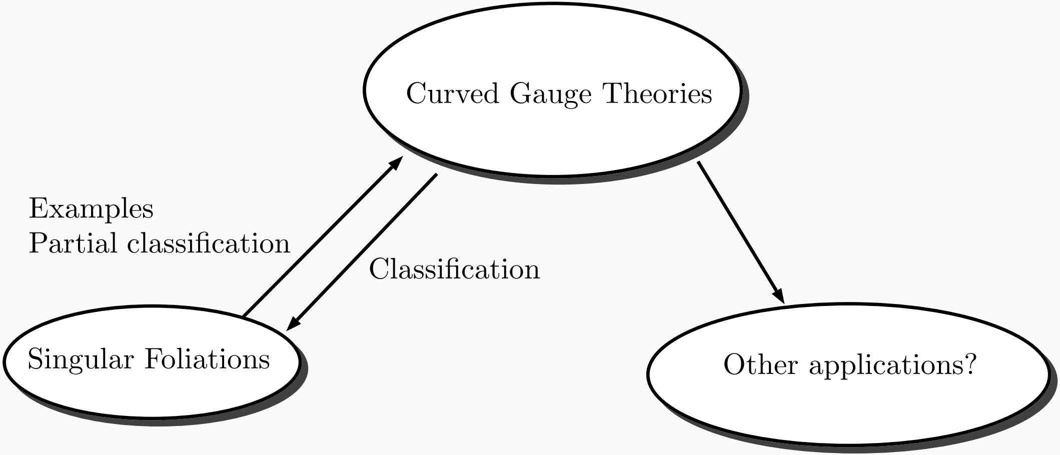
Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux (Université de Lorraine)

Simon-Raphael Fischer

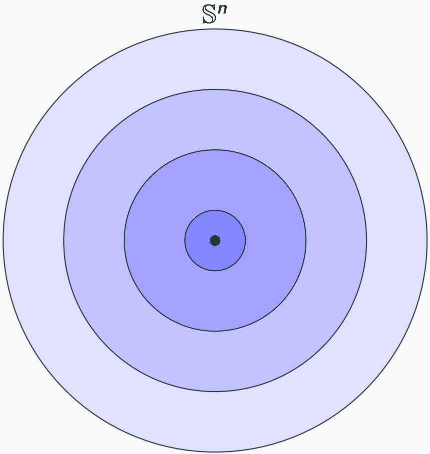
GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

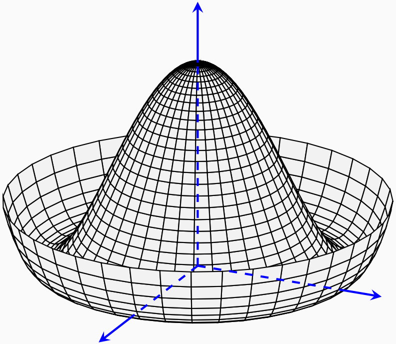


Remarks (Based on the following work)

S.-R. F. and Camille Laurent-Gengoux, *Classification of neighborhoods around leaves of singular foliations*, arXiv:2401.05966, (2024).

Singular Foliations





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- . . .

First idea

Definition (Partitionifolds)
Let M be a smooth manifold. A **partitionifold of** M is a partition of immersed connected submanifolds, which we call *leaves*.

Remarks

We will denote a partitionifold by $L_\bullet$, $p \mapsto L_p$, where L_p is the leaf through $p \in M$.

Right diagram made by Mark J.D. Hamilton.

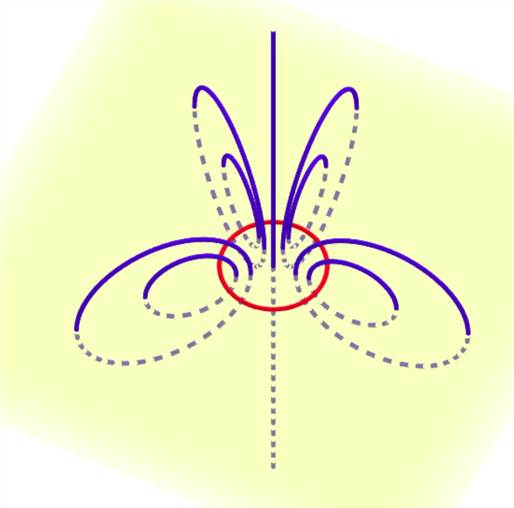


Figure 1: The magnetic partition

Remarks

A partitionifold with:

- All leaves are of the same dimension.
- **But:** It lacks regularity!

