

Integrating curved Yang-Mills gauge theories

Gauge theories related to principal bundles equipped with Lie group bundle actions

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Abstract[†]

In this paper we construct a gauge theory based on principal bundles \mathcal{P} equipped with a Lie group bundle action instead of a Lie group action. Due to the fact that pushforwards of right-translation act now only on the vertical structure of \mathcal{P} we fix a connection on \mathcal{E} , which modifies the pushforward via right translation by subtracting the fundamental vector field generated by the Darboux derivative of a section of \mathcal{E} .

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[†]Abbreviations used in this paper: **LGB** for Lie group bundle, **LAB** for Lie algebra bundle.

Contents

1. Introduction	1
1.1. Basic notations and remarks	1
1.2. Assumed background knowledge	2
2. Basic definitions	3
3. Lie group bundles (LGBs)	3
3.1. Definition	3
3.2. Associated Lie group bundles	6
4. LGB actions, part I	12
4.1. Definition	12
4.2. Examples of LGB actions	18
5. Lie algebra bundles (LABs)	25
5.1. Definition	25
5.2. From LGBs to LABs	28
5.3. Vertical Maurer-Cartan form of LGBs	35
5.4. Exponential map of LGBs	37
5.5. LABs of pullback LGBs	40
6. LGB actions, part II	41
6.1. LGB and LAB representations	41
6.2. Fundamental vector fields	49
6.3. Differential of smooth LGB actions	50
7. Connections and curvature on LGB principal bundles	55
7.1. Principal bundles with structural LGB	55
7.1.1. Definition	55
7.1.2. Examples	57
7.1.3. Morphism of principal LGB-bundles	58
7.2. Generalized distributions and connections	67
7.2.1. Idea and motivation	69
7.2.2. Darboux derivative on LGBs	72
7.2.3. First step towards associated bundles	81
7.2.4. Generalized connection 1-forms on principal bundles	83
7.3. Gauge transformations	91
7.4. Generalized curvature/field strength	91
7.5. Generalized covariant derivative/minimal coupling	91

8. Curved Yang-Mills gauge theory	91
9. Conclusion	91
List of References	91
A. Double tangent bundle and its canonical flip map	92
B. Other Interpretations of the compatibility conditions	100
B.1. Minimal coupling	100
B.2. Axiomatic Yang-Mills gauge theories	106

1. Introduction

1.1. Basic notations and remarks

- The appendix serves for providing extra information or background knowledge which did not fit in the flow of this paper's text. We will sometimes refer to the appendix, but the experienced reader may be able to ignore the appendix.
- In the main text we usually repeat and reintroduce needed objects for the statements (like "**Let M be a manifold [...]**" in every statement) (almost) allowing to just read the statements without having read the text introducing it, while the appendix is written as a continuous text which has to be read as a whole in order to understand the essential statements.
- With f^*F we denote the pullback/pull-back of the fibre bundles $F \rightarrow M$ under a smooth map $f : N \rightarrow M$. Similarly we denote the pullbacks of sections of a fibre bundle.
- For $V \rightarrow M$ a vector bundle over M do not confuse the pull-back of sections with the pull-back of forms $\omega \in \Omega^l(M; V)$ ($l \in \mathbb{N}_0$), here denoted by $f^!\omega$, which is an element of $\Gamma\left(\left(\bigwedge_{m=1}^l T^*M\right) \otimes f^*V\right) \cong \Omega^l(M; f^*V)$, and not of $\Gamma\left(\left(\bigotimes_{m=1}^l (f^*TN)^*\right) \otimes f^*V\right)$ like $f^*\omega$.
- Let $F \xrightarrow{\pi_F} M$ and $G \xrightarrow{\pi_G} N$ be two fibre bundles over smooth manifolds M and N , and let $\phi : N \rightarrow M$ be a smooth map. Furthermore, let us assume we have a morphism $\Phi : G \rightarrow F$ of fibre bundles over ϕ , that is, Φ is a smooth map such that the following diagram commutes

$$\begin{array}{ccc}
 G & \xrightarrow{\Phi} & F \\
 \downarrow \pi_G & & \downarrow \pi_F \\
 N & \xrightarrow{\phi} & M
 \end{array}$$

especially, $\pi_F \circ \Phi = \phi \circ \pi_G$. We make often use of that such morphisms have a 1:1 correspondence to **base-preserving** fibre bundle morphisms $\tilde{\Phi} : G \rightarrow \phi^*F$, *i.e.* $\tilde{\Phi}$ is a smooth map with $\phi^*\pi_F \circ \tilde{\Phi} = \pi_G$. For $p \in N$ the morphism $\tilde{\Phi}$ has the form

$$\tilde{\Phi}_p = (p, \Phi_p),$$

that is,

$$\tilde{\Phi}_p(g) = (p, \Phi_p(g))$$

for all $g \in G_p$, which is well-defined since $\Phi_p(g) \in F_{\phi(p)}$. The map $\tilde{\Phi} \mapsto \Phi := \text{pr}_2 \circ \tilde{\Phi}$ is then a bijective map between base-preserving morphisms $G \rightarrow \phi^*F$ and morphisms $G \rightarrow F$ over ϕ , where pr_2 is the projection onto the second component.

In total, $\tilde{\Phi}$ is a base-preserving morphism if and only if Φ is a morphism over ϕ ; in fact, one defines pullback bundles in such a way that this equivalence holds. Observe that $\tilde{\Phi}$ is an isomorphism (diffeomorphism) if and only if Φ is a fibre-wise isomorphism (diffeomorphism).

One can extend all of this similarly for more specific types of morphisms like vector bundle-morphisms.

Most of the time we will not mention this 1:1 correspondence explicitly, it should be clear by context. Hence, we will also denote $\tilde{\Phi}$ by Φ . In fact, we usually calculate with $\tilde{\Phi}$, while Φ and its diagram may only arise to give an illustration about the geometry.

- When we differentiate maps γ depending on just one parameter $t \in \mathbb{R}$, then we may write

$$\frac{d}{dt}\gamma(t) := \left. \frac{d}{dt}[t \mapsto \gamma] \right|_t.$$

1.2. Assumed background knowledge

It is highly recommended to have basic knowledge about differential geometry and gauge theory as presented in [1, especially Chapter 1 to 5], and we will follow the style and labeling as in [1] when we generalize certain notions; however, sometimes we will still give explicit references to help with more technical details. It can be useful to have knowledge about Lie algebra and Lie group bundles, and even Lie algebroids and Lie groupoids, but we will introduce their basic notions such that it is not necessarily needed to have knowledge about these upfront.

We also often give references about Lie group bundles (LGBs), but the given references are often about Lie groupoids. If the reader has no knowledge about Lie groupoids, then it is important to know that LGBs are a special example of Lie groupoids; Lie groupoids carry "two projections", called **source** and **target**. An LGB is a special example of a Lie groupoid whose source equals the target.¹ If you look into such a reference, then the source and target are often

¹But not every Lie groupoid with equal source and target is an LGB, they're in general bundles of Lie groups which is not completely the same; this nuance will not be important here.

denoted by α and β , or by s and t ; simply put both to be the same and identify these with our bundle projection which we often denote by π . In that way it should be possible to read the references without the need to know Lie groupoids. However, we try to re-prove the needed statements such that these types of references could be avoided by the reader.

See also the previous subsection about notions we assume to be known.

2. Basic definitions

3. Lie group bundles (LGBs)

3.1. Definition

Definition 3.1: Lie group bundle, [2, §1.1, Def. 1.1.19; p. 11]

Let G, \mathcal{G}, M be smooth manifolds. A fibre bundle

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{G} \\ & & \downarrow \pi \\ & & M \end{array}$$

is called a **Lie group bundle** if:

1. G and each fibre $\mathcal{G}_x := \pi^{-1}(\{x\})$, $x \in M$, are Lie groups;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} := \text{pr}_2 \circ \phi_i|_{\mathcal{G}_x} : \mathcal{G}_x \rightarrow G$$

are Lie group isomorphisms, where I is an (index) set, U_i are open sets covering M , $\phi_i : \mathcal{G}|_{U_i} \rightarrow U_i \times G$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie group bundle atlas** or **LGB atlas**.

We also often say that \mathcal{G} is an **LGB (over M)**, whose structural Lie group is either clear by context or not explicitly needed; and we may also denote LGBs by $G \rightarrow \mathcal{G} \xrightarrow{\pi} M$.

Remark 3.2: Principal and Lie group bundles

Beware, a Lie group bundle is **not** the same as a principal bundle $P \rightarrow M$ with the same fibre type G . First of all, the fibres of P are just diffeomorphic to a Lie group, a priori they carry no Lie group structure, while the fibres of \mathcal{G} carry a Lie group structure.

Second, on P we have a multiplication given as an action of G on P

$$P \times G \rightarrow P,$$

preserving the fibres P_x ($x \in M$) and simply transitive on them. Restricted on P_x we

have

$$P_x \times G \rightarrow P_x.$$

For \mathcal{G} we have canonically a multiplication over x given by

$$\mathcal{G}_x \times \mathcal{G}_x \rightarrow \mathcal{G}_x,$$

also clearly simply transitive. Observe, the second factor is not "constant", *i.e.* we do not have $\mathcal{G}_x \times G \rightarrow \mathcal{G}_x$ in general. Hence, there is in general no well-defined product $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$.

All of that is also resembled in the existence of sections. The existence of a section of P has a 1:1 correspondence to trivializations of P , which is why P in general only admits sections locally; see *e.g.* [1, §4.2, Thm. 4.2.19; page 219f.]. \mathcal{G} clearly admits always a global section, even if \mathcal{G} is non-trivial; just take the section which assigns each base point the neutral element of its fibre.

If M is a point we recover the notion of Lie groups, and, as usual, we have the notion of trivial LGBs:

Example 3.3: Trivial examples

The **trivial LGB** is given as the product manifold $M \times G \rightarrow M$ with canonical multiplication $(x, g) \cdot (x, q) := (x, gq)$, and we recover the notion of a Lie group in the case of $M = \{*\}$.

We are of course also interested into LGB bundle morphisms:

Definition 3.4: LGB morphism,

[2, §1.2, special situation of Def. 1.2.1 & 1.2.3, page 12]

Let $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ and $\mathcal{H} \xrightarrow{\pi_{\mathcal{H}}} N$ be two LGBs over two smooth manifolds M and N . An **LGB morphism** is a pair of smooth maps $F : \mathcal{H} \rightarrow \mathcal{G}$ and $f : N \rightarrow M$ such that

$$\pi_{\mathcal{G}} \circ F = f \circ \pi_{\mathcal{H}}, \tag{1}$$

$$F(gq) = F(g) F(q) \tag{2}$$

for all $g, q \in \mathcal{H}$ with $\pi_{\mathcal{H}}(g) = \pi_{\mathcal{H}}(q)$. We then also say that F is an **LGB morphism over f** . If $N = M$ and $f = \text{id}_M$, then we often omit mentioning f explicitly and just write that F is a **(base-preserving) LGB morphism**.

We speak of an **LGB isomorphism (over f)** if F is a diffeomorphism.

Remarks 3.5.

- The right hand side of Eq. (2) is well-defined because of Eq. (1).
- It is clear that condition 2 in Def. 3.1 is equivalent to say that \mathcal{G} is locally isomorphic to a trivial LGB; as one may have expected already.
- If F is a diffeomorphism, then also f : By Eq. (1) surjectivity of f is clear; for $y \in M$ just take any $g \in \mathcal{G}_y$, and since F is a bijective, we have a $q \in \mathcal{H}_x$ for some $x \in N$ with $F(q) = g$. By Eq. (1) we have $y = \pi_{\mathcal{G}}(F(q)) \stackrel{(1)}{=} f(x)$, thence, surjectivity follows. For injectivity we know by Eq. (2) and (1) that $F(e_x^{\mathcal{H}}) = e_{f(x)}^{\mathcal{G}}$, where $e_x^{\mathcal{H}}$ and $e_{f(x)}^{\mathcal{G}}$ denote the unique neutral elements of \mathcal{H}_x and $\mathcal{G}_{f(x)}$, respectively. Assume that there are $x, x' \in N$ with $f(x) = f(x')$, then we can derive

$$F(e_x^{\mathcal{H}}) = e_{f(x)}^{\mathcal{G}} = e_{f(x')}^{\mathcal{G}} = F(e_{x'}^{\mathcal{H}}).$$

Then we have $e_x^{\mathcal{H}} = e_{x'}^{\mathcal{H}}$ due to that F is bijective, and hence $x = x'$. Therefore f is bijective. Finally, F^{-1} is by assumption also a diffeomorphism, Eq. (2) clearly carries over, and Eq. (1) is clearly w.r.t. f^{-1} , that is

$$\pi_{\mathcal{H}} \circ F^{-1} = f^{-1} \circ \pi_{\mathcal{G}}.$$

Since $\pi_{\mathcal{H}} \circ F^{-1}$ is smooth and $\pi_{\mathcal{G}}$ is a smooth surjective submersion, it follows that f^{-1} is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We can conclude that f is a diffeomorphism. Observe that we also concluded that F^{-1} is an LGB isomorphism, too.

Similar to the case of Lie groups, *the* example of an LGB are the automorphisms of a vector bundle.

Example 3.6: Automorphisms of a vector bundle,
[2, §1.1, special situation of Ex. 1.1.12, page 8]

Let $V \rightarrow M$ be a vector bundle and $\text{Aut}(V) \rightarrow M$ its bundle of fibre-wise automorphisms (not to be confused with the sections of $\text{Aut}(V)$ which are the base-preserving automorphisms of V). Denote with W the structural vector space of V , then $\text{Aut}(V)$ is an LGB with structural Lie group $\text{Aut}(W)$. It is clear that each fibre of $\text{Aut}(V)$ is a Lie group, and the LGB atlas is directly inherited by a vector bundle atlas $\{(U_i, L_i)\}_{i \in I}$ of V , where we use a similar notation as for LGB atlases, especially we have vector bundle trivializations $L_i : V|_{U_i} \rightarrow U_i \times W$. Then define an LGB atlas over the same open covering $(U_i)_i$ by

$$\text{Aut}(V)|_{U_i} \rightarrow U_i \times \text{Aut}(W),$$

$$T \mapsto L_i \circ T \circ L_i^{-1}|_{\{x\} \times W},$$

where $T \in \text{Aut}(V)|_x = \text{Aut}(V_x)$, and $U_i \times \text{Aut}(W)$ acts canonically on $U_i \times W$ in a fibre-wise sense. Then it is trivial to check that these give local trivializations such that $\text{Aut}(V)$ carries the structure as an LGB.

3.2. Associated Lie group bundles

For another important example recall that there is the notion of associated fibre bundles; following and stating the results of [2, §1, Construction 1.3.8, page 20] and [1, §4.7, page 237ff.; see also Rem. 4.7.8, page 242f.]: Let $P \xrightarrow{\pi_P} M$ be a principal bundle with structural Lie group G , a smooth manifold N and a smooth left G -action Ψ given by

$$G \times N \rightarrow N,$$

$$(g, v) \mapsto \Psi(g, v) := g \cdot v.$$

Then we have a right G -action on $P \times N$ given by

$$(P \times N) \times G \rightarrow P \times N,$$

$$(p, v, g) \mapsto (p \cdot g, g^{-1} \cdot v),$$

and one can show that the quotient under this action, $P \times_{\Psi} N := (P \times N)/G$, yields the structure of a fibre bundle

$$\begin{array}{ccc} N & \longrightarrow & P \times_{\Psi} N \\ & & \downarrow \pi_{P \times_{\Psi} N} \\ & & M \end{array}$$

such that the projection $P \times N \rightarrow P \times_{\Psi} N$ is a smooth surjective submersion, where the projection $\pi_{P \times_{\Psi} N} : P \times_{\Psi} N \rightarrow M$ is given by

$$\pi_{P \times_{\Psi} N}([p, v]) := \pi_P(p)$$

for all $[p, v] \in P \times_{\Psi} N$, denoting equivalence classes of (p, v) by square brackets. For $x \in M$, the fibre $(P \times_{\Psi} N)_x$ is given by $(P_x \times N)/G = P_x \times_{\Psi} N$, and the fibre is diffeomorphic to N by $N \ni v \mapsto [p, v] \in (P \times_{\Psi} N)_x$ for a fixed $p \in P_x$. We will frequently use this diffeomorphism in the following without further notice.

A very important example are of course associated vector bundles, related to N being a vector space. We need a similar concept for Lie groups.

Definition 3.7: Lie group representation on Lie groups,

[2, special situation of the comment after Ex. 1.7.14, page 47]

Let G, H be Lie groups. Then a **Lie group representation of G on H** is a smooth left action ψ of G on H

$$G \times H \rightarrow H,$$

$$(g, h) \mapsto \psi_g(h) := \psi(g, h)$$

such that

$$\psi_g(hq) = \psi_g(h) \psi_g(q) \quad (3)$$

for all $g \in G$ and $h, q \in H$.

Remark 3.8: Note about labeling

Observe that we have by the definition of group actions

$$\psi_{gg'} = \psi_g \circ \psi_{g'}$$

for all $g, g' \in G$, viewing ψ_g as a map $H \rightarrow H$. Therefore we can view the action ψ as a homomorphism

$$G \rightarrow \text{Aut}(H),$$

where $\text{Aut}(H)$ is the set of Lie group automorphisms. The similarity to Lie group representations on vector spaces is obvious, thence the name.

This definition is of course also motivated by various references pointing out that Lie group representations define Lie group actions with extra properties; see for example [1, §3, Ex. 3.4.2, page 143f.]. In [2, comments after Ex. 1.7.14, page 47] this definition is also called *action by Lie group isomorphisms*.

With this we can discuss and define associated Lie group bundles; the following definition is clearly motivated by the definition of associated vector bundles as provided in [1, §4, Thm. 4.7.2, page 239f.].

Theorem 3.9: Associated Lie group bundle as quotient

Let G, H be Lie groups, $P \xrightarrow{\pi_P} M$ a principal G -bundle over a smooth manifold M , and ψ a G -representation on H . Then $\mathcal{H} := P \times_{\psi} H$ is an LGB

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{H} \\ & & \downarrow \pi \\ & & M \end{array}$$

with projection π given by

$$\mathcal{H} \rightarrow M,$$

$$[p, h] \mapsto \pi_P(p), \quad (4)$$

and fibres

$$\mathcal{H}_x = P_x \times_{\psi} H \quad (5)$$

for all $x \in M$, which are isomorphic to H as Lie groups. The Lie group structure on each fibre \mathcal{H}_x is defined by

$$[p, h] \cdot [p, q] := [p, hq] \quad (6)$$

for all $h, q \in H$ and $p_x \in P_x$, where $\pi_P(p) = x$.

Remark 3.10: Neutral and inverse elements

The neutral element for \mathcal{H}_x ($x \in M$) is given by

$$e_x = [p, e],$$

where $p \in P_x$ is arbitrary and e is the neutral element of H . This is clearly independent of the choice of p due to

$$[p, e] = [p \cdot g, \psi_{g^{-1}}(e)] = [p \cdot g, e]$$

for all $g \in G$. Thence, the fact that e_x is the neutral element follows immediately.

By Def. (6) the inverse of $[p, h] \in \mathcal{H}_x$ is clearly given by

$$([p, h])^{-1} = [p, h^{-1}].$$

Proof.

• That π is the well-defined projection and that the fibres are precisely $P_x \times_\psi H$ for all $x \in M$ is well-known, see our discussion before Def. 3.7 and the references therein; it is also very straightforward to check. We also discussed that \mathcal{H} is a fibre bundle with structural fibre H . Hence, if one knows that the proposed group structure in Def. (6) is well-defined, then the smoothness of the group structure is implied by the smoothness structures of H and \mathcal{H} . Thence, let us check whether Def. (6) is well-defined. Let $x \in M$, $p \in P_x$ and $p' := p \cdot g'$ be another element of P_x , where $g' \in G$. Also let $[p_1, h_1], [p_2, h_2] \in P_x \times_\psi H$; then we have unique elements q_i, q'_i of G such that ($i \in \{1, 2\}$)

$$p_i = p \cdot q_i, \quad p_i = p' \cdot q'_i,$$

especially, it follows $q_i = g' q'_i$. On the one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, h_1] \cdot [p_2, h_2] = [p, \psi_{q_1}(h_1)] \cdot [p, \psi_{q_2}(h_2)] = [p, \psi_{q_1}(h_1) \psi_{q_2}(h_2)], \quad (7)$$

on the other hand, using Def. 3.7 and $p' = p \cdot g'$ instead of p ,

$$[p_1, h_1] \cdot [p_2, h_2] = [p \cdot g', \psi_{q'_1}(h_1) \psi_{q'_2}(h_2)]$$

$$\begin{aligned}
&= \left[p, \underbrace{\psi_{g'}(\psi_{q'_1}(h_1) \psi_{q'_2}(h_2))}_{=\psi_{g'}(\psi_{q'_1}(h_1)) \psi_{g'}(\psi_{q'_2}(h_2))} \right] \\
&= \left[p, \psi_{g'q'_1}(h_1) \psi_{g'q'_2}(h_2) \right] \\
&= [p, \psi_{q_1}(h_1) \psi_{q_2}(h_2)],
\end{aligned}$$

which implies that Def. (6) is well-defined, and thus defines a Lie group structure on each fibre of \mathcal{H} .

- That the fibres \mathcal{H}_x are isomorphic to H as Lie groups for all $x \in M$ also quickly follows. Recall by our discussion before Def. 3.7 that the fibres are diffeomorphic to H by $H \ni h \mapsto [p, h] \in \mathcal{H}_x$ for a fixed $p \in P_x$. By Def. (6) it is clear that this map is a Lie group homomorphism and hence a Lie group isomorphism.

- Let us now construct an LGB atlas for \mathcal{H} by using a principal bundle atlas for P . That is, for some $U \subset M$ open and a trivialization $\varphi_U : P|_U \rightarrow U \times G$ we write

$$\varphi_U(p) = (\pi_P(p), \beta_U(p))$$

for all $p \in P$, where $\beta_U : P|_U \rightarrow G$ is an equivariant map, *i.e.* $\beta_U(p \cdot g) = \beta_U(p) g$ for all $g \in G$. Then define ϕ_U as a map by

$$\mathcal{H}|_U \rightarrow U \times H,$$

$$[p, h] \mapsto (\pi_P(p), \psi_{\beta_U(p)}(h)).$$

ϕ_U is well-defined: Let $[p', h'] \in \mathcal{H}|_U$ with $[p', h'] = [p, h]$. Then there is a $g \in G$ such that

$$(p', h') = (p \cdot g, \psi_{g^{-1}}(h)),$$

hence, using the equivariance of β_U and Def. 3.7,

$$\begin{aligned}
\phi_U([p', h']) &= \left(\underbrace{\pi_P(p \cdot g)}_{=\pi_P(p)}, \underbrace{(\psi_{\beta_U(p \cdot g)} \circ \psi_{g^{-1}})(h)}_{=\psi_{\beta_U(p)} \circ \psi_g \circ \psi_{g^{-1}}}(h) \right) = (\pi_P(p), \psi_{\beta_U(p)}(h)) = \phi_U([p, h]),
\end{aligned}$$

which proves that ϕ_U is well-defined. Denote the projection onto equivalence classes $P \times H \rightarrow \mathcal{H}$ by ϖ , then observe

$$\phi_U \circ \varpi = L,$$

where $L_U : P|_U \times H \rightarrow U \times H$ is given by $L_U(p, h) := (\pi_P(p), \psi_{\beta_U(p)}(h))$ for all $(p, h) \in P|_U \times H$. L_U is clearly smooth and recall that ϖ is a smooth surjective submersion, therefore ϕ_U is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We define a candidate of the inverse $\phi_U^{-1} : U \times H \rightarrow \mathcal{H}|_U$ by

$$\phi_U^{-1}(x, h) = [\varphi_U^{-1}(x, e), h]$$

for all $(x, h) \in U \times H$, where e is the neutral element of G . By the definition of φ_U we immediately get

$$(\varphi_U \circ \varphi_U^{-1})(x, e) = \left(\pi_P(\varphi_U^{-1}(x, e)), \beta_U(\varphi_U^{-1}(x, e)) \right) = (x, e),$$

for all $x \in U$, and, also using again the equivariance of β_U ,

$$\begin{aligned} \varphi_U^{-1}(\pi_P(p), e) &= \varphi_U^{-1}\left(\pi_P(p \cdot \beta_U^{-1}(p)), \beta_U(p \cdot \beta_U^{-1}(p))\right) \\ &= \varphi_U^{-1}\left(\pi_P(p \cdot \beta_U^{-1}(p)), \beta_U(p \cdot \beta_U^{-1}(p))\right) \\ &= (\varphi_U^{-1} \circ \varphi_U)(p \cdot \beta_U^{-1}(p)) \\ &= p \cdot \beta_U^{-1}(p) \end{aligned}$$

for all $p \in P|_U$. Then

$$(\phi_U \circ \phi_U^{-1})(x, h) = \left(\pi_P(\varphi_U^{-1}(x, e)), \psi_{\beta_U(\varphi_U^{-1}(x, e))}(h) \right) = (x, \psi_e(h)) = (x, h),$$

for all $(x, h) \in U \times H$, and

$$\begin{aligned} (\phi_U^{-1} \circ \phi_U)([p, h]) &= \left[\underbrace{\varphi_U^{-1}(\pi_P(p), e)}_{=p \cdot \beta_U^{-1}(p)}, \psi_{\beta_U(p)}(h) \right] \\ &= [p, h] \end{aligned}$$

for all $[p, h] \in \mathcal{H}|_U$. Thus, ϕ_U is bijective; additionally observe

$$\phi_U^{-1}(x, h) = \varpi(\varphi_U^{-1}(x, e), h)$$

such that ϕ_U^{-1} is clearly smooth as the composition of smooth maps, and we therefore conclude that ϕ_U is a diffeomorphism. Finally, derive with Def. 3.7 and Eq. (7) that

$$\begin{aligned} (\text{pr}_2 \circ \phi_U)([p_1, h_1] \cdot [p_2, h_2]) &= (\text{pr}_2 \circ \phi_U)([p, \psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)]) \\ &= \psi_{\beta_U(p)}(\psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)) \\ &= \underbrace{\psi_{\beta_U(p)}(\psi_{q_1}(h_1))}_{=\psi_{\beta_U(p) \cdot q_1}(h)} \cdot \psi_{\beta_U(p)}(\psi_{q_2}(h_2)) \\ &= \psi_{\beta_U(p_1)}(h) \cdot \psi_{\beta_U(p_2)}(h) \\ &= (\text{pr}_2 \circ \phi_U)([p_1, h_1]) \cdot (\text{pr}_2 \circ \phi_U)([p_2, h_2]) \end{aligned}$$

for all $[p_1, h_1], [p_2, h_2] \in \mathcal{H}_x$, where we used again the equivariance of β_U and the same notation as introduced for Eq. (7), and pr_2 denotes the projection onto the second factor. Thence, $\text{pr}_2 \circ \phi_U$ induces Lie group isomorphisms $\mathcal{H}_x \rightarrow H$ for all $x \in U$; by Def. 3.1 we can finally conclude that \mathcal{H} is an LGB. ■

Hence, we define:

Definition 3.11: Associated Lie group bundle,
labeling similar to [1, §4.7, Def. 4.7.3, page 240]

Let G, H be Lie groups, $P \xrightarrow{\pi_P} M$ a principal G -bundle over a smooth manifold M , and ψ a G -representation on H . Then we call the LGB

$$\mathcal{H} := P \times_{\psi} H = (P \times H) / G$$

the **Lie group bundle (LGB) associated** to the principal bundle P and the representation ψ on H :

$$\begin{array}{ccc} H & \longrightarrow & P \times_{\psi} H \\ & & \downarrow \pi_{\mathcal{H}} \\ & & M \end{array}$$

The special situation of $H = G$ is already an important example:

Example 3.12: Inner group bundle,
[2, §1, paragraph after Def. 1.1.19, page 11; comment after Construction 1.3.8, page 20]

The **inner group bundle** or **inner LGB** of a principal bundle $P \rightarrow M$, denoted by $c_G(P)$, is defined by

$$c_G(P) := P \times_{c_G} G, \tag{8}$$

where $c_G : G \times G \rightarrow G$ is the left action of G on itself given by the very well-known **conjugation**

$$c_G(g, h) := c_g(h) = (L_g \circ R_{g^{-1}})(h) = ghg^{-1} \tag{9}$$

for all $g, h \in G$, where we also denote left- and right-multiplications (with g) by L_g and R_g , respectively; see *e.g.* [1, beginning of §1.5.2, page 40f.] for its common properties. It is well-known that c_G satisfies the properties of a Lie group representation of G on itself in the sense of Def. 3.7.

$c_G(P)$ is an LGB by Thm. 3.9.

4. LGB actions, part I

4.1. Definition

As for Lie groups, we are interested into their actions. The idea is the following, similar to [2, §1.6, discussion around Def. 1.6.1, page 34]: We have an LGB $\mathcal{G} \rightarrow M$ over a smooth manifold M , and we want to construct an action of \mathcal{G} on another smooth manifold N . Each fibre of \mathcal{G} is a Lie group, and we have a notion of Lie groups actions on manifold N . Therefore one could define an LGB action as a collection of Lie group actions, that is, only sections of \mathcal{G} act on N ; however, one then expects that the general outcome of a product of $\Gamma(\mathcal{G})$ on N would be smooth maps from M to N . In order to recover a typical structure of action one could instead introduce a "multiplication rule", *i.e.* each point $p \in N$ can only be multiplied with elements of a specific fibre of \mathcal{G} . This "multiplication rule" will be described by a smooth map $f : N \rightarrow M$ in the sense of that the fibre over $f(p)$ will act on p .

For this recall that there is the notion of pullbacks of fibre bundles, see *e.g.* [1, §4.1.4, page 203ff.; especially Thm. 4.1.17, page 204f.]. That is, if we additionally have a smooth manifold N and a smooth map $f : M \rightarrow N$, then we have the pullback $f^*\mathcal{G}$ of \mathcal{G} as a fibre bundle defined as usual by

$$f^*\mathcal{G} := \{(x, g) \in N \times \mathcal{G} \mid f(x) = \pi(g)\}. \quad (10)$$

It is an embedded submanifold of $N \times \mathcal{G}$, and the structural fibre is the same Lie group as for \mathcal{G} . That is, the following diagram commutes

$$\begin{array}{ccc} f^*\mathcal{G} & \xrightarrow{\pi_2} & \mathcal{G} \\ \downarrow \pi_1 & & \downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

where π_1 and π_2 are the projections onto the first and second factor, respectively, of $N \times \mathcal{G}$. Actually, $f^*\mathcal{G}$ carries a natural structure as an LGB.

Corollary 4.1: Pullbacks of LGBs are LGBs,

[2, §2.3, simplified situation of the discussion around Prop. 2.3.1, page 63ff.]

Let M, N be smooth manifolds, $\mathcal{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \rightarrow M$ a smooth map. Then $f^\mathcal{G}$ has a unique (up to isomorphisms) LGB structure such that the projection $\pi_2 : f^*\mathcal{G} \rightarrow \mathcal{G}$ onto the second factor is an LGB morphism over f with $\pi_2|_x : (f^*\mathcal{G})_x \rightarrow \mathcal{G}_{f(x)}$ being a Lie group isomorphism for all $x \in N$.*

Remarks 4.2.

The mentioned reference, [2, §2.3, discussion around Prop. 2.3.1, page 63ff.], is rather general, formulated for Lie groupoids. If the reader is only interested into LGBs, then see *e.g.* [3, §3, Thm. 3.1].

Proof.

By construction, the structural fibre of $f^*\mathcal{G}$ is the same Lie group G as for \mathcal{G} , and for all $x \in N$ we have $(f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)}$, thence, the fibres are Lie groups and the fibrewise group multiplication has the form

$$(x, g) \cdot (x, q) = (x, gq) \quad (11)$$

for all $x \in N$ and $g, q \in (f^*\mathcal{G})_x$. We are left to show the existence of an LGB atlas. For this fix an LGB atlas $\{(U_i, \phi_i)\}_{i \in I}$ of \mathcal{G} , where I is an (index) set, $(U_i)_{i \in I}$ an open covering of M , and $\phi_i : \mathcal{G}|_{U_i} \rightarrow U_i \times G$ are LGB isomorphisms. Then $f^{-1}(U_i)$ gives rise to an open covering of N , and we get

$$\begin{aligned} f^*\phi_i : f^*\mathcal{G}|_{f^{-1}(U_i)} &\rightarrow f^{-1}(U_i) \times G, \\ (x, g) &\mapsto (x, \phi_{i, f(x)}(g)), \end{aligned}$$

where $\phi_{i, f(x)} : \mathcal{G}_{f(x)} \rightarrow G$ are the Lie group isomorphisms as defined in Def. 3.1. It is immediate by construction that this gives an LGB atlas.

That this is the unique (up to isomorphisms) LGB structure such that $\pi_2 : f^*\mathcal{G} \rightarrow \mathcal{G}$ is an LGB morphism over f inducing a Lie group isomorphism on each fibre simply follows by construction; observe for all $x \in N$ that $\pi_2|_x$ is clearly bijective. Furthermore, LGB morphisms need to be homomorphisms, which means here

$$\pi_2((x, g) \cdot (x, q)) \stackrel{!}{=} \pi_2((x, g)) \cdot \pi_2((x, q)) = gq = \pi_2((x, gq))$$

for all $x \in N$ and $g, q \in (f^*\mathcal{G})_x$. By using the bijectivity of $\pi_2|_x$, the group structure leading to this is uniquely the one provided in Eq. (11). Assume we have another underlying LGB atlas in sense of Cor. 4.1 with an LGB chart ψ_i on $f^*\mathcal{G}|_{f^{-1}(U_i)}$ (or just w.r.t. a subset of U_i), then

$$\phi_i \circ \pi_2 \circ \psi_i^{-1} = \underbrace{\phi_i \circ \pi_2 \circ (f^*\phi_i)^{-1}}_{=(f, \mathbb{1}_G)} \circ f^*\phi_i \circ \psi_i^{-1} = (f, \mathbb{1}_G) \circ f^*\phi_i \circ \psi_i^{-1},$$

If evaluating this at $x \in f^{-1}(U_i)$, then $(f, \mathbb{1}_G)$ is a Lie group automorphism of $\{x\} \times G$, and thus the condition about $\pi_2|_x$ being a Lie group isomorphism enforces that we can write

$$(f^*\phi_i \circ \psi_i^{-1})(x, g) = (x, L_i(x, g))$$

for all $(x, g) \in f^{-1}(U) \times G$, where $L_i(x, \cdot) : G \rightarrow G$ is a Lie group automorphism. So, in total

$$\phi_i \circ \pi_2 \circ \psi_i^{-1} = (f, L),$$

and since f and the left hand side are smooth, L has to be smooth. We can conclude that ψ_i gives rise to an LGB atlas compatible with the one defined by $f^*\phi_i$. This finalizes the proof. ■

Definition 4.3: Pullback LGB

Let M, N be smooth manifolds, $\mathcal{G} \rightarrow M$ an LGB over M and $f : N \rightarrow M$ a smooth map. Then we call the LGB structure on $f^*\mathcal{G}$ as given in Cor. 4.1 the **pullback LGB of \mathcal{G} (under f)**.

We will refer to this structure often without further mention.

Let us now define \mathcal{G} -actions.

Definition 4.4: Lie group bundle actions,

[2, §1.6, special case of Def. 1.6.1, page 34]

Let M, N be smooth manifolds, $\mathcal{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \rightarrow M$ a smooth map. Then a **right-action of \mathcal{G} on N** is a smooth map

$$f^*\mathcal{G} \rightarrow N,$$

$$(p, g) \mapsto p \cdot g,$$

satisfying the following properties:

$$f(p \cdot g) = \pi(g), \tag{12}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{13}$$

$$p \cdot e_{f(p)} = p \tag{14}$$

for all $p \in N$ and $g, h \in \mathcal{G}_{f(p)}$, where $e_{f(p)}$ is the neutral element of $\mathcal{G}_{f(p)}$.

We analogously define left-actions, and we often write (left or right) **\mathcal{G} -action on N** . Furthermore, in order to increase readability as long as the dependency on f is not important, we introduce the notation

$$N * \mathcal{G} := f^*\mathcal{G}, \tag{15}$$

such that the action's notation has the typical shape $N * \mathcal{G} \rightarrow N$; one may also write $N * \mathcal{G} = N \times_M \mathcal{G}$. For left actions similarly $\mathcal{G} * N \rightarrow N$; even though $\mathcal{G} * N$ is the same pullback LGB as for right-actions, we also change the order of notation in this case, that is, $(g, p) \in \mathcal{G} * N$ reads $g \in \mathcal{G}_{f(p)}$ and $p \in N$.

Remark 4.5: Relation to the structure of the canonical pullback Lie group bundle over N

Observe that by the definition of $f^*\mathcal{G}$ we can also write

$$f(p \cdot g) = f(p),$$

so, the \mathcal{G} -action is defined in such a way that f is invariant under it. If f is the projection of N as a bundle over M , then this means that an LGB action is fibre-preserving. Moreover, the fibre-wise group structure on \mathcal{G} naturally defines a \mathcal{G} -action on \mathcal{G} ; in this situation f would be π itself, see also Ex. 4.12. This is mainly a technical condition. On one hand, having $M = \{*\}$ already recovers the notion of a Lie group action and condition (12) is then trivial, and on the other hand the mentioned reference, [2, §1.6, Def. 1.6.1, page 34], actually generalizes this condition making use of the structure of groupoids.

Furthermore, the other conditions are the typical conditions for actions, especially such that we get a \mathcal{G} -action on $f^*\mathcal{G}$ by

$$(p, g) \cdot q := (p \cdot q, q^{-1}g) \quad (16)$$

for all $p \in N$ and $g, q \in \mathcal{G}_{f(p)}$.^a As usual, this gives rise to an equivalence relation, whose set of equivalence classes denoted by $f^*\mathcal{G}/\mathcal{G}$ is isomorphic to N (as a set) by $[p, g] \mapsto p \cdot g$, where we denote equivalence classes of $(p, g) \in f^*\mathcal{G}$ by $[p, g]$. All of this is straight-forward to check. Finally, observe the similarity to associated fibre bundles.

^aIn alignment to Def. 4.4, this action is a map $(f \circ \pi_1)^*\mathcal{G} \rightarrow f^*\mathcal{G}$, where π_1 is the projection onto the first factor in $f^*\mathcal{G}$.

