Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux (Université de Lorraine)

Simon-Raphael Fischer

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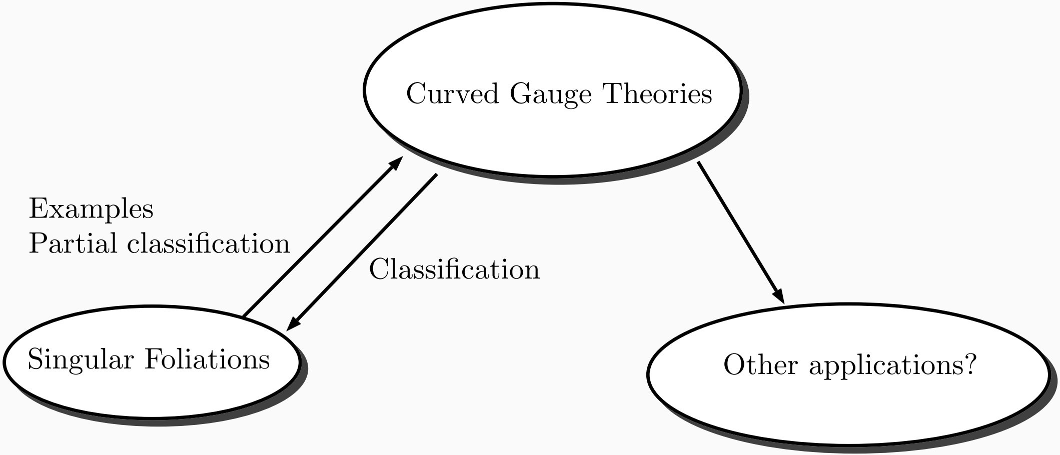
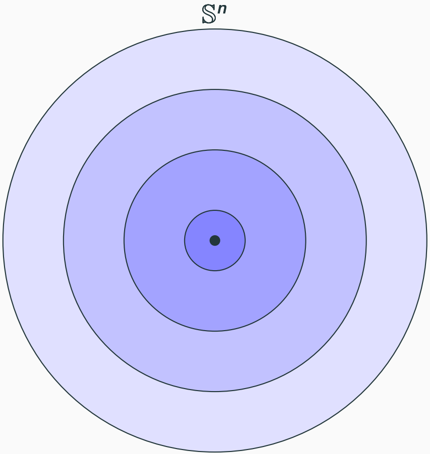
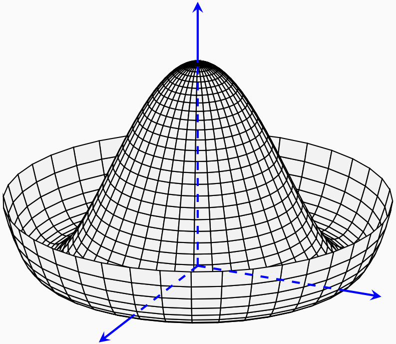
Singular Foliations



Remarks (Based on the following work)

S.-R. F. and Camille Laurent-Gengoux, *Classification of neighborhoods around leaves of singular foliations*, arXiv:2401.05966, (2024).





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold M is a submodule of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **locally finitely generated**.

Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

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- it is **involutive**, *i.e.* $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$,
- it is **locally finitely generated**, *i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $\left(X^i\right)_i$ ($X^i \in \mathscr{F}$) such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on U,*

