Integrating curved Yang-Mills gauge theories

Gauge theories related to principal bundles equipped with Lie group bundle actions

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Abstract†

In this paper we construct a gauge theory based on principal bundles $\mathscr{P}$ equipped with a Lie group bundle action instead of a Lie group action. Due to the fact that pushforwards of right-translation act now only on the vertical structure of $\mathscr{P}$ we fix a connection on $\mathscr{G}$, which modifies the pushforward via right translation by subtracting the fundamental vector field generated by the Darboux derivative of a section of $\mathscr{G}$.

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†Abbreviations used in this paper: **LGB** for Lie group bundle, **LAB** for Lie algebra bundle.

Contents

1. Introduction

1.1. Basic notations and remarks

- The appendix serves for providing extra information or background knowledge which did not fit in the flow of this paper's text. We will sometimes refer to the appendix, but the experienced reader may be able to ignore the appendix.

- In the main text we usually repeat and reintroduce needed objects for the statements (like "**Let** M **be a manifold [...]**" in every statement) (almost) allowing to just read the statements without having read the text introducing it, while the appendix is written as a continuous text which has to be read as a whole in order to understand the essential statements.

- As usual, there will be definitions of certain objects depending on other elements, and for keeping notations simple we will not always explicitly denote all dependencies. It will be clear by context on which it is based on, that is for example, if we define an object A using the notion of Lie algebra actions γ and we write "Let X be an object A", then it will be clear by context which Lie algebra action is going to be used, for example given in a previous sentence writing "Let γ be a Lie algebra action".

- Throughout this work we always use Einstein's sum convention if suitable.

- With f^*F we denote the pullback/pull-back of the fibre bundles $F \to M$ under a smooth map $f : N \to M$. Similarly we denote the pullbacks of sections of a fibre bundle.

- For $V \to M$ a vector bundle over M do not confuse the pull-back of sections with the pull-back of forms $\omega \in \Omega^l(M;V)$ ($l \in \mathbb{N}_0$), here denoted by $f^!\omega$, which is an element of $\Gamma\left(\left(\bigwedge_{m=1}^l \mathrm{T}^*M\right) \otimes f^*V\right) \cong \Omega^l(M; f^*V)$, and not of $\Gamma\left(\left(\bigotimes_{m=1}^l (f^*\mathrm{T}N)^*\right) \otimes f^*V\right)$ like $f^*\omega$.

- Let $F \overset{\pi_F}{\to} M$ and $G \overset{\pi_G}{\to} N$ be two fibre bundles over smooth manifolds M and N, and let $\phi : N \to M$ be a smooth map. Furthermore, let us assume we have a morphism $\Phi : G \to F$ of fibre bundles over ϕ, that is, Φ is a smooth map such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & F \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_F} \\ N & \xrightarrow{\phi} & M \end{array}$$

especially, $\pi_F \circ \Phi = \phi \circ \pi_G$. We make often use of that such morphisms have a 1:1 correspondence to **base-preserving** fibre bundle morphisms $\widetilde{\Phi} : G \to \phi^* F$, *i.e.* $\widetilde{\Phi}$ is a smooth map with $\phi^* \pi_F \circ \widetilde{\Phi} = \pi_G$. For $p \in N$ the morphism $\widetilde{\Phi}$ has the form

$$\widetilde{\Phi}_p = (p, \Phi_p),$$

that is,

$$\widetilde{\Phi}_p(g) = \big(p, \Phi_p(g)\big)$$

for all $g \in G_p$, which is well-defined since $\Phi_p(g) \in F_{\phi(p)}$. The map $\widetilde{\Phi} \mapsto \Phi \coloneqq \mathrm{pr}_2 \circ \widetilde{\Phi}$ is then a bijective map between base-preserving morphisms $G \to \phi^* F$ and morphisms $G \to F$ over ϕ, where pr_2 is the projection onto the second component.

In total, $\widetilde{\Phi}$ is a base-preserving morphism if and only if Φ is a morphism over ϕ; in fact, one defines pullback bundles in such a way that this equivalence holds. Observe that $\widetilde{\Phi}$ is an isomorphism (diffeomorphism) if and only if Φ is a fibre-wise isomorphism (diffeomorphism).

One can extend all of this similarly for more specific types of morphisms like vector bundle-morphisms.

Most of the time we will not mention this 1:1 correspondence explicitly, it should be clear by context. Hence, we will also denote $\widetilde{\Phi}$ by Φ. In fact, we usually calculate with $\widetilde{\Phi}$, while Φ and its diagram may only arise to give an illustration about the geometry.

- When we differentiate maps γ depending on just one parameter $t \in \mathbb{R}$, then we may shortly write

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}[t \mapsto \gamma]\bigg|_t.$$

- We will often have connections on a bundle, that is, a splitting into a horizontal and vertical bundle of its tangent bundle. When we write "Let π be the projection onto the vertical/horizontal bundle", then this will be always w.r.t. to the fixed connection even though we are not explicitly mention it again.

- We also need to extend contractions of tensors to graded extensions:

Definition 1.1: Graded extension of products, [1, generalization of Definition 5.5.3; page 275]

Let $l \in \mathbb{N}$ and $E_1, \ldots E_{l+1} \to M$ be vector bundles over a smooth manifold M, and $F \in \Gamma\left(\left(\bigotimes_{m=1}^{l} E_m^*\right) \otimes E_{l+1}\right)$. Then we define the **graded extension of F** as

$$\Omega^{k_1}(M; E_1) \times \cdots \times \Omega^{k_l}(M; E_l) \to \Omega^k(M; E_{l+1}),$$

$$(A_1, \ldots, A_l) \mapsto F\big(A_1 \stackrel{\wedge}{,} \ldots \stackrel{\wedge}{,} A_l\big),$$

where $k := k_1 + \ldots k_l$ and $k_i \in \mathbb{N}_0$ for all $i \in \{1, \ldots, l\}$. $F\big(A_1 \stackrel{\wedge}{,} \ldots \stackrel{\wedge}{,} A_l\big)$ is defined as an element of $\Omega^k(M; E_{l+1})$ by

$$F\big(A_1 \stackrel{\wedge}{,} \ldots \stackrel{\wedge}{,} A_l\big)(Y_1, \ldots, Y_k) :=$$

$$\frac{1}{k_1! \cdot \cdots \cdot k_l!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\ F\left(A_1\left(Y_{\sigma(1)}, \ldots, Y_{\sigma(k_1)}\right), \ldots, A_l\left(Y_{\sigma(k-k_l+1)}, \ldots, Y_{\sigma(k)}\right)\right)$$

for all $Y_1, \ldots, Y_k \in \mathfrak{X}(M)$, where S_k is the group of permutations of $\{1, \ldots, k\}$ and $\mathrm{sgn}(\sigma)$ the signature of a given permutation σ.
$\stackrel{\wedge}{,}$ may be written just as a comma when a zero-form is involved.

In case of antisymmetric tensors we of course get:

Proposition 1.2: Graded extensions of antisymmetric tensors

Let $E_1, E_2 \to N$ be real vector bundles of finite rank over a smooth manifold N, $F \in \Omega^2(E_1; E_2)$. Then

$$F\big(A \stackrel{\wedge}{,} B\big) = -(-1)^{km} F\big(B \stackrel{\wedge}{,} A\big) \tag{1}$$

for all $A \in \Omega^k(N; E_1)$ and $B \in \Omega^m(N; E_2)$ ($k, m \in \mathbb{N}_0$). Similarly extended to all $F \in \Omega^l(E_1; E_2)$.

Remarks 1.3.
This is a generalization of similar relations just using the Lie algebra bracket $[\cdot,\cdot]_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$, see [1, §5, first statement of Exercise 5.15.14; page 316].

Proof.
Trivial, for example by using local coordinates. ■

1.2. Assumed background knowledge

It is highly recommended to have basic knowledge about differential geometry and gauge theory as presented in [1, especially Chapter 1 to 5], and we will follow the style and labeling as in

[1] when we generalize certain notions; however, sometimes we will still give explicit references to help with more technical details. It can be useful to have knowledge about Lie algebra and Lie group bundles, and even Lie algebroids and Lie groupoids, but we will introduce their basic notions such that it is not necessarily needed to have knowledge about these upfront.

We also often give references about Lie group bundles (LGBs), but the given references are often about Lie groupoids. If the reader has no knowledge about Lie groupoids, then it is important to know that LGBs are a special example of Lie groupoids; Lie groupoids carry "two projections", called **source** and **target**. An LGB is a special example of a Lie groupoid whose source equals the target.[1] If you look into such a reference, then the source and target are often denoted by α and β, or by s and t; simply put both to be the same and identify these with our bundle projection which we often denote by π. In that way it should be possible to read the references without the need to know Lie groupoids. However, we try to re-prove the needed statements such that these types of references could be avoided by the reader.

See also the previous subsection about notions we assume to be known.

2. Lie group bundles (LGBs)

2.1. Definition

[1] But not every Lie groupoid with equal source and target is an LGB, they're in general bundles of Lie groups which is not completely the same; this nuance will not be important here.

Definition 2.1: Lie group bundle, [2, §1.1, Def. 1.1.19; p. 11]

Let $G, \mathscr{G}, M$ be smooth manifolds. A fibre bundle

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{G} \\ & & \big\downarrow \pi \\ & & M \end{array}$$

is called a **Lie group bundle** if:

1. G and each fibre $\mathscr{G}_x \coloneqq \pi^{-1}(\{x\})$, $x \in M$, are Lie groups;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} \coloneqq \mathrm{pr}_2 \circ \phi_i|_{\mathscr{G}_x} : \mathscr{G}_x \to G$$

are Lie group isomorphisms, where I is an (index) set, U_i are open sets covering M, $\phi_i : \mathscr{G}|_U \to U \times G$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie group bundle atlas** or **LGB atlas**.

We also often say that $\mathscr{G}$ **is an LGB (over** M**)**, whose structural Lie group is either clear by context or not explicitly needed; and we may also denote LGBs by $G \to \mathscr{G} \xrightarrow{\pi} M$. The global section e of $\mathscr{G}$ so that e_p is the neutral element of $\mathscr{G}_p$ will be labelled as the **neutral (element) section (of** $\mathscr{G}$**)**; we may also denote it by $e^{\mathscr{G}}$.

Remark 2.2: Principal and Lie group bundles

Beware, a Lie group bundle is **not** the same as a principal bundle $P \to M$ with the same fibre type G. First of all, the fibres of P are just diffeomorphic to a Lie group, a priori they carry no Lie group structure, while the fibres of $\mathscr{G}$ carry a Lie group structure.

Second, on P we have a multiplication given as an action of G on P

$$P \times G \to P,$$

preserving the fibres P_x ($x \in M$) and simply transitive on them. Restricted on P_x we have

$$P_x \times G \to P_x.$$

For $\mathscr{G}$ we have canonically a multiplication over x given by

$$\mathscr{G}_x \times \mathscr{G}_x \to \mathscr{G}_x,$$

also clearly simply transitive. Observe, the second factor is not "constant", *i.e.* we do not have $\mathscr{G}_x \times G \to \mathscr{G}_x$ in general. Hence, there is in general no well-defined product $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ or $\mathscr{G} \times G \to \mathscr{G}$.

All of that is also resembled in the existence of sections. The existence of a section of P has a 1:1 correspondence to trivializations of P, which is why P in general only admits sections locally; see *e.g.* [1, §4.2, Thm. 4.2.19; page 219f.]. $\mathscr{G}$ clearly admits always a global section, even if $\mathscr{G}$ is non-trivial; just take the section which assigns each base point the neutral element of its fibre.

If M is a point we recover the notion of Lie groups, and, as usual, we have the notion of trivial LGBs:

Example 2.3: Trivial LGB

The **trivial LGB** is given as the product manifold $M \times G \to M$ with canonical multiplication $(x, g) \cdot (x, q) := (x, gq)$.

We are also interested into LGB bundle morphisms:

Definition 2.4: LGB morphism, [2, §1.2, special situation of Def. 1.2.1 & 1.2.3, page 12]

Let $\mathscr{H} \stackrel{\pi_{\mathscr{H}}}{\to} N$ and $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be two LGBs over two smooth manifolds N and M. An **LGB morphism F over f** is a pair of smooth maps $F : \mathscr{H} \to \mathscr{G}$ and $f : N \to M$ such that

$$\pi_{\mathscr{G}} \circ F = f \circ \pi_{\mathscr{H}}, \tag{2}$$

$$F(gq) = F(g)\ F(q) \tag{3}$$

for all $g, q \in \mathscr{H}$ with $\pi_{\mathscr{H}}(g) = \pi_{\mathscr{H}}(q)$. We then also say that F **is an LGB morphism over f**. If $N = M$ and $f = \mathrm{id}_M$, then we often omit mentioning f explicitly and just write that F **is a (base-preserving) LGB morphism**.
We speak of an **LGB isomorphism (over f)** if F is a diffeomorphism.

Remarks 2.5.

- The right hand side of Eq. (3) is well-defined because of Eq. (2).
- It is clear that condition 2 in Def. 2.1 is equivalent to say that $\mathscr{G}$ is locally isomorphic to a trivial LGB; as one may have expected already.
- If F is a diffeomorphism, then also f: By Eq. (2) surjectivity of f is clear; for $y \in M$ just take any $g \in \mathscr{G}_y$, and since F is a bijective, we have a $q \in \mathscr{H}_x$ for some $x \in N$ with $F(q) = g$. By Eq. (2) we have $y = \pi_{\mathscr{G}}(F(q)) \stackrel{(2)}{=} f(x)$, thence, surjectivity follows. For injectivity we know by Eq. (3) and (2) that $F\left(e^{\mathscr{H}}_x\right) = e^{\mathscr{G}}_{f(x)}$, where $e^{\mathscr{H}}_x$ and $e^{\mathscr{G}}_{f(x)}$ denote the unique neutral elements of $\mathscr{H}_x$ and $\mathscr{G}_{f(x)}$, respectively. Assume that there are $x, x' \in N$ with $f(x) = f(x')$, then we can derive

$$F\left(e^{\mathscr{H}}_x\right) = e^{\mathscr{G}}_{f(x)} = e^{\mathscr{G}}_{f(x')} = F\left(e^{\mathscr{H}}_{x'}\right).$$

Then we have $e_x^{\mathscr{H}} = e_{x'}^{\mathscr{H}}$ due to that F is bijective, and hence $x = x'$. Therefore f is bijective. Finally, F^{-1} is by assumption also a diffeomorphism, Eq. (3) clearly carries over, and Eq. (2) is w.r.t. f^{-1}, that is

$$\pi_{\mathscr{H}} \circ F^{-1} = f^{-1} \circ \pi_{\mathscr{G}}.$$

Since $\pi_{\mathscr{H}} \circ F^{-1}$ is smooth and $\pi_{\mathscr{G}}$ is a smooth surjective submersion, it follows that f^{-1} is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We can conclude that f is a diffeomorphism. Observe that we also concluded that F^{-1} is an LGB isomorphism, too.

Similar to the case of Lie groups, *the* example of an LGB are the automorphisms of a vector bundle.

Example 2.6: Automorphisms of a vector bundle, [2, §1.1, special situation of Ex. 1.1.12, page 8]

Let $V \to M$ be a vector bundle and $\mathrm{Aut}(V) \to M$ its bundle of fibre-wise automorphisms (not to be confused with the sections of $\mathrm{Aut}(V)$ which are the base-preserving automorphisms of V). Denote with W the structural vector space of V, then $\mathrm{Aut}(V)$ is an LGB with structural Lie group $\mathrm{Aut}(W)$. It is clear that each fibre of $\mathrm{Aut}(V)$ is a Lie group, and the LGB atlas is directly inherited by a vector bundle atlas $\{(U_i, L_i)\}_{i \in I}$ of V, where we use a similar notation as for LGB atlases, especially we have vector bundle trivializations $L_i : V|_{U_i} \to U_i \times W$. Then define an LGB atlas over the same open covering $(U_i)_i$ by

$$\mathrm{Aut}(V)|_{U_i} \to U_i \times \mathrm{Aut}(W),$$

$$T \mapsto L_i \circ T \circ L_i^{-1}|_{\{x\} \times W},$$

where $T \in \mathrm{Aut}(V)|_x = \mathrm{Aut}(V_x)$, and $U_i \times \mathrm{Aut}(W)$ acts canonically on $U_i \times W$ in a fibre-wise sense. Then it is trivial to check that these give local trivializations such that $\mathrm{Aut}(V)$ carries the structure as an LGB.

2.2. Associated Lie group bundles

For another important example recall that there is the notion of associated fibre bundles; following and stating the results of [2, §1, Construction 1.3.8, page 20] and [1, §4.7, page 237ff.; see also Rem. 4.7.8, page 242f.]: Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G over a smooth manifold M, N another smooth manifold, and Ψ a smooth left G-action denoted by

$$G \times N \to N,$$

$$(g, v) \mapsto \Psi(g, v) := g \cdot v.$$

Then we have a right G-action on $P \times N$ given by

$$(P \times N) \times G \to P \times N,$$

$$(p, v, g) \mapsto \left(p \cdot g, g^{-1} \cdot v\right),$$

and one can show that the quotient under this action, $P \times_\Psi N \coloneqq (P \times N) \big/ G$, yields the structure of a fibre bundle

$$\begin{tikzcd} N \arrow[r] & P \times_\Psi N \arrow[d, "\pi_{P \times_\Psi N}"] \\ & M \end{tikzcd}$$

such that the map to the equivalence classes $P \times N \to P \times_\Psi N$, $(p, v) \mapsto [p, v]$, is a smooth surjective submersion, where the projection $\pi_{P \times_\Psi N} : P \times_\Psi N \to M$ is given by

$$\pi_{P \times_\Psi N}([p, v]) \coloneqq \pi_P(p)$$

for all $[p, v] \in P \times_\Psi N$. For $x \in M$, the fibre $(P \times_\Psi N)_x$ is given by $(P_x \times N) \big/ G = P_x \times_\Psi N$, and the fibre is diffeomorphic to N by $N \ni v \mapsto [p, v] \in (P \times_\Psi N)_x$ for a fixed $p \in P_x$. We will frequently use this diffeomorphism in the following without further notice.

A very important example are of course associated vector bundles, related to N being a vetor space. We need a similar concept for Lie groups.

Definition 2.7: Lie group representation on Lie groups, [2, special situation of the comment after Ex. 1.7.14, page 47]

Let G, H be Lie groups. Then a **Lie group representation of G on H** is a smooth left action ψ of G on H

$$G \times H \to H,$$

$$(g, h) \mapsto \psi_g(h) \coloneqq \psi(g, h)$$

such that

$$\psi_g(hq) = \psi_g(h)\ \psi_g(q) \tag{4}$$

for all $g \in G$ and $h, q \in H$.

Remark 2.8: Note about labeling

Observe that we have by the definition of group actions

$$\psi_{gg'} = \psi_g \circ \psi_{g'}$$

for all $g, g' \in G$, viewing ψ_g as a map $H \to H$. Therefore we can view the action ψ as a group homomorphism

$$G \to \mathrm{Aut}(H),$$

where $\mathrm{Aut}(H)$ is the set of Lie group automorphisms. The similarity to Lie group representations on vector spaces is obvious, thence the name. In fact, these integrate Lie group representations on Lie algebras: Observe that $\mathrm{D}_e\psi_g$ is a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{h}$ due to that ψ_g is a Lie group homomorphism, where $\mathfrak{h}$ is the Lie algebra of H and e is the neutral element of H. $\mathrm{D}_e\psi_g$ is an automorphism because ψ_g is, and it is a homomorphism in g due to that $\psi_{gg'} = \psi_g \circ \psi_{g'}$ implies $\mathrm{D}_e\psi_{gg'} = \mathrm{D}_e\psi_g \circ \mathrm{D}_e\psi_{g'}$, and thus $g \mapsto \mathrm{D}_e\psi_g$ is a G-representation on $\mathfrak{h}$ with values in automorphisms of Lie algebras (not just vector spaces).

This definition is of course also motivated by various references pointing out that Lie group representations define Lie group actions with extra properties; see for example [1, §3, Ex. 3.4.2, page 143f.]. In [2, comments after Ex. 1.7.14, page 47] this definition is also called *action by Lie group isomorphisms.*

With this we can discuss and define associated Lie group bundles; the following definition is clearly motivated by the definition of associated vector bundles as *e.g.* provided in [1, §4, Thm. 4.7.2, page 239f.].

Theorem 2.9: Associated Lie group bundle as quotient

Let G, H be Lie groups, $P \overset{\pi_P}{\to} M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then $\mathscr{H} := P \times_\psi H$ is an LGB

$$\begin{tikzcd} H \arrow[r] & \mathscr{H} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

with projection π given by

$$\mathscr{H} \to M,$$

$$[p, h] \mapsto \pi_P(p), \tag{5}$$

and fibres

$$\mathscr{H}_x = P_x \times_\psi H \tag{6}$$

for all $x \in M$, which are isomorphic to H as Lie groups. The Lie group structure on each fibre $\mathscr{H}_x$ is defined by

$$[p, h] \cdot [p, q] := [p, hq] \tag{7}$$

for all $h, q \in H$ and $p \in P_x$ (that is, $\pi_P(p) = x$).

Remark 2.10: Neutral and inverse elements

The neutral element for $\mathscr{H}_x$ ($x \in M$) is given by

$$e_x = [p, e],$$

where $p \in P_x$ is arbitrary and e is the neutral element of H. This is clearly independent of the choice of p due to

$$[p, e] = \left[p \cdot g, \psi_{g^{-1}}(e)\right] = [p \cdot g, e]$$

for all $g \in G$. Thence, the fact that e_x is the neutral element follows immediately. By Def. (7) the inverse of $[p, h] \in \mathscr{H}_x$ is clearly given by

$$([p, h])^{-1} = \left[p, h^{-1}\right].$$

Proof.

- That π is the well-defined projection and that the fibres are precisely $P_x \times_\psi H$ for all $x \in M$ is well-known, see our discussion before Def. 2.7 and the references therein; it is also very straightforward to check. We also discussed that $\mathscr{H}$ is a fibre bundle with structural fibre H. Hence, if one knows that the proposed group structure in Def. (7) is well-defined, then the smoothness of the group structure is implied by the smoothness structures of H and $\mathscr{H}$. Thence, let us check whether Def. (7) is well-defined. Let $x \in M$, $p \in P_x$ and $p' := p \cdot g'$ be another element of P_x, where $g' \in G$. Also let $[p_1, h_1], [p_2, h_2] \in P_x \times_\psi H$; then we have unique elements q_i, q_i' of G such that ($i \in \{1, 2\}$)

$$p_i = p \cdot q_i, \qquad\qquad p_i = p' \cdot q_i',$$

especially, it follows $q_i = g' q_i'$. On the one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, h_1] \cdot [p_2, h_2] = [p, \psi_{q_1}(h_1)] \cdot [p, \psi_{q_2}(h_2)] = [p, \psi_{q_1}(h_1)\ \psi_{q_2}(h_2)], \tag{8}$$

on the other hand, using Def. 2.7 and $p' = p \cdot g'$ instead of p,

$$\begin{aligned}
[p_1, h_1] \cdot [p_2, h_2] &= \left[p \cdot g', \psi_{q_1'}(h_1)\ \psi_{q_2'}(h_2)\right] \\
&= \Bigg[p, \underbrace{\psi_{g'}\Big(\psi_{q_1'}(h_1)\ \psi_{q_2'}(h_2)\Big)}_{=\psi_{g'}\left(\psi_{q_1'}(h_1)\right)\ \psi_{g'}\left(\psi_{q_2'}(h_2)\right)}\Bigg] \\
&= \left[p, \psi_{g'q_1'}(h_1)\ \psi_{g'q_2'}(h_2)\right] \\
&= [p, \psi_{q_1}(h_1)\ \psi_{q_2}(h_2)],
\end{aligned}$$

which implies that Def. (7) is well-defined, and thus defines a Lie group structure on each fibre of $\mathscr{H}$.

• That the fibres $\mathscr{H}_x$ are isomorphic to H as Lie groups for all $x \in M$ also quickly follows. Recall by our discussion before Def. 2.7 that the fibres are diffeormorphic to H by $H \ni h \mapsto [p,h] \in \mathscr{H}_x$ for a fixed $p \in P_x$. By Def. (7) it is clear that this map is a Lie group homomorphism and hence a Lie group isomorphism.

• Let us now construct an LGB atlas for $\mathscr{H}$, denoting its maps by ϕ_U w.r.t. an open subset $U \subset M$. For this we will use a principal bundle atlas for P, that is, for some $U \subset M$ open and a local trivialization $\varphi_U : P_U \to U \times G$ of P we write

$$\varphi_U(p) = \big(\pi_P(p), \beta_U(p)\big)$$

for all $p \in P$, where $\beta_U : P_U \to G$ is an equivariant map, *i.e.* $\beta_U(p \cdot g) = \beta_U(p)\, g$ for all $g \in G$. Then define ϕ_U as a map by

$$\mathscr{H}_U \to U \times H,$$

$$[p,h] \mapsto \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big).$$

ϕ_U is well-defined: Let $[p',h'] \in \mathscr{H}_U$ with $[p',h'] = [p,h]$. Then there is a $g \in G$ such that

$$\big(p',h'\big) = \big(p \cdot g, \psi_{g^{-1}}(h)\big),$$

hence, using the equivariance of β_U and Def. 2.7,

$$\phi_U\big([p',h']\big) = \Big(\underbrace{\pi_P(p\cdot g)}_{=\pi_P(p)}, \underbrace{\big(\psi_{\beta_U(p\cdot g)} \circ \psi_{g^{-1}}\big)}_{=\psi_{\beta_U(p)}\circ\psi_g\circ\psi_{g^{-1}}}(h)\Big) = \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big) = \phi_U\big([p,h]\big),$$

which proves that ϕ_U is well-defined. Denote the projection onto equivalence classes $P \times H \to \mathscr{H}$ by ϖ, then observe

$$\phi_U \circ \varpi = L,$$

where $L_U : P_U \times H \to U \times H$ is given by $L_U(p,h) \coloneqq \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big)$ for all $(p,h) \in P_U \times H$. L_U is clearly smooth and recall that ϖ is a smooth surjective submersion, therefore ϕ_U is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We define a candidate of the inverse $\phi_U^{-1} : U \times H \to \mathscr{H}_U$ by

$$\phi_U^{-1}(x,h) = \big[\varphi_U^{-1}(x,e), h\big]$$

for all $(x,h) \in U \times H$, where e is the neutral element of G. By the definition of φ_U we immediately get

$$(x,e) = \big(\varphi_U \circ \varphi_U^{-1}\big)(x,e) = \Big(\pi_P\big(\varphi_U^{-1}(x,e)\big), \beta_U\big(\varphi_U^{-1}(x,e)\big)\Big),$$

for all $x \in U$, and, also using again the equivariance of β_U,

$$\begin{aligned}
\varphi_U^{-1}(\pi_P(p), e) &= \varphi_U^{-1}\Big(\pi_P\big(p \cdot \beta_U^{-1}(p)\big), \beta_U(p)\ \beta_U^{-1}(p)\Big) \\
&= \varphi_U^{-1}\Big(\pi_P\big(p \cdot \beta_U^{-1}(p)\big), \beta_U\big(p \cdot \beta_U^{-1}(p)\big)\Big) \\
&= \big(\varphi_U^{-1} \circ \varphi_U\big)\big(p \cdot \beta_U^{-1}(p)\big) \\
&= p \cdot \beta_U^{-1}(p)
\end{aligned}$$

for all $p \in P_U$. Then altogether

$$\big(\phi_U \circ \phi_U^{-1}\big)(x, h) = \Big(\pi_P\big(\varphi_U^{-1}(x, e)\big), \psi_{\beta_U\left(\varphi_U^{-1}(x,e)\right)}(h)\Big) = \big(x, \psi_e(h)\big) = (x, h),$$

for all $(x, h) \in U \times H$, and

$$\big(\phi_U^{-1} \circ \phi_U\big)([p, h]) = \big[\underbrace{\varphi_U^{-1}(\pi_P(p), e)}_{=p\cdot\beta_U^{-1}(p)}, \psi_{\beta_U(p)}(h)\big] = [p, h]$$

for all $[p, h] \in \mathscr{H}_U$. Thus, ϕ_U is bijective; additionally observe

$$\phi_U^{-1}(x, h) = \varpi\big(\varphi_U^{-1}(x, e), h\big)$$

such that ϕ_U^{-1} is clearly smooth as the composition of smooth maps, and we therefore conclude that ϕ_U is a diffeomorphism. Finally, derive with Def. 2.7 and Eq. (8) that

$$\begin{aligned}
(\mathrm{pr}_2 \circ \phi_U)\big([p_1, h_1] \cdot [p_2, h_2]\big) &= (\mathrm{pr}_2 \circ \phi_U)\big([p, \psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)]\big) \\
&= \psi_{\beta_U(p)}\big(\psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)\big) \\
&= \underbrace{\psi_{\beta_U(p)}\big(\psi_{q_1}(h_1)\big)}_{=\psi_{\beta_U(p)\cdot q_1}(h_1)} \cdot\ \psi_{\beta_U(p)}\big(\psi_{q_2}(h_2)\big) \\
&= \psi_{\beta_U(p_1)}(h_1) \cdot \psi_{\beta_U(p_2)}(h_2) \\
&= (\mathrm{pr}_2 \circ \phi_U)\big([p_1, h_1]\big) \cdot (\mathrm{pr}_2 \circ \phi_U)\big([p_1, h_1]\big)
\end{aligned}$$

for all $[p_1, h_1], [p_2, h_2] \in \mathscr{H}_x$, where we used again the equivariance of β_U and the same notation as introduced for Eq. (8), and pr_2 denotes the projection onto the second factor. Thence, $\mathrm{pr}_2 \circ \phi_U$ induces Lie group isomorphisms $\mathscr{H}_x \to H$ for all $x \in U$; by Def. 2.1 we can finally conclude that $\mathscr{H}$ is an LGB. ■

Hence, we define:

Definition 2.11: Associated Lie group bundle,
labeling similar to [1, §4.7, Def. 4.7.3, page 240]

Let G, H be Lie groups, $P \overset{\pi_P}{\to} M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then we call the LGB

$$\mathscr{H} := (P \times H)\big/G := P \times_{\psi} H$$

the **Lie group bundle (LGB) associated** to the principal bundle P and the representation ψ on H:

$$\begin{array}{ccc} H & \longrightarrow & P \times_{\psi} H \\ & & \big\downarrow{\scriptstyle \pi_{\mathscr{H}}} \\ & & M \end{array}$$

The special situation of $H = G$ is already an important example:

Example 2.12: Inner group bundle,
[2, §1, paragraph after Def. 1.1.19, page 11; comment after Construction 1.3.8, page 20]

The **inner group bundle** or **inner LGB** of a principal bundle $P \to M$, denoted by $c_G(P)$, is defined by

$$c_G(P) := P \times_{c_G} G, \tag{9}$$

where $c_G : G \times G \to G$ is the left action of G on itself given by the very well-known **conjugation**

$$c_G(g, h) := c_g(h) = \left(L_g \circ R_{g^{-1}}\right)(h) = ghg^{-1} \tag{10}$$

for all $g, h \in G$, where we also denote left- and right-multiplications (with g) by L_g and R_g, respectively; see *e.g.* [1, beginning of §1.5.2, page 40f.] for its common properties. It is well-known that c_G satsfies the properties of a Lie group representation of G on itself in the sense of Def. 2.7.
$c_G(P)$ is an LGB by Thm. 2.9.

3. LGB actions, part I

3.1. Definition

As for Lie groups, we are interested into their actions. The idea is the following, similar to [2, §1.6, discussion around Def. 1.6.1, page 34]: We have an LGB $\mathscr{G} \to M$ over a smooth manifold

M, and we want to construct an action of $\mathscr{G}$ on another smooth manifold N. Each fibre of $\mathscr{G}$ is a Lie group, and we have a notion of Lie groups acting on a manifold N. Therefore one could define an LGB action as a collection of Lie group actions, that is, only sections of $\mathscr{G}$ act on N; however, one then expects that the general outcome of a product of $\Gamma(\mathscr{G})$ on N would be smooth maps from M to N. In order to recover a typical structure of action one could instead introduce a "multiplication rule", *i.e.* each point $p \in N$ can only be multiplied with elements of a specific fibre of $\mathscr{G}$. This "multiplication rule" will be described by a smooth map $f : N \to M$ in the sense of that the fibre over $f(p)$ will act on p.

For this recall that there is the notion of pullbacks of fibre bundles, see *e.g.* [1, §4.1.4, page 203ff.; especially Thm. 4.1.17, page 204f.]. That is, if we additionally have a smooth manifold N and a smooth map $f : M \to N$, then we have the pullback $f^*\mathscr{G}$ of $\mathscr{G}$ as a fibre bundle defined as usual by

$$f^*\mathscr{G} \coloneqq \{(x, g) \in N \times \mathscr{G} \mid f(x) = \pi(g)\}\,. \tag{11}$$

It is an embedded submanifold of $N \times \mathscr{G}$, and the structural fibre is the same Lie group as for $\mathscr{G}$. That is, the following diagram commutes

$$\begin{tikzcd} f^*\mathscr{G} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathscr{G} \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

where π_1 and π_2 are the projections onto the first and second factor, respectively, of $N \times \mathscr{G}$. Actually, $f^*\mathscr{G}$ carries a natural structure as an LGB.

Corollary 3.1: Pullbacks of LGBs are LGBs, [2, §2.3, simplified situation of the discussion around Prop. 2.3.1, page 63ff.]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Then $f^*\mathscr{G}$ has a unique (up to isomorphisms) LGB structure such that the projection $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$ onto the second factor is an LGB morphism over f with $\pi_2|_x : (f^*\mathscr{G})_x \to \mathscr{G}_{f(x)}$ being a Lie group isomorphism for all $x \in N$.

Remarks 3.2.
The mentioned reference, [2, §2.3, discussion around Prop. 2.3.1, page 63ff.], is rather general, formulated for Lie groupoids. If the reader is only interested into LGBs, then see *e.g.* [3, §3, Thm. 3.1].

Proof.
By construction, the structural fibre of $f^*\mathscr{G}$ is the same Lie group G as for $\mathscr{G}$, and for all $x \in N$ we have $(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)}$, thence, the fibres are Lie groups and the fibrewise group multiplication has the form

$$(x, g) \cdot (x, q) = (x, gq) \tag{12}$$

for all $x \in N$ and $g, q \in (f^*\mathscr{G})_x$. We are left to show the existence of an LGB atlas. For this fix an LGB atlas $\{(U_i, \phi_i\}_{i\in I}$ of $\mathscr{G}$, where I is an (index) set, $(U_i)_{i\in I}$ an open covering of M, and $\phi_i : \mathscr{G}|_{U_i} \to U_i \times G$ are LGB isomorphisms. Then $f^{-1}(U_i)$ gives rise to an open covering of N, and we get

$$f^*\phi_i : f^*\mathscr{G}|_{f^{-1}(U_i)} \to f^{-1}(U_i) \times G,$$

$$(x, g) \mapsto \big(x, \phi_{i,f(x)}(g)\big),$$

where $\phi_{i,f(x)} : \mathscr{G}_{f(x)} \to G$ are the Lie group isomorphisms as defined in Def. 2.1. It is immediate by construction that this gives an LGB atlas.

That this is the unique (up to isomorphisms) LGB structure such that $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$ is an LGB morphism over f inducing a Lie group isomorphism on each fibre simply follows by construction; observe for all $x \in N$ that $\pi_2|_x$ is clearly bijective. Furthermore, LGB morphisms need to be homomorphisms, which means here

$$\pi_2\big((x,g)\cdot(x,q)\big) \stackrel{!}{=} \pi_2\big((x,g)\big)\cdot\pi_2\big((x,q)\big) = gq = \pi_2\big((x,gq)\big)$$

for all $x \in N$ and $g, q \in (f^*\mathscr{G})_x$. By using the bijectivity of $\pi_2|_x$, the group structure leading to this equation is uniquely the one provided in Eq. (12). Assume we have another underlying LGB atlas in sense of Cor. 3.1 with an LGB chart ψ_i on $f^*\mathscr{G}|_{f^{-1}(U_i)}$ (or just w.r.t. a subset of U_i), then

$$\phi_i \circ \pi_2 \circ \psi_i^{-1} = \underbrace{\phi_i \circ \pi_2 \circ (f^*\phi_i)^{-1}}_{=(f,\mathbb{1}_G)} \circ f^*\phi_i \circ \psi_i^{-1} = (f, \mathbb{1}_G) \circ f^*\phi_i \circ \psi_i^{-1},$$

$(f, \mathbb{1}_G)$ is clearly a Lie group isomorphism $f^{-1}(U_i) \times G \cong U_i \times G$ of trivial LGBs, and thus the condition about $\pi_2|_x$ being a Lie group isomorphism enforces that we can write

$$\big(f^*\phi_i \circ \psi_i^{-1}\big)(x, g) = \big(x, L_i(x, g)\big)$$

for all $(x, g) \in f^{-1}(U) \times G$, where $L_i(x, \cdot) : G \to G$ is a Lie group automorphism. So, in total

$$\phi_i \circ \pi_2 \circ \psi_i^{-1} = \big(f, L\big),$$

and since f and the left hand side are smooth, L has to be smooth. We can conclude that ψ_i gives rise to an LGB atlas compatible with the one defined by $f^*\phi_i$. This finalizes the proof. ■

Definition 3.3: Pullback LGB

Let M, N be smooth manifolds, $\mathscr{G} \to M$ an LGB over M and $f : N \to M$ a smooth map. Then we call the LGB structure on $f^*\mathscr{G}$ as given in Cor. 3.1 the **pullback LGB of $\mathscr{G}$ (under f)**.

With writing $f^*\mathscr{G}$ we will often refer to this structure without further mention.

Let us now define $\mathscr{G}$-actions.

Definition 3.4: Lie group bundle actions, [2, §1.6, special case of Def. 1.6.1, page 34]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f: N \to M$ a smooth map. Then a **right-action of $\mathscr{G}$ on N** is a smooth map

$$f^*\mathscr{G} \to N,$$

$$(p, g) \mapsto p \cdot g,$$

satisfying the following properties:

$$f(p \cdot g) = \pi(g), \tag{13}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{14}$$

$$p \cdot e_{f(p)} = p \tag{15}$$

for all $p \in N$ and $g, h \in \mathscr{G}_{f(p)}$, where e is the neutral section of $\mathscr{G}$.