Graphic by Camille Laurent-Gengoux, Ruben Louis, and Leonid Ryvkin.

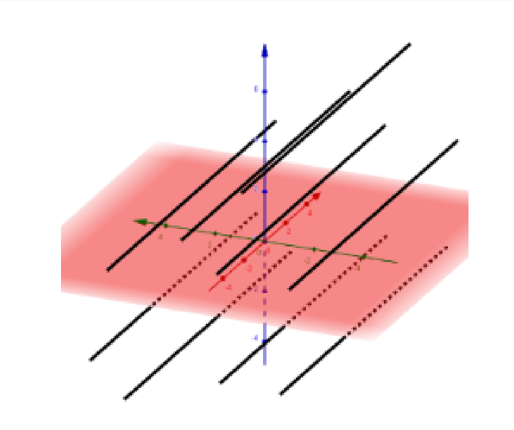


Figure 2: Isolated lasagna in a spaghetti dish

Remarks

A partitionifold with:

- Dimension is now different.
- **But:** Also no regularity!

Remarks

Isolated spaghetti in a lasagna dish: Regularity!

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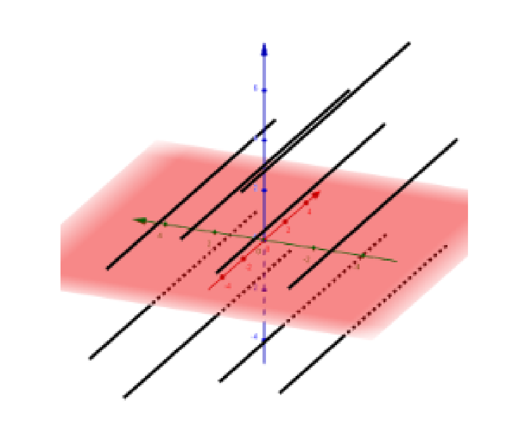

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Isolated spaghetti in a lasagna dish:

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Definition (Smooth partitionifold)
A smooth partitionifold $L_\bullet$ is smooth, if there is for all $p \in M$ and every vector $u \in \mathrm{T}_p L_p$ a vector field X tangent to $L_\bullet$ with

$$X_p = u.$$

Remarks

This definition is okay, but not widely used: It still has a problem...

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Example (Vector fields not necessarily finitely generated)
Consider the following smooth partitionifold:

- $M = \mathbb{R}$;
- 0-dimensional leaves:
$$\{1\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \dots, \left\{\frac{1}{n}\right\}, \dots, \{0\};$$
- 1-dimensional leaves: Remaining open intervals.

Remarks

One has a sort of "infinitesimal leaf" next to $\{0\}$.

Technically: Tangent vectors of $L_\bullet$ are locally not finitely generated around 0.

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Recall:

Theorem (Frobenius Theorem)
Every integrable subbundle E of $\mathrm{T}M$ corresponds to a regular foliation in M.

Remarks ($\Gamma(E)$ is involutive)

Integrable:

$$[X, Y] \in \Gamma(E)$$

for all $X, Y \in \Gamma(E)$.

Remarks

Alternatively: An involutive submodule of $\mathfrak{X}(M)$, or equivalently of $\mathfrak{X}_c(M)$.

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A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold M is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
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Right diagram made by Mark J.D. Hamilton.

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- it is **locally finitely generated**, *i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $\left(X^i\right)_i^r$ ($X^i \in \mathscr{F}$) such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on U.*

$$X = \sum_i f_i X^i.$$

Remarks (Leaves)