$$X = \sum_i f_i X^i.$$

Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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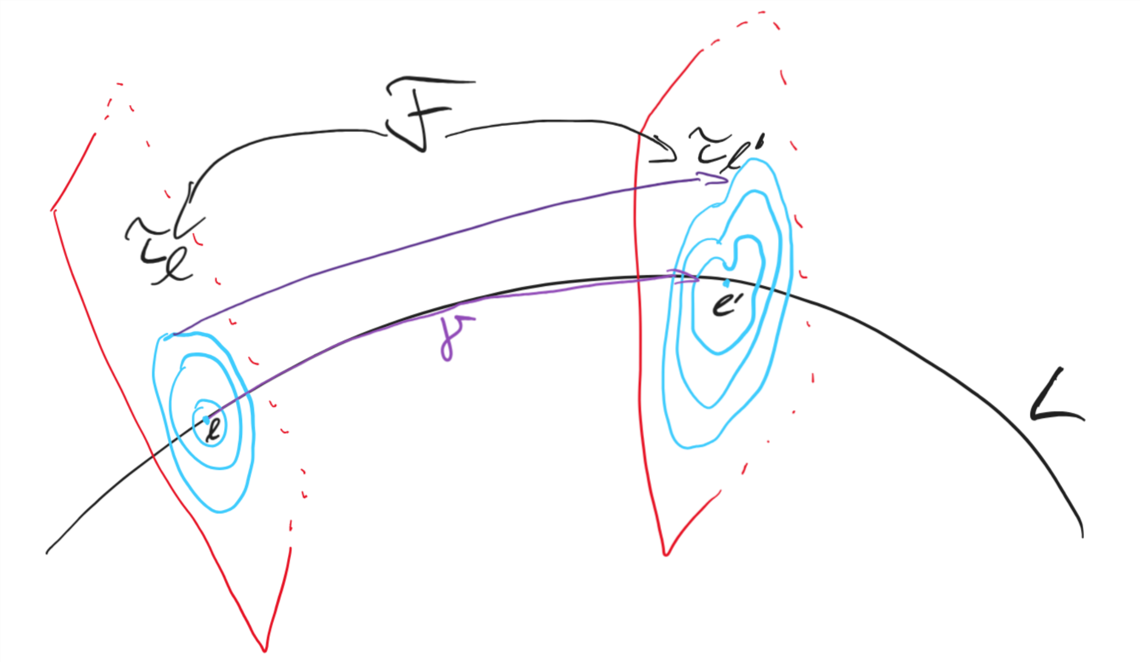
First step towards a classification


$\mathcal{F}$
τ_ℓ
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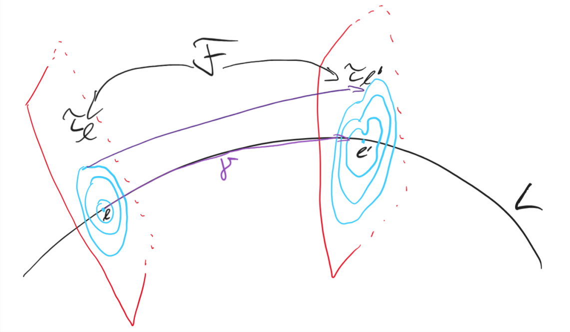
L



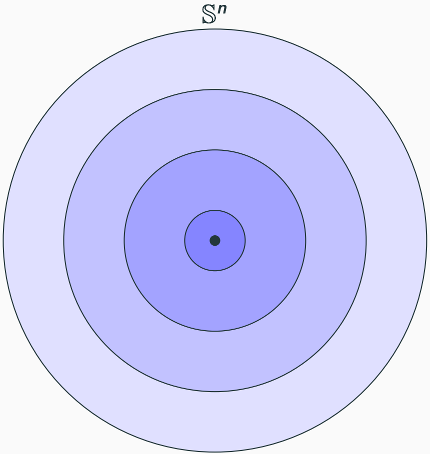
Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport* PT_γ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop γ_0 at l:* PT_{γ_0} *values in* $\mathrm{Inner}(\tau_l)$.

Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Example of a transverse foliation τ in $\mathbb{R}^d$:



Remarks

- Inner(τ_I) maps each circle to itself
- Sym(τ_I) allows to exchange circles
- Both preserve τ_I and fix the origin

