Curved Yang-Mills gauge theories

Infinitesimal and integrated gauge theory

Simon-Raphael Fischer*

April 7, 2022

National Center for Theoretical Sciences, Mathematics Division, National Taiwan University
No. 1, Sec. 4, Roosevelt Rd., Taipei City 106, Taiwan Room 503, Cosmology Building, Taiwan

Abstract†

2020 MSC: Primary 53D17; Secondary 81T13, 17B99.
Keywords: `Mathematical Gauge Theory`, Differential Geometry, High Energy Physics - Theory, Mathematical Physics

*Email: sfischer@ncts.tw; ORCiD: 0000-0002-5859-2825

†Abbreviations used in this paper: **LGB** for Lie group bundle, **LAB** for Lie algebra bundle.

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1. Introduction

1.1. Basic notations

- With f^*F we denote the pullback/pull-back of the fibre bundles $F \to M$ under a smooth map $f : N \to M$. Similarly we denote the pullbacks of sections of a fibre bundle.

- Let $F \overset{\pi_F}{\to} M$ and $G \overset{\pi_G}{\to} N$ be two fibre bundles over smooth manifolds M and N, and let $\phi : N \to M$ be a smooth map. Furthermore, let us assume we have a morphism $\Phi : G \to F$ of fibre bundles over ϕ, that is, Φ is a smooth map such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d, "\pi_G"] & F \arrow[d, "\pi_F"] \\ N \arrow[r, "\phi"] & M \end{tikzcd}$$

especially, $\pi_F \circ \Phi = \phi \circ \pi_G$. We make often use of that such morphisms have a 1:1 correspondence to **base-preserving** fibre bundle morphisms $\widetilde{\Phi} : G \to \phi^*F$, *i.e.* $\widetilde{\Phi}$ is a smooth map with $\phi^*\pi_F \circ \widetilde{\Phi} = \pi_G$. For $p \in N$ the morphism $\widetilde{\Phi}$ has the form

$$\widetilde{\Phi}_p = (p, \Phi_p),$$

that is,

$$\widetilde{\Phi}_p(g) = \big(p, \Phi_p(g)\big)$$

for all $g \in G_p$, which is well-defined since $\Phi_p(g) \in F_{\phi(p)}$. The map $\widetilde{\Phi} \mapsto \Phi := \mathrm{pr}_2 \circ \widetilde{\Phi}$ is then a bijective map between base-preserving morphisms $G \to \phi^*F$ and morphisms $G \to F$ over ϕ, where pr_2 is the projection onto the second component.

In total, $\widetilde{\Phi}$ is a base-preserving morphism if and only if Φ is a morphism over ϕ; in fact, one defines pullback bundles in such a way that this equivalence holds. Observe that $\widetilde{\Phi}$ is an isomorphism (diffeomorphism) if and only if Φ is a fibre-wise isomorphism (diffeomorphism).

One can extend all of this similarly for more specific types of morphisms like vector bundle-morphisms.

Most of the time we will not mention this 1:1 correspondence explicitly, it should be clear by context. Hence, we will also denote $\widetilde{\Phi}$ by Φ. In fact, we usually calculate with $\widetilde{\Phi}$, while Φ and its diagram may only arise to give an illustration about the geometry.

1.2. Assumed background knowledge

It is highly recommended to have basic knowledge about differential geometry and gauge theory as presented in [1, especially Chapter 1 to 5]; however, sometimes we will still give explicit references to help with more technical details. It can be useful to have knowledge about Lie algebra and Lie group bundles, and even Lie algebroids and Lie groupoids, but we will introduce their basic notions such that it is not necessarily needed to have knowledge about these upfront.

We also often give references about Lie group bundles (LGBs), but the given references are often about Lie groupoids. If the reader has no knowledge about Lie groupoids, then it is important to know that LGBs are a special example of Lie groupoids; Lie groupoids carry "two projections", called **source** and **target**. An LGB is a special example of a Lie groupoid whose source equals the target.[1] If you look into such a reference, then the source and target are often denoted by α and β, or by s and t; simply put both to be the same and identify these with our bundle projection which we often denote by π. In that way it should be possible to read the references without the need to know Lie groupoids. However, we try to re-prove the needed statements such that these types of references could be avoided by the reader.

See also the previous subsection about notions we assume to be known.

2. Basic definitions

Notation as in [1]

- $\widetilde{G}$ Lie group with Lie algebra $\mathfrak{g}$
- M smooth manifold (usually also a spacetime). An open subset of M is usually denoted by U; typically small enough that "everything works out" (especially without further mentioning intersections of given open sets and so on)
- $P \to M$ a principal bundle, a (local) gauge is usually denoted by s, an element of $\Gamma(P)$, sections of P
- V a vector space
- ρ a Lie group representation on V, ρ_* the induced Lie algebra representation on V
- $K \coloneqq P \times_\rho V$ the associated vector bundle induced by P and ρ on V. An element Φ of K is denoted by by $[p, \phi]$ for $p \in P$ and $\phi \in V$, where $[\cdot, \cdot]$ denotes the equivalence class with respect to the equivalence

  $$(p, \phi) \sim \big(pg, \rho\big(g^{-1}\big) \cdot \phi\big)$$

  for all $g \in \widetilde{G}$; pg denotes the canonical group action (from the right) $P \times \widetilde{G} \to P$ and $\cdot$ the action of $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ on V.

[1] But not every Lie groupoid with equal source and target is an LGB, they're in general bundles of Lie groups which is not completely the same; this nuance will not be important here.

- Especially if fixing a local gauge $s : U \to P$ we can write for sections $\Phi \in \Gamma(K)$ locally

$$\Phi|_U = [s, \phi],$$

where $\phi : U \to V$, *i.e.* a local section of the trivial vector bundle $M \times V \to M$.

- We especially focus on $V = \mathfrak{g}$ and $\rho = \mathrm{Ad}$ the adjoint representation of $\widetilde{G}$ on $\mathfrak{g}$.

The field of gauge bosons A is a connection on the principal bundle, *i.e.* an element of $\Omega^1(P; \mathfrak{g})$ with

$$r_g^! A = \mathrm{Ad}_{g^{-1}}(A) \coloneqq \mathrm{Ad}_{g^{-1}} \circ A,$$

$$A\left(\widetilde{X}\right) = X$$

for all $g \in \widetilde{G}$ and $X \in \mathfrak{g}$, where $r_g^!$ is the pullback of forms via the right $\widetilde{G}$-multiplication on P, and $\widetilde{X}$ the fundamental vector field of X on P.

Typically, a lot of the formalism of gauge theory comes from how to define the minimal coupling. So, let us look at this and reinvent it a bit. Usually the covariant derivative/minimal coupling ∇^A of A and $\Phi \in \Gamma(K)$ is locally (w.r.t. to a gauge s) defined by

$$\nabla^A \Phi \coloneqq \left[s, \nabla^A \phi\right],$$

where

$$\nabla^A \phi \coloneqq \mathrm{d}\phi + \rho_*(A_s) \cdot \phi, \tag{1}$$

where $A_s \coloneqq s^! A \in \Omega^1(U; \mathfrak{g})$ (local pullback as a form of A via s) and $\mathrm{d}\phi \coloneqq \nabla^0 \phi$, ∇^0 the canonical flat connection on $M \times V$.

The explicit definition of the field strength F of A is then usually motivated by looking at the curvature R_{∇^A} of ∇^A, that is

$$R_{\nabla^A}(\cdot, \cdot)\Phi|_U = [s, \rho_*(F_s) \cdot \phi],$$

where

$$F_s \coloneqq \mathrm{d}A_s + \frac{1}{2}\left[A_s \overset{\wedge}{,} A_s\right]_{\mathfrak{g}}$$

is the typical local definition of $F_s \in \Omega^2(U; \mathfrak{g})$ with

$$\left[A_s \overset{\wedge}{,} A_s\right]_{\mathfrak{g}}(X, Y) = 2\ [A_s(X), A_s(Y)]_{\mathfrak{g}}$$

for all $X, Y \in \mathfrak{X}(U)$. (The notation F_s is of course due to the fact that $F_s = s^! F$, where F is the curvature of A. But I want to avoid that for now because of what we are about doing to do.) We shortly could denote this also as

$$R_{\nabla^A} \phi = \rho_*(F_s) \cdot \phi \tag{2}$$

3. Motivation

Now: One could question why using $\mathrm{d}\phi = \nabla^0\phi$ in Eq. (1). Thence, let us assume that we have a general vector bundle connection $\widehat{\nabla}$ on the trivial vector bundle $M \times V \to M$. We are going to redefine ∇^A and F locally w.r.t. a gauge s, then discuss how the gauge transformations have to look like to receive definitions independent of the chosen gauge s. This also means that the following discussion is now often local by fixing a gauge without further mentioning it.

Let us first locally redefine $\nabla^A\phi$:

$$\nabla^A\phi \coloneqq \widehat{\nabla}\phi + \rho_*(A_s) \cdot \phi. \tag{3}$$

Motivated by Eq. (2), we want to identify the field strength with the curvature of ∇^A. One can check that we have

$$R_{\nabla^A} = R_{\widehat{\nabla}} + \mathrm{d}^{\widehat{\nabla}}\big(\rho_*(A_s)\big) + \rho_*(A_s) \wedge \rho_*(A_s), \tag{4}$$

where $\mathrm{d}^{\widehat{\nabla}}$ is the exterior covariant derivative of $\widehat{\nabla}$ canonically extended to $\mathrm{End}(V)$, viewing $\rho_*(A_s)$ as an element of $\Omega^1(U; \mathrm{End}(V))$, and where $\rho_*(A_s) \wedge \rho_*(A_s)$ is an element of $\Omega^2(U; \mathrm{End}(V))$ given by

$$\begin{aligned}
\big(\rho_*(A_s) \wedge \rho_*(A_s)\big)(X, Y) &\coloneqq \rho_*\big(A_s(X)\big) \circ \rho_*\big(A_s(Y)\big) - \rho_*\big(A_s(Y)\big) \circ \rho_*\big(A_s(X)\big) \\
&= \big[\rho_*\big(A_s(X)\big), \rho_*\big(A_s(Y)\big)\big]_{\mathrm{End}(V)} \\
&= \rho_*\Big([A_s(X), A_s(Y)]_{\mathfrak{g}}\Big) \\
&= \rho_*\bigg(\frac{1}{2}\big[A_s \stackrel{\wedge}{,} A_s\big]_{\mathfrak{g}}\bigg)(X, Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$.

In order to have a similar shape as in Eq. (2), we now assume that $\widehat{\nabla}$ satisfies the following **compatibility conditions**:

Remark 3.1: Comaptibility conditions

$$R_{\widehat{\nabla}} = \rho_*(\zeta), \tag{5}$$

$$\widehat{\nabla} \circ \rho_* = \rho_* \circ \nabla \tag{6}$$

for some $\zeta \in \Omega^2(M; \mathfrak{g})$ and ∇ a vector bundle connection on the trivial vector bundle $M \times \mathfrak{g} \to M$.

If we want that Eq. (4) has a shape like Eq. (2), it is obvious why we require (5); (6) is needed for the second summand in Eq. (4). Hence, let us study (6), that is

$$\widehat{\nabla}\big(\rho_*(\nu)\big) = \rho_*(\nabla\nu)$$

for all $\nu \in \Gamma(M \times \mathfrak{g})$,[2] especially $\widehat{\nabla}$ is again extended to $\mathrm{End}(V)$ on the left hand side. With this we get

$$\begin{aligned}
\mathrm{d}^{\widehat{\nabla}}\big(\rho_*(A_s)\big)(X,Y) &= \widehat{\nabla}_X\big(\rho_*(A_s(Y))\big) - \widehat{\nabla}_Y\big(\rho_*(A_s(X))\big) - \rho_*\big(A_s([X,Y])\big) \\
&= \rho_*\big(\nabla_X\big(A_s(Y)\big)\big) - \rho_*\big(\nabla_Y\big(A_s(X)\big)\big) - \rho_*\big(A_s([X,Y])\big) \\
&= \rho_*\big(\nabla_X\big(A_s(Y)\big) - \nabla_Y\big(A_s(X)\big) - A_s([X,Y])\big) \\
&= \rho_*\big(\mathrm{d}^{\nabla} A_s\big)(X,Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$. Collecting everything, Eq. (4) has now the following form

$$R_{\nabla^A} = \rho_*\bigg(\mathrm{d}^{\nabla} A_s + \frac{1}{2}\big[A_s \stackrel{\wedge}{,} A_s\big]_{\mathfrak{g}} + \zeta\bigg).$$

So, we have a new form of the field strength, assuming that ∇ and ζ satisfy the compatibility conditions in Remark 3.1. This is precisely the definition of the field strength as in the gauge theory of Thomas and Alexei, that is, we have a new field strength

$$G \coloneqq \mathrm{d}^{\nabla} A_s + \frac{1}{2}\big[A_s \stackrel{\wedge}{,} A_s\big]_{\mathfrak{g}} + \zeta.$$

Furthermore, if we are interested into Yang-Mills gauge theories, then we'd have $K = P \times_{\mathrm{Ad}} \mathfrak{g}$ (the adjoint bundle), and so also $\rho_* = \mathrm{ad}$. In this case we can put $\widehat{\nabla} = \nabla$ and then the compatibility conditions in Remark 3.1 read

$$R_{\nabla} = \mathrm{ad}(\zeta),$$

$$\nabla \circ \mathrm{ad} = \mathrm{ad} \circ \nabla.$$

The second condition precisely gives after a short calculation

$$\nabla\Big([\mu, \nu]_{\mathfrak{g}}\Big) = [\nabla\mu, \nu]_{\mathfrak{g}} + [\mu, \nabla\nu]_{\mathfrak{g}}$$

for all $\mu, \nu \in \Gamma(M \times \mathfrak{g})$, so, ∇ has to be a Lie bracket derivation. So, in this case the compatibility conditions in Remark 3.1 precisely reduce to the compatibility conditions of Alexei's and Thomas's theory! (in the case of Lie algebra bundles; the general theory is more general, formulated on general Lie algebroids)

As a summary:

Remark 3.2: Summary

We have

$$R_{\nabla} = \mathrm{ad}(\zeta), \tag{7}$$

[2]Elements of $\mathfrak{g}$ are viewed as constant sections of $M \times \mathfrak{g}$.

$$\nabla \circ \mathrm{ad} = \mathrm{ad} \circ \nabla, \tag{8}$$

$$G = \mathrm{d}^\nabla A_s + \frac{1}{2}\left[A_s \stackrel{\wedge}{,} A_s\right]_{\mathfrak{g}} + \zeta. \tag{9}$$

In fact, the compatibility conditions lead to a gauge invariant theory: Fix an ad-invariant scalar product κ on $\mathfrak{g}$; then define the Lagrangian by

$$\mathfrak{L}_{\mathrm{YM}} \coloneqq -\frac{1}{2}\kappa\left(G \stackrel{\wedge}{,} *G\right) \tag{10}$$

where $*$ is the Hodge star operator w.r.t. some spacetime metric. (In short, the typical definition, but replace F with G) It is easier to look at the infinitesimal version of the gauge transformations, hence everything with respect to a gauge s now.

In order to derive a formula for these, let us again look at ∇^A. Fix an $\varepsilon \in \Gamma(M \times \mathfrak{g})$, then the infinitesimal gauge transformation $\delta_\varepsilon \phi$ of $\phi \in \Gamma(M \times \mathfrak{g})$ is usually defined by

$$\delta_\varepsilon \phi \coloneqq \rho_*(\varepsilon) \cdot \phi.$$

We fix the infinitesimal gauge trafo $\delta_\varepsilon A$ of A by looking at the gauge trafo of $\nabla^A \phi$ via

$$\begin{aligned}
\delta_\varepsilon \nabla^A \phi &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \left(\nabla^{A + t\delta_\varepsilon A}(\phi + t\delta_\varepsilon \phi)\right) \\
&= \underbrace{\widehat{\nabla}(\delta_\varepsilon \phi)}_{=\left(\widehat{\nabla}(\rho_*(\varepsilon))\right)\cdot\phi + \rho_*(\varepsilon)\cdot\widehat{\nabla}\phi} + \rho_*(\delta_\varepsilon A_s) \cdot \phi + \rho_*(A_s) \cdot \delta_\varepsilon \phi \\
&= \left(\rho_*(\nabla\varepsilon + \delta_\varepsilon A_s) + \rho_*(A_s) \cdot \rho_*(\varepsilon)\right) \cdot \phi + \rho_*(\varepsilon) \cdot \widehat{\nabla}\phi
\end{aligned}$$

using Remark 3.1. We want $\delta_\varepsilon \nabla^A \phi = \rho_*(\varepsilon) \cdot \nabla^A \phi$ which gives

$$\rho_*(\varepsilon) \cdot \nabla^A \phi = \rho_*(\varepsilon) \cdot \widehat{\nabla}\phi + \rho_*(\varepsilon) \cdot \rho_*(A_s) \cdot \phi.$$

Imposing $\delta_\varepsilon \nabla^A \phi = \rho_*(\varepsilon) \cdot \nabla^A \phi$ we get

$$\rho_*\left(\delta_\varepsilon A_s + \nabla\varepsilon + [A_s, \varepsilon]_{\mathfrak{g}}\right) = 0$$

using again that ρ_* is a Lie algebra representation. If we require that this shall work for all ρ_*, we may say

$$\delta_\varepsilon A_s \coloneqq -\nabla\varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}. \tag{11}$$

This is precisely the infinitesimal gauge trafo of A as in the theory of Thomas and Alexei! Hence, we achieve infinitesimal gauge invariance of $\mathfrak{L}_{\mathrm{YM}}$. For completeness, let us check the gauge trafo of G using Def. (11) and Remark 3.2, it is very similar to the "classical" calculation due to Remark 3.2 which is why I skip some straightforward calculations to keep it short,

$$\delta_\varepsilon G = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \left(\mathrm{d}^\nabla(A_s + t\delta_\varepsilon A_s) + \frac{1}{2}\left[A_s + t\delta_\varepsilon A_s \stackrel{\wedge}{,} A_s + t\delta_\varepsilon A_s\right]_{\mathfrak{g}} + \zeta\right)$$

$$= \mathrm{d}^\nabla\left(-\nabla\varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}\right) + \left[A_s \overset{\wedge}{,} -\nabla\varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}$$

$$= \underbrace{-\left(\mathrm{d}^\nabla\right)^2\varepsilon}_{=-R_\nabla\varepsilon=[\varepsilon,\zeta]_{\mathfrak{g}}} + \left[\nabla\varepsilon \overset{\wedge}{,} A_s\right]_{\mathfrak{g}} + \left[\varepsilon, \mathrm{d}^\nabla A_s\right]_{\mathfrak{g}} + \underbrace{\left[A_s \overset{\wedge}{,} -\nabla\varepsilon\right]_{\mathfrak{g}}}_{=-\left[\nabla\varepsilon\overset{\wedge}{,}A_s\right]_{\mathfrak{g}}} + \left[A_s \overset{\wedge}{,} [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}$$

$$= \left[\varepsilon, \mathrm{d}^\nabla A_s + \zeta\right]_{\mathfrak{g}} + \left[A_s \overset{\wedge}{,} [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}$$

and, using the Jacobi identity,

$$\begin{aligned}\left[A_s \overset{\wedge}{,} [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}(X,Y) &= \left[A_s(X), [\varepsilon, A_s(Y)]_{\mathfrak{g}}\right]_{\mathfrak{g}} - \left[A_s(Y), [\varepsilon, A_s(X)]_{\mathfrak{g}}\right]_{\mathfrak{g}} \\ &= \left[\varepsilon, [A_s(X), A_s(Y)]_{\mathfrak{g}}\right]_{\mathfrak{g}} \\ &= \left[\varepsilon, \frac{1}{2}\left[A_s \overset{\wedge}{,} A_s\right]_{\mathfrak{g}}\right]_{\mathfrak{g}}(X,Y)\end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$. Altogether

$$\delta_\varepsilon G = \left[\varepsilon, \mathrm{d}^\nabla A_s + \frac{1}{2}\left[A_s \overset{\wedge}{,} A_s\right]_{\mathfrak{g}} + \zeta\right]_{\mathfrak{g}} = [\varepsilon, G]_{\mathfrak{g}}.$$

Hence, the field strength transforms with the adjoin of ε; since κ is ad-invariant, we can derive that $\mathfrak{L}_{\mathrm{YM}}$ is invariant under the infinitesimal gauge trafo in Def. (11)!