Remark 4.6: Localizing LGB actions

We can actually localize the LGB action, but in general not with respect to any open neighbourhood of N since that is in general not possible in a non-trivial way, *i.e.* the action cannot be brought into the form $(N * \mathcal{G})|_U \rightarrow U$ for arbitrary U because $p \cdot g$ may for example leave an neighbourhood U of p . However, with respect to M this is possible: Fix any open neighbourhood U of M . Then $f^{-1}(U)$ is an open neighbourhood of N , and we can restrict the action to $f^{-1}(U)$, resulting into a map

$$(N * \mathcal{G})|_{f^{-1}(U)} \rightarrow f^{-1}(U),$$

because of $f(p \cdot g) \stackrel{\text{Eq. (12)}}{=} \pi(g) = f(p)$, that is, if $(p, g) \in (N * \mathcal{G})|_{f^{-1}(U)}$, then $p \in f^{-1}(U)$ and so $p \cdot g \in f^{-1}(U)$. In fact, by the definition of $N * \mathcal{G}$ as $f^*\mathcal{G}$, this describes a $\mathcal{G}|_U$ -action on $f^{-1}(U)$

$$f^{-1}(U) * \mathcal{G}|_U := (f|_U)^*(\mathcal{G}|_U) \rightarrow f^{-1}(U).$$

If $x \in M$ is a regular value of f , then $f^{-1}(\{x\})$ is an embedded submanifold of N due to the regular value theorem (for this see *e.g.* [1, §A.1, Thm. A.1.32, page 611]). In that case one could apply the same arguments to restrict the action on $f^{-1}(\{x\})$. Since \mathcal{G}_x is a Lie group one actually gets a typical Lie group action on $f^{-1}(\{x\})$. For more details about this see Ex. 4.14 later.

Remark 4.7: Left- and right-actions

In the following we usually define everything with respect to right-actions; however, one can of course define the same for left actions in a similar manner. If we ever speak of a left action, then we assume precisely this. Some subtle changes like a sign change will be pointed out though.

One can probably see that it is straightforward to extend a lot of the typical notions of Lie group actions to LGB actions; hence, we mainly focus on the definitions and properties which we need in this paper.

Definition 4.8: Left and right translations,

[1, §3.2, notation similar to Def. 3.2.3, page 131]

[2, §1.4, special situation of Def. 1.4.1 and its discussion, page 22]

Let M, N be smooth manifolds, $\mathcal{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \rightarrow M$ a smooth map. Furthermore assume that we have a right action $N * \mathcal{G} \rightarrow N$. We define the **right translation** over $x \in M$ with $g \in \mathcal{G}$ as a map r_g defined by

$$f^{-1}(\{x\}) \rightarrow f^{-1}(\{x\}),$$

$$p \mapsto p \cdot g,$$

and we define the **orbit map** through $p \in f^{-1}(\{x\})$ as a map Φ_p given by

$$\mathcal{G}_x \rightarrow N,$$

$$g \mapsto p \cdot g.$$

For $\sigma \in \Gamma(\mathcal{G})$ we define the **right translation** on N as a map r_σ by

$$N \rightarrow N,$$

$$p \mapsto p \cdot \sigma_{f(p)}.$$

If one has a section of f , *i.e.* a smooth map $\tau : M \rightarrow N$, $x \mapsto \tau_x$, with $f \circ \tau = \mathbb{1}_M$, then we can define the **orbit map** through τ as a map Φ_τ given by

$$\mathcal{G} \rightarrow N,$$

$$g \mapsto \tau_{\pi(g)} \cdot g.$$

Remark 4.9: Left action and translation

Similarly we define left translations for left actions, which we similarly denote by l_g and l_σ . By Rem. 4.6 we can define r_σ (and l_σ) also for local sections $\sigma \in \Gamma(\mathcal{G}|_U)$ by restricting N onto $f^{-1}(U)$, where U is some open subset of M ; then $r_\sigma, l_\sigma : f^{-1}(U) \rightarrow f^{-1}(U)$. In the same manner one achieves a restriction for $\Phi_\tau : \mathcal{G}|_U \rightarrow f^{-1}(U)$, if $\tau : U \rightarrow f^{-1}(U)$. In case of N being \mathcal{G} itself we will denote right (and left) translations via capital letters.

Remark 4.10: Group action on sections

Assume that N is a fibre bundle over M with f as its projection. Also observe that $\Gamma(\mathcal{G})$ is clearly a group and it may be possible to endow it with an infinite-dimensional Lie group structure. In the same fashion as before, we can define a $\Gamma(\mathcal{G})$ -action on $\Gamma(N)$ given by

$$(\tau \cdot \sigma)_x := \tau_x \cdot \sigma_x$$

for all $\tau \in \Gamma(N)$, $\sigma \in \Gamma(\mathcal{G})$, and $x \in M$. It is straightforward to show that this is well-defined. We will make use of this action without further mention.

Remarks 4.11.

Similar to the arguments in [1, §3.2, discussion after Def. 3.2.3, page 131], Φ_p is given by the composition of smooth maps

$$\mathcal{G}_x \rightarrow N * \mathcal{G} \rightarrow N,$$

$$g \mapsto (p, g) \mapsto p \cdot g.$$

The second arrow/map is smooth due to the fact that we have a smooth action; the first one is smooth because $N * \mathcal{G}$ is the pullback LGB $f^*\mathcal{G}$ and the first arrow is precisely the embedding of \mathcal{G}_x into $f^*\mathcal{G}$ as a fibre over p ; recall *e.g.* Cor. 4.1.

For the right-translation r_σ we have a similar argument, namely, r_σ is a composition of smooth maps

$$N \rightarrow N * \mathcal{G} \rightarrow N,$$

$$p \mapsto (p, \sigma_{f(p)}) \mapsto p \cdot \sigma_{f(p)}.$$

The first map describes now a section of $N * \mathcal{G} = f^*\mathcal{G}$, and thus an embedding. Thus, smoothness follows again. Similarly, Φ_τ is the composition of maps

$$\mathcal{G} \rightarrow N * \mathcal{G} \rightarrow N,$$

$$g \mapsto (\tau_{\pi(g)}, g) \mapsto \tau_{\pi(g)} \cdot g,$$

and the first arrow is clearly a smooth map $\mathcal{G} \rightarrow N \times \mathcal{G}$ with values in $f^*\mathcal{G} = N * \mathcal{G}$ which is an embedded submanifold of $N \times \mathcal{G}$. Thence, smoothness follows as usual. In fact, $\Phi_\tau|_{\mathcal{G}_x} = \Phi_{\tau_x}$.

However, for r_g smoothness can only be discussed if $f^{-1}(\{x\})$ is a smooth manifold. That is for example the case if x is a regular value of f ; recall the regular value theorem as cited in [1, §A.1, Thm. A.1.32, page 611]. This would be the case if *e.g.* f is a submersion. If x is a regular value, then $f^{-1}(\{x\})$ is an embedded submanifold of N , and r_g is a similar composition of smooth maps as for r_σ but restricted to $f^{-1}(\{x\})$

$$f^{-1}(\{x\}) \rightarrow N * \mathcal{G}|_{f^{-1}(\{x\})} \rightarrow f^{-1}(\{x\}),$$

$$p \mapsto (p, g) \mapsto p \cdot g.$$

Since $f^{-1}(\{x\})$ is an embedded submanifold, $N * \mathcal{G}|_{f^{-1}(\{x\})}$ is also a fibre bundle, see for example [1, §4.1, Lemma 4.1.16, page 204], and trivially an embedded submanifold of $N * \mathcal{G}$. Altogether, the same arguments as for r_σ apply.

Last but not least, r_σ is clearly a diffeomorphism with inverse $r_{\sigma^{-1}}$, where $(\sigma^{-1})_x = (\sigma_x)^{-1} =: \sigma_x^{-1}$. Similarly for r_g if $f^{-1}(\{x\})$ is an embedded submanifold (otherwise r_g is just a bijection). Analogously for l_σ and l_g in case of a left action.

Motivated by the previous remark, it might be hence useful to require that f is a submersion, or that N is actually some bundle over M and f its projection. In fact, this will be later the case.

4.2. Examples of LGB actions

If M is a point or f a constant map, then we recover the typical notion of a Lie group action acting on N . Additionally, we have the following examples, the last two of which will be important in this paper. The notation will be as in Def. 4.8.

Example 4.12: LGB acting on itself

For each LGB $\mathcal{G} \xrightarrow{\pi} M$ acts on itself from the left and right, having $N := M$ and $f := \pi$,

$$\mathcal{G} * \mathcal{G} := \pi^*\mathcal{G} \rightarrow \mathcal{G},$$

$$(g, h) \mapsto gh.$$

That this satisfies all properties for an LGB action is clear by definition of LGBs; however, let us give a note about the smoothness of this action. Recall that an LGB is locally isomorphic to a trivial LGB $U \times G$ (U an open subset of M) with its canonical group multiplication, $(x, g) \cdot (x, q) = (x, gq)$. Hence, using that the multiplication of G is smooth and using local LGB trivializations of \mathcal{G} and $f^*\mathcal{G}$ (recall Cor. 4.1 and its proof to show that $f^*\mathcal{G}$ is locally diffeomorphic to the product manifold $U \times G \times G$), we achieve smoothness of the \mathcal{G} -action on itself because it is locally of the form

$$U \times G \times G \rightarrow U \times G,$$

$$(x, g, q) \mapsto (x, gq).$$

We will also call the \mathcal{G} -action on itself the **multiplication in \mathcal{G}** .

Similarly one can argue that the **inverse map** $\mathcal{G} \rightarrow \mathcal{G}$, $g \mapsto g^{-1}$, is smooth, too.

Example 4.13: Trivial action, [2, §1.6, special situation of Ex. 1.6.3, page 35]

The projection π_1 onto the first factor of $f^*\mathcal{G} \xrightarrow{\pi_1} N$ satisfies the properties of a right \mathcal{G} -action on N , that is, the action is given by

$$N * \mathcal{G} \rightarrow N,$$

$$(p, g) \mapsto p \cdot g := p.$$

That this action satisfies the properties of an action for all f is trivial, hence we call it the **trivial action**.

Example 4.14: Actions of trivial LGBs

Assume that \mathcal{G} is trivial, that is $\mathcal{G} \cong M \times G$ as LGBs, where G is the structural Lie group of \mathcal{G} . In that case, for $p \in N$, the product $p \cdot q$ is only defined if $q \in \mathcal{G}$ is of the form $(f(p), g)$, where $g \in G$; for this recall that (p, q) needs to be an element of $N * \mathcal{G} = f^*\mathcal{G}$ in order to define $(p, q) \mapsto p \cdot q$. We also have

$$f^*\mathcal{G} \cong N \times G,$$

the trivial G -LGB over N . Hence, let us define a map by

$$N \times G \rightarrow N,$$

$$(p, g) \mapsto p \cdot g := p \cdot (f(p), g),$$

which is clearly a smooth map since it is a composition of the \mathcal{G} -action on N and

$$N \times G \rightarrow f^*\mathcal{G},$$

$$(p, g) \mapsto (p, f(p), g).$$

The latter is smooth because $(p, g) \mapsto (p, f(p), g)$ is a smooth map $N \times G \rightarrow N \times M \times G$ and $f^*\mathcal{G}$ is an embedded submanifold of $N \times M \times G$. Using Def. 4.4, It is trivial to see that $p \cdot e = p$, and

$$\begin{aligned} p \cdot (gh) &= p \cdot \underbrace{(f(p), gh)}_{=(f(p),g) \cdot (f(p),h)} = (p \cdot (f(p), g)) \cdot (f(p), h) = (p \cdot g) \cdot h \end{aligned}$$

for all $g, h \in G$. Hence, we have a G -action on N , and by construction it is equivalent to the \mathcal{G} -action on N . Due to the discussion in Rem. 4.6 we can therefore conclude that every \mathcal{G} -action is locally a typical Lie group action. If f is a submersion, then the action is also a G -action on each fibre.

Observe, that one can therefore recover the notion of Lie group actions not only via $M = \{*\}$, the point manifold, but also via trivial LGBs. Translations with constant sections of $M \times G$ w.r.t. the action $N * \mathcal{G} \rightarrow N$ are trivially to be seen as translations with an element of G w.r.t. the action $N \times G \rightarrow N$.

Hence, if one wants something "truly" new, then one has to look at global structures of LGBs and their actions. In fact, the following example will provide such a new structure, which we will understand later once we have introduced the physical theory.

Example 4.15: Inner group bundle acting on associated fibre bundles,
[2, §1.6, simplified version of Ex. 1.6.4, page 35]

Let $P \xrightarrow{\pi_P} M$ be a principal bundle with structural Lie group G over a smooth manifold M , and recall Ex. 3.12. Furthermore, let F be another smooth manifold, equipped with a smooth left G -action $\Psi : G \times F \rightarrow F$. In total we have two associated bundles over M :

$$\begin{array}{ccc} G & \longrightarrow & c_G(P) \\ & & \downarrow \pi_{c_G(P)} \\ & & M \end{array} \qquad \begin{array}{ccc} F & \longrightarrow & \mathcal{F} := P \times_{\Psi} F \\ & & \downarrow \pi_{\mathcal{F}} \\ & & M \end{array}$$

the inner group bundle of P and an associated F -bundle, respectively.

Then we have a right $c_G(P)$ -action on \mathcal{F} given by

$$\begin{aligned} \mathcal{F} * c_G(P) &:= \pi_{\mathcal{F}}^* c_G(P) \rightarrow \mathcal{F}, \\ ([p, v], [p, g]) &\mapsto [p, \Psi(g, v)] = [p \cdot g, v] \end{aligned}$$

for all $p \in P_x$ ($x \in M$), $g \in G$ and $v \in F$.

Proof.

• We first check again that the action is well-defined, that is, we are going to prove that the action is independent of the choice of fixed point in P_x . Thence, let $x \in M$, $p \in P_x$ and $p' := p \cdot g'$ be another element of P_x , where $g' \in G$. Also let $[p_1, v] \in \mathcal{F}_x$ and $[p_2, g] \in c_G(P)_x$; then we have unique elements q_i, q'_i of G such that ($i \in \{1, 2\}$)

$$p_i = p \cdot q_i, \qquad p_i = p' \cdot q'_i,$$

especially, it follows $q_i = g' q'_i$.

On one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, v] \cdot [p_2, g] = [p, \Psi(q_1, v)] \cdot [p, c_{q_2}(g)] = [p \cdot c_{q_2}(g), \Psi(q_1, v)] = [p \cdot c_{q_2}(g) q_1, v].$$

On the other hand, using p' as a fixed element, we derive, using $q'_i = (g')^{-1}q_i$,

$$[p_1, v] \cdot [p_2, h] = [p' \cdot c_{q'_2}(g) q'_1, v] = [p \cdot g' q'_2 g (q'_2)^{-1} q'_1, v] = [p \cdot q_2 g q_2^{-1} q_1, v] = [p \cdot c_{q_2}(g) q_1, v],$$

which finalizes the argument needed to show that the action is well-defined.

- Let us now quickly check that the conditions in Def. 4.4 are satisfied. We have

$$\pi_{\mathcal{F}}([p, v] \cdot [p, g]) = \pi_{\mathcal{F}}([p, \Psi(g, v)]) = \pi_P(p) = \pi_{c_G(P)}([p, g])$$

for all $p \in P_x$ ($x \in M$), $v \in F$ and $g \in G$; similarly, having additionally $h \in G$,

$$([p, v] \cdot [p, g]) \cdot [p, h] = [p \cdot g, v] \cdot [p, h] = [p \cdot gh, v] = [p, v] \cdot [p, gh] = [p, v] \cdot ([p, g] [p, h]),$$

and

$$[p, v] \cdot [p, e] = [p \cdot e, v] = [p, v].$$

Therefore this describes an action. ■

Remark 4.16: Relation to automorphisms of principal bundles and gauge transformations

Recall that gauge transformations have a strong relation to principal bundle automorphisms f of the principal bundle P ; see *e.g.* [1, §5.3, Def. 5.3.1, page 256f.] and [1, §5.4, Thm. 5.4.4, page 273]. That is, f is a diffeomorphism $P \rightarrow P$ with

$$\pi_P \circ f = \mathbb{1}_M,$$

$$f(p \cdot g) = f(p) \cdot g$$

for all $p \in P$ and $g \in G$. The group of such maps will be denoted by $\mathcal{Aut}(P)$. One can identify such automorphisms with certain G -valued maps on P , following [1, §5.3, Def. 5.3.2 & Prop. 5.3.3, page 266f.]: We define the following set of smooth maps $P \rightarrow G$ by

$$C^\infty(P; G)^G := \{ \sigma : P \rightarrow G \text{ smooth} \mid \sigma(p \cdot g) = c_{g^{-1}}(\sigma(p)) \text{ for all } p \in P, g \in G \}.$$

It is straightforward to check that this is a group w.r.t. pointwise multiplication. Furthermore, there is a group isomorphism

$$\mathcal{Aut}(P) \rightarrow C^\infty(P; G)^G,$$

$$f \mapsto \sigma_f,$$

where σ_f is defined by

$$f(p) = p \cdot \sigma_f(p)$$

for all $p \in P$; one can prove that this is well-defined.

As argued in [1, §5.3, Thm. 5.3.8, page 269; formulated as left action there, which is why we have an inverse here], $\mathcal{Aut}(P)$ acts (on the right) on associated fibre bundles $\mathcal{F} = P \times_{\Psi} F$ by

$$[p, v] \cdot f := [f^{-1}(p), v] = [p \cdot \sigma_f(p)^{-1}, v]$$

for all $[p, v] \in \mathcal{F}_x$ ($x \in M$) and $f \in \mathcal{Aut}(P)$. σ_f can also be just locally defined, therefore one could investigate whether there is also an action just with an element g of G , basically the restriction of σ_f onto the fibre P_x . However, the action given by $[p, v] \cdot g = [p \cdot g^{-1}, v]$ for $g \in G$ is in general clearly only well-defined w.r.t. a change of the representative of $[p, v] = [p \cdot q, \Psi_{q^{-1}}(v)]$ ($q \in G$), if G is abelian. But one can resolve this by looking at it carefully: The rough idea is that g basically comes from $\sigma_f(p)$ in this context, but

$$\sigma_f(p \cdot q) = c_{q^{-1}}(\sigma_f(p)).$$

Roughly, while p is multiplied with g^{-1} , $p \cdot q$ has to be multiplied with $q^{-1}g^{-1}q$. It is easy to check that this resolves that issue, and the result is precisely the action described in Ex. 4.15. In fact, we have the following proposition:

For the following proposition observe that the (local) sections of an LGB have a group structure given by pointwise multiplication.

Proposition 4.17: Gauge transformations as sections of the inner LGB,
[2, §1.4, (the last sentence of) Ex. 1.4.7, page 25]

Let $P \xrightarrow{\pi_P} M$ be a principal bundle with structural Lie group G over a smooth manifold M . Then there is a group isomorphism

$$\mathcal{Aut}(P) \rightarrow \Gamma(c_G(P)),$$

$$f \mapsto q_f$$

where $q_f \in \Gamma(c_G(P))$ is defined by

$$q_f|_x := [p, \sigma_f(p)]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$, and σ_f is the element of $C^\infty(P; G)^G$ corresponding to f as introduced in Rem. 4.16.

Remarks 4.18.

As one may guess, $\Gamma(c_G(P))$ is the analogue of $C^\infty(M; G)^G$ such that one could ask for a more direct analogue to $\mathcal{Aut}(P)$. Indeed, as argued in [2, §1.3, Prop. 1.3.9, page 20], $c_G(P)$ is actually isomorphic to $(P \times_M P) / G$, where $P \times_M P := \pi_P^* P$, and the G -action is the diagonal action

on $P \times P$. One can prove that an isomorphism is given by

$$c_G(P) \rightarrow (P \times_M P) / G,$$

$$[p, g] \mapsto [p, p \cdot g].$$

It is also argued in [2, §1.4, Ex. 1.4.7, page 25] that $\mathcal{A}ut(P)$ is then isomorphic to $\Gamma((P \times_M P) / G)$ by

$$\mathcal{A}ut(P) \rightarrow \Gamma((P \times_M P) / G),$$

$$f \mapsto L_f,$$

where $L_f \in \Gamma((P \times_M P) / G)$ is given by

$$L_f|_x := [p, f(p)] = [p, p \cdot \sigma_f(p)]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$. This is clearly well-defined, and, so, while $c_G(P)$ is the bundle-analogue of $C^\infty(P; G)^G$ one can think of $(P \times_M P) / G$ as the bundle-analogue of $\mathcal{A}ut(P)$.

However, this description often arises if one wants to use the formalism of groupoids and algebroids, here especially using the **gauge groupoid** and **Atiyah algebroid** induced by P . These would allow an even more elegant version of the gauge transformations, however, we intend to write this paper in such a way that there is no need that the reader has knowledge about those bundle structures. See the cited references for more details in that regard.

Proof of Prop. 4.17.

• Let us first quickly check whether $g_f \in \Gamma(c_G(P))$ is well-defined for all $f \in \mathcal{A}ut(P)$. For $p \in P_x$ ($x \in M$) we have

$$q_f|_x = [p, \sigma_f(p)],$$

If $p' = p \cdot g$ ($g \in G$) is another element of P_x , then, using p' to define q_f ,

$$q_f|_x = [p \cdot g, \sigma_f(p \cdot g)] = [p \cdot g, c_{g^{-1}}(\sigma_f(p))] = [p, \sigma_f(p)],$$

also using the definition of $c_G(P)$, recall Ex. 3.12. It follows that q_f is well-defined, and it is clear that q_f is smooth.

• We want to show that $f \mapsto q_f$ is a group isomorphism by using that it is a composition of the group isomorphisms $\mathcal{A}ut \rightarrow C^\infty(P; G)^G$ as in Rem. 4.16 and

$$C^\infty(P; G)^G \rightarrow \Gamma(c_G(P)),$$

$$\sigma \mapsto q_\sigma, \tag{17}$$

where q_σ is effectively the same definition as q_f , that is $q_\sigma|_x = [p, \sigma(p)]$ which is well-defined by the very same reasons as before. It is only left to show that $C^\infty(P; G)^G \rightarrow \Gamma(c_G(P))$ is a group isomorphism. For injectivity let σ' be another element of $C^\infty(P; G)^G$ and assume $[p, \sigma(p)] = [p, \sigma'(p)]$. Then

$$e_x = [p, e] = [p, \sigma(p)] \cdot \underbrace{([p, \sigma'(p)])^{-1}}_{=[p, (\sigma'(p))^{-1}]} = [p, \sigma(p)(\sigma'(p))^{-1}],$$

such that

$$\sigma(p)(\sigma'(p))^{-1} = e,$$

so $\sigma = \sigma'$ and hence injectivity follows. For surjectivity observe that for a section $q \in \Gamma(c_G(P))$ we can define a map $\sigma : P \rightarrow G$ by

$$q_x = [p, \sigma(p)].$$

This map satisfies

$$[p, \sigma(p)] = [p \cdot g, c_{g^{-1}}(\sigma(p))] = [p \cdot g, \sigma(p \cdot g)]$$

for all $g \in G$; the last equality implies $\sigma(p \cdot g) = c_{g^{-1}}(\sigma(p))$, which is precisely what we need for $C^\infty(P; G)^G$. It is only left to show smoothness of σ . For an open neighbourhood $U \subset M$ of x fix a trivialization $\varphi_U : P|_U \rightarrow U \times G$, and we denote

$$\varphi_U(p') = (\pi_P(p'), \beta_U(p'))$$

for all $p' \in P$, where $\beta_U : P|_U \rightarrow G$ is an equivariant map, *i.e.* $\beta_U(p' \cdot g) = \beta_U(p') \cdot g$ for all $g \in G$. As shown in the proof of Thm. 3.9, we have a trivialization of $c_G(P)$ given by

$$c_G(P)|_U \rightarrow U \times G,$$

$$[p', g] \mapsto (\pi_P(p'), \psi_{\beta_U(p')}(g)).$$

Applying that trivialization to q we derive that

$$[p' \mapsto \psi_{\beta_U(p')}(\sigma(p'))]$$

is smooth, because q is smooth. Since $\psi_{\beta_U(p')}$ is smooth and bijective, we conclude that σ is smooth. Hence, $\sigma \in C^\infty(P; G)^G$, so, Def. (17) is also surjective and thence bijective.

Finally let us show that Def. (17) is a group isomorphism. Let σ, σ' be elements of $C^\infty(P; G)^G$, then use Def. (17) to derive

$$\sigma\sigma' \mapsto q_{\sigma\sigma'}$$

with

$$q_{\sigma\sigma'}|_x = [p, \sigma(p) \sigma'(p)] = [p, \sigma(p)] \cdot [p, \sigma'(p)] = q_\sigma|_x \cdot q_{\sigma'}|_x,$$

such that Def. (17) satisfies

$$\sigma\sigma' \mapsto q_\sigma \cdot q_{\sigma'}.$$

This concludes the proof. ■

Associated fibre bundles are motivated by making the invariance of gauge theory under local gauge transformations (that is, the change of gauge/local section of P) an inherent part of the bundle, similar to typical manifold coordinates; while the action of more global transformations "remain", similar to diffeomorphisms of a manifold. This procedure of "reducing" the action onto these is reflected in the quotient bundle $c_G(P)$.

5. Lie algebra bundles (LABs)

5.1. Definition

Lie algebras are the infinitesimal version of Lie groups, hence, we expect something similar for LGBs, the Lie algebra bundles:

Definition 5.1: Lie algebra bundle (LAB), [2, §3.3, Definition 3.3.8, page 104]

Let \mathfrak{g} be a Lie algebra, and \mathcal{G}, M be smooth manifolds. A vector bundle

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathcal{G} \\ & & \downarrow \pi \\ & & M \end{array}$$

is called a **Lie algebra bundle** if:

1. \mathfrak{g} and each fibre \mathcal{G}_x , $x \in M$, are Lie algebras;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} := \text{pr}_2 \circ \phi_i|_{\mathcal{G}_x} : \mathcal{G}_x \rightarrow \mathfrak{g}$$

are Lie algebra isomorphisms, where I is an (index) set, U_i are open sets covering M , $\phi_i : \mathcal{G}|_{U_i} \rightarrow U_i \times \mathfrak{g}$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie algebra bundle atlas** or **LAB atlas**.

We often say that \mathcal{G} is an **LAB (over M)**, whose structural Lie algebra is either clear by context or not explicitly needed; and we may also denote LABs by $\mathfrak{g} \rightarrow \mathcal{G} \xrightarrow{\pi} M$.

Of course, we have the typical trivial examples:

Example 5.2: Trivial examples

We recover the notion of a Lie algebra, if M consist of just one point. Moreover, the **trivial LAB** is given as the product manifold $\mathcal{Q} := M \times \mathfrak{g} \rightarrow M$. We have obviously a canonical smooth field of Lie brackets on this bundle $[\cdot, \cdot]_{\mathcal{Q}} : \Gamma(\mathcal{Q}) \times \Gamma(\mathcal{Q}) \rightarrow \Gamma(\mathcal{Q})$, i.e. $[\cdot, \cdot]_{\mathcal{Q}} \in \Gamma(\wedge^2 \mathcal{Q}^* \otimes \mathcal{Q})$ which restricts to the Lie algebra bracket $[\cdot, \cdot]_{\mathfrak{g}}$ of \mathfrak{g} on each fibre. The bracket is given by

$$[(x, X), (x, Y)]_{\mathcal{Q}} := (x, [X, Y]_{\mathfrak{g}})$$

for all $(x, X), (x, Y) \in M \times \mathfrak{g}$. Smoothness is an immediate consequence.

The definition of LAB morphisms is straight-forward:

Definition 5.3: LAB morphism,

[2, §. 4.3, simplified version of Def. 4.3.1, page 158]

Let $\mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} M$ and $\mathcal{H} \xrightarrow{\pi_{\mathcal{H}}} N$ be two LGBs over two smooth manifolds M and N . An **LAB morphism** is a pair of smooth maps $F : \mathcal{H} \rightarrow \mathcal{Q}$ and $f : N \rightarrow M$ such that

$$\pi_{\mathcal{Q}} \circ F = f \circ \pi_{\mathcal{H}}, \quad (18)$$

$$F \text{ linear}, \quad (19)$$

$$F([g, q]_{\mathcal{H}_p}) = [F(g), F(q)]_{\mathcal{Q}_{f(p)}} \quad (20)$$

for all $g, q \in \mathcal{H}_p$ ($p \in N$), where $[\cdot, \cdot]_{\mathcal{H}_p}$ and $[\cdot, \cdot]_{\mathcal{Q}_{f(p)}}$ are Lie brackets of \mathcal{H}_p and $\mathcal{Q}_{f(p)}$, respectively. We also say that F is an **LAB morphism over f** . If $N = M$ and $f = \text{id}_M$, then we often omit mentioning f explicitly and just write that F is a **(base-preserving) LAB morphism**.

We speak of an **LAB isomorphism (over f)** if F is a diffeomorphism.

Remark 5.4: Smooth field of Lie brackets

We have similar remarks as in Rem. 3.5. Additionally, we have locally a canonical smooth field of Lie brackets which restricts to a Lie bracket on each fibre because every LAB is locally isomorphic to a trivial LAB as in Ex. 5.2. Define a field of Lie brackets $[\cdot, \cdot]_{\mathcal{Q}} : \Gamma(\mathcal{Q}) \times \Gamma(\mathcal{Q}) \rightarrow \Gamma(\mathcal{Q})$, i.e. $[\cdot, \cdot]_{\mathcal{Q}} \in \Gamma(\wedge^2 \mathcal{Q}^* \otimes \mathcal{Q})$, by

$$[X, Y]_{\mathcal{Q}} := [X, Y]_{\mathcal{Q}_x} \quad (21)$$

for all $X, Y \in \mathcal{Q}_x$ ($x \in M$). Using a local trivialization, this bracket is locally of the form as in Rem. 3.5 such that smoothness follows.

In fact, as also argued in [4, §16.2, Example 2, page 114; but speaking in the context of Lie algebroids there whose are a generalization of LABs], every vector bundle equipped

with a smooth field of Lie brackets is an LAB.

Endomorphisms of a vector bundle are of course another important example of LABs.

Example 5.5: Endomorphisms of a vector bundle, [2, §3.3, part of Ex. 3.3.4]

Let $V \rightarrow M$ be a vector bundle, and denote with $\text{End}(V) \rightarrow M$ its bundle of fibre-wise endomorphisms (its sections are the base-preserving bundle endomorphisms of V). This is clearly an LAB whose field of Lie brackets is given by the commutator.

We have the notion of structure constants on all fibres, such that we get now structure functions after fixing a frame.²

Definition 5.6: Structure functions, [4, §16.5, page 119]

Let $\mathcal{g} \rightarrow M$ be an LAB over a smooth manifold M , and $(e_a)_a$ be a local frame of \mathcal{g} over some open subset $U \subset M$. Then the **structure functions** $C_{bc}^a \in C^\infty(U)$ are defined by

$$[e_b, e_c]_{\mathcal{g}} = C_{bc}^a e_a.$$

Remark 5.7: Properties of the structure functions

Of course, one has the typical properties for such structure functions as for structure constants due to that C_{bc}^a restrict to typical structure constants on each fibre. Similar to [1, §1.4, discussion after Def. 1.4.17, page 38],

$$C_{bc}^a = -C_{cb}^a,$$

$$0 = C_{ac}^d C_{bc}^e + C_{be}^d C_{ca}^e + C_{ce}^d C_{ab}^e$$

for all a, b, c, d ; the former due to antisymmetry, the latter because of the Jacobi identity.

As we have seen it for LGBs, the pullback of LABs is again an LAB.

Corollary 5.8: Pullbacks of LABs are LABs, [3, §3, Thm. 3.2]

Let M, N be smooth manifolds, $\mathcal{g} \xrightarrow{\pi} M$ an LAB over M and $f : N \rightarrow M$ a smooth map. Then the pullback vector bundle $f^*\mathcal{g}$ has a unique (up to isomorphisms) LAB structure such that the projection $\pi_2 : f^*\mathcal{g} \rightarrow \mathcal{g}$ onto the second factor is an LAB morphism over f with $\pi_2|_x : (f^*\mathcal{g})_x \rightarrow \mathcal{g}_{f(x)}$ being a Lie algebra isomorphism for all $x \in N$.

Proof.

Either prove this similarly as Cor. 4.1 (by also using a similar statement already known for

²However, observe that $\Gamma(\mathcal{g})$ is an infinite-dimensional Lie algebra, such that there is still a notion structure constants, but an inconvenient one due to that these constants are in general not finitely many.

vector bundles), or observe that the pullback $f^*([\cdot, \cdot]_{\mathcal{G}})$ of the field of Lie brackets $[\cdot, \cdot]_{\mathcal{G}}$ on \mathcal{G} as a section is clearly also a smooth field of Lie brackets on $f^*\mathcal{G}$ with same structural Lie algebra \mathfrak{g} . ■

Definition 5.9: Pullback LAB

Let M, N be smooth manifolds, $\mathcal{G} \rightarrow M$ an LGB over M and $f : N \rightarrow M$ a smooth map. Then we call the LAB structure on $f^*\mathcal{G}$ as given in Cor. 5.8 the **pullback LAB of \mathcal{G} (under f)**.

We will refer to this structure often without further mention.

5.2. From LGBs to LABs

Let us now quickly discuss how LGBs and LABs are related; it is very similar to the relation of Lie groups and algebras, now somewhat fibre-wise. We will follow the style of [1, §1.5.2, page 40ff.] and [2, §3.5, page 119ff.]; our approach will be using left-invariant vector fields but the mentioned latter reference actually uses right-invariant vector fields.

Let us start with introducing the basic notations needed.

Definition 5.10: Left and right translation and conjugation, [1, §1.5, similar notation to Def. 1.5.3, page 40]

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M . For $g \in \mathcal{G}_x$ ($x \in M$) we define the following maps:

- **Left translation** given by

$$L_g : \mathcal{G}_x \rightarrow \mathcal{G}_x,$$

$$h \mapsto gh.$$

- **Right translation** given by

$$R_g : \mathcal{G}_x \rightarrow \mathcal{G}_x,$$

$$h \mapsto hg.$$

- **Conjugation** given by

$$c_g : \mathcal{G}_x \rightarrow \mathcal{G}_x,$$

$$h \mapsto ghg^{-1}.$$

Remarks 5.11.

By definition of \mathcal{G} , all these maps are smooth. Furthermore, they clearly satisfy the typical

properties as known for these maps since \mathcal{G}_x is a Lie group for all $x \in M$; for reference about their basic properties see for example [1, §1.5, Lemma 1.5.5, page 40f.].

The left and right translations of Def. 5.10 and 4.8 align, and thus the smoothness concerns as mentioned in the last part of Rem. 4.11 for right translations $r_g = R_g$ ($g \in \mathcal{G}_x$, $x \in M$) do not arise. Moreover, while the conjugation c_g is a Lie group automorphism of \mathcal{G}_x , it describes an LGB automorphism of \mathcal{G} if extended to sections; following [2, §1.4, Def. 1.4.6 and its discussion afterwards, page 24f.]. That is for $\sigma \in \Gamma(\mathcal{G})$ we define the conjugation c_σ as a smooth map by

$$\mathcal{G} \rightarrow \mathcal{G},$$

$$q \mapsto c_\sigma(q) := (L_\sigma \circ R_{\sigma^{-1}})(q) = (R_{\sigma^{-1}} \circ L_\sigma)(q) = \sigma_{\pi(q)} \cdot q \cdot \sigma_{\pi(q)}^{-1}.$$

It is clear that $c_\sigma(gq) = c_\sigma(g) \cdot c_\sigma(q)$ for all $g, q \in \mathcal{G}$ with $\pi(g) = \pi(q)$, and that a smooth inverse is given by $c_{\sigma^{-1}}$; thence, c_σ is an LGB isomorphism of \mathcal{G} on itself, an automorphism, in sense of Def. 3.4. It is also trivial to check that we have $c_{\sigma \cdot \tau} = c_\sigma \circ c_\tau$, where τ is another section of \mathcal{G} .

Analogously we define R_σ as r_σ of Def. 4.8; with the capital letter we put an emphasis on that the \mathcal{G} -action acts on \mathcal{G} itself. Similarly for left translations.

Since these are diffeomorphism of the fibres, it makes sense to say that a left-invariant vector field of \mathcal{G} has to be a vertical vector field, that is, it is in the kernel of $D\pi$, the total differential/tangent map of the projection of $\mathcal{G} \xrightarrow{\pi} M$. For this recall that there is the notion of a **vertical bundle** for fibre bundles $F \xrightarrow{\varpi} M$ (as *e.g.* introduced in [1, §5.1.1, for principal bundles, but it is straightforward to extend the definitions; page 258ff.]), which is defined as a subbundle $VF \rightarrow F$ of the tangent bundle $TF \rightarrow F$ given as the kernel of $D\varpi : TF \rightarrow TM$. The fibres $V_v F$ of VF at $v \in F$ are then given by

$$V_v F = T_v F_x,$$

where $x := \varpi(v) \in M$ and F_x is the fibre of F at x . F_x is an embedded submanifold of F , thence, by definition a section $X \in \Gamma(VF)$ restricts to a vector field on the fibres, that is,

$$X|_{F_x} \in \mathfrak{X}(F_x).$$

In our case $F = \mathcal{G}$, in this case $F_x = \mathcal{G}_x$ is then a Lie group, so, the vertical bundle just consists of the tangent bundles of Lie groups of all fibres. All of these are generated by their Lie algebra at e_x , the identity element of \mathcal{G}_x . Hence, it is natural to guess that the LAB for \mathcal{G} will be $V\mathcal{G}|_{e_M}$, where e_M is the image of M under the identity section of \mathcal{G} , thus, an embedding of M into \mathcal{G} . Therefore $V\mathcal{G}|_{e_M}$ is a fibre bundle by [1, §4.1, Lemma 4.1.16, page 204], and clearly a vector bundle. Equivalently, since the identity section e is an embedding, we think of $V\mathcal{G}|_{e_M}$ as the pullback vector bundle $e^*V\mathcal{G}$, which is conveniently a vector bundle over M .

Hence, let us now show that \mathcal{G} will be related to $e^*V\mathcal{G}$ similar to how a Lie group will be related to its Lie algebra.

Definition 5.12: Left-invariant vector fields on LGBs,**[2, §3.5, special situation of Def. 3.5.2, page 120]**

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M . A vector field $X \in \mathfrak{X}(\mathcal{G})$ is a **left-invariant vector field** if

1. X is vertical, that is,

$$X \in \Gamma(\mathbf{V}\mathcal{G}),$$

2. X is invariant under the left-multiplication on each fibre, *i.e.*

$$\mathbf{D}_g L_q(X_g) = X_{qg}$$

for all $q, g \in \mathcal{G}_x$, where $x := \pi(g) = \pi(q)$.