We analogously define left-actions, and we often write (left or right) **$\mathscr{G}$-action on N**. Furthermore, in order to increase readability as long as the dependency on f is not important, we introduce the notation

$$N * \mathscr{G} := f^*\mathscr{G}, \tag{16}$$

such that the action's notation has the typical shape $N * \mathscr{G} \to N$; one may also write $N * \mathscr{G} = N \times_M \mathscr{G}$. For left actions similarly $\mathscr{G} * N \to N$; even though $\mathscr{G} * N$ is the same pullback LGB as for right-actions; we also change the order of notation in this case, that is, $(g, p) \in \mathscr{G} * N$ reads $g \in \mathscr{G}_{f(p)}$ and $p \in N$.

Remark 3.5: Relation to the structure of the canonical pullback Lie group bundle over N

Observe that by the definition of $f^*\mathscr{G}$ we can also write

$$f(p \cdot g) = f(p),$$

so, the $\mathscr{G}$-action is defined in such a way that f is invariant under it. If f is the projection of N as a bundle over M, then this means that an LGB action is fibre-preserving. Moreover, the fibre-wise group structure on $\mathscr{G}$ naturally defines a $\mathscr{G}$-action on $\mathscr{G}$; in this situation f would be π itself, see also Ex. 3.12 later. Furthermore, having $M = \{*\}$ recovers the notion of a Lie group action and condition (13) is then trivial.

The other conditions are the typical conditions for actions, especially such that we get a $\mathscr{G}$-action on $f^*\mathscr{G}$ by

$$(p, g) \cdot q := \left(p \cdot q, q^{-1} g\right) \tag{17}$$

for all $p \in N$ and $g, q \in \mathscr{G}_{f(p)}$.[a] As usual, this gives rise to an equivalence relation, whose set of equivalence classes denoted by $f^*\mathscr{G} \big/ \mathscr{G}$ is isomorphic to N (as a set) by $[p, g] \mapsto p \cdot g$, where we denote equivalence classes of $(p, g) \in f^*\mathscr{G}$ by $[p, g]$. All of this is straight-forward to check. Finally, observe the similarity to associated fibre bundles.

[a] In alignment to Def. 3.4, this action is a map $(f \circ \pi_1)^*\mathscr{G} \to f^*\mathscr{G}$, where π_1 is the projection onto the first factor in $f^*\mathscr{G}$.

Remark 3.6: Localizing LGB actions

We can actually localize the LGB action, but in general not with respect to any open neighbourhood of N since that is in general not possible in a non-trivial way, *i.e.* the action cannot be brought into the form $(N * \mathscr{G})|_U \to U$ for arbitrary U because $p \cdot g$ may for example leave an neighbourhood U of p. However, with respect to M this is possible: Fix any open neighbourhood U of M. Then $f^{-1}(U)$ is an open neighbourhood of N, and we can restrict the action to $f^{-1}(U)$, resulting into a map

$$(N * \mathscr{G})|_{f^{-1}(U)} \to f^{-1}(U),$$

because of $f(p \cdot g) \overset{\text{Eq. (13)}}{=} \pi(g) = f(p)$, that is, if $(p, g) \in (N * \mathscr{G})|_{f^{-1}(U)}$, then $p \in f^{-1}(U)$ and so $p \cdot g \in f^{-1}(U)$. In fact, by the definition of $N * \mathscr{G}$ as $f^*\mathscr{G}$, this describes a $\mathscr{G}|_U$-action on $f^{-1}(U)$

$$f^{-1}(U) * \mathscr{G}|_U := \left(f|_{f^{-1}(U)}\right)^* (\mathscr{G}|_U) = (N * \mathscr{G})|_{f^{-1}(U)} \to f^{-1}(U).$$

If $x \in M$ is a regular value of f, then $f^{-1}(\{x\})$ is an embedded submanifold of N due to the regular value theorem (for this see *e.g.* [1, §A.1, Thm. A.1.32, page 611]). In that case one could apply the same arguments to restrict the action on $f^{-1}(\{x\})$. Since $\mathscr{G}_x$ is a Lie group one actually gets a typical Lie group action on $f^{-1}(\{x\})$. For more details about this see Ex. 3.15 later.

Remark 3.7: Left- and right-actions

In the following we usually define everything with respect to right-actions; however, one can of course define the following notions for left actions in a similar manner. If we ever speak of a left action, then we assume precisely this. Some subtle changes like a sign change will be pointed out though.

One can probably see that it is straightforward to extend a lot of the typical notions of Lie group actions to LGB actions; hence, we mainly focus on the definitions and properties which we need in this paper.

Definition 3.8: Left and right translations, [1, §3.2, notation similar to Def. 3.2.3, page 131] [2, §1.4, special situation of Def. 1.4.1 and its discussion, page 22]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Furthermore assume that we have a right action $N * \mathscr{G} \to N$. We define the **right translation** over $x \in M$ with $g \in \mathscr{G}_x$ as a map r_g defined by

$$f^{-1}(\{x\}) \to f^{-1}(\{x\}),$$

$$p \mapsto p \cdot g,$$

and we define the **orbit map** through $p \in f^{-1}(\{x\})$ as a map Φ_p given by

$$\mathscr{G}_x \to N,$$

$$g \mapsto p \cdot g.$$

For $\sigma \in \Gamma(\mathscr{G})$ we define the **right translation** on N as a map r_σ by

$$N \to N,$$

$$p \mapsto p \cdot \sigma_{f(p)}.$$

If one has a section of f, *i.e.* a smooth map $\tau : M \to N$, $x \mapsto \tau_x$, with $f \circ \tau = \mathbb{1}_M$, then we can define the **orbit map** through τ as a map Φ_τ given by

$$\mathscr{G} \to N,$$

$$g \mapsto \tau_{\pi(g)} \cdot g.$$

Remark 3.9: Left action and translation

Similarly we define left translations for left actions, which we similarly denote by l_g and l_σ. By Rem. 3.6 we can define r_σ (and l_σ) also for local sections $\sigma \in \Gamma(\mathscr{G}|_U)$ by restricting N onto $f^{-1}(U)$, where U is some open subset of M; then $r_\sigma, l_\sigma : f^{-1}(U) \to f^{-1}(U)$. In the same manner one achieves a restriction for $\Phi_\tau : \mathscr{G}|_U \to f^{-1}(U)$, if $\tau : U \to f^{-1}(U)$. In case of N being $\mathscr{G}$ itself we will introduce right (and left) translations later via capital letters.

Remark 3.10: Group action on sections

Assume that N is a fibre bundle over M with f as its projection. Also observe that $\Gamma(\mathscr{G})$ is clearly a group and it may be possible to endow it with an infinite-dimensional Lie group structure (but this is not important here now). In the same fashion as before, we can define a $\Gamma(\mathscr{G})$-action on $\Gamma(N)$ given by

$$(\tau \cdot \sigma)_x \coloneqq \tau_x \cdot \sigma_x$$

for all $\tau \in \Gamma(N)$, $\sigma \in \Gamma(\mathscr{G})$, and $x \in M$. It is straightforward to show that this is well-defined. We will make use of this action without further mention.

Remarks 3.11.
Similar to the arguments in [1, §3.2, discussion after Def. 3.2.3, page 131], Φ_p is given by the composition of smooth maps

$$\mathscr{G}_x \to N * \mathscr{G} \to N,$$

$$g \mapsto (p, g) \mapsto p \cdot g.$$

The second arrow/map is smooth due to the fact that we have a smooth action; the first one is smooth because $N * \mathscr{G}$ is the pullback LGB $f^*\mathscr{G}$ and the first arrow is precisely the embedding of $\mathscr{G}_x$ into $f^*\mathscr{G}$ as a fibre over p; recall *e.g.* Cor. 3.1.

For the right-translation r_σ we proceed in a similar fashion, namely, r_σ is a composition of smooth maps

$$N \to N * \mathscr{G} \to N,$$

$$p \mapsto \left(p, \sigma_{f(p)}\right) \mapsto p \cdot \sigma_{f(p)}.$$

The first map describes now a section of $N * \mathscr{G} = f^*\mathscr{G}$ as LGB over N, and thence an embedding. Thus, smoothness follows again as previously. Similarly, Φ_τ is the composition of maps

$$\mathscr{G} \to N * \mathscr{G} \to N,$$

$$g \mapsto \left(\tau_{\pi(g)}, g\right) \mapsto \tau_{\pi(g)} \cdot g,$$

and the first arrow is clearly a smooth map $\mathscr{G} \to N \times \mathscr{G}$ with values in $f^*\mathscr{G} = N * \mathscr{G}$ which is an embedded submanifold of $N \times \mathscr{G}$. Thence, smoothness follows as usual. In fact, $\Phi_\tau|_{\mathscr{G}_x} = \Phi_{\tau_x}$.

However, for r_g smoothness can only be discussed if $f^{-1}(\{x\})$ is a smooth manifold. That is for example the case if x is a regular value of f; recall the regular value theorem as cited in [1, §A.1, Thm. A.1.32, page 611]. This would be the case if *e.g.* f is a submersion. If x is a regular value, then $f^{-1}(\{x\})$ is an embedded submanifold of N, and r_g is a similar composition of smooth maps as for r_σ but restricted to $f^{-1}(\{x\})$

$$f^{-1}(\{x\}) \to (N * \mathscr{G})|_{f^{-1}(\{x\})} \to f^{-1}(\{x\}),$$

$$p \mapsto (p, g) \mapsto p \cdot g.$$

Since $f^{-1}(\{x\})$ is an embedded submanifold, $(N * \mathscr{G})|_{f^{-1}(\{x\})}$ is also a fibre bundle, see for example [1, §4.1, Lemma 4.1.16, page 204], and trivially an embedded submanifold of $N * \mathscr{G}$. Altogether, the same arguments as for r_σ apply.

Last but not least, r_σ is clearly a diffeomorphism with inverse $r_{\sigma^{-1}}$, where $\left(\sigma^{-1}\right)_x := \sigma_x^{-1} := (\sigma_x)^{-1}$. Similarly for r_g, if $f^{-1}(\{x\})$ is an embedded submanifold (otherwise r_g is just a bijection). Analogously for l_σ and l_g in case of a left action.

Motivated by the previous remark, it might be useful to require that f is a submersion, or that N is actually some bundle over M and f its projection. In fact, this will be the case later.

3.2. Examples of LGB actions

If M is a point or f a constant map, then we recover the typical notion of a Lie group action acting on N. Additionally, we have the following examples, all will be important in this paper. The notation will be as in Def. 3.8.

Example 3.12: LGB acting on itself

Each LGB $\mathscr{G} \xrightarrow{\pi} M$ acts on itself from the left and right, having $N := \mathscr{G}$ and $f := \pi$,

$$\mathscr{G} * \mathscr{G} := \pi^* \mathscr{G} \to \mathscr{G},$$

$$(g, h) \mapsto gh.$$

That this satisfies all properties for an LGB action is clear by definition of LGBs; however, let us give a note about the smoothness of this action. Recall that an LGB is locally isomorphic to a trivial LGB $U \times G$ (U an open subset of M) with its canonical group multiplication, $(x, g) \cdot (x, q) = (x, gq)$. Hence, using that the multiplication of G is smooth and using local LGB trivializations of $\mathscr{G}$ and $\pi^* \mathscr{G}$ (recall Cor. 3.1 and its proof to show that $\pi^* \mathscr{G}$ is locally diffeomorphic to the product manifold $U \times G \times G$), we achieve smoothness of the $\mathscr{G}$-action on itself because it is locally of the form

$$U \times G \times G \to U \times G,$$

$$(x, g, q) \mapsto (x, gq).$$

We will also call the $\mathscr{G}$-action on itself the **multiplication in $\mathscr{G}$**.

Remarks 3.13.
Similarly one can argue that the **inverse map** $\mathscr{G} \to \mathscr{G}$, $g \mapsto g^{-1}$, is smooth, too.

Example 3.14: Trivial action, [2, §1.6, special situation of Ex. 1.6.3, page 35]

The projection π_1 onto the first factor of $f^*\mathscr{G} \overset{\pi_1}{\to} N$ satisfies the properties of a right $\mathscr{G}$-action on N, that is, the action is given by

$$N * \mathscr{G} \to N,$$

$$(p, g) \mapsto p \cdot g := p.$$

That this action satisfies the properties of an action for all f is trivial, hence we call it the **trivial action**.

Example 3.15: Actions of trivial LGBs

Assume that $\mathscr{G}$ is trivial, that is $\mathscr{G} \cong M \times G$ as LGBs, where G is the structural Lie group of $\mathscr{G}$. In that case, for $p \in N$, the product $p \cdot q$ is only defined if $q \in \mathscr{G}$ is of the form $(f(p), g)$, where $g \in G$; for this recall that (p, q) needs to be an element of $N * \mathscr{G} = f^*\mathscr{G}$ in order to define $(p, q) \mapsto p \cdot q$. We also have

$$f^*\mathscr{G} \cong N \times G,$$

the trivial G-LGB over N. Hence, let us define a map by

$$N \times G \to N,$$

$$(p, g) \mapsto p \cdot g := p \cdot (f(p), g),$$

which is clearly a smooth map since it is a composition of the $\mathscr{G}$-action on N and

$$N \times G \to f^*\mathscr{G},$$

$$(p, g) \mapsto (p, f(p), g).$$

The latter is smooth because $(p, g) \mapsto (p, f(p), g)$ is a smooth map $N \times G \to N \times M \times G$ and $f^*\mathscr{G}$ is an embedded submanifold of $N \times M \times G$. Using Def. 3.4, it is trivial to see that $p \cdot e = p$, and

$$p \cdot (gh) = p \cdot \underbrace{(f(p), gh)}_{=(f(p),g)\cdot(f(p),h)} = \big(p \cdot (f(p), g)\big) \cdot (f(p), h) = (p \cdot g) \cdot h$$

for all $g, h \in G$. Hence, we have a G-action on N, and by construction it is equivalent to the $\mathscr{G}$-action on N. Due to the discussion in Rem. 3.6 we can therefore conclude that every $\mathscr{G}$-action is locally a typical Lie group action. If f is a submersion, then the action is also a G-action on each fibre of f.

Observe, that one can therefore recover the notion of Lie group actions not only via $M = \{*\}$, the point manifold, but also via trivial LGBs. Translations with constant sections of $M \times G$ w.r.t. the action $N * \mathscr{G} \to N$ are trivially to be seen as translations with an element of G w.r.t. the action $N \times G \to N$.

Hence, if one wants something "truly" new regarding LGB actions, then one has to look at global structures of LGBs and their actions. In fact, the following example will provide such a new structure, which we will understand later once we have introduced the physical theory.

Example 3.16: Inner group bundle acting on associated fibre bundles, [2, §1.6, simplified version of Ex. 1.6.4, page 35]

Let $P \overset{\pi_P}{\to} M$ be a principal G-bundle with structural Lie group G over a smooth manifold M, and recall Ex. 2.12. Furthermore, let F be another smooth manifold, equipped with a smooth left G-action $\Psi : G \times F \to F$. In total we have two associated bundles over M:

$$\begin{tikzcd} G \arrow[r] & c_G(P) \arrow[d, "\pi_{c_G(P)}"] & & F \arrow[r] & \mathscr{F} := P \times_\Psi F \arrow[d, "\pi_{\mathscr{F}}"] \\ & M & & & M \end{tikzcd}$$

the inner group bundle of P and an associated F-bundle, respectively.
Then we have a right $c_G(P)$-action on $\mathscr{F}$ given by

$$\mathscr{F} * c_G(P) := \pi_{\mathscr{F}}^* c_G(P) \to \mathscr{F},$$

$$\big([p, v], [p, g]\big) \mapsto [p, \Psi(g, v)] = [p \cdot g, v]$$

for all $p \in P_x$ $(x \in M)$, $g \in G$ and $v \in F$.

Proof.

• We first check again that the action is well-defined, that is, we are going to prove that the action is independent of the choice of fixed point in P_x. Thence, let $x \in M$, $p \in P_x$ and $p' := p \cdot g'$ be another element of P_x, where $g' \in G$. Also let $[p_1, v] \in \mathscr{F}_x$ and $[p_2, g] \in c_G(P)_x$; then we have unique elements q_i, q_i' of G such that $(i \in \{1, 2\})$

$$p_i = p \cdot q_i, \qquad\qquad p_i = p' \cdot q_i',$$

especially, it follows $q_i = g' q_i'$.

On one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, v] \cdot [p_2, g] = [p, \Psi(q_1, v)] \cdot [p, c_{q_2}(g)] = [p \cdot c_{q_2}(g), \Psi(q_1, v)] = [p \cdot c_{q_2}(g)\ q_1, v].$$

On the other hand, using p' as a fixed element, we derive, using $q_i' = (g')^{-1} q_i$,

$$[p_1, v] \cdot [p_2, h] = \Big[p' \cdot c_{q_2'}(g)\ q_1', v\Big] = \Big[p \cdot g' q_2' g \big(q_2'\big)^{-1} q_1', v\Big] = \big[p \cdot q_2 g q_2^{-1} q_1, v\big] = [p \cdot c_{g_2}(g)\ q_1, v],$$

which finalizes the argument needed to show that the action is well-defined.

• Let us now quickly check that the conditions in Def. 3.4 are satisfied. We have

$$\pi_{\mathscr{F}}\big([p, v] \cdot [p, g]\big) = \pi_{\mathscr{F}}\big([p, \Psi(g, v)]\big) = \pi_P(p) = \pi_{c_G(P)}\big([p, g]\big)$$

for all $p \in P_x$ $(x \in M)$, $v \in F$ and $g \in G$; similarly, having additionally $h \in G$,

$$\big([p,v] \cdot [p,g]\big) \cdot [p,h] = [p \cdot g, v] \cdot [p,h] = [p \cdot gh, v] = [p,v] \cdot [p,gh] = [p,v] \cdot \big([p,g]\ [p,h]\big),$$

and

$$[p,v] \cdot [p,e] = [p \cdot e, v] = [p,v].$$

Therefore this describes an action. ■

Remark 3.17: Relation to automorphisms of principal bundles and gauge transformations

Recall that gauge transformations are related to principal bundle automorphisms f of the principal bundle P; see *e.g.* [1, §5.3, Def. 5.3.1, page 256f.] and [1, §5.4, Thm. 5.4.4, page 273]. That is, f is a diffeomorphism $P \to P$ with

$$\pi_P \circ f = \mathbb{1}_M,$$

$$f(p \cdot g) = f(p) \cdot g$$

for all $p \in P$ and $g \in G$. The group of such maps will be denoted by $\mathscr{Aut}(P)$. One can identify such automorphisms with certain G-valued maps on P, following [1, §5.3, Def. 5.3.2 & Prop. 5.3.3, page 266f.]: We define the following set of smooth maps $P \to G$ by

$$C^\infty(P;G)^G \coloneqq \left\{\sigma : P \to G \text{ smooth} \;\middle|\; \sigma(p \cdot g) = c_{g^{-1}}\big(\sigma(p)\big) \text{ for all } p \in P, g \in G\right\}.$$

It is straightforward to check that this is a group w.r.t. pointwise multiplication. Furthermore, there is a group isomorphism

$$\mathscr{Aut}(P) \to C^\infty(P;G)^G,$$

$$f \mapsto \sigma_f,$$

where σ_f is defined by

$$f(p) = p \cdot \sigma_f(p)$$

for all $p \in P$; one can prove that this is well-defined.

As argued in [1, §5.3, Thm. 5.3.8, page 269; formulated as left action there, which is why we have an inverse here], $\mathscr{Aut}(P)$ acts (on the right) on associated fibre bundles $\mathscr{F} = P \times_\Psi F$ by

$$[p,v] \cdot f \coloneqq \left[f^{-1}(p), v\right] = \left[p \cdot \sigma_f(p)^{-1}, v\right]$$

for all $[p,v] \in \mathscr{F}_x$ $(x \in M)$ and $f \in \mathscr{Aut}(P)$. σ_f can also be just locally defined, therefore one could investigate whether there is also an action just with an element g of G, basically the restriction of σ_f onto the fibre P_x. However, the action given by $[p,v]\cdot g = \left[p\cdot g^{-1}, v\right]$ for $g \in G$ is in general clearly only well-defined w.r.t. a change of the representative of $[p,v] = \left[p\cdot q, \Psi_{q^{-1}}(v)\right]$ $(q \in G)$, if G is abelian. But one can resolve this by looking at it carefully: The rough idea is that g basically comes from $\sigma_f(p)$ in this context, but

$$\sigma_f(p\cdot q) = c_{q^{-1}}\big(\sigma_f(p)\big).$$

Roughly, while p is multiplied with g^{-1}, $p\cdot q$ has to be multiplied with $q^{-1}g^{-1}q$. It is easy to check that this resolves that issue, and the result is precisely the action described in Ex. 3.16. In fact, we have the following proposition:

For the following proposition observe that the (local) sections of an LGB have a group structure given by pointwise multiplication.

Proposition 3.18: Gauge transformations as sections of the inner LGB, [2, §1.4, (the last sentence of) Ex. 1.4.7, page 25]

Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G over a smooth manifold M. Then there is a group isomorphism

$$\mathscr{A}ut(P) \to \Gamma\big(c_G(P)\big),$$

$$f \mapsto q_f$$

where $q_f \in \Gamma\big(c_G(P)\big)$ is defined by

$$q_f\big|_x \coloneqq \big[p, \sigma_f(p)\big]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$, and σ_f is the element of $C^\infty(P;G)^G$ corresponding to f as introduced in Rem. 3.17.

Remarks 3.19.
As one may guess, $\Gamma\big(c_G(P)\big)$ is the analogue of $C^\infty(M;G)^G$ such that one could ask for a more direct analogue to $\mathscr{A}ut(P)$. Indeed, as argued in [2, §1.3, Prop. 1.3.9, page 20], $c_G(P)$ is isomorphic to $(P\times_M P)\big/G$, where $P\times_M P \coloneqq \pi_P^*P$, and the G-action is the diagonal action on $P\times P$. One can prove that an isomorphism is given by

$$c_G(P) \to (P\times_M P)\big/G,$$

$$[p,g] \mapsto [p, p\cdot g].$$

It is also argued in [2, §1.4, Ex. 1.4.7, page 25] that $\mathscr{A}ut(P)$ is isomorphic to $\Gamma\Big((P\times_M P)\big/G\Big)$

by

$$\mathscr{Aut}(P) \to \Gamma\Big((P \times_M P)\Big/G\Big),$$

$$f \mapsto L_f,$$

where $L_f \in \Gamma\Big((P \times_M P)\Big/G\Big)$ is given by

$$L_f|_x := [p, f(p)] = [p, p \cdot \sigma_f(p)]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$. This is clearly well-defined, and, so, while $c_G(P)$ is the bundle-analogue of $C^\infty(P;G)^G$ one can think of $(P \times_M P)\Big/G$ as the bundle-analogue of $\mathscr{Aut}(P)$.

However, this description often arises if one wants to use the formalism of groupoids and algebroids, here especially using the **gauge groupoid** and **Atiyah algebroid** induced by P. These would allow an even more elegant version of the gauge transformations, however, we intend to write this paper in such a way that there is no need that the reader has knowledge about those bundle structures. See the cited references for more details in that regard.

Proof of Prop. 3.18.

• Let us first quickly check whether $q_f \in \Gamma\big(c_G(P)\big)$ is well-defined for all $f \in \mathscr{Aut}(P)$. For $p \in P_x$ $(x \in M)$ we have

$$q_f|_x = [p, \sigma_f(p)],$$

If $p' = p \cdot g$ $(g \in G)$ is another element of P_x, then, using p' to define $q_f|_x$,

$$q_f|_x = [p \cdot g, \sigma_f(p \cdot g)] = \big[p \cdot g, c_{g^{-1}}\big(\sigma_f(p)\big)\big] = [p, \sigma_f(p)],$$

also using the definition of $c_G(P)$, recall Ex. 2.12. It follows that q_f is well-defined, and it is clear that q_f is smooth.

• We want to show that $f \mapsto q_f$ is a group isomorphism by using that it is a composition of the group isomorphisms $\mathscr{Aut}(P) \to C^\infty(P;G)^G$ as in Rem. 3.17 and

$$C^\infty(P;G)^G \to \Gamma\big(c_G(P)\big),$$

$$\sigma \mapsto q_\sigma, \tag{18}$$

where q_σ is effectively the same definition as q_f, that is $q_\sigma|_x = [p, \sigma(p)]$ which is well-defined by the very same reasons as before. It is only left to show that $C^\infty(P;G)^G \to \Gamma\big(c_G(P)\big)$ is a group isomorphism. For injectivity let σ' be another element of $C^\infty(P;G)^G$ and assume $[p, \sigma(p)] = [p, \sigma'(p)]$. Then

$$e_x = [p, e] = [p, \sigma(p)] \cdot \underbrace{\big([p, \sigma'(p)]\big)^{-1}}_{=\left[p, (\sigma'(p))^{-1}\right]} = \Big[p, \sigma(p)\big(\sigma'(p)\big)^{-1}\Big],$$

such that

$$\sigma(p)\big(\sigma'(p)\big)^{-1} = e,$$

so $\sigma = \sigma'$ and hence injectivity follows. For surjectivity observe that for a section $q \in \Gamma\big(c_G(P)\big)$ we can define a map $\sigma : P \to G$ by

$$q_x = [p, \sigma(p)].$$

This map satisfies

$$[p, \sigma(p)] = \big[p \cdot g, c_{g^{-1}}\big(\sigma(p)\big)\big] = [p \cdot g, \sigma(p \cdot g)]$$

for all $g \in G$; the last equality implies $\sigma(p \cdot g) = c_{g^{-1}}\big(\sigma(p)\big)$, which is precisely what we need for $C^\infty(P; G)^G$. It is only left to show smoothness of σ. For an open neighbourhood $U \subset M$ of x fix a trivialization $\varphi_U : P|_U \to U \times G$, and we denote

$$\varphi_U(p') = \big(\pi_P(p'), \beta_U(p')\big)$$

for all $p' \in P$, where $\beta_U : P|_U \to G$ is an equivariant map, *i.e.* $\beta_U(p' \cdot g) = \beta_U(p')\ g$ for all $g \in G$. As shown in the proof of Thm. 2.9, we have a trivialization of $c_G(P)$ given by

$$\begin{aligned} c_G(P)|_U &\to U \times G, \\ [p', g] &\mapsto \big(\pi_P(p'), \psi_{\beta_U(p')}(g)\big). \end{aligned}$$

Applying that trivialization to q we derive that

$$\big[p' \mapsto \psi_{\beta_U(p')}\big(\sigma(p')\big)\big]$$

is smooth, because q is smooth. Since $\psi_{\beta_U(p')}$ is smooth and bijective, we conclude that σ is smooth. Hence, $\sigma \in C^\infty(P; G)^G$, so, Def. (18) is also surjective and thence bijective.

Finally let us show that Def. (18) is a group isomorphism. Let σ, σ' be elements of $C^\infty(P; G)^G$, then Def. (18) reads

$$\sigma\sigma' \mapsto q_{\sigma\sigma'}$$

with

$$q_{\sigma\sigma'}|_x = \big[p, \sigma(p)\ \sigma'(p)\big] = [p, \sigma(p)] \cdot \big[p, \sigma'(p)\big] = q_\sigma|_x \cdot q_{\sigma'}|_x,$$

such that Def. (18) satisfies

$$\sigma\sigma' \mapsto q_\sigma \cdot q_{\sigma'}.$$

This concludes the proof. ■

4. Lie algebra bundles (LABs)

4.1. Definition

Lie algebras are the infinitesimal version of Lie groups, hence, we expect something similar for LGBs, the Lie algebra bundles:

Definition 4.1: Lie algebra bundle (LAB), [2, §3.3, Definition 3.3.8, page 104]

Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{g}, M$ be smooth manifolds. A vector bundle

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathcal{g} \\ & & \big\downarrow \pi \\ & & M \end{array}$$

is called a **Lie algebra bundle** if:

1. $\mathfrak{g}$ and each fibre $\mathcal{g}_x$, $x \in M$, are Lie algebras;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} := \mathrm{pr}_2 \circ \phi_i|_{\mathcal{g}_x} : \mathcal{g}_x \to \mathfrak{g}$$

   are Lie algebra isomorphisms, where I is an (index) set, U_i are open sets covering M, $\phi_i : \mathcal{g}|_U \to U \times \mathfrak{g}$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie algebra bundle atlas** or **LAB atlas**.

We often say that $\mathcal{g}$ **is an LAB (over** M**)**, whose structural Lie algebra is either clear by context or not explicitly needed; and we may also denote LABs by $\mathfrak{g} \to \mathcal{g} \xrightarrow{\pi} M$.

Of course, we have the typical trivial examples:

Example 4.2: Trivial examples

We recover the notion of a Lie algebra, if M consist of just one point. Moreover, the **trivial LAB** is given as the product manifold $\mathcal{g} := M \times \mathfrak{g} \to M$. We have obviously a canonical smooth field of Lie brackets on this bundle $[\cdot,\cdot]_{\mathcal{g}} : \Gamma(\mathcal{g}) \times \Gamma(\mathcal{g}) \to \Gamma(\mathcal{g})$, *i.e.* $[\cdot,\cdot]_{\mathcal{g}} \in \Gamma\left(\bigwedge^2 \mathcal{g}^* \otimes \mathcal{g}\right)$ which restricts to the Lie algebra bracket $[\cdot,\cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$ on each fibre. The bracket is given by

$$[(x,X),(x,Y)]_{\mathcal{g}} := \left(x, [X,Y]_{\mathfrak{g}}\right)$$

for all $(x,X),(x,Y) \in M \times \mathfrak{g}$. Smoothness is an immediate consequence.

The definition of LAB morphisms is straight-forward:

Definition 4.3: LAB morphism,
[2, §. 4.3, simplified version of Def. 4.3.1, page 158]

Let $\mathscr{g} \xrightarrow{\pi_{\mathscr{g}}} M$ and $\mathscr{h} \xrightarrow{\pi_{\mathscr{h}}} N$ be two LGBs over two smooth manifolds M and N. An **LAB morphism** is a pair of smooth maps $F : \mathscr{h} \to \mathscr{g}$ and $f : N \to M$ such that

$$\pi_{\mathscr{g}} \circ F = f \circ \pi_{\mathscr{h}}, \tag{19}$$

$$F \text{ linear}, \tag{20}$$

$$F\Big([g, q]_{\mathscr{h}_p}\Big) = [F(g), F(q)]_{\mathscr{g}_{f(p)}} \tag{21}$$

for all $g, q \in \mathscr{h}_p$ ($p \in N$), where $[\cdot,\cdot]_{\mathscr{h}_p}$ and $[\cdot,\cdot]_{\mathscr{g}_{f(p)}}$ are Lie brackets of $\mathscr{h}_p$ and $\mathscr{g}_{f(p)}$, respectively. We also say that F **is an LAB morphism over** f. If $N = M$ and $f = \mathrm{id}_M$, then we often omit mentioning f explicitly and just write that F **is a (base-preserving) LAB morphism**.
We speak of an **LAB isomorphism (over f)** if F is a diffeomorphism.

Remark 4.4: Smooth field of Lie brackets

We have similar remarks as in Rem. 2.5. Additionally, we have locally a canonical smooth field of Lie brackets which restricts to a Lie bracket on each fibre because every LAB is locally isomorphic to a trivial LAB as in Ex. 4.2. Define a field of Lie brackets $[\cdot,\cdot]_{\mathscr{g}} : \Gamma(\mathscr{g}) \times \Gamma(\mathscr{g}) \to \Gamma(\mathscr{g})$, *i.e.* $[\cdot,\cdot]_{\mathscr{g}} \in \Gamma\Big(\bigwedge^2 \mathscr{g}^* \otimes \mathscr{g}\Big)$, by

$$[X, Y]_{\mathscr{g}} \coloneqq [X, Y]_{\mathscr{g}_x} \tag{22}$$

for all $X, Y \in \mathscr{g}_x$ ($x \in M$). Using a local trivialization, this bracket is locally of the form as in Rem. 2.5 such that smoothness follows.
In fact, as also argued in [4, §16.2, Example 2, page 114; but speaking in the context of Lie algebroids there whose are a generalization of LABs], every vector bundle equipped with a smooth field of Lie brackets is an LAB.

Endomorphisms of a vector bundle are of course another important example of LABs.

Example 4.5: Endomorphisms of a vector bundle, [2, §3.3, part of Ex. 3.3.4]

Let $V \to M$ be a vector bundle, and denote with $\mathrm{End}(V) \to M$ its bundle of fibre-wise endomorphisms (its sections are the base-preserving bundle endomorphisms of V). This is clearly an LAB whose field of Lie brackets is given by the commutator.

As we have seen it for LGBs, the pullback of LABs is again an LAB.

Corollary 4.6: Pullbacks of LABs are LABs, [3, §3, Thm. 3.2]

Let M, N be smooth manifolds, $\mathcal{g} \xrightarrow{\pi} M$ an LAB over M and $f : N \to M$ a smooth map. Then the pullback vector bundle $f^*\mathcal{g}$ has a unique (up to isomorphisms) LAB structure such that the projection $\pi_2 : f^*\mathcal{g} \to \mathcal{g}$ onto the second factor is an LAB morphism over f with $\pi_2|_x : (f^*\mathcal{g})_x \to \mathcal{g}_{f(x)}$ being a Lie algebra isomorphism for all $x \in N$.

Proof.
Either prove this similarly as Cor. 3.1 (by also using a similar statement already known for the pullbacks of vector bundles), or observe that the pullback $f^*\left([\cdot,\cdot]_{\mathcal{g}}\right)$ of the field of Lie brackets $[\cdot,\cdot]_{\mathcal{g}}$ on $\mathcal{g}$ as a section is clearly also a smooth field of Lie brackets on $f^*\mathcal{g}$ with same structural Lie algebra $\mathfrak{g}$. ■

Definition 4.7: Pullback LAB

Let M, N be smooth manifolds, $\mathcal{g} \to M$ an LGB over M and $f : N \to M$ a smooth map. Then we call the LAB structure on $f^*\mathcal{g}$ as given in Cor. 4.6 the **pullback LAB of $\mathcal{g}$ (under f)**.
With writing $f^*\mathcal{g}$ we will often refer to this structure without further mention.

4.2. From LGBs to LABs

Let us now quickly discuss how LGBs and LABs are related; it is very similar to the relation of Lie groups and algebras, now somewhat fibre-wise. We will follow the style of [1, §1.5.2, page 40ff.] and [2, §3.5, page 119ff.]; our approach will be using left-invariant vector fields but the mentioned latter reference actually uses right-invariant vector fields.

Let us start with introducing the basic notations needed.

Definition 4.8: Left and right translation and conjugation, [1, §1.5, similar notation to Def. 1.5.3, page 40]

Let $\mathcal{G} \to M$ be an LGB over a smooth manifold M. For $g \in \mathcal{G}_x$ ($x \in M$) we define the following maps:

- **Left translation** given by
$$L_g : \mathcal{G}_x \to \mathcal{G}_x,$$
$$h \mapsto gh.$$
- **Right translation** given by
$$R_g : \mathcal{G}_x \to \mathcal{G}_x,$$
$$h \mapsto hg.$$

- **Conjugation** given by

$$c_g : \mathscr{G}_x \to \mathscr{G}_x,$$
$$h \mapsto ghg^{-1}.$$

Remarks 4.9.
By definition of $\mathscr{G}$, all these maps are smooth. Furthermore, they clearly satisfy the typical properties as known for these maps since $\mathscr{G}_x$ is a Lie group for all $x \in M$; for reference about their basic properties see for example [1, §1.5, Lemma 1.5.5, page 40f.].

The left and right translations of Def. 4.8 and 3.8 align, and thus the smoothness concerns as mentioned in the last part of Rem. 3.11 for right translations $r_g = R_g$ ($g \in \mathscr{G}_x$, $x \in M$) do not arise. Moreover, while the conjugation c_g is a Lie group automorphism of $\mathscr{G}_x$, it describes an LGB automorphism of $\mathscr{G}$ if extended to sections; following [2, §1.4, Def. 1.4.6 and its discussion afterwards, page 24f.]. That is for $\sigma \in \Gamma(\mathscr{G})$ we define the conjugation c_σ as a smooth map by

$$\mathscr{G} \to \mathscr{G},$$
$$q \mapsto c_\sigma(q) := (L_\sigma \circ R_{\sigma^{-1}})(q) = (R_{\sigma^{-1}} \circ L_\sigma)(q) = \sigma_{\pi(q)} \cdot q \cdot \sigma^{-1}_{\pi(q)}.$$

It is clear that $c_\sigma(gq) = c_\sigma(g) \cdot c_\sigma(q)$ for all $g, q \in \mathscr{G}$ with $\pi(g) = \pi(q)$, and that a smooth inverse is given by $c_{\sigma^{-1}}$; thence, c_σ is an LGB isomorphism of $\mathscr{G}$ on itself, an automorphism, in sense of Def. 2.4. It is also trivial to check that we have $c_{\sigma \cdot \tau} = c_\sigma \circ c_\tau$, where τ is another section of $\mathscr{G}$.

Analogously we define R_σ as r_σ of Def. 3.8; with the capital letter we put an emphasis on that the $\mathscr{G}$-action acts on $\mathscr{G}$ itself. Similarly for left translations.