We have an induced smooth partitionifold $L_\bullet$,

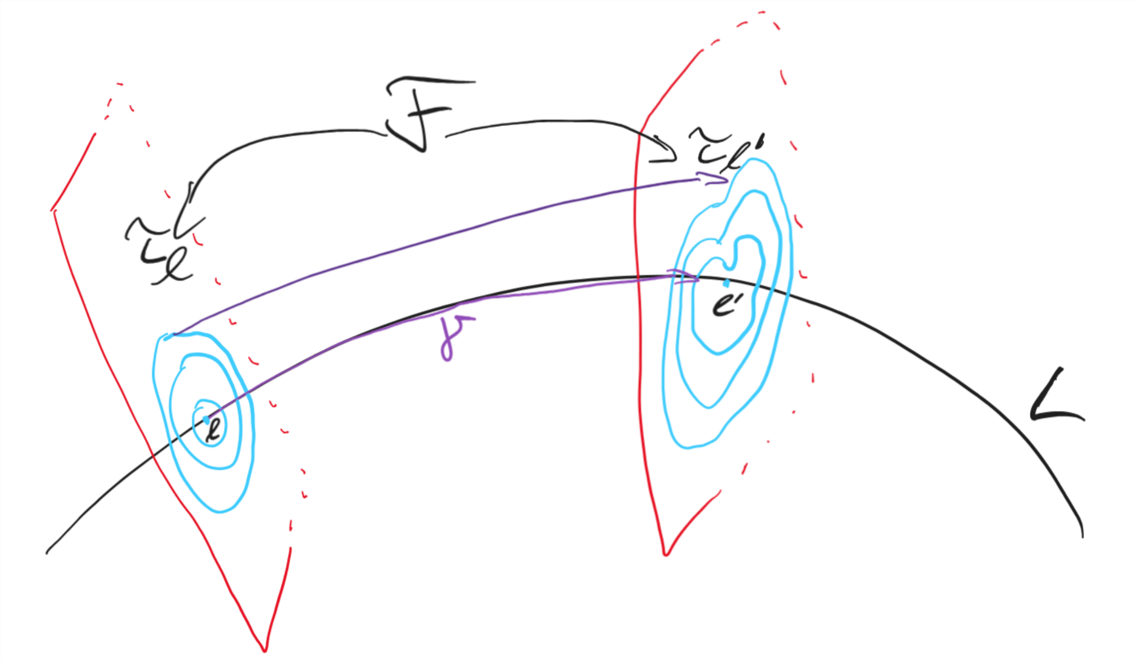
$$\mathscr{F} \Rightarrow L_\bullet,$$

but $\mathscr{F}$ also encodes the information about the generators,

$$\begin{tikzcd} \mathscr{F}_1 \arrow[dr, Rightarrow] & \\ & L_\bullet \\ \mathscr{F}_2 \arrow[ur, Rightarrow] & \end{tikzcd}$$

Right diagram made by Mark J.D. Hamilton.

First step towards classification



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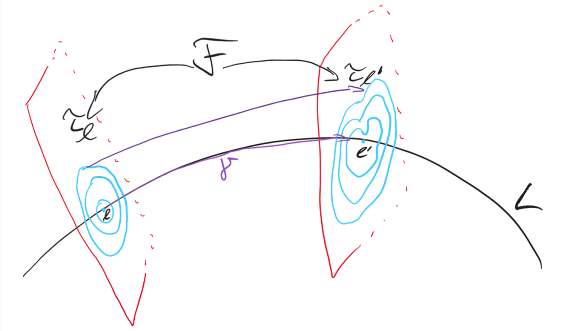
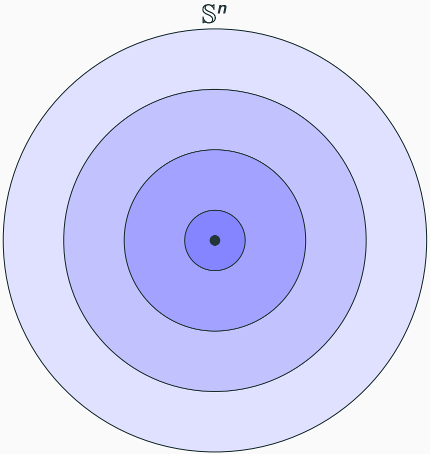


Theorem ($\mathscr{F}$-connections)
There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport* PT_γ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop* γ_0 *at l:* PT_{γ_0} *values in* $\mathrm{Inner}(\tau_l)$.

Right diagram made by Mark J.D. Hamilton.

Example of a transverse foliation τ in $\mathbb{R}^d$:



Remarks

- $\mathrm{Inner}(\tau_I)$ maps each circle to itself
- $\mathrm{Sym}(\tau_I)$ allows to exchange circles
- Both preserve τ_I and fix the origin