The set of all left-invariant vector fields on \mathcal{G} will be denoted by $L(\mathcal{G})$.

Remarks 5.13.

Observe that the second point in the definition is well-defined because X is a vertical vector field; that is, recall that $L_q : \mathcal{G}_x \rightarrow \mathcal{G}_x$ such that $\mathbf{D}_g L_q(X_g) : \mathbf{T}_g \mathcal{G}_x \rightarrow \mathbf{T}_{qg} \mathcal{G}_x$, hence, $\mathbf{D}_g L_q(X_g) : \mathbf{V}_g \mathcal{G} \rightarrow \mathbf{V}_{qg} \mathcal{G}$.

Remark 5.14: Abstract notation 1

Since X is vertical, recall that we can view the restriction of X onto a fibre as a vector field on that fibre, *i.e.*

$$X|_{\mathcal{G}_x} \in \mathfrak{X}(\mathcal{G}_x).$$

\mathcal{G}_x is a Lie group and left translations are diffeomorphisms on it, hence, the left-invariance can also be written as

$$\mathbf{D}L_q(X|_{\mathcal{G}_x}) = L_q^*(X|_{\mathcal{G}_x}). \quad (22)$$

For this recall that $\mathbf{D}L_q \in \Omega^1(\mathcal{G}_x; L_q^* \mathbf{T}\mathcal{G}_x)$ for the left hand side, and that L_q^* is the pullback of sections on the right hand side, that is, $L_q^*(X|_{\mathcal{G}_x}) \in \Gamma(L_q^* \mathbf{T}\mathcal{G}_x)$. Furthermore, $X|_{\mathcal{G}_x}$ is therefore a left-invariant vector on \mathcal{G}_x . Which is why one may also define the left-invariance of X as a vector field on \mathcal{G} by saying that it has to restrict to a left-invariant vector field on each fibre in the usual sense of Lie groups.

One quickly shows that this is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$.

Lemma 5.15: Closure of Lie bracket for left-invariant vector fields,
[2, §3.5, special situation of Lemma 3.5.5, page 122]

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M . Then $L(\mathcal{G})$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$.

Proof.

Let $X, Y \in L(\mathcal{G})$, then we need to show

$$D\pi([X, Y]) \equiv 0,$$

and if this holds, then we also need to derive

$$D_g L_q([X, Y]|_g) = [X, Y]|_{qg}.$$

One can either immediately show these directly by using statements like [1, Proposition A.1.49; page 615], which essentially describes how the Lie bracket of vector fields react under push-forwards. Or use the knowledge about Lie groups, recall Rem. 5.14: Each fibre \mathcal{G}_x is an embedded submanifold of \mathcal{G} and both, $X|_{\mathcal{G}_x}$ and $Y|_{\mathcal{G}_x}$, are vector fields of this submanifold. Thus, $[X|_{\mathcal{G}_x}, Y|_{\mathcal{G}_x}]|_p$ has values in $T_p \mathcal{G}_x$ for all $p \in \mathcal{G}_x$. Especially,

$$[X|_{\mathcal{G}_x}, Y|_{\mathcal{G}_x}]|_{\mathcal{G}_x} \in \mathfrak{X}(\mathcal{G}_x).$$

Because of this and since $X|_{\mathcal{G}_x}$ and $Y|_{\mathcal{G}_x}$ are left-invariant vector fields of \mathcal{G}_x (a Lie group), left-invariance of $[X|_{\mathcal{G}_x}, Y|_{\mathcal{G}_x}]|_{\mathcal{G}_x}$ follows, and thus the statement. ■

Of course, elements of $L(\mathcal{G})$ are determined by their values at e_M , as already suggested previously. Let us show this now; starting with a small auxiliary result.

Corollary 5.16: $L(\mathcal{G})$ a $C^\infty(M)$ -module,

[2, §3.5, comment before Lemma 3.5.5, page 122]

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M . Then $L(\mathcal{G})$ is a $C^\infty(M)$ -module under the multiplication

$$fX := \pi^* f X$$

for all $f \in C^\infty(M)$ and $X \in L(\mathcal{G})$.

Proof.

Obviously, $fX \in \Gamma(\mathcal{V}\mathcal{G})$ since

$$D\pi(fX) = D\pi(\pi^* f X) = \pi^* f D\pi(X) = 0.$$

Furthermore, $fX|_{\mathcal{G}_x}$ ($x \in M$) is left-invariant over \mathcal{G}_x since $X|_{\mathcal{G}_x}$ is left-invariant and $f|_{\mathcal{G}_x} \equiv f(x) \in \mathbb{R}$. Thence, $fX \in L(\mathcal{G})$. ■

Corollary 5.17: $L(\mathcal{G})$ as sections of $e^*\mathcal{V}\mathcal{G}$,

[2, §3.5, comment before Lemma 3.5.5, page 122; parts of Cor. 3.5.4, page 121]

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M , and denote with e the identity section of \mathcal{G} . Then we have an isomorphism of $C^\infty(M)$ -modules

$$L(\mathcal{G}) \rightarrow \Gamma(e^*\mathcal{V}\mathcal{G}),$$

$$X \mapsto e^*X.$$

The inverse of this map is given by

$$\Gamma(e^*\mathcal{V}\mathcal{G}) \rightarrow L(\mathcal{G}),$$

$$\nu \mapsto X_\nu,$$

where X_ν is given by

$$X_\nu|_g := D_{e_x}L_g(\text{pr}_2(\nu_x))$$

for all $g \in \mathcal{G}$ and $x := \pi(g)$, where pr_2 is the projection onto the second component in $e^*\mathcal{V}\mathcal{G}$.

Remark 5.18: Abstract notation 2

Since $e^*\mathcal{V}\mathcal{G} \cong \mathcal{V}\mathcal{G}|_{eM}$ is a very natural isomorphism, we will often just write

$$X_\nu|_g = D_{e_x}L_g(\nu_x),$$

omitting pr_2 and using that natural isomorphism without further mention.

Also observe that we can actually define a left translations by (local) sections of \mathcal{G} , i.e. for $\sigma \in \Gamma(\mathcal{G})$ we define the left translation L_σ as a map by

$$\mathcal{G} \rightarrow \mathcal{G},$$

$$q \mapsto \sigma_{\pi(q)} \cdot q.$$

This map is a diffeomorphism, and restricts to the fibres \mathcal{G}_x as embedded submanifolds to the map L_{σ_x} ; we discussed this in more generality in Rem. 4.11. Observe that for vertical vector fields $Y \in \Gamma(\mathcal{V}\mathcal{G})$ we have

$$D_q L_\sigma(Y_q) = \left. \frac{d}{dt} \right|_{t=0} (L_\sigma \circ \gamma) \equiv \left. \frac{d}{dt} \right|_{t=0} (L_{\sigma_{\pi(q)}} \circ \gamma) = D_q L_g(Y_q)$$

where $g := \sigma_{\pi(q)}$ and $\gamma : I \rightarrow \mathcal{G}_{\pi(q)}$ (I an open interval containing 0) is a curve with $\gamma(0) = q$ and $d/dt|_{t=0} \gamma = Y_q$. Therefore L_σ restricts onto vertical vector fields and is then

just the left translation via an element in the fibre over a fixed base point. In total one can then introduce the brief notation

$$X_\nu \circ \sigma = DL_\sigma(\nu) = DL_\sigma|_{e_M}(\nu) = DL_\sigma \circ \nu.$$

However, be careful, in general one cannot simply replace L_g with L_σ , even if $\sigma_{\pi(g)} = g$. This only works with respect to vertical tangent vectors; once horizontal parts play a role things change, L_g is a priori not even defined then. Once we turn to the definition of horizontal distributions we will come back to this.

Proof of Cor. 5.17.

This map is clearly $C^\infty(M)$ -linear, especially due to

$$e^*(fX) = e^*(\pi^* f X) = (f \circ \underbrace{\pi \circ e}_{=\mathbb{1}_M}) e^* X = f e^* X$$

for all $f \in C^\infty(M)$ and $X \in L(\mathcal{G})$; for this recall Cor. 5.16.

We essentially only need to show that the suggested inverse $\nu \mapsto X_\nu$ is well-defined. First of all, that X_ν is vertical and left-invariant is clear by construction; ν_x ($x \in M$) is an element of the Lie algebra of \mathcal{G}_x , and thus $X_\nu|_{\mathcal{G}_x}$ is a left-invariant vector field on \mathcal{G}_x . X is therefore an element of $L(\mathcal{G})$ once we know that X is smooth. We show smoothness similar as in [1, §1.5, proof of Lemma 1.5.13, page 42]: Denote the multiplication in \mathcal{G} by $\mu : \mathcal{G} * \mathcal{G} \rightarrow \mathcal{G}$. Then observe that $(0_g, \nu_x)$ ($0_g \in T_g \mathcal{G}$, $g \in \mathcal{G}_x$, the zero vector field 0 on $T\mathcal{G}$) is an element of

$$\begin{aligned} & T(\mathcal{G} * \mathcal{G}) \\ &= \{(Y, Z) \mid Y \in T_q \mathcal{G}, Z \in T_h \mathcal{G} \text{ with } D_q \pi(Y) = D_h \pi(Z), \text{ where } q, h \in \mathcal{G} \text{ with } \pi(q) = \pi(h)\} \end{aligned}$$

because $\nu_x \in V_{e_x} \mathcal{G}$.³ Therefore we can calculate

$$D_{(g, e_x)} \mu(0_g, \nu_x) = \left. \frac{d}{dt} \right|_{t=0} (g \cdot \gamma) = \left. \frac{d}{dt} \right|_{t=0} (L_g \circ \gamma) = D_{e_x} L_g(\nu_x) = X_\nu|_g,$$

where $\gamma : I \rightarrow \mathcal{G}_x$ (I an open interval containing 0) is a curve with $\gamma(0) = e_x$ and $d/dt|_{t=0} \gamma = \nu_x$. Since μ is smooth, $D\mu$ is smooth, and thus

$$\begin{aligned} & \mathcal{G} \rightarrow T(\mathcal{G} * \mathcal{G}), \\ & g \mapsto (D\mu \circ (0, \nu_\pi))|_g = D_{(g, e_x)} \mu(0_g, \nu_{\pi(g)}) = X_\nu|_g \end{aligned}$$

is smooth, also using the smoothness of ν , π and $g \mapsto 0_g$.

³If it is not clear how to derive the tangent bundle of $\mathcal{G} * \mathcal{G}$, then see later when we will discuss it in a more general manner. However, essentially recall that $\mathcal{G} * \mathcal{G} = \pi^* \mathcal{G}$.

Finally, that $\phi : L(\mathcal{G}) \rightarrow \Gamma(e^*\mathcal{V}\mathcal{G})$, $X \mapsto e^*X$, is bijective is also clear, similar to typical gauge theory; we know that $X|_{\mathcal{G}_x}$ is a left-invariant vector field on \mathcal{G}_x by Rem. 5.14. Hence, for $g \in \mathcal{G}_x$,

$$X|_{\mathcal{G}_x} = D_{e_x} L_g(X_{e_x}) = D_{e_x} L_g(e^*X|_x).$$

This is precisely the structure of the suggested inverse, that is,

$$X = X_{e^*X} = (\psi \circ \phi)(X),$$

where $\psi : \Gamma(e^*\mathcal{V}\mathcal{G}) \rightarrow L(\mathcal{G})$, $\nu \mapsto X_\nu$. Hence, injectivity follows; surjectivity simply follows similarly by

$$(\phi \circ \psi)(\nu)|_x = e^*X_\nu|_x = \underbrace{D_{e_x} L_{e_x}}_{= \mathbb{1}_{\mathcal{V}_{e_x}\mathcal{G}}}(\nu_x) = \nu_x$$

for all $\nu \in \Gamma(e^*\mathcal{V}\mathcal{G})$ and $x \in M$. This finishes the proof. ■

This result shows the typical statement about that elements of $L(\mathcal{G})$ are uniquely determined by their values at e_M . It immediately follows, too, that:

Corollary 5.19: LGBs induce an LAB structure,

[2, §3.5, simplified version of the discussion after Cor. 3.5.4, page 121ff.]

Let $G \rightarrow \mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and denote with e the identity section of \mathcal{G} . Then $\mathcal{G} := e^\mathcal{V}\mathcal{G} \rightarrow M$ admits the structure as an LAB with structural Lie algebra \mathfrak{g} , the Lie algebra of G , and the fibres \mathcal{G}_x ($x \in N$) are the Lie algebras of \mathcal{G}_x . The field of Lie algebra brackets $[\cdot, \cdot]_{\mathcal{G}}$ is given by*

$$[\nu, \mu]_{\mathcal{G}} := e^*([X_\nu, X_\mu])$$

for all $\nu, \mu \in \Gamma(e^\mathcal{V}\mathcal{G})$, where X_ν, X_μ are elements of $L(\mathcal{G})$ as given in Cor. 5.17. Point-wise*

$$[\nu_x, \mu_x]_{\mathcal{G}} = [X_\nu, X_\mu]|_{e_x}$$

for all $x \in M$.

Proof.

As already discussed $e^*\mathcal{V}\mathcal{G}$ is a vector bundle. The fibres are given by

$$\mathcal{G}_x = T_{e_x}\mathcal{G}_x \cong T_e G = \mathfrak{g}$$

for all $x \in M$, where we used that \mathcal{G}_x is isomorphic to G as a Lie group. All fibres are Lie algebras of the fibre Lie group, isomorphic to \mathfrak{g} . By construction, the Lie bracket is precisely

the Lie bracket isomorphic to the one of \mathfrak{g} , and $[\cdot, \cdot]_{\mathcal{G}}$ is smooth. Therefore we conclude that \mathcal{G} is an LAB with structural Lie algebra \mathfrak{g} . Alternatively see [2, §3.5, Ex. 3.5.12, page 126] for an explicit construction of an LAB atlas. ■

Definition 5.20: The LAB of an LGB,

[2, §3.5, special situation of Def. 3.5.1, page 120]

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and denote with e the identity section of \mathcal{G} . Then we define the **LAB \mathcal{G} of \mathcal{G}** as the vector bundle $e^*V\mathcal{G}$.

In the following \mathcal{G} will usually denote the LAB of \mathcal{G} , even if we only introduced \mathcal{G} .

Example 5.21: Endomorphisms of a vector bundle as LAB of fibre-wise automorphisms

Recall Ex. 3.6 and 5.5. For a vector bundle $V \rightarrow M$ one can show that the LAB of $\text{Aut}(V)$ is given by $\text{End}(V)$; the proof is precisely as for the automorphisms and endomorphisms of a vector space as in [1, §1.5.4, page 45ff.], just canonically extended to a bundle language.

5.3. Vertical Maurer-Cartan form of LGBs

As one may expect, the last result gives hints about the tangent bundle structure of \mathcal{G} ; this can be shown with the Maurer-Cartan form on LGBs, which we will call vertical Maurer-Cartan form. It will be clear later why we choose to add this adjective; however, as a first argument recall Rem. 5.18, especially the last paragraph.

Corollary 5.22: Well-definedness of the vertical Maurer-Cartan form

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M . Define the following map

$$(\mu_{\mathcal{G}})_g(v) := (D_g L_{g^{-1}})(v)$$

for all $g \in \mathcal{G}$ and $v \in V_g \mathcal{G}$. Then this map is an element of $\Gamma(V^* \mathcal{G} \otimes \pi^* \mathcal{G})$, where $V^* \mathcal{G}$ is the dual bundle of $V\mathcal{G}$.

Proof.

Observe that

$$D_g L_{g^{-1}} : V_g \mathcal{G} \rightarrow V_{e_x} \mathcal{G} \cong (\pi^* \mathcal{G})_g$$

where $x := \pi(g)$ and e_x is the neutral element of \mathcal{G}_x . Smoothness follows similarly to the smoothness of left-invariant vector fields, that is, denote with Φ the multiplication $\mathcal{G} * \mathcal{G} \rightarrow \mathcal{G}$ and recall the arguments and the notation in the proof of Cor. 5.17. We have

$$T(\mathcal{G} * \mathcal{G})$$

$$= \{(Y, Z) \mid Y \in T_q \mathcal{G}, Z \in T_h \mathcal{G} \text{ with } D_q \pi(Y) = D_h \pi(Z), \text{ where } q, h \in \mathcal{G} \text{ with } \pi(q) = \pi(h)\}$$

and thus

$$(0_{g^{-1}}, v) \in T(\mathcal{G} * \mathcal{G})$$

where 0 is the zero vector field, $v \in V_g \mathcal{G}$ and $g \in \mathcal{G}$. Therefore we can calculate

$$D_{(g^{-1}, g)} \Phi(0_{g^{-1}}, v) = \frac{d}{dt} \Big|_{t=0} (g^{-1} \cdot \gamma) = \frac{d}{dt} \Big|_{t=0} (L_{g^{-1}} \circ \gamma) = D_g L_{g^{-1}}(v) = \mu_{\mathcal{G}}(v),$$

where $\gamma : I \rightarrow \mathcal{G}_x$ (I an open interval containing 0 , and $x := \pi(g)$) is a curve with $\gamma(0) = g$ and $d/dt|_{t=0} \gamma = v$. Denote with 0^{-1} the vector field on \mathcal{G} given by $g \mapsto 0_{g^{-1}}$ and with $\iota_{0^{-1}}$ the contraction with 0^{-1} , that is,

$$(\iota_{0^{-1}} D\Phi)|_g := D_{(g^{-1}, g)} \Phi(0_{g^{-1}}, \cdot) = \left[V_g \mathcal{G} \ni v \mapsto D_{(g^{-1}, g)} \Phi(0_{g^{-1}}, v) \right] = \mu_{\mathcal{G}}$$

for all $g \in \mathcal{G}$, using the structure of $T(\mathcal{G} * \mathcal{G})$. Thus, we get in total that

$$\mu_{\mathcal{G}} = \iota_{0^{-1}} D\Phi \in \Gamma(V^* \mathcal{G} \otimes \pi^* \mathcal{G}),$$

using the smoothness of all involved parts, especially that Φ is smooth, hence also $D\Phi$ is smooth as an element of $\Omega^1(\mathcal{G} * \mathcal{G}; \Phi^* \mathcal{G})$. ■

Definition 5.23: Vertical Maurer-Cartan form of LGBs,

[1, generalization of Def. 3.5.2, page 148]

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M . The map defined in Cor. 5.22 is the **vertical Maurer-Cartan form** $\mu_{\mathcal{G}}$, i.e. defined to be an element of $\Gamma(V^* \mathcal{G} \otimes \pi^* \mathcal{G})$ given by

$$(\mu_{\mathcal{G}})_g(v) := (D_g L_{g^{-1}})(v)$$

for all $g \in \mathcal{G}$ and $v \in V_g \mathcal{G}$, where $V^* \mathcal{G}$ is the dual bundle of $V\mathcal{G}$.

Remark 5.24: Recovering of the classical definition

Observe that $\mu_{\mathcal{G}}|_{\mathcal{G}_x}$ ($x \in M$) is the typical Maurer-Cartan form of \mathcal{G}_x , hence, $\mu_{\mathcal{G}}$ restricts to the Maurer-Cartan form of Lie groups on each fibre.

Also recall Subsection 1.1, we have a 1:1 correspondence of $\mu_{\mathcal{G}}$ to the following commuting diagram

$$\begin{array}{ccc} V\mathcal{G} & \xrightarrow{\mu_{\mathcal{G}}} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{\pi} & M \end{array}$$

which is the same diagram as in [2, §3.5, special situation of Prop. 3.5.3, page 121].

We can finally finish the discussion about the vertical bundle of an LGB.

Corollary 5.25: Vertical tangent space of \mathcal{G} ,**[2, §3.5, a reformulation of Prop. 3.5.3, page 121]**

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M . Then we have an isomorphism of vector bundles

$$V\mathcal{G} \cong \pi^*\mathfrak{g}.$$

Remarks 5.26.

Observe that by Cor. 5.8 we know that $V\mathcal{G}$ admits a unique LAB structure such that $V\mathcal{G} \cong \pi^*\mathfrak{g}$ is an isomorphism of LABs. This statement is also not in contradiction with $\mathfrak{g} = e^*V\mathcal{G}$ (e the identity section of \mathcal{G}), because

$$e^*V\mathcal{G} = e^*\pi^*\mathfrak{g} = (\pi \circ e)^*\mathfrak{g} = \mathfrak{g}.$$

By this we also know that $V\mathcal{G}$ is trivial if and only if \mathfrak{g} is trivial; as also argued in [2, §3.5, discussion after Cor. 3.5.4, page 121]. Compare this result with $TG \cong G \times \mathfrak{g}$, where G is a Lie group with Lie algebra \mathfrak{g} . We recover this result by restricting to the Lie group fibres \mathcal{G}_x ($x \in M$), that is,

$$V\mathcal{G}|_{\mathcal{G}_x} = T\mathcal{G}_x \cong \mathcal{G}_x \times \mathfrak{g}_x = \pi^*\mathfrak{g}|_{\mathcal{G}_x}.$$

Last but not least, sections of $V\mathcal{G}$ are therefore generated by sections of \mathfrak{g} , the left-invariant vector fields.

Proof of Cor. 5.25.

This can be quickly shown by recalling Rem. 5.24, that is, we have the following commuting diagram

$$\begin{array}{ccc} V\mathcal{G} & \xrightarrow{\mu_{\mathcal{G}}} & \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{\pi} & M \end{array}$$

where $\mu_{\mathcal{G}}$ is defined as in Def. 5.23, and $\mu_{\mathcal{G}}$ restricts to the Maurer-Cartan form of \mathcal{G}_x ($x \in M$) on each fibre of \mathcal{G} ; especially, $\mu_{\mathcal{G}} : V\mathcal{G} \rightarrow \mathfrak{g}$ is a fibre-wise isomorphism (since $D_g L_{g^{-1}}$ is an isomorphism). Hence, as described in Subsection 1.1, $\mu_{\mathcal{G}}$ as an element of $\Gamma(V^*\mathcal{G} \otimes \pi^*\mathfrak{g})$, i.e. a vector bundle morphism $V\mathcal{G} \rightarrow \pi^*\mathfrak{g}$ (linearity of μ_G is clear), is a vector bundle isomorphism. This finishes the proof. \blacksquare

5.4. Exponential map of LGBs

By Remark 5.14 it is clear that we have a natural exponential map, just given by the fibre-wise exponential. If one is interested into a general exponential map, then see [2, §3.6, page 132ff.]. However, since our situation is much simpler, we quickly finish this discussion just making use

of the already existing exponential map in each fibre; a straightforward generalization on results as provided in [1, §1.7, page 55ff.].

Definition 5.27: Exponential map,

[2, §3.6, second part of Ex. 3.6.2, page 133f.]

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and $\mathcal{g} \rightarrow M$ its LAB. Then we define the **exponential map** $\exp : \mathcal{g} \rightarrow \mathcal{G}$ by

$$\exp(X) := e^X := \exp_{\mathcal{G}_x}(X)$$

for all $x \in M$ and $X \in \mathcal{g}_x$, where $\exp_{\mathcal{G}_x}$ is the exponential map of the Lie group \mathcal{G}_x as e.g. provided in [1, §1.7, Def. 1.7.6, page 57]. Its extension to sections, also denoted by $\exp : \Gamma(\mathcal{g}) \rightarrow \Gamma(\mathcal{G})$, is canonically given by

$$\exp(\nu)|_x := \exp(\nu_x)$$

for all $x \in M$ and $\nu \in \Gamma(\mathcal{g})$.

Usually, the LGB is given by context, otherwise we will denote the exponential map by $\exp_{\mathcal{G}}$ instead.

The exponential map is well-defined, especially it is smooth because it describes the flow of left-invariant vector fields as it also happens for Lie groups; see for example [1, Prop. 1.7.12, page 58] for the Lie group statement. Also recall Cor. 5.17, we denote left-invariant vector fields by X_ν where $\nu \in \Gamma(\mathcal{g})$ due to 1:1 correspondence of $L(\mathcal{G})$ and $\Gamma(\mathcal{g})$.

Corollary 5.28: The exponential map as flow of $L(\mathcal{G})$,

[2, discussion at the beginning of §3.6, Prop. 3.6.1 and its discussion afterwards; page 132f.]

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M , and $\mathcal{g} \rightarrow M$ its LAB. Then the flow of a left-invariant vector field $X_\nu \in L(\mathcal{G})$ ($\nu \in \Gamma(\mathcal{g})$) is a complete flow $\phi : \mathbb{R} \times \mathcal{G} \rightarrow \mathcal{G}$ given by

$$\phi(t, g) = g \cdot e^{t\nu_{\pi(g)}}$$

for all $t \in \mathbb{R}$ and $g \in \mathcal{G}$. Especially, the map

$$\mathbb{R} \times M \rightarrow \mathcal{G},$$

$$(t, x) \mapsto e^{t\nu_x}$$

is smooth.

Proof.

As mentioned in Remark 5.14, X_ν is a vertical vector field so that $X_\nu|_{\mathcal{G}_x}$ is a left-invariant vector field of the Lie group \mathcal{G}_x for all $x \in M$. $X_\nu|_{\mathcal{G}_x}$ is the left-invariant vector field in \mathcal{G}_x related to $\nu_x \in \mathcal{G}_x$ by Cor. 5.17, and the flow ϕ_x of $X_\nu|_{\mathcal{G}_x}$ is well-known, as *e.g.* in [1, §1.7, Prop. 1.7.12, page 58], that is,

$$\mathbb{R} \times \mathcal{G}_x \rightarrow \mathcal{G}_x,$$

$$(t, g) \mapsto \phi_x(t, g) = g \cdot \exp_{\mathcal{G}_x}(t\nu_x),$$

where $\exp_{\mathcal{G}_x}$ is the exponential map of \mathcal{G}_x . Since this works for all fibres \mathcal{G}_x and by Def. 5.27 we get that the flow ϕ of X_ν is complete and given by

$$\phi(t, g) = g \cdot e^{t\nu_{\pi(g)}}$$

for all $t \in \mathbb{R}$ and $g \in \mathcal{G}$. We also have

$$e^{t\nu_x} = e_x \cdot e^{t\nu_{\pi(e_x)}} = \phi(t, e_x)$$

for all $t \in \mathbb{R}$ and $x \in M$, such that smoothness of $(t, x) \mapsto e^{t\nu_x}$ follows as composition of smooth maps. ■

Remark 5.29: Simplifying notation related to the exponential map

Due to Def. 5.27 we recover a lot of the typical properties of the exponential map as in [1, §1.7, Prop. 1.7.9, page 57], if we understand these properties point-wise. Recall Remark 4.10, then for example

$$e^{(t+s)\nu} = e^{t\nu} \cdot e^{s\nu}$$

for all $t, s \in \mathbb{R}$ and $\nu \in \Gamma(\mathcal{G})$. As in [2, §3.6, discussion after Prop. 3.6.1, page 133], we say that $\mathbb{R} \ni t \mapsto e^{t\nu}$ is smooth in the sense of the second statement of Cor. 5.28, and by construction we have

$$\nu_x = \left. \frac{d}{dt} \right|_{t=0} e^{t\nu_x} := \left. \frac{d}{dt} \right|_{t=0} [t \mapsto e^{t\nu_x}]$$

for all $x \in M$, so that we also write

$$\nu = \left. \frac{d}{dt} \right|_{t=0} e^{t\nu}.$$

Since all of this is rather natural, we will make use of that without further mention.

This discussion also highlights that one could understand the infinite-dimensional Lie algebra $\Gamma(\mathcal{G})$ as the Lie algebra of the infinite-dimensional Lie group $\Gamma(\mathcal{G})$; thus, one could construct the LABs of LGBs by starting in that fashion, and then \mathcal{G} is constructed by making use of the 1:1 correspondence of vector bundles and locally free sheaf of modules of constant rank.

5.5. LABs of pullback LGBs

We are going to define LGB representations and corresponding LAB representations. Since group representation are a special form of actions, we will have something similar in the case of LGB representations. Since actions are defined as maps on a pullback of an LGB \mathcal{G} , which is also an LGB by Cor. 4.1, we expect that the corresponding LAB representation is related to the LAB of the pullback of \mathcal{G} . It is natural to think of this LAB as the pullback of \mathcal{g} , which is also an LAB by Cor. 5.8:

Corollary 5.30: LAB of pullback LGB is pullback LAB,
[3, §3, Thm. 3.5, page 21]

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and let $f : N \rightarrow M$ be a smooth map defined on another smooth manifold N . Then the LAB of $f^\mathcal{G}$ is isomorphic to the pullback LAB $f^*\mathcal{g}$.*

Proof.

By Cor. 4.1 we know that $\pi_2 : f^*\mathcal{G} \rightarrow \mathcal{G}$, the projection onto the second factor, is an LGB morphism over f ,

$$\begin{array}{ccc} f^*\mathcal{G} & \xrightarrow{\pi_2} & \mathcal{G} \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

and it is fibre-wise a Lie group isomorphism such that

$$D_{(p, e_x)}\pi_2 : \mathcal{h}_p \rightarrow \mathcal{g}_x$$

is a Lie algebra isomorphism for all $p \in N$, where e_x is the neutral element of \mathcal{G}_x for $x := f(p)$, and \mathcal{h} and \mathcal{g} are the LABs of $f^*\mathcal{G}$ and \mathcal{G} , respectively; for all of that recall that $(f^*\mathcal{G})_p$ and \mathcal{G}_x are Lie groups. Hence, we have a vector bundle morphism over f given by the following commuting diagram

$$\begin{array}{ccc} \mathcal{h} & \xrightarrow{D_{(\mathbb{1}_N, e_f)}\pi_2} & \mathcal{g} \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

which describes fibre-wise a Lie algebra isomorphism, where

$$D_{(\mathbb{1}_N, e_f)}\pi_2 := D\pi_2 \circ (\mathbb{1}_N, e_f) = \left[N \ni p \mapsto D_{(p, e_{f(p)})}\pi_2 \right].$$

By our notes in Subsection 1.1, we therefore achieve an LAB isomorphism $\mathcal{h} \rightarrow f^*\mathcal{g}$. ■

Remark 5.31: LAB of $f^*\mathcal{G}$

Since this isomorphism is very natural, we always use that identification and will refer to $f^*\mathcal{g}$ as *the* LAB of $f^*\mathcal{G}$.

With this we can quickly show the following familiar result.

Corollary 5.32: Differentials of LGB morphisms are LAB morphisms,
[2, §3.5, section about morphisms, page 124f.]

Let $\mathcal{H} \rightarrow N$ and $\mathcal{G} \rightarrow M$ be two LGBs over two smooth manifolds N and M , and we denote with \mathcal{h} and \mathcal{g} the LABs of \mathcal{H} and \mathcal{G} , respectively. Furthermore, assume that we have an LGB morphism $F : \mathcal{H} \rightarrow \mathcal{G}$ over a smooth map $f : N \rightarrow M$. Then

$$DF|_{\mathcal{h}} : \mathcal{h} \rightarrow \mathcal{g}$$

is an LAB morphism over f .

Proof.

Again by our notes in Subsection 1.1, we can view F as a base-preserving LGB morphism $F : \mathcal{H} \rightarrow f^*\mathcal{G}$, since $f^*\mathcal{G}$ is an LGB whose structure is naturally inherited by \mathcal{G} as given in Cor. 4.1; similarly for its LAB by Cor. 5.8. Thus, it is fibre-wise a Lie group morphism, and its tangent map restricted to $\mathcal{h} = e^*\mathcal{V}\mathcal{H}$ (e the identity section of \mathcal{H}) gives therefore fibre-wise a Lie algebra morphism. Thus, $DF|_{\mathcal{h}} : \mathcal{h} \rightarrow f^*\mathcal{g}$ is an LAB morphism (using Cor. 5.30), and can be seen as an LAB morphism $\mathcal{h} \rightarrow \mathcal{g}$ over f . Alternatively, it is straightforward and trivial to show it directly. ■

6. LGB actions, part II

Finally, we come to the last part of the *basics* for LGBs and their notions needed in this paper and needed to formulate a new and more general form of gauge theory.

6.1. LGB and LAB representations

As usual, representations are a special type of group action, with an infinitesimal analogue.

Definition 6.1: LGB representations,

[2, §1.7, special situation of the remark before Def. 1.7.1, page 43]

Let $\mathcal{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M , $V \xrightarrow{p} M$ be a vector bundle, and assume that we have a left \mathcal{G} -action on V , $\Psi : \mathcal{G} * V := p^*\mathcal{G} \rightarrow V$. Then we say that Ψ

is a **\mathcal{G} -representation on V** if it is linear, that is,

$$\Psi(g, \alpha v) = \alpha \Psi(g, v),$$

$$\Psi(g, v + w) = \Psi(g, v) + \Psi(g, w)$$

for all $\alpha \in \mathbb{R}$, and $(g, v), (g, w) \in \mathcal{G} * V$. In alignment with previous notations we may also write $\Psi(g, v) = \Psi_g(v)$, or $\Psi(g, v) = g \cdot v$.

Remarks 6.2.

Observe that $(g, v) \in \mathcal{G} * V$ means that $p(v) = \pi(g)$, same for (g, w) . Hence, given a base point in $x \in M$, the pairs (g, v) in $\mathcal{G} * V|_{p^{-1}(\{x\})}$ are given by elements $g \in \mathcal{G}_x$ and $v \in V_x$. By Def. 4.4 we also have

$$p(\Psi(g, v)) = \pi(g) = p(v) = x.$$

Thence, linearity of Ψ is well-defined. In fact, observe that $\mathcal{G} * V = p^*\mathcal{G} \cong \pi^*V$ as fibre bundle, therefore $\mathcal{G} * V$ carries not only the structure of an LGB but also of a vector bundle. That is, we have the following commuting diagram

$$\begin{array}{ccc} \mathcal{G} * V & \xrightarrow{\text{pr}_1} & \mathcal{G} \\ \downarrow \text{pr}_2 & & \downarrow \pi \\ V & \xrightarrow{p} & M \end{array}$$

the horizontal arrows describe the vector bundle structure (viewing $\mathcal{G} * V$ as the vector bundle π^*V), and the vertical ones the LGB structure (viewing $\mathcal{G} * V$ as the LGB $p^*\mathcal{G}$), where pr_i ($i \in \{1, 2\}$) are the projections onto the i -th component.

By fixing a base point $x \in M$ we clearly have the typical notion of a \mathcal{G}_x -representation on V_x . Equivalently, we could therefore obviously define LGB representations as a base-preserving LGB morphism $\Psi : \mathcal{G} \rightarrow \text{Aut}(V)$ as also in [2, §1.7, Def. 1.7.1, page 43].

Corollary 6.3: LGB representations as LGB morphisms,

[2, §1.7, Prop. 1.7.2, page 43]

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , $V \rightarrow M$ be a vector bundle. Then every \mathcal{G} -representation $\Psi : \mathcal{G} * V \rightarrow V$ is equivalent to a base-preserving LGB morphism $\tilde{\Psi} : \mathcal{G} \rightarrow \text{Aut}(V)$, related by

$$\Psi(g, v) = \tilde{\Psi}(g)(v)$$

for all $g \in \mathcal{G}_x$ ($x \in M$) and $v \in V_x$.

Remarks 6.4.

We will usually denote both interpretations with the same notation.

Proof of Cor. 6.3.

This is an immediate consequence of Def. 6.1 and 4.4. Ψ is smooth if and only if $\tilde{\Psi}$ is smooth; this is due to the fact that the LGB atlas of $\text{Aut}(V)$ is inherited by an atlas of V as constructed in Ex. 3.6, and due to that $\mathcal{G} * V$ is an embedded submanifold of $\mathcal{G} \times V$. ■

Example 6.5: Recovering Lie group representations on vector spaces

Similarly as to Ex. 4.14, if $\mathcal{G} = M \times G$ is a trivial LGB over M , G its structural Lie group, and $V = M \times W$ a trivial vector bundle with structural vector space W , then Ψ is equivalent to a G -representation on W . Backwards, every Lie group representation gives a representation of trivial LGBs on trivial vector bundles.

Also recall Ex. 3.6: We can now similarly construct another LGB needed for the adjoint representation.

Example 6.6: Another LGB example: Automorphisms of LABs, [2, §1.7, special situation of Ex. 1.7.12, page 46]

Let \mathcal{g} be an LAB over the smooth manifold M . We denote with $\text{Aut}(\mathcal{g}) \rightarrow M$ the bundle of fibre-wise Lie algebra automorphisms of \mathcal{g} , where the sections of $\text{Aut}(\mathcal{g})$ are the base-preserving LAB automorphisms of \mathcal{g} . As in Ex. 3.6 one can show $\text{Aut}(\mathcal{g})$ is an LGB by using LAB trivializations of \mathcal{g} instead of vector bundle trivializations.

The notation of $\text{Aut}(\mathcal{g})$ may be confusing with the notation as for vector bundles. We usually refer to this LAB automorphism when speaking of LABs and state it explicitly if we just mean the vector bundle version.

For the next major example of an LGB representation recall Def. 5.10.

Example 6.7: Adjoint LGB representation, [2, §3.5, special situation of Prop. 3.5.20, page 131]

For $g \in \mathcal{G}_x$ ($x \in M$) we have the conjugation c_g which is a \mathcal{G}_x -automorphism. In the usual way we define the **adjoint representation of \mathcal{G}** as a base-preserving LGB morphism $\mathcal{G} \rightarrow \text{Aut}(\mathcal{g})$ by

$$\text{Ad}_g := D_{e_x} c_g$$

for all $g \in \mathcal{G}_x$ ($x \in M$).