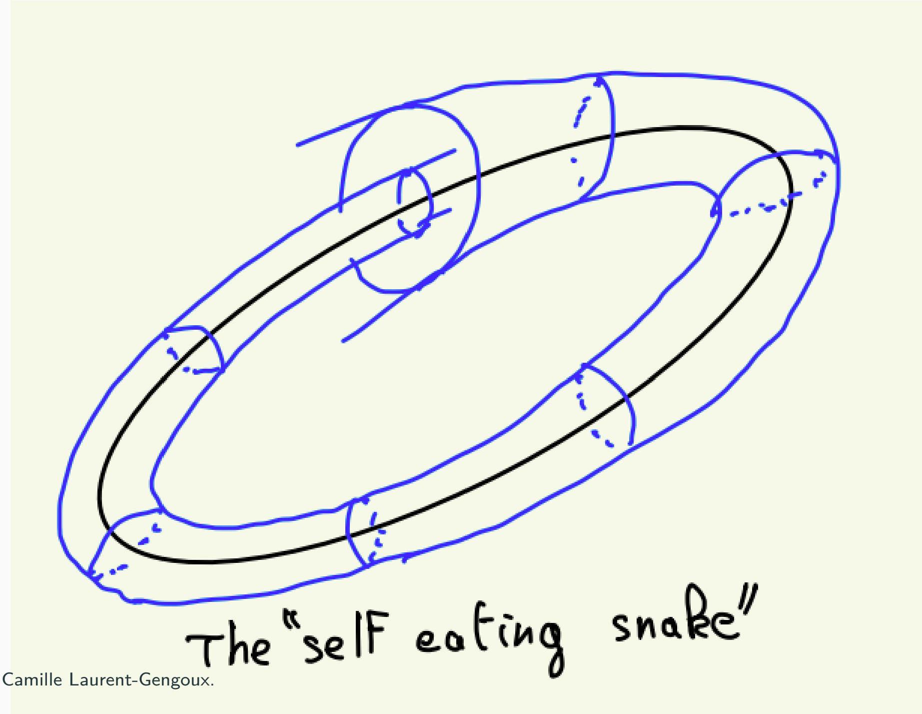


Diagram drawn by Camille Laurent-Gengoux.

Other example: Regular foliation

Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts

Remarks

Inner(τ_I): Trivial.

Sym(τ_I): We essentially need the image of a group morphism $\pi_1(L) \to \mathrm{Diff}\left(\mathbb{R}^d, 0\right)$.

Idea

We guess:

$$\mathscr{F} = \left\{ \begin{array}{c} \text{Some map } \pi_1(L) \to \text{Diff}\left(\mathbb{R}^d, 0\right) \text{ (at l)} \\ \text{Bundle structure by } \tau_l, \text{Inner}(\tau_l), \text{Sym}(\tau_l), \dots \end{array} \right\}$$

Thus, we want to classify $\mathscr{F}$ with given L and τ_l (for a fixed $l \in L$).

Danger

But $\text{Sym}(\tau_l)$ and $\text{Inner}(\tau_l)$ are in general infinite-dimensional, so that we have to deal with infinite-dimensional geometry!

Thus: We assume that τ_l is a formal singular foliation.

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Our aim

Remarks (Our assumptions)

- τ_l a formal singular foliation.
- L a manifold (connected immersed submanifold of M).

$$
\begin{tikzcd}
\text{Foliation } \mathscr{F} \text{ with } L \text{ and } \tau_l \arrow[r, longleftrightarrow] \arrow[r, bend left, "\text{formal}"] & \text{Classification} \arrow[l, bend left, "\text{does not need to be formal}"]
\end{tikzcd}
$$

Remarks (Avoiding formal setting)

Either

- add real-analyticity conditions to the classification,

or also

- assume embedded L and real-analytic $\mathscr{F}$.

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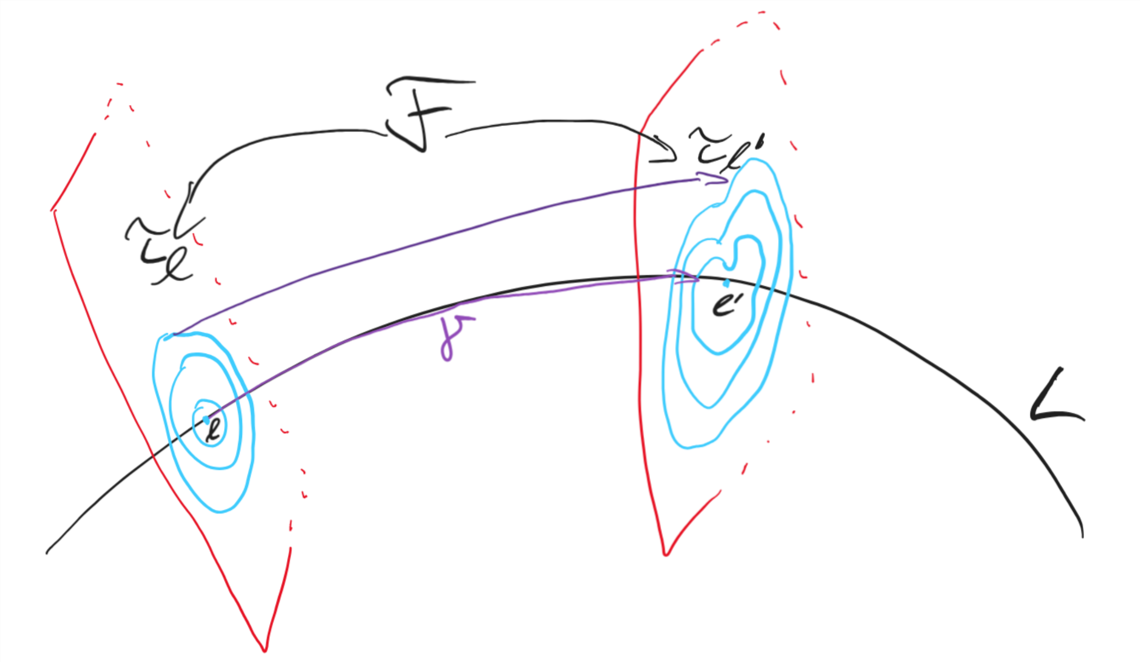
Multiplicative Yang-Mills connections