Since these are diffeomorphism of the fibres, it makes sense to say that a left-invariant vector field of $\mathscr{G}$ has to be a vertical vector field, that is, it is in the kernel of $\mathrm{D}\pi$, the total differential/tangent map of the projection of $\mathscr{G} \xrightarrow{\pi} M$. For this recall that there is the notion of a **vertical bundle** for fibre bundles $F \xrightarrow{\varpi} M$ (as *e.g.* introduced in [1, §5.1.1, for principal bundles, but it is straightforward to extend the definitions; page 258ff.]), which is defined as a subbundle $\mathrm{V}F \to F$ of the tangent bundle $\mathrm{T}F \to F$ given as the kernel of $\mathrm{D}\varpi : \mathrm{T}F \to \mathrm{T}M$. The fibres V_vF of $\mathrm{V}F$ at $v \in F$ are then given by

$$\mathrm{V}_vF = \mathrm{T}_vF_x,$$

where $x := \varpi(v) \in M$ and F_p is the fibre of F at x. F_x is an embedded submanifold of F, thence, by definition a section $X \in \Gamma(\mathrm{V}F)$ restricts to a vector field on the fibres, that is,

$$X|_{F_x} \in \mathfrak{X}(F_x).$$

In our case $F = \mathscr{G}$, in this case $F_x = \mathscr{G}_x$ is then a Lie group, so, the vertical bundle just consists of the tangent bundles of Lie groups of all fibres. All of these are generated by their Lie algebra at e_x, the identity element of $\mathscr{G}_x$. Hence, it is natural to guess that the LAB for $\mathscr{G}$ will be $\mathrm{V}\mathscr{G}|_{e_M}$, where e_M is the image of M under the identity section of $\mathscr{G}$, thus, an embedding of M into $\mathscr{G}$. Therefore $\mathrm{V}\mathscr{G}|_{e_M}$ is a fibre bundle by [1, §4.1, Lemma 4.1.16, page 204], and clearly a vector bundle. Equivalently, since the identity section e is an embedding, we think of $\mathrm{V}\mathscr{G}|_{e_M}$ as the pullback vector bundle $e^*\mathrm{V}\mathscr{G}$, which is conveniently a vector bundle over M.

Hence, let us now show that $\mathscr{G}$ will be related to $e^*\mathrm{V}\mathscr{G}$ similar to how a Lie group will be related to its Lie algebra.

Definition 4.10: Left-invariant vector fields on LGBs, [2, §3.5, special situation of Def. 3.5.2, page 120]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. A vector field $X \in \mathfrak{X}(\mathscr{G})$ is a **left-invariant vector field** if

1. X is vertical, that is,
$$X \in \Gamma(\mathrm{V}\mathscr{G}),$$
2. X is invariant under the left-multiplication on each fibre, *i.e.*
$$\mathrm{D}_g L_q(X_g) = X_{qg}$$
for all $q, g \in \mathscr{G}_x$, where $x \coloneqq \pi(g) = \pi(q)$.

The set of all left-invariant vector fields on $\mathscr{G}$ will be denoted by $L(\mathscr{G})$.

Remarks 4.11.
Observe that the second point in the definition is well-defined because X is a vertical vector field; that is, recall that $L_q : \mathscr{G}_x \to \mathscr{G}_x$ such that $\mathrm{D}_g L_q(X_g) : \mathrm{T}_g\mathscr{G}_x \to \mathrm{T}_{qg}\mathscr{G}_x$, hence, $\mathrm{D}_g L_q(X_g) : \mathrm{V}_g\mathscr{G} \to \mathrm{V}_{qg}\mathscr{G}$.

Remark 4.12: Abstract notation 1

Since X is vertical, recall that we can view the restriction of X onto a fibre as a vector field on that fibre, *i.e.*
$$X|_{\mathscr{G}_x} \in \mathfrak{X}(\mathscr{G}_x).$$
$\mathscr{G}_x$ is a Lie group and left translations are diffeomorphisms on it, hence, the left-invariance can also be written as
$$\mathrm{D}L_q(X|_{\mathscr{G}_x}) = L_q^*(X|_{\mathscr{G}_x}). \tag{23}$$
For this recall that $\mathrm{D}L_q \in \Omega^1\left(\mathscr{G}_x; L_q^*\mathrm{T}\mathscr{G}_x\right)$ for the left hand side, and that L_q^* is the pullback of sections on the right hand side, that is, $L_q^*(X|_{\mathscr{G}_x}) \in \Gamma\left(L_q^*\mathrm{T}\mathscr{G}_x\right)$. Furthermore,

$X|_{\mathscr{G}_x}$ is therefore a left-invariant vector on $\mathscr{G}_x$. Which is why one may also define the left-invariance of X as a vector field on $\mathscr{G}$ by saying that it has to restrict to a left-invariant vector field on each fibre in the usual sense of Lie groups.

One quickly shows that this is a Lie subalgebra of $\mathfrak{X}(\mathscr{G})$.

Lemma 4.13: Closure of Lie bracket for left-invariant vector fields, [2, §3.5, special situation of Lemma 3.5.5, page 122]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Then $L(\mathscr{G})$ is a Lie subalgebra of $\mathfrak{X}(\mathscr{G})$.

Proof.
Let $X, Y \in L(\mathscr{G})$, then we need to show

$$\mathrm{D}\pi\big([X,Y]\big) \equiv 0,$$

and if this holds, then we also need to derive

$$\mathrm{D}_g L_q\Big(\left.[X,Y]\right|_g\Big) = \left.[X,Y]\right|_{qg}.$$

One can either immediately show these directly by using statements like [1, Proposition A.1.49; page 615], which essentially describes how the Lie bracket of vector fields react under push-forwards. Or use the knowledge about Lie groups, recall Rem. 4.12: Each fibre $\mathscr{G}_x$ is an embedded submanifold of $\mathscr{G}$ and both, $X|_{\mathscr{G}_x}$ and $Y|_{\mathscr{G}_x}$, are vector fields of this submanifold. Thus, $\left.\big[X|_{\mathscr{G}_x}, Y|_{\mathscr{G}_x}\big]\right|_p$ has values in $\mathrm{T}_p\mathscr{G}_x$ for all $p \in \mathscr{G}_x$. Especially,

$$\left.\big[X|_{\mathscr{G}_x}, Y|_{\mathscr{G}_x}\big]\right|_{\mathscr{G}_x} \in \mathfrak{X}(\mathscr{G}_x).$$

Because of this and since $X|_{\mathscr{G}_x}$ and $Y|_{\mathscr{G}_x}$ are left-invariant vector fields of $\mathscr{G}_x$ (a Lie group), left-invariance of $\left.\big[X|_{\mathscr{G}_x}, Y|_{\mathscr{G}_x}\big]\right|_{\mathscr{G}_x}$ follows, and thus the statement. ■

Of course, elements of $L(\mathscr{G})$ are determined by their values at e_M, as already suggested previously. Let us show this now; starting with a small auxiliary result.

Corollary 4.14: $L(\mathscr{G})$ a $C^\infty(M)$-module, [2, §3.5, comment before Lemma 3.5.5, page 122]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Then $L(\mathscr{G})$ is a $C^\infty(M)$-module under the multiplication

$$fX := \pi^* f \ X$$

for all $f \in C^\infty(M)$ and $X \in L(\mathscr{G})$.

Proof.
Obviously, $fX \in \Gamma(\mathrm{V}\mathscr{G})$ since

$$\mathrm{D}\pi(fX) = \mathrm{D}\pi(\pi^* f\ X) = \pi^* f\ \mathrm{D}\pi(X) = 0.$$

Furthermore, $fX|_{\mathscr{G}_x}$ ($x \in M$) is left-invariant over $\mathscr{G}_x$ since $X|_{\mathscr{G}_x}$ is left-invariant and $f|_{\mathscr{G}_x} \equiv f(x) \in \mathbb{R}$. Thence, $fX \in L(\mathscr{G})$. ∎

Corollary 4.15: $L(\mathscr{G})$ as sections of $e^*\mathrm{V}\mathscr{G}$, [2, §3.5, comment before Lemma 3.5.5, page 122; parts of Cor. 3.5.4, page 121]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M, and denote with e the identity section of $\mathscr{G}$. Then we have an isomorphism of $C^\infty(M)$-modules

$$L(\mathscr{G}) \to \Gamma(e^*\mathrm{V}\mathscr{G}),$$

$$X \mapsto e^*X.$$

The inverse of this map is given by

$$\Gamma(e^*\mathrm{V}\mathscr{G}) \to L(\mathscr{G}),$$

$$\nu \mapsto X_\nu,$$

where X_ν is given by

$$X_\nu\big|_g := \mathrm{D}_{e_x} L_g\big(\mathrm{pr}_2(\nu_x)\big)$$

for all $g \in \mathscr{G}$ and $x := \pi(g)$, where pr_2 is the projection onto the second component in $e^*\mathrm{V}\mathscr{G}$.

Remark 4.16: Abstract notation 2

Since $e^*\mathrm{V}\mathscr{G} \cong \mathrm{V}\mathscr{G}|_{e_M}$ is a very natural isomorphism, we will often just write

$$X_\nu\big|_g = \mathrm{D}_{e_x} L_g(\nu_x),$$

omitting pr_2 and using that natural isomorphism without further mention.
Also observe that we can actually define a left translations by (local) sections of $\mathscr{G}$, *i.e.* for $\sigma \in \Gamma(\mathscr{G})$ we define the left translation L_σ as a map by

$$\mathscr{G} \to \mathscr{G},$$

$$q \mapsto \sigma_{\pi(q)} \cdot q.$$

This map is a diffeomorphism, and restricts to the fibres $\mathscr{G}_x$ as embedded submanifolds to the map L_{σ_x}; we discussed this in more generality in Rem. 3.11. Observe that for vertical vector fields $Y \in \Gamma(\mathrm{V}\mathscr{G})$ we have

$$\mathrm{D}_q L_\sigma(Y_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (L_\sigma \circ \gamma) \equiv \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(L_{\sigma_{\pi(q)}} \circ \gamma\right) = \mathrm{D}_q L_g(Y_q)$$

where $g := \sigma_{\pi(q)}$ and $\gamma : I \to \mathscr{G}_{\pi(q)}$ (I an open interval containing 0) is a curve with $\gamma(0) = q$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = Y_q$. Therefore L_σ restricts onto vertical vector fields and is then just the left translation via an element in the fibre over a fixed base point. In total one can then introduce the brief notation

$$X_\nu \circ \sigma = \mathrm{D}L_\sigma(\nu) = \mathrm{D}L_\sigma|_{e_M}(\nu) = \mathrm{D}L_\sigma \circ \nu.$$

However, be careful, in general one cannot simply replace L_g with L_σ, even if $\sigma_{\pi(g)} = g$. This only works with respect to vertical tangent vectors; once horizontal parts play a role things change, L_g is a priori not even defined then. Once we turn to the definition of horizontal distributions we will come back to this.

Proof of Cor. 4.15.
This map is clearly $C^\infty(M)$-linear, especially due to

$$e^*(fX) = e^*(\pi^* f\ X) = (f \circ \underbrace{\pi \circ e}_{=\mathbb{1}_M})\ e^* X = f\ e^* X$$

for all $f \in C^\infty(M)$ and $X \in L(\mathscr{G})$; for this recall Cor. 4.14.

We essentially only need to show that the suggested inverse $\nu \mapsto X_\nu$ is well-defined. First of all, that X_ν is vertical and left-invariant is clear by construction; ν_x ($x \in M$) is an element of the Lie algebra of $\mathscr{G}_x$, and thus $X_\nu|_{\mathscr{G}_x}$ is a left-invariant vector field on $\mathscr{G}_x$. X is therefore an element of $L(\mathscr{G})$ once we know that X is smooth. We show smoothness similar as in [1, §1.5, proof of Lemma 1.5.13, page 42]: Denote the multiplication in $\mathscr{G}$ by $\mu : \mathscr{G} * \mathscr{G} \to \mathscr{G}$. Then observe that $(0_g, \nu_x)$ ($0_g \in \mathrm{T}_g\mathscr{G}$, $g \in \mathscr{G}_x$, the zero vector field 0 on $\mathrm{T}\mathscr{G}$) is an element of

$$\begin{aligned}&\mathrm{T}(\mathscr{G} * \mathscr{G})\\ &= \{(Y, Z) \mid Y \in \mathrm{T}_q\mathscr{G}, Z \in \mathrm{T}_h\mathscr{G} \text{ with } \mathrm{D}_q\pi(Y) = \mathrm{D}_h\pi(Z),\ \text{where } q, h \in \mathscr{G} \text{ with } \pi(q) = \pi(h)\}\end{aligned}$$

because $\nu_x \in \mathrm{V}_{e_x}\mathscr{G}$.[2] Therefore we can calculate

$$\mathrm{D}_{(g,e_x)}\mu(0_g, \nu_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g \cdot \gamma) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (L_g \circ \gamma) = \mathrm{D}_{e_x} L_g(\nu_x) = X_\nu|_g,$$

where $\gamma : I \to \mathscr{G}_x$ (I an open interval containing 0) is a curve with $\gamma(0) = e_x$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = \nu_x$. Since μ is smooth, $\mathrm{D}\mu$ is smooth, and thus

$$\mathscr{G} \to \mathrm{T}(\mathscr{G} * \mathscr{G}),$$

[2] If it is not clear how to derive the tangent bundle of $\mathscr{G} * \mathscr{G}$, then see later when we will discuss it in a more general manner. However, essentially recall that $\mathscr{G} * \mathscr{G} = \pi^*\mathscr{G}$.

$$g \mapsto \left(\mathrm{D}\mu \circ (0, \nu_\pi)\right)\big|_g = \mathrm{D}_{(g,e_x)}\mu\left(0_g, \nu_{\pi(g)}\right) = X_\nu|_g$$

is smooth, also using the smoothness of ν, π and $g \mapsto 0_g$.

Finally, that $\phi : L(\mathscr{G}) \to \Gamma(e^*\mathrm{V}\mathscr{G})$, $X \mapsto e^*X$, is bijective is also clear, similar to typical gauge theory; we know that $X|_{\mathscr{G}_x}$ is a left-invariant vector field on $\mathscr{G}_x$ by Rem. 4.12. Hence, for $g \in \mathscr{G}_x$,

$$X|_{\mathscr{G}_x} = \mathrm{D}_{e_x} L_g(X_{e_x}) = \mathrm{D}_{e_x} L_g(e^*X|_x).$$

This is precisely the structure of the suggested inverse, that is,

$$X = X_{e^*X} = (\psi \circ \phi)(X),$$

where $\psi : \Gamma(e^*\mathrm{V}\mathscr{G}) \to L(\mathscr{G}), \nu \mapsto X_\nu$. Hence, injectivity follows; surjectivity simply follows similarly by

$$(\phi \circ \psi)(\nu)|_x = e^*X_\nu|_x = \underbrace{\mathrm{D}_{e_x} L_{e_x}}_{=\mathbb{1}_{\mathrm{V}_{e_x}\mathscr{G}}}(\nu_x) = \nu_x$$

for all $\nu \in \Gamma(e^*\mathrm{V}\mathscr{G})$ and $x \in M$. This finishes the proof. ■

This result shows the typical statement about that elements of $L(\mathscr{G})$ are uniquely determined by their values at e_M. It immediately follows, too, that:

Corollary 4.17: LGBs induce an LAB structure, [2, §3.5, simplified version of the discussion after Cor. 3.5.4, page 121ff.]

Let $G \to \mathscr{G} \to M$ be an LGB over a smooth manifold M, and denote with e the identity section of $\mathscr{G}$. Then $\mathscr{g} := e^*\mathrm{V}\mathscr{G} \to M$ admits the structure as an LAB with structural Lie algebra $\mathfrak{g}$, the Lie algebra of G, and the fibres $\mathscr{g}_x$ ($x \in N$) are the Lie algebras of $\mathscr{G}_x$. The field of Lie algebra brackets $[\cdot,\cdot]_{\mathscr{g}}$ is given by

$$[\nu, \mu]_{\mathscr{g}} := e^*\left([X_\nu, X_\mu]\right)$$

for all $\nu, \mu \in \Gamma(e^*\mathrm{V}\mathscr{G})$, where X_ν, X_μ are elements of $L(\mathscr{G})$ as given in Cor. 4.15. Pointwise

$$[\nu_x, \mu_x]_{\mathscr{g}} = [X_\nu, X_\mu]\big|_{e_x}$$

for all $x \in M$.

Proof.
As already discussed $e^*\mathrm{V}\mathscr{G}$ is a vector bundle. The fibres are given by

$$\mathscr{g}_x = \mathrm{T}_{e_x}\mathscr{G}_x \cong \mathrm{T}_e G = \mathfrak{g}$$

for all $x \in M$, where we used that $\mathscr{G}_x$ is isomorphic to G as a Lie group. All fibres are Lie algebras of the fibre Lie group, isomorphic to $\mathfrak{g}$. By construction, the Lie bracket is precisely the Lie bracket isomorphic to the one of $\mathfrak{g}$, and $[\cdot,\cdot]_{\mathscr{g}}$ is smooth. Therefore we conclude that $\mathscr{g}$ is an LAB with structural Lie algebra $\mathfrak{g}$. Alternatively see [2, §3.5, Ex. 3.5.12, page 126] for an explicit construction of an LAB atlas. ■

Definition 4.18: The LAB of an LGB, [2, §3.5, special situation of Def. 3.5.1, page 120]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and denote with e the identity section of $\mathscr{G}$. Then we define the **LAB $\mathscr{g}$ of $\mathscr{G}$** as the vector bundle $e^*\mathrm{V}\mathscr{G}$.

In the following $\mathscr{g}$ will usually denote the LAB of $\mathscr{G}$.

Example 4.19: Endomorphisms of a vector bundle as LAB of fibre-wise automorphisms

Recall Ex. 2.6 and 4.5. For a vector bundle $V \to M$ one can show that the LAB of $\mathrm{Aut}(V)$ is given by $\mathrm{End}(V)$; the proof is precisely as for the automorphisms and endomorphisms of a vector space as in [1, §1.5.4, page 45ff.], just canonically extended to a bundle language.

Example 4.20: Associated LAB

We discussed associated LGBs in Subsection 2.2. So, let G, H be Lie groups, $P \to M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then we have the associated LGB $\mathscr{H} := P \times_\psi H$.
Also recall Remark 2.8, $\psi_* := [g \mapsto \mathrm{D}_{e_G}\psi_g]$ is a G-representation on $\mathfrak{h}$, the Lie algebra of H, where e_G is the neutral element of G. Hence, we also have the associated bundle $\mathscr{h} := P \times_{\psi_*} \mathfrak{h}$.
It is now natural to guess that $\mathscr{h}$ is the LAB of $\mathscr{H}$, and this is indeed the case, we will give a sketch; the proof's construction is similar to [2, §3.1, Prop. 3.1.4, page 90]; also recall the short introduction of associated bundles at the beginning of Subsection 2.2. For $x \in M$ we have the fibre $\mathscr{H}_x = P_x \times_\psi H$ which is isomorphic to H as a Lie group by $H \ni h \mapsto [p, h]$ for a fixed $p \in P_x$. Thus the complete flow of an LAB element of $\mathscr{H}$ (in the fibre over x) is equivalent to $[p, \gamma]$ where $\gamma : \mathbb{R} \to H$, $t \mapsto \mathrm{e}^{tX}$, for an $X \in \mathfrak{h}$. Thus, we can already conclude that the fibre of the LAB of $\mathscr{H}$ over x is isomorphic to $\mathfrak{h} = \mathscr{h}_x$ as Lie algebra. By the definition of $\mathscr{H}$ we also have

$$[p, \gamma] = [p \cdot g, \psi_g \circ \gamma]$$

for all $g \in G$. Denoting $[\cdot,\cdot] =: \pi$ we get infinitesimally

$$\mathrm{D}_{(p,e_H)}\pi(p, X) = \mathrm{D}_{(p\cdot g,e_H)}\pi\big(p \cdot g, \mathrm{D}_e\psi_g(X)\big),$$

where e_H is the neutral element of H. One can trivially check that this gives an equivalence relation on $P_x \times \mathfrak{h}$, precisely the one we need for $\mathscr{h}$. It finally follows that the LAB of $\mathscr{H}$ is given by $\mathscr{h}$.
As a special example, the LAB of the inner group bundle $c_G(P)$, Ex. 2.12, is the **adjoint bundle** $\mathrm{Ad}(P)$ **of** P, $\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{g}$, where Ad is the adjoint representation of G. This is also a special example of the generalized Atiyah sequence (of both, a short exact sequence of Lie groupoids and algebroids) in [2, §3.5, Def. 3.5.19, page 130].

4.3. Vertical Maurer-Cartan form of LGBs

As one may expect, the last result gives hints about the tangent bundle structure of $\mathscr{G}$; this can be shown with the Maurer-Cartan form on LGBs, which we will call vertical Maurer-Cartan form. It will be clear later why we choose to add this adjective; however, as a first argument recall Rem. 4.16, especially the last paragraph.

Corollary 4.21: Well-definedness of the vertical Maurer-Cartan form

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Define the following map

$$(\mu_{\mathscr{G}})_g(v) := \left(\mathrm{D}_g L_{g^{-1}}\right)(v)$$

for all $g \in \mathscr{G}$ and $v \in V_g\mathscr{G}$. Then this map is an element of $\Gamma(\mathrm{V}^*\mathscr{G} \otimes \pi^*\mathscr{g})$, where $\mathrm{V}^*\mathscr{G}$ is the dual bundle of $\mathrm{V}\mathscr{G}$.

Proof.
Observe that

$$\mathrm{D}_g L_{g^{-1}} : \mathrm{V}_g\mathscr{G} \to \mathrm{V}_{e_x}\mathscr{G} \cong (\pi^*\mathscr{g})_g$$

where $x := \pi(g)$ and e_x is the neutral element of $\mathscr{G}_x$. Smoothness follows similarly to the smoothness of left-invariant vector fields, that is, denote with Φ the multiplication $\mathscr{G} * \mathscr{G} \to \mathscr{G}$ and recall the arguments and the notation in the proof of Cor. 4.15. We have

$$\mathrm{T}(\mathscr{G} * \mathscr{G})$$
$$= \{(Y, Z) \mid Y \in \mathrm{T}_q\mathscr{G}, Z \in \mathrm{T}_h\mathscr{G} \text{ with } \mathrm{D}_q\pi(Y) = \mathrm{D}_h\pi(Z), \text{ where } q, h \in \mathscr{G} \text{ with } \pi(q) = \pi(h)\}$$

and thus

$$\left(0_{g^{-1}}, v\right) \in \mathrm{T}(\mathscr{G} * \mathscr{G})$$

where 0 is the zero vector field, $v \in \mathrm{V}_g\mathscr{G}$ and $g \in \mathscr{G}$. Therefore we can calculate

$$\mathrm{D}_{(g^{-1},g)}\Phi\left(0_{g^{-1}}, v\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(g^{-1} \cdot \gamma\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(L_{g^{-1}} \circ \gamma\right) = \mathrm{D}_g L_{g^{-1}}(v) = \mu_{\mathscr{G}}(v),$$

where $\gamma : I \to \mathscr{G}_x$ (I an open interval containing 0, and $x := \pi(g)$) is a curve with $\gamma(0) = g$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = v$. Denote with 0^{-1} the vector field on $\mathscr{G}$ given by $g \mapsto 0_{g^{-1}}$ and with $\iota_{0^{-1}}$ the contraction with 0^{-1}, that is,

$$\left(\iota_{0^{-1}}\mathrm{D}\Phi\right)\big|_g := \mathrm{D}_{(g^{-1},g)}\Phi\big(0_{g^{-1}},\cdot\big) = \Big[\mathrm{V}_g\mathscr{G} \ni v \mapsto \mathrm{D}_{(g^{-1},g)}\Phi\big(0_{g^{-1}}, v\big)\Big] = \mu_{\mathscr{G}}$$

for all $g \in \mathscr{G}$, using the structure of $\mathrm{T}(\mathscr{G} * \mathscr{G})$. Thus, we get in total that

$$\mu_{\mathscr{G}} = \iota_{0^{-1}}\mathrm{D}\Phi \in \Gamma(V^*\mathscr{G} \otimes \pi^*\mathscr{g}),$$

using the smoothness of all involved parts, especially that Φ is smooth, hence also $\mathrm{D}\Phi$ is smooth as an element of $\Omega^1(\mathscr{G} * \mathscr{G}; \Phi^*\mathscr{G})$. ■

Definition 4.22: Vertical Maurer-Cartan form of LGBs, [1, generalization of Def. 3.5.2, page 148]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. The map defined in Cor. 4.21 is the **vertical Maurer-Cartan form** $\mu_{\mathscr{G}}$, *i.e.* defined to be an element of $\Gamma(\mathrm{V}^*\mathscr{G} \otimes \pi^*\mathscr{g})$ given by

$$\left(\mu_{\mathscr{G}}\right)_g(v) := \left(\mathrm{D}_g L_{g^{-1}}\right)(v)$$

for all $g \in \mathscr{G}$ and $v \in V_g\mathscr{G}$, where $\mathrm{V}^*\mathscr{G}$ is the dual bundle of $\mathrm{V}\mathscr{G}$.

Remark 4.23: Recovering of the classical definition

Observe that $\mu_{\mathscr{G}}|_{\mathscr{G}_x}$ ($x \in M$) is the typical Maurer-Cartan form of $\mathscr{G}_x$, hence, $\mu_{\mathscr{G}}$ restricts to the Maurer-Cartan form of Lie groups on each fibre.
Also recall Subsection 1.1, we have a 1:1 correspondence of $\mu_{\mathscr{G}}$ to the following commuting diagram

$$\begin{array}{ccc} \mathrm{V}\mathscr{G} & \xrightarrow{\mu_{\mathscr{G}}} & \mathscr{g} \\ \downarrow & & \downarrow \\ \mathscr{G} & \xrightarrow{\pi} & M \end{array}$$

which is the same diagram as in [2, §3.5, special situation of Prop. 3.5.3, page 121].

We can finally finish the discussion about the vertical bundle of an LGB.

Corollary 4.24: Vertical tangent space of $\mathscr{G}$, [2, §3.5, a reformulation of Prop. 3.5.3, page 121]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M. Then we have an isomorphism of vector bundles

$$\mathrm{V}\mathscr{G} \cong \pi^*\mathscr{g}.$$

Remarks 4.25.
Observe that by Cor. 4.6 we know that $\mathrm{V}\mathscr{G}$ admits a unique LAB structure such that $\mathrm{V}\mathscr{G} \cong \pi^*\mathscr{g}$ is an isomorphism of LABs. This statement is also not in contradiction with $\mathscr{g} = e^*\mathrm{V}\mathscr{G}$ (e the identity section of $\mathscr{G}$), because

$$e^*\mathrm{V}\mathscr{G} = e^*\pi^*\mathscr{g} \cong (\pi \circ e)^*\mathscr{g} = \mathscr{g}.$$

By this we also know that $\mathrm{V}\mathscr{G}$ is trivial if and only if $\mathscr{g}$ is trivial; as also argued in [2, §3.5, discussion after Cor. 3.5.4, page 121]. Compare this result with $\mathrm{T}G \cong G \times \mathfrak{g}$, where G is a Lie group with Lie algebra $\mathfrak{g}$. We recover this result by restricting to the Lie group fibres $\mathscr{G}_x$ ($x \in M$), that is,

$$\mathrm{V}\mathscr{G}|_{\mathscr{G}_x} = \mathrm{T}\mathscr{G}_x \cong \mathscr{G}_x \times \mathscr{g}_x = \pi^*\mathscr{g}|_{\mathscr{G}_x}.$$

Last but not least, sections of $\mathrm{V}\mathscr{G}$ are therefore generated by sections of $\mathscr{g}$, the left-invariant vector fields.

Proof of Cor. 4.24.
This can be quickly shown by recalling Rem. 4.23, that is, we have the following commuting diagram

$$\begin{tikzcd}
\mathrm{V}\mathscr{G} \arrow[r, "\mu_{\mathscr{G}}"] \arrow[d] & \mathscr{g} \arrow[d] \\
\mathscr{G} \arrow[r, "\pi"] & M
\end{tikzcd}$$

where $\mu_{\mathscr{G}}$ is defined as in Def. 4.22, and $\mu_{\mathscr{G}}$ restricts to the Maurer-Cartan form of $\mathscr{G}_x$ ($x \in M$) on each fibre of $\mathscr{G}$; especially, $\mu_{\mathscr{G}} : \mathrm{V}\mathscr{G} \to \mathscr{g}$ is a fibre-wise isomorphism (since $\mathrm{D}_g L_{g^{-1}}$ is an isomorphism). Hence, as described in Subsection 1.1, $\mu_{\mathscr{G}}$ as an element of $\Gamma(\mathrm{V}^*\mathscr{G} \otimes \pi^*\mathscr{G})$, *i.e.* a vector bundle morphism $\mathrm{V}\mathscr{G} \to \pi^*\mathscr{g}$ (linearity of μ_G is clear), is a vector bundle isomorphism. This finishes the proof. ∎

4.4. Exponential map of LGBs

By Remark 4.12 it is clear that we have a natural exponential map, just given by the fibre-wise exponential. If one is interested into a general exponential map, then see [2, §3.6, page 132ff.]. However, since our situation is much simpler, we quickly finish this discussion just making use of the already existing exponential map in each fibre; a straightforward generalization on results as provided in [1, §1.7, page 55ff.].

Definition 4.26: Exponential map,
[2, §3.6, second part of Ex. 3.6.2, page 133f.]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathscr{g} \to M$ its LAB. Then we define the **exponential map** $\exp : \mathscr{g} \to \mathscr{G}$ by

$$\exp(X) := \mathrm{e}^X := \exp_{\mathscr{G}_x}(X)$$

for all $x \in M$ and $X \in \mathscr{g}_x$, where $\exp_{\mathscr{G}_x}$ is the exponential map of the Lie group $\mathscr{G}_x$ as *e.g.* provided in [1, §1.7, Def. 1.7.6, page 57]. Its extension to sections, also denoted by $\exp : \Gamma(\mathscr{g}) \to \Gamma(\mathscr{G})$, is canonically given by

$$\exp(\nu)|_x := \exp(\nu_x)$$

for all $x \in M$ and $\nu \in \Gamma(\mathscr{g})$.
Usually, the LGB is given by context, otherwise we will denote the exponential map by $\exp_{\mathscr{G}}$ instead.

The exponential map is well-defined, especially it is smooth because it describes the flow of left-invariant vector fields as it also happens for Lie groups; see for example [1, Prop. 1.7.12, page 58] for the Lie group statement. Also recall Cor. 4.15, we denote left-invariant vector fields by X_ν where $\nu \in \Gamma(\mathscr{g})$ due to 1:1 correspondence of $L(\mathscr{G})$ and $\Gamma(\mathscr{g})$.

Corollary 4.27: The exponential map as flow of $L(\mathscr{G})$,
[2, discussion at the beginning of §3.6, Prop. 3.6.1 and its discussion afterwards; page 132f.]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M, and $\mathscr{g} \to M$ its LAB. Then the flow of a left-invariant vector field $X_\nu \in L(\mathscr{G})$ ($\nu \in \Gamma(\mathscr{g})$) is a complete flow $\phi : \mathbb{R} \times \mathscr{G} \to \mathscr{G}$ given by

$$\phi(t, g) = g \cdot \mathrm{e}^{t\nu_{\pi(g)}}$$

for all $t \in \mathbb{R}$ and $g \in \mathscr{G}$. Especially, the map

$$\mathbb{R} \times M \to \mathscr{G},$$
$$(t, x) \mapsto \mathrm{e}^{t\nu_x}$$

is smooth.

Proof.
As mentioned in Remark 4.12, X_ν is a vertical vector field so that $X_\nu|_{\mathscr{G}_x}$ is a left-invariant vector field of the Lie group $\mathscr{G}_x$ for all $x \in M$. $X_\nu|_{\mathscr{G}_x}$ is the left-invariant vector field in $\mathscr{G}_x$ related to

$\nu_x \in \mathscr{g}_x$ by Cor. 4.15, and the flow ϕ_x of $X_\nu|_{\mathscr{G}_x}$ is well-known, as *e.g.* in [1, §1.7, Prop. 1.7.12, page 58], that is,

$$\mathbb{R} \times \mathscr{G}_x \to \mathscr{G}_x,$$

$$(t, g) \mapsto \phi_x(t, g) = g \cdot \exp_{\mathscr{G}_x}(t\nu_x),$$

where $\exp_{\mathscr{G}_x}$ is the exponential map of $\mathscr{G}_x$. Since this works for all fibres $\mathscr{G}_x$ and by Def. 4.26 we get that the flow ϕ of X_ν is complete and given by

$$\phi(t, g) = g \cdot \mathrm{e}^{t\nu_{\pi(g)}}$$

for all $t \in \mathbb{R}$ and $g \in \mathscr{G}$. We also have

$$\mathrm{e}^{t\nu_x} = e_x \cdot \mathrm{e}^{t\nu_{\pi(e_x)}} = \phi(t, e_x)$$

for all $t \in \mathbb{R}$ and $x \in M$, such that smoothness of $(t, x) \mapsto \mathrm{e}^{t\nu_x}$ follows as composition of smooth maps. ■

Remark 4.28: Simplifying notation related to the exponential map

Due to Def. 4.26 we recover a lot of the typical properties of the exponential map as in [1, §1.7, Prop. 1.7.9, page 57], if we understand these properties point-wise. Recall Remark 3.10, then for example

$$\mathrm{e}^{(t+s)\nu} = \mathrm{e}^{t\nu} \cdot \mathrm{e}^{s\nu}$$

for all $t, s \in \mathbb{R}$ and $\nu \in \Gamma(\mathscr{g})$. As in [2, §3.6, discussion after Prop. 3.6.1, page 133], we say that $\mathbb{R} \ni t \mapsto \mathrm{e}^{t\nu}$ is smooth in the sense of the second statement of Cor. 4.27, and by construction we have

$$\nu_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{t\nu_x} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[t \mapsto \mathrm{e}^{t\nu_x}\right]$$

for all $x \in M$, so that we also write

$$\nu = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{t\nu}.$$

Since all of this is rather natural, we will make use of that without further mention.

This discussion also highlights that one could understand the infinite-dimensional Lie algebra $\Gamma(\mathscr{g})$ as the Lie algebra of the infinite-dimensional Lie group $\Gamma(\mathscr{G})$; thus, one could construct the LABs of LGBs by starting in that fashion, and then $\mathscr{g}$ is constructed by making use of the 1:1 correspondence of vector bundles and locally free sheaf of modules of constant rank.

4.5. LABs of pullback LGBs

We are going to define LGB representations and corresponding LAB representations. Since group representation are a special form of actions, we will have something similar in the case of LGB representations. Since actions are defined as maps on a pullback of an LGB $\mathscr{G}$, which is also an LGB by Cor. 3.1, we expect that the corresponding LAB representation is related to the LAB of the pullback of $\mathscr{G}$. It is natural to think of this LAB as the pullback of $\mathscr{g}$, which is also an LAB by Cor. 4.6:

Corollary 4.29: LAB of pullback LGB is pullback LAB, [3, §3, Thm. 3.5, page 21]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and let $f : N \to M$ be a smooth map defined on another smooth manifold N. Then the LAB of $f^*\mathscr{G}$ is isomorphic to the pullback LAB $f^*\mathscr{g}$.

Proof.
By Cor. 3.1 we know that $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$, the projection onto the second factor, is an LGB morphism over f,

$$\begin{tikzcd} f^*\mathscr{G} \arrow[r, "\pi_2"] \arrow[d] & \mathscr{G} \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

and it is fibre-wise a Lie group isomorphism such that

$$\mathrm{D}_{(p,e_x)}\pi_2 : \mathscr{h}_p \to \mathscr{g}_x$$

is a Lie algebra isomorphism for all $p \in N$, where e_x is the neutral element of $\mathscr{G}_x$ for $x := f(p)$, and $\mathscr{h}$ and $\mathscr{g}$ are the LABs of $f^*\mathscr{G}$ and $\mathscr{G}$, respectively; for all of that recall that $(f^*\mathscr{G})_p$ and $\mathscr{G}_x$ are Lie groups. Hence, we have a vector bundle morphism over f given by the following commuting diagram

$$\begin{tikzcd} \mathscr{h} \arrow[r, "\mathrm{D}_{(\mathbb{1}_N, e_f)}\pi_2"] \arrow[d] & \mathscr{g} \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

which describes fibre-wise a Lie algebra isomorphism, where

$$\mathrm{D}_{(\mathbb{1}_N, e_f)}\pi_2 := \mathrm{D}\pi_2 \circ (\mathbb{1}_N, e_f) = \left[N \ni p \mapsto \mathrm{D}_{(p, e_{f(p)})}\pi_2 \right].$$

By our notes in Subsection 1.1, we therefore achieve an LAB isomorphism $\mathscr{h} \to f^*\mathscr{g}$. ■

Remark 4.30: LAB of $f^*\mathscr{G}$

Since this isomorphism is very natural, we always use that identification and will refer to $f^*\mathscr{g}$ as *the* LAB of $f^*\mathscr{G}$.

With this we can quickly show the following familiar result.

Corollary 4.31: Differentials of LGB morphisms are LAB morphisms, [2, §3.5, section about morphisms, page 124f.]

Let $\mathscr{H} \to N$ and $\mathscr{G} \to M$ be two smooth LGBs over two smooth manifolds N and M, and we denote with $\mathscr{h}$ and $\mathscr{g}$ the LABs of $\mathscr{H}$ and $\mathscr{G}$, respectively. Furthermore, assume that we have an LGB morphism $F : \mathscr{H} \to \mathscr{G}$ over a smooth map $f : N \to M$. Then

$$\mathrm{D}F|_{\mathscr{h}} : \mathscr{h} \to \mathscr{g}$$

is an LAB morphism over f.

Proof.
Again by our notes in Subsection 1.1, we can view F as a base-preserving LGB morphism $F : \mathscr{H} \to f^*\mathscr{G}$, since $f^*\mathscr{G}$ is an LGB whose structure is naturally inherited by $\mathscr{G}$ as given in Cor. 3.1; similarly for its LAB by Cor. 4.6. Thus, it is fibre-wise a Lie group morphism, and its tangent map restricted to $\mathscr{h} = e^*\mathrm{V}\mathscr{H}$ (e the identity section of $\mathscr{H}$) gives therefore fibre-wise a Lie algebra morphism. Thus, $\mathrm{D}F|_{\mathscr{h}} : \mathscr{h} \to f^*\mathscr{g}$ is an LAB morphism (using Cor. 4.29), and can be seen as an LAB morphism $\mathscr{h} \to \mathscr{g}$ over f. Alternatively, it is straightforward and trivial to show it directly. ∎

5. LGB actions, part II

Finally, we come to the last part of the *basics* for LGBs and their notions needed in this paper and needed to formulate a new and more general form of gauge theory.

5.1. LGB and LAB representations

As usual, representations are a special type of group action, with an infinitesimal analogue.

