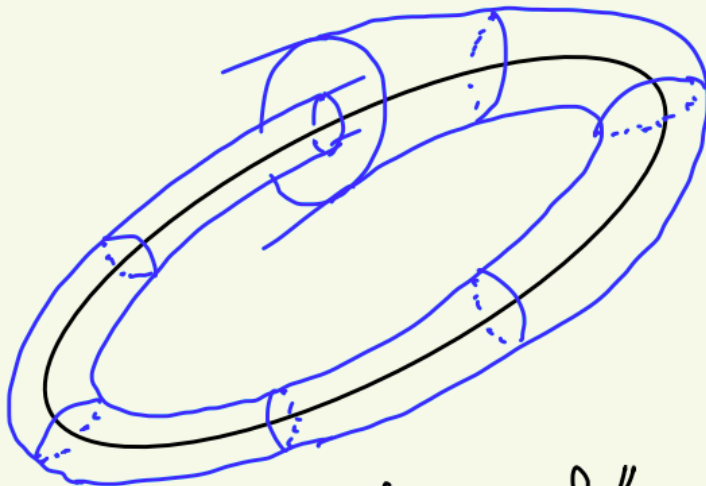
Idea: Relation to gauge theory

Example of a transverse foliation τ :



Remarks

- $\text{Inner}(\tau_I)$ maps each circle to itself
- $\text{Sym}(\tau_I)$ allows to exchange circles
- Both preserve τ_I and fix the origin



The "self eating snake"

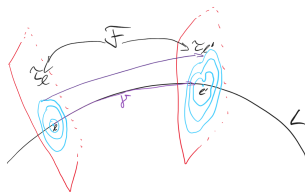
Idea: Relation to gauge theory

Other example: Regular foliation



Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts



Remarks (\mathcal{F} -connection)

For $\phi \in \text{Sym}(\tau_I)$ we have an induced parallel transport

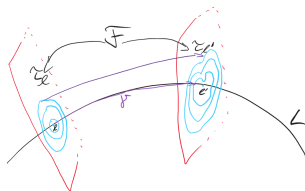
$$\text{PT}_\gamma^{\text{Sym}}(\phi) := \text{PT}_\gamma \circ \phi \circ \text{PT}_\gamma^{-1}.$$

Then, on the normal bundle $\pi: \mathcal{T} \rightarrow L$,

$$\text{PT}_\gamma(\phi \cdot p) = \text{PT}_\gamma^{\text{Sym}}(\phi) \cdot \text{PT}_\gamma(p)$$

$$\text{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all $p \in \mathcal{T}_I$, $\phi \in \text{Sym}(\tau_I)$, and for some $\varphi \in \text{Inner}(\tau_I)$.



Remarks (Sym-connection)

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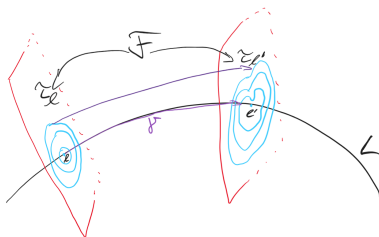
Then

$$\text{PT}_{\gamma}^{\text{Sym}}(\phi \circ \phi') = \text{PT}_{\gamma}^{\text{Sym}}(\phi) \circ \text{PT}_{\gamma}^{\text{Sym}}(\phi')$$

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for all $\phi, \phi' \in \text{Sym}(\tau_I)$, and for some $\varphi \in \text{Inner}(\tau_I)$.

Idea

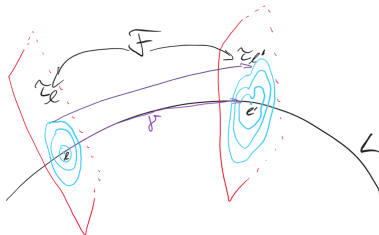


Idea

Generators of \mathcal{F} given by:

$$\mathbb{H}(X) + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift,
 $\nu \in \Gamma(\text{inner}(\tau))$ and $\bar{\nu}$ its fundamental vector field.

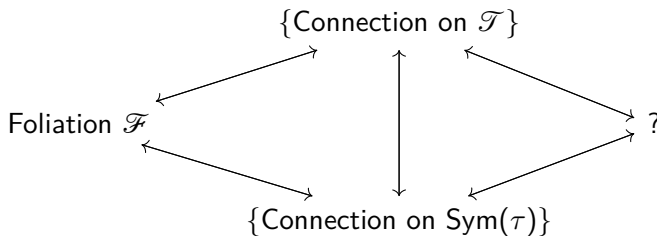


Idea

Fix I and given τ_I : Reconstruct \mathcal{F} .

$$\begin{aligned}
 [\mathbb{H}(X) + \bar{\nu}, \mathbb{H}(X') + \bar{\mu}] &= \mathbb{H}([X, X']) + \dots \\
 &= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{curvature}} \\
 &\quad + \underbrace{[\mathbb{H}(X), \bar{\mu}] - [\mathbb{H}(X'), \bar{\nu}]}_{\rightsquigarrow \text{connection}} + [\bar{\nu}, \bar{\mu}]
 \end{aligned}$$

Summary



Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given \mathcal{F} !