That this is a \mathcal{G} -representation on \mathcal{g} follows by the following proof: By construction $\text{Ad}_g \in \text{Aut}(\mathcal{g}_x) = \text{Aut}(\mathcal{g})|_x$ so that the adjoint representation is well-defined. As in the Lie group case, due to the fact that $c_{gq} = c_g \circ c_q$ for all $g, q \in \mathcal{G}_x$ and $c_g(e_x) = e_x$ we have

$$\text{Ad}_{gq} = D_{e_x}(c_g \circ c_q) = D_{e_x} c_g \circ D_{e_x} c_q = \text{Ad}_g \circ \text{Ad}_q.$$

Smoothness could either be shown in a bit more generalized way than the proof in [1, §2.1, Thm. 2.1.45, page 101f.], but that would be a bit tedious; instead let us use the

already-known smoothness of the adjoint representation in each fibre. Fix an open subset U of M such that \mathcal{G} and \mathcal{Q} are trivial, that is, $\mathcal{G}|_U \cong U \times G$ and $\mathcal{Q}|_U \cong U \times \mathfrak{g}$ as LGB and LAB, respectively, where G is the structural Lie group with its Lie algebra \mathfrak{g} . Then also clearly $\text{Aut}(\mathcal{Q})|_U \cong U \times \text{Aut}(\mathfrak{g})$ as LGBs. By construction we then have w.r.t. these trivializations

$$\text{Ad}_{(x,g)} = \left(x, D_{e_G} c_g^G\right) = \left(x, \text{Ad}_g^G\right)$$

for all $(x, g) \in U \times G$, where $\text{Ad}_g^G : G \rightarrow \text{Aut}(\mathfrak{g})$ is the adjoint representation of G on \mathfrak{g} . Smoothness now follows trivially by the canonical manifold structure of product manifolds and the smoothness of Ad^G .

Infinitesimally, we have LAB actions and representations; recall Cor. 5.8.

Definition 6.8: LAB actions,

[2, §4.1, reformulated version for LABs of Def. 4.1.1, page 149]

Let M, N be smooth manifolds, $\mathcal{Q} \xrightarrow{\pi} M$ an LAB, and $f : N \rightarrow M$ a smooth map. Then a **\mathcal{Q} -action on N** is a base-preserving vector bundle morphism

$$f^* \mathcal{Q} \rightarrow \text{TN},$$

$$\nu \mapsto \rho(\nu),$$

satisfying

$$Df \circ \rho = 0 \tag{23}$$

and such that the induced map

$$\Gamma(\mathcal{Q}) \rightarrow \Gamma(f^* \mathcal{Q}) \rightarrow \mathfrak{X}(N),$$

$$\mu \mapsto f^* \mu \mapsto \rho(f^* \mu)$$

is a homomorphism of Lie algebras, that is,

$$\rho\left(f^*\left([\mu, \nu]_{\mathcal{Q}}\right)\right) = [\rho(f^* \mu), \rho(f^* \nu)] \tag{24}$$

for all $\mu, \nu \in \Gamma(\mathcal{Q})$.

Remarks 6.9.

For the readers familiar with Lie algebroids, especially action Lie algebroid structures on trivial LABs, Eq. (24) should look familiar; in fact, as shown in [2, §4.1, Prop. 4.1.2, page 149f.], given an LAB action one can construct a Lie algebroid structure on $f^* \mathcal{Q}$. This leads to an action Lie

algebroid structure on a possibly non-trivial bundle.

As for Lie group and algebra actions, an LGB action induces an LAB action.

Lemma 6.10: LGB actions induce LAB actions,

[2, §4.1, special situation of Thm. 4.1.6, page 152]

Let M, N be smooth manifolds, $\mathcal{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \rightarrow M$ a smooth map. Then any right \mathcal{G} -action $\Phi : N * \mathcal{G} = f^*\mathcal{G} \rightarrow N$ on N induces a \mathcal{G} -action ρ on N by

$$\rho := D\Phi|_{f^*\mathcal{G}},$$

i.e.

$$\rho(\eta) = D_{(p, e_{f(p)})} \Phi(\eta)$$

for all $\eta \in (f^*\mathcal{G})_p$ ($p \in N$).

Remarks 6.11.

Similarly to the discussion as in [1, §3.4, page 141ff.], one can show the same for left actions, but one has to use the multiplication with the inverse, that is,

$$\rho := D\Phi'|_{f^*\mathcal{G}},$$

where

$$\Phi' : \mathcal{G} * N \rightarrow N,$$

$$(g, p) \mapsto g^{-1} \cdot p.$$

Proof of Lemma 6.10.

We generalize the proof as provided in [1, §3.4, proof of Prop. 3.4.4, page 144f.]. Observe that we have

$$\rho(p, \nu) = D_{(p, e_{f(p)})} \Phi(p, \nu) \in T_p N$$

for all $(p, \nu) \in f^*\mathcal{G}$, and thus describes a base-preserving vector bundle morphism $f^*\mathcal{G} \rightarrow TN$. We can rewrite Eq. (12) to

$$f \circ \Phi = \pi \circ \text{pr}_2,$$

where $\text{pr}_2^{f^*\mathcal{G}}$ is the projection onto the second factor in $f^*\mathcal{G} \subset N \times \mathcal{G}$. Thence, we have

$$Df \circ \rho = Df \circ D\Phi|_{f^*\mathcal{G}} = D(f \circ \Phi)|_{f^*\mathcal{G}} = \left(D\pi \circ D\text{pr}_2^{f^*\mathcal{G}} \right)|_{f^*\mathcal{G}} = D\pi|_{\mathcal{G}} \circ \text{pr}_2^{f^*\mathcal{G}} = 0,$$

making use of that \mathcal{G} consists of vertical vectors, and where $\text{pr}_2^{f^*\mathcal{G}}$ is the projection onto the second factor in $f^*\mathcal{G} \subset N \times \mathcal{G}$.

The proof of Eq. (24) is as straightforward as usual, as a quick argument for the experienced reader observe that Eq. (14) and (13) imply that Φ induces a Lie group homomorphism $\Gamma(\mathcal{G}) \rightarrow \text{Diff}(M)$ (Diff the group of diffeomorphisms) so that we have the desired Lie algebra homomorphism on an infinitesimal scale.⁴ Let $\mu, \nu \in \Gamma(\mathcal{G})$, then we know that

$$[\text{D}\Phi(f^*\mu), \text{D}\Phi(f^*\nu)] = \text{D}\Phi([f^*\mu, f^*\nu]_{f^*\mathcal{G}}),$$

if we can show that $\text{D}\Phi(f^*\mu)$ and $\text{D}\Phi(f^*\nu)$ are Φ -related to $f^*\mu$ and $f^*\nu$, respectively; this is a common procedure, see for example [1, §A.1, Prop. A.1.49, page 615]. That is, we need to show now that

$$\text{D}\Phi(f^*\mu)|_{\Phi(p,g)=p \cdot g} \stackrel{!}{=} \text{D}_{(p,g)}\Phi\left((f^*\mu)|_{(p,g)}\right) \quad (25)$$

for all $(p, g) \in N * \mathcal{G}$, similarly for ν ; here we understand the LABs of LGBs as left-invariant vector fields as in Cor. 5.17. Observe that we have in general for $\eta \in \Gamma(f^*\mathcal{G})$

$$\text{D}\phi(\eta)|_{p \cdot g} = \text{D}_{(p \cdot g, e_x)}\Phi(\eta_{(p \cdot g, e_x)}) = \left. \frac{d}{dt} \right|_{t=0} \Phi\left(p \cdot g, e^{t\xi_{p \cdot g}}\right) = \left. \frac{d}{dt} \right|_{t=0} \left(p \cdot g e^{t\xi_{p \cdot g}}\right)$$

for all $(p, g) \in N * \mathcal{G}$, where $t \in \mathbb{R}$ and we write in general $\eta_{(p, e_x)} = (p, \xi_p)$ with $\xi_p \in \mathcal{G}_x$ ($x := f(p)$), and the exponential e is the one of \mathcal{G} . But we also have similarly

$$\begin{aligned} \text{D}_{(p,g)}\Phi(\underbrace{\eta_{(p,g)}}_{\eta_{(p,e_x)}}) &= \text{D}_{(p,e_x)}(\Phi \circ L_{(p,g)})(p, \xi_p) = \left. \frac{d}{dt} \right|_{t=0} \Phi\left(p, g e^{t\xi_p}\right) = \left. \frac{d}{dt} \right|_{t=0} \left(p \cdot g e^{t\xi_p}\right), \\ &= \text{D}_{(p,e_x)}L_{(p,g)}(\eta_{(p,e_x)}) \end{aligned}$$

where L is the left-multiplication in $f^*\mathcal{G}$. So, in order to achieve Eq. (25) we now set $\eta = f^*\mu$. That is, $\eta_{(p,e_x)} = (p, \mu_{f(p)})$, therefore $\xi_p = \mu_{f(p)}$, and due to Eq. (12) and Rem. 4.5 we derive $\xi_p = \xi_{p \cdot g}$. In total, we see that Eq. (25) is satisfied.

Finally, as argued earlier, we can prove

$$[\text{D}\Phi(f^*\mu), \text{D}\Phi(f^*\nu)] = \text{D}\Phi([f^*\mu, f^*\nu]_{f^*\mathcal{G}}) = \text{D}\Phi\left(f^*\left([\mu, \nu]_{\mathcal{G}}\right)\right),$$

making use of that the field of Lie brackets of $f^*\mathcal{G}$ is the f -pullback of $[\cdot, \cdot]_{\mathcal{G}}$ as a section. This finishes the proof. ■

Remark 6.12: Variants of the LAB action as Lie algebra homomorphism

It is clear that one has a local version of Eq. (24). In fact, as one sees in the proof of Lemma 6.10, the argument for Eq. (24) works pointwise in M , so, one may also think of a homomorphism of Lie algebras

$$\begin{aligned} \mathcal{G}_x &\rightarrow \mathfrak{X}(f^{-1}(\{x\})), \\ \nu &\mapsto [f^{-1}(\{x\}) \ni p \mapsto \rho(p, \nu)] \end{aligned}$$

⁴We use the notion of sections to avoid the possible lack of smooth structure on the preimages of f .

for all $x \in M$, where we also used Eq. (23); that is, we expect that an LAB action is fibre-wise a Lie algebra action. Due to a possible lack of manifold structure on $f^{-1}(\{x\})$ we did not restrict to $x \in M$ to avoid technical difficulties. However, f will be the projection of a bundle later, and in that case the fibre $f^{-1}(\{x\})$ is an embedded submanifold of N . As argued in Remark 4.6, the \mathcal{G} -action is a \mathcal{G}_x -action restricted on $f^{-1}(\{x\})$. By what we know about Lie group actions we immediately know that we have the aforementioned Lie algebra homomorphism, and so also Eq. (24), in total a vector bundle morphism $f^*\mathcal{g} \rightarrow \text{TN}$ which gives fibre-wise over x rise to a Lie algebra action.

An important example will be fundamental vector fields which we will introduce later. Another example are LAB representations induced by LGB representations; hence let us introduce LAB representations. We will introduce them in a reverted order than the LGB representations, that is, first defining them as certain LAB morphisms, and then trivially concluding a relation to LAB actions; for the following recall Ex. 5.5.

Definition 6.13: LAB representations, [2, §3.3, Def. 3.3.13, page 107]

Let $\mathcal{g} \rightarrow M$ be an LAB over a smooth manifold M , and $V \rightarrow M$ be a vector bundle. A **\mathcal{g} -representation on V** is an LAB morphism $\psi : \mathcal{g} \rightarrow \text{End}(V)$.

Corollary 6.14: LAB representations are specific LAB actions, [2, §4.1, special consequence of Prop. 4.1.7 but we do not assume integrability of the LAB, page 153]

Let $\mathcal{g} \rightarrow M$ be an LAB over a smooth manifold M , $V \xrightarrow{p} M$ be a vector bundle, and $\psi : \mathcal{g} \rightarrow \text{End}(V)$ a \mathcal{g} -representation on V . Then ψ defines an LAB action $\tilde{\psi} : p^*\mathcal{g} \rightarrow \text{TV}$ by

$$\tilde{\psi}(v, \nu) := -\psi(\nu)(v)$$

for all $(v, \nu) \in p^*\mathcal{g}$, where one makes use of the identification $T_v V_x = V_v V \cong V_x$ ($x := p(v)$), so that $V_x \subset T_v V$.

Remarks 6.15.

In fact, similar to LGBs we could have defined LAB representations as a certain type of LAB action with values in certain *linear vector fields*. However, it would exceed this work to introduce these vector fields; start with [2, §3.4, page 110] for more elaborated details on how to do this.

As for LGB representations and linear actions, we denote both, LAB representation and its associated sense of action, in the same fashion.

This is a known relationship in the case of Lie algebra representations on vector spaces and their associated actions; usually this is proven by assuming a Lie group representation, however, one can show it without assuming integrability of the Lie algebra action. See for example [5,

§2.1, proof of Prop. 2.1.16, page 22]. Hence, we will just show that the proof of Cor. 6.14 breaks down to proving it for Lie algebras acting on vector spaces and refer to this reference for the remaining part of the proof.

Proof of Cor. 6.14.

By construction we have that $\tilde{\psi}$ has values in the vertical bundle VV of V , hence Eq. (23) follows. Making use of $V_v V \cong V_x$ for all $v \in V_x$ ($x \in M$), it is clear that $p^*V \cong VV$ as vector bundles, and so it is trivial to see that $\tilde{\psi}$ as a map $p^*\mathcal{G} \rightarrow p^*V$ is a smooth and base-preserving vector bundle morphism due to that ψ is a smooth map with values in $\text{End}(V)$ (making again use of that $p^*\mathcal{G}$ is an embedded submanifold of $V \times \mathcal{G}$, similarly for p^*V as embedded submanifold of $V \times V$).

As argued in Rem. 6.12, since \mathcal{G} acts via a representation on a bundle V , $\tilde{\psi} : p^*\mathcal{G} \rightarrow VV$ will be an LAB action, if it induces a Lie algebra action over each base point $x \in M$ via

$$\begin{aligned} \mathcal{G}_x &\rightarrow \mathfrak{X}(V_x), \\ \nu &\mapsto \left[V_x \ni v \mapsto \tilde{\psi}(v, \nu) \right]. \end{aligned}$$

Since ψ is just a \mathcal{G}_x -representation on V_x over x we know by [5, §2.1, proof of Prop. 2.1.16, page 22] that ψ indeed gives rise to a \mathcal{G}_x -action on V_x by

$$\tilde{\psi}(v, \nu) = -\psi(\nu)(v)$$

for all $v \in V_x$ and $\nu \in \mathcal{G}_x$, which is precisely the form of $\tilde{\psi}$. Thus, $\tilde{\psi}$ is an LAB action. ■

Usually statements like Cor. 6.14 are proven by assuming integrability: LGB representations as linear actions $V * \mathcal{G} \rightarrow V$ and as LGB morphisms $\mathcal{G} \rightarrow \text{Aut}(V)$ are the same by Cor. 6.3; the former induces LAB actions $p^*\mathcal{G} \rightarrow TV$ by Lemma 6.10, and in the same manner the latter implies LAB morphisms $\mathcal{G} \rightarrow \text{End}(V)$ by Cor. 5.32. Thence, LAB representations can be viewed as certain LAB actions.

The adjoint representation is an important example inherited by an LGB representation; this example will conclude this subsection.

Example 6.16: Adjoint LAB representations,

[2, §3.3, special situation of Ex. 3.3.15, page 108]

Let us assume the same situation as in Ex. 6.7, especially we have the adjoint representation of the LGB $\mathcal{G} \rightarrow M$, $\text{Ad} : \mathcal{G} \rightarrow \text{Aut}(\mathcal{G})$. As discussed earlier, its infinitesimal version is a \mathcal{G} -representation on itself, the **adjoint representation** ad of \mathcal{G} . By construction,

$$\text{ad}_\nu(\mu) := \text{ad}(\nu)(\mu) = [\nu, \mu]_{\mathcal{G}_x}$$

for all $\nu, \mu \in \mathcal{G}_x$ ($x \in M$).

6.2. Fundamental vector fields

The LAB actions inherited by LGB actions are actually so important that they have a separate label; we are following [1, §3.4, generalization of Def. 3.4.1, page 143] for the labeling:

Definition 6.17: Fundamental vector fields

Let M and N be two smooth manifolds, $\mathcal{G} \rightarrow M$ an LGB over M , $f : N \rightarrow M$ a smooth map, and assume we have a right \mathcal{G} -action on N , $N * \mathcal{G} \rightarrow N$. For $\nu \in \Gamma(\mathcal{G})$ we define its induced **fundamental vector field** $\tilde{\nu}$ as an element of $\mathfrak{X}(N)$ by

$$\tilde{\nu}_p := D_{e_{f(p)}} \Phi_p(\nu_{f(p)})$$

for all $p \in N$, where Φ_p is the orbit map defined in Def. 4.8 and $e_{f(p)}$ the neutral element of $\mathcal{G}_{f(p)}$.

For left actions we define fundamental vector fields similarly by

$$\tilde{\nu}_p := D_{e_{f(p)}} \Phi'_p(\nu_{f(p)})$$

for all $p \in N$, where Φ'_p is a slightly adjusted orbit map given by

$$\mathcal{G}_{f(p)} \rightarrow N,$$

$$g \mapsto g^{-1} \cdot p.$$

Remark 6.18: Notation

Again, point-wise over $x := f(p)$ this is just the typical definition of a fundamental vector field with respect to $\nu_x \in \mathcal{G}_x$ (except that $f^{-1}(\{x\})$ may not be a manifold). Hence, one has also a point-wise definition which we will also denote similarly by $\tilde{\nu}_x$. If $f^{-1}(\{x\})$ is not a manifold, then $\tilde{\nu}_x|_p$ is just a formal notation, and it only defines an element of $T_p N$ which may not be related to a vector field on $f^{-1}(\{x\})$, not even to a vector field on N if ν_x does not formally come from a fixed section of \mathcal{G} .

However, if f is *e.g.* the projection of a bundle, then we have a \mathcal{G} -action on the fibre $f^{-1}(\{x\})$ as manifold, and therefore $\tilde{\nu}|_{f^{-1}(\{x\})}$ and $\tilde{\nu}_x$ give rise to a vector field on $f^{-1}(\{x\})$. In other words, fundamental vector fields are vertical vector fields as expected and fibre-wise fundamental vector fields coming from a Lie group action.

For long expressions we use the following different font

$$\tilde{\nu}$$

instead of $\tilde{\nu}$.

Remark 6.19: Map to fundamental vector fields an LAB action

As anticipated, this is related to the LAB action coming from the right \mathcal{G} -action $\Phi : N * \mathcal{G} \rightarrow N$. The following can also be shown for left actions in a similar manner by recalling Rem. 6.11.

By Lemma 6.10 we have an LAB action $\rho : f^*\mathcal{G} \rightarrow TN$ given by $\rho := D\Phi|_{f^*\mathcal{G}}$. We have

$$\rho(p, \nu) = \left. \frac{d}{dt} \right|_{t=0} \Phi(p, e^{t\nu}) = \left. \frac{d}{dt} \right|_{t=0} (p \cdot e^{t\nu}) = D_{e_{f(p)}} \Phi_p(\nu) = \tilde{\nu}_p$$

for all $(p, \nu) \in f^*\mathcal{G}$, where $t \in \mathbb{R}$, e is the exponential map of \mathcal{G} , and $e_{f(p)}$ the neutral element of $\mathcal{G}_{f(p)}$. Thus, the map to fundamental vector fields

$$\Gamma(\mathcal{G}) \mapsto \mathfrak{X}(N),$$

$$\nu \mapsto \tilde{\nu},$$

is equivalent to the map

$$\Gamma(\mathcal{G}) \rightarrow \mathfrak{X}(N),$$

$$\nu \mapsto \rho(f^*\nu)$$

induced by ρ as in Def. 6.8, which also implies that the map to the fundamental vector fields is a homomorphism of Lie algebras. Last but not least, by Def. 6.8 it follows that fundamental vector fields are in the kernel of Df .

6.3. Differential of smooth LGB actions**Lemma 6.20: Tangent bundle of pullback fibre bundles**

Let M and N be two smooth manifolds, $F \xrightarrow{\pi} M$ a fibre bundle over M , and $f : N \rightarrow M$ a smooth map. Then we have for its tangent spaces

$$T_{(p,v)}(f^*F) = \{(X, Y) \mid X \in T_p N, Y \in T_v F \text{ with } D_p f(X) = D_v \pi(Y)\}$$

for all $(p, v) \in f^*F$.

Proof.

Recall that $(p, v) \in f^*F$ implies that

$$f(p) = \pi(v)$$

such that we can immediately derive its infinitesimal version as

$$D_p f(X) = D_v \pi(Y)$$

for all $X \in T_p N, Y \in T_v F$. Hence, we have derived that $T_{(p,v)}(f^* F)$ is a subset of the set of such pairs (X, Y) . That this is an equivalent description quickly follows by the fact that f and π are transversal to each other (trivially, because π is a surjective submersion). This means, the following linear map

$$\begin{aligned} T_p N \times T_v F &\rightarrow T_{f(p)} M, \\ (X, Y) &\mapsto D_p f(X) - D_v \pi(Y) \end{aligned}$$

is surjective because π is a submersion; it is also well-defined because of $f(p) = \pi(v)$. Hence, the dimension of the kernel of this map has the dimension

$$\dim(N) + \dim(F) - \dim(M) = \dim(N) + \text{rk}(F) = \dim(f^* N),$$

where \dim denotes the dimension as a manifold and rk the rank of a bundle (the dimension of its structural fibre). Its dimension is precisely the dimension of $f^* F$, and since it is about finite dimensions we can therefore identify $T_{(p,v)}(f^* F)$ with this kernel. This concludes the proof due to the fact that the kernel consists of (X, Y) with $D_p f(X) = D_v \pi(Y)$. ■

With this we can finally show the following theorem; also recall the notations introduced in Def. 4.8, 5.10, 5.23 and 6.17.

Theorem 6.21: Differential of smooth LGB actions

Let M and N be two smooth manifolds, $\mathcal{G} \xrightarrow{\pi} M$ an LGB over M , $f : N \rightarrow M$ a smooth map, and assume we have a right \mathcal{G} -action on N , $\Phi : N * \mathcal{G} \rightarrow N$. Then we have

$$D_{(p,g)} \Phi(X, Y) = D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}})_g(Y - D_{e_x} R_\sigma(D_x e(\omega)))} \Big|_{p,g} \quad (26)$$

for all $(p, g) \in N * \mathcal{G}$ and $(X, Y) \in T_{(p,g)}(N * \mathcal{G})$, where $x := f(p) = \pi(g)$, σ is any (local) section of \mathcal{G} with $\sigma_x = g$, e is the identity section of \mathcal{G} , and ω is an element of $T_x M$ given by

$$\omega := D_p f(X) = D_g \pi(Y).$$

We may omit the natural embedding of ω into $T_{e_x} \mathcal{G}$ via $D_x e$, writing instead

$$D_{(p,g)} \Phi(X, Y) = D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}})_g(Y - D_x \sigma(\omega))} \Big|_{p,g}. \quad (27)$$

If f is a surjective submersion, then we can also write

$$D_{(p,g)} \Phi(X, Y) = D_p r_\sigma(X) + D_g \Phi_\tau(Y) - D_{e_x}(\Phi_\tau \circ R_\sigma)(D_x e(\omega)) \quad (28)$$

and

$$D_{(p,g)} \Phi(X, Y) = D_p r_g(X - D_{e_x} \Phi_\tau(D_x e(\omega))) + D_g \Phi_p(Y - D_{e_x} R_\sigma(D_x e(\omega)))$$

$$+ D_{e_x}(\Phi_\tau \circ R_\sigma)(D_x e(\omega)), \quad (29)$$

where τ is any additional (local) section^a of f with $\tau_x = p$.

^aThat is, $f \circ \tau = 1_M$ (locally).

Remarks 6.22.

The assumption about f being a surjective submersion is being stated in order to assure the existence of τ and the manifold structure on $f^{-1}(\{x\})$ as an embedded submanifold of N ; see the proof for more details. If the existence of τ and the embedded submanifold structure is known otherwise, then those equations can still be derived. Following the proof, one may also just need the structure of an immersed submanifold.

Proof of Thm. 6.21.

We want to calculate the derivative of Ψ , and due to $N * \mathcal{G} = f^* \mathcal{G}$ we are going to use Lemma 6.20. That is, fix $(p, g) \in N * \mathcal{G}$ and $X \in T_p N$, $Y \in T_g \mathcal{G}$ with

$$D_p f(X) = D_g \pi(Y) =: \omega \in T_{f(p)} M.$$

Recall that we can localize LGB actions in sense of Rem. 4.6; so, let $x := f(p) = \pi(g)$, and fix a trivialization of \mathcal{G} around x . Then it is clear that there is a (local)⁵ section $\sigma \in \Gamma(\mathcal{G})$ with $\sigma_x = g$. Observe that we can write

$$Y = Y - D_x \sigma(\omega) + D_x \sigma(\omega).$$

The two first summands result into a vertical tangent vector due to

$$D_g \pi(Y - D_x \sigma(\omega)) = D_g \pi(Y) - D_x \underbrace{(\pi \circ \sigma)}_{=1_M}(\omega) = D_g \pi(Y) - \omega = 0.$$

For the following recall the notations introduced in Def. 4.8 and 5.10; we can derive that

$$D_{e_x} R_\sigma \circ D_x e = D_x \underbrace{(R_\sigma \circ e)}_{x \mapsto e_x \sigma_x = \sigma_x} = D_x \sigma.$$

So, let us write

$$Y^v := Y - D_x \sigma(\omega) = Y - D_{e_x} R_\sigma(D_x e(\omega)).$$

We have proven that $Y^v \in V_g \mathcal{G}$, and thus $(0_p, Y^v) \in T_{(p,g)}(f^* \mathcal{G})$ by Lemma 6.20, where 0_p is the zero tangent vector. In the same fashion we also have

$$(X, D_{e_x} R_\sigma(D_x e(\omega))) \in T_{(p,g)}(f^* \mathcal{G})$$

⁵For simplicity of notation we omit the notation of restricting on some open subset of M .

because of

$$D_g \pi \left(D_{e_x} R_\sigma (D_x e(\omega)) \right) = D_g \pi (Y - Y^v) = D_g \pi (Y) = D_p f(X),$$

thence we can write

$$(X, Y) = (X, D_{e_x} R_\sigma (D_x e(\omega)) + Y^v) = (X, D_{e_x} R_\sigma (D_x e(\omega))) + (0_p, Y^v).$$

Hence also

$$D_{(p,g)} \Phi(X, Y) = D_{(p,g)} \Phi(X, D_{e_x} R_\sigma (D_x e(\omega))) + D_{(p,g)} \Phi(0_p, Y^v)$$

The first summand is quickly calculated as

$$D_{(p,g)} \Phi(0_p, Y^v) = \frac{d}{dt} \Big|_{t=0} \underbrace{(p \cdot \gamma)}_{\Phi_p(\gamma)} = D_g \Phi_p(Y^v) \quad (30)$$

where $\gamma : I \rightarrow \mathcal{G}_x$ (I open interval of \mathbb{R} containing 0) is a curve with $\gamma(0) = g$ and $d/dt|_{t=0} \gamma = Y^v$; by the verticality of Y^v , $D_g \Phi_p(Y^v)$ is well-defined since Φ_p is a map $\mathcal{G}_x \rightarrow N$. Now recall Def. 5.23 and 6.17, and observe that we can write

$$D_g \Phi_p(Y^v) = (D_g \Phi_p \circ D_{e_x} L_g \circ D_g L_{g^{-1}})(Y^v) = D_{e_x} \underbrace{(\Phi_p \circ L_g)}_{\mathcal{G} \ni q \mapsto p \cdot g q = \Phi_{p \cdot g}(q)}((\mu_{\mathcal{G}})_g(Y^v)) = \overline{(\mu_{\mathcal{G}})_g(Y^v)} \Big|_{p \cdot g} \quad (31)$$

making use of the verticality of Y^v such that DL_g can act on Y^v and of $(\mu_{\mathcal{G}})_g(Y^v) \in \mathcal{G}_x$.

For the second summand in Eq. (30) we use

$$\begin{aligned} D_{(p,g)} \Phi(X, D_{e_x} R_\sigma (D_x e(\omega))) &= D_{(p,e_x)} \underbrace{(\Phi \circ (\mathbb{1}_N, R_\sigma))}_{N * \mathcal{G} \ni (p,g) \mapsto p \cdot g \sigma_x = r_\sigma(p \cdot g)}(X, D_x e(\omega)) \\ &= D_{(p,e_x)}(r_\sigma \circ \Phi)(X, D_x e(\omega)) \\ &= D_p r_\sigma \left(D_{(p,e_x)} \Phi(X, D_x e(\omega)) \right) \\ &= D_p r_\sigma \left(D_{(p,e_x)} \Phi(X, D_p(e \circ f)(X)) \right) \\ &= D_p r_\sigma \left(D_{(p,p)} (\Phi \circ (\mathbb{1}_N, e \circ f))(X, X) \right), \end{aligned}$$

but

$$\left[N \times N \ni (p, p) \mapsto (\Phi \circ (\mathbb{1}_N, e \circ f))(p, p) = \Phi(p, e_{f(p)}) = p \cdot e_{f(p)} = p \right] = [p \mapsto \mathbb{1}_N(p)],$$

hence,

$$D_{(p,p)} (\Phi \circ (\mathbb{1}_N, e \circ f))(X, X) = D_p \mathbb{1}_N(X) = \mathbb{1}_{T_p N}(X) = X.$$

So, we get in total

$$D_{(p,g)}\Phi\left(X, D_{e_x}R_\sigma(D_x e(\omega))\right) = D_p r_\sigma(X),$$

therefore

$$D_{(p,g)}\Phi(X, Y) = D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}})_g(Y^v)} \Big|_{p \cdot g}.$$

If f is a surjective submersion, then x is a regular value and thus $f^{-1}(\{x\})$ is an embedded submanifold, and we can assure the existence of a smooth local section $\tau : U \rightarrow N$ of f (U an open neighbourhood of $x \in M$), *i.e.* $f \circ \tau = \mathbb{1}_U$ with $\tau_x = p$; in case of doubt, this can be shown as in [1, §3.7, Lemma 3.7.4, page 152f.] via the Regular Point Theorem. Recalling the arguments of Rem. 5.18, we can rewrite Eq. (30) to

$$\begin{aligned} D_g \Phi_p(Y^v) &= D_g \Phi_\tau(Y^v) \\ &= D_g \Phi_\tau(Y) - D_g \Phi_\tau(D_{e_x} R_\sigma(D_x e(\omega))) \\ &= D_g \Phi_\tau(Y) - D_{e_x}(\Phi_\tau \circ R_\sigma)(D_x e(\omega)) \end{aligned}$$

making use of that $D_g \Phi_\tau$ is linear map $T_g \mathcal{G} \rightarrow T_{p \cdot g} N$. In that case we would get in total

$$D_{(p,g)}\Phi(X, Y) = D_p r_\sigma(X) + D_g \Phi_\tau(Y) - D_{e_x}(\Phi_\tau \circ R_\sigma)(D_x e(\omega)).$$

Alternatively, we can keep the form of Eq. (30) and instead apply the same trick to X as for Y , that is,

$$X = X^v + D_{e_x} \Phi_\tau(D_x e(\omega)),$$

where

$$X^v := X - D_x \tau(\omega) = X - D_{e_x} \Phi_\tau(D_x e(\omega)),$$

and X^v is vertical, too, that is,

$$D_p f(X^v) = D_p f(X) - D_x(f \circ \tau)(\omega) = \omega - \omega = 0,$$

especially $X^v \in T_p(f^{-1}(\{x\}))$ and so we can apply a similar argument as in Rem. 5.18 to derive

$$\begin{aligned} D_p r_\sigma(X) &= D_p r_\sigma(X^v) + D_p r_\sigma(D_{e_x} \Phi_\tau(D_x e(\omega))) \\ &= D_p r_g(X^v) + D_{e_x}(r_\sigma \circ \Phi_\tau)(D_x e(\omega)). \end{aligned}$$

Finally, using these expressions, the total formula would look like

$$\begin{aligned} D_{(p,g)}\Phi(X, Y) &= D_p r_g(X^v) + D_g \Phi_p(Y^v) + D_{e_x} \underbrace{(r_\sigma \circ \Phi_\tau)}_{\mathcal{G} \ni g \mapsto \tau_{\pi(g)} \cdot g \sigma_{\pi(g)} = (\Phi_\tau \circ R_\sigma)(g)}(D_x e(\omega)) \\ &= D_p r_g(X^v) + D_g \Phi_p(Y^v) + D_{e_x}(\Phi_\tau \circ R_\sigma)(D_x e(\omega)). \end{aligned}$$

■

Remarks 6.23.

Eq. (26) is very similar to the "classical" formula used in gauge theory, see *e.g.* [1, §3.5, Prop. 3.5.4, page 146]: In the case of a Lie group action on N (so, \mathcal{G} a Lie group G and M a point) we have

$$D_{(p,g)}\Phi(X, Y) = D_p r_g(X) + \overline{(\mu_G)_g(Y)} \Big|_{p \cdot g}$$

for all $p \in N$, $g \in G$, $X \in T_p N$ and $Y \in T_g G$. However, in our general case the vector Y is deformed by ω , due to the fact that the action Φ has no "constant" Lie group factor anymore. This will be important later when we are going to derive the gauge transformations. Furthermore, already the first summand is different than the classical formula, because we need to use LGB sections in order to define the push-forward of tangent vectors which are not vertical, that is, X may not be a tangent vector of $f^{-1}(\{x\})$ (which is in the general case not even an embedded submanifold) such that $D_p r_g(X)$ is in general not well-defined anymore.

The other two equations in the case of f being a surjective submersion are mainly for reference; the last equation, Eq. (29), emphasises the contribution of non-vertical vectors measured by ω . While the first two summands are the classical product rule on the vertical parts, the third summand shows the deformation of the product rule because of the new structure of an action without a "constant" Lie group factor.

Eq. (28) may be the most elegant formulation, making use of local sections σ and τ and their advantage that these can act on all tangent vectors, not just the vertical ones; the reader who knows Lie groupoids may recognize this equation's structure with the one as given in [2, §1.4, Thm. 1.4.14, page 28], where it is about the induced multiplication structure on the tangent bundle of a Lie groupoid making use of bisections playing a similar role like σ and τ .

Those equations additionally show that we have a more general Leibniz rule with a third summand. However, by Ex. 4.14 we expect still a typical Leibniz rule once a trivialization of \mathcal{G} around g is fixed. Indeed, as we will understand and see also later, this is the case; fix such a trivialization, equip it with a canonical flat connection, and take σ to be a parallel section, that is, g as a constant section. Then one can calculate that the typical Leibniz rule is recovered. Hence, one can view this Leibniz rule as the "covariantized" version of the Leibniz rule, independent of a choice of "coordinate"/section on \mathcal{G} .

7. Connections and curvature on LGB principal bundles

7.1. Principal bundles with structural LGB

7.1.1. Definition

The principal bundle \mathcal{P} we are interested into is still a fibre bundle related to the same Lie group as the one behind the LGB \mathcal{G} , especially also $\dim(\mathcal{P}) = \dim(\mathcal{G})$, but it is equipped with an LGB action.

Definition 7.1: Principal bundles with structural LGB,**[6, simplification of the beginning of §5.7, page 144f.]**

Let G be a Lie group, M a smooth manifold, and an LGB $\mathcal{G} \rightarrow M$ which acts on the right on another G -fibre bundle $\mathcal{P} \rightarrow M$

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{P} & \xleftarrow{\quad} & \mathcal{G} & \longleftarrow & G \\ & & \downarrow \pi_{\mathcal{P}} & & \downarrow \pi_{\mathcal{G}} & & \\ & & M & & M & & \end{array}$$

where the right-action is defined on $\mathcal{P} * \mathcal{G}$ given by $\pi_{\mathcal{P}}^* \mathcal{G}$. Then we call \mathcal{P} a **principal \mathcal{G} -bundle** if

1. The right \mathcal{G} -action on \mathcal{P} is **simply transitive on the fibres**, that is, the restriction of $\mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$ on $x \in M$

$$\mathcal{P}_x \times \mathcal{G}_x \rightarrow \mathcal{P}_x$$

induces bijective orbit maps

$$\mathcal{G}_x \rightarrow \mathcal{P}_x,$$

$$g \mapsto p \cdot g$$

for $p \in \mathcal{P}_x$.

2. There exist base-preserving **\mathcal{G} -equivariant diffeomorphisms** $\varphi_i : \mathcal{P}|_{U_i} \rightarrow \mathcal{G}|_{U_i}$ subordinate to an open covering $(U_i)_i$ of M , that is,

$$\pi_{\mathcal{G}} \circ \varphi_i = \pi_{\mathcal{P}},$$

$$\varphi_i(p \cdot g) = \varphi_i(p) \cdot g,$$

where the multiplication on the right hand side is inherited by $\mathcal{G} * \mathcal{G} \rightarrow \mathcal{G}$, the multiplication on \mathcal{G} .

The LGB \mathcal{G} is the **structural LGB** of the principal bundle \mathcal{P} .

Remark 7.2: Discussion about the definition of \mathcal{G} -principal bundles

There are several things in need to be discussed:

1. The mentioned reference, [6, simplification of the beginning of §5.7, page 144f.], introduces these principal bundles in a different manner; we will come back later to this in Remark 7.13.
2. By Remark 4.5 the \mathcal{G} -action on \mathcal{P} is fibre-preserving, thus, the restriction of $\mathcal{P} * \mathcal{G} \rightarrow$

\mathcal{P} is indeed a map $\mathcal{P}_x \times \mathcal{G}_x \rightarrow \mathcal{P}_x$.

3. The \mathcal{G} -equivariant diffeomorphisms φ_i give rise to a **bundle atlas of \mathcal{G} -equivariant bundle charts** $\xi_i : \mathcal{P}|_{U_i} \rightarrow U_i \times G$. Assume w.l.o.g. that U_i is small enough so that there is an LGB chart $\phi_i : \mathcal{G}|_{U_i} \rightarrow U_i \times G$ of \mathcal{G} ; recall Def. 3.1. Then $\xi_i := \phi_i \circ \varphi_i$, which clearly satisfies

$$\xi_i(p \cdot g) = \xi_i(p) \cdot \phi_i(g),$$

where the multiplication on the right hand side is the canonical one for $U_i \times G$ as a trivial LGB (recall Ex. 3.3), and this can be viewed as \mathcal{G} -equivariance (under the trivialization induced by ϕ_i). We usually then speak of **\mathcal{G} -equivariance w.r.t. the LGB morphism ϕ_i** .

We will call such an atlas and its charts a **principal bundle atlas** and **principal bundle charts** for \mathcal{P} , respectively. For simplicity we may also refer to φ_i as principal bundle chart giving rise to the principal bundle atlas.