Definition 5.1: LGB representations, [2, §1.7, special situation of the remark before Def. 1.7.1, page 43]

Let $\mathscr{G} \xrightarrow{\pi} M$ be an LGB over a smooth manifold M, $V \xrightarrow{p} M$ be a vector bundle, and assume that we have a left $\mathscr{G}$-action on V, $\Psi : \mathscr{G} * V := p^*\mathscr{G} \to V$. Then we say that Ψ

is a **$\mathscr{G}$-representation on** V if it is linear, that is,

$$\Psi(g, \alpha v) = \alpha \Psi(g, v),$$

$$\Psi(g, v + w) = \Psi(g, v) + \Psi(g, w)$$

for all $\alpha \in \mathbb{R}$, and $(g, v), (g, w) \in \mathscr{G} * V$. In alignment with previous notations we may also write $\Psi(g, v) = \Psi_g(v)$, or $\Psi(g, v) = g \cdot v$.

Remarks 5.2.
Observe that $(g, v) \in \mathscr{G} * V$ means that $p(v) = \pi(g)$, same for (g, w). Hence, given a base point in $x \in M$, the pairs (g, v) in $\mathscr{G} * V|_{p^{-1}(\{x\})}$ are given by elements $g \in \mathscr{G}_x$ and $v \in V_x$. By Def. 3.4 we also have

$$p\big(\Psi(g, v)\big) = \pi(g) = p(v) = x.$$

Thence, linearity of Ψ is well-defined. In fact, observe that $\mathscr{G} * V = p^*\mathscr{G} \cong \pi^*V$ as fibre bundle, therefore $\mathscr{G} * V$ carries not only the structure of an LGB but also of a vector bundle. That is, we have the following commuting diagram

$$\begin{tikzcd} \mathscr{G} * V \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathscr{G} \arrow[d, "\pi"] \\ V \arrow[r, "p"] & M \end{tikzcd}$$

the horizontal arrows describe the vector bundle structure (viewing $\mathscr{G} * V$ as the vector bundle π^*V), and the vertical ones the LGB structure (viewing $\mathscr{G} * V$ as the LGB $p^*\mathscr{G}$), where pr_i $(i \in \{1, 2\})$ are the projections onto the i-th component.

By fixing a base point $x \in M$ we clearly have the typical notion of a $\mathscr{G}_x$-representation on V_x. Equivalently, we could therefore obviously define LGB representations as a base-preserving LGB morphism $\Psi : \mathscr{G} \to \mathrm{Aut}(V)$ as also in [2, §1.7, Def. 1.7.1, page 43].

Corollary 5.3: LGB representations as LGB morphisms, [2, §1.7, Prop. 1.7.2, page 43]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, $V \to M$ be a vector bundle. Then every $\mathscr{G}$-representation $\Psi : \mathscr{G} * V \to V$ is equivalent to a base-preserving LGB morphism $\widetilde{\Psi} : \mathscr{G} \to \mathrm{Aut}(V)$, related by

$$\Psi(g, v) = \widetilde{\Psi}(g)(v)$$

for all $g \in \mathscr{G}_x$ $(x \in M)$ and $v \in V_x$.

Remarks 5.4.
We will usually denote both interpretations with the same notation.

Proof of Cor. 5.3.
This is an immediate consequence of Def. 5.1 and 3.4. Ψ is smooth if and only if $\widetilde{\Psi}$ is smooth; this is due to the fact that the LGB atlas of $\mathrm{Aut}(V)$ is inherited by an atlas of V as constructed in Ex. 2.6, and due to that $\mathcal{G} * V$ is an embedded submanifold of $\mathcal{G} \times V$. ∎

Example 5.5: Recovering Lie group representations on vector spaces

Similarly as to Ex. 3.15, if $\mathcal{G} = M \times G$ is a trivial LGB over M, G its structural Lie group, and $V = M \times W$ a trivial vector bundle with structural vector space W, then Ψ is equivalent to a G-representation on W. Backwards, every Lie group representation gives a representation of trivial LGBs on trivial vector bundles.

Also recall Ex. 2.6: We can now similarly construct another LGB needed for the adjoint representation.

Example 5.6: Another LGB example: Automorphisms of LABs, [2, §1.7, special situation of Ex. 1.7.12, page 46]

Let $\mathscr{g}$ be an LAB over the smooth manifold M. We denote with $\mathrm{Aut}(\mathscr{g}) \to M$ the bundle of fibre-wise Lie algebra automorphisms of $\mathscr{g}$, where the sections of $\mathrm{Aut}(\mathscr{g})$ are the base-preserving LAB automorphisms of $\mathscr{g}$. As in Ex. 2.6 one can show $\mathrm{Aut}(\mathscr{g})$ is an LGB by using LAB trivializations of $\mathscr{g}$ instead of vector bundle trivializations.
The notation of $\mathrm{Aut}(\mathscr{g})$ may be confusing with the notation as for vector bundles. We usually refer to this LAB automorphism when speaking of LABs and state it explicitly if we just mean the vector bundle version.

For the next major example of an LGB representation recall Def. 4.8.

Example 5.7: Adjoint LGB representation, [2, §3.5, special situation of Prop. 3.5.20, page 131]

For $g \in \mathcal{G}_x$ ($x \in M$) we have the conjugation c_g which is a $\mathcal{G}_x$-automorphism. In the usual way we define the **adjoint representation of** $\mathcal{G}$ as a base-preserving LGB morphism $\mathcal{G} \to \mathrm{Aut}(\mathscr{g})$ by

$$\mathrm{Ad}_g \coloneqq \mathrm{D}_{e_x} c_g$$

for all $g \in \mathcal{G}_x$ ($x \in M$).
That this is a $\mathcal{G}$-representation on $\mathscr{g}$ follows by the following proof: By construction $\mathrm{Ad}_g \in \mathrm{Aut}(\mathscr{g}_x) = \mathrm{Aut}(\mathscr{g})|_x$ so that the adjoint representation is well-defined. As in the Lie group case, due to the fact that $c_{gq} = c_g \circ c_q$ for all $g, q \in \mathcal{G}_x$ and $c_g(e_x) = e_x$ we have

$$\mathrm{Ad}_{gq} = \mathrm{D}_{e_x}(c_g \circ c_q) = \mathrm{D}_{e_x} c_g \circ \mathrm{D}_{e_x} c_q = \mathrm{Ad}_g \circ \mathrm{Ad}_q.$$

Smoothness could either be shown in a bit more generalized way than the proof in [1, §2.1, Thm. 2.1.45, page 101f.], but that would be a bit tedious; instead let us use the already-known smoothness of the adjoint representation in each fibre. Fix an open subset U of M such that $\mathscr{G}$ and $\mathscr{g}$ are trivial, that is, $\mathscr{G}|_U \cong U \times G$ and $\mathscr{g}|_U \cong U \times \mathfrak{g}$ as LGB and LAB, respectively, where G is the structural Lie group with its Lie algebra $\mathfrak{g}$. Then also clearly $\mathrm{Aut}(\mathscr{g})|_U \cong U \times \mathrm{Aut}(\mathfrak{g})$ as LGBs. By construction we then have w.r.t. these trivializations

$$\mathrm{Ad}_{(x,g)} = \left(x, \mathrm{D}_{e_G} c_g^G\right) = \left(x, \mathrm{Ad}_g^G\right)$$

for all $(x, g) \in U \times G$, where $\mathrm{Ad}_g^G : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G on $\mathfrak{g}$. Smoothness now follows trivially by the canonical manifold structure of product manifolds and the smoothness of Ad^G.

Infinitesimally, we have LAB actions and representations; recall Cor. 4.6.

Definition 5.8: LAB actions, [2, §4.1, reformulated version for LABs of Def. 4.1.1, page 149]

Let M, N be smooth manifolds, $\mathscr{g} \xrightarrow{\pi} M$ an LAB, and $f : N \to M$ a smooth map. Then a **$\mathscr{g}$-action on** N is a base-preserving vector bundle morphism

$$f^*\mathscr{g} \to \mathrm{T}N,$$

$$\nu \mapsto \rho(\nu),$$

satisfying

$$\mathrm{D}f \circ \rho = 0 \tag{24}$$

and such that the induced map

$$\Gamma(\mathscr{g}) \to \Gamma(f^*\mathscr{g}) \to \mathfrak{X}(N),$$

$$\mu \mapsto f^*\mu \mapsto \rho(f^*\mu)$$

is a homomorphism of Lie algebras, that is,

$$\rho\Big(f^*\Big([\mu, \nu]_{\mathscr{g}}\Big)\Big) = \big[\rho(f^*\mu), \rho(f^*\nu)\big] \tag{25}$$

for all $\mu, \eta \in \Gamma(\mathscr{g})$.

Remarks 5.9.

For the readers familiar with Lie algebroids, especially action Lie algebroid structures on trivial

LABs, Eq. (25) should look familiar; in fact, as shown in [2, §4.1, Prop. 4.1.2, page 149f.], given an LAB action one can construct a Lie algebroid structure on $f^*\mathscr{g}$. This leads to an action Lie algebroid structure on a possibly non-trivial bundle.

As for Lie group and algebra actions, an LGB action induces an LAB action.

Lemma 5.10: LGB actions induce LAB actions, [2, §4.1, special situation of Thm. 4.1.6, page 152]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Then any right $\mathscr{G}$-action $\Phi : N * \mathscr{G} = f^*\mathscr{G} \to N$ on N induces a $\mathscr{g}$-action ρ on N by

$$\rho \coloneqq \mathrm{D}\Phi|_{f^*\mathscr{g}},$$

i.e.

$$\rho(\eta) = \mathrm{D}_{\left(p, e_{f(p)}\right)}\Phi(\eta)$$

for all $\eta \in (f^*\mathscr{g})_p$ ($p \in N$).

Remarks 5.11.
Similarly to the discussion as in [1, §3.4, page 141ff.], one can show the same for left actions, but one has to use the multiplication with the inverse, that is,

$$\rho \coloneqq \mathrm{D}\Phi'|_{f^*\mathscr{g}},$$

where

$$\Phi' : \mathscr{G} * N \to N,$$
$$(g, p) \mapsto g^{-1} \cdot p.$$

Proof of Lemma 5.10.
We generalize the proof as provided in [1, §3.4, proof of Prop. 3.4.4, page 144f.]. Observe that we have

$$\rho(p, \nu) = \mathrm{D}_{\left(p, e_{f(p)}\right)}\Phi(p, \nu) \in \mathrm{T}_pN$$

for all $(p, \nu) \in f^*\mathscr{g}$, and thus describes a base-preserving vector bundle morphism $f^*\mathscr{g} \to \mathrm{T}N$. We can rewrite Eq. (13) to

$$f \circ \Phi = \pi \circ \mathrm{pr}_2,$$

where $\mathrm{pr}_2^{f^*\mathscr{G}}$ is the projection onto the second factor in $f^*\mathscr{G} \subset N \times \mathscr{G}$. Thence, we have

$$\mathrm{D}f \circ \rho = \mathrm{D}f \circ \mathrm{D}\Phi|_{f^*\mathscr{g}} = \mathrm{D}(f \circ \Phi)|_{f^*\mathscr{g}} = \left(\mathrm{D}\pi \circ \mathrm{D}\mathrm{pr}_2^{f^*\mathscr{G}}\right)\Big|_{f^*\mathscr{g}} = \mathrm{D}\pi|_{\mathscr{g}} \circ \mathrm{pr}_2^{f^*\mathscr{g}} = 0,$$

making use of that $\mathcal{g}$ consists of vertical vectors, and where $\mathrm{pr}_2^{f^*\mathcal{g}}$ is the projection onto the second factor in $f^*\mathcal{g} \subset N \times \mathcal{g}$.

The proof of Eq. (25) is as straightforward as usual, as a quick argument for the experienced reader observe that Eq. (15) and (14) imply that Φ induces a Lie group homomorphism $\Gamma(\mathscr{G}) \to \mathrm{Diff}(M)$ (Diff the group of diffeomorphisms) so that we have the desired Lie algebra homomorphism on an infinitesimal scale.[3] Let $\mu, \nu \in \Gamma(\mathcal{g})$, then we know that

$$\left[\mathrm{D}\Phi(f^*\mu), \mathrm{D}\Phi(f^*\nu)\right] = \mathrm{D}\Phi\Big([f^*\mu, f^*\nu]_{f^*\mathcal{g}}\Big),$$

if we can show that $\mathrm{D}\Phi(f^*\mu)$ and $\mathrm{D}\Phi(f^*\nu)$ are Φ-related to $f^*\mu$ and $f^*\nu$, respectively; this is a common procedure, see for example [1, §A.1, Prop. A.1.49, page 615]. That is, we need to show now that

$$\mathrm{D}\Phi(f^*\mu)|_{\Phi(p,g)=p\cdot g} \overset{!}{=} \mathrm{D}_{(p,g)}\Phi\Big((f^*\mu)|_{(p,g)}\Big) \tag{26}$$

for all $(p,g) \in N * \mathscr{G}$, similarly for ν; here we understand the LABs of LGBs as left-invariant vector fields as in Cor. 4.15. Observe that we have in general for $\eta \in \Gamma(f^*\mathcal{g})$

$$\mathrm{D}\phi(\eta)|_{p\cdot g} = \mathrm{D}_{(p\cdot g, e_x)}\Phi\big(\eta_{(p\cdot g, e_x)}\big) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi\Big(p\cdot g, \mathrm{e}^{t\xi_{p\cdot g}}\Big) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(p \cdot g\mathrm{e}^{t\xi_{p\cdot g}}\Big)$$

for all $(p,g) \in N * \mathscr{G}$, where $t \in \mathbb{R}$ and we write in general $\eta_{(p,e_x)} = (p, \xi_p)$ with $\xi_p \in \mathcal{g}_x$ ($x := f(p)$), and the exponential e is the one of $\mathscr{G}$. But we also have similarly

$$\mathrm{D}_{(p,g)}\Phi\big(\underbrace{\eta_{(p,g)}}_{=\mathrm{D}_{(p,e_x)}L_{(p,g)}(\eta_{(p,e_x)})}\big) = \mathrm{D}_{(p,e_x)}\big(\Phi \circ L_{(p,g)}\big)(p, \xi_p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi\Big(p, g\mathrm{e}^{t\xi_p}\Big) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(p \cdot g\mathrm{e}^{t\xi_p}\Big),$$

where L is the left-multiplication in $f^*\mathscr{G}$. So, in order to achieve Eq. (26) we now set $\eta = f^*\mu$. That is, $\eta_{(p,e_x)} = \big(p, \mu_{f(p)}\big)$, therefore $\xi_p = \mu_{f(p)}$, and due to Eq. (13) and Rem. 3.5 we derive $\xi_p = \xi_{p\cdot g}$. In total, we see that Eq. (26) is satisfied.

Finally, as argued earlier, we can prove

$$\left[\mathrm{D}\Phi(f^*\mu), \mathrm{D}\Phi(f^*\nu)\right] = \mathrm{D}\Phi\Big([f^*\mu, f^*\nu]_{f^*\mathcal{g}}\Big) = \mathrm{D}\Phi\Big(f^*\Big([\mu,\nu]_{\mathcal{g}}\Big)\Big),$$

making use of that the field of Lie brackets of $f^*\mathcal{g}$ is the f-pullback of $[\cdot,\cdot]_{\mathcal{g}}$ as a section. This finishes the proof. ■

Remark 5.12: Variants of the LAB action as Lie algebra homomorphism

It is clear that one has a local version of Eq. (25). In fact, as one sees in the proof of Lemma 5.10, the argument for Eq. (25) works pointwise in M, so, one may also think of a homomorphism of Lie algebras

$$\mathcal{g}_x \to \mathfrak{X}\big(f^{-1}(\{x\})\big),$$

[3] We use the notion of sections to avoid the possible lack of smooth structure on the preimages of f.

$$\nu \mapsto \left[f^{-1}(\{x\}) \ni p \mapsto \rho(p, \nu)\right]$$

for all $x \in M$, where we also used Eq. (24); that is, we expect that an LAB action is fibre-wise a Lie algebra action. Due to a possible lack of manifold structure on $f^{-1}(\{x\})$ we did not restrict to $x \in M$ to avoid technical difficulties. However, f will be the projection of a bundle later, and in that case the fibre $f^{-1}(\{x\})$ is an embedded submanifold of N. As argued in Remark 3.6, the $\mathscr{G}$-action is a $\mathscr{G}_x$-action restricted on $f^{-1}(\{x\})$. By what we know about Lie group actions we immediately know that we have the aforementioned Lie algebra homomorphism, and so also Eq. (25), in total a vector bundle morphism $f^*\mathscr{g} \to \mathrm{T}N$ which gives fibre-wise over x rise to a Lie algebra action.

An important example will be fundamental vector fields which we will introduce later. Another example are LAB representations induced by LGB representations; hence let us introduce LAB representations. We will introduce them in a reverted order than the LGB representations, that is, first defining them as certain LAB morphisms, and then trivially concluding a relation to LAB actions; for the following recall Ex. 4.5.

Definition 5.13: LAB representations, [2, §3.3, Def. 3.3.13, page 107]

Let $\mathscr{g} \to M$ be an LAB over a smooth manifold M, and $V \to M$ be a vector bundle. A **$\mathscr{g}$-representation on** V is an LAB morphism $\psi : \mathscr{g} \to \mathrm{End}(V)$.

Corollary 5.14: LAB representations are specific LAB actions, [2, §4.1, special consequence of Prop. 4.1.7 but we do not assume integrability of the LAB, page 153]

Let $\mathscr{g} \to M$ be an LAB over a smooth manifold M, $V \xrightarrow{p} M$ be a vector bundle, and $\psi : \mathscr{g} \to \mathrm{End}(V)$ a $\mathscr{g}$-representation on V. Then ψ defines an LAB action $\widetilde{\psi} : p^*\mathscr{g} \to \mathrm{T}V$ by

$$\widetilde{\psi}(v, \nu) := -\psi(\nu)(v)$$

for all $(v, \nu) \in p^*\mathscr{g}$, where one makes use of the identification $\mathrm{T}_v V_x = \mathrm{V}_v V \cong V_x$ ($x := p(v)$), so that $V_x \subset \mathrm{T}_v V$.

Remarks 5.15.
In fact, similar to LGBs we could have defined LAB representations as a certain type of LAB action with values in certain *linear vector fields*. However, it would exceed this work to introduce these vector fields; start with [2, §3.4, page 110] for more elaborated details on how to do this.

As for LGB representations and linear actions, we denote both, LAB representation and its associated sense of action, in the same fashion.

This is a known relationship in the case of Lie algebra representations on vector spaces and their associated actions; usually this is proven by assuming a Lie group representation, however, one can show it without assuming integrability of the Lie algebra action. See for example [5, §2.1, proof of Prop. 2.1.16, page 22]. Hence, we will just show that the proof of Cor. 5.14 breaks down to proving it for Lie algebras acting on vector spaces and refer to this reference for the remaining part of the proof.

Proof of Cor. 5.14.
By construction we have that $\widetilde{\psi}$ has values in the vertical bundle $\mathrm{V}V$ of V, hence Eq. (24) follows. Making use of $\mathrm{V}_v V \cong V_x$ for all $v \in V_x$ ($x \in M$), it is clear that $p^*V \cong \mathrm{V}V$ as vector bundles, and so it is trivial to see that $\widetilde{\psi}$ as a map $p^*\mathcal{g} \to p^*V$ is a smooth and base-preserving vector bundle morphism due to that ψ is a smooth map with values in $\mathrm{End}(V)$ (making again use of that $p^*\mathcal{g}$ is an embedded submanifold of $V \times \mathcal{g}$, similarly for p^*V as embedded submanifold of $V \times V$).

As argued in Rem. 5.12, since $\mathcal{g}$ acts via a representation on a bundle V, $\widetilde{\psi} : p^*\mathcal{g} \to \mathrm{V}V$ will be an LAB action, if it induces a Lie algebra action over each base point $x \in M$ via

$$\mathcal{g}_x \to \mathfrak{X}(V_x),$$

$$\nu \mapsto \left[V_x \ni v \mapsto \widetilde{\psi}(v, \nu) \right].$$

Since ψ is just a $\mathcal{g}_x$-representation on V_x over x we know by [5, §2.1, proof of Prop. 2.1.16, page 22] that ψ indeed gives rise to a $\mathcal{g}_x$-action on V_x by

$$\widetilde{\psi}(v, \nu) = -\psi(\nu)(v)$$

for all $v \in V_x$ and $\nu \in \mathcal{g}_x$, which is precisely the form of $\widetilde{\psi}$. Thus, $\widetilde{\psi}$ is an LAB action. ■

Usually statements like Cor. 5.14 are proven by assuming integrability: LGB representations as linear actions $V * \mathcal{G} \to V$ and as LGB morphisms $\mathcal{G} \to \mathrm{Aut}(V)$ are the same by Cor. 5.3; the former induces LAB actions $p^*\mathcal{g} \to \mathrm{T}V$ by Lemma 5.10, and in the same manner the latter implies LAB morphisms $\mathcal{g} \to \mathrm{End}(V)$ by Cor. 4.31. Thence, LAB representations can be viewed as certain LAB actions.

The adjoint representation is an important example inherited by an LGB representation; this example will conclude this subsection.

Example 5.16: Adjoint LAB representations, [2, §3.3, special situation of Ex. 3.3.15, page 108]

Let us assume the same situation as in Ex. 5.7, especially we have the adjoint representation of the LGB $\mathcal{G} \to M$, $\mathrm{Ad} : \mathcal{G} \to \mathrm{Aut}(\mathcal{g})$. As discussed earlier, its infinitesimal version is a $\mathcal{g}$-representation on itself, the **adjoint representation** ad **of** $\mathcal{g}$. By construction,

$$\mathrm{ad}_\nu(\mu) := \mathrm{ad}(\nu)(\mu) = [\nu, \mu]_{\mathcal{g}_x}$$

for all $\nu, \mu \in \mathcal{g}_x$ ($x \in M$).

5.2. Fundamental vector fields

The LAB actions inherited by LGB actions are actually so important that they have a separate label; we are following [1, §3.4, generalization of Def. 3.4.1, page 143] for the labeling:

Definition 5.17: Fundamental vector fields

Let M and N be two smooth manifolds, $\mathcal{G} \to M$ an LGB over M, $f : N \to M$ a smooth map, and assume we have a right $\mathcal{G}$-action on N, $N * \mathcal{G} \to N$. For $\nu \in \Gamma(\mathscr{g})$ we define its induced **fundamental vector field** $\widetilde{\nu}$ as an element of $\mathfrak{X}(N)$ by

$$\widetilde{\nu}_p := \mathrm{D}_{e_{f(p)}} \Phi_p \big(\nu_{f(p)}\big)$$

for all $p \in N$, where Φ_p is the orbit map defined in Def. 3.8 and $e_{f(p)}$ the neutral element of $\mathcal{G}_{f(p)}$.

For left actions we define fundamental vector fields similarly by

$$\widetilde{\nu}_p := \mathrm{D}_{e_{f(p)}} \Phi'_p \big(\nu_{f(p)}\big)$$

for all $p \in N$, where Φ'_p is a slightly adjusted orbit map given by

$$\mathcal{G}_{f(p)} \to N,$$

$$g \mapsto g^{-1} \cdot p.$$

Remark 5.18: Notation

Again, point-wise over $x := f(p)$ this is just the typical definition of a fundamental vector field with respect to $\nu_x \in \mathscr{g}_x$ (except that $f^{-1}(\{x\})$ may not be a manifold). Hence, one has also a point-wise definition which we will also denote similarly by $\widetilde{\nu_x}$. If $f^{-1}(\{x\})$ is not a manifold, then $\widetilde{\nu_x}|_p$ is just a formal notation, and it only defines an element of T_pN which may not be related to a vector field on $f^{-1}(\{x\})$, not even to a vector field on N if ν_x does not formally come from a fixed section of $\mathscr{g}$.

However, if f is *e.g.* the projection of a bundle, then we have a $\mathcal{G}$-action on the fibre $f^{-1}(\{x\})$ as manifold, and therefore $\widetilde{\nu}|_{f^{-1}(\{x\})}$ and $\widetilde{\nu_x}$ give rise to a vector field on $f^{-1}(\{x\})$. In other words, fundamental vector fields are vertical vector fields as expected and fibre-wise fundamental vector fields coming from a Lie group action.

For long expressions we use the following different font

$$\underset{\sim}{\nu}$$

instead of $\widetilde{\nu}$.

Remark 5.19: Map to fundamental vector fields an LAB action

As anticipated, this is related to the LAB action coming from the right $\mathscr{G}$-action $\Phi : N * \mathscr{G} \to N$. The following can also be shown for left actions in a similar manner by recalling Rem. 5.11.
By Lemma 5.10 we have an LAB action $\rho : f^*\mathscr{g} \to \mathrm{T}N$ given by $\rho := \mathrm{D}\Phi|_{f^*\mathscr{g}}$. We have

$$\rho(p, \nu) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi\big(p, \mathrm{e}^{t\nu}\big) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(p \cdot \mathrm{e}^{t\nu}\big) = \mathrm{D}_{e_{f(p)}}\Phi_p(\nu) = \widetilde{\nu}_p$$

for all $(p, \nu) \in f^*\mathscr{g}$, where $t \in \mathbb{R}$, e is the exponential map of $\mathscr{G}$, and $e_{f(p)}$ the neutral element of $\mathscr{G}_{f(p)}$. Thus, the map to fundamental vector fields

$$\Gamma(\mathscr{g}) \mapsto \mathfrak{X}(N),$$

$$\nu \mapsto \widetilde{\nu},$$

is equivalent to the map

$$\Gamma(\mathscr{g}) \to \mathfrak{X}(N),$$

$$\nu \mapsto \rho(f^*\nu)$$

induced by ρ as in Def. 5.8, which also implies that the map to the fundamental vector fields is a homomorphism of Lie algebras. Last but not least, by Def. 5.8 it follows that fundamental vector fields are in the kernel of $\mathrm{D}f$.

5.3. Differential of smooth LGB actions

Lemma 5.20: Tangent bundle of pullback fibre bundles

Let M and N be two smooth manifolds, $F \xrightarrow{\pi} M$ a fibre bundle over M, and $f : N \to M$ a smooth map. Then we have for its tangent spaces

$$\mathrm{T}_{(p,v)}(f^*F) = \big\{(X, Y) \bigm| X \in \mathrm{T}_pN, Y \in \mathrm{T}_vF \text{ with } \mathrm{D}_pf(X) = \mathrm{D}_v\pi(Y)\big\}$$

for all $(p, v) \in f^*F$.

Proof.
Recall that $(p, v) \in f^*F$ implies that

$$f(p) = \pi(v)$$

such that we can immediately derive its infinitesimal version as

$$\mathrm{D}_pf(X) = \mathrm{D}_v\pi(Y)$$

for all $X \in \mathrm{T}_p N, Y \in \mathrm{T}_v F$. Hence, we have derived that $\mathrm{T}_{(p,v)}(f^* F)$ is a subset of the set of such pairs (X, Y). That this is an equivalent description quickly follows by the fact that f and π are transversal to each other (trivially, because π is a surjective submersion). This means, the following linear map

$$\mathrm{T}_p N \times \mathrm{T}_v F \to \mathrm{T}_{f(p)} M,$$

$$(X, Y) \mapsto \mathrm{D}_p f(X) - \mathrm{D}_v \pi(Y)$$

is surjective because π is a submersion; it is also well-defined because of $f(p) = \pi(v)$. Hence, the dimension of the kernel of this map has the dimension

$$\dim(N) + \dim(F) - \dim(M) = \dim(N) + \mathrm{rk}(F) = \dim(f^* N),$$

where dim denotes the dimension as a manifold and rk the rank of a bundle (the dimension of its structural fibre). Its dimension is precisely the dimension of $f^* F$, and since it is about finite dimensions we can therefore identify $\mathrm{T}_{(p,v)}(f^* F)$ with this kernel. This concludes the proof due to the fact that the kernel consists of (X, Y) with $\mathrm{D}_p f(X) = \mathrm{D}_v \pi(Y)$. ■

With this we can finally show the following theorem; also recall the notations introduced in Def. 3.8, 4.8, 4.22 and 5.17.

Theorem 5.21: Differential of smooth LGB actions

Let M and N be two smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M, $f : N \to M$ a smooth map, and assume we have a right $\mathscr{G}$-action on N, $\Phi : N * \mathscr{G} \to N$. Then we have

$$\mathrm{D}_{(p,g)} \Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \left.\overbrace{(\mu_{\mathscr{G}})_g \big(Y - \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big)}\right|_{p \cdot g} \tag{27}$$

for all $(p, g) \in N * \mathscr{G}$ and $(X, Y) \in \mathrm{T}_{(p,g)}(N * \mathscr{G})$, where $x := f(p) = \pi(g)$, σ is any (local) section of $\mathscr{G}$ with $\sigma_x = g$, e is the identity section of $\mathscr{G}$, and ω is an element of $\mathrm{T}_x M$ given by

$$\omega := \mathrm{D}_p f(X) = \mathrm{D}_g \pi(Y).$$

We may omit the natural embedding of ω into $\mathrm{T}_{e_x} \mathscr{G}$ via $\mathrm{D}_x e$, writing instead

$$\mathrm{D}_{(p,g)} \Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \left.\overbrace{(\mu_{\mathscr{G}})_g \big(Y - \mathrm{D}_x \sigma(\omega)\big)}\right|_{p \cdot g}. \tag{28}$$

If f is a surjective submersion, then we can also write

$$\mathrm{D}_{(p,g)} \Phi(X, Y) = \mathrm{D}_p r_\sigma(X) + \mathrm{D}_g \Phi_\tau(Y) - \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big) \tag{29}$$

and

$$\mathrm{D}_{(p,g)} \Phi(X, Y) = \mathrm{D}_p r_g\big(X - \mathrm{D}_{e_x} \Phi_\tau\big(\mathrm{D}_x e(\omega)\big)\big) + \mathrm{D}_g \Phi_p\big(Y - \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big)$$

$$+ \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big), \tag{30}$$

where τ is any (local) section[a] of f with $\tau_x = p$.

[a] That is, $f \circ \tau = \mathbb{1}_M$ (locally).

Remarks 5.22.
The assumption about f being a surjective submersion is being stated in order to assure the existence of τ and the manifold structure on $f^{-1}(\{x\})$ as an embedded submanifold of N; see the proof for more details. If the existence of τ and the embedded submanifold structure is known otherwise, then those equations can still be derived. Following the proof, one may also just need the structure of an immersed submanifold.

Proof of Thm. 5.21.
We want to calculate the derivative of Ψ, and due to $N * \mathscr{G} = f^*\mathscr{G}$ we are going to use Lemma 5.20. That is, fix $(p, g) \in N * \mathscr{G}$ and $X \in \mathrm{T}_p N$, $Y \in \mathrm{T}_g \mathscr{G}$ with

$$\mathrm{D}_p f(X) = \mathrm{D}_g \pi(Y) =: \omega \in \mathrm{T}_{f(p)} M.$$

Recall that we can localize LGB actions in sense of Rem. 3.6; so, let $x := f(p) = \pi(g)$, and fix a trivialization of $\mathscr{G}$ around x. Then it is clear that there is a (local)[4] section $\sigma \in \Gamma(\mathscr{G})$ with $\sigma_x = g$. Observe that we can write

$$Y = Y - \mathrm{D}_x \sigma(\omega) + \mathrm{D}_x \sigma(\omega).$$

The two first summands result into a vertical tangent vector due to

$$\mathrm{D}_g \pi\big(Y - \mathrm{D}_x \sigma(\omega)\big) = \mathrm{D}_g \pi(Y) - \mathrm{D}_x \underbrace{(\pi \circ \sigma)}_{=\mathbb{1}_M}(\omega) = \mathrm{D}_g \pi(Y) - \omega = 0.$$

For the following recall the notations introduced in Def. 3.8 and 4.8; we can derive that

$$\mathrm{D}_{e_x} R_\sigma \circ \mathrm{D}_x e = \mathrm{D}_x \underbrace{(R_\sigma \circ e)}_{x \mapsto e_x \sigma_x = \sigma_x} = \mathrm{D}_x \sigma.$$

So, let us write

$$Y^v := Y - \mathrm{D}_x \sigma(\omega) = Y - \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big).$$

We have proven that $Y^v \in \mathrm{V}_g \mathscr{G}$, and thus $(0_p, Y^v) \in \mathrm{T}_{(p,g)}(f^*\mathscr{G})$ by Lemma 5.20, where 0_p is the zero tangent vector. In the same fashion we also have

$$\big(X, \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) \in \mathrm{T}_{(p,g)}(f^*\mathscr{G})$$

[4] For simplicity of notation we omit the notation of restricting on some open subset of M.

because of

$$\mathrm{D}_g\pi\Big(\mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) = \mathrm{D}_g\pi(Y - Y^v) = \mathrm{D}_g\pi(Y) = \mathrm{D}_p f(X),$$

thence we can write

$$(X, Y) = \big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big) + Y^v\big) = \big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) + (0_p, Y^v).$$

Hence also

$$\mathrm{D}_{(p,g)}\Phi(X, Y) = \mathrm{D}_{(p,g)}\Phi\big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) + \mathrm{D}_{(p,g)}\Phi(0_p, Y^v)$$

The first summand is quickly calculated as

$$\mathrm{D}_{(p,g)}\Phi(0_p, Y^v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \underbrace{(p \cdot \gamma)}_{\Phi_p(\gamma)} = \mathrm{D}_g\Phi_p(Y^v) \tag{31}$$

where $\gamma : I \to \mathscr{G}_x$ (I open interval of $\mathbb{R}$ containing 0) is a curve with $\gamma(0) = g$ and $\mathrm{d}/\mathrm{d}t|_{t=0}\gamma = Y^v$; by the verticality of Y^v, $\mathrm{D}_g\Phi_p(Y^v)$ is well-defined since Φ_p is a map $\mathscr{G}_x \to N$. Now recall Def. 4.22 and 5.17, and observe that we can write

$$\mathrm{D}_g\Phi_p(Y^v) = \big(\mathrm{D}_g\Phi_p \circ \mathrm{D}_{e_x}L_g \circ \mathrm{D}_g L_{g^{-1}}\big)(Y^v) = \mathrm{D}_{e_x}\underbrace{(\Phi_p \circ L_g)}_{\mathscr{G}\ni q \mapsto p\cdot gq = \Phi_{p\cdot g}(q)}\Big(\left(\mu_{\mathscr{G}}\right)_g(Y^v)\Big) = \left.\overline{\left(\mu_{\mathscr{G}}\right)_g(Y^v)}\right|_{p\cdot g} \tag{32}$$

making use of the verticality of Y^v such that $\mathrm{D}L_g$ can act on Y^v and of $\left(\mu_{\mathscr{G}}\right)_g(Y^v) \in \mathscr{g}_x$.

For the second summand in Eq. (31) we use

$$\begin{aligned}
\mathrm{D}_{(p,g)}\Phi\Big(X, \mathrm{D}_{e_x}R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) &= \mathrm{D}_{(p,e_x)}\underbrace{\big(\Phi \circ (\mathbb{1}_N, R_\sigma)\big)}_{N*\mathscr{G}\ni(p,g)\mapsto p\cdot g\sigma_x = r_\sigma(p\cdot g)}\big(X, \mathrm{D}_x e(\omega)\big) \\
&= \mathrm{D}_{(p,e_x)}(r_\sigma \circ \Phi)\big(X, \mathrm{D}_x e(\omega)\big) \\
&= \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{(p,e_x)}\Phi\big(X, \mathrm{D}_x e(\omega)\big)\Big) \\
&= \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{(p,e_x)}\Phi\big(X, \mathrm{D}_p(e \circ f)(X)\big)\Big) \\
&= \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{(p,p)}\big(\Phi \circ (\mathbb{1}_N, e \circ f)\big)(X, X)\Big),
\end{aligned}$$

but

$$\Big[N \times N \ni (p,p) \mapsto \big(\Phi \circ (\mathbb{1}_N, e \circ f)\big)(p,p) = \Phi\big(p, e_{f(p)}\big) = p \cdot e_{f(p)} = p\Big] = \big[p \mapsto \mathbb{1}_N(p)\big],$$

hence,

$$\mathrm{D}_{(p,p)}\big(\Phi \circ (\mathbb{1}_N, e \circ f)\big)(X, X) = \mathrm{D}_p\mathbb{1}_N(X) = \mathbb{1}_{\mathrm{T}_pN}(X) = X.$$

So, we get in total

$$\mathrm{D}_{(p,g)}\Phi\Big(X, \mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\Big) = \mathrm{D}_p r_\sigma(X),$$

therefore

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_\sigma(X) + \left.\overbrace{\left(\mu_{\mathscr{G}}\right)_g(Y^v)}\right|_{p\cdot g}.$$

If f is a surjective submersion, then x is a regular value and thus $f^{-1}(\{x\})$ is an embedded submanifold, and we can assure the existence of a smooth local section $\tau : U \to N$ of f (U an open neighbourhood of $x \in M$), *i.e.* $f \circ \tau = \mathbb{1}_U$ with $\tau_x = p$; in case of doubt, this can be shown as in [1, §3.7, Lemma 3.7.4, page 152f.] via the Regular Point Theorem. Recalling the arguments of Rem. 4.16, we can rewrite Eq. (31) to

$$\begin{aligned}
\mathrm{D}_g\Phi_p(Y^v) &= \mathrm{D}_g\Phi_\tau(Y^v) \\
&= \mathrm{D}_g\Phi_\tau(Y) - \mathrm{D}_g\Phi_\tau\big(\mathrm{D}_{e_x} R_\sigma\big(\mathrm{D}_x e(\omega)\big)\big) \\
&= \mathrm{D}_g\Phi_\tau(Y) - \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big)
\end{aligned}$$

making use of that $\mathrm{D}_g\Phi_\tau$ is linear map $\mathrm{T}_g\mathscr{G} \to \mathrm{T}_{p\cdot g}N$. In that case we would get in total

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_\sigma(X) + \mathrm{D}_g\Phi_\tau(Y) - \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big).$$

Alternatively, we can keep the form of Eq. (31) and instead apply the same trick to X as for Y, that is,

$$X = X^v + \mathrm{D}_{e_x}\Phi_\tau\big(\mathrm{D}_x e(\omega)\big),$$

where

$$X^v \coloneqq X - \mathrm{D}_x\tau(\omega) = X - \mathrm{D}_{e_x}\Phi_\tau\big(\mathrm{D}_x e(\omega)\big),$$

and X^v is vertical, too, that is,

$$\mathrm{D}_p f(X^v) = \mathrm{D}_p f(X) - \mathrm{D}_x(f \circ \tau)(\omega) = \omega - \omega = 0,$$

especially $X^v \in \mathrm{T}_p\big(f^{-1}(\{x\})\big)$ and so we can apply a similar argument as in Rem. 4.16 to derive

$$\begin{aligned}
\mathrm{D}_p r_\sigma(X) &= \mathrm{D}_p r_\sigma(X^v) + \mathrm{D}_p r_\sigma\Big(\mathrm{D}_{e_x}\Phi_\tau\big(\mathrm{D}_x e(\omega)\big)\Big) \\
&= \mathrm{D}_p r_g(X^v) + \mathrm{D}_{e_x}(r_\sigma \circ \Phi_\tau)\big(\mathrm{D}_x e(\omega)\big).
\end{aligned}$$

Finally, using these expressions, the total formula would look like

$$\begin{aligned}
\mathrm{D}_{(p,g)}\Phi(X,Y) &= \mathrm{D}_p r_g(X^v) + \mathrm{D}_g\Phi_p(Y^v) + \mathrm{D}_{e_x}\underbrace{(r_\sigma \circ \Phi_\tau)}_{\mathscr{G}\ni g\mapsto \tau_{\pi(g)}\cdot g\sigma_{\pi(g)} = (\Phi_\tau \circ R_\sigma)(g)}\big(\mathrm{D}_x e(\omega)\big) \\
&= \mathrm{D}_p r_g(X^v) + \mathrm{D}_g\Phi_p(Y^v) + \mathrm{D}_{e_x}(\Phi_\tau \circ R_\sigma)\big(\mathrm{D}_x e(\omega)\big).
\end{aligned}$$

■

Remarks 5.23.
Eq. (27) is very similar to the "classical" formula used in gauge theory, see *e.g.* [1, §3.5, Prop. 3.5.4, page 146]: In the case of a Lie group action on N (so, $\mathscr{G}$ a Lie group G and M a point) we have

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathrm{D}_p r_g(X) + \left.\overbrace{(\mu_G)_g(Y)}\right|_{p\cdot g}$$

for all $p \in N$, $g \in G$, $X \in \mathrm{T}_pN$ and $Y \in \mathrm{T}_gG$. However, in our general case the vector Y is deformed by ω, due to the fact that the action Φ has no "constant" Lie group factor anymore. This will be important later when we are going to derive the gauge transformations. Furthermore, already the first summand is different than the classical formula, because we need to use LGB sections in order to define the push-forward of tangent vectors which are not vertical, that is, X may not be a tangent vector of $f^{-1}(\{x\})$ (which is in the general case not even an embedded submanifold) such that $\mathrm{D}_p r_g(X)$ is in general not well-defined anymore.