Multiplicative Yang-Mills connections

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle \mathcal{G}

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{G} \\ & & \downarrow \\ & & L \end{array}$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

\rightsquigarrow We will use a general connection ∇ instead of ∇^0 , and ∇ may not be flat.

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Definition (LGB actions)

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 \swarrow & & \downarrow \pi_{\mathcal{G}} \\
 \mathcal{T} & \xrightarrow{\pi} & L
 \end{array}$$

A **right-action** of \mathcal{G} on \mathcal{T} is a smooth map

$\mathcal{T} * \mathcal{G} := \mathcal{T} \times_{\pi \times \pi_{\mathcal{G}}} \mathcal{G} \rightarrow \mathcal{T}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

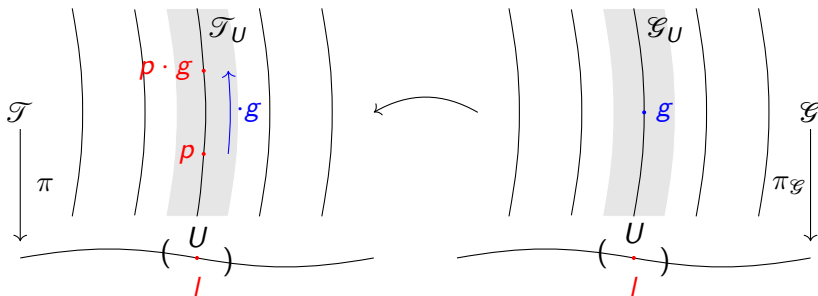
$$\pi(p \cdot g) = \pi(p), \quad (1)$$

$$(p \cdot g) \cdot h = p \cdot (gh), \quad (2)$$

$$p \cdot e_{\pi(p)} = p \quad (3)$$

for all $p \in \mathcal{T}$ and $g, h \in \mathcal{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathcal{G}_{\pi(p)}$.

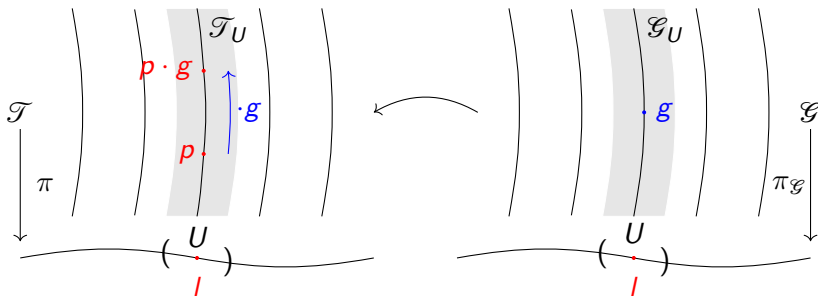
Connection on \mathcal{T} : Idea



But:

$$\Rightarrow \begin{aligned} & r_g: \mathcal{T}_I \rightarrow \mathcal{T}_I \\ & D_p r_g \text{ only defined on vertical structure} \end{aligned}$$

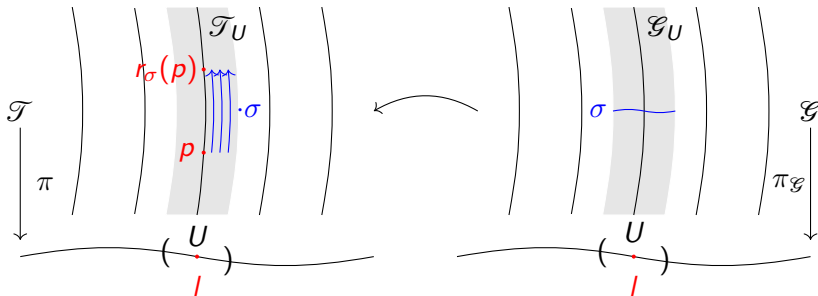
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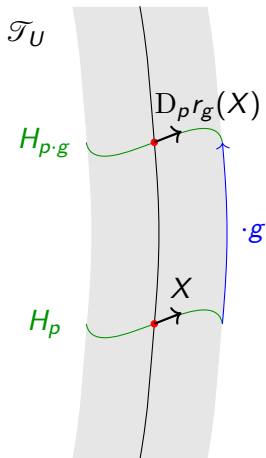
Connection on \mathcal{T} : Idea



Use $\sigma \in \Gamma(\mathcal{G})$: $r_\sigma(p) := p \cdot \sigma_l$

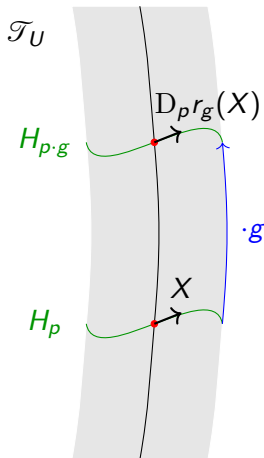
Connection on \mathcal{T} : Revisiting the classical setup