4. Due to the existence of φ_i one does not need to claim upfront that \mathcal{P} is a G -fibre bundle. However, for readability, we decided to structure the definition like this. We kept a similar style as for "typical" principal bundles as provided in [1, §4.2, Def. 4.2.1, page 207f.].
5. Since $\pi_{\mathcal{P}}$ is a surjective submersion we know by Remark 4.11 that right-translations r_g ($g \in \mathcal{G}_x$) are diffeomorphisms on \mathcal{P}_x . Furthermore, following [1, §4.2, discussion after Def. 4.2.1, page 208f.], by definition we have a simply transitive \mathcal{G}_x -action (as a Lie group) on \mathcal{P}_x , and the isotropy group for each $p \in \mathcal{P}_x$ is trivial; the isotropy group consists in general of $g \in \mathcal{G}_x$ with $p \cdot g = p$, see e.g. [1, §3.2, third part of Def. 3.2.4, page 132]. Therefore, and by [1, §3.8, Thm. 3.8.8, page 165], the orbit map Φ_p gives rise to a \mathcal{G}_x -equivariant diffeomorphism

$$\mathcal{G}_x \rightarrow \mathcal{P}_x,$$

where the action on \mathcal{G}_x is the right-action on itself regarding the \mathcal{G}_x -equivariance.

6. Similarly by definition, for each $x \in U_i$ we know that $(\varphi_i)|_x : \mathcal{P}_x \rightarrow \mathcal{G}_x$ is a \mathcal{G}_x -equivariant diffeomorphism. In fact, together with $\pi_{\mathcal{G}} \circ \varphi_i = \pi_{\mathcal{P}}$ this clearly gives an equivalent definition of φ_i which we may make use in the following without further mention.

7.1.2. Examples

Let us provide examples of such principal bundles.

Example 7.3: The "pre-classical" principal bundle

G -principal bundles themselves are $\mathcal{G} := M \times G$ -principal bundles. Recall Ex. 4.14, the LGB action of a trivial LGB $\mathcal{G} \cong M \times G$ is equivalent to a G -action, the right-translation with an element $g \in G$ is then either the right-translation of the corresponding constant section in \mathcal{G} or the right-translation with (x, g) pointwise; this action is clearly simply transitive. The \mathcal{G} -principal bundle atlas is then inherited by the existing G -principal bundle atlas.

We often refer to principal bundles related to trivial LGBs as **pre-classical principal bundles**, or, as usual, principal G -bundle. We will later clarify why "pre-classical" as adjective instead of "classical"; however, at this point both adjectives are permutable.

Example 7.4: The "trivial" principal \mathcal{G} -bundle

\mathcal{G} itself is a \mathcal{G} -principal bundle, equipped with its canonical right-action inherited by its multiplication $\mathcal{G} * \mathcal{G} \rightarrow \mathcal{G}$. The principal bundle atlas then just consists of the identity map $\mathbb{1}_{\mathcal{G}}$.

By Ex. 7.3, it may be natural to call this the **trivial \mathcal{G} -principal bundle** even if \mathcal{G} itself might not be trivial. It will be clearer later why one may choose to do so.

The remarkable property is that this example shows that we are going to define a gauge theory for which LGBs themselves are allowed as principal bundles. One might have wondered why classical gauge theory uses pre-classical principal bundles instead of LGBs, especially including non-trivial LGBs, because LGBs could be viewed as a more natural choice due to the fact that they are an analogue to how vector bundles are the "bundle-construction" of vector spaces. The problems described in Subsection 7.2.1 show why it was easier to choose pre-classical principal bundles, since these avoid the difficulties regarding the definition of a connection; however, we are going to solve these problems in such a way that one can either use LGBs or pre-classical bundles or even something more general.

Our main example will be the inner bundle of a pre-classical principal bundle, recall Ex. 3.12. To explain the "triviality" of Ex. 7.4 we need to introduce morphisms of principal bundles.

7.1.3. Morphism of principal LGB-bundles

Let us define morphisms of LGB-principal bundles

Definition 7.5: Morphism of principal bundles with structural LGB

Let M and N be smooth manifolds, $\mathcal{H} \xrightarrow{\pi_{\mathcal{H}}} N$ and $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ LGBs, and $\mathcal{H} \rightarrow \mathcal{P}' \xrightarrow{\pi'} N$ and $\mathcal{G} \rightarrow \mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{H} - and \mathcal{G} -bundle, respectively. A **principal bundle morphism** between \mathcal{P}' and \mathcal{P} is a triple of smooth maps $F : \mathcal{H} \rightarrow \mathcal{G}$, $f : N \rightarrow M$ and

$H : \mathcal{P}' \rightarrow \mathcal{P}$ such that the pair (F, f) is an LGB morphism as in Def. 3.4 and

$$\pi \circ H = f \circ \pi', \quad (32)$$

$$H(p \cdot h) = H(p) \cdot F(h) \quad (33)$$

for all $(p, h) \in \mathcal{P}' * \mathcal{H} = (\pi')^* \mathcal{H}$. We also speak of a **principal bundle morphism over f w.r.t. the LGB morphism F** .

We speak of a **principal bundle isomorphism (over f , w.r.t. F)** if H is a diffeomorphism.

If H is a base-preserving isomorphism $\mathcal{P} \rightarrow \mathcal{P}$ w.r.t. $F = \mathbb{1}_{\mathcal{G}}$, then we say that H is a **(global) gauge transformation or principal bundle automorphism**, and the set of all such automorphisms is denoted by $\text{Aut}(\mathcal{P})$.

Remarks 7.6.

- Observe that the right hand side of Eq. (33) is well-defined because of Eq. (32), $(p, h) \in \mathcal{P}' * \mathcal{H}$ and Def. 3.4, that is,

$$(\pi_{\mathcal{G}} \circ F)(h) \stackrel{3.4}{=} (f \circ \pi_{\mathcal{H}})(h) \stackrel{\pi'(p) = \pi_{\mathcal{H}}(h)}{=} (f \circ \pi')(p) \stackrel{(32)}{=} (\pi \circ H)(p),$$

thus,

$$(H(p), F(h)) \in \mathcal{P} * \mathcal{G} = \pi^* \mathcal{G}.$$

Furthermore, by additionally using Remark 7.2 we know that LGB actions preserve the fibres of the principal bundles, therefore both, $H(p \cdot h)$ and $H(p) \cdot F(h)$, are over the same base point $(f \circ \pi')(p)$, so that Eq. (33) as a whole is well-defined.

- Also observe that one can conclude that (F, f) has to be an LGB morphism in order to have a satisfied and well-defined Eq. (33), assuming that H is a map over f . To well-define the right hand side, F is a morphism over f , since H is defined over f . Furthermore, for $h' \in \mathcal{H}_{\pi'(p)}$ we have

$$H(p \cdot h'h) = H(p) \cdot F(h'h),$$

but we also get by associativity

$$H(p \cdot h'h) = H(p) \cdot F(h') \cdot F(h).$$

Using that \mathcal{G} acts simply transitive on \mathcal{P} , we derive

$$F(h'h) = F(h') \cdot F(h).$$

- If H is a diffeomorphism, then F is an LGB isomorphism. For this we only have to show that F is a diffeomorphism by Def. 3.4; by Remark 3.5, also f is then a diffeomorphism. Thence,

let us show that F is a diffeomorphism. This follows by the fact that the LGB actions are simply transitive on the fibres of \mathcal{P}' and \mathcal{P} , *i.e.* orbit maps are diffeomorphisms, also recall Remark 7.2. Denoting the orbit maps inherited by the action on \mathcal{P}' and \mathcal{P} by $\Phi^{\mathcal{P}'}$ and $\Phi^{\mathcal{P}}$, respectively, we can write

$$F(h) = \left(\left(\Phi_{H(p)}^{\mathcal{P}} \right)^{-1} \circ H \circ \Phi_p^{\mathcal{P}'} \right)(h)$$

for arbitrary p by rewriting Eq. (33). Hence, F is a diffeomorphism as the composition of diffeomorphisms.

Last but not least, it follows that H^{-1} is then \mathcal{G} -equivariant in the sense of

$$H^{-1}(q \cdot g) = H^{-1}(q) \cdot F^{-1}(g)$$

for all $(q, g) \in \mathcal{P} * \mathcal{G}$, which is well-defined by similar arguments as before, especially because the inverses are maps over f^{-1} . To prove this, observe that there is a unique $(p, h) \in \mathcal{P}' * \mathcal{H}$ such that $H(p) = q$ and $F(h) = g$ due to the fact that both are bijective maps over f .⁶ Then

$$H^{-1}(q \cdot g) = H^{-1}(\underbrace{H(p) \cdot F(h)}_{=H(p \cdot h)}) = p \cdot h = H^{-1}(q) \cdot F^{-1}(g).$$

Thence, Def. 7.5 is a valid definition for a principal bundle morphism, because it is easy to check that the principal bundle atlas on \mathcal{P} (consisting of φ_i related to an open covering $(U_i)_i$ of M) is related to the principal bundle atlas on \mathcal{P}' by

$$(F^{-1} \circ \varphi_i \circ H)|_{f^{-1}(U_i)}.$$

- Finally, observe that we have locally an isomorphism of every principal bundle \mathcal{P} to \mathcal{G} : Fix a principal bundle chart $\varphi : \mathcal{P}|_U \rightarrow \mathcal{G}|_U$ (U some open subset of M). Then take $H := \varphi$, $F := \mathbb{1}_{\mathcal{G}|_U}$ and $f := \mathbb{1}_U$; by the definition of a principal bundle chart this gives a principal bundle isomorphism. Therefore one could say that every principal bundle is locally "trivial" in the sense of being an LGB; recall Ex. 7.4.

Additionally using Remark 7.2 we have locally an isomorphism of \mathcal{P} to a trivial LGB; keeping the same notation as in Remark 7.2, choose $H := \xi_i$, $F := \phi_i$ and $f := \mathbb{1}_{U_i}$. Hence, locally every principal bundle is also pre-classical in the sense of Ex. 7.3.

As expected, there is a natural isomorphism induced by local sections of \mathcal{P} , which are, however, no trivializations in general. In other words, by Remark 7.2 we know that orbit maps through the fibres of \mathcal{P} are equivariant diffeomorphisms, we want to show the same for the orbit map through a section of \mathcal{P} .

⁶In fact, it is easy to prove that $(H, F) : \mathcal{P}' * \mathcal{H} \rightarrow \mathcal{P} * \mathcal{G}$ is an LGB isomorphism (over H); also recall Cor. 4.1.

Lemma 7.7: Local sections of principal bundles induce isomorphisms to the structural LGB

Let $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ be an LGB over a smooth manifold M , and $\mathcal{P} \rightarrow M$ a principal \mathcal{G} -bundle. Let $s : U \rightarrow \mathcal{P}$ be a smooth local section of \mathcal{P} over an open subset U of M . Then the orbit map Φ_s through s , given as in Def. 4.8 by

$$\mathcal{G}|_U \rightarrow \mathcal{P}|_U,$$

$$g \mapsto s_{\pi_{\mathcal{G}}(g)} \cdot g,$$

is a base-preserving principal bundle isomorphism w.r.t. $\mathbb{1}_{\mathcal{G}|_U}$.

Remarks 7.8.

As in the typical formulation of gauge theory, we have an isomorphism induced by sections, but it is not necessarily a trivialization as fibre bundle, so that we do not necessarily also have a "pre-classical" bundle; it is a trivialization in the sense of Ex. 7.4. Due to the similarity with the "classical" statement, we therefore were speaking of the "trivial" principal bundle in Ex. 7.4. Of course, since every LGB is locally trivial, we can find a typical trivialization by taking a "local-enough" section.

The latter about the typical trivialization is also stated in [6, §5.7, fourth part of Remark 5.34, page 145] for groupoid-based principal bundles.

Proof of Lemma 7.7.

The proof is similar to the "classical" statement as *e.g.* given in [1, §4.2, proof of Lemma 4.2.7, page 210ff.], but generalized since we have to treat a possible non-triviality of $\mathcal{G}|_U$. As already discussed, Φ_s is well-defined because of $\pi_{\mathcal{P}} \circ s = \mathbb{1}_U$ such that $(s_{\pi_{\mathcal{G}}(g)}, g) \in \mathcal{P} * \mathcal{G}$. Let us denote with $\Phi : \mathcal{P} * \mathcal{G} \rightarrow \mathcal{G}$ the right \mathcal{G} -action $(p, g) \mapsto p \cdot g$ on \mathcal{P} , then

$$\Phi_s(g) = \Phi(s_{\pi_{\mathcal{G}}(g)}, g) = (\Phi \circ (\pi_{\mathcal{G}}^* s, \mathbb{1}_{\mathcal{G}}))(g, g). \quad (34)$$

Thence, Φ_s is clearly a smooth map as the composition of smooth maps; the second map $(\pi_{\mathcal{G}}^* s, \mathbb{1}_{\mathcal{G}}) : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is given by $(g, q) \mapsto (s_{\pi_{\mathcal{G}}(g)}, q)$ and is clearly smooth if restricted to the diagonal as embedded submanifold. By Def. 4.4 we have

$$(\pi_{\mathcal{P}} \circ \Phi_s)(g) = \pi_{\mathcal{P}}(s_{\pi_{\mathcal{G}}(g)} \cdot g) = \pi_{\mathcal{G}}(g),$$

thus, Φ_s is base-preserving, similarly it follows that Φ_s is \mathcal{G} -equivariant w.r.t. $\mathbb{1}_{\mathcal{G}}$ by making use of $\pi_{\mathcal{G}}(gq) = \pi_{\mathcal{G}}(g)$ for all $g, q \in \mathcal{G} * \mathcal{G}$. Furthermore, Φ_s is bijective since the right \mathcal{G} -action on \mathcal{P} is simply transitive, that is, when we restrict Φ_s to an $x \in U$, then it is a map

$$\mathcal{G}_x \rightarrow \mathcal{P}_x,$$

$$g \mapsto s_x \cdot g,$$

thence, as expected is the orbit map Φ_{s_x} through s_x which is a \mathcal{G}_x -equivariant diffeomorphism by Remark 7.2. So, Φ_s is fibre-wise bijective and therefore bijective as a whole since it is base-preserving.

Now we want to use the inverse function theorem to show that its inverse is also smooth. Once we know that the tangent map/total derivative $D_g\Phi_s : T_g\mathcal{G} \rightarrow T_{s_x \cdot g}\mathcal{P}$ is an isomorphism of vector spaces for all $g \in \mathcal{G}|_U$, then we know by the inverse function theorem that Φ_s^{-1} is smooth. Hence, we will now show that $D_g\Phi_s$ is injective, then it has to be bijective by dimensional reasons ($\dim(\mathcal{G}) = \dim(\mathcal{P})$) so that we are done. This differential is given by Thm. 6.21, making use of Eq. (34),

$$\begin{aligned} D_g\Phi_s(Y) &= D_{s_x}r_\sigma(D_xs(\omega)) + \overline{(\mu_{\mathcal{G}})_g(Y - D_x\sigma(\omega))} \Big|_{s_x \cdot g} \\ &= D_x(r_\sigma \circ s)(\omega) + \overline{(\mu_{\mathcal{G}})_g(Y - D_x\sigma(\omega))} \Big|_{s_x \cdot g} \end{aligned}$$

for all $Y \in T_g\mathcal{G}$, where $x := \pi_{\mathcal{G}}(g)$, $\sigma \in \Gamma(\mathcal{G}|_U)$ ⁷ with $\sigma_x = g$ and $\omega := D_g\pi_{\mathcal{G}}(Y) \in T_xM$. We want to decompose Φ_s now. Observe that $Y - D_x\sigma(\omega) \in V_g\mathcal{G}$, i.e. it is vertical in \mathcal{G} due to the fact that

$$D_g\pi_{\mathcal{G}}(Y - D_x\sigma(\omega)) = \underbrace{D_g\pi_{\mathcal{G}}(Y)}_{=\omega} - \underbrace{D_g\pi_{\mathcal{G}}(D_x\sigma(\omega))}_{=\omega} = 0 \quad (35)$$

because of

$$\pi_{\mathcal{G}} \circ \sigma = \mathbb{1}_U$$

such that

$$D\pi_{\mathcal{G}} \circ D\sigma = \mathbb{1}_{TM|_U}. \quad (36)$$

Usually, we just make use of that without further mention; we repeat this trivial fact in order to emphasize that $D\sigma$ is injective because $\mathbb{1}_{TM|_U}$ bijective and that we have

$$\text{Im}(D_x\sigma) \cap \text{Ker}(D_g\pi_{\mathcal{G}}) = \{0\}$$

(the image of $D_x\sigma$ intersects trivially with the kernel of $D_g\pi_{\mathcal{G}}$). The injectivity of $D\sigma$ implies that the dimension of its image satisfies

$$\dim(\text{Im}(D_x\sigma)) = \dim(M),$$

so that, in total, we know by dimensional reasons

$$T_g\mathcal{G} \cong \text{Im}(D_x\sigma) \oplus \text{Ker}(D_g\pi_{\mathcal{G}}) = \text{Im}(D_x\sigma) \oplus V_g\mathcal{G}.$$

⁷W.l.o.g. we assume that σ is defined on U , otherwise "make U smaller".

We can decompose Y accordingly by Eq. (35),

$$Y = \underbrace{D_x \sigma(\omega)}_{=: Y^h} + \underbrace{Y - D_x \sigma(\omega)}_{=: Y^v} = Y^h + Y^v,$$

v stands for the vertical part and h for its complementary part ("horizontal"). In fact, as it is well-known, a section σ of a bundle induces a splitting of the short exact sequence of vector bundles

$$\sigma^* V\mathcal{G}|_U \hookrightarrow \sigma^* T\mathcal{G}|_U \xrightarrow{D\pi_{\mathcal{G}}} TM|_U,$$

and $D\sigma$ is a splitting/section of this sequence so that we have $\sigma^* T\mathcal{G} \cong \text{Im}(D\sigma) \oplus \sigma^* V\mathcal{G}$.⁸ Furthermore, again due to Eq. (36), we have $\text{Im}(D_x \sigma) \cong T_x M$ as vector spaces by $Y^h \mapsto D_g \pi_{\mathcal{G}}(Y^h) = D_g \pi_{\mathcal{G}}(Y) = \omega$ since Y^v is vertical. Hence, we can view $D_g \Phi_s$ as a map

$$T_x M \oplus V_g \mathcal{G} \rightarrow T_{s_x \cdot g} \mathcal{P},$$

$$(\omega, Y^v) \mapsto D_x(r_\sigma \circ s)(\omega) + \overline{(\mu_{\mathcal{G}})_g(Y^v)} \Big|_{s_x \cdot g}.$$

These arguments apply to any section of a bundle, hence, we repeat this now. Observe that $r_\sigma \circ s$ is also a section of $\mathcal{P}|_U$ due to Def. 4.4, that is,

$$\pi_{\mathcal{P}}(s_y \cdot \sigma_y) = \pi_{\mathcal{P}}(s_y) = y$$

for all $y \in M$. As before, we split

$$T_{s_x \cdot g} \mathcal{P}|_U \cong \text{Im}(D_x(r_\sigma \circ s)) \oplus V_{s_x \cdot g} \mathcal{P},$$

and we identify $\text{Im}(D_x(r_\sigma \circ s)) \cong T_x M$ as vector spaces, now via $\omega \mapsto D_x(r_\sigma \circ s)(\omega)$ due to Eq. (36) but with $r_\sigma \circ s$ playing the role as section. Additionally by Def. 6.17 and Remark 6.18 we know that fundamental vector fields are vertical, and so we can view $D_g \Phi_s$ as a map

$$T_x M \oplus V_g \mathcal{G} \rightarrow T_x M \oplus V_{s_x \cdot g} \mathcal{P},$$

$$(\omega, Y^v) \mapsto \left(\omega, \overline{(\mu_{\mathcal{G}})_g(Y^v)} \Big|_{s_x \cdot g} \right).$$

Thus, $D_g \Phi_s$ is the pair of two linear independent maps $T_x M \rightarrow T_x M$ and $V_g \mathcal{G} \rightarrow V_{s_x \cdot g} \mathcal{P}$. The former is clearly an isomorphism, and the latter is by definition of fundamental vector fields and the Maurer-Cartan form of the shape as in Eq. (31), *i.e.*

$$Y^v \mapsto D_g \Phi_{s_x}(Y^v).$$

By Remark 7.2, Φ_{s_x} is a \mathcal{G}_x -equivariant diffeomorphism $\mathcal{G}_x \rightarrow \mathcal{P}_x$, thence, $D_g \Phi_{s_x} : T_g \mathcal{G}_x = V_g \mathcal{G} \rightarrow T_{s_x \cdot g} \mathcal{P}_x = V_{s_x \cdot g} \mathcal{P}_x$ is an isomorphism of vector spaces. Finally we can conclude that $D_g \Phi_s$ is a linear isomorphism as the sum of two linear independent isomorphisms. As argued earlier, the inverse function theorem finishes now the proof. \blacksquare

⁸ σ is actually an embedding, and thus $\text{Im}(D\sigma)$ is a well-defined subbundle isomorphic to TM .

Usually, one likes to think about the choice of sections as a choice of a coordinate transformation as in [1, §4.2, Remark 4.2.21, page 220]. This is due to that the gauge theory usually corresponds to a formulation via a trivial LGB, which we will understand later, but already have an idea of by *e.g.* Ex. 4.14 and 7.3. Then we have a local trivialization of \mathcal{P} such that one usually thinks of a gauge as a choice of coordinate system.

However, we now learned that on a more general scale this is not completely what is happening. The idea of LGBs and principal bundles are very similar; both are fibre bundles related to a Lie group and they carry an action which also restricts on each fibre. But the fibres of an LGB are Lie groups themselves, while the fibres of a principal bundle are "just" diffeomorphic to a Lie group in an equivariant way as outlined in Remark 7.2 and as given in their definition Def. 7.1. One could view the fibres of \mathcal{P} as having an "almost" Lie group structure, a Lie group structure without a designated neutral element.

Lemma 7.7 shows that the choice of a section of \mathcal{P} is actually the choice of a designated neutral element, naturally inducing a Lie group structure in each fibre and thus an LGB structure, which may not be trivial. Aligning the more general definition of principal bundles with the definition of LGBs. For example set $\mathcal{P} = \mathcal{G} = c_G(P)$ as in Ex. 3.12, where P is a non-trivial pre-classical principal bundle and the underlying Lie group G is non-abelian; such an LGB is likely non-trivial but always carries a global section. We will see such an example later.

We want to use this now in order to study a certain pullback of principal bundles. For this we need to introduce general pullbacks of principal bundles; also recall again Cor. 4.1.

Corollary 7.9: Pullbacks of principal LGB-bundles are principal LGB-bundles

Let M, N be smooth manifolds, $f : N \rightarrow M$ a smooth map, $\mathcal{G} \rightarrow M$ an LGB, and $\mathcal{P} \rightarrow M$ a principal \mathcal{G} -bundle. Then there is a unique (up to isomorphisms) principal $f^\mathcal{G}$ -bundle structure on $f^*\mathcal{P}$ such that the projection $\pi_2 : f^*\mathcal{P} \rightarrow \mathcal{P}$ onto the second factor is a principal bundle morphism (over f) w.r.t. the projection $\pi_2^{\mathcal{G}} : f^*\mathcal{G} \rightarrow \mathcal{G}$ onto the second factor as LGB morphism and such that $\pi_2|_x : (f^*\mathcal{P})_x \rightarrow \mathcal{P}_{f(x)}$ is a \mathcal{G}_x -equivariant diffeomorphism w.r.t. the LGB isomorphism $\pi_2^{\mathcal{G}}|_x$.*

Remarks 7.10.

This was also stated in [6, §5.7, second argument in Remark 5.34, page 145] for an even more general type of principal bundle, but without proof.

Proof.

As in the previous statements about pullback structures, this is a rather trivial and canonical construction. We have a right $f^*\mathcal{G}$ -action on $f^*\mathcal{P}$ defined by

$$(x, p) \cdot (x, g) := (x, p \cdot g)$$

for all $(x, p) \in f^*\mathcal{P}$ and $(x, g) \in f^*\mathcal{G}$, that is, $x \in N$, $p \in \mathcal{P}_{f(x)}$ and $g \in \mathcal{G}_{f(x)}$. This is clearly an action $f^*\mathcal{P} * f^*\mathcal{G} := \pi_1^* f^*\mathcal{G} \rightarrow f^*\mathcal{P}$, where π_1 is the canonical projection of $f^*\mathcal{P} \rightarrow N$ as

fibre bundle. This action's restriction onto the fibres is simply transitive: A fibre of $f^*\mathcal{P}$ at x is $\{x\} \times \mathcal{P}_{f(x)} \cong \mathcal{P}_{f(x)}$, similarly $(f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)}$. Hence, the restriction of that action at $x \in N$ is the right $\mathcal{G}_{f(x)}$ -action on $\mathcal{P}_{f(x)}$ such that the action is simply transitive.

A principal bundle atlas can be constructed by a pullback of principal bundle charts of \mathcal{P} , that is, let $(U_i)_i$ be an open covering of M over which we have \mathcal{G} -equivariant diffeomorphisms $\varphi_i : \mathcal{P}|_{U_i} \rightarrow \mathcal{G}|_{U_i}$. Then define

$$f^*\mathcal{P}|_{f^{-1}(U_i)} \rightarrow f^*\mathcal{G}|_{f^{-1}(U_i)},$$

$$(x, p) \mapsto (f^*\varphi_i)(x, p) := \left(x, (\varphi_i)|_{f(x)}(p) \right),$$

which is well-defined and by construction a base-preserving $f^*\mathcal{G}$ -equivariant smooth map, and this map is equivalent to $(\mathbf{1}_{f^{-1}(U_i)}, \varphi_i) : N \times \mathcal{P} \rightarrow N \times \mathcal{G}$ restricted onto $f^*\mathcal{P}$ as an embedded submanifold of $N \times \mathcal{P}$. Therefore it is clearly a diffeomorphism, so that we can conclude that $f^*\mathcal{P}$ admits the structure as a principal $f^*\mathcal{G}$ -bundle.

The last part of the proof about the uniqueness of the structure is precisely as in the proof of Cor. 4.1, just replace the property of being an LGB morphism with being a principal bundle morphism w.r.t. the LGB morphism $\pi_2^{\mathcal{G}}$ (essentially, replace homomorphism with equivariance). Keeping the same notation as in the proof of Cor. 4.1, we get analogously

$$\varphi_i \circ \pi_2 \circ \psi_i^{-1} = \pi_2^{\mathcal{G}} \circ f^*\varphi_i \circ \psi_i^{-1}.$$

Then start by making use of the point-wise behaviour of π_2 and $\pi_2^{\mathcal{G}}$ and proceed similarly as in the proof of Cor. 4.1 to conclude the proof. \blacksquare

Definition 7.11: Pullback principal bundle

Let M, N be smooth manifolds, $f : N \rightarrow M$ a smooth map, $\mathcal{G} \rightarrow M$ an LGB, and $\mathcal{P} \rightarrow M$ a principal \mathcal{G} -bundle. Then we call the principal $f^*\mathcal{G}$ -bundle structure on $f^*\mathcal{P}$ as given in Cor. 7.9 the **pullback principal bundle of \mathcal{P} (under f)**.

We will refer to this structure often without further mention.

We can make a pullback with the projection of the principal bundle $\mathcal{P} \xrightarrow{\pi} M$ itself to show that $\mathcal{P} * \mathcal{G}$ is not only $\pi^*\mathcal{G}$ by definition but also $\pi^*\mathcal{P}$. For this recall Ex. 7.4, i.e. LGBs are "trivially" also principal bundles, hence we know that $\mathcal{P} * \mathcal{G}$ is a principal $\pi^*\mathcal{G}$ -bundle.

Corollary 7.12: $\mathcal{P} * \mathcal{G}$ is the pullback of \mathcal{P} along its projection

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle. Then we have a base-preserving principal bundle isomorphism

$$\mathcal{P} * \mathcal{G} \cong \pi^*\mathcal{P}$$

w.r.t. the LGB isomorphism given as the identity map on $\mathcal{P} * \mathcal{G}$.

Proof.

We have a global section of the pullback principal bundle $\pi^*\mathcal{P} \rightarrow \mathcal{P}$ given by

$$\mathcal{P} \rightarrow \pi^*\mathcal{P},$$

$$p \mapsto (p, p).$$

This is clearly a well-defined global section of $\pi^*\mathcal{P}$, so that by Lemma 7.7 we achieve the desired isomorphism $\pi^*\mathcal{P} \cong \pi^*\mathcal{G} = \mathcal{P} * \mathcal{G}$. ■

Remarks 7.13.

By Lemma 7.7 this isomorphism is explicitly given by

$$\mathcal{P} * \mathcal{G} \rightarrow \pi^*\mathcal{P},$$

$$(p, g) \mapsto (p, p \cdot g).$$

In fact, the cited reference of Def. 7.1, [6, simplification of the beginning of §5.7, page 144f.], takes the existence of such a diffeomorphism as the essential sole part of defining principal bundles. Such an approach essentially avoids the second part of Def. 7.1.

Remark 7.14: Why "principal"?

The last result and remark also outline why we are speaking of a *principal* bundle; this is similar to [1, §4.2.2, page 212ff.]. As in [1, §3.7, Def. 3.7.24, page 159] one could say that a **principal** \mathcal{G} -action on a manifold N (recall Def. 4.4) is a **free** action, *i.e.* orbit maps are injective, such that

$$N * \mathcal{G} \rightarrow N \times N,$$

$$(p, g) \mapsto (p, p \cdot g)$$

is a closed map. In our case, $N = \mathcal{P}$, we clearly have a free action, and we just have shown that that map is closed, because we have the composition of maps

$$\mathcal{P} * \mathcal{G} \rightarrow \pi^*\mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P},$$

$$(p, g) \mapsto (p, p \cdot g) \mapsto (p, p \cdot g).$$

By Remark 7.13 the first arrow is a diffeomorphism and thence a closed map. The second arrow is the inclusion, an embedding because $\pi^*\mathcal{P}$ is a closed embedded submanifold of $\mathcal{P} \times \mathcal{P}$ as the restriction of the fibre bundle $\mathcal{P} \times \mathcal{P} \xrightarrow{1_{\mathcal{P}} \times \pi} \mathcal{P} \times M$ along the graph of π ; see *e.g.* [1, §4.1, proof of Thm. 4.1.17, page 204ff.]. Additionally, by the continuity of π , the graph Γ of π is closed and thus $\pi^*\mathcal{P} = (1_{\mathcal{P}} \times \pi)^{-1}(\Gamma)$, too. Using this, it is a quick exercise to show that a set closed in $\pi^*\mathcal{P}$ is also closed in $\mathcal{P} \times \mathcal{P}$, thus the second arrow as inclusion is also closed. Hence, the whole composition is closed.

This knowledge should allow us to carry over a lot of similar results related to principal actions, as in [1, §3.7.5, page 159ff.] and [1, §4.2.2, page 212ff.]. Essentially Remark 7.13 allows us to look at the quotient \mathcal{P}/\mathcal{G} by using Godement's Theorem as given in [1, §3.7, Thm. 3.7.10, page 155], leading to a manifold structure on \mathcal{P}/\mathcal{G} and maybe similarly leading to a principal bundle structure on $\mathcal{P} \rightarrow \mathcal{P}/\mathcal{G}$.

Let us conclude our discussion about principal bundles by introducing a typical label.

Definition 7.15: Gauges of a principal bundle

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and $\mathcal{P} \rightarrow M$ a principal \mathcal{G} -bundle. A **gauge** of \mathcal{P} is a section of \mathcal{P} . If the section is globally defined on M , then we speak of a **global gauge**, otherwise we may just say **local gauge** or just gauge.

By Lemma 7.7, a gauge corresponds to a " \mathcal{G} -ization" of \mathcal{P} , not necessarily a trivialization.

7.2. Generalized distributions and connections

Remark 7.16: References for connections on principal LGB-bundles

It is important to mention beforehand that there is another great paper, [7], which also discusses the notion of connections on principal LGB-bundles. When I wrote this subsection about the connection, I was not aware of this preprint so that there will be some shared results which I found independently of the other authors. This will be obvious due to the fact that our approach is different. Both, this and the other paper, lead to the same sense of connection (based on the same idea of the behaviour of the parallel transport), but are explicitly not directly the same formulation. More like equivalent definitions of the same object, two sides of the same coin, so that it should be worth it to read both research works. [7] sole purpose was in defining connections, while we want to discuss this sense of connection in the context of Yang-Mills gauge theory, also pinpointing what happens for Yang-Mills-Higgs gauge theories. Hence, our approach will be different, we will use sections of the LGB to formulate connections, simplifying certain explicit formulas; while [7] avoids using sections and rather uses formulations just looking on the vertical parts of a vector or by rephrasing the notions via the total derivative of the LGB action. Another example of difference is that we are also introducing a generalisation of the Darboux derivative and phrasing important formulas with this derivative. Furthermore, we are not going to fix a specific connection on \mathcal{G} for the principal bundle's connection (other than a horizontal distribution), while [7] did fix a more specific type of connection on \mathcal{G} , and our curvature will be different and more general as already pointed out in research like [8] and [5].

Finally, let us now define the gauge theory, starting with horizontal distributions. For read-

ability, we start with a very easy toy model which should help in understanding the general definitions. We expect a basic understanding of horizontal distributions and their relationship to what we call connections (on principal and vector bundles). The bare-bones start with horizontal distributions.

Definition 7.17: Horizontal distribution,

[1, §5.1.2, Def. 5.1.6, page 260; without the symmetry along right-translations here]

Let $F \rightarrow M$ be a fibre bundle over a smooth manifold M . Then a **horizontal distribution/bundle of F** is a smooth subbundle HF of TF with

$$TF = HF \oplus VF.$$

For $p \in M$ the fibre is denoted by $H_p F$, which we may call a **horizontal tangent space**.

As usual for gauge theory we will understand connections as horizontal distributions with a certain symmetry along the fibres in order to assure a certain behaviour of the gauge transformation of what physicists call minimal coupling; in mathematical words, in order to assure to be able to define a connection on associated vector bundles. To do so this symmetry has to be similar to the symmetry carried in the vertical structure; recall Def. 6.17 and its remark 6.18. The following is a straightforward generalization of what one knows in the typical formulation of gauge theory, see *e.g.* [1, §5.1, part 2 to 4 of Prop. 5.1.3, page 258f].

Corollary 7.18: The natural invariance of the vertical bundle of \mathcal{P}

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , \mathcal{g} its LAB, and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle. Then

$$D_{pr_g}(V_p \mathcal{P}) = V_{p \cdot g} \mathcal{P} \quad (37)$$

for all $(p, g) \in \mathcal{P} * \mathcal{G}$, and we have an isomorphism of vector bundles

$$V\mathcal{P} \cong \pi^* \mathcal{g} \quad (38)$$

given by

$$\begin{aligned} \pi^* \mathcal{g} &\rightarrow V\mathcal{P}, \\ (p, \nu) &\mapsto \tilde{\nu}_p. \end{aligned} \quad (39)$$

Proof.

Recall Rem. 7.2 for this proof. By definition it is clear that each fibre \mathcal{P}_x ($x \in M$) is a principal \mathcal{G}_x -bundle over $\{x\}$ whose \mathcal{G}_x -action is the \mathcal{G} -action restricted to x , and thus we know

$$D_{pr_g}(V_p \mathcal{P}_x) = V_{p \cdot g} \mathcal{P}_x$$

for all $p \in \mathcal{P}_x$ and $g \in \mathcal{G}_x$; see *e.g.* [1, §5.1, fourth part of Prop. 5.1.3, page 258f.] for such statements about principal Lie group bundles. Due to that \mathcal{P}_x is a bundle over a point, we have $V\mathcal{P}_x = T\mathcal{P}_x = V\mathcal{P}|_{\mathcal{P}_x}$. Thus,

$$D_p r_g(V_p \mathcal{P}) = V_{p \cdot g} \mathcal{P}.$$

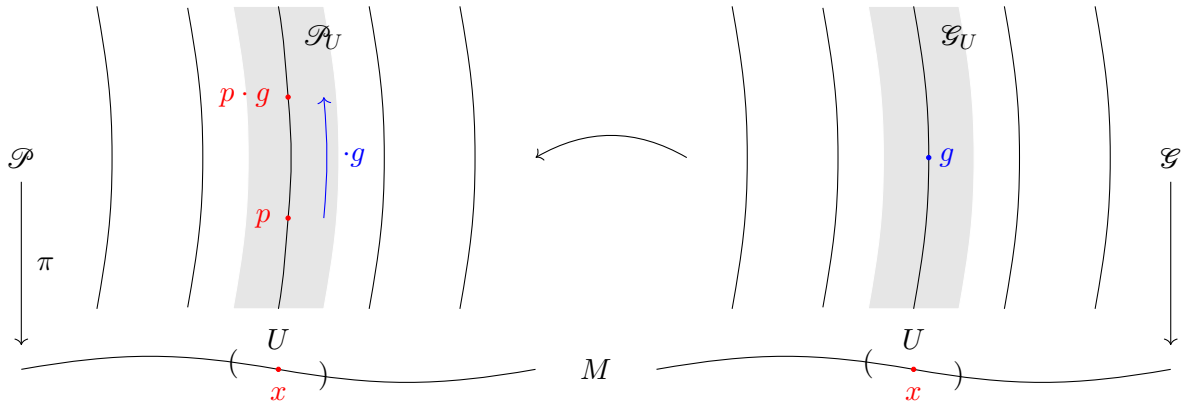
Due to the fact that the \mathcal{G} -action is simply transitive we can derive that its induced \mathcal{G} -action ρ is a vector bundle isomorphism: By Rem. 6.19 this LAB action ρ is precisely (39) and ρ has values in $V\mathcal{P}$ by Eq. (23). We know that the orbit maps $\Phi_p : \mathcal{G}_x \rightarrow \mathcal{P}_x$ through $p \in \mathcal{P}_x$ are \mathcal{G}_x -equivariant diffeomorphisms, so that $D_{e_x} \Phi_p : \mathcal{G}_x \rightarrow V_p \mathcal{P}$ is a vector space isomorphism. But we also have

$$\rho(p, \nu) = D_{e_x} \Phi_p(\nu)$$

for all $\nu \in \mathcal{G}_x$, and therefore ρ is fibre-wise an isomorphism of vector spaces such that it is an isomorphism of vector bundles. \blacksquare

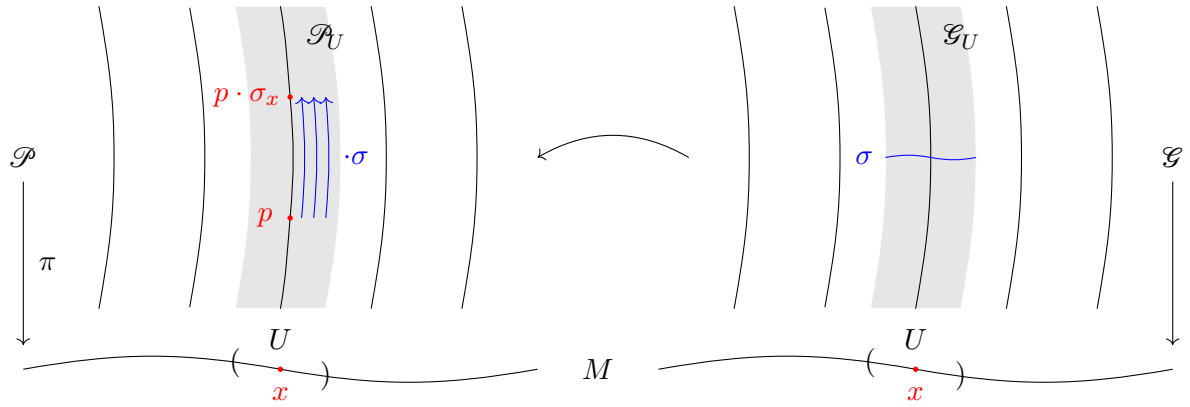
By this result, we would like to have Eq. (37) also for a chosen horizontal distribution. However, its formulation leads to certain problems which we now want to discuss.