The other two equations in the case of f being a surjective submersion are mainly for reference; the last equation, Eq. (30), emphasises the contribution of non-vertical vectors measured by ω. While the first two summands are the classical product rule on the vertical parts, the third summand shows the deformation of the product rule because of the new structure of an action without a "constant" Lie group factor.

Eq. (29) may be the most elegant formulation, making use of local sections σ and τ and their advantage that these can act on all tangent vectors, not just the vertical ones; the reader who knows Lie groupoids may recognize this equation's structure with the one as given in [2, §1.4, Thm. 1.4.14, page 28], where it is about the induced multiplication structure on the tangent bundle of a Lie groupoid making use of bisections playing a similar role like σ and τ.

Those equations additionally show that we have a more general Leibniz rule with a third summand. However, by Ex. 3.15 we expect still a typical Leibniz rule once a trivialization of $\mathscr{G}$ around g is fixed. Indeed, as we will understand and see also later, this is the case; fix such a trivialization, equip it with a canonical flat connection, and take σ to be a parallel section, that is, g as a constant section. Then one can calculate that the typical Leibniz rule is recovered. Hence, one can view this Leibniz rule as the "covariantized" version of the Leibniz rule, independent of a choice of "coordinate"/section on $\mathscr{G}$.

6. Connections and curvature on LGB principal bundles

6.1. Principal bundles with structural LGB

6.1.1. Definition

The principal bundle $\mathscr{P}$ we are interested into is still a fibre bundle related to the same Lie group as the one behind the LGB $\mathscr{G}$, especially also $\dim(\mathscr{P}) = \dim(\mathscr{G})$, but it is equipped with an LGB action.

Definition 6.1: Principal bundles with structural LGB, [6, simplification of the beginning of §5.7, page 144f.]

Let G be a Lie group, M a smooth manifold, and an LGB $\mathscr{G} \to M$ which acts on the right on another G-fibre bundle $\mathscr{P} \to M$

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] & G \arrow[l] \\ & M & M & \end{tikzcd}$$

where the right-action is defined on $\mathscr{P} * \mathscr{G}$ given by $\pi^*_{\mathscr{P}}\mathscr{G}$. Then we call $\mathscr{P}$ a **principal $\mathscr{G}$-bundle** if

1. The right $\mathscr{G}$-action on $\mathscr{P}$ **is simply transitive on the fibres**, that is, the restriction of $\mathscr{P} * \mathscr{G} \to \mathscr{P}$ on $x \in M$

   $$\mathscr{P}_x \times \mathscr{G}_x \to \mathscr{P}_x$$

   induces bijective orbit maps

   $$\mathscr{G}_x \to \mathscr{P}_x,$$

   $$g \mapsto p \cdot g$$

   for $p \in \mathscr{P}_x$.

2. There exist base-preserving **$\mathscr{G}$-equivariant diffeomorphisms** $\varphi_i : \mathscr{P}|_{U_i} \to \mathscr{G}|_{U_i}$ subordinate to an open covering $(U_i)_i$ of M, that is,

   $$\pi_{\mathscr{G}} \circ \varphi_i = \pi_{\mathscr{P}},$$

   $$\varphi_i(p \cdot g) = \varphi_i(p) \cdot g,$$

   where the multiplication on the right hand side is inherited by $\mathscr{G} * \mathscr{G} \to \mathscr{G}$, the multiplication on $\mathscr{G}$.

The LGB $\mathscr{G}$ is the **structural LGB** of the principal bundle $\mathscr{P}$.

Remark 6.2: Discussion about the definition of $\mathscr{G}$-principal bundles

There are several things in need to be discussed:

1. The mentioned reference, [6, simplification of the beginning of §5.7, page 144f.], introduces these principal bundles in a different manner; we will come back later to this in Remark 6.13.

2. By Remark 3.5 the $\mathscr{G}$-action on $\mathscr{P}$ is fibre-preserving, thus, the restriction of $\mathscr{P} * \mathscr{G} \to \mathscr{P}$ is indeed a map $\mathscr{P}_x \times \mathscr{G}_x \to \mathscr{P}_x$.

3. The $\mathscr{G}$-equivariant diffeomorphisms φ_i give rise to a **bundle atlas of $\mathscr{G}$-equivariant bundle charts** $\xi_i : \mathscr{P}|_{U_i} \to U_i \times G$. Assume w.l.o.g. that U_i is small enough so that there is an LGB chart $\phi_i : \mathscr{G}|_{U_i} \to U_i \times G$ of $\mathscr{G}$; recall Def. 2.1. Then $\xi_i \coloneqq \phi_i \circ \varphi_i$, which clearly satisfies

$$\xi_i(p \cdot g) = \xi_i(p) \cdot \phi_i(g),$$

where the multiplication on the right hand side is the canonical one for $U_i \times G$ as a trivial LGB (recall Ex. 2.3), and this can be viewed as $\mathscr{G}$-equivariance (under the trivialization induced by ϕ_i). We usually then speak of **$\mathscr{G}$-equivariance w.r.t. the LGB morphism** ϕ_i.

We will call such an atlas and its charts a **principal bundle atlas** and **principal bundle charts** for $\mathscr{P}$, respectively. For simplicity we may also refer to φ_i as principal bundle chart giving rise to the principal bundle atlas.

4. Due to the existence of φ_i one does not need to claim upfront that $\mathscr{P}$ is a G-fibre bundle. However, for readability, we decided to structure the definition like this. We kept a similar style as for "typical" principal bundles as provided in [1, §4.2, Def. 4.2.1, page 207f.].

5. Since $\pi_{\mathscr{P}}$ is a surjective submersion we know by Remark 3.11 that right-translations r_g ($g \in \mathscr{G}_x$) are diffeomorphisms on $\mathscr{P}_x$. Furthermore, following [1, §4.2, discussion after Def. 4.2.1, page 208f.], by definition we have a simply transitive $\mathscr{G}_x$-action (as a Lie group) on $\mathscr{P}_x$, and the isotropy group for each $p \in \mathscr{P}_x$ is trivial; the isotropy group consists in general of $g \in \mathscr{G}_x$ with $p \cdot g = p$, see *e.g.* [1, §3.2, third part of Def. 3.2.4, page 132]. Therefore, and by [1, §3.8, Thm. 3.8.8, page 165], the orbit map Φ_p gives rise to a $\mathscr{G}_x$-equivariant diffeomorphism

$$\mathscr{G}_x \to \mathscr{P}_x,$$

where the action on $\mathscr{G}_x$ is the right-action on itself regarding the $\mathscr{G}_x$-equivariance.

6. Similarly by definition, for each $x \in U_i$ we know that $(\varphi_i)|_x : \mathscr{P}_x \to \mathscr{G}_x$ is a $\mathscr{G}_x$-equivariant diffeomorphism. In fact, together with $\pi_{\mathscr{G}} \circ \varphi_i = \pi_{\mathscr{P}}$ this clearly gives an equivalent definition of φ_i which we may make use in the following without further mention.

6.1.2. Examples

Let us provide examples of such principal bundles.

Example 6.3: The "classical" principal bundle

G-principal bundles themselves are $\mathscr{G} := M \times G$-principal bundles. Recall Ex. 3.15, the LGB action of a trivial LGB $\mathscr{G} \cong M \times G$ is equivalent to a G-action, the right-translation with an element $g \in G$ is then either the right-translation of the corresponding constant section in $\mathscr{G}$ or the right-translation with (x, g) pointwise; this action is clearly simply transitive. The $\mathscr{G}$-principal bundle atlas is then inherited by the existing G-principal bundle atlas.
We often refer to principal bundles related to trivial LGBs as **typical** or **classical principal bundles**, or, as usual, principal G-bundle.

Example 6.4: The "trivial" principal $\mathscr{G}$-bundle

$\mathscr{G}$ itself is a $\mathscr{G}$-principal bundle, equipped with its canonical right-action inherited by its multiplication $\mathscr{G} * \mathscr{G} \to \mathscr{G}$. The principal bundle atlas then just consists of the identity map $\mathbb{1}_{\mathscr{G}}$.
By Ex. 6.3, it may be natural to call this the **trivial $\mathscr{G}$-principal bundle** even if $\mathscr{G}$ itself might not be trivial. It will be clearer later why one may choose to do so.
The remarkable property is that this example shows that we are going to define a gauge theory for which LGBs themselves are allowed as principal bundles. One might have wondered why classical gauge theory uses classical principal bundles instead of LGBs, especially including non-trivial LGBs, because LGBs could be viewed as a more natural choice due to the fact that they are an analogue to how vector bundles are the "bundle-construction" of vector spaces. The problems described in Subsection 6.2.1 show why it was easier to choose classical principal bundles, since these avoid the difficulties regarding the definition of a connection; however, we are going to solve these problems in such a way that one can either use LGBs or classical bundles or even something more general.

Our main example will be the inner bundle of a classical principal bundle, recall Ex. 2.12. To explain the "triviality" of Ex. 6.4 we need to introduce morphisms of principal bundles.

6.1.3. Morphism of principal LGB-bundles

Let us define morphisms of LGB-principal bundles

Definition 6.5: Morphism of principal bundles with structural LGB

Let M and N be smooth manifolds, $\mathscr{H} \stackrel{\pi_{\mathscr{H}}}{\to} N$ and $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ LGBs, and $\mathscr{H} \to \mathscr{P}' \stackrel{\pi'}{\to} N$ and $\mathscr{G} \to \mathscr{P} \stackrel{\pi}{\to} M$ a principal $\mathscr{H}$- and $\mathscr{G}$-bundle, respectively. A **principal bundle morphism** between $\mathscr{P}'$ and $\mathscr{P}$ is a triple of smooth maps $F : \mathscr{H} \to \mathscr{G}$, $f : N \to M$ and

$H : \mathscr{P}' \to \mathscr{P}$ such that the pair (F, f) is an LGB morphism as in Def. 2.4 and

$$\pi \circ H = f \circ \pi', \tag{33}$$

$$H(p \cdot h) = H(p) \cdot F(h) \tag{34}$$

for all $(p, h) \in \mathscr{P}' * \mathscr{H} = (\pi')^* \mathscr{H}$. We also speak of a **principal bundle morphism over f w.r.t. the LGB morphism F.**

We speak of a **principal bundle isomorphism (over f, w.r.t. F)** if H is a diffeomorphism.

If H is a base-preserving isomorphism $\mathscr{P} \to \mathscr{P}$ w.r.t. $F = \mathbb{1}_{\mathscr{G}}$, then we say that H is a **(global) gauge transformation** or **principal bundle automorphism**, and the set of all such automorphisms is denoted by $\mathscr{Aut}(\mathscr{P})$. If H is defined on an open subset of M, then we may also speak of a **local gauge transformation**.

Remarks 6.6.

• Observe that the right hand side of Eq. (34) is well-defined because of Eq. (33), $(p, h) \in \mathscr{P}' * \mathscr{H}$ and Def. 2.4, that is,

$$(\pi_{\mathscr{G}} \circ F)(h) \overset{2.4}{=} (f \circ \pi_{\mathscr{H}})(h) \overset{\pi'(p)=\pi_{\mathscr{H}}(h)}{=} \big(f \circ \pi'\big)(p) \overset{(33)}{=} (\pi \circ H)(p),$$

thus,

$$\big(H(p), F(h)\big) \in \mathscr{P} * \mathscr{G} = \pi^* \mathscr{G}.$$

Furthermore, by additionally using Remark 6.2 we know that LGB actions preserve the fibres of the principal bundles, therefore both, $H(p \cdot h)$ and $H(p) \cdot F(h)$, are over the same base point $(f \circ \pi')(p)$, so that Eq. (34) as a whole is well-defined.

• Also observe that one can conclude that (F, f) has to be an LGB morphism in order to have a satisfied and well-defined Eq. (34), assuming that H is a map over f. To well-define the right hand side, F is a morphism over f, since H is defined over f. Furthermore, for $h' \in \mathscr{H}_{\pi'(p)}$ we have

$$H\big(p \cdot h'h\big) = H(p) \cdot F\big(h'h\big),$$

but we also get by associativity

$$H\big(p \cdot h'h\big) = H(p) \cdot F\big(h'\big)\ F(h).$$

Using that $\mathscr{G}$ acts simply transitive on $\mathscr{P}$, we derive

$$F\big(h'h\big) = F\big(h'\big)\ F(h).$$

• If H is a diffeomorphism, then F is an LGB isomorphism. For this we only have to show that F is a diffeomorphism by Def. 2.4; by Remark 2.5, also f is then a diffeomorphism. Thence, let us show that F is a diffeomorphism. This follows by the fact that the LGB actions are simply transitive on the fibres of $\mathscr{P}'$ and $\mathscr{P}$, *i.e.* orbit maps are diffeomorphisms, also recall Remark 6.2. Denoting the orbit maps inherited by the action on $\mathscr{P}'$ and $\mathscr{P}$ by $\Phi^{\mathscr{P}'}$ and $\Phi^{\mathscr{P}}$, respectively, we can write

$$F(h) = \left(\left(\Phi^{\mathscr{P}}_{H(p)} \right)^{-1} \circ H \circ \Phi^{\mathscr{P}'}_p \right)(h)$$

for arbitrary p by rewriting Eq. (34). Hence, F is a diffeomorphism as the composition of diffeomorphisms.

Last but not least, it follows that H^{-1} is then $\mathscr{G}$-equivariant in the sense of

$$H^{-1}(q \cdot g) = H^{-1}(q) \cdot F^{-1}(g)$$

for all $(q, g) \in \mathscr{P} * \mathscr{G}$, which is well-defined by similar arguments as before, especially because the inverses are maps over f^{-1}. To prove this, observe that there is a unique $(p, h) \in \mathscr{P}' * \mathscr{H}$ such that $H(p) = q$ and $F(h) = g$ due to the fact that both are bijective maps over f.[5] Then

$$H^{-1}(q \cdot g) = H^{-1}\big(\underbrace{H(p) \cdot F(h)}_{=H(p \cdot h)}\big) = p \cdot h = H^{-1}(q) \cdot F^{-1}(g).$$

Thence, Def. 6.5 is a valid definition for a principal bundle morphism, because it is easy to check that the principal bundle atlas on $\mathscr{P}$ (consisting of φ_i related to an open covering $(U_i)_i$ of M) is related to the principal bundle atlas on $\mathscr{P}'$ by

$$\left(F^{-1} \circ \varphi_i \circ H\right)|_{f^{-1}(U_i)}.$$

• Finally, observe that we have locally an isomorphism of every principal bundle $\mathscr{P}$ to $\mathscr{G}$: Fix a principal bundle chart $\varphi : \mathscr{P}|_U \to \mathscr{G}|_U$ (U some open subset of M). Then take $H := \varphi$, $F := \mathbb{1}_{\mathscr{G}|_U}$ and $f := \mathbb{1}_U$; by the definition of a principal bundle chart this gives a principal bundle isomorphism. Therefore one could say that every principal bundle is locally "trivial" in the sense of being an LGB; recall Ex. 6.4.

Additionally using Remark 6.2 we have locally an isomorphism of $\mathscr{P}$ to a trivial LGB; keeping the same notation as in Remark 6.2, choose $H := \xi_i$, $F := \phi_i$ and $f := \mathbb{1}_{U_i}$. Hence, locally every principal bundle is also classical in the sense of Ex. 6.3.

As expected, there is a natural isomorphism induced by local sections of $\mathscr{P}$, which are, however, no trivializations in general. In other words, by Remark 6.2 we know that orbit maps through the fibres of $\mathscr{P}$ are equivariant diffeomorphisms, we want to show the same for the orbit map through a section of $\mathscr{P}$.

[5] In fact, it is easy to prove that $(H, F) : \mathscr{P}' * \mathscr{H} \to \mathscr{P} * \mathscr{G}$ is an LGB isomorphism (over H); also recall Cor. 3.1.

Lemma 6.7: Local sections of principal bundles induce isomorphisms to the structural LGB

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M, and $\mathscr{P} \to M$ a principal $\mathscr{G}$-bundle. Let $s : U \to \mathscr{P}$ be a smooth local section of $\mathscr{P}$ over an open subset U of M. Then the orbit map Φ_s through s, given as in Def. 3.8 by

$$\mathscr{G}|_U \to \mathscr{P}|_U,$$

$$g \mapsto s_{\pi_{\mathscr{G}}(g)} \cdot g,$$

is a base-preserving principal bundle isomorphism w.r.t. $\mathbb{1}_{\mathscr{G}|_U}$.

Remarks 6.8.
As in the typical formulation of gauge theory, we have an isomorphism induced by sections, but it is not necessarily a trivialization as fibre bundle, so that we do not necessarily also have a "classical" bundle; it is a trivialization in the sense of Ex. 6.4. Due to the similarity with the "classical" statement, we therefore were speaking of the "trivial" principal bundle in Ex. 6.4. Of course, since every LGB is locally trivial, we can find a typical trivialization by taking a "local-enough" section.

The latter about the typical trivialization is also stated in [6, §5.7, fourth part of Remark 5.34, page 145] for groupoid-based principal bundles.

Proof of Lemma 6.7.
The proof is similar to the "classical" statement as *e.g.* given in [1, §4.2, proof of Lemma 4.2.7, page 210ff.], but generalized since we have to treat a possible non-triviality of $\mathscr{G}|_U$. As already discussed, the orbit map Φ_s is well-defined because of $\pi_{\mathscr{P}} \circ s = \mathbb{1}_U$ such that $\left(s_{\pi_{\mathscr{G}}(g)}, g\right) \in \mathscr{P} * \mathscr{G}$. Let us denote with $\Phi : \mathscr{P} * \mathscr{G} \to \mathscr{G}$ the right $\mathscr{G}$-action $(p, g) \mapsto p \cdot g$ on $\mathscr{P}$, then

$$\Phi_s(g) = \Phi\left(s_{\pi_{\mathscr{G}}(g)}, g\right) = \left(\Phi \circ (\pi_{\mathscr{G}}^* s, \mathbb{1}_{\mathscr{G}})\right)(g, g). \tag{35}$$

Thence, Φ_s is clearly a smooth map as the composition of smooth maps; the second map $\left(\pi_{\mathscr{G}}^* s, \mathbb{1}_{\mathscr{G}}\right) : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ is given by $(g, q) \mapsto \left(s_{\pi_{\mathscr{G}}(g)}, q\right)$ and is clearly smooth if restricted to the diagonal as embedded submanifold. By Def. 3.4 we have

$$(\pi_{\mathscr{P}} \circ \Phi_s)(g) = \pi_{\mathscr{P}}\left(s_{\pi_{\mathscr{G}}(g)} \cdot g\right) = \pi_{\mathscr{G}}(g),$$

thus, Φ_s is base-preserving, similarly it follows that Φ_s is $\mathscr{G}$-equivariant w.r.t. $\mathbb{1}_{\mathscr{G}}$ by making use of $\pi_{\mathscr{G}}(gq) = \pi_{\mathscr{G}}(g)$ for all $g, q \in \mathscr{G} * \mathscr{G}$. Furthermore, Φ_s is bijective since the right $\mathscr{G}$-action on $\mathscr{P}$ is simply transitive, that is, when we restrict Φ_s to an $x \in U$, then it is a map

$$\mathscr{G}_x \to \mathscr{P}_x,$$

$$g \mapsto s_x \cdot g,$$

thence, as expected is is the orbit map Φ_{s_x} through s_x which is a $\mathscr{G}_x$-equivariant diffeomorphism by Remark 6.2. So, Φ_s is fibre-wise bijective and therefore bijective as a whole since it is base-preserving.

Now we want to use the inverse function theorem to show that its inverse is also smooth. Once we know that the tangent map/total derivative $\mathrm{D}_g\Phi_s : \mathrm{T}_g\mathscr{G} \to \mathrm{T}_{s_x \cdot g}\mathscr{P}$ is an isomorphism of vector spaces for all $g \in \mathscr{G}|_U$, then we know by the inverse function theorem that Φ_s^{-1} is smooth. Hence, we will now show that $\mathrm{D}_g\Phi_s$ is injective, then it has to bijective by dimensional reasons $(\dim(\mathscr{G}) = \dim(\mathscr{P}))$ so that we are done. This differential is given by Thm. 5.21, making use of Eq. (35),

$$\begin{aligned}\mathrm{D}_g\Phi_s(Y) &= \mathrm{D}_{s_x} r_\sigma\big(\mathrm{D}_x s(\omega)\big) + \left.\overline{(\mu_{\mathscr{G}})_g\big(Y - \mathrm{D}_x\sigma(\omega)\big)}\right|_{s_x \cdot g} \\ &= \mathrm{D}_x(r_\sigma \circ s)(\omega) + \left.\overline{(\mu_{\mathscr{G}})_g\big(Y - \mathrm{D}_x\sigma(\omega)\big)}\right|_{s_x \cdot g}\end{aligned}$$

for all $Y \in \mathrm{T}_g\mathscr{G}$, where $x := \pi_{\mathscr{G}}(g)$, $\sigma \in \Gamma(\mathscr{G}|_U)$[6] with $\sigma_x = g$ and $\omega := \mathrm{D}_g\pi_{\mathscr{G}}(Y) \in \mathrm{T}_xM$. We want to decompose Φ_s now. Observe that $Y - \mathrm{D}_x\sigma(\omega) \in \mathrm{V}_g\mathscr{G}$, *i.e.* it is vertical in $\mathscr{G}$ due to the fact that

$$\mathrm{D}_g\pi_{\mathscr{G}}\big(Y - \mathrm{D}_x\sigma(\omega)\big) = \underbrace{\mathrm{D}_g\pi_{\mathscr{G}}(Y)}_{=\omega} - \underbrace{\mathrm{D}_g\pi_{\mathscr{G}}\big(\mathrm{D}_x\sigma(\omega)\big)}_{=\omega} = 0 \tag{36}$$

because of

$$\pi_{\mathscr{G}} \circ \sigma = \mathbb{1}_U$$

such that

$$\mathrm{D}\pi_{\mathscr{G}} \circ \mathrm{D}\sigma = \mathbb{1}_{\mathrm{T}M|_U}. \tag{37}$$

Usually, we just make use of that without further mention; we repeat this trivial fact in order to emphasize that $\mathrm{D}\sigma$ is injective because $\mathbb{1}_{\mathrm{T}M|_U}$ bijective and that we have

$$\mathrm{Im}(\mathrm{D}_x\sigma) \cap \mathrm{Ker}(\mathrm{D}_g\pi_{\mathscr{G}}) = \{0\}$$

(the image of $\mathrm{D}_x\sigma$ intersects trivially with the kernel of $\mathrm{D}_g\pi_{\mathscr{G}}$). The injectivity of $\mathrm{D}\sigma$ implies that the dimension of its image satisfies

$$\dim\big(\mathrm{Im}(\mathrm{D}_x\sigma)\big) = \dim(M),$$

so that, in total, we know by dimensional reasons

$$\mathrm{T}_g\mathscr{G} \cong \mathrm{Im}(\mathrm{D}_x\sigma) \oplus \mathrm{Ker}(\mathrm{D}_g\pi_{\mathscr{G}}) = \mathrm{Im}(\mathrm{D}_x\sigma) \oplus \mathrm{V}_g\mathscr{G}.$$

[6] W.l.o.g. we assume that σ is defined on U, otherwise "make U smaller".

We can decompose Y accordingly by Eq. (36),

$$Y = \underbrace{\mathrm{D}_x\sigma(\omega)}_{=:Y^h} + \underbrace{Y - \mathrm{D}_x\sigma(\omega)}_{=:Y^v} = Y^h + Y^v,$$

v stands for the vertical part and h for its complementary part ("horizontal"). In fact, as it is well-known, a section σ of a bundle induces a splitting of the short exact sequence of vector bundles

$$\sigma^*\mathrm{V}\mathscr{G}|_U \hookrightarrow \sigma^*\mathrm{T}\mathscr{G}|_U \xrightarrow{\mathrm{D}\pi_{\mathscr{G}}} \mathrm{T}M|_U,$$

and $\mathrm{D}\sigma$ is a splitting/section of this sequence so that we have $\sigma^*\mathrm{T}\mathscr{G} \cong \mathrm{Im}(\mathrm{D}\sigma) \oplus \sigma^*\mathrm{V}\mathscr{G}$.[7] Furthermore, again due to Eq. (37), we have $\mathrm{Im}(\mathrm{D}_x\sigma) \cong \mathrm{T}_xM$ as vector spaces by $Y^h \mapsto \mathrm{D}_g\pi_{\mathscr{G}}\left(Y^h\right) = \mathrm{D}_g\pi_{\mathscr{G}}(Y) = \omega$ since Y^v is vertical. Hence, we can view $\mathrm{D}_g\Phi_s$ as a map

$$\mathrm{T}_xM \oplus \mathrm{V}_g\mathscr{G} \to \mathrm{T}_{s_x\cdot g}\mathscr{P},$$

$$(\omega,\ Y^v) \mapsto \mathrm{D}_x(r_\sigma \circ s)(\omega) + \left.\overline{(\mu_{\mathscr{G}})_g\left(Y^v\right)}\right|_{s_x\cdot g}.$$

These arguments apply to any section of a bundle, hence, we repeat this now. Observe that $r_\sigma \circ s$ is also a section of $\mathscr{P}|_U$ due to Def. 3.4, that is,

$$\pi_{\mathscr{P}}(s_y \cdot \sigma_y) = \pi_{\mathscr{P}}(s_y) = y$$

for all $y \in M$. As before, we split

$$\mathrm{T}_{s_x\cdot g}\mathscr{P}|_U \cong \mathrm{Im}\big(\mathrm{D}_x(r_\sigma \circ s)\big) \oplus \mathrm{V}_{s_x\cdot g}\mathscr{P},$$

and we identify $\mathrm{Im}\big(\mathrm{D}_x(r_\sigma \circ s)\big) \cong \mathrm{T}_xM$ as vector spaces, now via $\omega \mapsto \mathrm{D}_x(r_\sigma \circ s)(\omega)$ due to Eq. (37) but with $r_\sigma \circ s$ playing the role as section. Additionally by Def. 5.17 and Remark 5.18 we know that fundamental vector fields are vertical, and so we can view $\mathrm{D}_g\Phi_s$ as a map

$$\mathrm{T}_xM \oplus \mathrm{V}_g\mathscr{G} \to \mathrm{T}_xM \oplus \mathrm{V}_{s_x\cdot g}\mathscr{P},$$

$$(\omega,\ Y^v) \mapsto \left(\omega,\ \left.\overline{(\mu_{\mathscr{G}})_g\left(Y^v\right)}\right|_{s_x\cdot g}\right).$$

Thus, $\mathrm{D}_g\Phi_s$ is the pair of two linear independent maps $\mathrm{T}_xM \to \mathrm{T}_xM$ and $\mathrm{V}_g\mathscr{G} \to \mathrm{V}_{s_x\cdot g}\mathscr{P}$. The former is clearly an isomorphism, and the latter is by definition of fundamental vector fields and the Maurer-Cartan form of the shape as in Eq. (32), *i.e.*

$$Y^v \mapsto \mathrm{D}_g\Phi_{s_x}(Y^v).$$

By Remark 6.2, Φ_{s_x} is a $\mathscr{G}_x$-equivariant diffeomorphism $\mathscr{G}_x \to \mathscr{P}_x$, thence, $\mathrm{D}_g\Phi_{s_x} : \mathrm{T}_g\mathscr{G}_x = \mathrm{V}_g\mathscr{G} \to \mathrm{T}_{s_x\cdot g}\mathscr{P}_x = \mathrm{V}_{s_x\cdot g}\mathscr{P}_x$ is an isomorphism of vector spaces. Finally we can conclude that $\mathrm{D}_g\Phi_s$ is a linear isomorphism as the sum of two linear independent isomorphisms. As argued earlier, the inverse function theorem finishes now the proof. ■

[7] σ is actually an embedding, and thus $\mathrm{Im}(\mathrm{D}\sigma)$ is a well-defined subbundle isomorphic to $\mathrm{T}M$.

Usually, one likes to think about the choice of sections as a choice of a coordinate transformation as in [1, §4.2, Remark 4.2.21, page 220]. This is due to that the gauge theory usually corresponds to a formulation via a trivial LGB, which we will understand later, but already have an idea of by *e.g.* Ex. 3.15 and 6.3. Then we have a local trivialization of $\mathscr{P}$ such that one usually thinks of the choice of a section as a choice of coordinate system.

However, we now learned that on a more general scale this is not completely what is happening. The idea of LGBs and principal bundles are very similar; both are fibre bundles related to a Lie group and they carry an action which also restricts on each fibre. But the fibres of an LGB are Lie groups themselves, while the fibres of a principal bundle are "just" diffeomorphic to a Lie group in an equivariant way as outlined in Remark 6.2 and as given in their definition Def. 6.1. One could view the fibres of $\mathscr{P}$ as having an "almost" Lie group structure, a Lie group structure without a designated neutral element.

Lemma 6.7 shows that the choice of a section of $\mathscr{P}$ is actually the choice of a designated neutral element, naturally inducing a Lie group structure in each fibre and thus an LGB structure, which may not be trivial. Aligning the more general definition of principal bundles with the definition of LGBs. For example set $\mathscr{P} = \mathscr{G} = c_G(P)$ as in Ex. 2.12, where P is a non-trivial classical principal bundle and the underlying Lie group G is non-abelian; such an LGB is likely non-trivial but always carries a global section. We will see such an example later.

We want to use this now in order to study a certain pullback of principal bundles. For this we need to introduce general pullbacks of principal bundles; also recall again Cor. 3.1.

Corollary 6.9: Pullbacks of principal LGB-bundles are principal LGB-bundles

Let M, N be smooth manifolds, $f : N \to M$ a smooth map, $\mathscr{G} \to M$ an LGB, and $\mathscr{P} \to M$ a principal $\mathscr{G}$-bundle. Then there is a unique (up to isomorphisms) principal $f^*\mathscr{G}$-bundle structure on $f^*\mathscr{P}$ such that the projection $\pi_2 : f^*\mathscr{P} \to \mathscr{P}$ onto the second factor is a principal bundle morphism (over f) w.r.t. the projection $\pi_2^{\mathscr{G}} : f^*\mathscr{G} \to \mathscr{G}$ onto the second factor as LGB morphism and such that $\pi_2|_x : (f^*\mathscr{P})_x \to \mathscr{P}_{f(x)}$ is a $\mathscr{G}_x$-equivariant diffeomorphism w.r.t. the LGB isomorphism $\pi_2^{\mathscr{G}}\big|_x$.

Remarks 6.10.
This was also stated in [6, §5.7, second argument in Remark 5.34, page 145] for an even more general type of principal bundle, but without proof.

Proof.
As in the previous statements about pullback structures, this is a rather trivial and canonical construction. We have a right $f^*\mathscr{G}$-action on $f^*\mathscr{P}$ defined by

$$(x, p) \cdot (x, g) \coloneqq (x, p \cdot g)$$

for all $(x, p) \in f^*\mathscr{P}$ and $(x, g) \in f^*\mathscr{G}$, that is, $x \in N$, $p \in \mathscr{P}_{f(x)}$ and $g \in \mathscr{G}_{f(x)}$. This is clearly an action $f^*\mathscr{P} * f^*\mathscr{G} \coloneqq \pi_1^* f^*\mathscr{G} \to f^*\mathscr{P}$, where π_1 is the canonical projection of $f^*\mathscr{P} \to N$ as fibre bundle. This action's restriction onto the fibres is simply transitive: A fibre of $f^*\mathscr{P}$ at x is $\{x\} \times \mathscr{P}_{f(x)} \cong \mathscr{P}_{f(x)}$, similarly $(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)}$. Hence, the restriction of that action at $x \in N$ is the right $\mathscr{G}_{f(x)}$-action on $\mathscr{P}_{f(x)}$ such that the action is simply transitive.

A principal bundle atlas can be constructed by a pullback of principal bundle charts of $\mathscr{P}$, that is, let $(U_i)_i$ be an open covering of M over which we have $\mathscr{G}$-equivariant diffeomorphisms $\varphi_i : \mathscr{P}|_{U_i} \to \mathscr{G}|_{U_i}$. Then define

$$f^*\mathscr{P}|_{f^{-1}(U_i)} \to f^*\mathscr{G}|_{f^{-1}(U_i)},$$
$$(x, p) \mapsto (f^*\varphi_i)(x, p) := \Big(x, (\varphi_i)|_{f(x)}(p)\Big),$$

which is well-defined and by construction a base-preserving $f^*\mathscr{G}$-equivariant smooth map, and this map is equivalent to $\left(\mathbb{1}_{f^{-1}(U_i)}, \varphi_i\right) : N \times \mathscr{P} \to N \times \mathscr{G}$ restricted onto $f^*\mathscr{P}$ as an embedded submanifold of $N \times \mathscr{P}$. Therefore it is clearly a diffeomorphism, so that we can conclude that $f^*\mathscr{P}$ admits the structure as a principal $f^*\mathscr{G}$-bundle.

The last part of the proof about the uniqueness of the structure is precisely as in the proof of Cor. 3.1, just replace the property of being an LGB morphism with being a principal bundle morphism w.r.t. the LGB morphism $\pi_2^{\mathscr{G}}$ (essentially, replace homomorphism with equivariance). Keeping the same notation as in the proof of Cor. 3.1, we get analogously

$$\varphi_i \circ \pi_2 \circ \psi_i^{-1} = \pi_2^{\mathscr{G}} \circ f^*\varphi_i \circ \psi_i^{-1}.$$

Then start by making use of the point-wise behaviour of π_2 and $\pi_2^{\mathscr{G}}$ and proceed similarly as in the proof of Cor. 3.1 to conclude the proof. ∎

Definition 6.11: Pullback principal bundle

Let M, N be smooth manifolds, $f : N \to M$ a smooth map, $\mathscr{G} \to M$ an LGB, and $\mathscr{P} \to M$ a principal $\mathscr{G}$-bundle. Then we call the principal $f^*\mathscr{G}$-bundle structure on $f^*\mathscr{P}$ as given in Cor. 6.9 the **pullback principal bundle of $\mathscr{P}$ (under f)**.
We will refer to this structure often without further mention.

We can actually show that $\mathscr{P} * \mathscr{G}$ is not only $\pi^*\mathscr{G}$ by definition but it is also isomorphic to $\pi^*\mathscr{P}$. For this recall Ex. 6.4, *i.e.* LGBs are "trivially" also principal bundles, hence we know that $\mathscr{P} * \mathscr{G}$ is a principal $\pi^*\mathscr{G}$-bundle.

Corollary 6.12: $\mathscr{P} * \mathscr{G}$ is the pullback of $\mathscr{P}$ along its projection

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle. Then we have a base-preserving principal bundle isomorphism

$$\mathscr{P} * \mathscr{G} \cong \pi^*\mathscr{P}$$

w.r.t. the LGB isomorphism given as the identity map on $\mathscr{P} * \mathscr{G}$.

Proof.
We have a global section of the pullback principal bundle $\pi^*\mathscr{P} \to \mathscr{P}$ given by

$$\mathscr{P} \to \pi^*\mathscr{P},$$

$$p \mapsto (p, p).$$

This is clearly a well-defined global section of $\pi^*\mathscr{P}$, so that by Lemma 6.7 we achieve the desired isomorphism $\pi^*\mathscr{P} \cong \pi^*\mathscr{G} = \mathscr{P} * \mathscr{G}$. ■

Remarks 6.13.
By Lemma 6.7 this isomorphism is explicitly given by

$$\mathscr{P} * \mathscr{G} \to \pi^*\mathscr{P},$$

$$(p, g) \mapsto (p, p \cdot g).$$

In fact, the cited reference of Def. 6.1, [6, simplification of the beginning of §5.7, page 144f.], takes the existence of such a diffeomorphism as the essential sole part of defining principal bundles. Such an approach essentially avoids the second part of Def. 6.1.