Observe that by Remark 3.2 that ζ can be non-trivial even if we still use $\nabla = \nabla^0$, the canonical flat connection on $M \times \mathfrak{g}$, even though this whole discussion started with allowing more general connections.

If we minimise $\mathfrak{L}_{\mathrm{YM}}$, then one obvious way would be to search solutions with $G \equiv 0$ for an absolute minimum/maximum (because of the sign), doing so would result into that the classical Yang-Mills energy would have a bound which is non-zero. May this be an explanation for the mass gap? As shown in my thesis, every classical theory has a ζ after a field redefinition. Even though field redefinitions are an equivalence for the classical theories, one may argue that it does not describe an equivalence for the quantised theory, leading to a possible explanation of the mass gap? But that is just high hope right now :)

3.1. Integration

For an integrated version of Def. (11) we need to discuss when the new "minimal coupling" of Def. (3) behaves nicely under a change of the gauge s. That is, we now want to extend the new definition of ∇^A to a well-defined connection on $K = P \times_\rho V$, especially on the adjoint bundle $K = P \times_{\mathrm{Ad}} \mathfrak{g}$ in our case. (and later maybe generalise this to a $\widetilde{G}$-quotient of a general Lie algebra bundle over P)

Let s' be another (local) gauge such that we have a unique smooth map $g : U \to G$ such that

$$s' = sg,$$

then we want for well-definedness

$$\nabla^A \Phi = [s, \nabla\phi + \mathrm{ad}(A_s) \cdot \phi] \stackrel{!}{=} \left[s', \nabla\phi' + \mathrm{ad}(A_{s'}) \cdot \phi'\right], \tag{12}$$

where we have $\Phi = [s, \phi] = [s', \phi']$, especially

$$\phi' = \mathrm{Ad}\left(g^{-1}\right) \cdot \phi.$$

Since the new field strength G still transforms via the adjoin under δ_ε (see above), we make the following ansatz

$$A_{s'} = \mathrm{Ad}\left(g^{-1}\right) \cdot A_s + \mu, \tag{13}$$

where $\mu \in \Omega^1(U; \mathfrak{g})$. Usually, $\mu = g^! \mu_{\widetilde{G}}$, the pullback as a form of the Maurer-Cartan-Form $\mu_{\widetilde{G}}$ on $\widetilde{G}$. One can then check with some short calculation that Eq. (12) is equivalent to

$$\nabla\left(\mathrm{Ad}\left(g^{-1}\right) \cdot \phi\right) + \mathrm{ad}(\mu) \cdot \mathrm{Ad}\left(g^{-1}\right) \cdot \phi \stackrel{!}{=} \mathrm{Ad}\left(g^{-1}\right) \cdot \nabla\phi$$

using the definition of $P \times_{\mathrm{Ad}} \mathfrak{g}$. Equivalently,

$$\mathrm{ad}(\mu) = \mathrm{Ad}(g) \circ ()$$

4. Lie group bundles (LGBs)

4.1. Definition

Definition 4.1: Lie group bundle, [2, §1.1, Def. 1.1.19; p. 11]

Let $G, \mathscr{G}, M$ be smooth manifolds. A fibre bundle

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{G} \\ & & \downarrow \pi \\ & & M \end{array}$$

is called a **Lie group bundle** if:

1. G and each fibre $\mathscr{G}_x \coloneqq \pi^{-1}(\{x\})$, $x \in M$, are Lie groups;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} \coloneqq \mathrm{pr}_2 \circ \phi_i|_{\mathscr{G}_x} : \mathscr{G}_x \to G$$

are Lie group isomorphisms, where I is an (index) set, U_i are open sets covering M, $\phi_i : \mathscr{G}|_U \to U \times G$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie group bundle atlas** or **LGB atlas**.

We also often say that $\mathscr{G}$ **is an LGB (over** M**)**, whose structural Lie group is either clear by context or not explicitly needed; and we may also denote LGBs by $G \to \mathscr{G} \xrightarrow{\pi} M$.

Remark 4.2: Principal and Lie group bundles

Beware, a Lie group bundle is **not** the same as a principal bundle $P \to M$ with the same fibre type G. First of all, the fibres of P are just diffeomorphic to a Lie group, a priori they carry no Lie group structure, while the fibres of $\mathscr{G}$ carry a Lie group structure.

Second, on P we have a multiplication given as an action of G on P

$$P \times G \to P,$$

preserving the fibres P_x $(x \in M)$ and simply transitive on them. Restricted on P_x we have

$$P_x \times G \to P_x.$$

For $\mathscr{G}$ we have canonically a multiplication over x given by

$$\mathscr{G}_x \times \mathscr{G}_x \to \mathscr{G}_x,$$

also clearly simply transitive. Observe, the second factor is not "constant", *i.e.* we do not have $\mathscr{G}_x \times G \to \mathscr{G}_x$ in general. Hence, there is in general no well-defined product $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$.

All of that is also resembled in the existence of sections. The existence of a section of P has a 1:1 correspondence to trivializations of P, which is why P in general only admits sections locally; see *e.g.* [1, §4.2, Thm. 4.2.19; page 219f.]. $\mathscr{G}$ clearly admits always a global section, even if $\mathscr{G}$ is non-trivial; just take the section which assigns each base point the neutral element of its fibre.

If M is a point we recover the notion of Lie groups, and, as usual, we have the notion of trivial LGBs:

Example 4.3: Trivial examples

The **trivial LGB** is given as the product manifold $M \times G \to M$ with canonical multiplication $(x, g) \cdot (x, q) \coloneqq (x, gq)$, and we recover the notion of a Lie group in the case of $M = \{*\}$.

We are of course also interested into LGB bundle morphisms:

Definition 4.4: LGB morphism, [2, §1.2, special situation of Def. 1.2.1 & 1.2.3, page 12]

Let $\mathscr{G} \overset{\pi_{\mathscr{G}}}{\to} M$ and $\mathscr{H} \overset{\pi_{\mathscr{H}}}{\to} N$ be two LGBs over two smooth manifolds M and N. An **LGB morphism** is a pair of smooth maps $F : \mathscr{H} \to \mathscr{G}$ and $f : N \to M$ such that

$$\pi_{\mathscr{G}} \circ F = f \circ \pi_{\mathscr{H}}, \tag{14}$$

$$F(gq) = F(g)\ F(q) \tag{15}$$

for all $g, q \in \mathscr{H}$ with $\pi_{\mathscr{H}}(g) = \pi_{\mathscr{H}}(q)$. We then say that F **is an LGB morphism over** f. If $N = M$ and $f = \mathrm{id}_M$, then we often omit mentioning f explicitly and just write that F **is a (base-preserving) LGB morphism**.
We speak of an **LGB isomorphism (over** f**)** if F is a diffeomorphism.

Remarks 4.5.

- The right hand side of Eq. (15) is well-defined because of Eq. (14).
- It is clear that condition 2 in Def. 4.1 is equivalent to say that $\mathscr{G}$ is locally isomorphic to a trivial LGB; as one may have expected already.
- If F is a diffeomorphism, then also f: By Eq. (14) surjectivity of f is clear; for $y \in M$ just take any $g \in \mathscr{G}_y$, and since F is a bijective, we have a $q \in \mathscr{H}_x$ for some $x \in N$ with $F(q) = g$. By Eq. (14) we have $y = \pi_{\mathscr{G}}(F(q)) \overset{(14)}{=} f(x)$, thence, surjectivity follows. For injectivity we know by Eq. (15) and (14) that $F\big(e^{\mathscr{H}}_x\big) = e^{\mathscr{G}}_{f(x)}$, where $e^{\mathscr{H}}_x$ and $e^{\mathscr{G}}_{f(x)}$ denote the unique neutral elements of $\mathscr{H}_x$ and $\mathscr{G}_{f(x)}$, respectively. Assume that there are $x, x' \in N$ with $f(x) = f(x')$, then we can derive

$$F\left(e^{\mathscr{H}}_x\right) = e^{\mathscr{G}}_{f(x)} = e^{\mathscr{G}}_{f(x')} = F\left(e^{\mathscr{H}}_{x'}\right).$$

Then we have $e^{\mathscr{H}}_x = e^{\mathscr{H}}_{x'}$ due to that F is bijective, and hence $x = x'$. Therefore f is bijective. Finally, F^{-1} is by assumption also a diffeomorphism, Eq. (15) clearly carries over, and Eq. (14) is clearly w.r.t. f^{-1}, that is

$$\pi_{\mathscr{H}} \circ F^{-1} = f^{-1} \circ \pi_{\mathscr{G}}.$$

Since $\pi_{\mathscr{H}} \circ F^{-1}$ is smooth and $\pi_{\mathscr{G}}$ is a smooth surjective submersion, it follows that f^{-1} is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We can conclude that f is a diffeomorphism. Observe that we also concluded that F^{-1} is an LGB isomorphism, too.

4.2. Associated Lie group bundles

For another important example recall that there is the notion of associated fibre bundles; following and stating the results of [2, §1, Construction 1.3.8, page 20] and [1, §4.7, page 237ff.;

see also Rem. 4.7.8, page 242f.]: Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G, a smooth manifold N and a smooth left G-action Ψ given by

$$\begin{aligned} G \times N &\to N, \\ (g, v) &\mapsto \Psi(g, v) \coloneqq g \cdot v. \end{aligned}$$

Then we have a right G-action on $P \times N$ given by

$$\begin{aligned} (P \times N) \times G &\to P \times N, \\ (p, v, g) &\mapsto \left(p \cdot g, g^{-1} \cdot v\right), \end{aligned}$$

and one can show that the quotient under this action, $P \times_\Psi N \coloneqq (P \times N)\big/G$, yields the structure of a fibre bundle

$$\begin{array}{ccc} N & \longrightarrow & P \times_\Psi N \\ & & \downarrow{\scriptstyle \pi_{P \times_\Psi N}} \\ & & M \end{array}$$

such that the projection $P \times N \to P \times_\Psi N$ is a smooth surjective submersion, where the projection $\pi_{P \times_\Psi N} : P \times_\Psi N \to M$ is given by

$$\pi_{P \times_\Psi N}([p, v]) \coloneqq \pi_P(p)$$

for all $[p, v] \in P \times_\Psi N$, denoting equivalence classes of (p, v) by square brackets. For $x \in M$, the fibre $(P \times_\Psi N)_x$ is given by $(P_x \times N)\big/G = P_x \times_\Psi N$, and the fibre is diffeomorphic to N by $N \ni v \mapsto [p, v] \in (P \times_\Psi N)_x$ for a fixed $p \in P_x$. We will frequently use this diffeomorphism in the following without further notice.

A very important example are of course associated vector bundles, related to N being a vetor space. We need a similar concept for Lie groups.

Definition 4.6: Lie group representation on Lie groups, [2, special situation of the comment after Ex. 1.7.14, page 47]

Let G, H be Lie groups. Then a **Lie group representation of G on H** is a smooth left action ψ of G on H

$$\begin{aligned} G \times H &\to H, \\ (g, h) &\mapsto \psi_g(h) \coloneqq \psi(g, h) \end{aligned}$$

such that

$$\psi_g(hq) = \psi_g(h)\ \psi_g(q) \tag{16}$$

for all $g \in G$ and $h, q \in H$.

Remark 4.7: Note about labeling

Observe that we have by the definition of group actions

$$\psi_{gg'} = \psi_g \circ \psi_{g'}$$

for all $g, g' \in G$, viewing ψ_g as a map $H \to H$. Therefore we can view the action ψ as a homomorphism

$$G \to \mathrm{Aut}(H),$$

where $\mathrm{Aut}(H)$ is the set of Lie group automorphisms. The similarity to Lie group representations on vector spaces is obvious, thence the name.

This definition is of course also motivated by various references pointing out that Lie group representations define Lie group actions with extra properties; see for example [1, §3, Ex. 3.4.2, page 143f.]. In [2, comments after Ex. 1.7.14, page 47] this definition is also called *action by Lie group isomorphisms*.

With this we can discuss and define associated Lie group bundles.

Theorem 4.8: Associated Lie group bundle as quotient, [1, motivated by vector spaces as in §4, Thm. 4.7.2, page 239f.]

Let G, H be Lie groups, $P \xrightarrow{\pi_P} M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then $\mathscr{H} \coloneqq P \times_\psi H$ is an LGB

$$\begin{array}{ccc} H & \longrightarrow & \mathscr{H} \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

with projection π given by

$$\mathscr{H} \to M,$$

$$[p, h] \mapsto \pi_P(p), \tag{17}$$

and fibres

$$\mathscr{H}_x = P_x \times_\psi H \tag{18}$$

for all $x \in M$, which are isomorphic to H as Lie groups. The Lie group structure on each fibre $\mathscr{H}_x$ is defined by

$$[p, h] \cdot [p, q] \coloneqq [p, hq] \tag{19}$$

for all $h, q \in H$ and $p_x \in P_x$, where $\pi_P(p) = x$.

Remark 4.9: Neutral and inverse elements

The neutral element for $\mathscr{H}_x$ ($x \in M$) is given by

$$e_x = [p, e],$$

where $p \in P_x$ is arbitrary and e is the neutral element of H. This is clearly independent of the choice of p due to

$$[p, e] = \left[p \cdot g, \psi_{g^{-1}}(e)\right] = [p \cdot g, e]$$

for all $g \in G$. Thence, the fact that e_x is the neutral element follows immediately.
The inverse of $[p, h] \in \mathscr{H}_x$ is clearly given by

$$([p, h])^{-1} = \left[p, h^{-1}\right].$$

Proof.

- That π is the well-defined projection and that the fibres are precisely $P_x \times_\psi H$ for all $x \in M$ is well-known, see our discussion before Def. 4.6 and the references therein; it is also very straightforward to check. We also discussed that $\mathscr{H}$ is a fibre bundle with structural fibre H. Hence, if one knows that the proposed group structure in Def. (19) is well-defined, then the smoothness of the group structure is implied by the smoothness structures of H and $\mathscr{H}$. Thence, let us check whether Def. (19) is well-defined. Let $x \in M$, $p \in P_x$ and $p' := p \cdot g'$ be another element of P_x, where $g' \in G$. Also let $[p_1, h_1], [p_2, h_2] \in P_x \times_\psi H$; then we have unique elements q_i, q_i' of G such that ($i \in \{1, 2\}$)

$$p_i = p \cdot q_i, \qquad\qquad p_i = p' \cdot q_i',$$

especially, it follows $q_i = g' q_i'$. On the one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, h_1] \cdot [p_2, h_2] = [p, \psi_{q_1}(h_1)] \cdot [p, \psi_{q_2}(h_2)] = [p, \psi_{q_1}(h_1)\ \psi_{q_2}(h_2)], \tag{20}$$

on the other hand, using Def. 4.6 and $p' = p \cdot g'$ instead of p,

$$\begin{aligned}
[p_1, h_1] \cdot [p_2, h_2] &= \left[p \cdot g', \psi_{q_1'}(h_1)\ \psi_{q_2'}(h_2)\right] \\
&= \Bigg[p,\ \underbrace{\psi_{g'}\Big(\psi_{q_1'}(h_1)\ \psi_{q_2'}(h_2)\Big)}_{=\psi_{g'}\left(\psi_{q_1'}(h_1)\right)\ \psi_{g'}\left(\psi_{q_2'}(h_2)\right)}\ \Bigg] \\
&= \left[p, \psi_{g'q_1'}(h_1)\ \psi_{g'q_2'}(h_2)\right] \\
&= [p, \psi_{q_1}(h_1)\ \psi_{q_2}(h_2)],
\end{aligned}$$

which implies that Def. (19) is well-defined, and thus defines a Lie group structure on each fibre of $\mathscr{H}$.

• That the fibres $\mathscr{H}_x$ are isomorphic to H as Lie groups for all $x \in M$ also quickly follows. Recall by our discussion before Def. 4.6 that the fibres are diffeormorphic to H by $H \ni h \mapsto [p, h] \in \mathscr{H}_x$ for a fixed $p \in P_x$. By Def. (19) it is clear that this map is a Lie group homomorphism and hence a Lie group isomorphism.