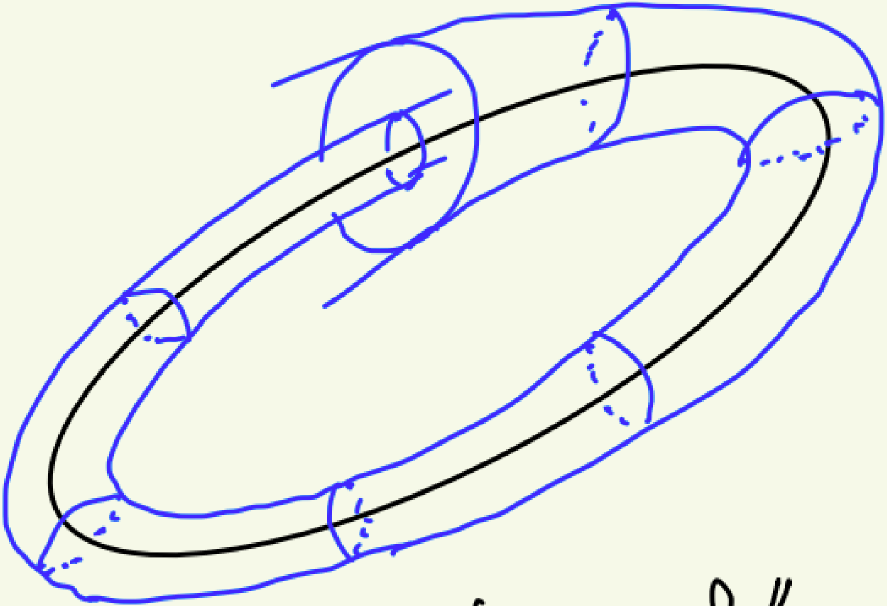
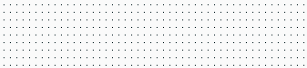


Diagram drawn by Camille Laurent-Gengoux

Sources

Other example: Regular foliation



Recovering the ordinary definition

Foliation	Connection
Regular	Flat lift
Singular	Family of possibly curved lifts

Remarks

Inner(τ_I): Trivial.

Sym(τ_I): We essentially need the image of a group morphism $\pi_1(L) \to \mathrm{Diff}\left(\mathbb{R}^d, 0\right)$.

Idea

We guess:

$$\mathscr{F} = \left\{ \begin{matrix} \text{Some map } \pi_1(L) \to \mathrm{Diff}\left(\mathbb{R}^d, 0\right) \ \text{(at l)} \\ \text{Bundle structure by } \tau_l, \mathrm{Inner}(\tau_l), \mathrm{Sym}(\tau_l), \ldots \end{matrix} \right\}$$

Thus, we want to classify $\mathscr{F}$ with given L and τ_l (for a fixed $l \in L$).

Danger

But $\mathrm{Sym}(\tau_l)$ and $\mathrm{Inner}(\tau_l)$ are in general infinite-dimensional, so that we have to deal with infinite-dimensional geometry!

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Definition (Formal foliation)

$X \in \mathscr{F}$ induces a derivation on $\hat{C} := C^\infty(M)/C_0^\infty(M)$, where $C_0^\infty(M)$ is the ideal of functions vanishing with all their derivatives along L. The image of $\mathscr{F}$ under this is the **formal singular foliation**.

Remarks

$f \in \hat{C}$ a formal power series, w.r.t. $(x_1, \ldots, x_d)$ as "normal coordinates":

$$f = \sum_{i_1, \ldots, i_d \geq 0} f_{i_1, \ldots, i_d} x_1^{i_1} \ldots x_d^{i_d}$$

with $f_{i_1, \ldots, i_d} \in C^\infty(L)$.

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Example (Canonical example of a formal foliation)
For embedded submanifolds L:

- Normal bundle $\mathscr{T}$: $\mathrm{T}M|_L/\mathrm{T}L$.
- Formal vector fields via tubular neighbourhood embedding, a Lie algebra morphism $\mathfrak{X}(M) \to \mathfrak{X}^{\text{formal}}$.

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Our aim

Remarks (Our assumptions)

- τ_l a formal singular foliation.
- L a manifold (connected immersed submanifold of M).

$$\begin{tikzcd}
\text{Foliation } \mathscr{F} \text{ with } L \text{ and } \tau_l \arrow[r, leftrightarrow] \arrow[r, bend left, "\text{formal}"] & \text{Classification} \arrow[l, bend left, "\text{does not need to be formal}"]
\end{tikzcd}$$

Remarks (Avoiding formal setting)

Either

- add real-analyticity conditions to the classification,

or also

- assume embedded L and real-analytic $\mathscr{F}$.