If \mathcal{G} is trivial, $\sigma \equiv g$ constant,
and H a connection:



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Remarks (Integrated case)

Parallel transport $\text{PT}_\gamma^{\mathcal{T}}$ in \mathcal{T} :

$$\text{PT}_\gamma^{\mathcal{T}}(p \cdot g) = \text{PT}_\gamma^{\mathcal{T}}(p) \cdot g$$

where $\gamma : I \rightarrow L$ is a base path

Connection on \mathcal{T} : General case

Remarks (Integrated case)

Ansatz: Introduce connection on \mathcal{G} ,

$$\mathrm{PT}_{\gamma}^{\mathcal{T}}(p \cdot g) = \mathrm{PT}_{\gamma}^{\mathcal{T}}(p) \cdot \mathrm{PT}_{\gamma}^{\mathcal{G}}(g).$$

Recovering the ordinary definition

- 1 $\mathcal{G} \cong L \times G$
- 2 Equip \mathcal{G} with canonical flat connection

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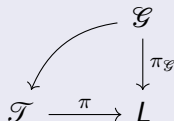
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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.]

A surjective submersion $\pi: \mathcal{T} \rightarrow L$ so that one has a commuting diagram

**1 Ehresmann connection:**

$$\text{PT}_{\gamma}^{\mathcal{T}}(p \cdot g) = \text{PT}_{\gamma}^{\mathcal{T}}(p) \cdot \text{PT}_{\gamma}^{\mathcal{G}}(g)$$

2 Yang-Mills connection: Additionally

$$\text{PT}_{\gamma_0}^{\mathcal{T}}(p) = p \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathcal{G}_{\pi(p)}^0$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On \mathcal{G} there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\begin{aligned} \mathrm{PT}_{\gamma}^{\mathcal{G}}(q \cdot g) &= \mathrm{PT}_{\gamma}^{\mathcal{G}}(q) \cdot \mathrm{PT}_{\gamma}^{\mathcal{G}}(g), \\ \mathrm{PT}_{\gamma_0}^{\mathcal{G}}(q) &= g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1} \end{aligned}$$

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Compare this with the Maurer-Cartan form and its curvature equation!

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On the Lie algebra bundle \mathfrak{g} we have a connection ∇ with

$$\begin{aligned}\nabla([\mu, \nu]_{\mathfrak{g}}) &= [\nabla\mu, \nu]_{\mathfrak{g}} + [\mu, \nabla\nu]_{\mathfrak{g}}, \\ R_{\nabla} &= \text{ad} \circ \zeta.\end{aligned}$$

Example

Consider the Atiyah sequence of a principal G -bundle P :

$$(P \times \mathfrak{g})/G \hookrightarrow TP/G \overset{\mathbb{H}}{\rightrightarrows} TL$$

with splitting $\mathbb{H}: TL \rightarrow E$, where \mathfrak{g} is the Lie algebra. Then

$$\begin{aligned}\nabla_X \nu &= [\mathbb{H}(X), \nu]_{TP/G}, \\ \zeta(X, X') &= [\mathbb{H}(X), \mathbb{H}(X')]_{TP/G} - \mathbb{H}([X, X']).\end{aligned}$$

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Going back to foliations

Theorem ([C. L.-G., S.-R. F.]

Given a multiplicative Yang-Mills connection on \mathcal{G} and a Yang-Mills connection \mathbb{H} on \mathcal{T} , then there is a natural foliation on \mathcal{T} generated by

$$\mathbb{H}(X) + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathfrak{g})$.

Proof.

We have

$$\begin{aligned} [\mathbb{H}(X), \bar{\nu}] &= \overline{\nabla_X \nu}, \\ [\mathbb{H}(X), \mathbb{H}(X')] &= \mathbb{H}([X, X']) + \overline{\zeta(X, X')}, \end{aligned}$$

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on \mathcal{G} is a multiplicative Yang-Mills connection and the one on \mathcal{T} is a corresponding Yang-Mills connection.

Remarks

Thus, we have a singular foliation on \mathcal{T} , which, by construction, admits L as a leaf and τ_l as transverse data.

Proposition ([C. L.-G., S.-R. F.]

The reconstructed foliation is independent of the choice of connection on P .

Proof.

- The adjoint bundle of P , $\text{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of \mathcal{G}
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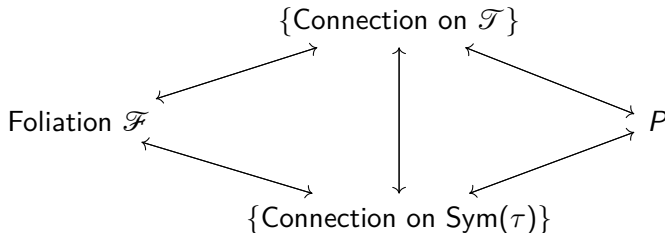
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Summary

Remarks ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- Singular foliations with leaf L and transverse model (\mathbb{R}^d, τ_l)
- Principal $\text{Inner}(\tau_l)$ -bundles P over L



Thank you!