• Let us now construct an LGB atlas for $\mathscr{H}$ by using a principal bundle atlas for P. That is, for some $U \subset M$ open and a trivialization $\varphi_U : P|_U \to U \times G$ we write

$$\varphi_U(p) = \big(\pi_P(p), \beta_U(p)\big)$$

for all $p \in P$, where $\beta_U : P|_U \to G$ is an equivariant map, *i.e.* $\beta_U(p \cdot g) = \beta_U(p)\ g$ for all $g \in G$. Then define ϕ_U as a map by

$$\mathscr{H}|_U \to U \times H,$$

$$[p, h] \mapsto \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big).$$

ϕ_U is well-defined: Let $[p', h'] \in \mathscr{H}|_U$ with $[p', h'] = [p, h]$. Then there is a $g \in G$ such that

$$(p', h') = \big(p \cdot g, \psi_{g^{-1}}(h)\big),$$

hence, using the equivariance of β_U and Def. 4.6,

$$\phi_U\big([p', h']\big) = \Big(\underbrace{\pi_P(p \cdot g)}_{=\pi_P(p)}, \underbrace{\big(\psi_{\beta_U(p \cdot g)} \circ \psi_{g^{-1}}\big)}_{=\psi_{\beta_U(p)} \circ \psi_g \circ \psi_{g^{-1}}}(h)\Big) = \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big) = \phi_U\big([p, h]\big),$$

which proves that ϕ_U is well-defined. Denote the projection onto equivalence classes $P \times H \to \mathscr{H}$ by ϖ, then observe

$$\phi_U \circ \varpi = L,$$

where $L_U : P|_U \times H \to U \times H$ is given by $L_U(p, h) := \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big)$ for all $(p, h) \in P|_U \times H$. L_U is clearly smooth and recall that ϖ is a smooth surjective submersion, therefore ϕ_U is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We define a candidate of the inverse $\phi_U^{-1} : U \times H \to \mathscr{H}|_U$ by

$$\phi_U^{-1}(x, h) = \big[\varphi_U^{-1}(x, e), h\big]$$

for all $(x, h) \in U \times H$, where e is the neutral element of G. By the definition of φ_U we immediately get

$$\big(\varphi_U \circ \varphi_U^{-1}\big)(x, e) = \Big(\pi_P\big(\varphi_U^{-1}(x, e)\big), \beta_U\big(\varphi_U^{-1}(x, e)\big)\Big) = (x, e),$$

for all $x \in U$, and, also using again the equivariance of β_U,

$$\varphi_U^{-1}(\pi_P(p), e) = \varphi_U^{-1}\Big(\pi_P\big(p \cdot \beta_U^{-1}(p)\big), \beta_U(p)\ \beta_U^{-1}(p)\Big)$$

$$
\begin{aligned}
&= \varphi_U^{-1}\Big(\pi_P\big(p \cdot \beta_U^{-1}(p)\big), \beta_U\big(p \cdot \beta_U^{-1}(p)\big)\Big) \\
&= \big(\varphi_U^{-1} \circ \varphi_U\big)\big(p \cdot \beta_U^{-1}(p)\big) \\
&= p \cdot \beta_U^{-1}(p)
\end{aligned}
$$

for all $p \in P|_U$. Then

$$
\big(\phi_U \circ \phi_U^{-1}\big)(x, h) = \Big(\pi_P\big(\varphi_U^{-1}(x, e)\big), \psi_{\beta_U\left(\varphi_U^{-1}(x,e)\right)}(h)\Big) = \big(x, \psi_e(h)\big) = (x, h),
$$

for all $(x, h) \in U \times H$, and

$$
\begin{aligned}
\big(\phi_U^{-1} \circ \phi_U\big)([p, h]) &= \big[\underbrace{\varphi_U^{-1}(\pi_P(p), e)}_{= p \cdot \beta_U^{-1}(p)}, \psi_{\beta_U(p)}(h)\big] \\
&= [p, h]
\end{aligned}
$$

for all $[p, h] \in \mathscr{H}|_U$. Thus, ϕ_U is bijective; additionally observe

$$
\phi_U^{-1}(x, h) = \varpi\big(\varphi_U^{-1}(x, e), h\big)
$$

such that ϕ_U^{-1} is clearly smooth as the composition of smooth maps, and we therefore conclude that ϕ_U is a diffeomorphism. Finally, derive with Def. 4.6 and Eq. (20) that

$$
\begin{aligned}
(\mathrm{pr}_2 \circ \phi_U)\big([p_1, h_1] \cdot [p_2, h_2]\big) &= (\mathrm{pr}_2 \circ \phi_U)\big([p, \psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)]\big) \\
&= \psi_{\beta_U(p)}\big(\psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)\big) \\
&= \underbrace{\psi_{\beta_U(p)}\big(\psi_{q_1}(h_1)\big)}_{= \psi_{\beta_U(p) \cdot q_1}(h)} \cdot \psi_{\beta_U(p)}\big(\psi_{q_2}(h_2)\big) \\
&= \psi_{\beta_U(p_1)}(h) \cdot \psi_{\beta_U(p_2)}(h) \\
&= (\mathrm{pr}_2 \circ \phi_U)\big([p_1, h_1]\big) \cdot (\mathrm{pr}_2 \circ \phi_U)\big([p_1, h_1]\big)
\end{aligned}
$$

for all $[p_1, h_1], [p_2, h_2] \in \mathscr{H}_x$, where we used again the equivariance of β_U and the same notation as introduced for Eq. (20), and pr_2 denotes the projection onto the second factor. Thence, $\mathrm{pr}_2 \circ \phi_U$ induces Lie group isomorphisms $\mathscr{H}_x \to H$ for all $x \in U$; by Def. 4.1 we can finally conclude that $\mathscr{H}$ is an LGB. ■

Hence, we define:

Definition 4.10: Associated Lie group bundle, labeling similar to [1, §4.7, Def. 4.7.3, page 240]

Let G, H be Lie groups, $P \overset{\pi_P}{\to} M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then we call the LGB

$$\mathscr{H} \coloneqq P \times_\psi H = (P \times H) \Big/ G$$

the **Lie group bundle (LGB) associated** to the principal bundle P and the representation ψ on H:

$$\begin{array}{ccc} H & \longrightarrow & P \times_\psi H \\ & & \downarrow{\scriptstyle \pi_{\mathscr{H}}} \\ & & M \end{array}$$

The special situation of $H = G$ is already an important example:

Example 4.11: Inner group bundle, [2, §1, paragraph after Def. 1.1.19, page 11; comment after Construction 1.3.8, page 20]

The **inner group bundle** or **inner LGB** of a principal bundle $P \to M$, denoted by $c_G(P)$, is defined by

$$c_G(P) \coloneqq P \times_{c_G} G, \tag{21}$$

where $c_G : G \times G \to G$ is the left action of G on itself given by the very well-known **conjugation**

$$c_G(g, h) \coloneqq c_g(h) = \left(L_g \circ R_{g^{-1}}\right)(h) = ghg^{-1} \tag{22}$$

for all $g, h \in G$, where we also denote left- and right-multiplications (with g) by L_g and R_g, respectively; see *e.g.* [1, beginning of §1.5.2, page 40f.] for its common properties. It is well-known that c_G satsfies the properties of a Lie group representation of G on itself in the sense of Def. 4.6.
$c_G(P)$ is an LGB by Thm. 4.8.

5. LGB actions, part I

5.1. Definition

As for Lie groups, we are interested into their actions. The idea is the following, similar to [2, §1.6, discussion around Def. 1.6.1, page 34]: We have an LGB $\mathscr{G} \to M$ over a smooth manifold

M, and we want to construct an action of $\mathscr{G}$ on another smooth manifold N. Each fibre of $\mathscr{G}$ is a Lie group, and we have a notion of Lie groups actions on manifold N. Therefore one could define an LGB action as a collection of Lie group actions, that is, only sections of $\mathscr{G}$ act on N; however, one then expects that the general outcome of a product of $\Gamma(\mathscr{G})$ on N would be smooth maps from M to N. In order to recover a typical structure of action one could instead introduce a "multiplication rule", *i.e.* each point $p \in N$ can only be multiplied with elements of a specific fibre of $\mathscr{G}$. This "multiplication rule" will be described by a smooth map $f : N \to M$ in the sense of that the fibre over $f(p)$ will act on p.

For this recall that there is the notion of pullbacks of fibre bundles, see *e.g.* [1, §4.1.4, page 203ff.; especially Thm. 4.1.17, page 204f.]. That is, if we additionally have a smooth manifold N and a smooth map $f : M \to N$, then we have the pullback $f^*\mathscr{G}$ of $\mathscr{G}$ as a fibre bundle defined as usual by

$$f^*\mathscr{G} \coloneqq \{(x, g) \in N \times \mathscr{G} \mid f(x) = \pi(g)\}. \tag{23}$$

It is an embedded submanifold of $N \times \mathscr{G}$, and the structural fibre is the same Lie group as for $\mathscr{G}$. That is, the following diagram commutes

$$\begin{tikzcd} f^*\mathscr{G} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathscr{G} \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

where π_1 and π_2 are the projections onto the first and second factor, respectively, of $N \times \mathscr{G}$. Actually, $f^*\mathscr{G}$ carries a natural structure as an LGB.

Corollary 5.1: Pullbacks of LGBs are LGBs, [2, §2.3, simplified situation of the discussion around Prop. 2.3.1, page 63ff.]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Then $f^*\mathscr{G}$ has a unique (up to isomorphisms) LGB structure such that the projection $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$ onto the second factor is an LGB morphism over f with $\pi_2|_x : (f^*\mathscr{G})_x \to \mathscr{G}_{f(x)}$ being a Lie group isomorphism for all $x \in N$.

Remarks 5.2.
The mentioned reference, [2, §2.3, discussion around Prop. 2.3.1, page 63ff.], is rather general, formulated for Lie groupoids. If the reader is only interested into LGBs, then see *e.g.* [3, §3, Thm. 3.1].

Proof.
By construction, the structural fibre of $f^*\mathscr{G}$ is the same Lie group G as for $\mathscr{G}$, and for all $x \in N$ we have $(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)}$, thence, the fibres are Lie groups and the fibrewise group multiplication has the form

$$(x, g) \cdot (x, q) = (x, gq) \tag{24}$$

for all $x \in N$ and $g, q \in (f^*\mathscr{G})_x$. We are left to show the existence of an LGB atlas. For this fix an LGB atlas $\{(U_i, \phi_i\}_{i\in I}$ of $\mathscr{G}$, where I is an (index) set, $(U_i)_{i\in I}$ an open covering of M, and $\phi_i : \mathscr{G}|_{U_i} \to U_i \times G$ are LGB isomorphisms. Then $f^{-1}(U_i)$ gives rise to an open covering of N, and we get

$$f^*\phi_i : f^*\mathscr{G}|_{f^{-1}(U_i)} \to f^{-1}(U_i) \times G,$$

$$(x, g) \mapsto \big(x, \phi_{i,f(x)}(g)\big),$$

where $\phi_{i,f(x)} : \mathscr{G}_{f(x)} \to G$ are the Lie group isomorphisms as defined in Def. 4.1. It is immediate by construction that this gives an LGB atlas.

That this is the unique (up to isomorphisms) LGB structure such that $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$ is an LGB morphism over f inducing a Lie group isomorphism on each fibre simply follows by construction; observe for all $x \in N$ that $\pi_2|_x$ is clearly bijective. Furthermore, LGB morphisms need to be homomorphisms, which means here

$$\pi_2\big((x, g) \cdot (x, q)\big) \stackrel{!}{=} \pi_2((x, g)) \cdot \pi_2((x, q)) = gq = \pi_2\big((x, gq)\big)$$

for all $x \in N$ and $g, q \in (f^*\mathscr{G})_x$. By using the bijectivity of $\pi_2|_x$, the group structure leading to this is uniquely the one provided in Eq. (24). Especially, π_2 is a homomorphism with the provided structure. Assume we have another LGB chart ψ_i on (a subset of) $f^{-1}(U_i)$, then

$$\phi_i \circ \pi_2 \circ \psi_i^{-1} = \underbrace{\phi_i \circ \pi_2 \circ (f^*\phi_i)^{-1}}_{=(f, \mathbb{1}_G)} \circ f^*\phi_i \circ \psi_i^{-1} = (f, \mathbb{1}_G) \circ f^*\phi_i \circ \psi_i^{-1},$$

If evaluating this at $x \in f^{-1}(U_i)$, then all parts are bijective, and thus the condition about $\pi_2|_x$ being a Lie group isomorphism enforces that ψ_i is an LGB atlas compatible with $f^*\phi_i$. This concludes the proof. ∎

Let us now define $\mathscr{G}$-actions.

Definition 5.3: Lie group bundle actions, [2, §1.6, special case of Def. 1.6.1, page 34]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Then a **right-action of $\mathscr{G}$ on** N is a smooth map

$$f^*\mathscr{G} \to N,$$

$$(p, g) \mapsto p \cdot g,$$

satisfying the following properties:

$$f(p \cdot g) = \pi(g), \tag{25}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{26}$$

$$p \cdot e_{f(p)} = p \tag{27}$$

for all $p \in N$ and $g, h \in \mathscr{G}_{f(p)}$, where $e_{f(p)}$ is the neutral element of $\mathscr{G}_{f(p)}$.

We analogously define left-actions, and we often write (left or right) **$\mathscr{G}$-action on** N. Furthermore, in order to increase readability as long as the dependency on f is not important, we introduce the notation

$$N * \mathscr{G} := f^* \mathscr{G}, \tag{28}$$

such that the action's notation has the typical shape $N * \mathscr{G} \to N$. For left actions similarly $\mathscr{G} * N \to N$.

Remark 5.4: Relation to the structure of the canonical pullback Lie group bundle over N

Observe that by the definition of $f^*\mathscr{G}$ we can also write

$$f(p \cdot g) = f(p),$$

so, the $\mathscr{G}$-action is defined in such a way that f is invariant under it. Moreover, the fibre-wise group structure on $\mathscr{G}$ naturally defines a $\mathscr{G}$-action on $\mathscr{G}$; in this situation f would be π itself. This is mainly a technical condition. On one hand, having $M = \{*\}$ already recovers the notion of a Lie group action and condition (25) is then trivial, and on the other hand the mentioned reference, [2, §1.6, Def. 1.6.1, page 34], actually generalizes this condition making use of the structure of groupoids.

Furthermore, the other conditions are the typical conditions for actions, especially such that we get a $\mathscr{G}$-action on $f^*\mathscr{G}$ by

$$(p, g) \cdot q := \left(p \cdot q, q^{-1} g\right) \tag{29}$$

for all $p \in N$ and $g, q \in \mathscr{G}_{f(p)}$.[a] As usual, this gives rise to an equivalence relation, whose set of equivalence classes $f^*\mathscr{G} \big/ G$ is isomorphic to N (as a set) by $[p, g] \mapsto p \cdot g$, where we denote equivalence classes of $(p, g) \in f^*\mathscr{G}$ by $[p, g]$. All of this is straight-forward to check. Finally, observe the similarity to associated fibre bundles.

[a] In alignment to Def. 5.3, this action is a map $(f \circ \pi_1)^*\mathscr{G} \to f^*\mathscr{G}$, where π_1 is the projection onto the first factor in $f^*\mathscr{G}$.

Remark 5.5: Localizing LGB actions

We can actually localize the LGB action, but in general not with respect to any open neighbourhood of N since that is in general not possible in a non-trivial way, *i.e.* the action cannot be brought into the form $(N * \mathscr{G})|_U \to U$ for arbitrary U because $p \cdot g$ may for example leave an neighbourhood U of p. However, with respect to M this is possible: Fix any open neighbourhood U of M. Then $f^{-1}(U)$ is an open neighbourhood of N, and we can restrict the action to $f^{-1}(U)$, resulting into a map

$$(N * \mathscr{G})|_{f^{-1}(U)} \to f^{-1}(U),$$

because of $f(p \cdot g) \overset{\text{Eq. (25)}}{=} \pi(g) = f(p)$, that is, if $(p, g) \in (N * \mathscr{G})|_{f^{-1}(U)}$, then $p \in f^{-1}(U)$ and so $p \cdot g \in f^{-1}(U)$. In fact, by the definition of $N * \mathscr{G}$ as $f^*\mathscr{G}$, this describes a $\mathscr{G}|_U$-action on $f^{-1}(U)$

$$f^{-1}(U) * \mathscr{G}|_U \coloneqq (f|_U)^*(\mathscr{G}|_U) \to f^{-1}(U).$$

If $x \in M$ is a regular value of f, then $f^{-1}(\{x\})$ is an embedded submanifold of N due to the regular value theorem (for this see *e.g.* [1, §A.1, Thm. A.1.32, page 611]). In that case one could apply the same arguments to restrict the action on $f^{-1}(\{x\})$. Since $\mathscr{G}_x$ is a Lie group one actually gets a typical Lie group action on $f^{-1}(\{x\})$. For more details about this see Ex. 5.12 later.

Remark 5.6: Left- and right-actions

In the following we usually define everything with respect to right-actions; however, one can of course define the same for left actions in a similar manner. If we ever speak of a left action, then we assume precisely this. Some subtle changes like a sign change will be pointed out though.

One can probably see that it is straightforward to extend a lot of the typical notions of Lie group actions to LGB actions; hence, we mainly focus on the definitions and properties which we need in this paper.

Definition 5.7: Left and right translations, [1, §3.2, notation similar to Def. 3.2.3, page 131] [2, §1.4, special situation of Def. 1.4.1 and its discussion, page 22]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f: N \to M$ a smooth map. Furthermore assume that we have a right action $N * \mathscr{G} \to N$. We define the **right**

translation over $x \in M$ with $g \in \mathscr{G}$ as a map r_g defined by

$$f^{-1}(\{x\}) \to f^{-1}(\{x\}),$$

$$p \mapsto p \cdot g,$$

and we define the **orbit map** through $p \in f^{-1}(\{x\})$ as a map Φ_p given by

$$\mathscr{G}_x \to N,$$

$$g \mapsto p \cdot g.$$

For $\sigma \in \Gamma(\mathscr{G})$ we define the **right translation** on N as a map r_σ by

$$N \to N,$$

$$p \mapsto p \cdot \sigma_{f(p)}.$$

If one has a smooth map $\tau : M \to N$, $x \mapsto \tau_x$, with $f \circ \tau = \mathbb{1}_M$, then we can define the **orbit map** through τ as a map Φ_τ given by

$$\mathscr{G} \to N,$$

$$g \mapsto \tau_{\pi(g)} \cdot g.$$

Remark 5.8: Left action and translation

Similarly we define left translations for left actions, which we similarly denote by l_g and l_σ. By Rem. 5.5 we can define r_σ (and l_σ) also for local sections $\sigma \in \Gamma(\mathscr{G}|_U)$ by restricting N onto $f^{-1}(U)$, where U is some open subset of M; then $r_\sigma, l_\sigma : f^{-1}(U) \to f^{-1}(U)$. In the same manner one achieves a restriction for $\Phi_\tau : \mathscr{G}|_U \to f^{-1}(U)$, if $\tau : U \to f^{-1}(U)$. In case of N being $\mathscr{G}$ itself we will denote right (and left) translations via capital letters.

Remarks 5.9.
Similar to the arguments in [1, §3.2, discussion after Def. 3.2.3, page 131], Φ_p is given by the composition of smooth maps

$$\mathscr{G}_x \to N * \mathscr{G} \to N,$$

$$g \mapsto (p, g) \mapsto p \cdot g.$$

The second arrow/map is smooth due to the fact that we have a smooth action; the first one is smooth because $N * \mathscr{G}$ is the pullback LGB $f^*\mathscr{G}$ and the first arrow is precisely the embedding of $\mathscr{G}_x$ into $f^*\mathscr{G}$ as a fibre over p; recall *e.g.* Cor. 5.1.

For the right-translation r_σ we have a similar argument, namely, r_σ is a composition of smooth maps

$$N \to N * \mathscr{G} \to N,$$

$$p \mapsto \left(p, \sigma_{f(p)}\right) \mapsto p \cdot \sigma_{f(p)}.$$

The first map describes now a section of $N * \mathscr{G} = f^* \mathscr{G}$, and thus an embedding. Thus, smoothness follows again. Similarly, Φ_τ is the composition of maps

$$\mathscr{G} \to N * \mathscr{G} \to N,$$

$$g \mapsto \left(\tau_{\pi(g)}, g\right) \mapsto \tau_{\pi(g)} \cdot g,$$

and the first arrow is clearly a smooth map $\mathscr{G} \to N \times \mathscr{G}$ with values in $f^* \mathscr{G} = N * \mathscr{G}$ which is an embedded submanifold of $N \times \mathscr{G}$. Thence, smoothness follows as usual. In fact, $\Phi_\tau|_{\mathscr{G}_x} = \Phi_{\tau_x}$.