7.2.1. Idea and motivation



For $\mathcal{G} \rightarrow M$ an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle we fix a point $p \in \mathcal{P}_x$ ($x \in M$) and can multiply that with an element $g \in \mathcal{G}_x$. Infinitesimally, we are interested into how this multiplication by g affects tangent vectors, especially non-vertical ones.

However, as we have seen in Def. 4.8, Rem. 5.18 and Thm. 6.21 (and its proof) the push-forward of horizontal vectors is not well-defined anymore on non-vertical vectors if one uses a fixed element of an LGB; r_g is just a map $\mathcal{P}_x \rightarrow \mathcal{P}_x$. In order to study push-forwards of non-vertical vectors, we need information of the \mathcal{G} -action in an open neighbourhood U around the fibres over x . Hence, we want to use an section $\sigma \in \Gamma(\mathcal{G}|_U)$ with $\sigma_x = g$ instead.



But there are in general a plethora of sections with $\sigma_x = g$, thenceforth one expects that a definition of connections based on that may depend on the choice of sections and thus leading to conflicts once one looks at push-forwards with all possible $g \in \mathcal{G}$. Especially the tangential behaviour of the section's image (as an embedding of the base) may contribute to the push-forward; given a horizontal distribution, a horizontal vector may be still horizontal after a push-forward with one LGB section but not with respect to another LGB section. A similar problem may arise if we would work with local trivializations instead in order to use Ex. 4.14. Thus, we need to adjust the typical definition of connections on principal bundles.

In order to understand what has to be changed, let us revisit the "typical" situation, that is, let \mathcal{P} be a pre-classical principal bundle as in Ex. 7.3, i.e. $\mathcal{G} = M \times G$ is trivial with Lie group G . That is, the \mathcal{G} -action is equivalent to a G -action on \mathcal{P} by Ex. 4.14, a push-forward with $g \in G$ w.r.t. the latter action is equivalent to the push-forward with a constant section in \mathcal{G} for which we may still simply write g .

Equip \mathcal{P} with a "typical" (Ehresmann) connection as in [1, §5.1, Def. 5.1.6, page 260], that is, a horizontal distribution $H\mathcal{P}$ of \mathcal{P} with

$$D_p r_g(H\mathcal{P}_p) = H\mathcal{P}_{p \cdot g} \quad (40)$$

for all $p \in \mathcal{P}$ and $g \in G$. Recall that a connection has a 1:1 correspondence to a parallel transport, as presented in [1, §5.8, page 286ff.]. This means corresponding to a piece-wise smooth base curve $\alpha : [0, t] \rightarrow M$ ($t > 0$) with $\alpha(0) = x$ we have a **parallel transport along** α as a map $PT_\alpha^\mathcal{P} : \mathcal{P}_x \rightarrow \Gamma(\alpha^*\mathcal{P})$ satisfying

$$PT_{\alpha * \alpha'}^\mathcal{P} \Big|_t = PT_{\alpha'}^\mathcal{P} \Big|_t \circ PT_\alpha^\mathcal{P} \Big|_t, \quad (41)$$

$$PT_{\alpha^-}^\mathcal{P} \Big|_t = \left(PT_\alpha^\mathcal{P} \right)^{-1} \Big|_t, \quad (42)$$

especially parallel transport is a diffeomorphism between the fibres, where α' is just another similarly defined base curve with $\alpha'(0) = \alpha(t)$, $\alpha * \alpha'$ their concatenation (α coming first and with a suitable parametrization such that $\alpha * \alpha' : [0, t] \rightarrow M$), and α^- denotes α traversed backwards. Moreover, Eq. (40) integrates to

$$PT_\alpha^\mathcal{P}(p \cdot g) \Big|_t = PT_\alpha^\mathcal{P}(p) \Big|_t \cdot g. \quad (43)$$

Thinking of the associated \mathcal{G} -action, g is an element of \mathcal{G}_x such that the right hand side is in general not well-defined⁹ anymore due to the fact that $\text{PT}_\alpha^\mathcal{P}(p)|_t \in \mathcal{P}_{\alpha(t)}$; it is well-defined if interpreting g as a constant section of $\mathcal{G} = M \times G$, denoted by $\tilde{g} \in \Gamma(\mathcal{G})$ for bookkeeping reasons. The left hand side uses $\tilde{g}_{\alpha(0)} = (x, g)$, the right hand side $\tilde{g}_{\alpha(t)} = (\alpha(t), g)$, and by Ex. 4.14 we rewrite the G -action to a \mathcal{G} -action:

$$p \cdot g = p \cdot (x, g) = p \cdot \tilde{g}_x, \quad \text{PT}_\alpha^\mathcal{P}(p) \cdot g = \text{PT}_\alpha^\mathcal{P}(p) \cdot (\alpha, g) = \text{PT}_\alpha^\mathcal{P}(p) \cdot \alpha^* \tilde{g},$$

where we recall that $\alpha^* \mathcal{P}$ is a principal $\alpha^* \mathcal{G}$ -bundle in sense of Def. 7.11.

Now we equip $\mathcal{G} \xrightarrow{\pi_\mathcal{G}} M$ with its **canonical flat connection** $\text{H}\mathcal{G} := \pi_\mathcal{G}^* \text{TM}$ which induces a parallel transport $\text{PT}_\alpha^\mathcal{G} : \mathcal{G}_x \rightarrow \Gamma(\alpha^* \mathcal{G})$ with $\text{PT}_\alpha^\mathcal{G}|_t(x, g) = (\alpha(t), g)$, especially $\text{PT}_\alpha^\mathcal{G}(\tilde{g}_x) = \alpha^* \tilde{g}$. In total, we can rewrite Eq. (43) to

$$\text{PT}_\alpha^\mathcal{P}(p \cdot \tilde{g}_x) = \text{PT}_\alpha^\mathcal{P}(p) \cdot \alpha^* \tilde{g} = \text{PT}_\alpha^\mathcal{P}(p) \cdot \text{PT}_\alpha^\mathcal{G}(\tilde{g}_x).$$

This opens a gateway to define Eq. (43) on general principal \mathcal{G} -bundles. That is, now let \mathcal{P} be again a general principal \mathcal{G} -bundle. Fix any horizontal distribution $\text{H}\mathcal{G}$ on \mathcal{G} inducing a parallel transport $\text{PT}_\alpha^\mathcal{G}$, then a connection on \mathcal{P} should be equivalent to a parallel transport $\text{PT}_\alpha^\mathcal{P}$ on \mathcal{P} satisfying Eq. (41), (42) and

$$\text{PT}_\alpha^\mathcal{P}(p \cdot g) = \text{PT}_\alpha^\mathcal{P}(p) \cdot \text{PT}_\alpha^\mathcal{G}(g) \quad (44)$$

for all $p \in \mathcal{P}_x$ and $g \in \mathcal{G}_x$. The right hand side is now well-defined since both, $\text{PT}_\alpha^\mathcal{P}(p)$ and $\text{PT}_\alpha^\mathcal{G}(g)$, are elements of $\Gamma(\alpha^* \mathcal{P})$ and $\Gamma(\alpha^* \mathcal{G})$, respectively. We now want to derive its infinitesimal analogue similar to Eq. (40). Recall that we can view the parallel transports like $\text{PT}_\alpha^\mathcal{P}(p)$ also as a map $[0, t] \rightarrow \mathcal{P}$ with $\pi \circ \text{PT}_\alpha^\mathcal{P}(p) = \alpha$ (recall Subsection 1.1; alternatively use Lemma 6.20 and project onto the second component for the following derivatives). Then by construction

$$Y := \left. \frac{d}{dt} \right|_{t=0} \text{PT}_\alpha^\mathcal{G}(g) \in \text{H}_g \mathcal{G}$$

for all $g \in \mathcal{G}$, that is, it is horizontal in \mathcal{G} ; similarly for the parallel transport on \mathcal{P} ,

$$X := \left. \frac{d}{dt} \right|_{t=0} \text{PT}_\alpha^\mathcal{P}(p) \in \text{H}_p \mathcal{P}.$$

We want to use Thm. 6.21 now in order to understand what Eq. (44) implies about the corresponding horizontal distribution of \mathcal{P} . For this we differentiate both sides of Eq. (44) with $d/dt|_{t=0}$; the left hand side implies that the right hand side is an element of $\text{H}_{p \cdot g} \mathcal{P}$, while the right hand side gives

$$\left. \frac{d}{dt} \right|_{t=0} \left(\text{PT}_\alpha^\mathcal{P}(p) \cdot \text{PT}_\alpha^\mathcal{G}(g) \right) = D_{(p, g)} \Phi(X, Y)$$

⁹Also here one could work with trivializations, especially since $\alpha^* \mathcal{P}$ is trivial as a fibre bundle due to the fact that the image of α is contractible. As before, this would just lead to other problems on a global scale, and we aim to provide a definition of connections on \mathcal{P} without making use of trivializations.

$$= D_p r_\sigma(X) + \overbrace{(\mu_{\mathcal{G}})_g(Y - D_x \sigma(\dot{\alpha}(0)))} \Big|_{p.g}$$

where $\Phi : \mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$ denotes the \mathcal{G} -action on \mathcal{P} , σ is any (local) section of \mathcal{G} with $\sigma_x = g$ and

$$\dot{\alpha}(0) = \frac{d}{dt} \Big|_{t=0} \alpha = D_p \pi(X)$$

because of $\pi \circ \text{PT}_\alpha^{\mathcal{P}}(p) = \alpha$. As we already have shown several times, for example recall the beginning of the proof of Thm. 6.21, we have

$$Y - D_x \sigma(\dot{\alpha}(0)) \in V_g \mathcal{G},$$

so that the canonical projection $\pi^{\text{vert}, \mathcal{G}} : T\mathcal{G} \rightarrow V\mathcal{G}$ onto the vertical structure of \mathcal{G} acts as identity on it. By making use of the horizontality of Y , we can write

$$\begin{aligned} (\mu_{\mathcal{G}})_g(Y - D_x \sigma(\dot{\alpha}(0))) &= (\mu_{\mathcal{G}} \circ \pi^{\text{vert}, \mathcal{G}})_g(Y - D_p(\sigma \circ \pi)(X)) \\ &= -(\mu_{\mathcal{G}} \circ \pi^{\text{vert}, \mathcal{G}})_g(D_p(\sigma \circ \pi)(X)). \end{aligned}$$

It may not surprise that $\mu_{\mathcal{G}} \circ \pi^{\text{vert}, \mathcal{G}}$ is by construction actually *the* connection 1-form on \mathcal{G} corresponding to $H\mathcal{G}$, therefore we will denote $\mu_{\mathcal{G}} \circ \pi^{\text{vert}, \mathcal{G}}$ as the **total Maurer-Cartan form** $\mu_{\mathcal{G}}^{\text{tot}}$ of \mathcal{G} . Hence we get in total (see also Figure 1)

$$D_p r_\sigma(X) - \overbrace{(\mu_{\mathcal{G}}^{\text{tot}})_g(D_p(\sigma \circ \pi)(X))} \Big|_{p.g} \in H_{p.g} \mathcal{P} \quad (45)$$

for all $X \in H_p \mathcal{P}$, and we would like. We are going to prove that this is independent of the chosen section σ , and thence this gives the fundamental formula for the following sections. Last but not least, our starting point was Eq. (37); there is no contradiction between the approaches of Eq. (37) and (45). If X is a vertical vector, then

$$D_p \pi(X) = 0,$$

and

$$D_p r_\sigma(X) = D_p r_g(X)$$

by Remark 5.18. Thence, the symmetry behind Eq. (45) will be compatible with the one of Eq. (37) on the vertical bundle. Let us first study the new arising term in Eq. (45) via the *Darboux derivative*.

7.2.2. Darboux derivative on LGBs

As we have seen, we need a slight adjustment of the vertical Maurer-Cartan as presented in Def. 5.23.

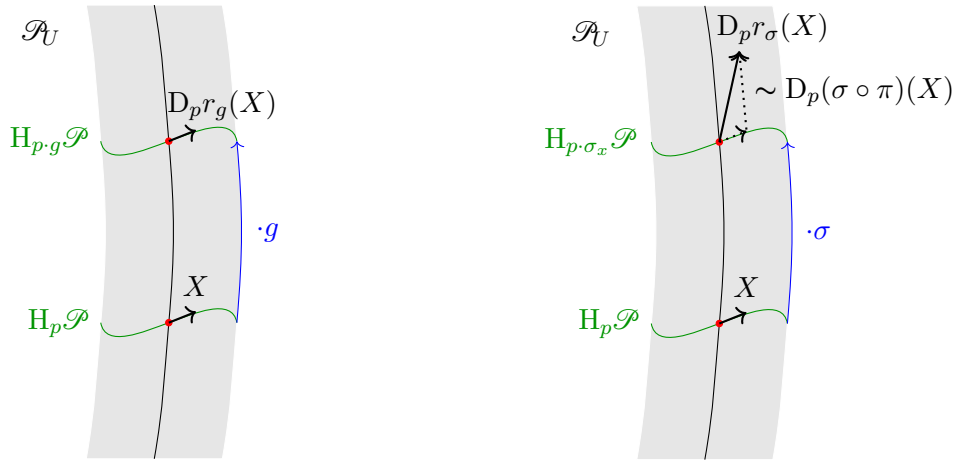


Figure 1: Push-forward of a horizontal tangent vector X with constant section (left) and general section (right), where \mathcal{P} is a pre-classical principal bundle as in Ex. 7.3 equipped with a "typical" connection $H\mathcal{P}$ of principal G -bundles (G the structural Lie group).

Definition 7.19: Total Maurer-Cartan form

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and $H\mathcal{G}$ be a horizontal distribution of \mathcal{G} , where we denote with $\pi^{\text{vert}} : T\mathcal{G} \rightarrow V\mathcal{G}$ the corresponding projection onto its vertical bundle. Then we define the **total Maurer-Cartan form** $\mu_{\mathcal{G}}^{\text{tot}} \in \Omega^1(\mathcal{G}; \pi^* \mathfrak{g})$ of \mathcal{G} as the connection 1-form corresponding to $H\mathcal{G}$, i.e.

$$(\mu_{\mathcal{G}}^{\text{tot}})_g(Y) := (\mu_{\mathcal{G}} \circ \pi^{\text{vert}})|_g(Y) = (D_g L_{g^{-1}})(\pi^{\text{vert}}(Y))$$

for all $g \in G$ and $Y \in T_g \mathcal{G}$.

This is clearly well-defined by construction; also recall our discussion of the vertical Maurer-Cartan form, especially Cor. 5.22.

Remark 7.20: Total Maurer-Cartan form just typical form on trivial LGBs

Let G be a Lie group with Lie algebra \mathfrak{g} , and $\mathcal{G} := M \times G \xrightarrow{\text{pr}_1} M$ be the trivial LGB equipped with its canonical flat connection $H\mathcal{G} := \text{pr}_1^* TM$, where pr_1 is the projection onto the first component in $M \times G$. Its LAB \mathfrak{g} is also trivial $M \times \mathfrak{g}$, and we have several identities (recall Cor. 5.25)

$$T\mathcal{G} \cong \text{pr}_1^* TM \oplus \text{pr}_2^* TG \cong \text{pr}_1^* TM \oplus \mathfrak{g} \cong \text{pr}_1^* TM \oplus \text{pr}_1^* \mathfrak{g} = H\mathcal{G} \oplus V\mathcal{G},$$

where $\text{pr}_2 : M \times G \rightarrow G$ is the projection onto the second component, and π^{vert} is then equivalent to $D\text{pr}_2 \in \Omega^1(\mathcal{G}; \text{pr}_2^* TG)$. We also have the typical projection properties

$$D\text{pr}_1 \circ D\text{pr}_2 = 0, \quad D\text{pr}_2 \circ D\text{pr}_2 = D\text{pr}_2,$$

and for $(x, g) \in M \times G$

$$L_{(x, g)}((x, q)) = (x, L_g(q))$$

for all $(x, g) \in M \times G$, where the left-translation on the right hand side is the one in G . Thence,

$$L_{(x,g)} = (\text{pr}_1, L_{\text{pr}_2(x,g)} \circ \text{pr}_2),$$

and hence

$$\begin{aligned} D_{(x,g)} L_{(x,g)}^{-1} &= (D_{(x,g)} \text{pr}_1, D_{\text{pr}_2(x,g)} L_{\text{pr}_2(x,g)^{-1}} \circ D_{(x,g)} \text{pr}_2) \\ &= (D_{(x,g)} \text{pr}_1, (\mu_G)_{\text{pr}_2(x,g)} \circ D_{(x,g)} \text{pr}_2), \end{aligned}$$

where μ_G is the Maurer-Cartan form of G . Altogether we get

$$\begin{aligned} (\mu_{\mathcal{G}}^{\text{tot}})_{(x,g)} &= D_{(x,g)} L_{(x,g)}^{-1} \circ D_{(x,g)} \text{pr}_2 \\ &= (D_{(x,g)} \text{pr}_1, (\mu_G)_{\text{pr}_2(x,g)} \circ D_{(x,g)} \text{pr}_2) \circ D_{(x,g)} \text{pr}_2 \\ &= (0, (\mu_G)_{\text{pr}_2(x,g)} \circ D_{(x,g)} \text{pr}_2) \\ &= \text{pr}_2^! \mu_G \Big|_{(x,g)}, \end{aligned}$$

where μ_G is the Maurer-Cartan form of G . In general we played around a bit with the isomorphisms provided at the beginning of this remark; it can be *e.g.* useful to think of the value of an element of TG as an element of $\text{pr}_2^* TG$: In the first line we view $D\text{pr}_2$ as a **base-preserving** morphism $T\mathcal{G} \rightarrow \text{pr}_2^* TG \subset T\mathcal{G}$, while later on as a morphism $T\mathcal{G} \rightarrow TG$ **over the base map** pr_2 . Since all of this is straightforward, such kind of stuff should have been clear by context.

The Maurer-Cartan form is important for gauge transformations because it induces a derivative, which we also need now.

Remark 7.21: Maurer-Cartan form inducing a natural derivative: Part I

If G is a Lie group and \mathfrak{g} its Lie algebra, then there is actually some stance that the typical Maurer-Cartan form $\mu_G \in \Omega^1(G; \mathfrak{g})$ describes *the* generalization of the total derivative of smooth maps $M \rightarrow \mathbb{R}^n$ ($n \in \mathbb{N}$), given by the **Darboux derivative** Δ as given in [2, §5.1, page 182ff.]. For a smooth section $\sigma : M \rightarrow G$ of the trivial LGB over M , this is $\Delta\sigma \in \Omega^1(M; \mathfrak{g})$ given by

$$\Delta\sigma := \sigma^! \mu_G,$$

that is,

$$(\Delta\sigma)_p(X) = D_{\sigma_x} L_{\sigma_x^{-1}} (D_x \sigma(X))$$

for all $p \in M$ and $X \in T_p M$. For $G = \mathbb{R}^n$ one usually shows that $D\sigma$ is the actual total derivative (Jacobian) by making use of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$; however, if one views the triviality of $T\mathbb{R}^n$ as the trivialization of general TG as given by $G \times \mathfrak{g}$, then it is more natural to think of the total derivative as $D\sigma$ followed by $D_{\sigma_x} L_{\sigma_x^{-1}}$ in order to receive information about the essential Lie algebra element. Then the classical total derivative of \mathbb{R}^n -valued maps is actually more naturally given by the Darboux derivative.

We now have a similar behaviour in our case but related to arbitrary connections on \mathbb{R}^n (as a trivial bundle over M).

Definition 7.22: Darboux derivative

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M , and $H\mathcal{G}$ be a horizontal distribution of \mathcal{G} . For $\sigma \in \Gamma(\mathcal{G})$ define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathfrak{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathcal{G}}^{\text{tot}} = (\sigma^* \mu_{\mathcal{G}}^{\text{tot}}) \circ D\sigma.$$

Remarks 7.23.

The notation with the pullback σ^* is only needed if one wants to view $\Delta\sigma$ as a $C^\infty(M)$ -linear map $\mathfrak{X}(M) \rightarrow \Gamma(\mathfrak{g})$; in this case it is a composition of base-preserving vector bundle morphisms $D\sigma : TM \rightarrow \sigma^* T\mathcal{G}$ and $\sigma^* \mu_{\mathcal{G}}^{\text{tot}} : \sigma^* T\mathcal{G} \rightarrow \sigma^* \pi^* \mathfrak{g} \cong \mathfrak{g}$ which can be extended to sections, where π is the projection of \mathcal{G} .

Of course one can view $\Delta\sigma$ as a map $TM \rightarrow \mathfrak{g}$ which one can also write as the composition of $D\sigma : TM \rightarrow T\mathcal{G}$ and $\mu_{\mathcal{G}}^{\text{tot}} : T\mathcal{G} \rightarrow \mathfrak{g}$, that is,

$$\Delta\sigma = \mu_{\mathcal{G}}^{\text{tot}} \circ D\sigma.$$

Remark 7.24: Maurer-Cartan form inducing a natural derivative: Part II

We have now something similar to Remark 7.21. Let $\mathcal{G} = M \times \mathbb{R}^n$ ($n \in \mathbb{N}$) be the trivial abelian LGB, and ∇ a vector bundle connection on $M \times \mathbb{R}^n$, for which we have the associated projection onto the vertical bundle $\pi^{\text{vert}} : T\mathcal{G} \rightarrow V\mathcal{G}$. For $\sigma \in \Gamma(\mathcal{G})$ we then have

$$\nabla_{X_x} \sigma = (\sigma^* \pi^{\text{vert}})(D_x \sigma(X_x)) \quad (46)$$

for all $x \in M$ and $X_x \in T_x M$, by making use of $\sigma^* V\mathcal{G} \cong M \times \mathbb{R}^n$ as vector bundles such that the right hand side has again values in \mathcal{G} due to "enough triviality" and equals the left hand side.^a Again one could use the isomorphism as given in Cor. 5.25. This is also more natural in the sense of that one wants that $\nabla_{X_x} \sigma$ is a section over M ; M can be viewed as the image of the neutral section e (the zero vector here), so that $\nabla_{X_x} \sigma$ should have values in $e^* V\mathcal{G} = \mathfrak{g}$. Henceforth one could say it is more natural to "pull

$(\sigma^* \pi^{\text{vert}})(D_x \sigma(X_x))$ back" to \mathfrak{g} by left-translation in order to define $\nabla_{X_x} \sigma$, *i.e.*

$$\begin{aligned} D_{\sigma_x} L_{\sigma_x^{-1}} \left((\sigma^* \pi^{\text{vert}})(D_x \sigma(X_x)) \right) &= (\sigma^* \mu_{\mathcal{G}} \circ \sigma^* \pi^{\text{vert}})|_x (D_x \sigma(X_x)) \\ &= (\Delta \sigma)|_x (X_x). \end{aligned}$$

Thus, as in Remark 7.21 one may say that $\Delta \sigma$ is *the* generalization of vector bundle connections to general LGBs \mathcal{G} . Furthermore, this argument is not based on a given (local) trivialization to handle the arising pullback in Eq. (46).

^aStrictly spoken, there is also a natural projection $V\mathcal{G} \rightarrow \mathfrak{g}$ involved which we omit for simplicity; recall Cor. 5.25.

Remark 7.25: Canonical flat Darboux derivative

Let G be a Lie group with Lie algebra \mathfrak{g} , and $\mathcal{G} := M \times G \xrightarrow{\text{pr}_1} M$ be the trivial LGB equipped with its canonical flat connection $H\mathcal{G} := \text{pr}_1^* TM$; also recall Remark 7.20 and its notation, especially $\pi^{\text{vert}} = D\text{pr}_2$ with $\text{pr}_2 : M \times G \rightarrow G$ the projection onto the second component.

For a constant section $\sigma \in \Gamma(\mathcal{G})$, *i.e.* $\sigma_x = (x, g)$ for all $x \in M$ ($g \in G$), we have

$$D_x \sigma(X) = (X, 0)$$

for all $x \in M$ and $X \in T_x M$. So that we get $\sigma^* \pi^{\text{vert}} \circ D\sigma = 0$ as expected. Therefore in total

$$\Delta \sigma = (\sigma^* \mu_{\mathcal{G}}^{\text{tot}}) \circ D\sigma = \sigma^* \mu_{\mathcal{G}} \circ \sigma^* \pi^{\text{vert}} \circ D\sigma = 0.$$

In sense of Remark 7.24 it makes sense to say that σ is **parallel w.r.t. Δ** . Thus, we then speak of the **canonical flat Darboux derivative**, and constant sections are its parallel sections.

As mentioned in Subsection 1.1, we can view general sections $\sigma \in \Gamma(\mathcal{G})$ (not necessarily constant) equivalently as a smooth map $\text{pr}_2 \circ \sigma : M \rightarrow G$, denoted by $\tilde{\sigma}$ for bookkeeping reasons. By Remark 7.20 we have

$$\Delta \sigma = (\sigma^* \text{pr}_2^! \mu_G) \circ D\sigma = ((\text{pr}_2 \circ \sigma)^* \mu_G) \circ D(\text{pr}_2 \circ \sigma) = \Delta \tilde{\sigma},$$

where Δ on the right hand side is the "classical" Darboux derivative as in Remark 7.21 related to μ_G . This emphasizes why we can speak of a canonical flat derivative due to the fact that μ_G is flat (Maurer-Cartan equation).

We can actually rewrite some important equations now.

Remark 7.26: Darboux derivative in the infinitesimal LGB action

Recall Thm. 6.21; keeping the same notation as in this theorem but denoting the projection of \mathcal{P} by π , we checked several times that $Y - D_x\sigma(\omega)$ is vertical, and thus we can now write

$$\begin{aligned}
D_{(p,g)}\Phi(X, Y) &= D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}})_g(Y - D_x\sigma(\omega))} \Big|_{p \cdot g} \\
&= D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}} \circ \pi^{\text{vert}})_g(Y - D_x\sigma(\omega))} \Big|_{p \cdot g} \\
&= D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}} \circ \pi^{\text{vert}})_g(Y)} \Big|_{p \cdot g} - \overline{(\mu_{\mathcal{G}} \circ \pi^{\text{vert}})_{\sigma_x}(D_x\sigma(\omega))} \Big|_{p \cdot g} \\
&= D_p r_\sigma(X) + \overline{(\mu_{\mathcal{G}}^{\text{tot}})_g(Y)} \Big|_{p \cdot g} - \overline{(\Delta\sigma)|_x(\omega)} \Big|_{p \cdot g} \\
&= D_p r_\sigma(X) - \overline{(\pi^! \Delta\sigma)|_p(X)} \Big|_{p \cdot g} + \overline{(\mu_{\mathcal{G}}^{\text{tot}})_g(Y)} \Big|_{p \cdot g}
\end{aligned}$$

If \mathcal{G} is a trivial LGB, then this emphasizes again that we recover the typical Leibniz rule by choosing a constant section σ , using Remarks 7.20 and 7.25.

Remark 7.27: Idea behind the notion of connection on principal bundles using the Darboux derivative

Recall the notation and discussion around the terms in Eq. (45) which will be important for our definition of a connection on principal LGB-bundles. The terms in Eq. (45) can be similarly rewritten to

$$D_p r_\sigma(X) - \overline{((\Delta\sigma)_x \circ D_p \pi)(X)} \Big|_{p \cdot g} = D_p r_\sigma(X) - \overline{(\pi^! \Delta\sigma)|_p(X)} \Big|_{p \cdot g}.$$

We will use this form for the definition of the principal bundle connection. As in Remark 7.26, if \mathcal{G} is trivial and σ a constant section corresponding to a Lie group element g , then these terms are just

$$D_p r_g(X),$$

making use of Ex. 4.14.

In fact, the Darboux derivative naturally induces a connection on \mathcal{G} as LAB of \mathcal{G} . One may have expected that since $H\mathcal{G}$ should infinitesimally induce a horizontal distribution on \mathcal{G} ; recall the exponential map introduced in Subsection 5.4. Also recall that by definition of vertical bundles we have for the LAB $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ that we have for the vertical bundle that $V\mathcal{G} \cong \pi_{\mathcal{G}}^* \mathcal{G}$,

making use of that LABs are vector bundles; in the following we will use the natural projection onto the second component of $\pi_{\mathcal{G}}^*$ but now defined on $V\mathcal{G}$, denoted by $\text{pr}_2 : V\mathcal{G} \rightarrow \mathcal{G}$.

Proposition 7.28: LGB connection induces LAB connection

Let $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ be an LGB over a smooth manifold M , and $H\mathcal{G}$ be a horizontal distribution of \mathcal{G} . Then the map $\nabla^{\mathcal{G}} : \Gamma(\mathcal{G}) \rightarrow \Omega^1(M; \mathcal{G})$, $\nu \mapsto \nabla^{\mathcal{G}}\nu$ denoted as an element of $\Omega^1(M; \mathcal{G})$ by $X \mapsto \nabla_X^{\mathcal{G}}\nu$, defined by

$$\nabla_X^{\mathcal{G}}\nu \Big|_x := \text{pr}_2 \left(\frac{d}{dt} \Big|_{t=0} \left((\Delta e^{t\nu})_x(X) \right) \right)$$

for all $x \in M$, $X \in T_x M$ and $\nu \in \Gamma(\mathcal{G})$, is a vector bundle connection on \mathcal{G} , where $t \in \mathbb{R}$, and $\text{pr}_2 : V\mathcal{G} \rightarrow \mathcal{G}$ is the projection onto the second component of $V\mathcal{G}$ naturally viewed as pullback bundle.

In order to prove this we need to apply Schwarz's Theorem in order to switch Δ with d/dt . To do this rigorously we need to introduce the canonical involution/flip on double tangent bundles; since this does not completely fit into this paper's subject and may be already known by the reader, you can learn about the double tangent bundle and its flip map in Appendix A if needed.

Proof of Prop. 7.28.

Let us begin with well-definedness. Similar to Diagram (A.1) we have the double vector bundle

$$\begin{array}{ccc} TT\mathcal{G} & \xrightarrow{D\pi_{T\mathcal{G}}} & T\mathcal{G} \\ \downarrow \pi_{TT\mathcal{G}} & & \downarrow \pi_{T\mathcal{G}} \\ T\mathcal{G} & \xrightarrow{\pi_{T\mathcal{G}}} & \mathcal{G} \end{array} \quad (47)$$

We also have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \left((\Delta e^{t\nu})_x(X) \right) &= \frac{d}{dt} \Big|_{t=0} \left((\mu_{\mathcal{G}}^{\text{tot}} \circ D_x e^{t\nu})(X) \right) \\ &= D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(\frac{d}{dt} \Big|_{t=0} \left(D_x e^{t\nu}(X) \right) \right) \\ &= D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(S_{\mathcal{G}}(D_x \nu(X)) \right) \\ &= D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} (\nu_T(X)) \\ &\in T_{\mu_{\mathcal{G}}^{\text{tot}}(D_x e(X))} \mathcal{G} \end{aligned} \quad (48)$$

for all $x \in M$ and $X \in T_x M$, where e is the neutral section of \mathcal{G} , and we viewed $\mu_{\mathcal{G}}^{\text{tot}}$ as a map $T\mathcal{G} \rightarrow \mathcal{G}$ when applying the chain rule; recall Remark 7.23. $S_{\mathcal{G}}$ is the linear canonical flip map on $TT\mathcal{G}$, especially see the last part of Remark A.2. We also introduced the notation ν_T similar to Remark A.5 for simplicity; in fact ν_T is a vector field on $T\mathcal{G}$ only defined over TM

which is canonically embedded into $T\mathcal{G}$ by the zero section while M is embedded into \mathcal{G} by e . Similarly we naturally embed $T\mathcal{G}$ into $TT\mathcal{G}$. In total, we will work with these embeddings now so that everything is embedded into $TT\mathcal{G}$ as the "total space"; hence, you will see e and its total derivative De acting as an embedding several times.

We get by Eq. (47)

$$\begin{aligned} D\pi_{T\mathcal{G}}\left(\nabla_X^\mathcal{G}\nu\Big|_x\right) &= D_{D_x e(X)}\underbrace{(\pi_{T\mathcal{G}} \circ \mu_\mathcal{G}^{\text{tot}})}_{=e \circ \pi_\mathcal{G} \circ \pi_{T\mathcal{G}}}(\nu_T(X)) \\ &= (D_x e \circ D_{e_x} \pi_\mathcal{G})(\nu_x) \\ &= 0 \in T_{e_x} \mathcal{G} \end{aligned}$$

viewing $S_\mathcal{G}$ as a base-preserving isomorphism from $\pi_{TT\mathcal{G}} : TT\mathcal{G} \rightarrow T\mathcal{G}$ to $D\pi_{T\mathcal{G}} : TT\mathcal{G} \rightarrow T\mathcal{G}$, and using that $\mathcal{G} = e^*V\mathcal{G}$. The projection of $\mathcal{G} \rightarrow M$ is canonically the restriction of $\pi_{T\mathcal{G}}$, hence we can conclude that

$$\frac{d}{dt}\Big|_{t=0} \left((\Delta e^{t\nu})_x(X) \right) \in V_{\mu_\mathcal{G}^{\text{tot}}(D_x e(X))} \mathcal{G},$$

and so we can derive that $\text{pr}_2 : V\mathcal{G} \rightarrow \mathcal{G}$ is defined on this; in total $\nabla^\mathcal{G}\nu$ is well-defined. By Eq. (48) we also trivially know that

$$\pi_{TT\mathcal{G}}\left(\frac{d}{dt}\Big|_{t=0} \left((\Delta e^{t\nu})_x(X) \right)\right) = \mu_\mathcal{G}^{\text{tot}}(D_x e(X)) \in \mathcal{G}_x.$$

Viewing $V\mathcal{G}$ naturally as the pullback of \mathcal{G} along its projection we can therefore write

$$D_{D_x e(X)} \mu_\mathcal{G}^{\text{tot}}(\nu_T(X)) \stackrel{\text{Eq. (48)}}{=} \frac{d}{dt}\Big|_{t=0} \left((\Delta e^{t\nu})_x(X) \right) = \left(\mu_\mathcal{G}^{\text{tot}}(D_x e(X)), \nabla_X^\mathcal{G}\nu\Big|_x \right),$$

so that smoothness of $\nabla^\mathcal{G}\nu$ follows. In order to understand whether $\nabla^\mathcal{G}$ is a vector bundle connection we are hence interested into the restriction of Diagram (47) onto

$$\begin{array}{ccc} V\mathcal{G} & \xrightarrow{D\pi_{T\mathcal{G}}} & \widetilde{M} \\ \downarrow \pi_{TT\mathcal{G}} & & \downarrow \cong \\ \mathcal{G} & \xrightarrow{\pi_{T\mathcal{G}}} & M \end{array}$$

where M is canonically embedded into \mathcal{G} by e , and \widetilde{M} is the further canonical embedding of this (the image of e) into $T\mathcal{G}$ by the zero section. As in Appendix A, the addition of vectors and the scalar multiplication of the left vertical arrow is denoted as usual, while the one of the upper horizontal arrow will be denoted by \oplus and \cdot , respectively. For $Y \in T_x M$ we now have

$$\begin{aligned} \nabla_{\lambda X + \kappa Y}^\mathcal{G}\nu\Big|_x &= \text{pr}_2\left(D_{D_x e(\lambda X + \kappa Y)} \mu_\mathcal{G}^{\text{tot}}(\nu_T(\lambda X + \kappa Y))\right) \\ &= \text{pr}_2\left(D_{D_x e(\lambda X + \kappa Y)} \mu_\mathcal{G}^{\text{tot}}(\lambda \cdot \nu_T(X) \oplus \kappa \cdot \nu_T(Y))\right) \end{aligned}$$

$$\begin{aligned}
&= \text{pr}_2 \left(\lambda \cdot D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}(\nu_T(X)) \blacktriangleleft \kappa \cdot D_{D_x e(Y)} \mu_{\mathcal{G}}^{\text{tot}}(\nu_T(Y)) \right) \\
&= \lambda \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}(\nu_T(X)) \right) + \kappa \text{pr}_2 \left(D_{D_x e(Y)} \mu_{\mathcal{G}}^{\text{tot}}(\nu_T(Y)) \right) \\
&= \lambda \nabla_X^{\mathcal{G}} \nu \Big|_x + \kappa \nabla_Y^{\mathcal{G}} \nu \Big|_x
\end{aligned}$$

for all $\lambda, \kappa \in \mathbb{R}$, using Remark A.3, A.4 and A.5. By these remarks we also derive for another section $\mu \in \Gamma(\mathcal{G})$

$$\begin{aligned}
\nabla_X^{\mathcal{G}}(\lambda\nu + \kappa\mu) \Big|_x &= \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(\underbrace{(\lambda\nu + \kappa\mu)_T(X)}_{=\lambda\nu_T + \kappa\mu_T} \right) \right) \\
&= \lambda \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}(\nu_T(X)) \right) + \kappa \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}(\mu_T(X)) \right) \\
&= \lambda \nabla_X^{\mathcal{G}} \nu \Big|_x + \kappa \nabla_X^{\mathcal{G}} \mu \Big|_x,
\end{aligned}$$

and

$$\begin{aligned}
\nabla_X^{\mathcal{G}}(f\nu) \Big|_x &= \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}((f\nu)_T(X)) \right) \\
&= \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(f(x) \nu_T(X) + X(f) \nu^a \frac{\partial}{\partial \xi^a} \Big|_{D_x e(X)} \right) \right) \\
&= f(x) \nabla_X^{\mathcal{G}}(f\nu) \Big|_x + X(f) \text{pr}_2 \left(D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(\nu^a \frac{\partial}{\partial \xi^a} \Big|_{D_x e(X)} \right) \right)
\end{aligned}$$

for all $f \in C^\infty(M)$, where $(\xi^a)_a$ are fibre coordinates of \mathcal{G} . It was well-defined to use the linearity of pr_2 , since we know by what we have shown earlier that both, $D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}((f\nu)_T(X))$ and $D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}(f\nu_T(X)) = f D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}}(\nu_T(X))$, are elements of $V_{\mathcal{G}}$. Thence,

$$D_{D_x e(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(\nu^a \frac{\partial}{\partial \xi^a} \Big|_{D_x e(X)} \right)$$

is vertical, too.