Remark 6.14: Why "principal"?

The last result and remark also outline why we are speaking of a *principal* bundle; this is similar to [1, §4.2.2, page 212ff.]. As in [1, §3.7, Def. 3.7.24, page 159] one could say that a **principal** $\mathscr{G}$-action on a manifold N (recall Def. 3.4) is a **free** action, *i.e.* orbit maps are injective, such that

$$N * \mathscr{G} \to N \times N,$$

$$(p, g) \mapsto (p, p \cdot g)$$

is a closed map. In our case, $N = \mathscr{P}$, we clearly have a free action, and we just have shown that that map is closed, because we have the composition of maps

$$\mathscr{P} * \mathscr{G} \to \pi^*\mathscr{P} \to \mathscr{P} \times \mathscr{P},$$

$$(p, g) \mapsto (p, p \cdot g) \mapsto (p, p \cdot g).$$

By Remark 6.13 the first arrow is a diffeomorphism and thence a closed map. The second arrow is the inclusion, an embedding because $\pi^*\mathscr{P}$ is a closed embedded submanifold of $\mathscr{P} \times \mathscr{P}$ as the restriction of the fibre bundle $\mathscr{P} \times \mathscr{P} \overset{\mathbb{1}_{\mathscr{P}} \times \pi}{\to} \mathscr{P} \times M$ along the graph of π; see *e.g.* [1, §4.1, proof of Thm. 4.1.17, page 204ff.]. Additionally, by the continuity of π, the graph Γ of π is closed and thus $\pi^*\mathscr{P} = (\mathbb{1}_{\mathscr{P}} \times \pi)^{-1}(\Gamma)$, too. Using this, it is a quick exercise to show that a set closed in $\pi^*\mathscr{P}$ is also closed in $\mathscr{P} \times \mathscr{P}$, thus the second arrow as inclusion is also closed. Hence, the whole composition is closed.

This knowledge should allow us to carry over a lot of similar results related to principal actions, as in [1, §3.7.5, page 159ff.] and [1, §4.2.2, page 212ff.]. Essentially Remark 6.13 allows us to look at the quotient $\mathscr{P}\big/\mathscr{G}$ by using Godement's Theorem as given in [1, §3.7, Thm. 3.7.10, page 155], leading to a manifold structure on $\mathscr{P}\big/\mathscr{G}$ and maybe similarly leading to a principal bundle structure on $\mathscr{P} \to \mathscr{P}\big/\mathscr{G}$.

Let us conclude our discussion about principal bundles by introducing a typical label.

Definition 6.15: Gauges of a principal bundle

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathscr{P} \to M$ a principal $\mathscr{G}$-bundle. A **gauge** of $\mathscr{P}$ is a section of $\mathscr{P}$. If the section is globally defined on M, then we speak of a **global gauge**, otherwise we may just say **local gauge** or just gauge.

By Lemma 6.7, a gauge corresponds to a "$\mathscr{G}$-ization" of $\mathscr{P}$, not necessarily a trivialization.

6.2. Generalized distributions and connections

Remark 6.16: References for connections on principal LGB-bundles

It is important to mention beforehand that there is another great paper, [7], which also discusses the notion of connections on principal LGB-bundles. When I wrote this subsection about the connection, I was not aware of this preprint so that there will be some shared results which I found independently of the other authors. This will be obvious due to the fact that our approach is different. Both, this and the other paper, lead to the same sense of connection (based on the same idea of the behaviour of the parallel transport), but are explicitly not directly the same formulation. More like equivalent definitions of the same object, two sides of the same coin, so that it should be worth it to read both research works. [7] sole purpose was in defining connections, while we want to discuss this sense of connection in the context of Yang-Mills gauge theory, also pinpointing what happens for Yang-Mills-Higgs gauge theories. Hence, our approach will be different, we will use sections of the LGB to formulate connections, simplifying certain explicit formulas; while [7] avoids using sections and rather uses formulations just looking on the vertical parts of a vector or by rephrasing the notions via the total derivative of the LGB action. Another example of difference is that we are also introducing a generalisation of the Darboux derivative and phrasing important formulas with this derivative. Furthermore, we are not going to fix a specific connection on $\mathscr{G}$ for the principal bundle's connection (other than a horizontal distribution), while [7] did fix a more specific type of connection on $\mathscr{G}$, and our curvature will be different and more general as already pointed out in research like [8] and [5].

Finally, let us now define the gauge theory, starting with horizontal distributions. For readability, we start with a very easy toy model which should help in understanding the general definitions. We expect a basic understanding of horizontal distributions and their relationship to what we call connections (on principal and vector bundles). The bare-bones start with horizontal distributions.

Definition 6.17: Horizontal distribution, [1, §5.1.2, Def. 5.1.6, page 260; without the symmetry along right-translations here]

Let $F \to M$ be a fibre bundle over a smooth manifold M. Then a **horizontal distribution/bundle of** F is a smooth subbundle $\mathrm{H}F$ of $\mathrm{T}F$ with

$$\mathrm{T}F = \mathrm{H}F \oplus \mathrm{V}F.$$

For $p \in M$ the fibre is denoted by H_pF, which we may call a **horizontal tangent space**.

As usual for gauge theory we will understand connections as horizontal distributions with a certain symmetry along the fibres in order to assure a certain behaviour of the gauge transformation of what physicists call minimal coupling; in mathematical words, in order to assure to be able to define a connection on associated vector bundles. To do so this symmetry has to be similar to the symmetry carried in the vertical structure; recall Def. 5.17 and its remark 5.18. The following is a straightforward generalization of what one knows in the typical formulation of gauge theory, see *e.g.* [1, §5.1, part 2 to 4 of Prop. 5.1.3, page 258f].

Corollary 6.18: The natural invariance of the vertical bundle of $\mathscr{P}$

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, $\mathscr{g}$ its LAB, and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle. Then

$$\mathrm{D}_p r_g(\mathrm{V}_p\mathscr{P}) = \mathrm{V}_{p\cdot g}\mathscr{P} \tag{38}$$

for all $(p, g) \in \mathscr{P} * \mathscr{G}$, and we have an isomorphism of vector bundles

$$\mathrm{V}\mathscr{P} \cong \pi^*\mathscr{g} \tag{39}$$

given by

$$\pi^*\mathscr{g} \to \mathrm{V}\mathscr{P},$$

$$(p, \nu) \mapsto \widetilde{\nu}_p. \tag{40}$$

Remark 6.19: Extending the notation of fundamental vector fields

For $\mu := (p, \nu)$ we may also write

$$\widetilde{\mu}_p := \widetilde{\nu}_p,$$

which simplifies the notation by avoiding to write elements or the values of sections of $\pi^* \mathscr{g}$ as a pair of elements as in (p, ν).

Proof.
Recall Rem. 6.2 for this proof. By definition it is clear that each fibre $\mathscr{P}_x$ $(x \in M)$ is a principal $\mathscr{G}_x$-bundle over $\{x\}$ whose $\mathscr{G}_x$-action is the $\mathscr{G}$-action restricted to x, and thus we know

$$\mathrm{D}_p r_g (\mathrm{V}_p \mathscr{P}_x) = \mathrm{V}_{p \cdot g} \mathscr{P}_x$$

for all $p \in \mathscr{P}_x$ and $g \in \mathscr{G}_x$; see *e.g.* [1, §5.1, fourth part of Prop. 5.1.3, page 258f.] for such statements about principal Lie group bundles. Due to that $\mathscr{P}_x$ is a bundle over a point, we have $\mathrm{V}\mathscr{P}_x = \mathrm{T}\mathscr{P}_x = \mathrm{V}\mathscr{P}|_{\mathscr{P}_x}$. Thus,

$$\mathrm{D}_p r_g (\mathrm{V}_p \mathscr{P}) = \mathrm{V}_{p \cdot g} \mathscr{P}.$$

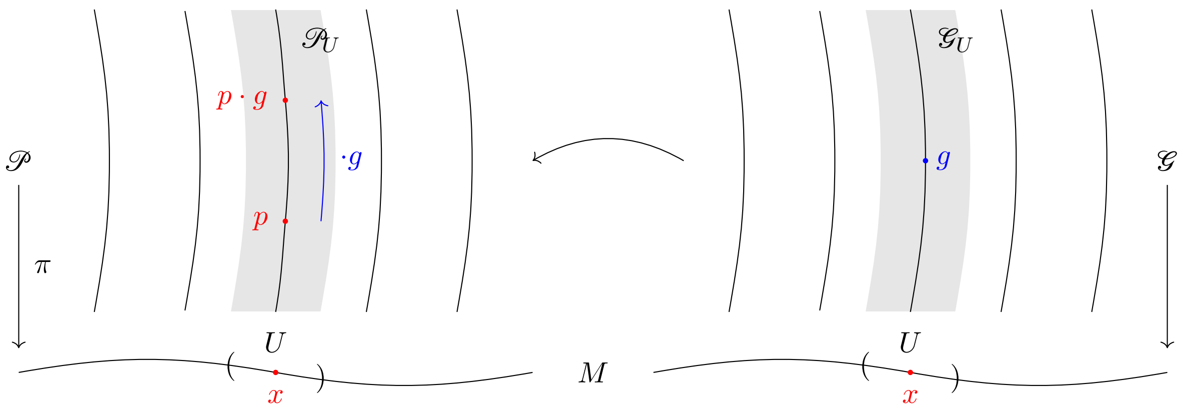
Due to the fact that the $\mathscr{G}$-action is simply transitive we can derive that its induced $\mathscr{g}$-action ρ is a vector bundle isomorphism: By Rem. 5.19 this LAB action ρ is precisely (40) and ρ has values in $\mathrm{V}\mathscr{P}$ by Eq. (24). We know that the orbit maps $\Phi_p : \mathscr{G}_x \to \mathscr{P}_x$ through $p \in \mathscr{P}_x$ are $\mathscr{G}_x$-equivariant diffeomorphisms, so that $\mathrm{D}_{e_x} \Phi_p : \mathscr{g}_x \to \mathrm{V}_p \mathscr{P}$ is a vector space isomorphism. But we also have

$$\rho(p, \nu) = \mathrm{D}_{e_x} \Phi_p(\nu)$$

for all $\nu \in \mathscr{g}_x$, and therefore ρ is fibre-wise an isomorphism of vector spaces such that it is an isomorphism of vector bundles. ■

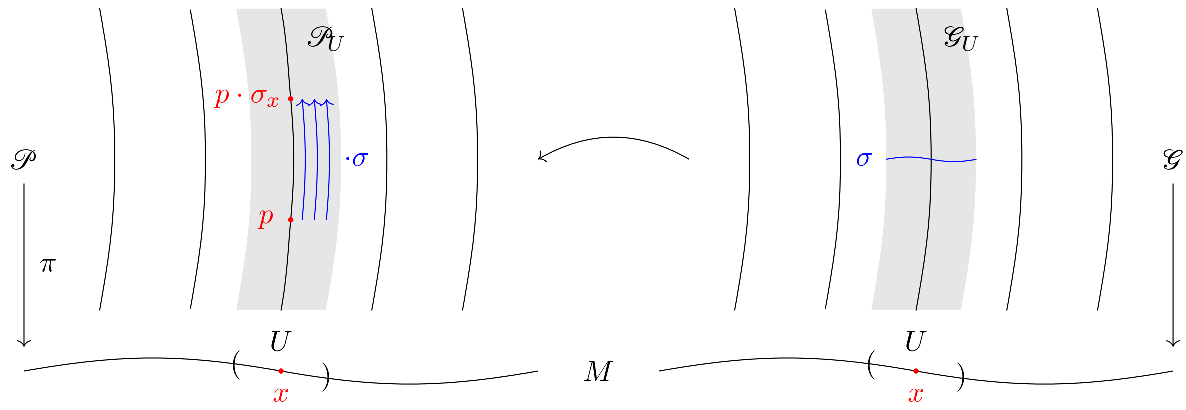
By this result, we would like to have Eq. (38) also for a chosen horizontal distribution. However, its formulation leads to certain problems which we now want to discuss.

6.2.1. Idea and motivation



For $\mathscr{G} \to M$ an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle we fix a point $p \in \mathscr{P}_x$ ($x \in M$) and can multiply that with an element $g \in \mathscr{G}_x$. Infinitesimally, we are interested into how this multiplication by g affects tangent vectors, especially non-vertical ones.

However, as we have seen in Def. 3.8, Rem. 4.16 and Thm. 5.21 (and its proof) the push-forward of horizontal vectors is not well-defined anymore on non-vertical vectors if one uses a fixed element of an LGB; r_g is just a map $\mathscr{P}_x \to \mathscr{P}_x$. In order to study push-forwards of non-vertical vectors, we need information of the $\mathscr{G}$-action in an open neighbourhood U around the fibres over x. Hence, we want to use a section $\sigma \in \Gamma(\mathscr{G}|_U)$ with $\sigma_x = g$ instead.



But there are in general a plethora of sections with $\sigma_x = g$, thenceforth one expects that a definition of connections based on that may depend on the choice of sections and thus leading to conflicts once one looks at push-forwards with all possible $g \in \mathscr{G}$. Especially the tangential behaviour of the section's image (as an embedding of the base) may contribute to the push-forward; given a horizontal distribution, a horizontal vector may be still horizontal after a push-forward with one LGB section but not with respect to another LGB section. A similar problem may arise if we would work with local trivializations instead in order to use Ex. 3.15. Thus, we need to adjust the typical definition of connections on principal bundles.

In order to understand what has to be changed, let us revisit the "typical" situation, that is, let $\mathscr{P}$ be a classical principal bundle as in Ex. 6.3, *i.e.* $\mathscr{G} = M \times G$ is trivial with Lie group G. That is, the $\mathscr{G}$-action is equivalent to a G-action on $\mathscr{P}$ by Ex. 3.15, a push-forward with $g \in G$ w.r.t. the latter action is equivalent to the push-forward with a constant section in $\mathscr{G}$ for which we may still simply write g.

Equip $\mathscr{P}$ with a "typical" (Ehresmann) connection as in [1, §5.1, Def. 5.1.6, page 260], that is, a horizontal distribution $\mathrm{H}\mathscr{P}$ of $\mathscr{P}$ with

$$\mathrm{D}_p r_g(\mathrm{H}\mathscr{P}_p) = \mathrm{H}\mathscr{P}_{p \cdot g} \tag{41}$$

for all $p \in \mathscr{P}$ and $g \in G$. Recall that a connection has a 1:1 correspondence to a parallel transport, as presented in [1, §5.8, page 286ff.]. This means corresponding to a piece-wise smooth base curve $\alpha : [0,t] \to M$ ($t > 0$) with $\alpha(0) = x$ we have a **parallel transport along**

α as a map $\mathrm{PT}^{\mathscr{P}}_{\alpha} : \mathscr{P}_x \to \Gamma(\alpha^*\mathscr{P})$ satisfying

$$\mathrm{PT}^{\mathscr{P}}_{\alpha*\alpha'}\Big|_t = \mathrm{PT}^{\mathscr{P}}_{\alpha'}\Big|_t \circ \mathrm{PT}^{\mathscr{P}}_{\alpha}\Big|_t, \tag{42}$$

$$\mathrm{PT}^{\mathscr{P}}_{\alpha^-}\Big|_t = \left(\mathrm{PT}^{\mathscr{P}}_{\alpha}\right)^{-1}\Big|_t, \tag{43}$$

especially parallel transport is a diffeomorphism between the fibres, where α' is just another similarly defined base curve with $\alpha'(0) = \alpha(t)$, $\alpha * \alpha'$ their concatenation (α coming first and with a suitable parametrization such that $\alpha * \alpha' : [0, t] \to M$), and α^- denotes α traversed backwards. Moreover, Eq. (41) integrates to

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot g)\Big|_t = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p)\Big|_t \cdot g. \tag{44}$$

Thinking of the associated $\mathscr{G}$-action, g is an element of $\mathscr{G}_x$ such that the right hand side is in general not well-defined[8] anymore due to the fact that $\mathrm{PT}^{\mathscr{P}}_{\alpha}(p)\big|_t \in \mathscr{P}_{\alpha(t)}$; it is well-defined if interpreting g as a constant section of $\mathscr{G} = M \times G$, denoted by $\widetilde{g} \in \Gamma(\mathscr{G})$ for bookkeeping reasons. The left hand side uses $\widetilde{g}_{\alpha(0)} = (x, g)$, the right hand side $\widetilde{g}_{\alpha(t)} = (\alpha(t), g)$, and by Ex. 3.15 we rewrite the G-action to a $\mathscr{G}$-action:

$$p \cdot g = p \cdot (x, g) = p \cdot \widetilde{g}_x, \qquad \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot g = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot (\alpha, g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \alpha^*\widetilde{g},$$

where we recall that $\alpha^*\mathscr{P}$ is a principal $\alpha^*\mathscr{G}$-bundle in sense of Def. 6.11.

Now we equip $\mathscr{G} \overset{\pi_{\mathscr{G}}}{\to} M$ with its **canonical flat connection** $\mathrm{H}\mathscr{G} := \pi^*_{\mathscr{G}}\mathrm{T}M$ which induces a parallel transport $\mathrm{PT}^{\mathscr{G}}_{\alpha} : \mathscr{G}_x \to \Gamma(\alpha^*\mathscr{G})$ with $\mathrm{PT}^{\mathscr{G}}_{\alpha}\big|_t(x, g) = (\alpha(t), g)$, especially $\mathrm{PT}^{\mathscr{G}}_{\alpha}(\widetilde{g}_x) = \alpha^*\widetilde{g}$. In total, we can rewrite Eq. (44) to

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot \widetilde{g}_x) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \alpha^*\widetilde{g} = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\alpha}(\widetilde{g}_x).$$

This opens a gateway to define Eq. (44) on general principal $\mathscr{G}$-bundles. That is, now let $\mathscr{P}$ be again a general principal $\mathscr{G}$-bundle. Fix any horizontal distribution $\mathrm{H}\mathscr{G}$ on $\mathscr{G}$ inducing a parallel transport $\mathrm{PT}^{\mathscr{G}}_{\alpha}$, then a connection on $\mathscr{P}$ should be equivalent to a parallel transport $\mathrm{PT}^{\mathscr{P}}_{\alpha}$ on $\mathscr{P}$ satisfying Eq. (42), (43) and

$$\mathrm{PT}^{\mathscr{P}}_{\alpha}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\alpha}(g) \tag{45}$$

for all $p \in \mathscr{P}_x$ and $g \in \mathscr{G}_x$. The right hand side is now well-defined since both, $\mathrm{PT}^{\mathscr{P}}_{\alpha}(p)$ and $\mathrm{PT}^{\mathscr{G}}_{\alpha}(g)$, are elements of $\Gamma(\alpha^*\mathscr{P})$ and $\Gamma(\alpha^*\mathscr{G})$, respectively. We now want to derive its infinitesimal analogue similar to Eq. (41). Recall that we can view the parallel transports like $\mathrm{PT}^{\mathscr{P}}_{\alpha}(p)$ also as a map $[0, t] \to \mathscr{P}$ with $\pi \circ \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) = \alpha$ (recall Subsection 1.1; alternatively use

[8] Also here one could work with trivializations, especially since $\alpha^*\mathscr{P}$ is trivial as a fibre bundle due to the fact that the image of α is contractible. As before, this would just lead to other problems on a global scale, and we aim to provide a definition of connections on $\mathscr{P}$ without making use of trivializations.

Lemma 5.20 and project onto the second component for the following derivatives). Then by construction

$$Y := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{PT}^{\mathscr{G}}_{\alpha}(g) \in \mathrm{H}_g\mathscr{G}$$

for all $g \in \mathscr{G}$, that is, it is horizontal in $\mathscr{G}$; similarly for the parallel transport on $\mathscr{P}$,

$$X := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) \in \mathrm{H}_p\mathscr{P}.$$

We want to use Thm. 5.21 now in order to understand what Eq. (45) implies about the corresponding horizontal distribution of $\mathscr{P}$. For this we differentiate both sides of Eq. (45) with $\mathrm{d}/\mathrm{d}t|_{t=0}$; the left hand side implies that the right hand side is an element of $\mathrm{H}_{p\cdot g}\mathscr{P}$, while the right hand side gives

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(\mathrm{PT}^{\mathscr{P}}_{\alpha}(p)\cdot \mathrm{PT}^{\mathscr{G}}_{\alpha}(g)\Big) &= \mathrm{D}_{(p,g)}\Phi(X,Y)\\ &= \mathrm{D}_p r_\sigma(X) + \left.\overbrace{(\mu_{\mathscr{G}})_g\Big(Y - \mathrm{D}_x\sigma\big(\dot\alpha(0)\big)\Big)}\right|_{p\cdot g}\end{aligned}$$

where $\Phi : \mathscr{P} * \mathscr{G} \to \mathscr{P}$ denotes the $\mathscr{G}$-action on $\mathscr{P}$, σ is any (local) section of $\mathscr{G}$ with $\sigma_x = g$ and

$$\dot\alpha(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha = \mathrm{D}_p\pi(X)$$

because of $\pi \circ \mathrm{PT}^{\mathscr{P}}_{\alpha}(p) = \alpha$. As we already have shown several times, for example recall the beginning of the proof of Thm. 5.21, we have

$$Y - \mathrm{D}_x\sigma\big(\dot\alpha(0)\big) \in \mathrm{V}_g\mathscr{G},$$

so that the canonical projection $\pi^{\mathrm{vert},\mathscr{G}} : \mathrm{T}\mathscr{G} \to \mathrm{V}\mathscr{G}$ onto the vertical structure of $\mathscr{G}$ acts as identity on it. By making use of the horizontality of Y, we can write

$$\begin{aligned}(\mu_{\mathscr{G}})_g\Big(Y - \mathrm{D}_x\sigma\big(\dot\alpha(0)\big)\Big) &= \Big(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert},\mathscr{G}}\Big)_g\big(Y - \mathrm{D}_p(\sigma\circ\pi)(X)\big)\\ &= -\Big(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert},\mathscr{G}}\Big)_g\big(\mathrm{D}_p(\sigma\circ\pi)(X)\big).\end{aligned}$$

It may not surprise that $\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert},\mathscr{G}}$ is by construction actually *the* connection 1-form on $\mathscr{G}$ corresponding to $\mathrm{H}\mathscr{G}$, therefore we will denote $\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert},\mathscr{G}}$ as the **total Maurer-Cartan form $\mu^{\mathrm{tot}}_{\mathscr{G}}$ of $\mathscr{G}$**. Hence we get in total (see also Figure 1)

$$\mathrm{D}_p r_\sigma(X) - \left.\overbrace{\big(\mu^{\mathrm{tot}}_{\mathscr{G}}\big)_g\big(\mathrm{D}_p(\sigma\circ\pi)(X)\big)}\right|_{p\cdot g} \in \mathrm{H}_{p\cdot g}\mathscr{P} \tag{46}$$

for all $X \in \mathrm{H}_p\mathscr{P}$, and we would like. We are going to prove that this is independent of the chosen section σ, and thence this gives the fundamental formula for the following sections. Last

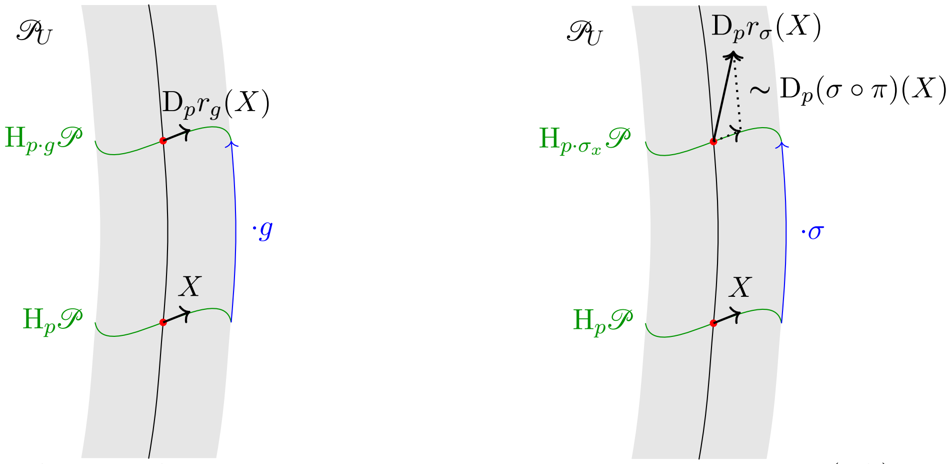


Figure 1: Push-forward of a horizontal tangent vector X with constant section (left) and general section(right), where $\mathscr{P}$ is a classical principal bundle as in Ex. 6.3 equipped with a "typical" connection $\mathrm{H}\mathscr{P}$ of principal G-bundles (G the structural Lie group).

but not least, our starting point was Eq. (38); there is no contradiction between the approaches of Eq. (38) and (46). If X is a vertical vector, then

$$\mathrm{D}_p\pi(X) = 0,$$

and

$$\mathrm{D}_p r_\sigma(X) = \mathrm{D}_p r_g(X)$$

by Remark 4.16. Thence, the symmetry behind Eq. (46) will be compatible with the one of Eq. (38) on the vertical bundle. Let us first study the new arising term in Eq. (46) via the *Darboux derivative*.

6.2.2. Darboux derivative on LGBs

By Cor. 6.18 we know that $\mathrm{V}\mathscr{P}$ is the pullback of $\mathscr{g}$. Since connections are projections onto the vertical bundle, we will put some remarks after some definitions in the following in order to discuss the situation of pullback LGBs. Hence let us start with a general remark about the pullback situation to which we will later refer in the other remarks regarding this situation.

Remark 6.20: Pullback LGBs and their connections

Let $\mathscr{G} \xrightarrow{\pi_\mathscr{G}} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$, where we denote with $\pi^{\mathrm{vert}} : \mathrm{T}\mathscr{G} \to \mathrm{V}\mathscr{G}$ the corresponding (base-preserving) projection onto its vertical bundle; we will view π^{vert} as an element of $\Omega^1(\mathscr{G};\mathrm{V}\mathscr{G})$. Furthermore, let $f : N \to M$ be a smooth map from another smooth manifold N. By Lemma 5.20 we know that $\mathrm{T}(f^*\mathscr{G})$ consists of pairs of tangent vectors $(X,Y) \in \mathrm{T}_{(p,g)}(f^*\mathscr{G})$ ($(p,g) \in f^*\mathscr{G}$) with

$X \in \mathrm{T}_p N$, $Y \in \mathrm{T}_g \mathscr{G}$ and

$$\mathrm{D}_p f(X) = \mathrm{D}_g \pi_{\mathscr{G}}(Y).$$

In the following we denote with pr_i ($i \in \{1, 2\}$) the projections onto the i-th component of $f^* \mathscr{G}$.

The projection of $f^* \mathscr{G} \to N$ is pr_1, thus $\mathrm{Dpr}_1(X, Y) = X$. The vertical bundle $\mathrm{V}(f^* \mathscr{G})$ as the kernel of Dpr_1 then consists of pairs $(X = 0, Y)$, and therefore $\mathrm{D}_g \pi_{\mathscr{G}}(Y) = 0$ which implies $Y \in \mathrm{V}_g \mathscr{G}$. Thence,

$$\mathrm{V}(f^* \mathscr{G}) \cong \mathrm{pr}_2^*(\mathrm{V}\mathscr{G}).$$

It is then trivial to check that

$$\mathrm{pr}_2^! \pi^{\mathrm{vert}} \in \Omega^1\big(f^* \mathscr{G}; \mathrm{pr}_2^*(\mathrm{V}\mathscr{G})\big) \cong \Omega^1\big(f^* \mathscr{G}; \mathrm{V}(f^* \mathscr{G})\big)$$

is a projection onto $\mathrm{V}(f^* \mathscr{G})$, and gives therefore rise to a horizontal distribution on $f^* \mathscr{G}$, the **pullback connection on** $f^* \mathscr{G}$. Especially we have

$$\left(\mathrm{pr}_2^! \pi^{\mathrm{vert}}\right)_{(p,g)}(X, Y) = \big(0, \pi_g^{\mathrm{vert}}(Y)\big),$$

and thus

$$\left((\mathrm{Dpr}_1, \mathrm{Dpr}_2) \circ \mathrm{pr}_2^! \pi^{\mathrm{vert}}\right)\Big|_{(p,g)}(X, Y) = \big(0, \pi_g^{\mathrm{vert}}(Y)\big) = \left(\mathrm{pr}_2^! \pi^{\mathrm{vert}}\right)_{(p,g)}(X, Y).$$

As we have seen, we need a slight adjustment of the vertical Maurer-Cartan as presented in Def. 4.22.

Definition 6.21: Total Maurer-Cartan form

Let $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$, where we denote with $\pi^{\mathrm{vert}} : \mathrm{T}\mathscr{G} \to \mathrm{V}\mathscr{G}$ the corresponding projection onto its vertical bundle. Then we define the **total Maurer-Cartan form** $\mu_{\mathscr{G}}^{\mathrm{tot}} \in \Omega^1\big(\mathscr{G}; \pi_{\mathscr{G}}^* \mathscr{g}\big)$ **of** $\mathscr{G}$ as the connection 1-form corresponding to $\mathrm{H}\mathscr{G}$, *i.e.*

$$\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Y) := \left(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert}}\right)\big|_g(Y) = \left(\mathrm{D}_g L_{g^{-1}}\right)\left(\pi^{\mathrm{vert}}(Y)\right)$$

for all $g \in G$ and $Y \in \mathrm{T}_g \mathscr{G}$.

This is clearly well-defined by construction; also recall our discussion of the vertical Maurer-Cartan form, especially Cor. 4.21.

Remark 6.22: Pullback situation: Part I

Given the situation as in Remark 6.20, then we have

$$L_{(p,g)}\big((p,q)\big) = \big(p, L_g(q)\big)$$

for all $(p,g),(p,q) \in f^*\mathscr{G}$, where the left-translation on the left and right hand side are the ones of $f^*\mathscr{G}$ and $\mathscr{G}$, respectively. Thence,

$$L_{(p,g)} = \big(\mathrm{pr}_1, L_{\mathrm{pr}_2(p,g)} \circ \mathrm{pr}_2\big),$$

and hence,

$$\begin{aligned}
\mathrm{D}_{(p,g)}L_{(p,g)^{-1}} &= \big(\mathrm{D}_{(p,g)}\mathrm{pr}_1, \mathrm{D}_{\mathrm{pr}_2(p,g)}L_{\mathrm{pr}_2(p,g^{-1})} \circ \mathrm{D}_{(p,g)}\mathrm{pr}_2\big)\big|_{\mathrm{V}_{(p,g)}(f^*\mathscr{G})} \\
&= \Big(0, (\mu_{\mathscr{G}})_{\mathrm{pr}_2(p,g)} \circ \mathrm{D}_{(p,g)}\mathrm{pr}_2\Big)\Big|_{\mathrm{V}_{(p,g)}(f^*\mathscr{G})},
\end{aligned}$$

making use of that $L_{(p,g)} : (f^*\mathscr{G})_p \to (f^*\mathscr{G})_p$ so that $\mathrm{D}_{(p,g)}L_{(p,g)^{-1}} : \mathrm{V}_{(p,g)}(f^*\mathscr{G}) \to \mathrm{V}_{(p,e_x)}(f^*\mathscr{G}) = (f^*\mathscr{g})_p$, where $x := f(p)$ and e_x is the neutral element of $\mathscr{G}_x$. Altogether we get

$$\begin{aligned}
\big(\mu^{\mathrm{tot}}_{f^*\mathscr{G}}\big)_{(p,g)} &= \mathrm{D}_{(p,g)}L_{(p,g)^{-1}} \circ \mathrm{pr}_2^! \pi^{\mathrm{vert}}\Big|_{(p,g)} \\
&= \Big(0, (\mu_{\mathscr{G}})_{\mathrm{pr}_2(p,g)} \circ \mathrm{D}_{(p,g)}\mathrm{pr}_2\Big) \circ \mathrm{pr}_2^! \pi^{\mathrm{vert}}\Big|_{(p,g)} \\
&= \Big(0, \big(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert}}\big)\big|_{\mathrm{pr}_2(p,g)} \circ \mathrm{D}_{(p,g)}\mathrm{pr}_2\Big) \\
&= \Big(0, \big(\mu^{\mathrm{tot}}_{\mathscr{G}}\big)_{\mathrm{pr}_2(p,g)} \circ \mathrm{D}_{(p,g)}\mathrm{pr}_2\Big) \\
&= \Big(\mathrm{pr}_2^! \mu^{\mathrm{tot}}_{\mathscr{G}}\Big)\Big|_{(p,g)}.
\end{aligned}$$

Remark 6.23: Total Maurer-Cartan form just typical form on trivial LGBs

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\mathscr{G} := M \times G \overset{\mathrm{pr}_1}{\to} M$ be the trivial LGB equipped with its canonical flat connection $\mathrm{H}\mathscr{G} := \mathrm{pr}_1^*\mathrm{T}M$, where pr_1 is the projection onto the first component in $M \times G$. Its LAB $\mathscr{g}$ is also a trivial bundle, $M \times \mathfrak{g}$, and we have several identities (recall Cor. 4.24)

$$\mathrm{T}\mathscr{G} \cong \mathrm{pr}_1^*\mathrm{T}M \oplus \mathrm{pr}_2^*\mathrm{T}G \cong \mathrm{pr}_1^*\mathrm{T}M \oplus \mathfrak{g} \cong \mathrm{pr}_1^*\mathrm{T}M \oplus \mathrm{pr}_1^*\mathscr{g} = \mathrm{H}\mathscr{G} \oplus \mathrm{V}\mathscr{G},$$

where $\mathrm{pr}_2 : M \times G \to G$ is the projection onto the second component, and the projection onto the vertical bundle is then equivalent to $\mathrm{Dpr}_2 \in \Omega^1(\mathscr{G}; \mathrm{pr}_2^*\mathrm{T}G)$.
Now let us view G as an LGB over a point $\{*\}$, then $\mathscr{G} = f^*G$, where $f : M \to \{*\}$.

Making use of the uniqueness of $\pi^{\mathrm{vert}} = \mathbb{1}_{\mathrm{T}G}$ for $G \to \{*\}$, we have for the pullback connection $\mathrm{pr}_2^! \pi^{\mathrm{vert}} = \mathrm{Dpr}_2$ which is precisely the projection for $\mathscr{G}$, and therefore we can use Remark 6.22 to derive

$$\mu_{\mathscr{G}}^{\mathrm{tot}} = \mathrm{pr}_2^! \mu_G,$$

where μ_G is the Maurer-Cartan form of G.

The Maurer-Cartan form is important for gauge transformations because it induces a derivative, which we also need now.

Remark 6.24: Maurer-Cartan form inducing a natural derivative: Part I

If G is a Lie group and $\mathfrak{g}$ its Lie algebra, then there is actually some stance that the typical Maurer-Cartan form $\mu_G \in \Omega^1(G; \mathfrak{g})$ describes *the* generalization of the total derivative of smooth maps $M \to \mathbb{R}^n$ ($n \in \mathbb{N}$), given by the **Darboux derivative** Δ as given in [2, §5.1, page 182ff.]. For a smooth section $\sigma : M \to G$ of the trivial LGB over M, this is $\Delta\sigma \in \Omega^1(M; \mathfrak{g})$ given by

$$\Delta\sigma := \sigma^! \mu_G,$$

that is,

$$(\Delta\sigma)_p(X) = \mathrm{D}_{\sigma_x} L_{\sigma_x^{-1}} \big(\mathrm{D}_x \sigma(X)\big)$$

for all $p \in M$ and $X \in \mathrm{T}_p M$. For $G = \mathbb{R}^n$ one usually shows that $\mathrm{D}\sigma$ is the actual total derivative (Jacobian) by making use of $\mathrm{T}\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$; however, if one views the triviality of $\mathrm{T}\mathbb{R}^n$ as the trivialization of general $\mathrm{T}G$ as given by $G \times \mathfrak{g}$, then it is more natural to think of the total derivative as $\mathrm{D}\sigma$ followed by $\mathrm{D}_{\sigma_x} L_{\sigma_x^{-1}}$ in order to receive information about the essential Lie algebra element. Then the classical total derivative of $\mathbb{R}^n$-valued maps is actually more naturally given by the Darboux derivative.

We now have a similar behaviour in our case but related to arbitrary connections on $\mathbb{R}^n$ (as a trivial bundle over M).

Definition 6.25: Darboux derivative

Let $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$. For $\sigma \in \Gamma(\mathscr{G})$ define the **Darboux derivative** $\Delta\sigma \in \Omega^1(M; \mathscr{g})$

$$\Delta\sigma = \sigma^! \mu_{\mathscr{G}}^{\mathrm{tot}} = \left(\sigma^* \mu_{\mathscr{G}}^{\mathrm{tot}}\right) \circ \mathrm{D}\sigma.$$

We may also write $\Delta^{\mathscr{G}}$ instead of Δ in order to accentuate the LGB.

Remarks 6.26.
The notation with the pullback σ^* is only needed if one wants to view $\Delta\sigma$ as a $C^\infty(M)$-linear map $\mathfrak{X}(M) \to \Gamma(\mathscr{g})$; in this case it is a composition of base-preserving vector bundle morphisms $\mathrm{D}\sigma : \mathrm{T}M \to \sigma^*\mathrm{T}\mathscr{G}$ and $\sigma^*\mu_{\mathscr{G}}^{\mathrm{tot}} : \sigma^*\mathrm{T}\mathscr{G} \to \sigma^*\pi_{\mathscr{G}}^*\mathscr{g} \cong \mathscr{g}$ which can be extended to sections, where π is the projection of $\mathscr{G}$.