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Multiplicative Yang-Mills connections

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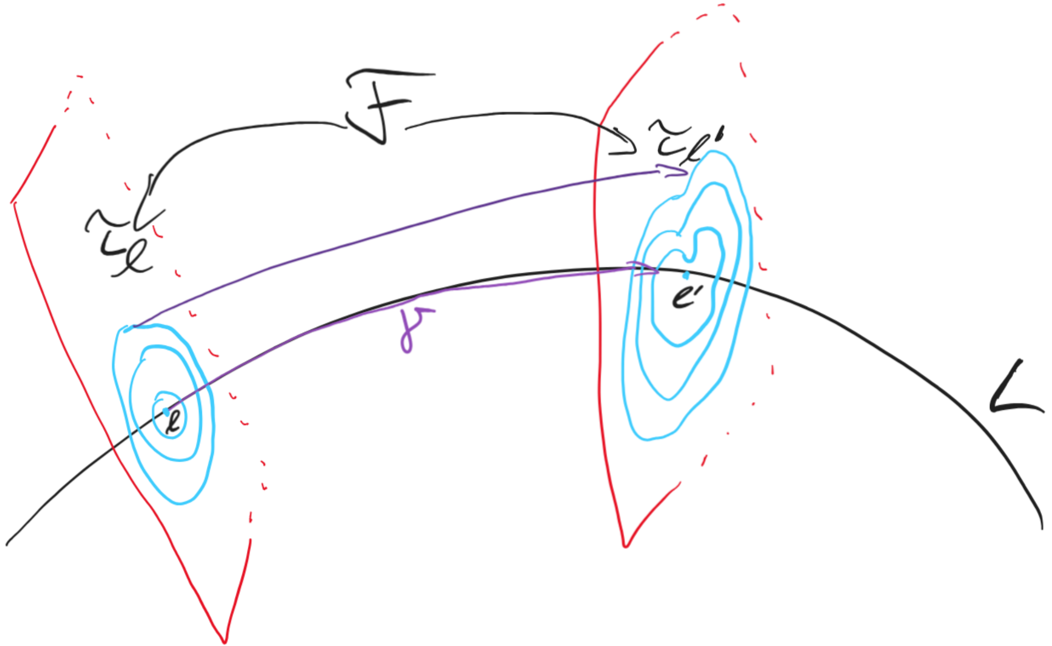
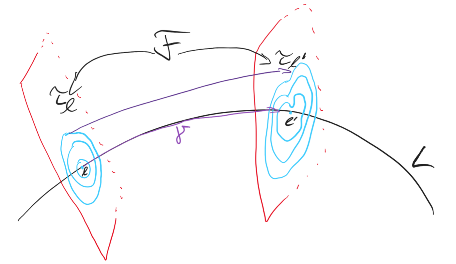


$\mathcal{F}$
$\tilde{\tau}_\ell$
$\tilde{\tau}_{\ell'}$
ℓ
ℓ'
γ
L



Remarks ($\mathscr{F}$-connection)

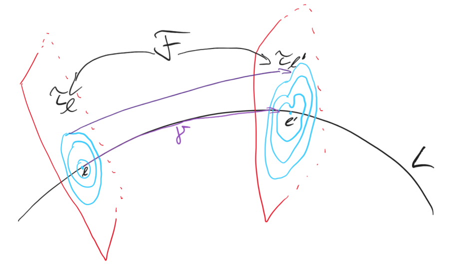
For $\phi \in \mathrm{Sym}(\tau_l)$ we have an induced parallel transport

$$\mathrm{PT}_\gamma^{\mathrm{Sym}}(\phi) := \mathrm{PT}_\gamma \circ \phi \circ \mathrm{PT}_\gamma^{-1}.$$

Then, on the normal bundle $\pi \colon \mathscr{T} \to L$,

$$\mathrm{PT}_\gamma(\phi \cdot p) = \mathrm{PT}_\gamma^{\mathrm{Sym}}(\phi) \cdot \mathrm{PT}_\gamma(p)$$
$$\mathrm{PT}_{\gamma_0}(p) = \varphi \cdot p$$

for all $p \in \mathscr{T}_l$, $\phi \in \mathrm{Sym}(\tau_l)$, and for some $\varphi \in \mathrm{Inner}(\tau_l)$.