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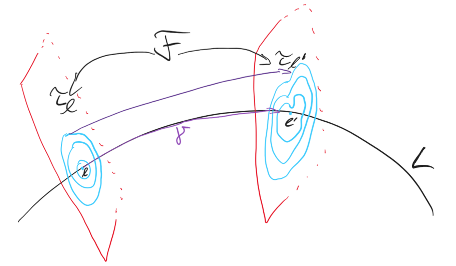


Remarks ($\mathscr{F}$-connection)

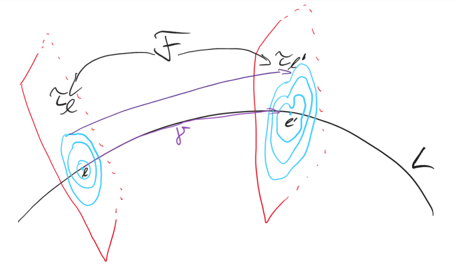
For $\phi \in \mathrm{Sym}(\tau_l)$ we have an induced parallel transport

$$\mathrm{PT}_\gamma^{\mathrm{Sym}}(\phi) := \mathrm{PT}_\gamma \circ \phi \circ \mathrm{PT}_\gamma^{-1}.$$

Then, on the normal bundle $\pi\colon \mathscr{T} \to L$,

$$\begin{aligned} \mathrm{PT}_\gamma(\phi \cdot p) &= \mathrm{PT}_\gamma^{\mathrm{Sym}}(\phi) \cdot \mathrm{PT}_\gamma(p) \\ \mathrm{PT}_{\gamma_0}(p) &= \varphi \cdot p \end{aligned}$$

for all $p \in \mathscr{T}_l$, $\phi \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.



Remarks (Sym-**connection)**

For $\phi \in \mathsf{Sym}(\tau_l)$ we have an induced parallel transport

$$\mathsf{PT}_\gamma^{\mathsf{Sym}}(\phi) := \mathsf{PT}_\gamma \circ \phi \circ \mathsf{PT}_\gamma^{-1}.$$

Then

$$\mathsf{PT}_\gamma^{\mathsf{Sym}}(\phi \circ \phi') = \mathsf{PT}_\gamma^{\mathsf{Sym}}(\phi) \circ \mathsf{PT}_\gamma^{\mathsf{Sym}}(\phi')$$
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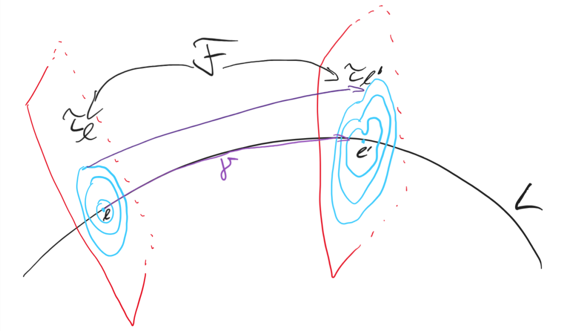
for all $\phi, \phi' \in \mathsf{Sym}(\tau_l)$, and for some $\varphi \in \mathsf{Inner}(\tau_l)$.