Making use of the aforementioned isomorphism $V_{\mathcal{G}} \cong \pi_{T\mathcal{G}}^* \mathcal{G}$, we can write similar as in Remark A.4

$$\frac{\partial}{\partial \xi^a} \Big|_{D_x e(X)} \cong (D_x e(X), e_a),$$

where $(e_a)_a$ is a local frame of \mathcal{G} dual to ξ^a . Thus,

$$\text{pr}_2 \left(\frac{\partial}{\partial \xi^a} \Big|_{D_x e(X)} \right) = e_a.$$

By definition, $\mu_{\mathcal{G}}^{\text{tot}}$ acts as identity on $\mathcal{G} = e^*V\mathcal{G}$, and so $D\mu_{\mathcal{G}}^{\text{tot}}$ is the identity on $T\mathcal{G}$. Thus, we derive

$$\text{pr}_2 \left(D_{D_{xe}(X)} \mu_{\mathcal{G}}^{\text{tot}} \left(\nu^a \frac{\partial}{\partial \xi^a} \Big|_{D_{xe}(X)} \right) \right) = \text{pr}_2 \left(\nu^a \frac{\partial}{\partial \xi^a} \Big|_{D_{xe}(X)} \right) = \nu^a e_a = \nu,$$

thus, finally,

$$\nabla_X^{\mathcal{G}}(f\nu) \Big|_x = f(x) \nabla_X^{\mathcal{G}}(f\nu) \Big|_x + X(f) \nu.$$

This finishes the proof. ■

Definition 7.29: LGB connection on its LAB

Let $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ be an LGB over a smooth manifold M , and $H\mathcal{G}$ be a horizontal distribution of \mathcal{G} . Then we call the vector bundle connection $\nabla^{\mathcal{G}}$ on \mathcal{G} of Prop. 7.28, shortly denoted by

$$\nabla^{\mathcal{G}} \nu = \frac{d}{dt} \Big|_{t=0} \Delta e^{t\nu}$$

for all $\nu \in \Gamma(\mathcal{G})$, the \mathcal{G} -connection (on its LAB \mathcal{G}).

7.2.3. First step towards associated bundles

In order to provide a concise definition of connection 1-forms on principal bundles, we will need to introduce some canonical form of action which will be obviously related to the action needed for an analogue to the notion of associated bundles related to typical principal bundles. However, we will neither discuss nor introduce a more general notion of associated bundles in this paper; so, there will be just the "first step". The following action can be seen as another canonical action on the pullback of a principal bundle, but coming from \mathcal{G} itself instead of its pullback, if \mathcal{G} acts on the manifold one makes a pullback to.

Proposition 7.30: Canonical LGB action on pullback vector bundles over principal LGB-bundles

Let $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle. Furthermore let N be another smooth manifold and $f : N \rightarrow M$ a smooth map on which \mathcal{G} acts on the left as in Def. 4.4. Then on the pullback manifold $\mathcal{P} \times_M N := f^*\mathcal{P}$, whose pairs of points are now reordered as in

$$\mathcal{P} \times_M N := \{(p, x) \in \mathcal{P} \times N \mid \pi(p) = f(x)\},$$

we have a right \mathcal{G} -action given by

$$(\mathcal{P} \times_M N) * \mathcal{G} \rightarrow \mathcal{P} \times_M N,$$

$$(p, x, g) \mapsto (p, x) \cdot g := (p \cdot g, g^{-1} \cdot x)$$

where $(\mathcal{P} \times_M N) * \mathcal{G}$ is given as pullback of \mathcal{G} w.r.t. the map $\tilde{\pi} : \mathcal{P} \times_M N \rightarrow M$, $(p, x) = \pi(p)$.

Proof.

By Def. 4.4 it is clear that this action is well-defined, due to

$$\pi(p \cdot g) = \pi_{\mathcal{G}}(g) = f(g^{-1} \cdot x)$$

for all $(p, x, g) \in (\mathcal{P} \times_M N) * \mathcal{G}$, and this also implies

$$\tilde{\pi}((p, x) \cdot g) = \pi(p \cdot g) = \pi_{\mathcal{G}}(g),$$

so that Eq. (12) is satisfied. By construction we also have a smooth action, since it is the composition of maps

$$(\mathcal{P} \times_M N) * \mathcal{G} \rightarrow (\mathcal{P} * \mathcal{G}) \times (\mathcal{G} * N) \rightarrow \mathcal{P} \times_M N,$$

$$(p, x, g) \mapsto ((p, g), (g, x)) \mapsto (p \cdot g, g^{-1} \cdot x).$$

The first arrow is clearly an embedding, and the second arrow is just the restriction of the smooth diagonal action,

$$(\mathcal{P} * \mathcal{G}) \times (\mathcal{G} * N) \rightarrow \mathcal{P} \times N,$$

$$((p, g), (q, p)) \mapsto (p \cdot g, q^{-1} \cdot p),$$

onto an embedded submanifold, and for its image make use of that $\mathcal{P} \times_M N$ is an embedded submanifold of $\mathcal{P} \times N$, as we already did several times for pullback bundles.

Associativity follows simply by

$$((p, x) \cdot g) \cdot q = (p \cdot g, g^{-1} \cdot x) \cdot q = (p \cdot gq, (gq)^{-1} \cdot x) = (p, x) \cdot (gq)$$

for all $g, q \in \mathcal{G}_x$ ($x := \pi(p) = f(x)$); it is trivial to check that $(p, x) \cdot e_x = (p, x)$. ■

As mentioned in Remark 7.14, constructions of quotients related to the LGB action may be possible here. Thus, using the last proposition, one should be able to construct associated bundles in this more general setting. We mainly need this proposition for the following example.

Example 7.31: Adjoint action on the vertical bundle of \mathcal{P}

Recall the adjoint representation of \mathcal{G} on its LAB $\mathcal{g} \xrightarrow{\pi_{\mathcal{G}}} M$, Ex. 6.7. Using the notation of Prop. 7.30 we have a right \mathcal{G} -action on $\mathcal{P} \times_M \mathcal{g} = \{(p, v) \in \mathcal{P} \times \mathcal{g} \mid \pi(p) = \pi_{\mathcal{g}}(v)\}$

given by

$$(p, v) \cdot g = (p \cdot g, \text{Ad}_{g^{-1}}(v))$$

for all $p \in \mathcal{P}_x$ ($x \in M$), $v \in \mathfrak{g}_x$ and $g \in \mathcal{G}_x$. Observe that $\mathcal{P} \times_M \mathfrak{g} = \pi^* \mathfrak{g}$ which is isomorphic to $V\mathcal{P}$ by Cor. 7.18. We will denote this action shortly by

$$\mathcal{A}d_{g^{-1}}(p, v) := (p, v) \cdot g,$$

the **adjoint representation of \mathcal{G} on $V\mathcal{P}$** ; not to be confused with the adjoint representation of $\pi^* \mathcal{G}$ on $\pi^* \mathfrak{g}$. In fact, it is trivial to check that $\mathcal{A}d : \mathcal{G} \rightarrow \text{Aut}(\pi^* \mathfrak{g})$, $g \mapsto \mathcal{A}d_g$, is a \mathcal{G} -representation on $\mathcal{P} \times_M \mathfrak{g} = \pi^* \mathfrak{g} \cong V\mathcal{P}$ in the sense of Cor. 6.3.

Let us finally define connections on principal bundles.

7.2.4. Generalized connection 1-forms on principal bundles

As also stated in [1, §5.1, Prop. 5.1.5, page 260], for a given horizontal distribution $H\mathcal{P}$ of a principal \mathcal{G} -bundle $\mathcal{P} \xrightarrow{\pi} M$ over a smooth manifold M we have by construction that

$$D_p \pi|_{H_p \mathcal{P}} : H_p \mathcal{P} \rightarrow T_x M$$

is a vector space isomorphism for all $x \in M$ and $p \in \mathcal{P}_x$; similarly for \mathcal{G} itself. Using that, one has some sort of identification between the horizontal tangent spaces; we want to provide another identification in the sense of Eq. (45), also recall Rem. 7.27. Hence, the following proposition and definition are needed.

Proposition 7.32: The right-pushforward modified by the Darboux derivative is a well-defined isomorphism

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle, and we have horizontal a distribution $H\mathcal{G}$ on \mathcal{G} . Furthermore, for $g \in \mathcal{G}_x$ ($x \in M$) define the map

$$T\mathcal{P}|_{\mathcal{P}_x} \rightarrow T\mathcal{P}|_{\mathcal{P}_x},$$

$$X \mapsto \mathfrak{r}_{g*}(X) := D_p r_\sigma(X) - \overline{\left(\pi^! \Delta \sigma \right) \Big|_p (X)} \Big|_{p \cdot g},$$

where $p \in \mathcal{P}_x$, $X \in T_p \mathcal{P}$, and σ is any (local) section of \mathcal{G} with $\sigma_x = g$. Then \mathfrak{r}_{g*} is independent of the choice of the local section σ , and it is a vector bundle automorphism over the right-translation r_g .

Furthermore, if we also have a horizontal distribution $H\mathcal{P}$ on \mathcal{P} , then $H_p \mathcal{P}$ is isomorphic via \mathfrak{r}_{g*} to a complement of $V_{p \cdot g} \mathcal{P}$ in $T_{p \cdot g} \mathcal{P}$ (this complement is not necessarily $H_{p \cdot g} \mathcal{P}$).

Proof.

In the following we have $(p, g) \in \mathcal{P} * \mathcal{G}$, $X \in T_p \mathcal{P}$, and σ is any (local) section of \mathcal{G} with $\sigma_x = g$ ($x := \pi(p)$).

- Let $\pi_{\mathcal{G}}$ be the projection of \mathcal{G} , then as mentioned before Prop. 7.32 there is a unique $Y \in H_g \mathcal{G}$ with

$$D_g \pi_{\mathcal{G}}(Y) = D_p \pi(X) \in T_x M.$$

By Remark 7.26 we can derive

$$D_{(p,g)} \Phi(X, Y) = \mathcal{R}_{g*}(X),$$

where we made use of that Y is horizontal in \mathcal{G} so that $(\mu_{\mathcal{G}}^{\text{tot}})_g(Y) = 0$, and where Φ denotes the right \mathcal{G} -action on \mathcal{P} as a map $\mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$. Therefore the independence of \mathcal{R}_{g*} w.r.t. the choice of σ follows.

- \mathcal{R}_{g*} is clearly linear by construction. Let us now show fibre-wise that \mathcal{R}_{g*} is injective, then it is also bijective by dimensional reasons. By Remark 4.11 we know that r_{σ} is a diffeomorphism, and thus $D_p r_{\sigma} : T_p \mathcal{P} \rightarrow T_{p,g} \mathcal{P}$ is a vector space isomorphism. Restricted onto the vertical subspace $V_p \mathcal{P}$ we have by Remark 5.18

$$D_p r_{\sigma}|_{V_p \mathcal{P}} = D_p r_g : V_p \mathcal{P} \rightarrow V_{p,g} \mathcal{P},$$

which is a vector space isomorphism by Cor. 7.18 and dimensional reasons. We rewrite $D_p r_{\sigma}$ as

$$H_p \mathcal{P} \oplus V_p \mathcal{P} \rightarrow T_{p,g} \mathcal{P},$$

$$(X^H, X^V) \mapsto D_p r_{\sigma}|_{H_p \mathcal{P}}(X^H) + D_p r_g(X^V).$$

By dimensional reasons and due to the bijectivity of $D_p r_{\sigma}$ and $D_p r_g$ the image $\text{Im}(D_p r_{\sigma}|_{H_p \mathcal{P}})$ of $D_p r_{\sigma}|_{H_p \mathcal{P}}$ has to be a complement subspace of $V_{p,g} \mathcal{P}$ in $T_{p,g} \mathcal{P}$; $\text{Im}(D_p r_{\sigma}|_{H_p \mathcal{P}})$ is not necessarily equal to $H_{p,g} \mathcal{P}$ in general but of the same dimension, especially of the same dimension as $H_p \mathcal{P}$. Thus, $D_p r_{\sigma}|_{H_p \mathcal{P}}$ is a vector space isomorphism onto its image which is complementary to $V_{p,g} \mathcal{P}$. Furthermore, observe

$$\left(\pi^! \Delta \sigma \right) \Big|_p (X) = (\Delta \sigma)|_x \left(D_p \pi(X^H + X^V) \right) = (\Delta \sigma)|_x \left(D_p \pi(X^H) \right) = \left(\pi^! \Delta \sigma \right) \Big|_p (X^H),$$

using that the vertical subbundle is given by the kernel of $D\pi$, and where we split $X = X^H + X^V$ for all $X \in T_p \mathcal{P} = H_p \mathcal{P} \oplus V_p \mathcal{P}$.

Hence, we get in total that \mathcal{R}_{g*} at p is equivalent to

$$H_p \mathcal{P} \oplus V_p \mathcal{P} \rightarrow \text{Im}(D_p r_{\sigma}|_{H_p \mathcal{P}}) \oplus V_{p,g} \mathcal{P},$$

$$(X^H, X^V) \mapsto \left(D_p r_{\sigma}|_{H_p \mathcal{P}}(X^H), D_p r_g(X^V) - \overline{\left(\pi^! \Delta \sigma \right) \Big|_p (X^H)} \Big|_{p,g} \right).$$

Let (X^H, X^V) be now in the kernel of \mathcal{r}_{g*} , then $X^H = 0$ by the bijectivity of $D_p r_\sigma|_{H_p \mathcal{P}}$ onto its image. The second component of \mathcal{r}_{g*} is then just $D_p r_g(X^V)$; again by the bijectivity of $D_p r_g : V_p \mathcal{P} \rightarrow V_{p \cdot g} \mathcal{P}$ we also get $X^V = 0$. Thus, \mathcal{r}_{g*} is injective and therefore defines vector space isomorphisms $T_p \mathcal{P} \rightarrow T_{p \cdot g} \mathcal{P}$. It follows that \mathcal{r}_{g*} is an automorphism of $T\mathcal{P}|_{\mathcal{P}_x}$ over r_g .

- It is then clear that $\mathcal{r}_{g*}|_{H_p \mathcal{P}}$, given by

$$H_p \mathcal{P} \rightarrow \text{Im}\left(D_p r_\sigma|_{H_p \mathcal{P}}\right) \oplus V_{p \cdot g} \mathcal{P},$$

$$X^H \mapsto \left(D_p r_\sigma|_{H_p \mathcal{P}}(X^H), -\overline{\left(\pi^! \Delta \sigma\right)\Big|_p(X^H)}\Big|_{p \cdot g} \right),$$

is also an isomorphism onto its image; its image is complementary to $V_{p \cdot g} \mathcal{P}$ because its intersection with $V_{p \cdot g} \mathcal{P}$ would require that

$$D_p r_\sigma|_{H_p \mathcal{P}}(X^H) = 0,$$

which implies that $X^H = 0$ due to the fact that $D_p r_\sigma|_{H_p \mathcal{P}}$ is a vector space isomorphism onto its image, as we discussed before. But then

$$\mathcal{r}_{g*}|_{H_p \mathcal{P}}(0) = 0,$$

which implies that the image of $\mathcal{r}_{g*}|_{H_p \mathcal{P}}$ is a complement of $V_{p \cdot g} \mathcal{P}$ in $T_{p \cdot g} \mathcal{P}$. This finishes the proof. ■

Hence, we formally define:

Definition 7.33: Modified pushforward via right-translation

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle, and we have a horizontal distribution $H\mathcal{G}$ on \mathcal{G} . Then we define the **modified right-pushforward**^a \mathcal{r}_{g*} (**with** $g \in \mathcal{G}_x$, $x \in M$) as the vector bundle isomorphism $T\mathcal{P}|_{\mathcal{P}_x} \rightarrow T\mathcal{P}|_{\mathcal{P}_x}$ over r_g as given in Prop. 7.32 by

$$\mathcal{r}_{g*}(X) := D_p r_\sigma(X) - \overline{\left(\pi^! \Delta \sigma\right)\Big|_p(X)}\Big|_{p \cdot g}$$

for all $p \in \mathcal{P}_x$ and $X \in T_p \mathcal{P}$, where σ is any (local) section of \mathcal{G} with $\sigma_x = g$.

Similarly, for a (local) section σ of \mathcal{G} we define the **modified right-pushforward** $\mathcal{r}_{\sigma*}$ **with** σ as a (local) map $T\mathcal{P} \rightarrow T\mathcal{P}$ by

$$\mathcal{r}_{\sigma*}(X) := r_{\sigma_x*}(X) = D_p r_\sigma(X) - \overline{\left(\pi^! \Delta \sigma\right)\Big|_p(X)}\Big|_{p \cdot \sigma_x}$$

for all $X \in T_p \mathcal{P}$ ($p \in \mathcal{P}_x$).

^aThe font of \mathcal{r} is a calligraphic r.

Remark 7.34: Restriction onto vertical bundle gives typical right-pushforward

It is trivial to check that we have

$$\mathcal{r}_{g*}(X) = D_p r_g(X) =: r_{g*}(X)$$

for all $X \in V_p \mathcal{P}$; we actually have proven this directly after Eq. (45). As also pinpointed in Remark 7.27, if \mathcal{G} is a trivial LGB equipped with its canonical flat connection and σ a constant section, then it is easy to see that $\mathcal{r}_{\sigma*} = \mathcal{r}_{g*} = r_{g*}$, where r_g has to be understood as the right-translation of the canonical Lie group action inherited by \mathcal{G} in the sense of Ex. 4.14.

Remarks 7.35.

Prop. 7.32 trivially extends to $\mathcal{r}_{\sigma*}$, that is $\mathcal{r}_{\sigma*}$ is a (local) automorphism of $T\mathcal{P}$ over r_σ . Thus, we can view $\mathcal{r}_{\sigma*}$ as an element of $\Omega^1(\mathcal{P}; r_\sigma^* T\mathcal{P})$.

Hence, the following definition makes sense, especially if thinking about what we discussed for Eq. (45), also recall Rem. 7.27.

Definition 7.36: Ehresmann connection on principal LGB-bundles

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle, and we have horizontal distributions $H\mathcal{G}$ and $H\mathcal{P}$ on \mathcal{G} and \mathcal{P} , respectively. We call $H\mathcal{P}$ an **Ehresmann connection** or a **connection on \mathcal{P}** if it is **right-invariant (w.r.t. modifier right-pushforward)**, *i.e.*

$$\mathcal{r}_{g*}(H_p \mathcal{P}) = H_{p \cdot g} \mathcal{P}$$

for all $p \in \mathcal{P}_x$ and $g \in \mathcal{G}_x$ ($x := \pi(p)$).

Remarks 7.37.

There is also a definition of connections on such and more general principal bundles in [6, §5.7, paragraph before Prop. 5.38, page 148]. However, this reference provides a different type of definition; translated to our situation, it is based on assuming that \mathcal{G} is defined over another base manifold N . In order to define the LGB action on \mathcal{P} this reference introduces a moment map $\mu : \mathcal{P} \rightarrow N$ so that the action of an element \mathcal{G}_y ($y \in N$) is defined on $\mu^{-1}(\{y\})$, especially the infinitesimal action r_g of a fixed LGB element g acts on tangent vectors of $\mu^{-1}(\{y\})$, not necessarily on the vertical structure of \mathcal{P} . Hence, in order to circumvent the problem we discussed in Subsubsection 7.2.1, the reference's definition of a connection is then based on assuming that the "fibres" $\mu^{-1}(\{y\})$ are complementary to the vertical structure of \mathcal{P} , so these fibres' tangent spaces define a horizontal distribution while the infinitesimal action r_g now acts well-defined on the horizontal structure. Henceforth, the reference does not need to look at using sections σ and their actions.

However, this is not a suitable definition for us, because our moment map is the projection of

\mathcal{P} itself such that r_g acts on the vertical structure. The reference's definition is rather restrictive, while our definition works for all principal \mathcal{G} -bundles by fixing a connection on \mathcal{G} .

Let us discuss several examples.

Example 7.38: Recovering of the classical definition

As we already discussed several times, especially recall Remark 7.34 and Ex. 7.3, we recover the typical definition of a connection on a principal bundle if \mathcal{P} is a pre-classical principal bundle and if the trivial LGB \mathcal{G} is equipped with its canonical flat connection $H\mathcal{G}$. We call such a connection a **pre-classical connection**.

Example 7.39: Associated LGBs

Recall Def. 3.11, and recall that LGBs themselves are principal bundles as in Ex. 7.4. That is, let G, H be Lie groups, $P \rightarrow M$ a principal G -bundle over a smooth manifold M , and ψ a G -representation on H . Then we have the LGB associated to the principal bundle P and the representation ψ on H

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{H} := P \times_{\psi} H \\ & & \downarrow \\ & & M \end{array}$$

Fix a typical pre-classical connection on P ; as also introduced in Subsubsection 7.2.1 we have an associated parallel transport $\text{PT}_{\alpha}^{\mathcal{P}} : \mathcal{P}_x \rightarrow \Gamma(\alpha^*\mathcal{P})$ along a curve $\alpha : [0, t] \rightarrow M$ ($t > 0$), where $\alpha(0) =: x$. As proven in [1, §5.9, Thm. 5.9.1, page 289f.], we have a canonical well-defined parallel transport $\text{PT}_{\alpha}^{\mathcal{H}} : \mathcal{H}_x \rightarrow \Gamma(\alpha^*\mathcal{H})$ given by

$$\text{PT}_{\alpha}^{\mathcal{H}}([p, h]) = [\text{PT}_{\alpha}^{\mathcal{P}}(p), h]$$

for all $[p, h] \in \mathcal{H}_x$. This has a 1:1 correspondence to a horizontal distribution $H\mathcal{H}$ on \mathcal{H} . Observe that we have

$$\begin{aligned} \text{PT}_{\alpha}^{\mathcal{H}}([p, h] \cdot [p, h']) &= \text{PT}_{\alpha}^{\mathcal{H}}([p, hh']) \\ &= [\text{PT}_{\alpha}^{\mathcal{P}}(p), hh'] \\ &= [\text{PT}_{\alpha}^{\mathcal{P}}(p), h] \cdot [\text{PT}_{\alpha}^{\mathcal{P}}(p), h'] \\ &= \text{PT}_{\alpha}^{\mathcal{H}}([p, h]) \cdot \text{PT}_{\alpha}^{\mathcal{H}}([p, h']) \end{aligned}$$

for all $[p, h], [p, h'] \in \mathcal{H}_x$. Now recall that the whole motivation behind Def. 7.36 comes from Eq. (45) (also recall Remark 7.27) which itself stems from Eq. (44). We see that that Eq. (44) is satisfied here, and thus Eq. (45) follows, *i.e.*

$$\mathcal{r}_{[p, h']^*}(X) \in H_{[p, h] \cdot [p, h']} \mathcal{H}$$

for all $X \in H_{[p,h]}\mathcal{H}$. By Prop. 7.32 and dimensional reasons (horizontal subspaces are of the same dimension) it follows immediately that $H\mathcal{H}$ is an (Ehresmann) connection on \mathcal{H} in sense of Def. 7.36.

We call such connections on LGBs associated to a pre-classical principal bundle an **associated connection**.

As expected, we have a corresponding connection 1-form. For this we need to formally define the pullback of 1-forms with respect to the modified right-pushforward.

Definition 7.40: The pullback of forms

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle, and we have a fixed horizontal distribution $H\mathcal{G}$ on \mathcal{G} . For $\omega \in \Omega^k(\mathcal{P}; \pi^*\mathcal{G})$ ($k \in \mathbb{N}_0$) we define the **pullback via the modified right-pushforward** $\mathcal{r}_g^!(\omega|_{\mathcal{P}_x})$ (with $g \in \mathcal{G}_x$, over \mathcal{P}_x , $x \in M$) as an element of $\Gamma(\Lambda^k T^*\mathcal{P}|_{\mathcal{P}_x} \otimes \mathcal{G}_x)$ by

$$\left(\mathcal{r}_g^!(\omega|_{\mathcal{P}_x})\right)\Big|_p(Y_1, \dots, Y_k) := \omega_{p \cdot g}(\mathcal{r}_{g*}(Y_1), \dots, \mathcal{r}_{g*}(Y_k))$$

for all $p \in \mathcal{P}_x$ and $Y_1, \dots, Y_k \in T_p\mathcal{P}$. Similarly we define the **pullback via the modified right-pushforward** $\mathcal{r}_\sigma^!\omega$ with a (local) section $\sigma \in \Gamma(\mathcal{G})$ as an element of $\Omega^1(\mathcal{P}; \pi^*\mathcal{G})$ by

$$\left(\mathcal{r}_\sigma^!\omega\right)\Big|_p(Y_1, \dots, Y_k) := \left(\mathcal{r}_{\sigma_x}^!(\omega|_{\mathcal{P}_x})\right)\Big|_p(Y_1, \dots, Y_k) = (r_\sigma^*\omega)|_p(\mathcal{r}_{\sigma*}(Y_1), \dots, \mathcal{r}_{\sigma*}(Y_k))$$

for all $p \in \mathcal{P}_x$ and $Y_1, \dots, Y_k \in T_p\mathcal{P}$.

Remarks 7.41.

By Prop. 7.32 (and Remark 7.35) these definitions are well-defined. A short note about the notation on the very right hand side of the second definition: This notation allows us to extend it to vector fields, that is,

$$\left(\mathcal{r}_\sigma^!\omega\right)(Y_1, \dots, Y_k) = (r_\sigma^*\omega)(\mathcal{r}_{\sigma*}(Y_1), \dots, \mathcal{r}_{\sigma*}(Y_k))$$

for all $p \in \mathcal{P}_x$ and $Y_1, \dots, Y_k \in \mathfrak{X}(\mathcal{P})$. Observe that we have $\mathcal{r}_{\sigma*}(Y_l) \in \Gamma(r_\sigma^*T\mathcal{P})$ ($l \in \{1, \dots, k\}$) and $r_\sigma^*\omega \in \Gamma(\Lambda^k r_\sigma^*T\mathcal{P} \otimes \pi^*\mathcal{G})$, using $r_\sigma^*\pi^*\mathcal{G} \cong (\pi \circ r_\sigma)^*\mathcal{G} = \pi^*\mathcal{G}$, such that the right hand side is well-defined.

For the following definition recall Ex. 6.7 and the isomorphism for $V\mathcal{P}$ in Cor. 7.18; also recall the adjoint \mathcal{G} -representation on $V\mathcal{P}$, Ex. 7.31.

Definition 7.42: Connection 1-forms on principal LGB-bundles

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle, and we have a fixed horizontal distribution $H\mathcal{G}$ on \mathcal{G} . A **connection 1-form** on \mathcal{P} is a 1-form $A \in \Omega^1(\mathcal{P}; \pi^*\mathcal{G})$ satisfying:

- (\mathcal{G} -equivariance, but w.r.t. modified right-pushforward)

$$\tau_\sigma^! A = \text{Ad}_{\sigma^{-1}} \circ A$$

for all (local) $\sigma \in \Gamma(\mathcal{G})$.

- (Identity on $V\mathcal{P}$)

$$A(\tilde{\nu}) = \pi^* \nu$$

for all (local) $\nu \in \Gamma(\mathfrak{g})$.

Remarks 7.43.

• Due to the fact that we formulated the \mathcal{G} -equivariance using Ex. 7.31, one may be able to show an analogue of the 1:1 correspondence of typical (pre-classical) connection 1-forms to connection reforms and splittings of the Atiyah sequence; that is, a generalization of [2, §3.2, page 90 ff.].

- The \mathcal{G} -equivariance reads point-wise for $g := \sigma_x$ ($x \in M$)

$$\left(\tau_g^! (A|_{\mathcal{P}_x}) \right) \Big|_p (X) = (p \cdot g, \text{Ad}_{g^{-1}}(A_p(X)))$$

for all $p \in \mathcal{P}_x$ and $X \in T_p \mathcal{P}$, where we wrote $A_p = (p, A_p)$ with $A_p \in \Gamma(T_p^* \mathcal{P} \otimes \mathfrak{g}_x)$, and Ad is the adjoint representation of \mathcal{G} . In the typical formulation of gauge theory the base point component is usually omitted due to that \mathfrak{g} is then trivial and so also $V\mathcal{P} \cong \pi^* \mathfrak{g}$.

Finally, we identify (Ehresmann) connections and connection 1-forms on principal LGB-bundles; for this recall Cor. 7.18.

Theorem 7.44: 1:1 correspondence of Ehresmann connections and connection 1-forms

Let $\mathcal{G} \rightarrow M$ be an LGB over a smooth manifold M and $\mathcal{P} \xrightarrow{\pi} M$ a principal \mathcal{G} -bundle, and let $H\mathcal{G}$ be a horizontal distribution on \mathcal{G} . Then there is a 1:1 correspondence between Ehresmann connections and connection 1-forms on \mathcal{P} :

- Let $H\mathcal{P}$ be an Ehresmann connection on \mathcal{P} . Then $H\mathcal{P}$ defines a connection 1-form $A \in \Omega^1(\mathcal{P}; \pi^* \mathfrak{g})$ by

$$A_p(\tilde{v}_p + X_p) = (p, v)$$

for all $p \in \mathcal{P}_x$ ($x \in M$), $v \in \mathfrak{g}_x$ and $X \in H_p \mathcal{P}$.

- Let $A \in \Omega^1(\mathcal{P}; \pi^* \mathfrak{g})$ be a connection 1-form on \mathcal{P} . Then A defines an Ehresmann connection $H\mathcal{P}$ on \mathcal{P} via its kernel $\text{Ker}(A)$, that is,

$$H_p \mathcal{P} = \text{Ker}(A_p)$$

for all $p \in \mathcal{P}$.

Proof.

The proof is very similar to the proof of "typical connections", as *e.g.* provided in [1, §5.2, Thm. 5.2.2, page 262].

• For the first bullet point we need to show that Def. 7.42 is satisfied, and the identity behaviour on $V\mathcal{P}$ quickly follows by definition: We have

$$A(\tilde{\nu})|_p = A_p(\tilde{\nu}_p) = (p, \nu_x) = (\pi^*\nu)|_p$$

for all $\nu \in \Gamma(\mathcal{G})$ and $p \in \mathcal{P}_x$ ($x \in M$). Therefore it is only left to show the \mathcal{G} -equivariance. As mentioned in Remark 6.18, for $v \in \mathcal{G}_x$ the vector field \tilde{v} on \mathcal{P}_x is a fundamental vector field coming from the \mathcal{G}_x -action on \mathcal{P}_x . Hence, we know by [1, §3.4, Prop. 3.4.6, page 145f.] that

$$\mathcal{r}_{g*}(\tilde{v}) = r_{g*}(\tilde{v}) = \overline{\text{Ad}_{g^{-1}}(v)}$$

for all $g \in \mathcal{G}_x$, also using Remark 7.34.¹⁰ Thus,

$$\begin{aligned} \left(\mathcal{r}_{\sigma}^! A \right) \Big|_p (\tilde{v}_p + X_p) &= A_{p \cdot \sigma_x} \left(\underbrace{\mathcal{r}_{\sigma_x*}(\tilde{v}_p)}_{= \mathcal{r}_{\sigma_x*}(\tilde{v})|_{p \cdot \sigma_x}} + \underbrace{\mathcal{r}_{\sigma_x*}(X_p)}_{\in H_{p \cdot \sigma_x} \mathcal{P}} \right) \\ &= (p \cdot \sigma_x, \text{Ad}_{\sigma_x^{-1}}(v)) \\ &= \mathcal{A} \mathcal{d}_{\sigma_x^{-1}}(p, v) \\ &= \mathcal{A} \mathcal{d}_{\sigma_x^{-1}}(A_p(\tilde{v}_p + X_p)) \\ &= (\mathcal{A} \mathcal{d}_{\sigma^{-1}} \circ A) \Big|_p (\tilde{v}_p + X_p) \end{aligned}$$

for all $p \in \mathcal{P}_x$, $\sigma \in \Gamma(\mathcal{G})$, $v \in \mathcal{G}_x$ and $X_p \in H_p \mathcal{P}$. This finishes the proof for the first bullet point.

• For the second bullet point we make use of Cor. 7.18 and

$$A(\tilde{\nu}) = \pi^*\nu$$

for all (local) sections $\nu \in \Gamma(\mathcal{G})$. This implies that A has not only values in $V\mathcal{P}$, but acts also as identity on $V\mathcal{P}$. Thus, $H\mathcal{P} = \text{Ker}(A_p)$ is a complementary subspace of $V_p \mathcal{P}$ for all $p \in \mathcal{P}$. In order to show that $H\mathcal{P}$ is a horizontal distribution of \mathcal{P} , we fix a local frame $(e_a)_a$ of \mathcal{G} over U (an open subset of M), so that $(\pi^*e_a)_a$ is a frame of $\pi^*\mathcal{G}|_{\pi^{-1}(U)}$. Therefore we write

$$A = A^a \otimes \pi^*e_a,$$

where $A^a \in \Omega^1(\mathcal{P}|_U)$. Due to $A(\tilde{\nu}) = \pi^*\nu$ we get

$$A^a(\tilde{e}_b) = \delta_b^a \pi^*e_a,$$

where δ_b^a is the Kronecker delta. By Cor. 7.18 $(A^a)_a$ is the dual frame of $(e_a)_a$, thus, the A^a are linear independent to each other. ■

¹⁰This is very straightforward to prove.

7.3. Gauge transformations**7.4. Generalized curvature/field strength****7.5. Generalized covariant derivative/minimal coupling****8. Curved Yang-Mills gauge theory****9. Conclusion**

Also mention that the new connection is not necessarily equivalent to a classical connection even though the related gauge theories are.

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A. Double tangent bundle and its canonical flip map

We will follow [2, §9.6, page 363]. For a smooth manifold M we denote the projection of its tangent bundle by $\pi_{TM} : TM \rightarrow M$; similarly we have the projection of the **double tangent bundle** $\pi_{TTM} : TTM \rightarrow TM$, the tangent bundle of TM . However, there is also $D\pi_{TM} : TTM \rightarrow TM$, and in fact there is another vector bundle structure on TTM rendering $D\pi_{TM}$ a projection; see *e.g.* [2, §3.4 *et seq.*; page 110ff.]. Let us give a very rough sketch:

Let $\xi, \eta \in TTM$ with

$$D_{X_0}\pi_{TM}(\xi) = D_{Y_0}\pi_{TM}(\eta) =: \omega,$$

where $X_0 := \pi_{TTM}(\xi)$ and $Y_0 := \pi_{TTM}(\eta)$, and due to the fact that $D\pi_{TM}$ is a vector bundle morphism over π_{TM} we get

$$p = \pi_{TM}(X_0) = \pi_{TM}(Y_0),$$

where $\pi_{TM}(\omega) =: p$. Thus, one can take curves $X, Y : I \rightarrow TM$ ($I \subset \mathbb{R}$ an open interval around 0) with

$$\begin{aligned} X(0) &= X_0, & \left. \frac{d}{dt} \right|_{t=0} X &= \xi, \\ Y(0) &= Y_0, & \left. \frac{d}{dt} \right|_{t=0} Y &= \eta, \end{aligned}$$

such that

$$\pi_{TM} \circ X = \pi_{TM} \circ Y,$$

because the condition on ξ and η imply on the base paths $\pi_{TM} \circ X, \pi_{TM} \circ Y : I \rightarrow M$ that

$$(\pi_{TM} \circ X)(0) = p = (\pi_{TM} \circ Y)(0),$$

$$\left. \frac{d}{dt} \right|_{t=0} (\pi_{TM} \circ X) = D_{X_0}\pi_{TM}(\xi) = \omega = D_{Y_0}\pi_{TM}(\eta) = \left. \frac{d}{dt} \right|_{t=0} (\pi_{TM} \circ Y).$$

Then the addition and scalar multiplication with $\lambda \in \mathbb{R}$ for $TTM \xrightarrow{D\pi_{TM}} TM$ is defined by

$$\xi \blacktriangleleft \eta := \left. \frac{d}{dt} \right|_{t=0} (X + Y),$$

$$\lambda \cdot \xi := \left. \frac{d}{dt} \right|_{t=0} (\lambda X),$$

where the addition of curves is well-defined because of $\pi_{TM} \circ X = \pi_{TM} \circ Y$ which implies $\pi_{TM} \circ (X + Y) = \pi_{TM} \circ X = \pi_{TM} \circ Y$; so, one can take the sum of the curves and

$$D\pi_{TM}(\xi \blacktriangleleft \eta) = \left. \frac{d}{dt} \right|_{t=0} \underbrace{(\pi_{TM} \circ (X + Y))}_{=\pi_{TM} \circ X} = D_{X_0} \pi_{TM}(\xi) = \omega.$$

The operations of the linear structure in $\text{T}TM \xrightarrow{\pi_{\text{T}TM}} TM$ is still denoted in the same manner as usual, and by definition one also gets

$$\pi_{\text{T}TM}(\xi \blacktriangleleft \eta) = \pi_{\text{T}TM}(\xi) + \pi_{\text{T}TM}(\eta),$$

$$\pi_{\text{T}TM}(\lambda \cdot \xi) = \lambda \pi_{\text{T}TM}(\xi).$$

In total, we have a double vector bundle given by the following commuting diagram

$$\begin{array}{ccc} \text{T}TM & \xrightarrow{D\pi_{TM}} & TM \\ \downarrow \pi_{\text{T}TM} & & \downarrow \pi_{TM} \\ TM & \xrightarrow{\pi_{TM}} & M \end{array} \quad (\text{A.1})$$

i.e. each horizontal and vertical line is a vector bundle so that the horizontal and vertical scalar multiplications on $\text{T}TM$ commute; see *e.g.* [9, §3ff.] or [2, §9.1, page 340ff.] for a definition on double vector bundles in general.