Of course one can view $\Delta\sigma$ as a map $\mathrm{T}M \to \mathscr{g}$ which one can also write as the composition of $\mathrm{D}\sigma : \mathrm{T}M \to \mathrm{T}\mathscr{G}$ and $\mu_{\mathscr{G}}^{\mathrm{tot}} : \mathrm{T}\mathscr{G} \to \mathscr{g}$, that is,

$$\Delta\sigma = \mu_{\mathscr{G}}^{\mathrm{tot}} \circ \mathrm{D}\sigma.$$

If one wants to emphasize base points of the involved pullback bundles, that is, one views $\mu_{\mathscr{G}}^{\mathrm{tot}}$ as a map $\mathrm{T}\mathscr{G} \to \pi_{\mathscr{G}}^*\mathscr{g}$, then one can write point-wise

$$\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{\sigma_x} \circ \mathrm{D}_x\sigma = \left(\sigma_x, \left.(\Delta\sigma)\right|_x\right)$$

for all $x \in M$.

We can actually rewrite some important equations now; recall the notation introduced in Remark 6.19.

Remark 6.27: Darboux derivative in the infinitesimal LGB action

Recall Thm. 5.21; keeping the same notation as in this theorem but denoting the projection of $\mathscr{P}$ by π, we checked several times that $Y - \mathrm{D}_x\sigma(\omega)$ is vertical, and thus we can now write

$$\begin{aligned}
\mathrm{D}_{(p,g)}\Phi(X,Y) &= \mathrm{D}_p r_\sigma(X) + \left.\overline{\left(\mu_{\mathscr{G}}\right)_g\left(Y - \mathrm{D}_x\sigma(\omega)\right)}\right|_{p\cdot g} \\
&= \mathrm{D}_p r_\sigma(X) + \left.\overline{\left(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert}}\right)_g\left(Y - \mathrm{D}_x\sigma(\omega)\right)}\right|_{p\cdot g} \\
&= \mathrm{D}_p r_\sigma(X) + \left.\overline{\left(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert}}\right)_g(Y)}\right|_{p\cdot g} - \left.\overline{\left(\mu_{\mathscr{G}} \circ \pi^{\mathrm{vert}}\right)_{\sigma_x}\left(\mathrm{D}_x\sigma(\omega)\right)}\right|_{p\cdot g} \\
&= \mathrm{D}_p r_\sigma(X) + \left.\overline{\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Y)}\right|_{p\cdot g} - \left.\overline{\left.(\Delta\sigma)\right|_x(\omega)}\right|_{p\cdot g} \\
&= \mathrm{D}_p r_\sigma(X) - \left.\overline{\left.\left(\pi^!\Delta\sigma\right)\right|_p(X)}\right|_{p\cdot g} + \left.\overline{\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Y)}\right|_{p\cdot g}
\end{aligned}$$

If $\mathscr{G}$ is a trivial LGB, then this emphasizes again that we recover the typical Leibniz rule by choosing a constant section σ, using Remarks 6.23 and 6.30.

Remark 6.28: Idea behind the notion of connection on principal bundles using the Darboux derivative

Recall the notation and discussion around the terms in Eq. (46) which will be important for our definition of a connection on principal LGB-bundles. The terms in Eq. (46) can be similarly rewritten to

$$\mathrm{D}_p r_\sigma(X) - \left.\overbrace{\left((\Delta\sigma)_x \circ \mathrm{D}_p\pi\right)(X)}\right|_{p\cdot g} = \mathrm{D}_p r_\sigma(X) - \left.\overbrace{\left(\left.\left(\pi^!\Delta\sigma\right)\right|_p(X)\right)}\right|_{p\cdot g}.$$

We will use this form for the definition of the principal bundle connection. As in Remark 6.27, if $\mathscr{G}$ is trivial and σ a constant section corresponding to a Lie group element g, then these terms are just

$$\mathrm{D}_p r_g(X),$$

making use of Ex. 3.15.

As a derivative we expect a Leibniz rule as for the classical Darboux derivative (see [2, §5.1, Eq. 2, page 182]). However, we cannot make a general statement about that yet, as long as we do not fix a more specific type of connection on $\mathscr{G}$. We will come back to this later; at this point just be sure of that there is of course a certain Leibniz rule, simply due to that it generalizes the "classical" Darboux derivative.

Let us now discuss the pullback situation for the Darboux derivative.

Remark 6.29: Pullback situation: Part II

Following Remark 6.22 and its notation and results (see Remark 6.20 for the initial setup), we can calculate $\Delta^{f^*\mathscr{G}}(f^*\sigma) \in \Omega^1(N; f^*\mathscr{g})$ for $\sigma \in \mathscr{G}$, first we write

$$f^*\sigma|_p = \left(p, \sigma_{f(p)}\right)$$

for all $p \in N$, and thus

$$\begin{aligned}
\Delta^{f^*\mathscr{G}}(f^*\sigma) &= (f^*\sigma)^! \mu^{\mathrm{tot}}_{f^*\mathscr{G}} \\
&= (f^*\sigma)^! \mathrm{pr}_2^! \mu^{\mathrm{tot}}_{\mathscr{G}} \\
&= (\mathrm{pr}_2 \circ f^*\sigma)^! \mu^{\mathrm{tot}}_{\mathscr{G}} \\
&= (\sigma \circ f)^! \mu^{\mathrm{tot}}_{\mathscr{G}} \\
&= \left((\sigma \circ f)^* \mu^{\mathrm{tot}}_{\mathscr{G}}\right) \circ \mathrm{D}(\sigma \circ f) \\
&= \underbrace{\left(f^*\sigma^*\mu^{\mathrm{tot}}_{\mathscr{G}}\right) \circ f^*(\mathrm{D}\sigma)}_{= f^*\left(\left(\sigma^*\mu^{\mathrm{tot}}_{\mathscr{G}}\right)\circ \mathrm{D}\sigma\right)} \circ \mathrm{D}f
\end{aligned}$$

$$= f^!\left(\left(\sigma^*\mu_{\mathscr{G}}^{\mathrm{tot}}\right) \circ \mathrm{D}\sigma\right)$$

$$= f^!\left(\Delta^{\mathscr{G}}\sigma\right), \tag{47}$$

where we rewrote the chain rule

$$\mathrm{D}(\sigma \circ f) = f^*(\mathrm{D}\sigma) \circ \mathrm{D}f,$$

but as in Remark 6.26 one can decide to omit this notation. Such Darboux derivatives related to the pullback connection on $f^*\mathscr{G}$ we call the **pullback Darboux derivative**, and similar to the notation of pullback vector bundle connections we write

$$f^*\Delta^{\mathscr{G}} := \Delta^{f^*\mathscr{G}}.$$

Remark 6.30: Canonical flat Darboux derivative

Given the situation as in Remark 6.23 we can use Remark 6.29 to derive

$$\Delta^{\mathscr{G}} = f^*\Delta^0,$$

where Δ^0 on the right hand side is the Darboux derivative of the Lie group G as a bundle over point. Trivially, $\Delta^0 g \equiv 0$ for all $g \in G$, and thus

$$\Delta^{\mathscr{G}}(f^*g) = f^!\left(\Delta^0 g\right) = 0$$

for all $g \in G$ viewed as a section of $G \to \{0\}$, so f^*g are the constant sections of $\mathscr{G}$. In sense of Remark 6.31 it makes sense to say that σ **is parallel w.r.t.** Δ. Since constant sections generate all sections of $\mathscr{G}$, we then speak of the **canonical flat Darboux derivative**, and constant sections are its parallel sections.

However, just because Δ^0 is a zero map, does not mean that also $\Delta^{\mathscr{G}}$ is zero. On one hand because of a Leibniz rule which we will discuss later in more detail, and on the other hand, as mentioned in Subsection 1.1, we can view general sections $\sigma \in \Gamma(\mathscr{G})$ (not necessarily constant) equivalently as a smooth map $\mathrm{pr}_2 \circ \sigma : M \to G$, denoted by $\widetilde{\sigma}$ for bookkeeping reasons. By Remark 6.23 we then get

$$\Delta^{\mathscr{G}}\sigma = \sigma^! \mathrm{pr}_2^! \mu_G = (\mathrm{pr}_2 \circ \sigma)^! \mu_G = \Delta^G \widetilde{\sigma}$$

for all $\sigma \in \Gamma(\mathscr{G})$, where Δ^G is the "classical" Darboux derivative as in Remark 6.24 related to μ_G. This emphasizes why we can speak of a canonical flat derivative due to the fact that μ_G is flat (Maurer-Cartan equation), and we may simply write $\Delta^{\mathscr{G}} = \Delta^G$.

Remark 6.31: Maurer-Cartan form inducing a natural derivative: Part II

We have now something similar to Remark 6.24. Let $\mathscr{G} = M \times \mathbb{R}^n$ ($n \in \mathbb{N}$) be the trivial abelian LGB, and ∇ a vector bundle connection on $M \times \mathbb{R}^n$, for which we have the associated projection onto the vertical bundle $\pi^{\mathrm{vert}} : \mathrm{T}\mathscr{G} \to \mathrm{V}\mathscr{G}$. For $\sigma \in \Gamma(\mathscr{G})$ we then have

$$\nabla_{X_x}\sigma = \left(\sigma^*\pi^{\mathrm{vert}}\right)\left(\mathrm{D}_x\sigma(X_x)\right) \tag{48}$$

for all $x \in M$ and $X_x \in \mathrm{T}_xM$, by making use of $\sigma^*\mathrm{V}\mathscr{G} \cong M \times \mathbb{R}^n$ as vector bundles such that the right hand side has again values in $\mathscr{G}$ due to "enough triviality" and equals the left hand side.[a] Again one could use the isomorphism as given in Cor. 4.24. This is also more natural in the sense of that one wants that $\nabla_{X_x}\sigma$ is a section over M; M can be viewed as the image of the neutral section e (the zero vector here), so that $\nabla_{X_x}\sigma$ should have values in $e^*\mathrm{V}\mathscr{G} = \mathscr{g}$. Henceforth one could say it is more natural to "pull $\left(\sigma^*\pi^{\mathrm{vert}}\right)\left(\mathrm{D}_x\sigma(X_x)\right)$ back" to $\mathscr{g}$ by left-translation in order to define $\nabla_{X_x}\sigma$, *i.e.*

$$\begin{aligned}\mathrm{D}_{\sigma_x}L_{\sigma_x^{-1}}\Big(\left(\sigma^*\pi^{\mathrm{vert}}\right)\left(\mathrm{D}_x\sigma(X_x)\right)\Big) &= \left(\sigma^*\mu_{\mathscr{G}} \circ \sigma^*\pi^{\mathrm{vert}}\right)\big|_x\left(\mathrm{D}_x\sigma(X_x)\right) \\ &= (\Delta\sigma)|_x(X_x).\end{aligned}$$

Thus, as in Remark 6.24 one may say that $\Delta\sigma$ is *the* generalization of vector bundle connections to general LGBs $\mathscr{G}$. Furthermore, this argument is not based on a given (local) trivialization to handle the arising pullback in Eq. (48).

[a] Strictly spoken, there is also a natural projection $\mathrm{V}\mathscr{G} \to \mathscr{g}$ involved which we omit for simplicity; recall Cor. 4.24.

In fact, the Darboux derivative naturally induces a connection on $\mathscr{g}$ as LAB of $\mathscr{G}$. One may have expected that since $\mathrm{H}\mathscr{G}$ should infinitesimally induce a horizontal distribution on $\mathscr{g}$; recall the exponential map introduced in Subsection 4.4. Also recall that by definition of vertical bundles we have for the LAB $\mathscr{g} \stackrel{\pi_{\mathscr{g}}}{\to} M$ that we have for the vertical bundle that $\mathrm{V}\mathscr{g} \cong \pi_{\mathscr{g}}^*\mathscr{g}$, making use of that LABs are vector bundles; in the following we will use the natural projection onto the second component of $\pi_{\mathscr{g}}^*\mathscr{g}$ but now defined on $\mathrm{V}\mathscr{g}$, denoted by $\mathrm{pr}_2 : \mathrm{V}\mathscr{g} \to \mathscr{g}$.

Proposition 6.32: LGB connection induces LAB connection

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$. Then the map $\nabla^{\mathscr{G}} : \Gamma(\mathscr{g}) \to \Omega^1(M;\mathscr{g})$, $\nu \mapsto \nabla^{\mathscr{G}}\nu$ denoted as an element of $\Omega^1(M;\mathscr{g})$ by $X \mapsto \nabla^{\mathscr{G}}_X\nu$, defined by

$$\nabla^{\mathscr{G}}_X\nu\Big|_x \coloneqq \mathrm{pr}_2\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Big(\left(\Delta \mathrm{e}^{t\nu}\right)_x(X)\Big)\right)$$

for all $x \in M$, $X \in \mathrm{T}_xM$ and $\nu \in \Gamma(\mathscr{g})$, is a vector bundle connection on $\mathscr{g}$, where $t \in \mathbb{R}$, *and* $\mathrm{pr}_2 : \mathrm{V}\mathscr{g} \to \mathscr{g}$ *is the projection onto the second component of* $\mathrm{V}\mathscr{g}$ *naturally viewed as pullback bundle.*

Remarks 6.33.
Usually, the notation of pr_2 is usually omitted in such constructions due to the fact that

$$t \mapsto \left(\Delta \mathrm{e}^{t\nu}\right)_x(X)$$

is a curve with values in $\mathscr{g}_x$, a vector space, so that one canonically uses the identification of tangent spaces of vector spaces with itself to show that $\nabla^{\mathscr{G}}_X \nu\big|_x \in \mathscr{g}_x$. We will keep pr_2 for the proof for the sake of rigorousness, but we will drop it after this proposition for simplicity without further mention.

In order to prove this we need to apply Schwarz's Theorem in order to switch Δ with $\mathrm{d}/\mathrm{d}t$. To do this rigorously we need to introduce the canonical involution/flip on double tangent bundles; since this does not completely fit into this paper's subject and may be already known by the reader, you can learn about the double tangent bundle and its flip map in Appendix A if needed.

Proof of Prop. 6.32.
Let us begin with well-definedness. Similar to Diagram (A.1) we have the double vector bundle

$$\begin{array}{ccc} \mathrm{TT}\mathscr{G} & \xrightarrow{\mathrm{D}\pi_{\mathrm{T}\mathscr{G}}} & \mathrm{T}\mathscr{G} \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{TT}\mathscr{G}}} & & \Big\downarrow{\scriptstyle \pi_{\mathrm{T}\mathscr{G}}} \\ \mathrm{T}\mathscr{G} & \xrightarrow{\pi_{\mathrm{T}\mathscr{G}}} & \mathscr{G} \end{array} \tag{49}$$

We also have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Big(\left(\Delta \mathrm{e}^{t\nu}\right)_x(X)\Big) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Big(\left(\mu^{\mathrm{tot}}_{\mathscr{G}} \circ \mathrm{D}_x \mathrm{e}^{t\nu}\right)(X)\Big) \\ &= \mathrm{D}_{\mathrm{D}_x e(X)}\mu^{\mathrm{tot}}_{\mathscr{G}}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Big(\mathrm{D}_x \mathrm{e}^{t\nu}(X)\Big)\right) \\ &= \mathrm{D}_{\mathrm{D}_x e(X)}\mu^{\mathrm{tot}}_{\mathscr{G}}\Big(S_{\mathscr{G}}\left(\mathrm{D}_x \nu(X)\right)\Big) \\ &= \mathrm{D}_{\mathrm{D}_x e(X)}\mu^{\mathrm{tot}}_{\mathscr{G}}\left(\nu_T(X)\right) \qquad (50) \\ &\in \mathrm{T}_{\mu^{\mathrm{tot}}_{\mathscr{G}}(\mathrm{D}_x e(X))}\mathscr{g} \end{aligned}$$

for all $x \in M$ and $X \in \mathrm{T}_x M$, where e is the neutral section of $\mathscr{G}$, and we viewed $\mu^{\mathrm{tot}}_{\mathscr{G}}$ as a map $\mathrm{T}\mathscr{G} \to \mathscr{g}$ when applying the chain rule; recall Remark 6.26. $S_{\mathscr{G}}$ is the linear canonical flip map on $\mathrm{TT}\mathscr{G}$, especially see the last part of Remark A.2. We also introduced the notation ν_T similar to Remark A.5 for simplicity; in fact ν_T is a vector field on $\mathrm{T}\mathscr{G}$ only defined over $\mathrm{T}M$ which is canonically embedded into $\mathrm{T}\mathscr{G}$ by the zero section while M is embedded into $\mathscr{G}$ by e. Similarly we naturally embed $\mathrm{T}\mathscr{g}$ into $\mathrm{TT}\mathscr{G}$. In total, we will work with these embeddings now so that everything is embedded into $\mathrm{T}\mathrm{T}\mathscr{G}$ as the "total space"; hence, you will see e and its total derivative $\mathrm{D}e$ acting as an embedding several times.

We get by Eq. (49)

$$\mathrm{D}\pi_{\mathrm{T}\mathscr{G}}\left(\nabla^{\mathscr{G}}_X \nu\Big|_x\right) = \mathrm{D}_{\mathrm{D}_x e(X)} \underbrace{\left(\pi_{\mathrm{T}\mathscr{G}} \circ \mu^{\mathrm{tot}}_{\mathscr{G}}\right)}_{=e\circ\pi_{\mathscr{G}}\circ\pi_{\mathrm{T}\mathscr{G}}}(\nu_T(X))$$

$$= (\mathrm{D}_x e \circ \mathrm{D}_{e_x}\pi_{\mathscr{G}})(\nu_x)$$

$$= 0 \in \mathrm{T}_{e_x}\mathscr{G}$$

viewing $S_{\mathscr{G}}$ as a base-preserving isomorphism from $\pi_{\mathrm{TT}G} : \mathrm{T}\mathrm{T}\mathscr{G} \to \mathrm{T}\mathscr{G}$ to $\mathrm{D}\pi_{\mathrm{T}\mathscr{G}} : \mathrm{T}\mathrm{T}\mathscr{G} \to \mathrm{T}\mathscr{G}$, and using that $\mathscr{g} = e^*\mathrm{V}\mathscr{G}$. The projection of $\mathscr{g} \to M$ is canonically the restriction of $\pi_{\mathrm{T}\mathscr{G}}$, hence we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\left(\Delta \mathrm{e}^{t\nu}\right)_x(X)\right) \in \mathrm{V}_{\mu^{\mathrm{tot}}_{\mathscr{G}}(\mathrm{D}_x e(X))}\mathscr{g},$$

and so we can derive that $\mathrm{pr}_2 : \mathrm{V}\mathscr{g} \to \mathscr{g}$ is defined on this; in total $\nabla^{\mathscr{G}}\nu$ is well-defined. By Eq. (50) we also trivially know that

$$\pi_{\mathrm{T}\mathrm{T}\mathscr{G}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\left(\Delta \mathrm{e}^{t\nu}\right)_x(X)\right)\right) = \mu^{\mathrm{tot}}_{\mathscr{G}}(\mathrm{D}_x e(X)) \in \mathscr{g}_x.$$

Viewing $\mathrm{V}\mathscr{g}$ naturally as the pullback of $\mathscr{g}$ along its projection we can therefore write

$$\mathrm{D}_{\mathrm{D}_x e(X)}\mu^{\mathrm{tot}}_{\mathscr{G}}(\nu_T(X)) \overset{\text{Eq. (50)}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\left(\Delta \mathrm{e}^{t\nu}\right)_x(X)\right) = \left(\mu^{\mathrm{tot}}_{\mathscr{G}}(\mathrm{D}_x e(X)),\, \nabla^{\mathscr{G}}_X\nu\Big|_x\right),$$

so that smoothness of $\nabla^{\mathscr{G}}\nu$ follows. In order to understand whether $\nabla^{\mathscr{G}}$ is a vector bundle connection we are hence interested into the restriction of Diagram (49) onto

$$\begin{array}{ccc}
\mathrm{V}\mathscr{g} & \xrightarrow{\mathrm{D}\pi_{\mathrm{T}\mathscr{G}}} & \widetilde{M} \\
\Big\downarrow{\scriptstyle \pi_{\mathrm{TT}\mathscr{G}}} & & \Big\downarrow{\scriptstyle \cong} \\
\mathscr{g} & \xrightarrow{\pi_{\mathrm{T}\mathscr{G}}} & M
\end{array}$$

where M is canonically embedded into $\mathscr{G}$ by e, and $\widetilde{M}$ is the further canonical embedding of this (the image of e) into $\mathrm{T}\mathscr{G}$ by the zero section. As in Appendix A, the addition of vectors and the scalar multiplication of the left vertical arrow is denoted as usual, while the one of the upper horizontal arrow will be denoted by $\boldsymbol{+}$ and $\boldsymbol{\cdot}$, respectively. For $Y \in \mathrm{T}_x M$ we now have

$$\nabla^{\mathscr{G}}_{\lambda X+\kappa Y}\nu\Big|_x = \mathrm{pr}_2\left(\mathrm{D}_{\mathrm{D}_x e(\lambda X+\kappa Y)}\mu^{\mathrm{tot}}_{\mathscr{G}}(\nu_T(\lambda X+\kappa Y))\right)$$

$$= \mathrm{pr}_2\left(\mathrm{D}_{\mathrm{D}_x e(\lambda X+\kappa Y)}\mu^{\mathrm{tot}}_{\mathscr{G}}(\lambda \cdot \nu_T(X) \boldsymbol{+} \kappa \cdot \nu_T(Y))\right)$$

$$= \mathrm{pr}_2\left(\lambda \cdot \mathrm{D}_{\mathrm{D}_x e(X)}\mu^{\mathrm{tot}}_{\mathscr{G}}(\nu_T(X)) \boldsymbol{+} \kappa \cdot \mathrm{D}_{\mathrm{D}_x e(Y)}\mu^{\mathrm{tot}}_{\mathscr{G}}(\nu_T(Y))\right)$$

$$= \lambda \, \mathrm{pr}_2\Big(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \big(\nu_T(X)\big)\Big) + \kappa \, \mathrm{pr}_2\Big(\mathrm{D}_{\mathrm{D}_x e(Y)} \mu_{\mathscr{G}}^{\mathrm{tot}} \big(\nu_T(Y)\big)\Big)$$

$$= \lambda \, \nabla^{\mathscr{G}}_X \nu\Big|_x + \kappa \, \nabla^{\mathscr{G}}_Y \nu\Big|_x$$

for all $\lambda, \kappa \in \mathbb{R}$, using Remark A.3, A.4 and A.5. By these remarks we also derive for another section $\mu \in \Gamma(\mathscr{g})$

$$\begin{aligned}
\nabla^{\mathscr{G}}_X (\lambda\nu + \kappa\mu)\Big|_x &= \mathrm{pr}_2\Big(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \big(\underbrace{(\lambda\nu + \kappa\mu)_T}_{=\lambda\nu_T + \kappa\mu_T}(X)\big)\Big) \\
&= \lambda \, \mathrm{pr}_2\Big(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \big(\nu_T(X)\big)\Big) + \kappa \, \mathrm{pr}_2\Big(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \big(\mu_T(X)\big)\Big) \\
&= \lambda \nabla^{\mathscr{G}}_X \nu\Big|_x + \kappa \nabla^{\mathscr{G}}_X \mu\Big|_x,
\end{aligned}$$

and

$$\begin{aligned}
\nabla^{\mathscr{G}}_X (f\nu)\Big|_x &= \mathrm{pr}_2\Big(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \big((f\nu)_T(X)\big)\Big) \\
&= \mathrm{pr}_2\left(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \left(f(x) \; \nu_T(X) + X(f) \; \nu^a \left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)}\right)\right) \\
&= f(x) \; \nabla^{\mathscr{G}}_X (f\nu)\Big|_x + X(f) \; \mathrm{pr}_2\left(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \left(\nu^a \left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)}\right)\right)
\end{aligned}$$

for all $f \in C^\infty(M)$, where $(\xi^a)_a$ are fibre coordinates of $\mathscr{g}$. It was well-defined to use the linearity of pr_2, since we know by what we have shown earlier that both, $\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}}\big((f\nu)_T(X)\big)$ and $\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}}\big(f\nu_T(X)\big) = f \; \mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}}\big(\nu_T(X)\big)$, are elements of $\mathrm{V}\mathscr{g}$. Thence,

$$\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \left(\nu^a \left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)}\right)$$

is vertical, too.

Making use of the aforementioned isomorphism $\mathrm{V}\mathscr{g} \cong \pi^*_{\mathrm{T}\mathscr{G}}\mathscr{g}$, we can write similar as in Remark A.4

$$\left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)} \cong (\mathrm{D}_x e(X), e_a),$$

where $(e_a)_a$ is a local frame of $\mathscr{g}$ dual to ξ^a. Thus,

$$\mathrm{pr}_2\left(\left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)}\right) = e_a.$$

By definition, $\mu_{\mathscr{G}}^{\mathrm{tot}}$ acts as identity on $\mathscr{g} = e^*\mathrm{V}\mathscr{G}$, and so $\mathrm{D}\mu_{\mathscr{G}}^{\mathrm{tot}}$ is the identity on $\mathrm{T}\mathscr{g}$. Thus, we derive

$$\mathrm{pr}_2\left(\mathrm{D}_{\mathrm{D}_x e(X)} \mu_{\mathscr{G}}^{\mathrm{tot}} \left(\nu^a \left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)}\right)\right) = \mathrm{pr}_2\left(\nu^a \left.\frac{\partial}{\partial \xi^a}\right|_{\mathrm{D}_x e(X)}\right) = \nu^a e_a = \nu,$$

thus, finally,

$$\nabla^{\mathscr{G}}_X(f\nu)\Big|_x = f(x)\ \nabla^{\mathscr{G}}_X(f\nu)\Big|_x + X(f)\ \nu.$$

This finishes the proof. ∎

Definition 6.34: LGB connection on its LAB

Let $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$. Then we call the vector bundle connection $\nabla^{\mathscr{G}}$ on $\mathscr{g}$ of Prop. 6.32, shortly denoted by

$$\nabla^{\mathscr{G}}\nu = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Delta \mathrm{e}^{t\nu}$$

for all $\nu \in \Gamma(\mathscr{g})$, the **$\mathscr{G}$-connection (on its LAB $\mathscr{g}$)**.

Remarks 6.35.
Such a construction of vector bundle connections on LABs also arises in [9, §4.5, Prop. 4.22], but w.r.t. a more specific $\mathrm{H}\mathscr{G}$, see Subsubsection 6.4.1 later; in this context that reference also shows that the parallel transport associated to $\nabla^{\mathscr{G}}$ is the infinitesimal version of the parallel transport associated with $\mathrm{H}\mathscr{G}$.

Example 6.36: Canonical flat $\mathscr{G}$-connection

If we again focus on trivial LGBs with their canonical flat connection as in Remark 6.23 and 6.30, we can quickly derive that $\nabla^{\mathscr{G}}$ is then the canonical flat connection on the trivial LAB $\mathscr{g} = M \times \mathfrak{g}$ of $\mathscr{G} = M \times G$, $\mathfrak{g}$ the Lie algebra of the Lie group G. Let ν be a constant section of $\mathscr{g}$, then $\mathrm{e}^{t\nu}$ is a constant section of $\mathscr{G}$ for all $t \in \mathbb{R}$, so that by Remark 6.30

$$\Delta^{\mathscr{G}} \mathrm{e}^{t\nu} = 0,$$

and thus

$$\nabla^{\mathscr{G}}\nu = 0$$

for all constant sections $\nu \in \Gamma(\mathscr{g})$. By the uniqueness of the canonical flat connection (w.r.t. a trivialisation) we conclude that $\nabla^{\mathscr{G}}$ is the canonical flat connection on $\mathscr{g} = M \times \mathfrak{g}$.

Remark 6.37: Pullback situation: Part III

Following Remark 6.29 (see Remark 6.20 for the initial setup) and all the involved notation we can again derive something related to the pullback LGB $f^*\mathscr{G}$. First of all, let us denote the exponential of $f^*\mathscr{G}$ and $\mathscr{G}$ by $\mathrm{e}_{f^*\mathscr{G}}$ and $\mathrm{e}_{\mathscr{G}}$, respectively, then we clearly have

$$\mathrm{e}_{f^*\mathscr{G}}^{(p,\nu_x)} = \exp_{(f^*\mathscr{G})_p}(p,\nu_x) = \big(p, \exp_{\mathscr{G}_x}(\nu_x)\big) = \big(p, \mathrm{e}_{\mathscr{G}}^{\nu_x}\big)$$

for all $(p,\nu_x) \in (f^*\mathscr{g})_p$, where $x := f(p)$ and so $\nu_x \in \mathscr{g}_x$, and where we made use of that $(f^*\mathscr{G})_p \cong \mathscr{G}_x$ as Lie groups via the projection pr_2 onto the second component (recall Cor. 3.1), that is, we have used the well-known relation

$$\mathrm{pr}_2\Big(\exp_{(f^*\mathscr{G})_p}(p,\nu_x)\Big) = \big(\exp_{\mathscr{G}_x} \circ \mathrm{D}_{(p,e_x)}\mathrm{pr}_2\big)(p,\nu_x) = \exp_{\mathscr{G}_x}(\nu_x),$$

see *e.g.* [1, §1.7, Thm. 1.7.16, page 59] for the general formula; then use that $\mathrm{pr}_2|_{(f^*\mathscr{G})_p}$ is bijective. Hence, we get

$$\mathrm{e}_{f^*\mathscr{G}}^{f^*\nu} = f^*(\mathrm{e}_{\mathscr{G}}^{\nu})$$

for all $\nu \in \Gamma(\mathscr{g})$. Due to clatch of notation we relabel the projection $\mathrm{V}\mathscr{g} \to \mathscr{g}$ in Prop. 6.32 to π_2, and then observe, by using Eq. (47),

$$\begin{aligned}
\nabla_Y^{f^*\mathscr{G}}(f^*\nu)\Big|_p &:= \pi_2\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\left(\Delta^{f^*\mathscr{G}}\mathrm{e}_{f^*\mathscr{G}}^{t\cdot f^*\nu}\right)_p(Y)\right)\right)\\
&= \pi_2\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\left(\Delta^{f^*\mathscr{G}}\big(f^*(\mathrm{e}_{\mathscr{G}}^{t\nu})\big)\right)_p(Y)\right)\right)\\
&= \pi_2\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\left(f^!\left(\Delta^{\mathscr{G}}\mathrm{e}_{\mathscr{G}}^{t\nu}\right)\right)_p(Y)\right)\right)\\
&= \pi_2\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(p, \left(\Delta^{\mathscr{G}}\mathrm{e}_{\mathscr{G}}^{t\nu}\right)_{f(p)}(\mathrm{D}_pf(Y))\right)\right)\\
&= \pi_2\left(0, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\left(\Delta^{\mathscr{G}}\mathrm{e}_{\mathscr{G}}^{t\nu}\right)_{f(p)}(\mathrm{D}_pf(Y))\right)\right)\\
&= \pi_2\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\left(\Delta^{\mathscr{G}}\mathrm{e}_{\mathscr{G}}^{t\nu}\right)_{f(p)}(\mathrm{D}_pf(Y))\right)\right)\\
&= \nabla^{\mathscr{G}}_{\mathrm{D}_pf(Y)}\nu\Big|_{f(p)}
\end{aligned}$$

for all $\nu \in \Gamma(\mathscr{g})$, $p \in N$ and $Y \in \mathrm{T}_pN$. By the uniqueness of pullback vector bundle connections we therefore get

$$\nabla^{f^*\mathscr{G}} = f^*\nabla^{\mathscr{G}}.$$

6.2.3. First step towards towards associated bundles

In order to provide a concise definition of connection 1-forms on principal bundles, we will need to introduce some canonical form of action which will be obviously related to the action needed for an analogue to the notion of associated bundles related to typical principal bundles. However, we will neither discuss nor introduce a more general notion of associated bundles in this paper; so, there will be just the "first step". The following action can be seen as another canonical action on the pullback of a principal bundle, but coming from $\mathscr{G}$ itself instead of its pullback, if $\mathscr{G}$ acts on the manifold one makes a pullback to.

Proposition 6.38: Canonical LGB action on pullback vector bundles over principal LGB-bundles

Let $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle. Furthermore let N be another smooth manifold and $f : N \to M$ a smooth map on which $\mathscr{G}$ acts on the left as in Def. 3.4. Then on the pullback manifold $\mathscr{P} \times_M N := f^*\mathscr{P}$, whose pairs of points are now reordered as in

$$\mathscr{P} \times_M N := \{(p, x) \in \mathscr{P} \times N \mid \pi(p) = f(x)\},$$

we have a right $\mathscr{G}$-action given by

$$(\mathscr{P} \times_M N) * \mathscr{G} \to \mathscr{P} \times_M N,$$

$$(p, x, g) \mapsto (p, x) \cdot g := \left(p \cdot g, g^{-1} \cdot x\right)$$

where $(\mathscr{P} \times_M N) * \mathscr{G}$ is given as pullback of $\mathscr{G}$ w.r.t. the map $\widetilde{\pi} : \mathscr{P} \times_M N \to M$, $(p, x) \mapsto \pi(p)$.

Proof.
By Def. 3.4 it is clear that this action is well-defined, due to

$$\pi(p \cdot g) = \pi_{\mathscr{G}}(g) = f\left(g^{-1} \cdot x\right)$$

for all $(p, x, g) \in (\mathscr{P} \times_M N) * \mathscr{G}$, and this also implies

$$\widetilde{\pi}\big((p, x) \cdot g\big) = \pi(p \cdot g) = \pi_{\mathscr{G}}(g),$$

so that Eq. (13) is satisfied. By construction we also have a smooth action, since it is the composition of maps

$$(\mathscr{P} \times_M N) * \mathscr{G} \to (\mathscr{P} * \mathscr{G}) \times (\mathscr{G} * N) \to \mathscr{P} \times_M N,$$

$$(p, x, g) \mapsto \big((p, g), (g, x)\big) \mapsto \left(p \cdot g, g^{-1} \cdot x\right).$$

The first arrow is clearly an embedding, and the second arrow is just the restriction of the smooth diagonal action,

$$
\begin{aligned}
(\mathscr{P} * \mathscr{G}) \times (\mathscr{G} * N) &\to \mathscr{P} \times N, \\
\big((p, g), (q, p)\big) &\mapsto \big(p \cdot g, q^{-1} \cdot p\big),
\end{aligned}
$$

onto an embedded submanifold, and for its image make use of that $\mathscr{P} \times_M N$ is an embedded submanifold of $\mathscr{P} \times N$, as we already did several times for pullback bundles.

Associativity follows simply by

$$
\big((p, x) \cdot g\big) \cdot q = \big(p \cdot g, g^{-1} \cdot x\big) \cdot q = \big(p \cdot gq, (gq)^{-1} \cdot x\big) = (p, x) \cdot (gq)
$$

for all $g, q \in \mathscr{G}_x$ ($x \coloneqq \pi(p) = f(x)$); it is trivial to check that $(p, x) \cdot e_x = (p, x)$. ■

As mentioned in Remark 6.14, constructions of quotients related to the LGB action may be possible here. Thus, using the last proposition, one should be able to construct associated bundles in this more general setting. We mainly need this proposition for the following examples.

Example 6.39: Adjoint action on the vertical bundle of $\mathscr{P}$

Recall the adjoint representation of $\mathscr{G}$ on its LAB $\mathscr{g} \xrightarrow{\pi_{\mathscr{g}}} M$, Ex. 5.7. Using the notation of Prop. 6.38 we have a right $\mathscr{G}$-action on $\mathscr{P} \times_M \mathscr{g} = \{(p, v) \in \mathscr{P} \times \mathscr{g} \mid \pi(p) = \pi_{\mathscr{g}}(v)\}$ given by

$$
(p, v) \cdot g \coloneqq \big(p \cdot g, \mathrm{Ad}_{g^{-1}}(v)\big)
$$

for all $p \in \mathscr{P}_x$ ($x \in M$), $v \in \mathscr{g}_x$ and $g \in \mathscr{G}_x$. Observe that $\mathscr{P} \times_M \mathscr{g} = \pi^* \mathscr{g}$ which is isomorphic to $\mathrm{V}\mathscr{P}$ by Cor. 6.18. We will denote this action shortly by

$$
\mathscr{Ad}_{g^{-1}}(p, v) \coloneqq (p, v) \cdot g,
$$

the **adjoint representation of $\mathscr{G}$ on** $\mathrm{V}\mathscr{P}$; not to be confused with the adjoint representation of $\pi^*\mathscr{G}$ on $\pi^*\mathscr{g}$. In fact, it is trivial to check that $\mathscr{Ad} : \mathscr{G} \to \mathrm{Aut}(\pi^*\mathscr{g})$, $g \mapsto \mathscr{Ad}_g$, is a $\mathscr{G}$-representation on $\mathscr{P} \times_M \mathscr{g} = \pi^*\mathscr{g} \cong \mathrm{V}\mathscr{P}$ in the sense of Cor. 5.3.
The quotient of $\mathscr{P} \times_M \mathscr{g}$ w.r.t. this group action should lead to a structure which is the generalization of the adjoint bundle in typical formulations of gauge theory.

Similarly, we can define the conjugation action on its integrated bundle; recall the conjugation defined in Def. 4.8. Especially observe that the conjugation defines a left $\mathscr{G}$-action on itself by

$$
\begin{aligned}
\mathscr{G} * \mathscr{G} &\to \mathscr{G}, \\
(g, q) &\mapsto c_g(q) = gqg^{-1}.
\end{aligned}
$$

Example 6.40: Conjugation action over $\mathscr{P}$

Let us look at $\mathscr{P} \times_M \mathscr{G} \cong \pi^*\mathscr{G}$. Then we have a right $\mathscr{G}$-action on $\mathscr{P} \times_M \mathscr{G}$ given by

$$(p, q) \cdot g := \left(p \cdot g, c_{g^{-1}}(q)\right) = \left(p \cdot g, g^{-1} q g\right)$$

for all $p \in \mathscr{P}_x$ ($x \in M$), and $q, g \in \mathscr{G}_x$. We will denote this also by

$$\mathscr{c}_{g^{-1}}(p, q) := (p, g) \cdot g,$$

the **conjugation of $\mathscr{G}$ on the integral of** $\mathrm{V}\mathscr{P}$.
Taking a quotient of $\mathscr{P} \times_M \mathscr{G}$ over this action should lead to a generalization of inner group bundles as introduced in Ex. 2.12.

Let us finally define connections on principal bundles.