However, for r_g smoothness can only be discussed if $f^{-1}(\{x\})$ is a smooth manifold. That is for example the case if x is a regular value of f; recall the regular value theorem as cited in [1, §A.1, Thm. A.1.32, page 611]. This would be the case if *e.g.* f is a submersion. If x is a regular value, then $f^{-1}(\{x\})$ is an embedded submanifold of N, and r_g is a similar composition of smooth maps as for r_σ but restricted to $f^{-1}(\{x\})$

$$f^{-1}(\{x\}) \to N * \mathscr{G}|_{f^{-1}(\{x\})} \to f^{-1}(\{x\}),$$

$$p \mapsto (p, g) \mapsto p \cdot g.$$

Since $f^{-1}(\{x\})$ is an embedded submanifold, $N * \mathscr{G}|_{f^{-1}(\{x\})}$ is also a fibre bundle, see for example [1, §4.1, Lemma 4.1.16, page 204], and trivially an embedded submanifold of $N * \mathscr{G}$. Altogether, the same arguments as for r_σ apply.

Last but not least, r_σ is clearly a diffeomorphism with inverse $r_{\sigma^{-1}}$, where $\left(\sigma^{-1}\right)_x = (\sigma_x)^{-1} =: \sigma_x^{-1}$. Similarly for r_g if $f^{-1}(\{x\})$ is an embedded submanifold (otherwise r_g is just a bijection). Analogously for l_σ and l_g in case of a left action.

Motivated by the previous remark, it might be hence useful to require that f is a submersion, or that N is actually some bundle over M and f its projection. In fact, this will be later the case.

5.2. Examples of LGB actions

If M is a point or f a constant map, then we recover the typical notion of a Lie group action acting on N. Additionally, we have the following examples, the last two of which will be important in this paper. The notation will be as in Def. 5.7.

Example 5.10: LGB acting on itself

For each LGB $\mathscr{G} \xrightarrow{\pi} M$ acts on itself from the left and right, having $N := M$ and $f := \pi$,

$$\mathscr{G} * \mathscr{G} := \pi^* \mathscr{G} \to \mathscr{G},$$

$$(g, h) \mapsto gh.$$

That this satisfies all properties for an LGB action is clear by definition of LGBs; however, let us give a note about the smoothness of this action. Recall that an LGB is locally isomorphic to a trivial LGB $U \times G$ (U an open subset of M) with its canonical group multiplication, $(x, g) \cdot (x, q) = (x, gq)$. Hence, using that the multiplication of G is smooth and using local LGB trivializations of $\mathscr{G}$ and $f^* \mathscr{G}$ (recall Cor. 5.1 and its proof to show that $f^* \mathscr{G}$ is locally diffeomorphic to the product manifold $U \times G \times G$), we achieve smoothness of the $\mathscr{G}$-action on itself because it is locally of the form

$$U \times G \times G \to U \times G,$$

$$(x, g, q) \mapsto (x, gq).$$

We will also call the $\mathscr{G}$-action on itself the **multiplication in $\mathscr{G}$**.

Example 5.11: Trivial action, [2, §1.6, special situation of Ex. 1.6.3, page 35]

The projection π_1 onto the first factor of $f^* \mathscr{G} \xrightarrow{\pi_1} N$ satisfies the properties of a right $\mathscr{G}$-action on N, that is, the action is given by

$$N * \mathscr{G} \to N,$$

$$(p, g) \mapsto p \cdot g := p.$$

That this action satisfies the properties of an action for all f is trivial, hence we call it the **trivial action**.

Example 5.12: Actions of trivial LGBs

Assume that $\mathscr{G}$ is trivial, that is $\mathscr{G} \cong M \times G$ as LGBs, where G is the structural Lie group of $\mathscr{G}$. In that case, for $p \in N$, the product $p \cdot q$ is only defined if $q \in \mathscr{G}$ is of the form $(f(p), g)$, where $g \in G$; for this recall that (p, q) needs to be an element of $N * \mathscr{G} = f^* \mathscr{G}$ in order to define $(p, q) \mapsto p \cdot q$. We also have

$$f^* \mathscr{G} \cong N \times G,$$

the trivial G-LGB over N. Hence, let us define a map by

$$\begin{aligned} N \times G &\to N, \\ (p, g) &\mapsto p \cdot g := p \cdot (f(p), g), \end{aligned}$$

which is clearly a smooth map since it is a composition of the $\mathscr{G}$-action on N and

$$\begin{aligned} N \times G &\to f^*\mathscr{G}, \\ (p, g) &\mapsto (p, f(p), g). \end{aligned}$$

The latter is smooth because $(p, g) \mapsto (p, f(p), g)$ is a smooth map $N \times G \to N \times M \times G$ and $f^*\mathscr{G}$ is an embedded submanifold of $N \times M \times G$. Using Def. 5.3, It is trivial to see that $p \cdot e = p$, and

$$p \cdot (gh) = p \cdot \underbrace{(f(p), gh)}_{=(f(p),g)\cdot(f(p),h)} = \big(p \cdot (f(p), g)\big) \cdot (f(p), h) = (p \cdot g) \cdot h$$

for all $g, h \in G$. Hence, we have a G-action on N, and by construction it is equivalent to the $\mathscr{G}$-action on N. Due to the discussion in Rem. 5.5 we can therefore conclude that every $\mathscr{G}$-action is locally a typical Lie group action. If f is a submersion, then the action is also a G-action on each fibre.
Observe, that one can therefore recover the notion of Lie group actions not only via $M = \{*\}$, the point manifold, but also via trivial LGBs. Translations with constant sections of $M \times G$ w.r.t. the action $N * \mathscr{G} \to N$ are trivially to be seen as translations with an element of G w.r.t. the action $N \times G \to N$.

Hence, if one wants something "truly" new, then one has to look at global structures of LGBs and their actions. In fact, the following example will provide such a new structure, which we will understand later once we have introduced the physical theory.

Example 5.13: Inner group bundle acting on associated fibre bundles, [2, §1.6, simplified version of Ex. 1.6.4, page 35]

Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G over a smooth manifold M, and recall Ex. 4.11. Furthermore, let F be another smooth manifold, equipped with a smooth left G-action $\Psi : G \times F \to F$. In total we have two associated bundles over M:

$$\begin{tikzcd} G \arrow[r] & c_G(P) \arrow[d, "\pi_{c_G(P)}"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r] & \mathscr{F} := P \times_\Psi F \arrow[d, "\pi_{\mathscr{F}}"] \\ & M \end{tikzcd}$$

the inner group bundle of P and an associated F-bundle, respectively.

Then we have a right $c_G(P)$-action on $\mathscr{F}$ given by

$$\mathscr{F} * c_G(P) \coloneqq \pi_{\mathscr{F}}^* c_G(P) \to \mathscr{F},$$

$$\big([p,v],[p,g]\big) \mapsto [p,\Psi(g,v)] = [p\cdot g, v]$$

for all $p \in P_x$ ($x \in M$), $g \in G$ and $v \in F$.

Proof.

• We first check again that the action is well-defined, that is, we are going to prove that the action is independent of the choice of fixed point in P_x. Thence, let $x \in M$, $p \in P_x$ and $p' \coloneqq p \cdot g'$ be another element of P_x, where $g' \in G$. Also let $[p_1, v] \in \mathscr{F}_x$ and $[p_2, g] \in c_G(P)_x$; then we have unique elements q_i, q_i' of G such that ($i \in \{1,2\}$)

$$p_i = p \cdot q_i, \qquad\qquad p_i = p' \cdot q_i',$$

especially, it follows $q_i = g' q_i'$.

On one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, v] \cdot [p_2, g] = [p, \Psi(q_1, v)] \cdot [p, c_{q_2}(g)] = [p \cdot c_{q_2}(g), \Psi(q_1, v)] = [p \cdot c_{q_2}(g)\ q_1, v].$$

On the other hand, using p' as a fixed element, we derive, using $q_i' = (g')^{-1} q_i$,

$$[p_1, v] \cdot [p_2, h] = \left[p' \cdot c_{q_2'}(g)\ q_1', v\right] = \left[p \cdot g' q_2' g \left(q_2'\right)^{-1} q_1', v\right] = \left[p \cdot q_2 g q_2^{-1} q_1, v\right] = [p \cdot c_{g_2}(g)\ q_1, v],$$

which finalizes the argument needed to show that the action is well-defined.

• Let us now quickly check that the conditions in Def. 5.3 are satisfied. We have

$$\pi_{\mathscr{F}}\big([p,v]\cdot[p,g]\big) = \pi_{\mathscr{F}}([p,\Psi(g,v)]) = \pi_P(p) = \pi_{c_G(P)}\big([p,g]\big)$$

for all $p \in P_x$ ($x \in M$), $v \in F$ and $g \in G$; similarly, having additionally $h \in G$,

$$\big([p,v]\cdot[p,g]\big)\cdot[p,h] = [p\cdot g, v]\cdot[p,h] = [p\cdot gh, v] = [p,v]\cdot[p,gh] = [p,v]\cdot\big([p,g]\ [p,h]\big),$$

and

$$[p,v]\cdot[p,e] = [p\cdot e, v] = [p,v].$$

Therefore this describes an action. ■

Remark 5.14: Relation to automorphisms of principal bundles and gauge transformations

Recall that gauge transformations have a strong relation to principal bundle automorphisms f of the principal bundle P; see *e.g.* [1, §5.3, Def. 5.3.1, page 256f.] and [1, §5.4, Thm. 5.4.4, page 273]. That is, f is a diffeomorphism $P \to P$ with

$$\pi_P \circ f = \mathbb{1}_M,$$

$$f(p \cdot g) = f(p) \cdot g$$

for all $p \in P$ and $g \in G$. The group of such maps will be denoted by $\mathscr{Aut}(P)$. One can identify such automorphisms with certain G-valued maps on P, following [1, §5.3, Def. 5.3.2 & Prop. 5.3.3, page 266f.]: We define the following set of smooth maps $P \to G$ by

$$C^\infty(P;G)^G := \left\{\sigma : P \to G \text{ smooth} \;\middle|\; \sigma(p \cdot g) = c_{g^{-1}}\big(\sigma(p)\big) \text{ for all } p \in P, g \in G\right\}.$$

It is straightforward to check that this is a group w.r.t. pointwise multiplication. Furthermore, there is a group isomorphism

$$\mathscr{Aut}(P) \to C^\infty(P;G)^G,$$

$$f \mapsto \sigma_f,$$

where σ_f is defined by

$$f(p) = p \cdot \sigma_f(p)$$

for all $p \in P$; one can prove that this is well-defined.
As argued in [1, §5.3, Thm. 5.3.8, page 269; formulated as left action there, which is why we have an inverse here], $\mathscr{Aut}(P)$ acts (on the right) on associated fibre bundles $\mathscr{F} = P \times_\Psi F$ by

$$[p, v] \cdot f := \left[f^{-1}(p), v\right] = \left[p \cdot \sigma_f(p)^{-1}, v\right]$$

for all $[p, v] \in \mathscr{F}_x$ ($x \in M$) and $f \in \mathscr{Aut}(P)$. σ_f can also be just locally defined, therefore one could investigate whether there is also an action just with an element g of G, basically the restriction of σ_f onto the fibre P_x. However, the action given by $[p, v] \cdot g = \left[p \cdot g^{-1}, v\right]$ for $g \in G$ is in general clearly only well-defined w.r.t. a change of the representative of $[p, v] = \left[p \cdot q, \Psi_{q^{-1}}(v)\right]$ ($q \in G$), if G is abelian. But one can resolve this by looking at it carefully: The rough idea is that g basically comes from $\sigma_f(p)$ in this context, but

$$\sigma_f(p \cdot q) = c_{q^{-1}}\big(\sigma_f(p)\big).$$

Roughly, while p is multiplied with g^{-1}, $p \cdot q$ has to be multiplied with $q^{-1}g^{-1}q$. It is easy to check that this resolves that issue, and the result is precisely the action described in Ex. 5.13. In fact, we have the following proposition:

For the following proposition observe that the (local) sections of an LGB have a group structure given by pointwise multiplication.

Proposition 5.15: Gauge transformations as sections of the inner LGB, [2, §1.4, (the last sentence of) Ex. 1.4.7, page 25]

Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G over a smooth manifold M. Then there is a group isomorphism

$$\mathscr{Aut}(P) \to \Gamma\big(c_G(P)\big),$$

$$f \mapsto q_f$$

where $g_f \in \Gamma\big(c_G(P)\big)$ is defined by

$$q_f\big|_x := \big[p, \sigma_f(p)\big]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$, and σ_f is the element of $C^\infty(P;G)^G$ corresponding to f as introduced in Rem. 5.14.

Remarks 5.16.
As one may guess, $\Gamma\big(c_G(P)\big)$ is the analogue of $C^\infty(M;G)^G$ such that one could ask for a more direct analogue to $\mathscr{Aut}(P)$. Indeed, as argued in [2, §1.3, Prop. 1.3.9, page 20], $c_G(P)$ is actually isomorphic to $(P \times_M P)\big/G$, where $P \times_M P := \pi_P^* P$, and the G-action is the diagonal action on $P \times P$. One can prove that an isomorphism is given by

$$c_G(P) \to (P \times_M P)\big/G,$$

$$[p, g] \mapsto [p, p \cdot g].$$

It is also argued in [2, §1.4, Ex. 1.4.7, page 25] that $\mathscr{Aut}(P)$ is then isomorphic to $\Gamma\Big((P \times_M P)\big/G\Big)$ by

$$\mathscr{Aut}(P) \to \Gamma\Big((P \times_M P)\big/G\Big),$$

$$f \mapsto L_f,$$

where $L_f \in \Gamma\Big((P \times_M P)\big/G\Big)$ is given by

$$L_f\big|_x := [p, f(p)] = [p, p \cdot \sigma_f(p)]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$. This is clearly well-defined, and, so, while $c_G(P)$ is the bundle-analogue of $C^\infty(P;G)^G$ one can think of $(P \times_M P)\big/G$ as the bundle-analogue of $\mathscr{Aut}(P)$.

However, this description often arises if one wants to use the formalism of groupoids and algebroids, here especially using the **gauge groupoid** and **Atiyah algebroid** induced by P.

These would allow an even more elegant version of the gauge transformations, however, we intend to write this paper in such a way that there is no need that the reader has knowledge about those bundle structures. See the cited references for more details in that regard.

Proof of Prop. 5.15.

• Let us first quickly check whether $g_f \in \Gamma\big(c_G(P)\big)$ is well-defined for all $f \in \mathscr{Aut}(P)$. For $p \in P_x$ $(x \in M)$ we have

$$q_f|_x = [p, \sigma_f(p)],$$

If $p' = p \cdot g$ $(g \in G)$ is another element of P_x, then, using p' to define q_f,

$$q_f|_x = [p \cdot g, \sigma_f(p \cdot g)] = \big[p \cdot g, c_{g^{-1}}\big(\sigma_f(p)\big)\big] = [p, \sigma_f(p)],$$

also using the definition of $c_G(P)$, recall Ex. 4.11. It follows that q_f is well-defined, and it is clear that q_f is smooth.

• We want to show that $f \mapsto q_f$ is a group isomorphism by using that it is a composition of the group isomorphisms $\mathscr{Aut} \to C^\infty(P;G)^G$ as in Rem. 5.14 and

$$C^\infty(P;G)^G \to \Gamma\big(c_G(P)\big),$$

$$\sigma \mapsto q_\sigma, \tag{30}$$

where q_σ is effectively the same definition as q_f, that is $q_\sigma|_x = [p, \sigma(p)]$ which is well-defined by the very same reasons as before. It is only left to show that $C^\infty(P;G)^G \to \Gamma\big(c_G(P)\big)$ is a group isomorphism. For injectivity let σ' be another element of $C^\infty(P;G)^G$ and assume $[p, \sigma(p)] = [p, \sigma'(p)]$. Then

$$e_x = [p, e] = [p, \sigma(p)] \cdot \underbrace{\big([p, \sigma'(p)]\big)^{-1}}_{=\left[p, (\sigma'(p))^{-1}\right]} = \Big[p, \sigma(p)\big(\sigma'(p)\big)^{-1}\Big],$$

such that

$$\sigma(p)\big(\sigma'(p)\big)^{-1} = e,$$

so $\sigma = \sigma'$ and hence injectivity follows. For surjectivity observe that for a section $q \in \Gamma\big(c_G(P)\big)$ we can define a map $\sigma : P \to G$ by

$$q_x = [p, \sigma(p)].$$

This map satisfies

$$[p, \sigma(p)] = \big[p \cdot g, c_{g^{-1}}\big(\sigma(p)\big)\big] = [p \cdot g, \sigma(p \cdot g)]$$

for all $g \in G$; the last equality implies $\sigma(p \cdot g) = c_{g^{-1}}\big(\sigma(p)\big)$, which is precisely what we need for $C^\infty(P;G)^G$. It is only left to show smoothness of σ. For an open neighbourhood $U \subset M$ of x fix a trivialization $\varphi_U : P|_U \to U \times G$, and we denote

$$\varphi_U(p') = \big(\pi_P(p'), \beta_U(p')\big)$$

for all $p' \in P$, where $\beta_U : P|_U \to G$ is an equivariant map, *i.e.* $\beta_U(p' \cdot g) = \beta_U(p')\ g$ for all $g \in G$. As shown in the proof of Thm. 4.8, we have a trivialization of $c_G(P)$ given by

$$\begin{aligned} c_G(P)|_U &\to U \times G, \\ [p', g] &\mapsto \left(\pi_P(p'), \psi_{\beta_U(p')}(g)\right). \end{aligned}$$

Applying that trivialization to q we derive that

$$\left[p' \mapsto \psi_{\beta_U(p')}\left(\sigma(p')\right)\right]$$

is smooth, because q is smooth. Since $\psi_{\beta_U(p')}$ is smooth and bijective, we conclude that σ is smooth. Hence, $\sigma \in C^\infty(P;G)^G$, so, Def. (30) is also surjective and thence bijective.

Finally let us show that Def. (30) is a group isomorphism. Let σ, σ' be elements of $C^\infty(P;G)^G$, then use Def. (30) to derive

$$\sigma\sigma' \mapsto q_{\sigma\sigma'}$$

with

$$q_{\sigma\sigma'}|_x = \left[p, \sigma(p)\ \sigma'(p)\right] = [p, \sigma(p)] \cdot \left[p, \sigma'(p)\right] = q_\sigma|_x \cdot q_{\sigma'}|_x,$$

such that Def. (30) satisfies

$$\sigma\sigma' \mapsto q_\sigma \cdot q_{\sigma'}.$$

This concludes the proof. ■

Associated fibre bundles are motivated by making the invariance of gauge theory under local gauge transformations (that is, the change of gauge/local section of P) an inherent part of the bundle, similar to typical manifold coordinates; while the action of more global transformations "remain", similar to diffeomorphisms of a manifold. This procedure of "reducing" the action onto these is reflected in the quotient bundle $c_G(P)$.