Now observe that the flow X of ξ has values in TM , that is, for all $t \in I$ we have a curve $\alpha_t : J \rightarrow M$ (J another open interval containing 0), $s \mapsto \alpha_t(s)$, so that

$$\alpha_t(0) = \pi_{TM}(X(t)),$$

$$\left. \frac{d}{ds} \right|_{s=0} \alpha_t = X(t),$$

and the first equation implies

$$\left. \frac{d}{dt} \right|_{t=0} \alpha_t(0) = D_{X_0} \pi_{TM}(\xi) = \omega.$$

So, in total we have for all $\xi \in \text{T}TM$ a smooth map $\alpha : I \times J \rightarrow M$, $(t, s) \mapsto \alpha(t, s) = \alpha_t(s)$, such that

$$\alpha(0, 0) = p = (\pi_{TM} \circ \pi_{\text{T}TM})(\xi) = (\pi_{TM} \circ D\pi_{TM})(\xi),$$

$$\left. \frac{d}{dt} \right|_{t=0} \alpha(t, 0) = \omega = D_{X_0} \pi_{TM}(\xi), \quad \left. \frac{d}{ds} \right|_{s=0} \alpha(0, s) = X_0 = \pi_{\text{T}TM}(\xi),$$

$$\left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \alpha = \xi.$$

As for tangent vectors, the class $[\alpha]$ of all such α uniquely describes ξ , in fact giving rise to an equivalence relation so that $[\alpha]$ is an equivalence class.

In the context of Schwarz's Theorem one may find it natural to define the **canonical involution (or flip) on TTM** as a map $S : \text{TTM} \rightarrow \text{TTM}$ by

$$S(\xi) := \frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} \alpha. \quad (\text{A.2})$$

By [2, §9.6, Thm. 9.6.1, page 363; but without proof] one has that S is an isomorphism of double vector bundles with certain "special" properties; we will not need the general definition of these. What we need of this, is that S is a base-preserving vector bundle isomorphism as both maps, from $\text{TTM} \xrightarrow{D\pi_{\text{TM}}} \text{TM}$ to $\text{TTM} \xrightarrow{\pi_{\text{TTM}}} \text{TM}$ and vice versa. Let us prove this, by starting with showing the well-definedness of S :

W.l.o.g. we can put $J = I$ for simplicity. By construction, the equivalence class $[\beta]$ of $S(\xi)$ is represented by $\beta : I^2 \rightarrow M$ given by $\beta(t, s) := \alpha(s, t)$ with

$$\beta(0, 0) = p,$$

$$D\pi_{\text{TM}}(S(\xi)) = \frac{d}{dt} \Big|_{t=0} \beta(t, 0) = \frac{d}{dt} \Big|_{t=0} \alpha(0, t) = \pi_{\text{TTM}}(\xi) = X_0,$$

$$\pi_{\text{TTM}}(S(\xi)) = \frac{d}{ds} \Big|_{s=0} \beta(0, s) = \frac{d}{ds} \Big|_{s=0} \alpha(s, 0) = D\pi_{\text{TM}}(\xi) = \omega,$$

$$S(\xi) = \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \beta.$$

With this, we can express $S(\xi)$ in coordinates: Fix local coordinates $(x^i)_i$ on M , then $(\pi_{\text{TM}}^* x^i)_i$ and $(dx^i)_i$ are local coordinates on TM . Then by chain rule

$$\xi(\pi_{\text{TM}}^* x^i) = \xi(x^i \circ \pi_{\text{TM}}) = d_p x^i (D\pi_{\text{TM}}(\xi)) = \omega^i,$$

and

$$\xi(dx^i) = \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha \right) (dx^i) = \frac{d}{dt} \Big|_{t=0} \underbrace{\left(dx^i \circ \frac{d}{ds} \Big|_{s=0} \alpha \right)}_{= \left(\frac{d}{ds} \Big|_{s=0} \alpha \right)(x^i)} = \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha^i,$$

thus we have in total

$$\xi = \omega^i \frac{\partial}{\partial(\pi_{\text{TM}}^* x^i)} \Big|_{X_0} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha^i \right) \frac{\partial}{\partial(dx^i)} \Big|_{X_0}. \quad (\text{A.3})$$

Similarly we can proceed with $S(\xi)$ and then apply the "classical" Schwarz' Theorem on the derivatives of α^i (recall the properties of β), so that we get

$$S(\xi) = X_0^i \frac{\partial}{\partial(\pi_{\text{TM}}^* x^i)} \Big|_{\omega} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \beta^i \right) \frac{\partial}{\partial(dx^i)} \Big|_{\omega}$$

$$\begin{aligned}
&= X_0^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\omega} + \left(\frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} \alpha^i \right) \frac{\partial}{\partial(dx^i)} \Big|_{\omega} \\
&= X_0^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\omega} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha^i \right) \frac{\partial}{\partial(dx^i)} \Big|_{\omega}.
\end{aligned} \tag{A.4}$$

It is now clear that $S(\xi)$ is independent of the choice of α and just depends on $[\alpha]$. Furthermore, due to what we have shown for β , it is also clear that S is a base-preserving map either from $\mathrm{TTM} \xrightarrow{D\pi_{TM}} \mathrm{TM}$ to $\mathrm{TTM} \xrightarrow{\pi_{\mathrm{TTM}}} \mathrm{TM}$ or vice versa. Let $[\gamma] \in \mathrm{TTM}$, then by the same calculation as for the properties of β we have $S([\tau]) = [\gamma]$, where $\tau(t, s) := \gamma(s, t)$; thence, surjectivity of S follows. Similarly, again by the properties of $\beta(t, s)$ above, if we have $S([\alpha]) = S([\alpha'])$, then we clearly get $[\alpha] = [\alpha']$, and hence injectivity. Alternatively, bijectivity follows by $S \circ S = \mathbb{1}_{\mathrm{TTM}}$.

It is only left to show that we have linearity for S as a map from $\mathrm{TTM} \xrightarrow{D\pi_{TM}} \mathrm{TM}$ to $\mathrm{TTM} \xrightarrow{\pi_{\mathrm{TTM}}} \mathrm{TM}$ and vice versa. For this we use the derived coordinate expressions. Let $\eta \in \mathrm{TTM}$ be defined as before with associated area function γ such that $\eta = [\gamma]$. Then one can derive by using what we have shown in the discussion about $\mathrm{TTM} \xrightarrow{D\pi_{TM}} \mathrm{TM}$ as vector bundle that

$$\lambda \cdot \xi \mathbin{\boxplus} \kappa \cdot \eta = \omega^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\lambda X_0 + \kappa Y_0} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} (\lambda \alpha^i + \kappa \gamma^i) \right) \frac{\partial}{\partial(dx^i)} \Big|_{\lambda X_0 + \kappa Y_0} \tag{A.5}$$

for all $\lambda, \kappa \in \mathbb{R}$, and so

$$\begin{aligned}
S(\lambda \cdot \xi \mathbin{\boxplus} \kappa \cdot \eta) &= (\lambda X_0^i + \kappa Y_0^i) \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\omega} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} (\lambda \alpha^i + \kappa \gamma^i) \right) \frac{\partial}{\partial(dx^i)} \Big|_{\omega} \\
&= \lambda S(\xi) + \kappa S(\eta).
\end{aligned}$$

In the same fashion, let $\zeta = [\delta] \in \mathrm{T}_{X_0} \mathrm{TM}$ with $D_{X_0} \pi_{\mathrm{TM}}(\zeta) = \varphi$, then $\lambda \xi + \kappa \zeta$ is just the typical sum of tangent vectors and we get

$$\begin{aligned}
S(\lambda \xi + \kappa \zeta) &= X_0^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\lambda \omega + \kappa \varphi} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} (\lambda \alpha^i + \kappa \delta^i) \right) \frac{\partial}{\partial(dx^i)} \Big|_{\lambda \omega + \kappa \varphi} \\
&= \lambda \cdot \left(X_0^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\omega} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha^i \right) \frac{\partial}{\partial(dx^i)} \Big|_{\omega} \right) \\
&\quad \mathbin{\boxplus} \kappa \cdot \left(X_0^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\varphi} + \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \delta^i \right) \frac{\partial}{\partial(dx^i)} \Big|_{\varphi} \right) \\
&= \lambda \cdot S(\xi) \mathbin{\boxplus} \kappa \cdot S(\zeta).
\end{aligned}$$

This finishes the proof, so we have:

Remark A.1: The canonical involution/flip map an isomorphism,
[2, §9.6, Thm. 9.6.1, page 363; but without proof]

The canonical involution/flip S is a base-preserving vector bundle isomorphism from $\mathrm{TTM} \xrightarrow{D\pi_{\mathrm{TM}}} \mathrm{TM}$ to $\mathrm{TTM} \xrightarrow{\pi_{\mathrm{TTM}}} \mathrm{TM}$, and similarly vice versa. We also write $S =: S_M$ to give an accentuation on M .

This isomorphism is basically now Schwarz's Theorem:

Remark A.2: Revisit: Schwarz's Theorem

By definition we have

$$\left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} \alpha = S_M \left(\left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \alpha \right)$$

for all smooth $\alpha : I \times J \rightarrow M$, where I and J are open intervals containing 0. Similarly this extends to smooths maps $F : M \rightarrow N$, where N is another smooth manifold. So let $\xi = [\alpha] \in \mathrm{TTM}$, and then we calculate for $\mathrm{DDF} : \mathrm{TTM} \rightarrow \mathrm{TTN}$ that

$$\begin{aligned} \mathrm{DDF}(\xi) &= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (F \circ \alpha) \\ &= S_N \left(\left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} (F \circ \alpha) \right) \\ &= S_N(\mathrm{DDF}(S_M(\xi))) \\ &= (S_N \circ \mathrm{DDF} \circ S_M)(\xi). \end{aligned}$$

Since the canonical involutions are clearly self-inverse, we can also write

$$S_N \circ \mathrm{DDF} = \mathrm{DDF} \circ S_M.$$

For a notation with base points in M , consider a special case with $M = M_1 \times M_2$, where M_i ($i \in \{1, 2\}$) are smooth manifolds. Thus, $\mathrm{TM} \cong \pi_1^* \mathrm{TM}_1 \oplus \pi_2^* \mathrm{TM}_2$, where $\pi_i : M_1 \times M_2 \rightarrow M_i$ are the projections onto the i -th component. Then let $p_i \in M_i$, $Y_i \in \mathrm{T}_{p_i} M_i$ and $\xi := (Y_1, Y_2)$. Denoting with $\gamma_1 : I \rightarrow M_1$ and $\gamma_2 : J \rightarrow M_2$ the curves with velocities Y_1 and Y_2 at 0, respectively, and so $\alpha(t, s) = (\gamma_1(s), \gamma_2(t))$; then we define $\mathrm{D}_{p_1} F(Y_1)$ as a map

$$M_2 \rightarrow \mathrm{TN},$$

$$p_2 \mapsto \mathrm{D}_{p_1} F(Y_1)|_{p_2} = \mathrm{D}_{(p_1, p_2)} F(Y_1, 0_{p_2}),$$

where 0_{p_2} is the zero tangent vector at $\mathrm{T}_{p_2} M_2$, in a similar fashion for $\mathrm{D}_{p_2} F(Y_2)$. Then

$$\mathrm{D}_{p_2} (\mathrm{D}_{p_1} F(Y_1))(Y_2) = \left. \frac{d}{dt} \right|_{t=0} \mathrm{D}_{(p_1, \gamma_2(t))} F(Y_1, 0_{\gamma_2(t)})$$

$$\begin{aligned}
&= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (F \circ \alpha(t, s)) \\
&= DDF(\xi),
\end{aligned}$$

similarly

$$\begin{aligned}
D_{p_1}(D_{p_2}F(Y_2))(Y_1) &= \left. \frac{d}{ds} \right|_{s=0} D_{(\gamma_1(s), p_2)} F(0_{\gamma_1(s)}, Y_2) \\
&= \left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} (F \circ \alpha(t, s)) \\
&= S_N(DDF(\xi)),
\end{aligned}$$

in total

$$D_{p_1}(D_{p_2}F(Y_2))(Y_1) = S_N(D_{p_2}(D_{p_1}F(Y_1))(Y_2)).$$

In fact, using the canonical flip/involution, we can construct and state other properties which can be useful for calculations related to second derivatives.

Remark A.3: Total derivatives of tangent bundle morphisms linear with respect to prolonged vertical structure

Consider a vector bundle morphism $L : TM \rightarrow TN$, where N is another smooth manifold. Then for $\xi, \eta \in TTM$ with their approximating curves X and Y , respectively, as previously, then

$$\begin{aligned}
DL(\lambda \cdot \xi \mathbin{\boxplus} \kappa \cdot \eta) &= \left. \frac{d}{dt} \right|_{t=0} (L(\lambda X + \kappa Y)) \\
&= \left. \frac{d}{dt} \right|_{t=0} (\lambda L(X) + \kappa L(Y)) \\
&= \lambda \cdot DL(\xi) \mathbin{\boxplus} \kappa \cdot DL(\eta)
\end{aligned}$$

for all $\lambda, \kappa \in \mathbb{R}$. Hence we achieve linearity with respect to both vector bundle structures of TTM . If we denote the base points as before, then this reads

$$D_{\lambda X_0 + \kappa Y_0} L(\lambda \cdot \xi \mathbin{\boxplus} \kappa \cdot \eta) = \lambda \cdot D_{X_0} L(\xi) \mathbin{\boxplus} \kappa \cdot D_{Y_0} L(\eta).$$

Remark A.4: Alignment of both vector bundle structures on TTM on the restricted vertical bundle

By definition, the vertical bundle VTM of TM is a subbundle of TTM . The zero section of TM is a natural embedding of M into TM ; this embedded submanifold will also be by \widetilde{M} . Then Diagram (A.1) restricts onto

$$\begin{array}{ccc} \mathrm{VTM}|_{\widetilde{M}} & \xrightarrow{D\pi_{\mathrm{TM}}} & \widetilde{M} \\ \downarrow \pi_{\mathrm{TTM}} & & \downarrow \cong \\ \widetilde{M} & \xrightarrow{\cong} & M \end{array}$$

and $S = S_M$ does not only restrict onto that, S is actually the identity on $\mathrm{VTM}|_M$, as also stated in [2, §9.6, Thm. 9.6.1, page 363; but without proof]. This follows simply by the coordinate expressions Eq. (A.3) and (A.4) (recall the involved notation). $\xi \in \mathrm{VTM}|_{\widetilde{M}}$ is in the kernel of both projections, $D\pi_{\mathrm{TM}}$ and π_{TTM} , thus $\omega = X_0 = 0 \in \mathrm{T}_p M$, therefore $S(\xi) = \xi$ by Eq. (A.3) and (A.4). By Remark A.1 the addition \blackplus and scalar multiplication \cdot align with the typical addition $+$ and scalar multiplication \cdot , respectively, of TTM as tangent bundle.

Motivated by this, we can actually recover something similar for VTM . In this case

$$\begin{array}{ccc} \mathrm{VTM} & \xrightarrow{D\pi_{\mathrm{TM}}} & \widetilde{M} \\ \downarrow \pi_{\mathrm{TTM}} & & \downarrow \cong \\ \mathrm{TM} & \xrightarrow{\pi_{\mathrm{TM}}} & M \end{array}$$

and, as already mentioned earlier, the vertical bundle of vector bundles is trivially the pullback of the bundle along itself, here $\mathrm{VTM} \cong \pi_{\mathrm{TM}}^* \mathrm{TM}$. Thus, we have a canonical projection $\mathrm{pr}_2 : \mathrm{VTM} \rightarrow \mathrm{TM}$ onto the second component, and we can write for $\xi = [\alpha] \in \mathrm{V}_{X_0} \mathrm{TM}$

$$\xi = \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha^i \right) \frac{\partial}{\partial(\mathrm{d}x^i)} \Big|_{X_0} \cong \left(X_0, \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \alpha^i \right) \frac{\partial}{\partial x^i} \Big|_p \right)$$

using the notation of Eq. (A.3) (especially $X_0 \in \mathrm{T}_p M$, $p \in M$). Similarly for $\eta = [\beta] \in \mathrm{V}_{Y_0} \mathrm{TM}$ (same notation as previously in this appendix, *i.e.* $Y_0 \in \mathrm{T}_p M$). Then by Eq. (A.5)

$$\lambda \cdot \xi \blackplus \kappa \cdot \eta \cong \left(\lambda X_0 + \kappa Y_0, \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} (\lambda \alpha^i + \kappa \beta^i) \right) \frac{\partial}{\partial x^i} \Big|_p \right)$$

for all $\lambda, \kappa \in \mathbb{R}$. Therefore

$$\mathrm{pr}_2(\lambda \cdot \xi \blackplus \kappa \cdot \eta) = \lambda \mathrm{pr}_2(\xi) + \kappa \mathrm{pr}_2(\eta).$$

For $\zeta \in \mathrm{V}_{X_0} \mathrm{TM}$ we clearly get $\mathrm{pr}_2(\lambda \xi + \kappa \zeta) = \lambda \mathrm{pr}_2(\xi) + \kappa \mathrm{pr}_2(\zeta)$, and so both linear structures on VTM align under pr_2 .

Remark A.5: Tangent lift, [10, §2.2, last paragraph in Subsection 2.2]

Let $X \in \mathfrak{X}(M)$, then its total derivative is a map $DX : TM \rightarrow TTM$ satisfying

$$D\pi_{TM} \circ DX = D(\underbrace{\pi_{TM} \circ X}_{=\mathbb{1}_M}) = \mathbb{1}_{TM},$$

so that DX is a section of $TTM \xrightarrow{D\pi_{TM}} M$. Due to the fact that $S = S_M$ is a base-preserving vector bundle isomorphism from $TTM \xrightarrow{D\pi_{TM}} M$ to $TTM \xrightarrow{\pi_{TM}} M$ (and vice versa), we have a vector field $X_T \in \mathfrak{X}(TM)$ given by

$$X_T := S \circ DX,$$

called the **tangent lift** of X . We then also have

$$D\pi_{TM} \circ X_T = \pi_{TTM} \circ DX = X,$$

thus the label as lift. We also have linearity properties (sort of) and a Leibniz rule, that is, we clearly get

$$X_T(\lambda Y + \kappa Z) = \lambda \cdot X_T(Y) \mathbf{+} \kappa \cdot X_T(Z).$$

for all $Y, Z \in T_p M$ and $\kappa, \lambda \in \mathbb{R}$, and we also have something similar w.r.t. X : Recall Eq. (A.3) and (A.4), including their notation w.r.t. to local coordinates $(x^i)_i$. That is, by definition of DX we can derive

$$D_p X(Y) = Y^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{X_p} + Y^j \frac{\partial X^i}{\partial x^j} \Big|_p \frac{\partial}{\partial(dx^i)} \Big|_{X_p},$$

thus,

$$\begin{aligned} D_p(\kappa X + \lambda W)(Y) &= Y^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{\lambda X_p + \kappa W_p} + Y^j \frac{\partial(\lambda X^i + \kappa W^i)}{\partial x^j} \Big|_p \frac{\partial}{\partial(dx^i)} \Big|_{\lambda X_p + \kappa W_p} \\ &= \lambda \cdot D_p X(Y) \mathbf{+} \kappa \cdot D_p W(Y) \end{aligned}$$

for all $X, W \in \mathfrak{X}(M)$ and $\lambda, \kappa \in \mathbb{R}$, and

$$\begin{aligned} D_p(fX)(Y) &= Y^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_{f(p)X_p} + f(p) Y^j \frac{\partial X^i}{\partial x^j} \Big|_p \frac{\partial}{\partial(dx^i)} \Big|_{f(p)X_p} + Y^j X^i(p) \frac{\partial f}{\partial x^j} \Big|_p \frac{\partial}{\partial(dx^i)} \Big|_{f(p)X_p} \end{aligned} \tag{A.6}$$

for all $f \in C^\infty(M)$. Therefore

$$(\lambda X + \kappa W)_T = \lambda X_T + \kappa W_T,$$

and by Eq. (A.4)

$$(fX)_T(Y) = f(p) X_p^i \frac{\partial}{\partial(\pi_{TM}^* x^i)} \Big|_Y + f(p) Y^j \frac{\partial X^i}{\partial x^j} \Big|_p \frac{\partial}{\partial(dx^i)} \Big|_Y + Y(f) X^i(p) \frac{\partial}{\partial(dx^i)} \Big|_Y$$

so

$$(fX)_T = f X_T + df \otimes X^i \frac{\partial}{\partial(dx^i)}.$$

Similar to Remark A.4, the vertical bundle of TM is canonically isomorphic to $\pi_{TM}^* TM$, one can think of the second term as $df \otimes \pi_{TM}^* X$.

B. Other Interpretations of the compatibility conditions

B.1. Minimal coupling

Notation as in [1]

- \tilde{G} Lie group with Lie algebra \mathfrak{g}
- M smooth manifold (usually also a spacetime). An open subset of M is usually denoted by U ; typically small enough that "everything works out" (especially without further mentioning intersections of given open sets and so on)
- $P \rightarrow M$ a principal bundle, a (local) gauge is usually denoted by s , an element of $\Gamma(P)$, sections of P
- V a vector space
- ρ a Lie group representation on V , ρ_* the induced Lie algebra representation on V
- $K := P \times_{\rho} V$ the associated vector bundle induced by P and ρ on V . An element Φ of K is denoted by $[p, \phi]$ for $p \in P$ and $\phi \in V$, where $[\cdot, \cdot]$ denotes the equivalence class with respect to the equivalence

$$(p, \phi) \sim (pg, \rho(g^{-1}) \cdot \phi)$$

for all $g \in \tilde{G}$; pg denotes the canonical group action (from the right) $P \times \tilde{G} \rightarrow P$ and \cdot the action of $\text{Aut}(V) \subset \text{End}(V)$ on V .

- Especially if fixing a local gauge $s : U \rightarrow P$ we can write for sections $\Phi \in \Gamma(K)$ locally

$$\Phi|_U = [s, \phi],$$

where $\phi : U \rightarrow V$, i.e. a local section of the trivial vector bundle $M \times V \rightarrow M$.

- We especially focus on $V = \mathfrak{g}$ and $\rho = \text{Ad}$ the adjoint representation of \tilde{G} on \mathfrak{g} .

The field of gauge bosons A is a connection on the principal bundle, *i.e.* an element of $\Omega^1(P; \mathfrak{g})$ with

$$r_g^! A = \text{Ad}_{g^{-1}}(A) := \text{Ad}_{g^{-1}} \circ A,$$

$$A(\tilde{X}) = X$$

for all $g \in \tilde{G}$ and $X \in \mathfrak{g}$, where $r_g^!$ is the pullback of forms via the right \tilde{G} -multiplication on P , and \tilde{X} the fundamental vector field of X on P .

Typically, a lot of the formalism of gauge theory comes from how to define the minimal coupling. So, let us look at this and reinvent it a bit. Usually the covariant derivative/minimal coupling ∇^A of A and $\Phi \in \Gamma(K)$ is locally (w.r.t. to a gauge s) defined by

$$\nabla^A \Phi := [s, \nabla^A \phi],$$

where

$$\nabla^A \phi := d\phi + \rho_*(A_s) \cdot \phi, \tag{B.7}$$

where $A_s := s^! A \in \Omega^1(U; \mathfrak{g})$ (local pullback as a form of A via s) and $d\phi := \nabla^0 \phi$, ∇^0 the canonical flat connection on $M \times V$.

The explicit definition of the field strength F of A is then usually motivated by looking at the curvature R_{∇^A} of ∇^A , that is

$$R_{\nabla^A}(\cdot, \cdot)\Phi|_U = [s, \rho_*(F_s) \cdot \phi],$$

where

$$F_s := dA_s + \frac{1}{2} [A_s \wedge A_s]_{\mathfrak{g}}$$

is the typical local definition of $F_s \in \Omega^2(U; \mathfrak{g})$ with

$$[A_s \wedge A_s]_{\mathfrak{g}}(X, Y) = 2 [A_s(X), A_s(Y)]_{\mathfrak{g}}$$

for all $X, Y \in \mathfrak{X}(U)$. (The notation F_s is of course due to the fact that $F_s = s^! F$, where F is the curvature of A . But I want to avoid that for now because of what we are about doing to do.) We shortly could denote this also as

$$R_{\nabla^A} \phi = \rho_*(F_s) \cdot \phi \tag{B.8}$$

Now: One could question why using $d\phi = \nabla^0 \phi$ in Eq. (B.7). Thence, let us assume that we have a general vector bundle connection $\widehat{\nabla}$ on the trivial vector bundle $M \times V \rightarrow M$. We are

going to redefine ∇^A and F locally w.r.t. a gauge s , then discuss how the gauge transformations have to look like to receive definitions independent of the chosen gauge s . This also means that the following discussion is now often local by fixing a gauge without further mentioning it.

Let us first locally redefine $\nabla^A\phi$:

$$\nabla^A\phi := \widehat{\nabla}\phi + \rho_*(A_s) \cdot \phi. \quad (\text{B.9})$$

Motivated by Eq. (B.8), we want to identify the field strength with the curvature of ∇^A . One can check that we have

$$R_{\nabla^A} = R_{\widehat{\nabla}} + d\widehat{\nabla}(\rho_*(A_s)) + \rho_*(A_s) \wedge \rho_*(A_s), \quad (\text{B.10})$$

where $d\widehat{\nabla}$ is the exterior covariant derivative of $\widehat{\nabla}$ canonically extended to $\text{End}(V)$, viewing $\rho_*(A_s)$ as an element of $\Omega^1(U; \text{End}(V))$, and where $\rho_*(A_s) \wedge \rho_*(A_s)$ is an element of $\Omega^2(U; \text{End}(V))$ given by

$$\begin{aligned} (\rho_*(A_s) \wedge \rho_*(A_s))(X, Y) &:= \rho_*(A_s(X)) \circ \rho_*(A_s(Y)) - \rho_*(A_s(Y)) \circ \rho_*(A_s(X)) \\ &= [\rho_*(A_s(X)), \rho_*(A_s(Y))]_{\text{End}(V)} \\ &= \rho_*([A_s(X), A_s(Y)]_{\mathfrak{g}}) \\ &= \rho_*\left(\frac{1}{2}[A_s \wedge A_s]_{\mathfrak{g}}\right)(X, Y) \end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$.

In order to have a similar shape as in Eq. (B.8), we now assume that $\widehat{\nabla}$ satisfies the following **compatibility conditions**:

Remark B.1: Comaptibility conditions

$$R_{\widehat{\nabla}} = \rho_*(\zeta), \quad (\text{B.11})$$

$$\widehat{\nabla} \circ \rho_* = \rho_* \circ \nabla \quad (\text{B.12})$$

for some $\zeta \in \Omega^2(M; \mathfrak{g})$ and ∇ a vector bundle connection on the trivial vector bundle $M \times \mathfrak{g} \rightarrow M$.

If we want that Eq. (B.10) has a shape like Eq. (B.8), it is obvious why we require (B.11); (B.12) is needed for the second summand in Eq. (B.10). Hence, let us study (B.12), that is

$$\widehat{\nabla}(\rho_*(\nu)) = \rho_*(\nabla\nu)$$

for all $\nu \in \Gamma(M \times \mathfrak{g})$,¹¹ especially $\widehat{\nabla}$ is again extended to $\text{End}(V)$ on the left hand side. With this we get

$$d\widehat{\nabla}(\rho_*(A_s))(X, Y) = \widehat{\nabla}_X(\rho_*(A_s(Y))) - \widehat{\nabla}_Y(\rho_*(A_s(X))) - \rho_*(A_s([X, Y]))$$

¹¹Elements of \mathfrak{g} are viewed as constant sections of $M \times \mathfrak{g}$.

$$\begin{aligned}
&= \rho_*(\nabla_X(A_s(Y))) - \rho_*(\nabla_Y(A_s(X))) - \rho_*(A_s([X, Y])) \\
&= \rho_*(\nabla_X(A_s(Y)) - \nabla_Y(A_s(X)) - A_s([X, Y])) \\
&= \rho_*(d^\nabla A_s)(X, Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$. Collecting everything, Eq. (B.10) has now the following form

$$R_{\nabla^A} = \rho_* \left(d^\nabla A_s + \frac{1}{2} [A_s \wedge A_s]_{\mathfrak{g}} + \zeta \right).$$

So, we have a new form of the field strength, assuming that ∇ and ζ satisfy the compatibility conditions in Remark B.1. This is precisely the definition of the field strength as in the gauge theory of Thomas and Alexei, that is, we have a new field strength

$$G := d^\nabla A_s + \frac{1}{2} [A_s \wedge A_s]_{\mathfrak{g}} + \zeta.$$

Furthermore, if we are interested into Yang-Mills gauge theories, then we'd have $K = P \times_{\text{Ad}} \mathfrak{g}$ (the adjoint bundle), and so also $\rho_* = \text{ad}$. In this case we can put $\widehat{\nabla} = \nabla$ and then the compatibility conditions in Remark B.1 read

$$R_{\nabla} = \text{ad}(\zeta),$$

$$\nabla \circ \text{ad} = \text{ad} \circ \nabla.$$

The second condition precisely gives after a short calculation

$$\nabla([\mu, \nu]_{\mathfrak{g}}) = [\nabla\mu, \nu]_{\mathfrak{g}} + [\mu, \nabla\nu]_{\mathfrak{g}}$$

for all $\mu, \nu \in \Gamma(M \times \mathfrak{g})$, so, ∇ has to be a Lie bracket derivation. So, in this case the compatibility conditions in Remark B.1 precisely reduce to the compatibility conditions of Alexei's and Thomas's theory! (in the case of Lie algebra bundles; the general theory is more general, formulated on general Lie algebroids)

As a summary:

Remark B.2: Summary

We have

$$R_{\nabla} = \text{ad}(\zeta), \tag{B.13}$$

$$\nabla \circ \text{ad} = \text{ad} \circ \nabla, \tag{B.14}$$

$$G = d^\nabla A_s + \frac{1}{2} [A_s \wedge A_s]_{\mathfrak{g}} + \zeta. \tag{B.15}$$

In fact, the compatibility conditions lead to a gauge invariant theory: Fix an ad-invariant scalar product κ on \mathfrak{g} ; then define the Lagrangian by

$$\mathfrak{L}_{\text{YM}} := -\frac{1}{2} \kappa(G \wedge *G) \tag{B.16}$$

where $*$ is the Hodge star operator w.r.t. some spacetime metric. (In short, the typical definition, but replace F with G) It is easier to look at the infinitesimal version of the gauge transformations, hence everything with respect to a gauge s now.

In order to derive a formula for these, let us again look at ∇^A . Fix an $\varepsilon \in \Gamma(M \times \mathfrak{g})$, then the infinitesimal gauge transformation $\delta_\varepsilon \phi$ of $\phi \in \Gamma(M \times \mathfrak{g})$ is usually defined by

$$\delta_\varepsilon \phi := \rho_*(\varepsilon) \cdot \phi.$$

We fix the infinitesimal gauge trafo $\delta_\varepsilon A$ of A by looking at the gauge trafo of $\nabla^A \phi$ via

$$\begin{aligned} \delta_\varepsilon \nabla^A \phi &= \left. \frac{d}{dt} \right|_{t=0} \left(\nabla^{A+t\delta_\varepsilon A}(\phi + t\delta_\varepsilon \phi) \right) \\ &= \widehat{\nabla}(\delta_\varepsilon \phi) + \underbrace{\rho_*(\delta_\varepsilon A_s) \cdot \phi + \rho_*(A_s) \cdot \delta_\varepsilon \phi}_{=(\widehat{\nabla}(\rho_*(\varepsilon))) \cdot \phi + \rho_*(\varepsilon) \cdot \widehat{\nabla} \phi} \\ &= (\rho_*(\nabla \varepsilon + \delta_\varepsilon A_s) + \rho_*(A_s) \cdot \rho_*(\varepsilon)) \cdot \phi + \rho_*(\varepsilon) \cdot \widehat{\nabla} \phi \end{aligned}$$

using Remark B.1. We want $\delta_\varepsilon \nabla^A \phi = \rho_*(\varepsilon) \cdot \nabla^A \phi$ which gives

$$\rho_*(\varepsilon) \cdot \nabla^A \phi = \rho_*(\varepsilon) \cdot \widehat{\nabla} \phi + \rho_*(\varepsilon) \cdot \rho_*(A_s) \cdot \phi.$$

Imposing $\delta_\varepsilon \nabla^A \phi = \rho_*(\varepsilon) \cdot \nabla^A \phi$ we get

$$\rho_* \left(\delta_\varepsilon A_s + \nabla \varepsilon + [A_s, \varepsilon]_{\mathfrak{g}} \right) = 0$$

using again that ρ_* is a Lie algebra representation. If we require that this shall work for all ρ_* , we may say

$$\delta_\varepsilon A_s := -\nabla \varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}. \quad (\text{B.17})$$

This is precisely the infinitesimal gauge trafo of A as in the theory of Thomas and Alexei! Hence, we achieve infinitesimal gauge invariance of \mathfrak{L}_{YM} . For completeness, let us check the gauge trafo of G using Def. (B.17) and Remark B.2, it is very similar to the "classical" calculation due to Remark B.2 which is why I skip some straightforward calculations to keep it short,

$$\begin{aligned} \delta_\varepsilon G &= \left. \frac{d}{dt} \right|_{t=0} \left(d^\nabla(A_s + t\delta_\varepsilon A_s) + \frac{1}{2} [A_s + t\delta_\varepsilon A_s \wedge A_s + t\delta_\varepsilon A_s]_{\mathfrak{g}} + \zeta \right) \\ &= d^\nabla \left(-\nabla \varepsilon + [\varepsilon, A_s]_{\mathfrak{g}} \right) + \left[A_s \wedge -\nabla \varepsilon + [\varepsilon, A_s]_{\mathfrak{g}} \right]_{\mathfrak{g}} \\ &= \underbrace{-(d^\nabla)^2 \varepsilon}_{=-R\nabla \varepsilon = [\varepsilon, \zeta]_{\mathfrak{g}}} + [\nabla \varepsilon \wedge A_s]_{\mathfrak{g}} + [\varepsilon, d^\nabla A_s]_{\mathfrak{g}} + \underbrace{[A_s \wedge -\nabla \varepsilon]_{\mathfrak{g}}}_{=-[\nabla \varepsilon \wedge A_s]_{\mathfrak{g}}} + [A_s \wedge [\varepsilon, A_s]_{\mathfrak{g}}]_{\mathfrak{g}} \\ &= [\varepsilon, d^\nabla A_s + \zeta]_{\mathfrak{g}} + [A_s \wedge [\varepsilon, A_s]_{\mathfrak{g}}]_{\mathfrak{g}} \end{aligned}$$

and, using the Jacobi identity,

$$\begin{aligned} \left[A_s \frown [\varepsilon, A_s]_{\mathfrak{g}} \right]_{\mathfrak{g}}(X, Y) &= \left[A_s(X), [\varepsilon, A_s(Y)]_{\mathfrak{g}} \right]_{\mathfrak{g}} - \left[A_s(Y), [\varepsilon, A_s(X)]_{\mathfrak{g}} \right]_{\mathfrak{g}} \\ &= \left[\varepsilon, [A_s(X), A_s(Y)]_{\mathfrak{g}} \right]_{\mathfrak{g}} \\ &= \left[\varepsilon, \frac{1}{2} [A_s \frown A_s]_{\mathfrak{g}} \right]_{\mathfrak{g}}(X, Y) \end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$. Altogether

$$\delta_\varepsilon G = \left[\varepsilon, d^\nabla A_s + \frac{1}{2} [A_s \frown A_s]_{\mathfrak{g}} + \zeta \right]_{\mathfrak{g}} = [\varepsilon, G]_{\mathfrak{g}}.$$

Hence, the field strength transforms with the adjoint of ε ; since κ is ad-invariant, we can derive that \mathfrak{L}_{YM} is invariant under the infinitesimal gauge trafo in Def. (B.17)!

Observe that by Remark B.2 that ζ can be non-trivial even if we still use $\nabla = \nabla^0$, the canonical flat connection on $M \times \mathfrak{g}$, even though this whole discussion started with allowing more general connections.

If we minimise \mathfrak{L}_{YM} , then one obvious way would be to search solutions with $G \equiv 0$ for an absolute minimum/maximum (because of the sign), doing so would result into that the classical Yang-Mills energy would have a bound which is non-zero. May this be an explanation for the mass gap? As shown in my thesis, every classical theory has a ζ after a field redefinition. Even though field redefinitions are an equivalence for the classical theories, one may argue that it does not describe an equivalence for the quantised theory, leading to a possible explanation of the mass gap? But that is just high hope right now :)

Integration (probably no need for the following)

For an integrated version of Def. (B.17) we need to discuss when the new "minimal coupling" of Def. (B.9) behaves nicely under a change of the gauge s . That is, we now want to extend the new definition of ∇^A to a well-defined connection on $K = P \times_\rho V$, especially on the adjoint bundle $K = P \times_{\text{Ad}} \mathfrak{g}$ in our case. (and later maybe generalise this to a \tilde{G} -quotient of a general Lie algebra bundle over P)

Let s' be another (local) gauge such that we have a unique smooth map $g : U \rightarrow G$ such that

$$s' = sg,$$

then we want for well-definedness

$$\nabla^A \Phi = [s, \nabla \phi + \text{ad}(A_s) \cdot \phi] \stackrel{!}{=} [s', \nabla \phi' + \text{ad}(A_{s'}) \cdot \phi'], \quad (\text{B.18})$$

where we have $\Phi = [s, \phi] = [s', \phi']$, especially

$$\phi' = \text{Ad}(g^{-1}) \cdot \phi.$$

Since the new field strength G still transforms via the adjoint under δ_ε (see above), we make the following ansatz

$$A_{s'} = \text{Ad}(g^{-1}) \cdot A_s + \mu, \quad (\text{B.19})$$

where $\mu \in \Omega^1(U; \mathfrak{g})$. Usually, $\mu = g^! \mu_{\tilde{G}}$, the pullback as a form of the Maurer-Cartan-Form $\mu_{\tilde{G}}$ on \tilde{G} . One can then check with some short calculation that Eq. (B.18) is equivalent to

$$\nabla(\text{Ad}(g^{-1}) \cdot \phi) + \text{ad}(\mu) \cdot \text{Ad}(g^{-1}) \cdot \phi \stackrel{!}{=} \text{Ad}(g^{-1}) \cdot \nabla \phi$$

using the definition of $P \times_{\text{Ad}} \mathfrak{g}$. Equivalently,

$$\text{ad}(\mu) = \text{Ad}(g) \circ ()$$

B.2. Axiomatic Yang-Mills gauge theories

Let us discuss where the compatibility conditions may arise from a certain axiomatic point of view.