Remarks (Sym-**connection)**

For $\phi \in \mathsf{Sym}(\tau_l)$ we have an induced parallel transport

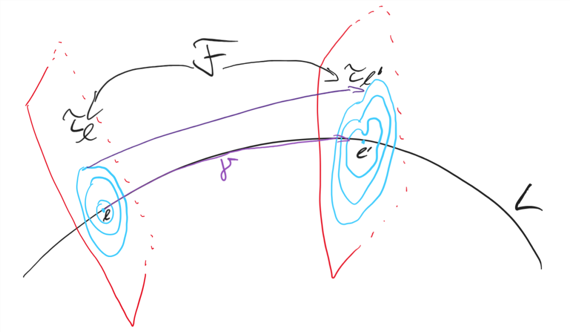
$$\mathsf{PT}^{\mathsf{Sym}}_{\gamma}(\phi) := \mathsf{PT}_{\gamma} \circ \phi \circ \mathsf{PT}_{\gamma}^{-1}.$$

Then

$$\mathsf{PT}^{\mathsf{Sym}}_{\gamma}(\phi \circ \phi') = \mathsf{PT}^{\mathsf{Sym}}_{\gamma}(\phi) \circ \mathsf{PT}^{\mathsf{Sym}}_{\gamma}(\phi')$$
$$\mathsf{PT}^{\mathsf{Sym}}_{\gamma_0}(\phi) = \varphi \circ \phi \circ \varphi^{-1}$$

for all $\phi, \phi' \in \mathsf{Sym}(\tau_l)$, and for some $\varphi \in \mathsf{Inner}(\tau_l)$.

Idea

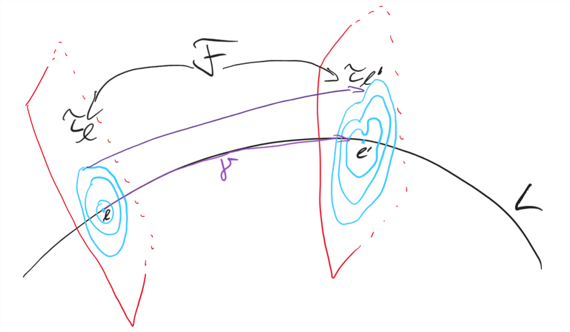


Idea

Generators of $\mathscr{F}$ given by:

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\mathbb{H}(X)$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.



Idea

Fix $l \in L$, given τ and $\mathbb{H}$. Reconstruct $\mathscr{F}$.

$$
\begin{aligned}
[\mathbb{H}(X) + \overline{\nu}, \mathbb{H}(X') + \overline{\mu}] &= \mathbb{H}([X, X']) + \dots \\
&= [\mathbb{H}(X), \mathbb{H}(X')] \\
&\qquad + [\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}] + \overline{[\nu, \mu]}
\end{aligned}
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&\qquad + \underbrace{[\mathbb{H}(X), \overline{\mu}] - [\mathbb{H}(X'), \overline{\nu}]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu,\mu]}
\end{aligned}
$$

Idea $(\dots \in \tau)$

We need:

1. Lie algebra bundle τ with structure τ_I
2. A horizontal lift $\mathbb{H}$ into $\mathscr{T}$ satisfying

$$
\begin{aligned}
\text{Curvature:} &\quad [\mathbb{H}(X), \mathbb{H}(X')] - \mathbb{H}([X,X']) \in \tau \\
\text{Connection:} &\quad [\mathbb{H}(X), \overline{\mu}] \in \tau
\end{aligned}
$$

Summary

$$\begin{tikzcd}
 & \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Foliation } \mathscr{F} \arrow[dr, leftrightarrow] & & ? \\
 & \{\text{Connection on } \mathrm{Sym}(\tau)\} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

Remarks

Connections

- preserve group bundle action,
- and their curvatures follow corresponding orbits.

The pair of connections may **not** be unique for a given $\mathscr{F}$!

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Motivation
What are Ehresmann connections, preserving $\mathscr{G}$-actions?

Definition (LGB actions)

$$\begin{tikzcd} \mathscr{G} \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\ L & \mathscr{T} \arrow[l, "\pi"] \end{tikzcd}$$

A **left-action of** $\mathscr{G}$ **on** $\mathscr{T}$ is a smooth map $\mathscr{G} * \mathscr{T} := \mathscr{G}\ {}_{\pi_{\mathscr{G}}}\!\times_{\pi} \mathscr{T} \to \mathscr{T}$, $(g, p) \mapsto g \cdot p$, satisfying the following properties:

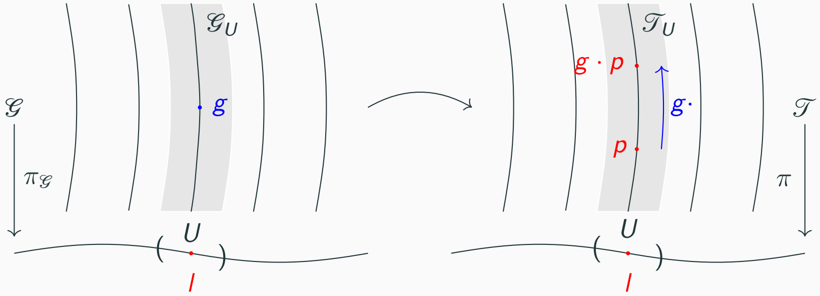
$$\pi(g \cdot p) = \pi(p),$$

$$h \cdot (g \cdot p) = (hg) \cdot p,$$

$$e_{\pi(p)} \cdot p = p$$

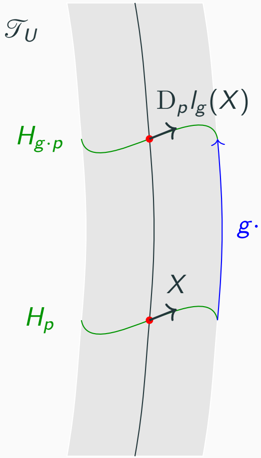
for all $p \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

Connection on $\mathscr{T}$: Idea



Connection on $\mathscr{T}$: Revisiting the classical setup

If $\mathscr{G}$ is trivial, and H a connection:



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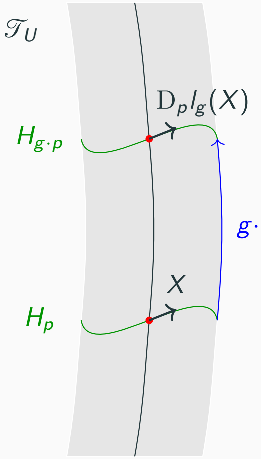

Remarks (Integrated case)

Parallel transport $\mathrm{PT}^{\mathscr{T}}_{\gamma}$ in $\mathscr{T}$:

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(g \cdot p) = g \cdot \mathrm{PT}^{\mathscr{T}}_{\gamma}(p),$$

where $\gamma : I \to L$ is a base path

Connection on $\mathscr{T}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(g \cdot p) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(g) \cdot \mathrm{PT}^{\mathscr{T}}_{\gamma}(p).$$

Recovering the ordinary definition

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])
A surjective submersion $\pi\colon \mathscr{T} \to L$ so that one has a commuting diagram

$$\begin{tikzcd} \mathscr{G} \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\ L & \mathscr{T} \arrow[l, "\pi"] \end{tikzcd}$$

1. **Ehresmann connection:**

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(g \cdot p) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(g) \cdot \mathrm{PT}^{\mathscr{T}}_{\gamma}(p)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(p) = g_{\gamma_0} \cdot p$$

for some $g_{\gamma_0} \in \mathscr{G}^0_{\pi(p)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])
On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathsf{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathsf{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathsf{PT}^{\mathscr{G}}_{\gamma}(g),$$
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On the Lie algebra bundle $\mathscr{g}$ we have a connection ∇ with

$$\nabla\left([\mu,\nu]_{\mathscr{g}}\right) = [\nabla\mu,\nu]_{\mathscr{g}} + [\mu,\nabla\nu]_{\mathscr{g}},$$
$$R_{\nabla} = \mathrm{ad}\circ\zeta.$$

Example

Consider the Atiyah sequence of a principal G-bundle P:

$$\begin{tikzcd} \mathscr{g} := (P\times\mathfrak{g})/G \arrow[r, hook] & \mathrm{T}P/G \arrow[r, two heads] & \mathrm{T}L \arrow[l, bend right, "\mathbb{H}"'] \end{tikzcd}$$

with splitting $\mathbb{H}\colon \mathrm{T}L\to\mathrm{T}P/G$, where $\mathfrak{g}$ is the Lie algebra. Then

$$\nabla_X\nu = [\mathbb{H}(X),\nu]_{\mathrm{T}P/G},$$
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Right diagram made by Mark J.D. Hamilton.

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Foliations and Yang-Mills connections

Going back to foliations

Theorem ([C. L.-G., S.-R. F.])
Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection $\mathbb{H}$ on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$\mathbb{H}(X) + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.
We have

$$[\mathbb{H}(X), \overline{\nu}] = \overline{\nabla_X \nu},$$
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Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

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Proposition ([C. L.-G., S.-R. F.])
The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Proof.
Recall

$$[p, g] \cdot [p, v] = [p, g \cdot v]$$

for all $[p, g] \in \mathscr{G}$ and $[p, v] \in \mathscr{T}$, and

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}([p, v]) = \left[\mathrm{PT}^{P}_{\gamma}(p), v\right].$$

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Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

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The reconstructed foliation is independent of the choice of connection on P.