6. Lie algebra bundles (LABs)

6.1. Definition

Lie algebras are the infinitesimal version of Lie groups, hence, we expect something similar for LGBs, the Lie algebra bundles:

Definition 6.1: Lie algebra bundle (LAB), [2, §3.3, Definition 3.3.8, page 104]

Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{g}, M$ be smooth manifolds. A vector bundle

$$\begin{tikzcd} \mathfrak{g} \arrow[r] & \mathcal{g} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

is called a **Lie algebra bundle** if:

1. $\mathfrak{g}$ and each fibre $\mathcal{g}_x$, $x \in M$, are Lie algebras;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} := \mathrm{pr}_2 \circ \phi_i|_{\mathcal{g}_x} : \mathcal{g}_x \to \mathfrak{g}$$

are Lie algebra isomorphisms, where I is an (index) set, U_i are open sets covering M, $\phi_i : \mathcal{g}|_U \to U \times \mathfrak{g}$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie algebra bundle atlas** or **LAB atlas**.

We often say that $\mathcal{g}$ **is an LAB (over** M**)**, whose structural Lie algebra is either clear by context or not explicitly needed; and we may also denote LABs by $\mathfrak{g} \to \mathcal{g} \xrightarrow{\pi} M$.

Of course, we have the typical trivial examples:

Example 6.2: Trivial examples

We recover the notion of a Lie algebra, if M consist of just one point. Moreover, the **trivial LAB** is given as the product manifold $\mathcal{g} := M \times \mathfrak{g} \to M$. We have obviously a canonical smooth field of Lie brackets on this bundle $[\cdot,\cdot]_{\mathcal{g}} : \Gamma(\mathcal{g}) \times \Gamma(\mathcal{g}) \to \Gamma(\mathcal{g})$, *i.e.* $[\cdot,\cdot]_{\mathcal{g}} \in \Gamma\left(\bigwedge^2 \mathcal{g}^* \otimes \mathcal{g}\right)$ which restricts to the Lie algebra bracket $[\cdot,\cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$ on each fibre. The bracket is given by

$$[(x, X), (x, Y)]_{\mathcal{g}} := \left(x, [X, Y]_{\mathfrak{g}}\right)$$

for all $(x, X), (x, Y) \in M \times \mathfrak{g}$. Smoothness is an immediate consequence.

The definition of LAB morphisms is straight-forward:

Definition 6.3: LAB morphism,
[2, §. 4.3, simplified version of Def. 4.3.1, page 158]

Let $\mathcal{g} \xrightarrow{\pi_{\mathcal{g}}} M$ and $\mathcal{h} \xrightarrow{\pi_{\mathcal{h}}} N$ be two LGBs over two smooth manifolds M and N. An **LAB morphism** is a pair of smooth maps $F : \mathcal{h} \to \mathcal{g}$ and $f : N \to M$ such that

$$\pi_{\mathcal{g}} \circ F = f \circ \pi_{\mathcal{h}}, \tag{31}$$

$$F \text{ linear}, \tag{32}$$

$$F\Big([g,q]_{\mathscr{h}_p}\Big) = [F(g), F(q)]_{\mathscr{g}_{f(p)}} \tag{33}$$

for all $g, q \in \mathscr{h}_p$ ($p \in N$), where $[\cdot,\cdot]_{\mathscr{h}_p}$ and $[\cdot,\cdot]_{\mathscr{g}_{f(p)}}$ are Lie brackets of $\mathscr{h}_p$ and $g_{f(p)}$, respectively. We then say that F **is an LAB morphism over** f. If $N = M$ and $f = \mathrm{id}_M$, then we often omit mentioning f explicitly and just write that F **is a (base-preserving) LAB morphism**.
We speak of an **LAB isomorphism (over** f**)** if F is a diffeomorphism.

Remark 6.4: Smooth field of Lie brackets

We have similar remarks as in Rem. 4.5. Additionally, we have locally a canonical smooth field of Lie brackets which restricts to a Lie bracket on each fibre because every LAB is locally isomorphic to a trivial LAB as in Ex. 6.2. Define a field of Lie brackets $[\cdot,\cdot]_{\mathscr{g}}$: $\Gamma(\mathscr{g}) \times \Gamma(\mathscr{g}) \to \Gamma(\mathscr{g})$, *i.e.* $[\cdot,\cdot]_{\mathscr{g}} \in \Gamma\left(\bigwedge^2 \mathscr{g}^* \otimes \mathscr{g}\right)$, by

$$[X,Y]_{\mathscr{g}} \coloneqq [X,Y]_{\mathscr{g}_x} \tag{34}$$

for all $X, Y \in \mathscr{g}_x$ ($x \in M$). Using a local trivialization, this bracket is locally of the form as in Rem. 4.5 such that smoothness follows.
In fact, as also argued in [4, §16.2, Example 2, page 114; but speaking in the context of Lie algebroids there whose are a generalization of LABs], every vector bundle equipped with a smooth field of Lie brackets is an LAB.

We have the notion of structure constants on all fibres, such that we get now structure functions after fixing a frame.[3]

Definition 6.5: Structure functions, [4, §16.5, page 119]

Let $\mathscr{g} \to M$ be an LAB over a smooth manifold M, and $(e_a)_a$ be a local frame of $\mathscr{g}$ over some open subset $U \subset M$. Then the **structure functions** $C^a_{bc} \in C^\infty(U)$ are defined by

$$[e_b, e_c]_{\mathscr{g}} = C^a_{bc} e_a.$$

Remark 6.6: Properties of the structure functions

Of course, one has the typical properties for such structure functions as for structure constants due to that C^a_{bc} restrict to typical structure constants on each fibre. Similar to [1, §1.4, discussion after Def. 1.4.17, page 38],

$$C^a_{bc} = -C^a_{cb},$$

[3] However, observe that $\Gamma(\mathscr{g})$ is an infinite-dimensional Lie algebra, such that there is still a notion structure constants, but an inconvenient one due to that these constants are in general not finitely many.

$$0 = C^d_{ae} C^e_{bc} + C^d_{be} C^e_{ca} + C^d_{ce} C^e_{ab}$$

for all a, b, c, d; the former due to antisymmetry, the latter because of the Jacobi identity.

As we have seen it for LGBs, the pullback of LABs is again an LAB.

Corollary 6.7: Pullbacks of LABs are LABs, [3, §3, Thm. 3.2]

Let M, N be smooth manifolds, $\mathscr{g} \xrightarrow{\pi} M$ an LAB over M and $f : N \to M$ a smooth map. Then the pullback vector bundle $f^*\mathscr{g}$ has a unique (up to isomorphisms) LAB structure such that the projection $\pi_2 : f^*\mathscr{g} \to \mathscr{g}$ onto the second factor is an LAB morphism over f with $\pi_2|_x : (f^*\mathscr{g})_x \to \mathscr{g}_{f(x)}$ being a Lie algebra isomorphism for all $x \in N$.

Proof.
Either prove this similarly as Cor. 5.1 (by also using a similar statement already known for vector bundles), or observe that the pullback $f^*\left([\cdot,\cdot]_{\mathscr{g}}\right)$ of the field of Lie brackets $[\cdot,\cdot]_{\mathscr{g}}$ on $\mathscr{g}$ as a section is clearly also a smooth field of Lie brackets on $f^*\mathscr{g}$ with same structural Lie algebra $\mathfrak{g}$. ■

6.2. From LGBs to LABs

Let us now quickly discuss how LGBs and LABs are related; it is very similar to the relation of Lie groups and algebras, now somewhat fibre-wise. We will follow the style of [1, §1.5.2, page 40ff.] and [2, §3.5, page 119ff.]; our approach will be using left-invariant vector fields but the mentioned latter reference actually uses right-invariant vector fields.

Let us start with introducing the basic notations needed.

Definition 6.8: Left and right translation and conjugation, [1, §1.5, similar notation to Def. 1.5.3, page 40]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M. For $g \in \mathscr{G}_x$ $(x \in M)$ we define the following maps:

- **Left translation** given by

$$\begin{aligned} L_g : \mathscr{G}_x &\to \mathscr{G}_x, \\ h &\mapsto gh. \end{aligned}$$

- **Right translation** given by

$$\begin{aligned} R_g : \mathscr{G}_x &\to \mathscr{G}_x, \\ h &\mapsto hg. \end{aligned}$$

- **Conjugation** given by

$$\begin{aligned} c_g : \mathscr{G}_x &\to \mathscr{G}_x, \\ h &\mapsto ghg^{-1}. \end{aligned}$$

Remarks 6.9.
By definition of $\mathscr{G}$, all these maps are smooth. Furthermore, they clearly satisfy the typical properties as known for these maps since $\mathscr{G}_x$ is a Lie group for all $x \in M$; for reference about their basic properties see for example [1, §1.5, Lemma 1.5.5, page 40f.].

The left and right translations of Def. 6.8 and 5.7 align, and thus the smoothness concerns as mentioned in the last part of Rem. 5.9 for right translations $r_g = R_g$ ($g \in \mathscr{G}_x$, $x \in M$) do not arise. Moreover, while the conjugation c_g is a Lie group automorphism of $\mathscr{G}_x$, it describes an LGB automorphism of $\mathscr{G}$ if extended to sections; following [2, §1.4, Def. 1.4.6 and its discussion afterwards, page 24f.]. That is for $\sigma \in \Gamma(\mathscr{G})$ we define the conjugation c_σ as a smooth map by

$$\begin{aligned} \mathscr{G} &\to \mathscr{G}, \\ q &\mapsto c_\sigma(q) \coloneqq (L_\sigma \circ R_{\sigma^{-1}})(q) = (R_{\sigma^{-1}} \circ L_\sigma)(q) = \sigma_{\pi(q)} \cdot q \cdot \sigma_{\pi(q)}^{-1}. \end{aligned}$$

It is clear that $c_\sigma(gq) = c_\sigma(g) \cdot c_\sigma(q)$ for all $g, q \in \mathscr{G}$ with $\pi(g) = \pi(q)$, and that a smooth inverse is given by $c_{\sigma^{-1}}$; thence, c_σ is an LGB isomorphism of $\mathscr{G}$ on itself, an automorphism, in sense of Def. 4.4. It is also trivial to check that we have $c_{\sigma \cdot \tau} = c_\sigma \circ c_\tau$, where τ is another section of $\mathscr{G}$.

Analogously we define R_σ as r_σ of Def. 5.7; with the capital letter we put an emphasis on that the $\mathscr{G}$-action acts on $\mathscr{G}$ itself. Similarly for left translations.

Since these are diffeomorphism of the fibres, it makes sense to say that a left-invariant vector field of $\mathscr{G}$ has to be a vertical vector field, that is, it is in the kernel of $\mathrm{D}\pi$, the total differential/tangent map of the projection of $\mathscr{G} \xrightarrow{\pi} M$. For this recall that there is the notion of a **vertical bundle** for fibre bundles $F \xrightarrow{\varpi} M$ (as *e.g.* introduced in [1, §5.1.1, for principal bundles, but it is straightforward to extend the definitions; page 258ff.]), which is defined as a subbundle $\mathrm{V}F \to F$ of the tangent bundle $\mathrm{T}F \to F$ given as the kernel of $\mathrm{D}\varpi : \mathrm{T}F \to \mathrm{T}M$. The fibres $\mathrm{V}_v F$ of $\mathrm{V}F$ at $v \in F$ are then given by

$$\mathrm{V}_v F = \mathrm{T}_v F_x,$$

where $x \coloneqq \varpi(v) \in M$ and F_p is the fibre of F at x. F_x is an embedded submanifold of F, thence, by definition a section $X \in \Gamma(\mathrm{V}F)$ restricts to a vector field on the fibres, that is,

$$X|_{F_x} \in \mathfrak{X}(F_x).$$

In our case $F = \mathscr{G}$, in this case $F_x = \mathscr{G}_x$ is then a Lie group, so, the vertical bundle just consists of the tangent bundles of Lie groups of all fibres. All of these are generated by their Lie algebra at e_x, the identity element of $\mathscr{G}_x$. Hence, it is natural to guess that the LAB for $\mathscr{G}$ will be $\mathrm{V}\mathscr{G}|_{e_M}$, where e_M is the image of M under the identity section of $\mathscr{G}$, thus, an embedding of M into $\mathscr{G}$. Therefore $\mathrm{V}\mathscr{G}|_{e_M}$ is a fibre bundle by [1, §4.1, Lemma 4.1.16, page 204], and clearly a vector bundle. Equivalently, since the identity section e is an embedding, we think of $\mathrm{V}\mathscr{G}|_{e_M}$ as the pullback vector bundle $e^*\mathrm{V}\mathscr{G}$, which is conveniently a vector bundle over M.

Hence, let us now show that $\mathscr{G}$ will be related to $e^*\mathrm{V}\mathscr{G}$ similar to how a Lie group will be related to its Lie algebra.

Definition 6.10: Left-invariant vector fields on LGBs, [2, §3.5, special situation of Def. 3.5.2, page 120]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. A vector field $X \in \mathfrak{X}(\mathscr{G})$ is a **left-invariant vector field** if

1. X is vertical, that is,
$$X \in \Gamma(\mathrm{V}\mathscr{G}),$$
2. X is invariant under the left-multiplication on each fibre, *i.e.*
$$\mathrm{D}_g L_q(X_g) = X_{qg}$$
for all $q, g \in \mathscr{G}_x$, where $x := \pi(g) = \pi(q)$.

The set of all left-invariant vector fields on $\mathscr{G}$ will be denoted by $L(\mathscr{G})$.

Remarks 6.11.
Observe that the second point in the definition is well-defined because X is a vertical vector field; that is, recall that $L_q : \mathscr{G}_x \to \mathscr{G}_x$ such that $\mathrm{D}_g L_q(X_g) : \mathrm{T}_g\mathscr{G}_x \to \mathrm{T}_{qg}\mathscr{G}_x$, hence, $\mathrm{D}_g L_q(X_g) : \mathrm{V}_g\mathscr{G} \to \mathrm{V}_{qg}\mathscr{G}$.

Remark 6.12: Abstract notation 1

Since X is vertical, recall that we can view the restriction of X onto a fibre as a vector field on that fibre, *i.e.*
$$X|_{\mathscr{G}_x} \in \mathfrak{X}(\mathscr{G}_x).$$
$\mathscr{G}_x$ is a Lie group and left translations are diffeomorphisms on it, hence, the left-invariance can also be written as
$$\mathrm{D}L_q(X|_{\mathscr{G}_x}) = L_q^*(X|_{\mathscr{G}_x}). \tag{35}$$
For this recall that $\mathrm{D}L_q \in \Omega^1\left(\mathscr{G}_x; L_q^*\mathrm{T}\mathscr{G}_x\right)$ for the left hand side, and that L_q^* is the pullback of sections on the right hand side, that is, $L_q^*(X|_{\mathscr{G}_x}) \in \Gamma\left(L_q^*\mathrm{T}\mathscr{G}_x\right)$. Furthermore,

$X|_{\mathscr{G}_x}$ is therefore a left-invariant vector on $\mathscr{G}_x$. Which is why one may also define the left-invariance of X as a vector field on $\mathscr{G}$ by saying that it has to restrict to a left-invariant vector field on each fibre in the usual sense of Lie groups.

One quickly shows that this is a Lie subalgebra of $\mathfrak{X}(\mathscr{G})$.

Lemma 6.13: Closure of Lie bracket for left-invariant vector fields, [2, §3.5, special situation of Lemma 3.5.5, page 122]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Then $L(\mathscr{G})$ is a Lie subalgebra of $\mathfrak{X}(\mathscr{G})$.

Proof.
Let $X, Y \in L(\mathscr{G})$, then we need to show

$$\mathrm{D}\pi\big([X,Y]\big) \equiv 0,$$

and if this holds, then we also need to derive

$$\mathrm{D}_g L_q\Big([X,Y]|_g\Big) = [X,Y]|_{qg}.$$

One can either immediately show these directly by using statements like [1, Proposition A.1.49; page 615], which essentially describes how the Lie bracket of vector fields react under push-forwards. Or use the knowledge about Lie groups, recall Rem. 6.12: Each fibre $\mathscr{G}_x$ is an embedded submanifold of $\mathscr{G}$ and both, $X|_{\mathscr{G}_x}$ and $Y|_{\mathscr{G}_x}$, are vector fields of this submanifold. Thus, $\big[X|_{\mathscr{G}_x}, Y|_{\mathscr{G}_x}\big]\big|_p$ has values in $\mathrm{T}_p\mathscr{G}_x$ for all $p \in \mathscr{G}_x$. Especially,

$$\big[X|_{\mathscr{G}_x}, Y|_{\mathscr{G}_x}\big]\big|_{\mathscr{G}_x} \in \mathfrak{X}(\mathscr{G}_x).$$

Because of this and since $X|_{\mathscr{G}_x}$ and $Y|_{\mathscr{G}_x}$ are left-invariant vector fields of $\mathscr{G}_x$ (a Lie group), left-invariance of $\big[X|_{\mathscr{G}_x}, Y|_{\mathscr{G}_x}\big]\big|_{\mathscr{G}_x}$ follows, and thus the statement. ■

Of course, elements of $L(\mathscr{G})$ are determined by their values at e_M, as already suggested previously. Let us show this now; starting with a small auxiliary result.

Corollary 6.14: $L(\mathscr{G})$ a $C^\infty(M)$-module, [2, §3.5, comment before Lemma 3.5.5, page 122]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Then $L(\mathscr{G})$ is a $C^\infty(M)$-module under the multiplication

$$fX \coloneqq \pi^* f \; X$$

for all $f \in C^\infty(M)$ and $X \in L(\mathscr{G})$.

Proof.
Obviously, $fX \in \Gamma(\mathrm{V}\mathscr{G})$ since

$$\mathrm{D}\pi(fX) = \mathrm{D}\pi(\pi^* f \ X) = \pi^* f \ \mathrm{D}\pi(X) = 0.$$

Furthermore, $fX|_{\mathscr{G}_x}$ $(x \in M)$ is left-invariant over $\mathscr{G}_x$ since $X|_{\mathscr{G}_x}$ is left-invariant and $f|_{\mathscr{G}_x} \equiv f(x) \in \mathbb{R}$. Thence, $fX \in L(\mathscr{G})$. ∎

Corollary 6.15: $L(\mathscr{G})$ as sections of $e^*\mathrm{V}\mathscr{G}$, [2, §3.5, comment before Lemma 3.5.5, page 122; parts of Cor. 3.5.4, page 121]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M, and denote with e the identity section of $\mathscr{G}$. Then we have an isomorphism of $C^\infty(M)$-modules

$$L(\mathscr{G}) \to \Gamma(e^*\mathrm{V}\mathscr{G}),$$

$$X \mapsto e^* X.$$

The inverse of this map is given by

$$\Gamma(e^*\mathrm{V}\mathscr{G}) \to L(\mathscr{G}),$$

$$\nu \mapsto X_\nu,$$

where X_ν is given by

$$X_\nu\big|_g \coloneqq \mathrm{D}_{e_x} L_g\big(\mathrm{pr}_2(\nu_x)\big)$$

for all $g \in \mathscr{G}$ and $x \coloneqq \pi(g)$, where pr_2 is the projection onto the second component in $e^*\mathrm{V}\mathscr{G}$.