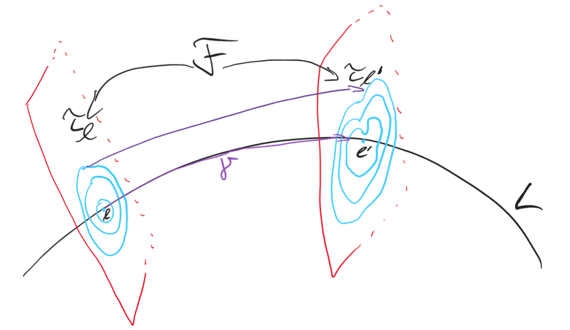


Idea

Generators of $\mathscr{F}$ given by:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.



Idea

Fix $l \in L$, given τ and $\mathbb{H}$. Reconstruct $\mathscr{F}$.

$$
\begin{aligned}
[\mathbb{H}(X) + \overline{\nu}, \mathbb{H}(X') + \overline{\mu}] &= \mathbb{H}([X, X']) + \dots \\
&= [\mathbb{H}(X), \mathbb{H}(X')] \\
&\qquad + [\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}] + \overline{[\nu, \mu]}
\end{aligned}
$$

Idea

Fix I and given τ_I: Reconstruct $\mathscr{F}$.

$$\begin{aligned}[\mathbb{H}(X)+\overline{\nu}, \mathbb{H}(X')+\overline{\mu}] &= \mathbb{H}([X,X'])+\dots \\ &= \underbrace{[\mathbb{H}(X), \mathbb{H}(X')]}_{\rightsquigarrow \text{ curvature}} \\ &\qquad + \underbrace{[\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu, \mu]}\end{aligned}$$

Idea ($\ldots \in \tau$)

We need:

1. Lie algebra bundle τ with structure τ_I
2. A horizontal lift $\mathbb{H}$ into $\mathscr{T}$ satisfying

$$\begin{aligned}\text{Curvature:} &\qquad [\mathbb{H}(X), \mathbb{H}(X')] - \mathbb{H}([X,X']) \in \tau \\ \text{Connection:} &\qquad [\mathbb{H}(X), \overline{\mu}] \in \tau\end{aligned}$$

Summary

$$
\begin{tikzcd}
& \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & ? \arrow[dl, leftrightarrow] \\
& \{\text{Connection on } \mathrm{Sym}(\tau)\} &
\end{tikzcd}
$$

Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given $\mathscr{F}$!

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Motivation
What are Ehresmann connections, preserving $\mathscr{G}$-actions?

Definition (LGB actions)

$$\begin{tikzcd} \mathscr{G} \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\ L & \mathscr{T} \arrow[l, "\pi"] \end{tikzcd}$$

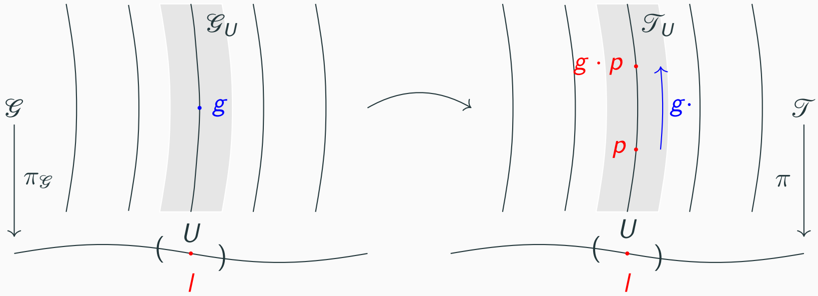
A **left-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{G} * \mathscr{T} := \mathscr{G} {}_{\pi_{\mathscr{G}}}\!\times_{\pi} \mathscr{T} \to \mathscr{T}$, $(g, p) \mapsto g \cdot p$, satisfying the following properties:

$$\begin{aligned} \pi(g \cdot p) &= \pi(p), \\ h \cdot (g \cdot p) &= (hg) \cdot p, \\ e_{\pi(p)} \cdot p &= p \end{aligned}$$

for all $p \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

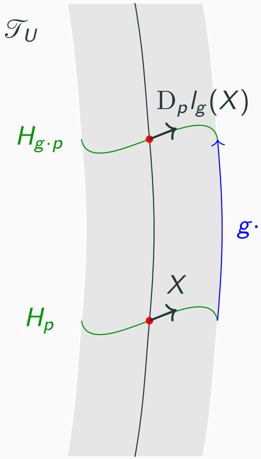
K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Connection on $\mathscr{T}$: Idea