Proof.

- The adjoint bundle of P, $\mathrm{Ad}(P) := (P \times \mathfrak{g})/G$, is the Lie algebra bundle of $\mathscr{G}$
- $\tau = \overline{\mathrm{Ad}(P)}$
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Generators take this already into account; recall:

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Main Theorems

Theorem ([C. L.-G., S.-R. F.])
In the simply connected case, the following are equivalent:

- *Singular foliations with leaf L and transverse model* $\left(\mathbb{R}^d, \tau_l\right)$
- *Principal* $\mathrm{Inner}(\tau_l)$*-bundles P over L*

$$\begin{tikzcd}
 & \{\text{Connection on } \mathscr{T}\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & \\
\text{Foliation } \mathscr{F} & & P \\
 & \{\text{Connection on Sym}(\tau)\} \arrow[ul, leftrightarrow] \arrow[ur, leftrightarrow] &
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Main Theorems

Theorem ([C. L.-G., S.-R. F.])
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P is now finite-dimensional!

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Corollary
L simply connected and τ_l *is made of vector fields vanishing quadratically at 0. Then the unique singular foliation is the trivial one, i.e. the trivial product of* $(L, \mathfrak{X}(L))$ *and* $\left(\mathbb{R}^d, \tau_l\right)$.

Right diagram made by Mark J.D. Hamilton.

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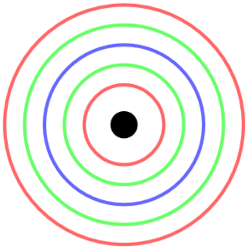
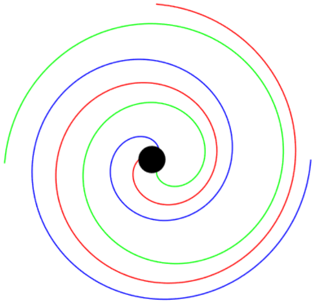
Corollary ([C. L.-G., S.-R. F.])
L simply connected. Then $\mathscr{F}$ *is the trivial foliation if and only if it admits a flat* $\mathscr{F}$*-connection.*

Examples [C. L.-G., S.-R. F.]

$L = \mathbb{S}^2$, $M = \mathrm{T}\mathbb{S}^2$. Let us consider two possible τ_l:

Name	Concentric circles	Spirals
Generator	$x\partial_y - y\partial_x$	$x\partial_y - y\partial_x + (x^2 + y^2)(x\partial_x + y\partial_y)$
Picture of the leaves		
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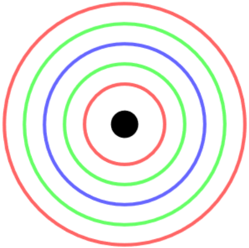
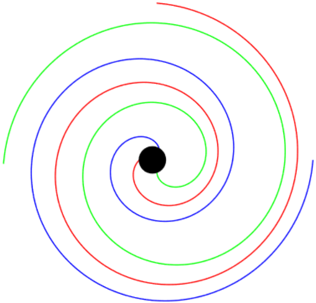
Picture of the leaves		
$\mathrm{Inner}(\tau_l)/\mathrm{Inner}(\tau_l)_{\geq 2}$	$\mathbb{S}^1$	$\mathbb{R}$
Foliation	☺	☹

Thank you! ☺

Total classification

Remarks

Inner(τ_I) is a normal subgroup of Sym(τ_I), thus we have a quotient:

$$\mathsf{Inner}(\tau_I) \hookrightarrow \mathsf{Sym}(\tau_I) \twoheadrightarrow \mathsf{Out}(\tau_I)$$

Theorem ([C. L.-G., S.-R. F.])
The following are equivalent:

- *Singular foliations with leaf L and transverse model* $\left(\mathbb{R}^d, \tau_I\right)$,
- 1. *a group morphism* $\Xi\colon \pi_1(L) \longrightarrow \mathrm{Out}(\tau_I)$, *and*
  2. *a finite-dimensional principal* $H/\mathsf{Inner}(\tau_I)_{\geq 2}$*-bundle, with H a subgroup of* Sym(τ_I) *containing* Inner(τ_I).