Remark 6.16: Abstract notation 2

Since $e^*\mathrm{V}\mathscr{G} \cong \mathrm{V}\mathscr{G}|_{e_M}$ is a very natural isomorphism, we will often just write

$$X_\nu\big|_g = \mathrm{D}_{e_x} L_g(\nu_x),$$

omitting pr_2 and using that natural isomorphism without further mention.
Also observe that we can actually define a left translations by (local) sections of $\mathscr{G}$, *i.e.* for $\sigma \in \Gamma(\mathscr{G})$ we define the left translation L_σ as a map by

$$\mathscr{G} \to \mathscr{G},$$

$$q \mapsto \sigma_{\pi(q)} \cdot q.$$

This map is a diffeomorphism, and restricts to the fibres $\mathscr{G}_x$ as embedded submanifolds to the map L_{σ_x}; we discussed this in more generality in Rem. 5.9. Observe that for vertical vector fields $Y \in \Gamma(\mathrm{V}\mathscr{G})$ we have

$$\mathrm{D}_q L_\sigma(Y_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (L_\sigma \circ \gamma) \equiv \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(L_{\sigma_{\pi(q)}} \circ \gamma\right) = \mathrm{D}_q L_g(Y_q)$$

where $g := \sigma_{\pi(q)}$ and $\gamma : I \to \mathscr{G}_{\pi(q)}$ (I an open interval containing 0) is a curve with $\gamma(0) = q$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = Y_q$. Therefore L_σ restricts onto vertical vector fields and is then just the left translation via an element in the fibre over a fixed base point. In total one can then introduce the brief notation

$$X_\nu \circ \sigma = \mathrm{D}L_\sigma(\nu) = \mathrm{D}L_\sigma|_{e_M}(\nu) = \mathrm{D}L_\sigma \circ \nu.$$

However, be careful, in general one cannot simply replace L_g with L_σ, even if $\sigma_{\pi(g)} = g$. This only works with respect to vertical tangent vectors; once horizontal parts play a role things change, L_g is a priori not even defined then. Once we turn to the definition of horizontal distributions we will come back to this.

Proof of Cor. 6.15.
This map is clearly $C^\infty(M)$-linear, especially due to

$$e^*(fX) = e^*(\pi^* f\ X) = (f \circ \underbrace{\pi \circ e}_{=\mathbb{1}_M})\ e^* X = f\ e^* X$$

for all $f \in C^\infty(M)$ and $X \in L(\mathscr{G})$; for this recall Cor. 6.14.

We essentially only need to show that the suggested inverse $\nu \mapsto X_\nu$ is well-defined. First of all, that X_ν is vertical and left-invariant is clear by construction; ν_x ($x \in M$) is an element of the Lie algebra of $\mathscr{G}_x$, and thus $X_\nu|_{\mathscr{G}_x}$ is a left-invariant vector field on $\mathscr{G}_x$. X is therefore an element of $L(\mathscr{G})$ once we know that X is smooth. We show smoothness similar as in [1, §1.5, proof of Lemma 1.5.13, page 42]: Denote the multiplication in $\mathscr{G}$ by $\mu : \mathscr{G} * \mathscr{G} \to \mathscr{G}$. Then observe that $(0_g, \nu_x)$ ($0_g \in \mathrm{T}_g\mathscr{G}$, $g \in \mathscr{G}_x$, the zero vector field 0 on $\mathrm{T}\mathscr{G}$) is an element of

$$\begin{aligned}&\mathrm{T}(\mathscr{G} * \mathscr{G})\\ &= \{(Y, Z) \mid Y \in \mathrm{T}_q\mathscr{G}, Z \in \mathrm{T}_h\mathscr{G} \text{ with } \mathrm{D}_q\pi(Y) = \mathrm{D}_h\pi(Z), \text{ where } q, h \in \mathscr{G} \text{ with } \pi(q) = \pi(h)\}\end{aligned}$$

because $\nu_x \in \mathrm{V}_{e_x}\mathscr{G}$.[4] Therefore we can calculate

$$\mathrm{D}_{(g,e_x)}\mu(0_g, \nu_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g \cdot \gamma) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (L_g \circ \gamma) = \mathrm{D}_{e_x} L_g(\nu_x) = X_\nu|_g,$$

where $\gamma : I \to \mathscr{G}_x$ (I an open interval containing 0) is a curve with $\gamma(0) = e_x$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = \nu_x$. Since μ is smooth, $\mathrm{D}\mu$ is smooth, and thus

$$\mathscr{G} \to \mathrm{T}(\mathscr{G} * \mathscr{G}),$$

[4] If it is not clear how to derive the tangent bundle of $\mathscr{G} * \mathscr{G}$, then see later when we will discuss it in a more general manner. However, essentially recall that $\mathscr{G} * \mathscr{G} = \pi^*\mathscr{G}$.

$$g \mapsto \left(\mathrm{D}\mu \circ (0, \nu_\pi)\right)\big|_g = \mathrm{D}_{(g,e_x)}\mu\left(0_g, \nu_{\pi(g)}\right) = X_\nu|_g$$

is smooth, also using the smoothness of ν, π and $g \mapsto 0_g$.

Finally, that $\phi : L(\mathscr{G}) \to \Gamma(e^*\mathrm{V}\mathscr{G})$, $X \mapsto e^*X$, is bijective is also clear, similar to typical gauge theory; we know that $X|_{\mathscr{G}_x}$ is a left-invariant vector field on $\mathscr{G}_x$ by Rem. 6.12. Hence, for $g \in \mathscr{G}_x$,

$$X|_{\mathscr{G}_x} = \mathrm{D}_{e_x} L_g(X_{e_x}) = \mathrm{D}_{e_x} L_g(e^*X|_x).$$

This is precisely the structure of the suggested inverse, that is,

$$X = X_{e^*X} = (\psi \circ \phi)(X),$$

where $\psi : \Gamma(e^*\mathrm{V}\mathscr{G}) \to L(\mathscr{G}), \nu \mapsto X_\nu$. Hence, injectivity follows; surjectivity simply follows similarly by

$$(\phi \circ \psi)(\nu)|_x = e^*X_\nu|_x = \underbrace{\mathrm{D}_{e_x} L_{e_x}}_{=\mathbb{1}_{\mathrm{V}_{e_x}\mathscr{G}}}(\nu_x) = \nu_x$$

for all $\nu \in \Gamma(e^*\mathrm{V}\mathscr{G})$ and $x \in M$. This finishes the proof. ■

This result shows the typical statement about that elements of $L(\mathscr{G})$ are uniquely determined by their values at e_M. It immediately follows, too, that:

Corollary 6.17: LGBs induce an LAB structure, [2, §3.5, simplified version of the discussion after Cor. 3.5.4, page 121ff.]

Let $G \to \mathscr{G} \to M$ be an LGB over a smooth manifold M, and denote with e the identity section of $\mathscr{G}$. Then $\mathscr{g} := e^*\mathrm{V}\mathscr{G} \to M$ admits the structure as an LAB with structural Lie algebra $\mathfrak{g}$, the Lie algebra of G, and the fibres $\mathscr{g}_x$ ($x \in N$) are the Lie algebras of $\mathscr{G}_x$. The field of Lie algebra brackets $[\cdot,\cdot]_{\mathscr{g}}$ is given by

$$[\nu, \mu]_{\mathscr{g}} := e^*\left([X_\nu, X_\mu]\right)$$

for all $\nu, \mu \in \Gamma(e^*\mathrm{V}\mathscr{G})$, where X_ν, X_μ are elements of $L(\mathscr{G})$ as given in Cor. 6.15. Pointwise

$$[\nu_x, \mu_x]_{\mathscr{g}} = [X_\nu, X_\mu]\big|_{e_x}$$

for all $x \in M$.

Proof.
As already discussed $e^*\mathrm{V}\mathscr{G}$ is a vector bundle. The fibres are given by

$$\mathscr{g}_x = \mathrm{T}_{e_x}\mathscr{G}_x \cong \mathrm{T}_e G = \mathfrak{g}$$

for all $x \in M$, where we used that $\mathscr{G}_x$ is isomorphic to G as a Lie group. All fibres are Lie algebras of the fibre Lie group, isomorphic to $\mathfrak{g}$. By construction, the Lie bracket is precisely the Lie bracket isomorphic to the one of $\mathfrak{g}$, and $[\cdot,\cdot]_{\mathscr{g}}$ is smooth. Therefore we conclude that $\mathscr{g}$ is an LAB with structural Lie algebra $\mathfrak{g}$. Alternatively see [2, §3.5, Ex. 3.5.12, page 126] for an explicit construction of an LAB atlas. ■

Definition 6.18: The LAB of an LGB, [2, §3.5, special situation of Def. 3.5.1, page 120]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and denote with e the identity section of $\mathscr{G}$. Then we define the **LAB $\mathscr{g}$ of $\mathscr{G}$** as the vector bundle $e^*\mathrm{V}\mathscr{G}$.

6.3. Vertical Maurer-Cartan form of LGBs

As one may expect, the last result gives hints about the tangent bundle structure of $\mathscr{G}$; this can be shown with the Maurer-Cartan form on LGBs, which we will call vertical Maurer-Cartan form. It will be clear later why we choose to add this adjective; however, as a first argument recall Rem. 6.16, especially the last paragraph.

Corollary 6.19: Well-definedness of the vertical Maurer-Cartan form

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Define the following map

$$(\mu_{\mathscr{G}})_g(v) := \left(\mathrm{D}_g L_{g^{-1}}\right)(v)$$

for all $g \in \mathscr{G}$ and $v \in V_g\mathscr{G}$. Then this map is an element of $\Gamma(\mathrm{V}^*\mathscr{G} \otimes \pi^*\mathscr{g})$, where $\mathrm{V}^*\mathscr{G}$ is the dual bundle of $\mathrm{V}\mathscr{G}$.

Proof.
Observe that

$$\mathrm{D}_g L_{g^{-1}} : \mathrm{V}_g\mathscr{G} \to \mathrm{V}_{e_x}\mathscr{G} \cong (\pi^*\mathscr{g})_g$$

where $x := \pi(g)$ and e_x is the neutral element of $\mathscr{G}_x$. Smoothness follows similarly to the smoothness of left-invariant vector fields, that is, denote with μ the multiplication $\mathscr{G} * \mathscr{G} \to \mathscr{G}$ and recall the arguments and the notation in the proof of Cor. 6.15. We have

$$\begin{aligned} &\mathrm{T}(\mathscr{G} * \mathscr{G}) \\ &= \{(Y,Z) \mid Y \in \mathrm{T}_q\mathscr{G}, Z \in \mathrm{T}_h\mathscr{G} \text{ with } \mathrm{D}_q\pi(Y) = \mathrm{D}_h\pi(Z), \text{ where } q,h \in \mathscr{G} \text{ with } \pi(q) = \pi(h)\} \end{aligned}$$

and thus

$$\left(0_{g^{-1}}, v\right) \in \mathrm{T}(\mathscr{G} * \mathscr{G})$$

where 0 is the zero vector field, $v \in \mathrm{V}_g \mathscr{G}$ and $g \in \mathscr{G}$. Therefore we can calculate

$$\mathrm{D}_{(g^{-1},g)}\mu(0_{g^{-1}}, v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(g^{-1} \cdot \gamma\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(L_{g^{-1}} \circ \gamma\right) = \mathrm{D}_g L_{g^{-1}}(v) = \mu_{\mathscr{G}}(v),$$

where $\gamma : I \to \mathscr{G}_x$ (I an open interval containing 0, and $x := \pi(g)$) is a curve with $\gamma(0) = g$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = v$. Denote with 0^{-1} the vector field on $\mathscr{G}$ given by $g \mapsto 0_{g^{-1}}$ and with $\iota_{0^{-1}}$ the contraction with 0^{-1}, that is,

$$\left.(\iota_{0^{-1}}\mathrm{D}\mu)\right|_g := \mathrm{D}_{(g^{-1},g)}\mu\left(0_{g^{-1}}, \cdot\right) = \left[\mathrm{V}_g\mathscr{G} \ni v \mapsto \mathrm{D}_{(g^{-1},g)}\mu\left(0_{g^{-1}}, v\right)\right] = \mu_{\mathscr{G}}$$

for all $g \in \mathscr{G}$, using the structure of $\mathrm{T}(\mathscr{G} * \mathscr{G})$. Thus, we get in total that

$$\mu_{\mathscr{G}} = \iota_{0^{-1}}\mathrm{D}\mu \in \Gamma(V^*\mathscr{G} \otimes \pi^*\mathscr{g}),$$

using the smoothness of all involved parts, especially that μ is smooth, hence also $\mathrm{D}\mu$ is smooth as an element of $\Omega^1(\mathscr{G} * \mathscr{G}; \mu^*\mathscr{G})$. ■

Definition 6.20: Vertical Maurer-Cartan form of LGBs, [1, generalization of Def. 3.5.2, page 148]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. The map defined in Cor. 6.19 is the **vertical Maurer-Cartan form** $\mu_{\mathscr{G}}$, *i.e.* defined to be an element of $\Gamma(\mathrm{V}^*\mathscr{G} \otimes \pi^*\mathscr{g})$ given by

$$(\mu_{\mathscr{G}})_g(v) := \left(\mathrm{D}_g L_{g^{-1}}\right)(v)$$

for all $g \in \mathscr{G}$ and $v \in V_g\mathscr{G}$, where $\mathrm{V}^*\mathscr{G}$ is the dual bundle of $\mathrm{V}\mathscr{G}$.

Remark 6.21: Recovering of the classical definition

Observe that $\mu_{\mathscr{G}}|_{\mathscr{G}_x}$ ($x \in M$) is the typical Maurer-Cartan form of $\mathscr{G}_x$, hence, $\mu_{\mathscr{G}}$ restricts to the Maurer-Cartan form of Lie groups on each fibre.
Also recall Subsection 1.1, we have a 1:1 correspondence of $\mu_{\mathscr{G}}$ to the following commuting diagram

$$\begin{array}{ccc} \mathrm{V}\mathscr{G} & \xrightarrow{\mu_{\mathscr{G}}} & \mathscr{g} \\ \downarrow & & \downarrow \\ \mathscr{G} & \xrightarrow{\pi} & M \end{array}$$

which is the same diagram as in [2, §3.5, special situation of Prop. 3.5.3, page 121].

We can finally finish the discussion about the vertical bundle of an LGB.

Corollary 6.22: Vertical tangent space of $\mathscr{G}$,
[2, §3.5, a reformulation of Prop. 3.5.3, page 121]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Then we have an isomorphism of vector bundles

$$\mathrm{V}\mathscr{G} \cong \pi^* \mathscr{g}.$$

Remarks 6.23.
Observe that by Cor. 6.7 we know that $\mathrm{V}\mathscr{G}$ admits a unique LAB structure such that $\mathrm{V}\mathscr{G} \cong \pi^*\mathscr{g}$ is an isomorphism of LABs. This statement is also not in contradiction with $\mathscr{g} = e^*\mathrm{V}\mathscr{G}$ (e the identity section of $\mathscr{G}$), because

$$e^*\mathrm{V}\mathscr{G} = e^*\pi^*\mathscr{g} = (\pi \circ e)^*\mathscr{g} = \mathscr{g}.$$

By this we also know that $\mathrm{V}\mathscr{G}$ is trivial if and only if $\mathscr{g}$ is trivial; as also argued in [2, §3.5, discussion after Cor. 3.5.4, page 121]. Compare this result with $\mathrm{T}G \cong G \times \mathfrak{g}$, where G is a Lie group with Lie algebra $\mathfrak{g}$. We recover this result by restricting to the Lie group fibres $\mathscr{G}_x$ ($x \in M$), that is,

$$\mathrm{V}\mathscr{G}|_{\mathscr{G}_x} = \mathrm{T}\mathscr{G}_x \cong \mathscr{G}_x \times \mathscr{g}_x = \pi^*\mathscr{g}|_{\mathscr{G}_x}.$$

Last but not least, sections of $\mathrm{V}\mathscr{G}$ are therefore generated by sections of $\mathscr{g}$, the left-invariant vector fields.

Proof of Cor. 6.22.
This can be quickly shown by recalling Rem. 6.21, that is, we have the following commuting diagram

$$\begin{tikzcd}
\mathrm{V}\mathscr{G} \arrow[r, "\mu_{\mathscr{G}}"] \arrow[d] & \mathscr{g} \arrow[d] \\
\mathscr{G} \arrow[r, "\pi"] & M
\end{tikzcd}$$

where $\mu_{\mathscr{G}}$ is defined as in Def. 6.20, and $\mu_{\mathscr{G}}$ restricts to the Maurer-Cartan form of $\mathscr{G}_x$ ($x \in M$) on each fibre of $\mathscr{G}$; especially, $\mu_{\mathscr{G}} : \mathrm{V}\mathscr{G} \to \mathscr{g}$ is a fibre-wise isomorphism (since $\mathrm{D}_g L_{g^{-1}}$ is an isomorphism). Hence, as described in Subsection 1.1, $\mu_{\mathscr{G}}$ as an element of $\Gamma(\mathrm{V}^*\mathscr{G} \otimes \pi^*\mathscr{G})$, *i.e.* a vector bundle morphism $\mathrm{V}\mathscr{G} \to \pi^*\mathscr{g}$ (linearity of μ_G is clear), is a vector bundle isomorphism. This finishes the proof. ■

6.4. LABs of pullback LGBs

We are going to define LGB representations and corresponding LAB representations. Since group representation are a special form of actions, we will have something similar in the case of LGB representations. Since actions are defined as maps on a pullback of an LGB $\mathscr{G}$, which is also an LGB by Cor. 5.1, we expect that the corresponding LAB representation is related to the LAB of the pullback of $\mathscr{G}$. It is natural to think of this LAB as the pullback of $\mathscr{g}$, which is also an LAB by Cor. 6.7:

Corollary 6.24: LAB of pullback LGB is pullback LAB, [3, §3, Thm. 3.5, page 21]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and let $f : N \to M$ be a smooth map defined on another smooth manifold N. Then the LAB of $f^*\mathscr{G}$ is isomorphic to the pullback LAB $f^*\mathscr{g}$.

Proof.
By Cor. 5.1 we know that $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$, the projection onto the second factor, is an LGB morphism over f,

$$\begin{tikzcd} f^*\mathscr{G} \arrow[r, "\pi_2"] \arrow[d] & \mathscr{G} \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

and it is fibre-wise a Lie group isomorphism such that

$$\mathrm{D}_{(p,e_x)}\pi_2 : \mathscr{h}_p \to \mathscr{g}_x$$

is a Lie algebra isomorphism for all $p \in N$, where e_x is the neutral element of $\mathscr{G}_x$ for $x := f(p)$, and $\mathscr{h}$ and $\mathscr{g}$ are the LABs of $f^*\mathscr{G}$ and $\mathscr{G}$, respectively; for all of that recall that $(f^*\mathscr{G})_p$ and $\mathscr{G}_x$ are Lie groups. Hence, we have a vector bundle morphism over f given by the following commuting diagram

$$\begin{tikzcd} \mathscr{h} \arrow[r, "\mathrm{D}_{\left(\mathbb{1}_N, e_f\right)}\pi_2"] \arrow[d] & \mathscr{g} \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

which describes fibre-wise a Lie algebra isomorphism, where

$$\mathrm{D}_{\left(\mathbb{1}_N, e_f\right)}\pi_2 := \mathrm{D}\pi_2 \circ (\mathbb{1}_N, e_f) = \left[N \ni p \mapsto \mathrm{D}_{\left(p, e_{f(p)}\right)}\pi_2\right].$$

By our notes in Subsection 1.1, we therefore achieve an LAB isomorphism $\mathscr{h} \to f^*\mathscr{g}$. ∎

Remark 6.25: LAB of $f^*\mathscr{G}$

Since this isomorphism is very natural, we always use that identification and will refer to $f^*\mathscr{g}$ as *the* LAB of $f^*\mathscr{G}$.