Connection on $\mathscr{T}$: Revisiting the classical setup

If $\mathscr{G}$ is trivial, and H a connection:



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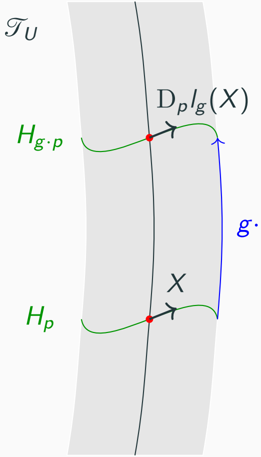

Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{T}}_{\gamma}$ in $\mathscr{T}$:

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(g \cdot p) = g \cdot \mathrm{PT}^{\mathscr{T}}_{\gamma}(p),$$

where $\gamma : I \to L$ is a base path

Connection on $\mathscr{T}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(g \cdot p) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(g) \cdot \mathrm{PT}^{\mathscr{T}}_{\gamma}(p).$$

Recovering the ordinary definition

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi \colon \mathscr{T} \to L$ so that one has a commuting diagram

$$\begin{tikzcd} \mathscr{G} \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\ L & \mathscr{T} \arrow[l, "\pi"] \end{tikzcd}$$

1. **Ehresmann connection:**

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(g \cdot p) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(g) \cdot \mathrm{PT}^{\mathscr{T}}_{\gamma}(p)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(p) = g_{\gamma_0} \cdot p$$

for some $g_{\gamma_0} \in \mathscr{G}^0_{\pi(p)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
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Compare this with the Maurer-Cartan form and its curvature equation!

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Compare this with the Maurer-Cartan form and its curvature equation!

Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection ∇ with

$$\nabla\left([\mu,\nu]_{\mathscr{g}}\right) = [\nabla\mu,\nu]_{\mathscr{g}} + [\mu,\nabla\nu]_{\mathscr{g}},$$
$$R_\nabla = \mathrm{ad}\circ\zeta.$$

Example

Consider the Atiyah sequence of a principal G-bundle P:

$$\begin{tikzcd} \mathscr{g} := (P \times \mathfrak{g})/G \arrow[r, hook] & \mathrm{T}P/G \arrow[r, two heads] & \mathrm{T}L \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

with splitting $\mathbb{H}\colon \mathrm{T}L \to \mathrm{T}P/G$, where $\mathfrak{g}$ is the Lie algebra. Then

$$\nabla_X\nu = [\mathbb{H}(X),\nu]_{\mathrm{T}P/G},$$
$$\zeta(X,X') = [\mathbb{H}(X),\mathbb{H}(X')]_{\mathrm{T}P/G} - \mathbb{H}([X,X']).$$

For the first equation: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

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Foliations and Yang-Mills connections

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.
We have

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Proof.
Recall

$$[p, g] \cdot [p, v] = [p, g \cdot v]$$

for all $[p, g] \in \mathscr{G}$ and $[p, v] \in \mathscr{T}$, and

$$\mathrm{PT}_\gamma^{\mathscr{T}}([p, v]) = \left[\mathrm{PT}_\gamma^{P}(p), v\right].$$

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

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Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

Proof.

- The adjoint bundle of P, $\mathrm{Ad}(P) \coloneqq (P \times \mathfrak{g})/G$, is the Lie algebra bundle of $\mathscr{G}$
- $\tau = \overline{\mathrm{Ad}(P)}$
- Difference of two connections on P has values in $\mathrm{Ad}(P)$

Generators take this already into account; recall:

$$\mathbb{H}(X) + \overline{\nu}$$

with $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathrm{Ad}(P))$.

Proposition ([C. L.-G., S.-R. F.])

The reconstructed foliation is independent of the choice of connection on P.

Proof.

- The adjoint bundle of P, $\mathrm{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of $\mathscr{G}$
- $\tau = \overline{\mathrm{Ad}(P)}$
- Difference of two connections on P has values in $\mathrm{Ad}(P)$

Generators take this already into account; recall:

$$\mathbb{H}(X) + \overline{\nu}$$

with $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathrm{Ad}(P))$.