With this we can quickly show the following familiar result.

Corollary 6.26: Differentials of LGB morphisms are LAB morphisms, [2, §3.5, section about morphisms, page 124f.]

Let $\mathscr{H} \to N$ and $\mathscr{G} \to M$ be two LGBs over two smooth manifolds N and M, and we denote with $\mathscr{h}$ and $\mathscr{g}$ the LABs of $\mathscr{H}$ and $\mathscr{G}$, respectively. Furthermore, assume that we have an LGB morphism $F : \mathscr{H} \to \mathscr{G}$ over a smooth map $f : N \to M$. Then

$$\mathrm{D}_e F : \mathscr{h} \to \mathscr{g}$$

is an LAB morphism over f, where e is the identity section of $\mathscr{H}$, that is,

$$\mathrm{D}_e F|_p \coloneqq \left(\mathrm{D}F|_{e_N}\right)\Big|_p = \mathrm{D}_{e_p} F : \mathscr{h}_p \to \mathscr{g}_{f(p)}$$

for all $p \in N$.

Proof.
Again by our notes in Subsection 1.1, we can view F as a base-preserving LGB morphism $F : \mathscr{H} \to f^*\mathscr{G}$, since $f^*\mathscr{G}$ is an LGB whose structure is naturally inherited by $\mathscr{G}$ as given in Cor. 5.1; similarly for its LAB by Cor. 6.7. Thus, it is fibre-wise a Lie group morphism, and its tangent map restricted to the identity section, $\mathrm{D}_e F$, gives fibre-wise a Lie algebra morphism. Thus, $\mathrm{D}_e F : \mathscr{h} \to f^*\mathscr{g}$ is an LAB morphism (using Cor. 6.24), and can be seen as an LAB morphism $\mathscr{h} \to \mathscr{g}$ over f. Alternatively, it is straightforward and trivial to show it directly. ■

Add 1-parameter subgroups, which are sections of LGB at each time.

7. LGB actions, part II

Finally, we come to the last part of the *basics* for LGBs and their notions needed in this paper and needed to formulate a new and more general form of gauge theory.

7.1. LGB and LAB representations

As usual, representations are a special type of group action, with an infinitesimal analogue.

Definition 7.1: LGB representation, [2, §1.7, special situation of Def. 1.7.1, page 43]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold, $V \xrightarrow{p} M$ be a vector bundle, and assume that we have a left $\mathscr{G}$-action on V, $\Psi : \mathscr{G} * V \coloneqq p^*\mathscr{G} \to V$. Then we say that Ψ is a **$\mathscr{G}$-representation on** V if it is linear, that is,

$$\Psi\big((g, \alpha v)\big) = \alpha \Psi\big((g, v)\big),$$

$$\Psi\big((g, v + w)\big) = \Psi\big((g, v)\big) + \Psi\big((g, w)\big)$$

for all $\alpha \in \mathbb{R}$, and $(g, v), (g, w) \in \mathscr{G} * V$. In alignment with previous notations we may also write $\Psi(g, v) = \Psi_g(v)$, or $\Psi(g, v) = g \cdot v$.

Remark 7.2: Well-definedness and fibrewise a typical representation

Observe that $(g, v) \in \mathscr{G} * V$ means that $p(v) = \pi(g)$, same for (g, w). Hence, given a base point in $x \in M$, the pairs (g, v) in $\mathscr{G} * V|_{p^{-1}(\{x\})}$ are given by elements $g \in \mathscr{G}_x$ and $v \in V_x$. By Def. 5.3 we also have

$$p\big(\Psi(g, v)\big) = \pi(g) = p(v) = x.$$

Thence, linearity of Ψ is well-defined. In fact, observe that $\mathscr{G} * V = p * \mathscr{G} \cong \pi^* V$ as fibre bundle, therefore $\mathscr{G} * V$ carries not only the structure of an LGB but also of a vector bundle. That is, we have the following commuting diagram

$$\begin{tikzcd} \mathscr{G} * V \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathscr{G} \arrow[d, "\pi"] \\ V \arrow[r, "p"] & M \end{tikzcd}$$

the horizontal arrows describe the vector bundle structure (viewing $\mathscr{G} * V$ as the vector bundle $\pi^* V$), and the vertical ones the LGB structure (viewing $\mathscr{G} * V$ as the LGB $p^* \mathscr{G}$); pr_i $(i \in \{1, 2\})$ is the projection onto the i-th component.
Last but not least, by fixing a base point $x \in M$ we clearly have the typical notion of a $\mathscr{G}_x$-representation on V_x.

Add induced representations, especially adjoint ones

7.2. Fundamental vector fields

Definition 7.3: Fundamental vector fields, [1, §3.4, generalization of Def. 3.4.1, page 143]

Let M and N be two smooth manifolds, $\mathscr{G} \to M$ an LGB over M, $f : N \to M$ a smooth map, and assume we have a right $\mathscr{G}$-action on N, $N * \mathscr{G} \to N$. For $\nu \in \Gamma(\mathscr{g})$ we define its induced **fundamental vector field** $\widetilde{\nu}$ as an element of $\mathfrak{X}(N)$ by

$$\widetilde{\nu}_p := \mathrm{D}_{e_{f(p)}} \Phi_p\big(\nu_{f(p)}\big)$$

for all $p \in N$, where Φ_p is the orbit map defined in Def. 5.7 and $e_{f(p)}$ the neutral element of $\mathscr{G}_{f(p)}$.
For left actions we define fundamental vector fields similarly by

$$\widetilde{\nu}_p := \mathrm{D}_{e_{f(p)}} \Phi'_p\big(\nu_{f(p)}\big)$$

for all $p \in N$, where Φ'_p is a slightly adjusted orbit map given by

$$\mathscr{G}_{f(p)} \to N,$$

$$g \mapsto g^{-1} \cdot p.$$

Remark 7.4: Notation

Again, point-wise this is just the typical definition of a fundamental vector field at $x := f(p)$ with respect to $\nu_x \in \mathscr{g}_x$ (except that $f^{-1}(\{x\})$ may not be a manifold). Hence, one has also a point-wise definition which we will also denote similarly by $\widetilde{\nu_x}$. If $f^{-1}(\{x\})$ is not a manifold, then $\widetilde{\nu_x}|_p$ is just a formal notation, and it only defines an element of $\mathrm{T}_p N$ which may not be related to a vector field on $f^{-1}(\{x\})$, not even to a vector field on N if ν_x does not formally come from a fixed section of $\mathscr{g}$.
For long expressions we use the following different font

$$\widetilde{\nu}$$

instead of $\widetilde{\nu}$.

Adding typical stuff about fundamental vector fields, like Prop 3.4.4 in Hamilton; and smoothness

7.3. Differential of smooth LGB actions

Lemma 7.5: Tangent bundle of pullback fibre bundles

Let M and N be two smooth manifolds, $F \xrightarrow{\pi} M$ a fibre bundle over M, and $f : N \to M$ a smooth map. Then we have for its tangent spaces

$$\mathrm{T}_{(p,v)}(f^*F) = \left\{(X, Y) \;\middle|\; X \in \mathrm{T}_p N, Y \in \mathrm{T}_v F \textit{ with } \mathrm{D}_p f(X) = \mathrm{D}_v \pi(Y)\right\}$$

for all $(p, v) \in f^*F$.

Proof.
Recall that $(p, v) \in f^*F$ implies that

$$f(p) = \pi(v)$$

such that we can immediately derive its infinitesimal version as

$$\mathrm{D}_p f(X) = \mathrm{D}_v \pi(Y)$$

for all $X \in \mathrm{T}_p N, Y \in \mathrm{T}_v F$. Hence, we have derived that $\mathrm{T}_{(p,v)}(f^*F)$ is a subset of the set of such pairs (X, Y). That this is an equivalent description quickly follows by the fact that f and

π are transversal to each other (trivially, because π is a surjective submersion). This means, the following linear map

$$\mathrm{T}_p N \times \mathrm{T}_v F \to \mathrm{T}_{f(p)} M,$$

$$(X, Y) \mapsto \mathrm{D}_p f(X) - \mathrm{D}_v \pi(Y)$$

is surjective because π is a submersion; it is also well-defined because of $f(p) = \pi(v)$. Hence, the dimension of the kernel of this map has the dimension

$$\dim(N) + \dim(F) - \dim(M) = \dim(N) + \mathrm{rk}(F) = \dim(f^* N),$$

where dim denotes the dimension as a manifold and rk the rank of a bundle (the dimension of its structural fibre). Its dimension is precisely the dimension of $f^* F$, and since it is about finite dimensions we can therefore identify $\mathrm{T}_{(p,v)}(f^* F)$ with this kernel. This concludes the proof due to the fact that the kernel consists of (X, Y) with $\mathrm{D}_p f(X) = \mathrm{D}_v \pi(Y)$. ■

With this we can finally show the following theorem; also recall the notations introduced in Def. 5.7, 6.8, 6.20 and 7.3.

Theorem 7.6: Differential of smooth LGB actions

Let M and N be two smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M, $f : N \to M$ a smooth map, and assume we have a right $\mathscr{G}$-action on N, $\Phi : N * \mathscr{G} \to N$. Then we have

$$\mathrm{D}_{(p,g)}\Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \left.\overleftarrow{(\mu_{\mathscr{G}})_g\big(Y - \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big)}\right|_{p\cdot g} \tag{36}$$

for all $(p, g) \in N * \mathscr{G}$ and $(X, Y) \in \mathrm{T}_{(p,g)}(N * \mathscr{G})$, where $x := f(p) = \pi(g)$, σ is any (local) section of $\mathscr{G}$ with $\sigma_x = g$, e is the identity section of $\mathscr{G}$, and ω is an element of $\mathrm{T}_x M$ given by

$$\omega := \mathrm{D}_p f(X) = \mathrm{D}_g \pi(Y).$$

We may omit the natural embedding of ω into $\mathrm{T}_{e_x}\mathscr{G}$ via $\mathrm{D}_x e$, just writing

$$\mathrm{D}_{(p,g)}\Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \left.\overleftarrow{(\mu_{\mathscr{G}})_g\big(Y - \mathrm{D}_{e_x} R_\sigma(\omega)\big)}\right|_{p\cdot g}. \tag{37}$$

If f is a surjective submersion, then we can also write

$$\mathrm{D}_{(p,g)}\Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \mathrm{D}_g \Phi_\tau(Y) - \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big) \tag{38}$$

and

$$\begin{aligned}\mathrm{D}_{(p,g)}\Phi(X, Y) = {} & \mathrm{D}_p r_g\big(X - \mathrm{D}_{e_x}\Phi_\tau\big(\mathrm{D}_x e(\omega)\big)\big) + \mathrm{D}_g \Phi_p\big(Y - \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) \\ & + \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big),\end{aligned} \tag{39}$$

where τ is any additional (local) section of f[a] with $\tau_x = p$.

[a] That is, $f \circ \tau = \mathbb{1}_M$.

Remarks 7.7.
The assumption about f being a surjective submersion is being stated in order to assure the existence of τ and the manifold structure on $f^{-1}(\{x\})$ as an embedded submanifold of N; see the proof for more details. If the existence of τ and the embedded submanifold structure is known otherwise, then those equations can still be derived. Following the proof, one may also just need the structure of an immersed submanifold.

Proof of Thm. 7.6.
We want to calculate the derivative of Ψ, and due to $N * \mathscr{G} = f^*\mathscr{G}$ we are going to use Lemma 7.5. That is, fix $(p, g) \in N * \mathscr{G}$ and $X \in \mathrm{T}_pN$, $Y \in \mathrm{T}_g\mathscr{G}$ with

$$\mathrm{D}_p f(X) = \mathrm{D}_g \pi(Y) =: \omega \in \mathrm{T}_{f(p)}M.$$

Recall that we can localize LGB actions in sense of Rem. 5.5; so, let $x := f(p) = \pi(g)$, and fix a trivialization of $\mathscr{G}$ around x. Then it is clear that there is a (local)[5] section $\sigma \in \Gamma(\mathscr{G})$ with $\sigma_x = g$. Observe that we can write

$$Y = Y - \mathrm{D}_x\sigma(\omega) + \mathrm{D}_x\sigma(\omega).$$

The two first summands result into a vertical tangent vector due to

$$\mathrm{D}_g\pi\big(Y - \mathrm{D}_x\sigma(\omega)\big) = \mathrm{D}_g\pi(Y) - \mathrm{D}_x \underbrace{(\pi \circ \sigma)}_{=\mathbb{1}_M}(\omega) = \mathrm{D}_g\pi(Y) - \omega = 0.$$

For the following recall the notations introduced in Def. 5.7 and 6.8; we can derive that

$$\mathrm{D}_{e_x} R_\sigma \circ \mathrm{D}_x e = \mathrm{D}_x \underbrace{(R_\sigma \circ e)}_{x \mapsto e_x\sigma_x = \sigma_x} = \mathrm{D}_x\sigma.$$

So, let us write

$$Y^v := Y - \mathrm{D}_x\sigma(\omega) = Y - \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big).$$

We have proven that $Y^v \in \mathrm{V}_g\mathscr{G}$, and thus $(0_p, Y^v) \in \mathrm{T}_{(p,g)}(f^*\mathscr{G})$ by Lemma 7.5, where 0_p is the zero tangent vector. In the same fashion we also have

$$\Big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) \in \mathrm{T}_{(p,g)}(f^*\mathscr{G})$$

because of

$$\mathrm{D}_g\pi\Big(\mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) = \mathrm{D}_g\pi(Y - Y^v) = \mathrm{D}_g\pi(Y) = \mathrm{D}_p f(X),$$

thence we can write

$$(X, Y) = \big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big) + Y^v\big) = \big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) + (0_p, Y^v).$$

[5] For simplicity of notation we omit the notation of restricting on some open subset of M.

Hence also

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_{(p,g)}\Phi\big(X, \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big). + \mathrm{D}_{(p,g)}\Phi(0_p, Y^v)$$

The first summand is quickly calculated as

$$\mathrm{D}_{(p,g)}\Phi(0_p, Y^v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \underbrace{(p \cdot \gamma)}_{\Phi_p(\gamma)} = \mathrm{D}_g\Phi_p(Y^v) \tag{40}$$

where $\gamma : I \to \mathscr{G}_x$ (I open interval of $\mathbb{R}$ containing 0) is a curve with $\gamma(0) = g$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = Y^v$; by the verticality of Y^v, $\mathrm{D}_g\Phi_p(Y^v)$ is well-defined since Φ_p is a map $\mathscr{G}_x \to N$. Now recall Def. 6.20 and 7.3, and observe that we can write

$$\mathrm{D}_g\Phi_p(Y^v) = \big(\mathrm{D}_g\Phi_p \circ \mathrm{D}_{e_x} L_g \circ \mathrm{D}_g L_{g^{-1}}\big)(Y^v) = \mathrm{D}_{e_x} \underbrace{(\Phi_p \circ L_g)}_{\mathscr{G} \ni q \mapsto p \cdot g q = \Phi_{p \cdot g}(q)}\Big(\big(\mu_{\mathscr{G}}\big)_g(Y^v)\Big) = \left.\overline{\big(\mu_{\mathscr{G}}\big)_g(Y^v)}\right|_{p \cdot g}$$

making use of the verticality of Y^v such that $\mathrm{D}L_g$ can act on Y^v and of $\big(\mu_{\mathscr{G}}\big)_g(Y^v) \in \mathscr{g}_x$.

For the second summand in Eq. (40) we use

$$\begin{aligned}
\mathrm{D}_{(p,g)}\Phi\Big(X, \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) &= \mathrm{D}_{(p,e_x)} \underbrace{\big(\Phi \circ (\mathbb{1}_N, R_\sigma)\big)}_{N * \mathscr{G} \ni (p,g) \mapsto p \cdot g\sigma_x = r_\sigma(p \cdot g)}\big(X, \mathrm{D}_x e(\omega)\big) \\
&= \mathrm{D}_{(p,e_x)}(r_\sigma \circ \Phi)\big(X, \mathrm{D}_x e(\omega)\big) \\
&= \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{(p,e_x)}\Phi\big(X, \mathrm{D}_x e(\omega)\big)\Big) \\
&= \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{(p,e_x)}\Phi\big(X, \mathrm{D}_p(e \circ f)(X)\big)\Big) \\
&= \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{(p,p)}\big(\Phi \circ (\mathbb{1}_N, e \circ f)\big)(X,X)\Big),
\end{aligned}$$

but

$$\Big[N \times N \ni (p,p) \mapsto \big(\Phi \circ (\mathbb{1}_N, e \circ f)\big)(p,p) = \Phi\big(p, e_{f(p)}\big) = p \cdot e_{f(p)} = p\Big] = \big[p \mapsto \mathbb{1}_N(p)\big],$$

hence,

$$\mathrm{D}_{(p,p)}\big(\Phi \circ (\mathbb{1}_N, e \circ f)\big)(X,X) = \mathrm{D}_p\mathbb{1}_N(X) = \mathbb{1}_{\mathrm{T}_pN}(X) = X.$$

So, we get in total

$$\mathrm{D}_{(p,g)}\Phi\Big(X, \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) = \mathrm{D}_p r_\sigma(X),$$

therefore

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_\sigma(X) + \left.\overline{\big(\mu_{\mathscr{G}}\big)_g(Y^v)}\right|_{p \cdot g}.$$

If f is a surjective submersion, then x is a regular value and thus $f^{-1}(\{x\})$ is an embedded submanifold, and we can assure the existence of a smooth local section $\tau : U \to N$ of f (U an open neighbourhood of $x \in M$), *i.e.* $f \circ \tau = \mathbb{1}_U$ with $\tau_x = p$; in case of doubt, this can be shown as in [1, §3.7, Lemma 3.7.4, page 152f.] via the Regular Point Theorem. Recalling the arguments of Rem. 6.16, we can rewrite Eq. (40) to

$$\begin{aligned}
\mathrm{D}_g \Phi_p(Y^v) &= \mathrm{D}_g \Phi_\tau(Y^v) \\
&= \mathrm{D}_g \Phi_\tau(Y) - \mathrm{D}_g \Phi_\tau\big(\mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) \\
&= \mathrm{D}_g \Phi_\tau(Y) - \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big)
\end{aligned}$$

making use of that $\mathrm{D}_g \Phi_\tau$ is linear map $\mathrm{T}_g \mathscr{G} \to \mathrm{T}_{p\cdot g} N$. In that case we would get in total

$$\mathrm{D}_{(p,g)} \Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \mathrm{D}_g \Phi_\tau(Y) - \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big).$$

Alternatively, we can keep the form of Eq. (40) and instead apply the same trick to X as for Y, that is,

$$X = X^v + \mathrm{D}_{e_x} \Phi_\tau\big(\mathrm{D}_x e(\omega)\big),$$

where

$$X^v := X - \mathrm{D}_x \tau(\omega) = X - \mathrm{D}_{e_x} \Phi_\tau\big(\mathrm{D}_x e(\omega)\big),$$

and X^v is vertical, too, that is,

$$\mathrm{D}_p f(X^v) = \mathrm{D}_p f(X) - \mathrm{D}_x(f \circ \tau)(\omega) = \omega - \omega = 0,$$

especially $X^v \in \mathrm{T}_p\big(f^{-1}(\{x\})\big)$ and so we can apply a similar argument as in Rem. 6.16 to derive

$$\begin{aligned}
\mathrm{D}_p r_\sigma(X) &= \mathrm{D}_p r_\sigma(X^v) + \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{e_x} \Phi_\tau\big(\mathrm{D}_x e(\omega)\big)\Big) \\
&= \mathrm{D}_p r_g(X^v) + \mathrm{D}_{e_x}(r_\sigma \circ \Phi_\tau)\big(\mathrm{D}_x e(\omega)\big).
\end{aligned}$$

Finally, using these expressions, the total formula would look like

$$\begin{aligned}
\mathrm{D}_{(p,g)} \Phi(X, Y) &= \mathrm{D}_p r_g(X^v) + \mathrm{D}_g \Phi_p(Y^v) + \mathrm{D}_{e_x} \underbrace{(r_\sigma \circ \Phi_\tau)}_{\mathscr{G} \ni g \mapsto \tau_{\pi(g)} \cdot g \sigma_{\pi(g)} = (\Phi_\tau \circ R_\sigma)(g)}\big(\mathrm{D}_x e(\omega)\big) \\
&= \mathrm{D}_p r_g(X^v) + \mathrm{D}_g \Phi_p(Y^v) + \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big).
\end{aligned}$$

■

Remarks 7.8.
Eq. (36) is very similar to the "classical" formula used in gauge theory, see *e.g.* [1, §3.5, Prop.

3.5.4, page 146]: In the case of a Lie group action on N (so, $\mathscr{G}$ a Lie group G and M a point) we have

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_g(X) + \left.\overbrace{(\mu_G)_g(Y)}\right|_{p\cdot g}$$

for all $p \in N$, $g \in G$, $X \in \mathrm{T}_pN$ and $Y \in \mathrm{T}_gG$. However, in our general case the vector Y is deformed by ω, due to the fact that the action Φ has no "constant" Lie group factor anymore. This will be important later when we are going to derive the gauge transformations. Furthermore, already the first summand is different than the classical formula, because we need to use LGB sections in order to define the push-forward of tangent vectors which are not vertical, that is, X may not be a tangent vector of $f^{-1}(\{x\})$ (which is in the general case not even an embedded submanifold) such that $\mathrm{D}_p r_g(X)$ is in general not well-defined anymore.

The other two equations in the case of f being a surjective submersion are mainly for reference; the last equation, Eq. (39), emphasises the contribution of non-vertical vectors measured by ω. While the first two summands are the classical product rule on the vertical parts, the third summand shows the deformation of the product rule because of the new structure of an action without a "constant" Lie group factor.

Eq. (38) may be the most elegant formulation, making use of local sections σ and τ and their advantage that these can act on all tangent vectors, not just the vertical ones; the reader who knows Lie groupoids may recognize this equation's structure with the one as given in [2, §1.4, Thm. 1.4.14, page 28], where it is about the induced multiplication structure on the tangent bundle of a Lie groupoid making use of bisections playing a similar role like σ and τ.

8. Connections and curvature on LGB principal bundles

Finally, let us now define the gauge theory, starting with horizontal distributions. For readability, we start with a very easy toy model which should help in understanding the general definitions. We expect a basic understanding of horizontal distributions and their relationship to what we call connections (on principal and vector bundles). The basis is the notion of an Ehresmann connection.

Definition 8.1: Ehresmann connection, [1, §5.1.2, Def. 5.1.6, page 260; without the symmetry along right-translations here]

Let $F \to M$ be a fibre bundle over a smooth manifold M. Then an **Ehresmann connection of** F is a smooth subbundle $\mathrm{H}F$ of $\mathrm{T}F$ with

$$\mathrm{T}F = \mathrm{H}F \oplus \mathrm{V}F.$$

We often call $\mathrm{H}F$ also a **horizontal distribution/bundle of** F, while the term Ehresmann connection may imply a further property which will then have been declared. For $p \in M$ the fibre is denoted by H_pF.

8.1. Toy model

As usual for gauge theory we will understand connections as horizontal distributions as a complement of the vertical bundle with a certain symmetry along the fibres to assure that connection forms have a relationship to a form defined on the spacetime/basis itself;[6] this symmetry is usually formulated by looking at push-forwards of horizontal vectors. However, as we have seen in Rem. 6.16 and Thm. 7.6 (and its proof) the push-forward of horizontal vectors is not well-defined anymore on non-vertical vectors if one uses a fixed element of an LGB. Hence, we were using sections of an LGB instead. Using sections carries the problem that the tangential behaviour of their image (as embedding of the base) may contribute; then a horizontal vector may be still horizontal after a push-forward with one LGB section but not with respect to another LGB section. Thus, we need to adjust the typical definition of connections on principal bundles.

In order to understand what has to be changed, let us look at a very trivial LGB and an action thereof; we will omit discussions about smoothness in the following. We will now discuss the trivial LGB $\mathscr{G}$ given by

$$\begin{tikzcd} \mathbb{R}^* \arrow[r] & \mathbb{R} \times \mathbb{R}^* \arrow[d, "\mathrm{pr}_1"] \\ & \mathbb{R} \end{tikzcd}$$

where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and pr_1 is the projection onto the first component, and whose multiplication is fibre-wise given by

$$(x, g) \cdot (x, q) := (x, gq)$$

for all $x \in \mathbb{R}$ and $g, q \in \mathbb{R}^*$. We are interested into an action of $\mathscr{G}$ on another bundle $\mathscr{P}$, and we define $\mathscr{P}$ to be the same bundle as $\mathscr{G}$ for simplicity. The action is a right-action $\Phi : \mathscr{P} * \mathscr{G} \to \mathscr{P}$ whose explicit form is precisely the same as the multiplication on $\mathscr{G}$. For bookkeeping reasons we keep the notation $\mathscr{P}$ because this bundle will later be interpreted as the principal bundle; that $\mathscr{P}$ is the LGB $\mathscr{G}$ itself makes it easier to express certain things explicitly in the following.

The tangent bundle $\mathrm{T}\mathscr{P}$ of $\mathscr{P}$ is canonically isomorphic to

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & \mathscr{P} \times \mathbb{R}^2 \arrow[d] \\ & \mathscr{P} \end{tikzcd}$$

where the projection is given by the projection onto the $\mathscr{P}$-component. The vertical bundle $\mathrm{V}\mathscr{P}$ of $\mathscr{P}$ is canonically isomorphic to

[6] Recall that connection 1-forms on principal bundles (in the classical gauge theory) can be viewed as a 1-form on the spacetime with values in the Atiyah bundle. We usually also want that the connection acting on a section X, denoted by ∇X in the case of vector bundles, is still a section of that vector bundle as a bundle over the spacetime/basis; one usually wants to avoid that ∇X is a section over the tangent bundle of the vector bundle, but this may be the case without further symmetry of the horizontal distribution.

$$\begin{tikzcd} \mathbb{R} \arrow[r] & \mathscr{P} \times \{0\} \times \mathbb{R} \arrow[d] \\ & \mathscr{P} \end{tikzcd}$$

as a natural subbundle of $\mathrm{T}\mathscr{P}$ via $\{0\} \times \mathbb{R} \subset \mathbb{R}^2$. In this case we have $\mathscr{P} = \mathscr{G}$, and thus we have $\mathrm{V}\mathscr{P} \cong \mathrm{pr}_1^* \mathscr{g}$ by Cor. 6.22; though one expects something also in general by using the notion of fundamental vector fields. Hence, we can think of the LAB $\mathscr{g}$ as the vertical bundle.

We want to define a horizontal bundle as a "suitable" complement to the vertical bundle. One way would be the ***canonical flat horizontal distribution/connection*** $\mathrm{H}^{\mathrm{flat}}\mathscr{P}$ of $\mathscr{P}$ given by

$$\begin{tikzcd} \mathbb{R} \arrow[r] & \mathscr{P} \times \mathbb{R} \times \{0\} \arrow[d] \\ & \mathscr{P} \end{tikzcd}$$

We clearly have $\mathrm{T}\mathscr{P} = \mathrm{H}^{\mathrm{flat}}\mathscr{P} \oplus \mathrm{V}\mathscr{P}$ (Whitney sum). So, we will denote elements of $\mathrm{T}_{(x,g)}\mathscr{P}$ ($(x, g) \in \mathscr{P}$) by $(v_x, v_g) \in \mathbb{R}^2 \cong \mathrm{T}_x\mathbb{R} \oplus \mathrm{T}_g\mathbb{R}^*$, and we interpret the second component as a vertical component (but not necessarily the projection onto the vertical part of (v_x, v_g)). In the case of $\mathrm{H}^{\mathrm{flat}}\mathscr{P}$, we have $(v_x, v_g) \in \mathrm{H}^{\mathrm{flat}}_{(x,g)}\mathscr{P}$ if and only if $v_g = 0$.

We want to describe horizontal distributions by a projection onto the vertical bundle, $\mathrm{T}\mathscr{P} \to \mathrm{V}\mathscr{P}$; for $\mathrm{H}^{\mathrm{flat}}\mathscr{P}$ this projection is simply given by $(v_x, v_g) \mapsto (0, v_g)$. However, one can think of course of other horizontal distributions $\mathrm{H}\mathscr{P}$ with $\mathrm{T}\mathscr{P} = \mathrm{H}\mathscr{P} \oplus \mathrm{V}\mathscr{P}$ given by a more general projection $\mathrm{T}\mathscr{P} \to \mathrm{V}\mathscr{P}$. Imagine we have

$$\begin{aligned} \pi_v : \mathrm{T}\mathscr{P} &\to \mathrm{V}\mathscr{P}, \\ (v_x, v_g) &\mapsto \Big(0, v_g + \mathrm{D}_{(x,e)} R_{(x,g)}\big(\omega_x(v_x)\big)\Big) \end{aligned} \tag{41}$$

for all $(x, g) \in \mathscr{P}$, where (x, e) is the neutral element of $\mathscr{G}_x$ and

$$\omega \in \Omega^1(M; \mathscr{g}).$$

By definition $\mathrm{D}_{(x,e)} R_{(x,g)} : \mathscr{g}_x \to \mathrm{V}_{(x,g)}\mathscr{P}$; that its image is vertical quickly follows by $\mathscr{g}_x = \mathrm{V}_{(x,e)}\mathscr{G}$ (Def. 6.18) and

$$\mathrm{D}_{(x,g)}\mathrm{pr}_1 \circ \mathrm{D}_{(x,e)} R_{(x,g)} = \mathrm{D}_{(x,e)} \underbrace{\left(\mathrm{pr}_1 \circ R_{(x,g)}\right)}_{=\mathrm{pr}_1} = \mathrm{D}_{(x,e)}\mathrm{pr}_1$$

such that vertical vectors as the kernel of Dpr_1 stay vertical under $\mathrm{D}_{(x,e)} R_{(x,g)}$; or simply recall the proof of Cor. 6.22. Hence, Def. (41) is well-defined. Since $\mathbb{R}^*$ is a (rather trivial) abelian matrix group one knows that $\mathrm{D}R_{(x,g)}$ is just the multiplication with (x, g), that is, we can write

$$\pi_v(v_x, v_g) = (0, v_g + w_x(v_x)\ g).$$

Let us now define a ***horizontal distribution*** $\mathrm{H}\mathscr{P}$ as the kernel of π_v. By $\pi_v(0, v_g) = (0, v_g)$ we know that π_v is surjective, and thus the dimension of the kernel is

$$\dim\Big(\mathrm{Ker}\Big(\pi_v|_{(x,g)}\Big)\Big) = \dim\big(\mathrm{T}_{(x,g)}\mathscr{P}\big) - \dim\big(\mathrm{V}_{(x,g)}\mathscr{P}\big) = \dim(M),$$

and its elements are of the form

$$(v_x, -\omega_x(v_x)\ g),$$

desribing a lift of v_x as a tangent vector of the basis. Therefore we constructed a subbundle $\mathrm{H}\mathscr{P}$ of $\mathrm{T}\mathscr{P}$ with

$$\mathrm{T}\mathscr{P} = \mathrm{H}\mathscr{P} \oplus \mathrm{V}\mathscr{P}.$$

Let us now investigate the behaviour of this horizontal distribution under push-forwards of right-multiplications; as we already discussed before, we are interested into such push-forwards using sections of $\mathscr{G}$. That is, let $\widetilde{\sigma}$ a (local) section of $\mathscr{G}$; for the section we also write $\widetilde{\sigma}_x = (x, \sigma_x)$, where σ is a (locally) defined map with values in $\mathbb{R}^*$; for simplicity we also write σ for $\widetilde{\sigma}$, it should be clear by context what we mean. Then

$$\begin{aligned}
\mathrm{D}_{(x,g)} r_\sigma(v_x, v_g) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big((x + tv_x, g + tv_g) \cdot (x + tv_x, \sigma_{x+tv_x})\big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(x + tv_x, (g + tv_g)\ \sigma_{x+tv_x}\big) \\
&= \big(v_x, v_g\sigma_x + g\ \mathrm{D}_x\sigma(v_x)\big).
\end{aligned}$$

Also let $X = (X_x, -\omega_x(X_x)\ g) \in \mathrm{H}_{(x,g)}\mathscr{P}$, then

$$\mathrm{D}_{(x,g)} r_\sigma(X) = \big(X_x, -\omega_x(X_x)\ g\sigma_x + g\ \mathrm{D}_x\sigma(X_x)\big). \tag{42}$$

In order to have $\mathrm{D}_{(x,g)} r_\sigma(X) \in \mathrm{H}_{(x,g\sigma_x)}\mathscr{P}$ we would need

$$\mathrm{D}_{(x,g)} r_\sigma(X) \stackrel{!}{=} \big(X_x, -\omega_x(X_x)\ g\sigma_x\big),$$

and we would like that for all X_x, hence, $\mathrm{D}_{(x,g)} r_\sigma(X) \in \mathrm{H}_{(x,g\sigma_x)}\mathscr{P}$ for all X_x holds if and only if $\mathrm{D}_x\sigma = 0$. Varying with respect to x, σ needs to be a constant section of $\mathscr{P} = \mathbb{R} \times \mathbb{R}^*$. A constant section is equivalent to an element of $\mathbb{R}^*$, also recall Ex. 5.12, and thus we are back at the classical formulation of horizontal distributions. However, due to the lack of a suitable notion of a constant section in the general case, especially if $\mathscr{G}$ is non-trivial, we need to work differently with Eq. (42).

Rewrite Eq. (42) to

$$\begin{aligned}
\mathrm{D}_{(x,g)} r_\sigma(X) &= \big(X_x, -\omega_x(X_x)\ g\sigma_x\big) + \big(0, g\ \mathrm{D}_x\sigma(X_x)\big) \\
&= \big(X_x, -\omega_x(X_x)\ g\sigma_x\big) + \left.\overbrace{\mu_{\mathscr{G}}^{\mathrm{tot}}\big(\mathrm{D}_x\sigma(X_x)\big)}\right|_{(x,g\sigma_x)},
\end{aligned}$$

where

$$\mu_{\mathscr{G}}^{\mathrm{tot}}(Y) := \mu_{\mathscr{G}}\Big(\pi^{\mathrm{flat}}(Y)\Big)$$

for all $Y \in \mathrm{T}\mathscr{G}$, were π^{flat} is the projection onto $\mathrm{V}\mathscr{G}$ along the canonically flat horizontal distribution. The notation $\mu^{\mathrm{tot}}_{\mathscr{G}}$ comes from that it is essentially the Maurer-Cartan form but defined on the **tot**al tangent bundle of $\mathscr{G}$; in fact, by construction, it is **the** connection-1-form corresponding to the canonical flat connection on $\mathscr{G}$. So, in total we get that

$$\mathrm{D}_{(x,g)} r_\sigma(X) - \left.\widehat{\mu^{\mathrm{tot}}_{\mathscr{G}}\big(\mathrm{D}_x\sigma(X_x)\big)}\right|_{(x, g\sigma_x)}$$

is horizontal.

Recall Ex. 5.12, observe that $\mathscr{P}$ clearly admits the structure of an $\mathbb{R}^*$-principal bundle, and we have proven that $\mathrm{H}\mathscr{P}$ is a "typical" horizontal distribution on $\mathscr{P}$,[7] such that we have a 1:1 correspondence to a field of gauge bosons corresponding to $\mathrm{H}\mathscr{P}$. Thus, this example has obviously a strong correspondence to the classical formulation which allows us to compare the notions.

That is, altogether, we have derived that the pushforward of a horizontal vector is horizontal after subtracting a form very similar to the Maurer-Cartan form making use of a horizontal distribution on $\mathscr{G}$ itself. This horizontal distribution is the canonical flat one on $\mathscr{G}$, and the "classical" notion of a connection on $\mathscr{P}$ uses this canonical flat horizontal distribution in order to remediate the contribution of σ to the horizontality of $\mathrm{D}_{(x,g)} r_\sigma(X)$.

Henceforth, one now asks what happens if one chooses a different connection on $\mathscr{G}$. This is the notion we need because such a notion also allows non-trivial LGBs $\mathscr{G}$. The answer is clearly provided by Thm. 7.6: Define $Y \coloneqq \mathrm{D}_x\sigma(X_x) = \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(X_x)\big)$, then

$$\mathrm{D}_{\sigma_x}\mathrm{pr}_1(Y) = X_x = \mathrm{D}_{(x,g)}\mathrm{pr}_1(X),$$

and by Thm. 7.6 we get

$$\mathrm{D}_{(x,g)} r_\sigma(X) = \mathrm{D}_{((x,g),\sigma_x)}\Phi(X, Y).$$

[7] We have shown that the typical symmetry of horizontal distributions applies w.r.t. constant sections; then argue with Ex. 5.12.

8.2. Principal fibre bundles with structural LGB

8.3. Generalized distributions and connections

8.4. Generalized connection 1-forms on principal bundles

8.5. Gauge transformations

8.6. Generalized curvature/field strength

8.7. Generalized covariant derivative/minimal coupling

9. Curved Yang-Mills gauge theory

10. Conclusion

Acknowledgements: I want to thank Mark John David Hamilton and Alessandra Frabetti for their great help and support in making this paper.

Funding: The paper is part of my post-doc fellowship at the National Center for Theoretical Sciences (NCTS), which is why I also want to thank the NCTS.

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A. Axiomatic Yang-Mills gauge theories

Let us discuss where the compatibility conditions may arise from a certain axiomatic point of view.