Main Theorems

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model* $\left(\mathbb{R}^d, \tau_l\right)$
- *Principal* $\mathrm{Inner}(\tau_l)$*-bundles P over L*

$$
\begin{tikzcd}
& \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & P \arrow[dl, leftrightarrow] \\
& \{\text{Connection on } \mathrm{Sym}(\tau)\} &
\end{tikzcd}
$$

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P is now finite-dimensional!

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Formal setting is here very important! Especially here is where one has to think about real-analytic conditions for the general case.

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Corollary

L simply connected and τ_l *is made of vector fields vanishing quadratically at 0. Then the unique singular foliation is the trivial one, i.e. the trivial product of (L,* $\mathfrak{X}(L)$*) and* $\left(\mathbb{R}^d, \tau_l\right)$*.*

Corollary by Camille Laurent-Gengoux and Leonid Ryvkin, The neighborhood of a singular leaf, *Journal de l'École Polytechnique*, Math. 8, 1037 - 1064, 2021.

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Corollary ([C. L.-G., S.-R. F.])

L contractible. Then the unique singular foliation is the trivial one.

Examples

Theorem ([C. L.-G., S.-R. F.])

In the simply connected case, the following are equivalent:

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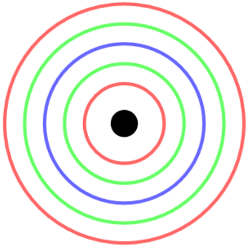
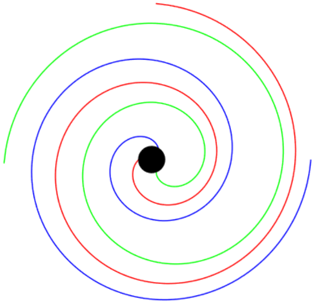
Corollary ([C. L.-G., S.-R. F.])

L simply connected. Then $\mathscr{F}$ *is the trivial foliation if and only if it admits a flat* $\mathscr{F}$*-connection.*

Examples [C. L.-G., S.-R. F.]

$L = \mathbb{S}^2$, $M = \mathrm{T}\mathbb{S}^2$. Let us consider two possible τ_l:

Name	Concentric circles	Spirals
Generator	$x\partial_y - y\partial_x$	$x\partial_y - y\partial_x + (x^2 + y^2)(x\partial_x + y\partial_y)$
Picture of the leaves		
$\mathrm{Inner}(\tau_l)/\mathrm{Inner}(\tau_l)_{\geq 2}$	$\mathbb{S}^1$	$\mathbb{R}$

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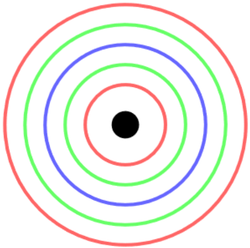
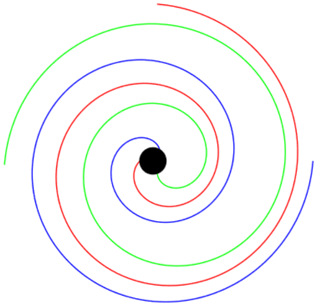
Picture of the leaves		
$\mathrm{Inner}(\tau_l)/\mathrm{Inner}(\tau_l)_{\geq 2}$	$\mathbb{S}^1$	$\mathbb{R}$
Foliation	☺	☹

Thank you! ☺

Total classification

Remarks

$\mathsf{Inner}(\tau_I)$ is a normal subgroup of $\mathsf{Sym}(\tau_I)$, thus we have a quotient:

$$\mathsf{Inner}(\tau_I) \hookrightarrow \mathsf{Sym}(\tau_I) \twoheadrightarrow \mathsf{Out}(\tau_I)$$

Theorem ([C. L.-G., S.-R. F.])

The following are equivalent:

- *Singular foliations with leaf L and transverse model $\left(\mathbb{R}^d, \tau_I\right)$,*
- 1. *a group morphism $\Xi\colon \pi_1(L) \longrightarrow \mathrm{Out}(\tau_I)$, and*
  2. *a finite-dimensional principal $H/\mathsf{Inner}(\tau_I)_{\geq 2}$-bundle, with H a subgroup of $\mathsf{Sym}(\tau_I)$ containing $\mathsf{Inner}(\tau_I)$.*