6.2.4. Generalized connection 1-forms on principal bundles

As also stated in [1, §5.1, Prop. 5.1.5, page 260], for a given horizontal distribution $\mathrm{H}\mathscr{P}$ of a principal $\mathscr{G}$-bundle $\mathscr{P} \xrightarrow{\pi} M$ over a smooth manifold M we have by construction that

$$\mathrm{D}_p\pi|_{\mathrm{H}_p\mathscr{P}} : \mathrm{H}_p\mathscr{P} \to \mathrm{T}_x M$$

is a vector space isomorphism for all $x \in M$ and $p \in \mathscr{P}_x$; similarly for $\mathscr{G}$ itself. Using that, one has some sort of identification between the horizontal tangent spaces; we want to provide another identification in the sense of Eq. (46), also recall Rem. 6.28; especially recall the Darboux derivative introduced in Subsubsection 6.2.2. The following proposition and definition will be needed.

Proposition 6.41: The right-pushforward modified by the Darboux derivative is a well-defined isomorphism

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, and we have horizontal a distribution $\mathrm{H}\mathscr{G}$ on $\mathscr{G}$. Furthermore, for $g \in \mathscr{G}_x$ ($x \in M$) define the map

$$\mathrm{T}\mathscr{P}|_{\mathscr{P}_x} \to \mathrm{T}\mathscr{P}|_{\mathscr{P}_x},$$

$$X \mapsto \mathscr{r}_{g*}(X) := \mathrm{D}_p r_\sigma(X) - \left.\overline{\left(\pi^!\Delta\sigma\right)\Big|_p(X)}\right|_{p\cdot g},$$

where $p \in \mathscr{P}_x$, $X \in \mathrm{T}_p\mathscr{P}$, and σ is any (local) section of $\mathscr{G}$ with $\sigma_x = g$. Then $\mathscr{r}_{g*}$ is independent of the choice of the local section σ, and it is a vector bundle automorphism over the right-translation r_g.
Furthermore, if we also have a horizontal distribution $\mathrm{H}\mathscr{P}$ on $\mathscr{P}$, then $\mathrm{H}_p\mathscr{P}$ is isomorphic via $\mathscr{r}_{g*}$ to a complement of $\mathrm{V}_{p\cdot g}\mathscr{P}$ in $\mathrm{T}_{p\cdot g}\mathscr{P}$ (this complement is not necessarily $\mathrm{H}_{p\cdot g}\mathscr{P}$).

Proof.
In the following we have $(p,g) \in \mathscr{P} * \mathscr{G}$, $X \in \mathrm{T}_p\mathscr{P}$, and σ is any (local) section of $\mathscr{G}$ with $\sigma_x = g$ ($x := \pi(p)$).

• Let $\pi_{\mathscr{G}}$ be the projection of $\mathscr{G}$, then as mentioned before Prop. 6.41 there is a unique $Y \in \mathrm{H}_g\mathscr{G}$ with

$$\mathrm{D}_g\pi_{\mathscr{G}}(Y) = \mathrm{D}_p\pi(X) \in \mathrm{T}_xM.$$

By Remark 6.27 we can derive

$$\mathrm{D}_{(p,g)}\Phi(X,Y) = \mathscr{r}_{g*}(X),$$

where we made use of that Y is horizontal in $\mathscr{G}$ so that $\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Y) = 0$, and where Φ denotes the right $\mathscr{G}$-action on $\mathscr{P}$ as a map $\mathscr{P} * \mathscr{G} \to \mathscr{P}$. Therefore the independence of $\mathscr{r}_{g*}$ w.r.t. the choice of σ follows.

• $\mathscr{r}_{g*}$ is clearly linear by construction. Let us now show fibre-wise that $\mathscr{r}_{g*}$ is injective, then it is also bijective by dimensional reasons. By Remark 3.11 we know that r_σ is a diffeomorphism, and thus $\mathrm{D}_pr_\sigma : \mathrm{T}_p\mathscr{P} \to \mathrm{T}_{p\cdot g}\mathscr{P}$ is a vector space isomorphism. Restricted onto the vertical subspace $\mathrm{V}_p\mathscr{P}$ we have by Remark 4.16

$$\mathrm{D}_pr_\sigma|_{\mathrm{V}_p\mathscr{P}} = \mathrm{D}_pr_g : \mathrm{V}_p\mathscr{P} \to \mathrm{V}_{p\cdot g}\mathscr{P},$$

which is a vector space isomorphism by Cor. 6.18 and dimensional reasons. We rewrite D_pr_σ as

$$\begin{aligned}
\mathrm{H}_p\mathscr{P} \oplus \mathrm{V}_p\mathscr{P} &\to \mathrm{T}_{p\cdot g}\mathscr{P},\\
\left(X^{\mathrm{H}},\ X^{\mathrm{V}}\right) &\mapsto \mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}\left(X^{\mathrm{H}}\right) + \mathrm{D}_pr_g\left(X^{\mathrm{V}}\right).
\end{aligned}$$

By dimensional reasons and due to the bijectivity of D_pr_σ and D_pr_g the image $\mathrm{Im}\left(\mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}\right)$ of $\mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}$ has to be a complement subspace of $\mathrm{V}_{p\cdot g}\mathscr{P}$ in $\mathrm{T}_{p\cdot g}\mathscr{P}$; $\mathrm{Im}\left(\mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}\right)$ is not necessarily equal to $\mathrm{H}_{p\cdot g}\mathscr{P}$ in general but of the same dimension, especially of the same dimension as $\mathrm{H}_p\mathscr{P}$. Thus, $\mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}$ is a vector space isomorphism onto its image which is complementary to $\mathrm{V}_{p\cdot g}\mathscr{P}$. Furthermore, observe

$$\left(\pi^!\Delta\sigma\right)\Big|_p(X) = (\Delta\sigma)|_x\left(\mathrm{D}_p\pi\left(X^{\mathrm{H}} + X^{\mathrm{V}}\right)\right) = (\Delta\sigma)|_x\left(\mathrm{D}_p\pi\left(X^{\mathrm{H}}\right)\right) = \left(\pi^!\Delta\sigma\right)\Big|_p\left(X^{\mathrm{H}}\right),$$

using that the vertical subbundle is given by the kernel of $\mathrm{D}\pi$, and where we split $X = X^{\mathrm{H}} + X^{\mathrm{V}}$ for all $X \in \mathrm{T}_p\mathscr{P} = \mathrm{H}_p\mathscr{P} \oplus \mathrm{V}_p\mathscr{P}$.

Hence, we get in total that $\mathscr{r}_{g*}$ at p is equivalent to

$$\begin{aligned}
\mathrm{H}_p\mathscr{P} \oplus \mathrm{V}_p\mathscr{P} &\to \mathrm{Im}\left(\mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}\right) \oplus \mathrm{V}_{p\cdot g}\mathscr{P},\\
\left(X^{\mathrm{H}},\ X^{\mathrm{V}}\right) &\mapsto \left(\mathrm{D}_pr_\sigma|_{\mathrm{H}_p\mathscr{P}}\left(X^{\mathrm{H}}\right),\ \mathrm{D}_pr_g\left(X^{\mathrm{V}}\right) - \widetilde{\left(\pi^!\Delta\sigma\right)\Big|_p\left(X^{\mathrm{H}}\right)}\Big|_{p\cdot g}\right).
\end{aligned}$$

Let $\left(X^{\mathrm{H}},\ X^{\mathrm{V}}\right)$ be now in the kernel of $\mathscr{r}_{g*}$, then $X^{\mathrm{H}} = 0$ by the bijectivity of $\mathrm{D}_p r_\sigma|_{\mathrm{H}_p\mathscr{P}}$ onto its image. The second component of $\mathscr{r}_{g*}$ is then just $\mathrm{D}_p r_g\left(X^{\mathrm{V}}\right)$; again by the bijectivity of $\mathrm{D}_p r_g : \mathrm{V}_p\mathscr{P} \to \mathrm{V}_{p\cdot g}\mathscr{P}$ we also get $X^{\mathrm{V}} = 0$. Thus, $\mathscr{r}_{g*}$ is injective and therefore defines vector space isomorphisms $\mathrm{T}_p\mathscr{P} \to \mathrm{T}_{p\cdot g}\mathscr{P}$. It follows that $\mathscr{r}_{g*}$ is an automorphism of $\mathrm{T}\mathscr{P}|_{\mathscr{P}_x}$ over r_g.

- It is then clear that $\mathscr{r}_{g*}|_{\mathrm{H}_p\mathscr{P}}$, given by

$$\begin{aligned}\mathrm{H}_p\mathscr{P} &\to \mathrm{Im}\left(\mathrm{D}_p r_\sigma|_{\mathrm{H}_p\mathscr{P}}\right) \oplus \mathrm{V}_{p\cdot g}\mathscr{P},\\ X^{\mathrm{H}} &\mapsto \left(\mathrm{D}_p r_\sigma|_{\mathrm{H}_p\mathscr{P}}\left(X^{\mathrm{H}}\right),\ -\left.\overbrace{\left.\left(\pi^!\Delta\sigma\right)\right|_p\left(X^{\mathrm{H}}\right)}\right|_{p\cdot g}\right),\end{aligned}$$

is also an isomorphism onto its image; its image is complementary to $\mathrm{V}_{p\cdot g}\mathscr{P}$ because its intersection with $\mathrm{V}_{p\cdot g}\mathscr{P}$ would require that

$$\mathrm{D}_p r_\sigma|_{\mathrm{H}_p\mathscr{P}}\left(X^{\mathrm{H}}\right) = 0,$$

which implies that $X^{\mathrm{H}} = 0$ due to the fact that $\mathrm{D}_p r_\sigma|_{\mathrm{H}_p\mathscr{P}}$ is a vector space isomorphism onto its image, as we discussed before. But then

$$\mathscr{r}_{g*}|_{\mathrm{H}_p\mathscr{P}}(0) = 0,$$

which implies that the image of $\mathscr{r}_{g*}|_{\mathrm{H}_p\mathscr{P}}$ is a complement of $\mathrm{V}_{p\cdot g}\mathscr{P}$ in $\mathrm{T}_{p\cdot g}\mathscr{P}$. This finishes the proof. ■

Hence, we formally define:

Definition 6.42: Modified pushforward via right-translation

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, and we have a horizontal distribution $\mathrm{H}\mathscr{G}$ on $\mathscr{G}$. Then we define the **modified right-pushforward**[a] $\mathscr{r}_{g*}$ **(with $g \in \mathscr{G}_x$, $x \in M$)** as the vector bundle isomorphism $\mathrm{T}\mathscr{P}|_{\mathscr{P}_x} \to \mathrm{T}\mathscr{P}|_{\mathscr{P}_x}$ over r_g as given in Prop. 6.41 by

$$\mathscr{r}_{g*}(X) := \mathrm{D}_p r_\sigma(X) - \left.\overbrace{\left.\left(\pi^!\Delta\sigma\right)\right|_p(X)}\right|_{p\cdot g}$$

for all $p \in \mathscr{P}_x$ and $X \in \mathrm{T}_p\mathscr{P}$, where σ is any (local) section of $\mathscr{G}$ with $\sigma_x = g$. Similarly, for a (local) section σ of $\mathscr{G}$ we define the **modified right-pushforward** $\mathscr{r}_{\sigma*}$ **with** σ as a (local) vector bundle isomorphism $\mathrm{T}\mathscr{P} \to \mathrm{T}\mathscr{P}$ by

$$\mathscr{r}_{\sigma*}(X) := \mathscr{r}_{\sigma_x*}(X) = \mathrm{D}_p r_\sigma(X) - \left.\overbrace{\left.\left(\pi^!\Delta\sigma\right)\right|_p(X)}\right|_{p\cdot\sigma_x}$$

for all $X \in \mathrm{T}_p\mathscr{P}$ $(p \in \mathscr{P}_x)$.

[a] The font of $\mathscr{r}$ is a calligraphic r.

Remark 6.43: Restriction onto vertical bundle gives typical right-pushforward

It is trivial to check that we have

$$\mathscr{r}_{g*}(X) = \mathrm{D}_p r_g(X) =: r_{g*}(X)$$

for all $X \in \mathrm{V}_p\mathscr{P}$; we actually have proven this directly after Eq. (46). As also pinpointed in Remark 6.28, if $\mathscr{G}$ is a trivial LGB equipped with its canonical flat connection and σ a constant section, then it is easy to see that $\mathscr{r}_{\sigma*} = \mathscr{r}_{g*} = r_{g*}$, where r_g has to be understood as the right-translation of the canonical Lie group action inherited by $\mathscr{G}$ in the sense of Ex. 3.15.

Remarks 6.44.
Prop. 6.41 trivially extends to $\mathscr{r}_{\sigma*}$, that is $\mathscr{r}_{\sigma*}$ is a (local) automorphism of $\mathrm{T}\mathscr{P}$ over r_σ. Thus, we can view $\mathscr{r}_{\sigma*}$ as an element of $\Omega^1(\mathscr{P}; r_\sigma^*\mathrm{T}\mathscr{P})$.

Hence, the following definition makes sense, especially if thinking about what we discussed for Eq. (46), also recall Rem. 6.28.

Definition 6.45: Ehresmann connection on principal LGB-bundles

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, and we have horizontal distributions $\mathrm{H}\mathscr{G}$ and $\mathrm{H}\mathscr{P}$ on $\mathscr{G}$ and $\mathscr{P}$, respectively. We call $\mathrm{H}\mathscr{P}$ an **Ehresmann connection** or a **connection on** $\mathscr{P}$ if it is **right-invariant (w.r.t. modifier right-pushforward)**, *i.e.*

$$\mathscr{r}_{g*}(\mathrm{H}_p\mathscr{P}) = \mathrm{H}_{p\cdot g}\mathscr{P}$$

for all $p \in \mathscr{P}_x$ and $g \in \mathscr{G}_x$ ($x := \pi(p)$).

Remarks 6.46.
There is also a definition of connections on such and more general principal bundles in [6, §5.7, paragraph before Prop. 5.38, page 148]. However, this reference provides a different type of definition; translated to our situation, it is based on assuming that $\mathscr{G}$ is defined over another base manifold N. In order to define the LGB action on $\mathscr{P}$ this reference introduces a moment map $\mu : \mathscr{P} \to N$ so that the action of an element $\mathscr{G}_y$ ($y \in N$) is defined on $\mu^{-1}(\{y\})$, especially the infinitesimal action r_g of a fixed LGB element g acts on tangent vectors of $\mu^{-1}(\{y\})$, not necessarily on the vertical structure of $\mathscr{P}$. Hence, in order to circumvent the problem we discussed in Subsubsection 6.2.1, the reference's definition of a connection is then based on assuming that the "fibres" $\mu^{-1}(\{y\})$ are complementary to the vertical structure of $\mathscr{P}$, so these fibres' tangent spaces define a horizontal distribution while the infinitesimal action r_g now acts well-defined on the horizontal structure. Henceforth, the reference does not need to look at using sections σ and their actions.

However, this is not a suitable definition for us, because our moment map is the projection of

$\mathscr{P}$ itself such that r_g acts on the vertical structure. The reference's definition is rather restrictive, while our definition works for all principal $\mathscr{G}$-bundles by fixing a connection on $\mathscr{G}$.

Let us discuss several examples.

Example 6.47: Recovering of the classical definition

As we already discussed several times, especially recall Remark 6.43 and Ex. 6.3, we recover the typical definition of a connection on a principal bundle if $\mathscr{P}$ is a classical principal bundle and if the trivial LGB $\mathscr{G}$ is equipped with its canonical flat connection $\mathrm{H}\mathscr{G}$. We call such a connection a **classical connection**.

Example 6.48: Associated LGBs

Recall Def. 2.11, and recall that LGBs themselves are principal bundles as in Ex. 6.4. That is, let G, H be Lie groups, $P \to M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then we have the LGB associated to the principal bundle P and the representation ψ on H

$$\begin{tikzcd} H \arrow[r] & \mathscr{H} := P \times_\psi H \arrow[d] \\ & M \end{tikzcd}$$

Fix a typical classical connection on P; as also introduced in Subsubsection 6.2.1 we have an associated parallel transport $\mathrm{PT}^P_\alpha : P_x \to \Gamma(\alpha^* P)$ along a curve $\alpha : [0,t] \to M$ $(t > 0)$, where $\alpha(0) =: x$. As proven in [1, §5.9, Thm. 5.9.1, page 289f.], we have a canonical well-defined parallel transport $\mathrm{PT}^{\mathscr{H}}_\alpha : \mathscr{H}_x \to \Gamma(\alpha^*\mathscr{H})$ given by

$$\mathrm{PT}^{\mathscr{H}}_\alpha([p,h]) = \left[\mathrm{PT}^P_\alpha(p), h\right]$$

for all $[p,h] \in \mathscr{H}_x$. This has a 1:1 correspondence to a horizontal distribution $\mathrm{H}\mathscr{H}$ on $\mathscr{H}$. Observe that we have

$$\begin{aligned} \mathrm{PT}^{\mathscr{H}}_\alpha\Big([p,h]\cdot[p,h']\Big) &= \mathrm{PT}^{\mathscr{H}}_\alpha\Big([p,hh']\Big) \\ &= \left[\mathrm{PT}^P_\alpha(p), hh'\right] \\ &= \left[\mathrm{PT}^P_\alpha(p), h\right]\cdot\left[\mathrm{PT}^P_\alpha(p), h'\right] \\ &= \mathrm{PT}^{\mathscr{H}}_\alpha([p,h])\cdot\mathrm{PT}^{\mathscr{H}}_\alpha\Big([p,h']\Big) \end{aligned}$$

for all $[p,h],[p,h'] \in \mathscr{H}_x$. Now recall that the whole motivation behind Def. 6.45 comes from Eq. (46) (also recall Remark 6.28) which itself stems from Eq. (45). We see that that Eq. (45) is satisfied here, and thus Eq. (46) follows, *i.e.*

$$\mathscr{r}_{[p,h']*}(X) \in \mathrm{H}_{[p,h]\cdot[p,h']}\mathscr{H}$$

for all $X \in \mathrm{H}_{[p,h]}\mathscr{H}$. By Prop. 6.41 and dimensional reasons (horizontal subspaces are of the same dimension) it follows immediately that $\mathrm{H}\mathscr{H}$ is an (Ehresmann) connection on $\mathscr{H}$ in sense of Def. 6.45.

We call such connections on LGBs associated to a classical principal bundle an **associated connection**.

As expected, we have a corresponding connection 1-form. For this we need to formally define the pullback of 1-forms with respect to the modified right-pushforward.

Definition 6.49: The pullback of forms

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, and we have a fixed horizontal distribution $\mathrm{H}\mathscr{G}$ on $\mathscr{G}$. For $\omega \in \Omega^k(\mathscr{P}; \pi^*\mathscr{g})$ ($k \in \mathbb{N}_0$) we define the **pullback via the modified right-pushforward** $\mathscr{r}_g^!\big(\omega|_{\mathscr{P}_x}\big)$ **(with** $g \in \mathscr{G}_x$**, over** $\mathscr{P}_x$**,** $x \in M$**)** as an element of $\Gamma\big(\Lambda^k \mathrm{T}^*\mathscr{P}|_{\mathscr{P}_x} \otimes \mathscr{g}_x\big)$ by

$$\Big(\mathscr{r}_g^!\big(\omega|_{\mathscr{P}_x}\big)\Big)\Big|_p(Y_1, \dots, Y_k) := \omega_{p\cdot g}\big(\mathscr{r}_{g*}(Y_1), \dots, \mathscr{r}_{g*}(Y_k)\big)$$

for all $p \in \mathscr{P}_x$ and $Y_1, \dots, Y_k \in \mathrm{T}_p\mathscr{P}$. Similarly we define the **pullback via the modified right-pushforward** $r_\sigma^!\omega$ **with a (local) section** $\sigma \in \Gamma(\mathscr{G})$ as an element of $\Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ by

$$\Big(\mathscr{r}_\sigma^!\omega\Big)\Big|_p(Y_1, \dots, Y_k) := \Big(\mathscr{r}_{\sigma_x}^!\big(\omega|_{\mathscr{P}_x}\big)\Big)\Big|_p(Y_1, \dots, Y_k) = (r_\sigma^*\omega)|_p\big(\mathscr{r}_{\sigma*}(Y_1), \dots, \mathscr{r}_{\sigma*}(Y_k)\big)$$

for all $p \in \mathscr{P}_x$ and $Y_1, \dots, Y_k \in \mathrm{T}_p\mathscr{P}$.

Remarks 6.50.
By Prop. 6.41 (and Remark 6.44) these definitions are well-defined. A short note about the notation on the very right hand side of the second definition: This notation allows us to extend it to vector fields, that is,

$$\Big(\mathscr{r}_\sigma^!\omega\Big)(Y_1, \dots, Y_k) = (r_\sigma^*\omega)\big(\mathscr{r}_{\sigma*}(Y_1), \dots, \mathscr{r}_{\sigma*}(Y_k)\big)$$

for all $p \in \mathscr{P}_x$ and $Y_1, \dots, Y_k \in \mathfrak{X}(\mathscr{P})$. Observe that we have $\mathscr{r}_{\sigma*}(Y_l) \in \Gamma(r_\sigma^*\mathrm{T}\mathscr{P})$ ($l \in \{1, \dots, k\}$) and $r_\sigma^*\omega \in \Gamma\big(\Lambda^k r_\sigma^*\mathrm{T}\mathscr{P} \otimes \pi^*\mathscr{g}\big)$, using $r_\sigma^*\pi^*\mathscr{g} \cong (\pi \circ r_\sigma)^*\mathscr{g} = \pi^*\mathscr{g}$, such that the right hand side is well-defined.

For the following definition recall the adjoint $\mathscr{G}$-representation on $\mathrm{V}\mathscr{P}$, Ex. 6.39, and the isomorphism for $\mathrm{V}\mathscr{P}$ in Cor. 6.18.

Definition 6.51: Connection 1-forms on principal LGB-bundles

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, and we have a fixed horizontal distribution $\mathrm{H}\mathscr{G}$ on $\mathscr{G}$. A **connection 1-form** or **gauge**

field on $\mathscr{P}$ is a 1-form $A \in \Omega^1(\mathscr{P}; \pi^* \mathscr{g})$ satisfying:

- **($\mathscr{G}$-equivariance, but w.r.t. modified right-pushforward)**

$$r^!_\sigma A = \mathscr{Ad}_{\sigma^{-1}} \circ A$$

for all (local) $\sigma \in \Gamma(\mathscr{G})$.

- **(Identity on** $\mathrm{V}\mathscr{P}$**)**

$$A(\widetilde{\nu}) = \pi^* \nu$$

for all (local) $\nu \in \Gamma(\mathscr{g})$.

Remarks 6.52.

• Due to the fact that we formulated the $\mathscr{G}$-equivariance using Ex. 6.39, one may be able to show an analogue of the 1:1 correspondence of typical (classical) connection 1-forms to connection reforms and splittings of the Atiyah sequence; that is, a generalization of [2, §3.2, page 90 ff.].

• The $\mathscr{G}$-equivariance reads point-wise for $g \coloneqq \sigma_x$ $(x \in M)$

$$\left(r^!_g (A|_{\mathscr{P}_x}) \right)\Big|_p (X) = \left(p \cdot g, \mathrm{Ad}_{g^{-1}} \left(\hat{A}_p(X) \right) \right)$$

for all $p \in \mathscr{P}_x$ and $X \in \mathrm{T}_p \mathscr{P}$, where we wrote $A_p = \left(p, \hat{A}_p \right)$ with $\hat{A}_p \in \Gamma\left(\mathrm{T}^*_p \mathscr{P} \otimes \mathscr{g}_x \right)$, and Ad is the adjoint representation of $\mathscr{G}$. In the typical formulation of gauge theory the base point component is usually omitted due to that $\mathscr{g}$ is then trivial and so also $\mathrm{V}\mathscr{P} \cong \pi^* \mathscr{g}$.

The identity on $\mathrm{V}\mathscr{P}$ reads pointwise

$$A_p(\widetilde{v_p}) = (p, v_p)$$

for all $v_p \in \mathscr{g}_x$. For readability we may also omit the basepoint information and just write

$$A_p(\widetilde{v_p}) \equiv v_p.$$

Furthermore, recall the notation introduced in Remark 6.19, then for $\mu \coloneqq (p, v_p)$ we can therefore write

$$A_p(\widetilde{\mu}_p) = \mu.$$

Finally, we identify (Ehresmann) connections and connection 1-forms on principal LGB-bundles in the typical way; for this recall Cor. 6.18.

Theorem 6.53: 1:1 correspondence of Ehresmann connections and connection 1-forms

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, and let $\mathrm{H}\mathscr{G}$ be a horizontal distribution on $\mathscr{G}$. Then there is a 1:1 correspondence between Ehresmann connections and connection 1-forms on $\mathscr{P}$:

- *Let $\mathrm{H}\mathscr{P}$ be an Ehresmann connection on $\mathscr{P}$. Then $\mathrm{H}\mathscr{P}$ defines a connection 1-form $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ by*

$$A_p(\widetilde{v}_p + X_p) = (p, v)$$

for all $p \in \mathscr{P}_x$ ($x \in M$), $v \in \mathscr{g}_x$ and $X \in \mathrm{H}_p\mathscr{P}$.

- *Let $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Then A defines an Ehresmann connection $\mathrm{H}\mathscr{P}$ on $\mathscr{P}$ via its kernel $\mathrm{Ker}(A)$, that is,*

$$\mathrm{H}_p\mathscr{P} = \mathrm{Ker}(A_p)$$

for all $p \in \mathscr{P}$.

Proof.
The proof is very similar to the proof of "typical connections", as *e.g.* provided in [1, §5.2, Thm. 5.2.2, page 262].

• For the first bullet point we need to show that Def. 6.51 is satisfied, and the identity behaviour on $\mathrm{V}\mathscr{P}$ quickly follows by definition: We have

$$A(\widetilde{\nu})\big|_p = A_p(\widetilde{\nu}_p) = (p, \nu_x) = (\pi^*\nu)\big|_p$$

for all $\nu \in \Gamma(\mathscr{g})$ and $p \in \mathscr{P}_x$ ($x \in M$). Therefore it is only left to show the $\mathscr{G}$-equivariance. As mentioned in Remark 5.18, for $v \in \mathscr{g}_x$ the vector field $\widetilde{v}$ on $\mathscr{P}_x$ is a fundamental vector field coming from the $\mathscr{G}_x$-action on $\mathscr{P}_x$. Hence, we know by [1, §3.4, Prop. 3.4.6, page 145f.] that

$$\mathscr{r}_{g*}(\widetilde{v}) = r_{g*}(\widetilde{v}) = \widetilde{\mathrm{Ad}_{g^{-1}}(v)}$$

for all $g \in \mathscr{G}_x$, also using Remark 6.43.[9] Thus,

$$\begin{aligned}
\left(\mathscr{r}^!_\sigma A\right)\Big|_p (\widetilde{v}_p + X_p) &= A_{p\cdot\sigma_x}\Big(\underbrace{\mathscr{r}_{\sigma_x *}(\widetilde{v}_p)}_{=\mathscr{r}_{\sigma_x *}(\widetilde{v})|_{p\cdot\sigma_x}} + \underbrace{\mathscr{r}_{\sigma_x *}(X_p)}_{\in \mathrm{H}_{p\cdot\sigma_x}\mathscr{P}}\Big) \\
&= \left(p\cdot\sigma_x, \mathrm{Ad}_{\sigma_x^{-1}}(v)\right) \\
&= \mathscr{Ad}_{\sigma_x^{-1}}(p, v)
\end{aligned}$$

[9] This is very straightforward to prove.

$$= \mathscr{A}\!d_{\sigma_x^{-1}}\big(A_p(\widetilde{v}_p + X_p)\big)$$

$$= (\mathscr{A}\!d_{\sigma^{-1}} \circ A)|_p(\widetilde{v}_p + X_p)$$

for all $p \in \mathscr{P}_x$, $\sigma \in \Gamma(\mathscr{G})$, $v \in \mathscr{g}_x$ and $X_p \in \mathrm{H}_p\mathscr{P}$. This finishes the proof for the first bullet point.

- For the second bullet point we make use of Cor. 6.18 and

$$A(\widetilde{\nu}) = \pi^*\nu$$

for all (local) sections $\nu \in \Gamma(\mathscr{g})$. This implies that A has not only values in $\mathrm{V}\mathscr{P}$, but acts also as identity on $\mathrm{V}\mathscr{P}$. Thus, $\mathrm{H}\mathscr{P} = \mathrm{Ker}(A_p)$ is a complementary subspace of $\mathrm{V}_p\mathscr{P}$ in $\mathrm{T}_p\mathscr{P}$ for all $p \in \mathscr{P}$. In order to show that $\mathrm{H}\mathscr{P}$ is a horizontal distribution of $\mathscr{P}$, we fix a local frame $(e_a)_a$ of $\mathscr{g}$ over U (an open subset of M), so that $(\pi^* e_a)_a$ is a frame of $\pi^*\mathscr{g}|_{\pi^{-1}(U)}$. Therefore we write

$$A = A^a \otimes \pi^* e_a,$$

where $A^a \in \Omega^1(\mathscr{P}|_U)$. Due to $A(\widetilde{\nu}) = \pi^*\nu$ we get

$$A^a(\widetilde{e_b}) = \delta^a_b\ \pi^* e_a,$$

where δ^a_b is the Kronecker delta. $(A^a)_a$ is the dual frame of $(\widetilde{e_a})_a$, which is a frame of $\mathrm{V}\mathscr{P}$ by Cor. 6.18, thus, the A^a are linear independent to each other. Fix an auxiliary fibre metric $\langle\cdot,\cdot\rangle$ on $\mathrm{T}\mathscr{P}$, and denote with $(V^a)_a$ the $\langle\cdot,\cdot\rangle$-dual frame to $(A^a)_a$, *i.e.*

$$A^a = \langle V^a, \cdot\rangle.$$

Due to the smoothness of A, $(V^a)_a$ are smooth (local) vector fields on $\mathscr{P}$, linear independent to each other. Observe that any orthogonal frame $(W^\alpha)_\alpha$ to $(V^a)_a$ satisfies

$$A^a(W^\alpha) = \langle V^a, W^\alpha\rangle = 0.$$

Thus, we derived that $\mathrm{Ker}(A)|_U$ is spanned by such frames $(W^\alpha)_\alpha$, in total, $\mathrm{Ker}(A)$ is spanned by a locally free sheaf of modules of constant rank. By the 1:1 correspondence of vector bundle and locally free sheaf of modules of constant rank, we can conclude that $\mathrm{H}\mathscr{P} = \mathrm{Ker}(A)$ is a subbundle of $\mathrm{T}\mathscr{P}$; complementary to $\mathrm{V}\mathscr{P}$ due to what we have shown earlier.

It is only left to show the right-invariance of $\mathrm{H}\mathscr{P}$. So let $X \in \mathrm{H}_p\mathscr{P}$ for $p \in \mathscr{P}$. Then

$$A_{p\cdot\sigma_x}\big(r_{\sigma*}(X)\big) = \Big(r_\sigma^!(A)\Big)_p(X) = \mathscr{A}\!d_{\sigma_x^{-1}}\big(A_p(X)\big) = 0$$

for all (local) $\sigma \in \Gamma(\mathscr{G})$, where $x := \pi(p)$. Thence, $r_{\sigma*}(X) = r_{\sigma_x*}(X) \in \mathrm{H}_{p\cdot\sigma_x}\mathscr{P}$, therefore we derive by Prop. 6.41 and dimensional reasons that $r_{\sigma_x*}(\mathrm{H}_p\mathscr{P}) = \mathrm{H}_{p\cdot\sigma_x}\mathscr{P}$, and so $\mathrm{H}\mathscr{P}$ is an Ehresmann connection. ■

We conclude this subsection with the following technical and useful corollary.

Corollary 6.54: Commutation of modified push-forward and projections

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \to M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$ on $\mathscr{G}$ and $\mathrm{H}\mathscr{P}$ an Ehresmann connection on $\mathscr{P}$; denote with π_h and π_v the associated projections on the horizontal and vertical bundle of $\mathscr{P}$, respectively. Then we have

$$\mathscr{r}_{g*} \circ \pi_h = \pi_h \circ \mathscr{r}_{g*},$$

$$\mathscr{r}_{g*} \circ \pi_v = \pi_v \circ \mathscr{r}_{g*}$$

for all $g \in \mathscr{G}$.

Remarks 6.55.
This extends of course to (local) sections $\sigma \in \Gamma(\mathscr{G})$, and as discussed in Remark 6.26 we can extend these equations to vector fields on $\mathscr{P}$ via pullbacks, that is,

$$\mathscr{r}_{\sigma*} \circ \pi_h = r_\sigma^* \pi_h \circ \mathscr{r}_{\sigma*},$$

$$\mathscr{r}_{\sigma*} \circ \pi_v = r_\sigma^* \pi_v \circ \mathscr{r}_{\sigma*},$$

making use of that $\mathscr{r}_{\sigma*}$ is an isomorphism over $r_{\sigma*}$, recall Prop. 6.41.

Proof.
By Cor. 6.18 (also recall Remark 6.43) and Def. 6.45 we have

$$\mathscr{r}_{g*}(\mathrm{V}_p\mathscr{P}) = \mathrm{V}_{p\cdot g}\mathscr{P},$$

$$\mathscr{r}_{g*}(\mathrm{H}_p\mathscr{P}) = \mathrm{H}_{p\cdot g}\mathscr{P},$$

thus, $\mathscr{r}_{g*}$ preserves the splitting $\mathrm{T}\mathscr{P} = \mathrm{H}\mathscr{P} \oplus \mathrm{V}\mathscr{P}$. This concludes the proof. ∎

6.3. Gauge transformations

Let us now look at how gauge transformations of A look like in this setting; for this recall the definition of gauge transformations in Def. 6.5. As in Remark 3.17 we expect a relationship between gauge transformations and certain LGB valued maps. For the following also recall Ex. 6.40.

Definition 6.56: LGB-valued conjugation maps

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle. Then we define the group $C^\infty(\mathscr{P};\mathscr{G})^\mathscr{G}$ **of $\mathscr{G}$-valued conjugation maps** as a set by

$$C^\infty(\mathscr{P};\mathscr{G})^\mathscr{G} = \left\{\sigma \in \Gamma(\pi^*\mathscr{G}) \;\middle|\; \sigma_{p\cdot g} = \mathscr{c}_{g^{-1}}(\sigma_p) \text{ for all } (p,g) \in \mathscr{P} * \mathscr{G}\right\}.$$

Its group structure is inherited by the point-wise group structure of $\Gamma(\pi^*\mathscr{G})$; recall Cor. 3.1.

Remark 6.57: Group structure on $C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$

It is trivial to check that $C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$ is indeed a subgroup of $\Gamma(\pi^*\mathscr{G})$.

$C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$ canonically acts on $\mathscr{P}$ on the right via the given $\mathscr{G}$-action: Let $\sigma \in C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$, then we define

$$p \cdot \sigma_p \coloneqq p \cdot \mathrm{pr}_2(\sigma_p) \tag{51}$$

for all $p \in \mathscr{P}$, where $\mathrm{pr}_2 : \pi^*\mathscr{G} \to \mathscr{G}$ is the projection onto the second component. In essence, we drop the notation of pr_2, as if we view σ as a map $\mathscr{P} \to \mathscr{G}$ of fibre bundles over π; recall Subsection 1.1. If we denote the right $\mathscr{G}$-action on $\mathscr{P}$ by Φ, then observe by $\sigma_p = (p, \mathrm{pr}_2(\sigma_p))$ that we could also write

$$p \cdot \sigma_p = \Phi\big(p, \mathrm{pr}_2(\sigma_p)\big) = \Phi(\sigma_p).$$

As in the case of typical/classical principal bundles, we have the following statement.

Proposition 6.58: Gauge transformations as $\mathscr{G}$-valued conjugation maps

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle. Then there is a well-defined group isomorphism of gauge transformations and $\mathscr{G}$-valued conjugation maps given by

$$\mathscr{Aut}(\mathscr{P}) \to C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}},$$

$$H \mapsto \sigma^H,$$

where $\sigma^H \in C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$ is defined by

$$H(p) = p \cdot \sigma^H_p$$

for all $p \in \mathscr{P}$.

Proof.
This result can be proven similarly as for Lie group based principal bundles; however smoothness needs to be discussed a bit. In the following we make use of that pullback manifolds like $\mathscr{P} * \mathscr{G}$ are embedded submanifolds of product manifolds like $\mathscr{P} \times \mathscr{G}$, we will not further mention it.

• First of all, due to the fact that H is base-preserving we know that $H(p)$ is in the same fibre as p ($p \in \mathscr{P}$), and due to that the $\mathscr{G}$-action on $\mathscr{P}$ is simply transitive there is a unique element of $\pi^*\mathscr{G}|_p$, denoted by σ^H_p, such that

$$H(p) = p \cdot \sigma^H_p = p \cdot \mathrm{pr}_2\big(\sigma^H_p\big),$$

where $\mathrm{pr}_2 : \mathscr{P} * \mathscr{G} = \pi^* \mathscr{G} \to \mathscr{G}$ is the smooth projection onto the second component. Let us first show smoothness of $p \mapsto \sigma_p^H$: We can write

$$H(p) = \Phi_p\big(\sigma_p^H\big),$$

where $\Phi_p : \mathscr{G}_x \to \mathscr{P}_x$ is the orbit map through $p \in \mathscr{P}_x$ ($x \in M$). By Remark 6.2, Φ_p is a $\mathscr{G}_x$-equivariant diffeomorphism, especially invertible, so that

$$\mathrm{pr}_2\big(\sigma_p^H\big) = (\Phi_p)^{-1}\big(H(p)\big).$$

Let us define the map

$$\pi^* \mathscr{P} \to \mathscr{G},$$

$$\big(p, p'\big) \mapsto \Phi^{-1}\big(p, p'\big) := (\Phi_p)^{-1}\big(p'\big)$$

which is a map over π by construction. If we can show smoothness of Φ^{-1}, then smoothness of σ^H follows additionally due to smoothness of H and

$$\sigma_p^H = \big(p, \mathrm{pr}_2\big(\sigma_p^H\big)\big) = \Big(p, (\Phi_p)^{-1}\big(H(p)\big)\Big) = \Big(p, \Phi^{-1}\big(p, H(p)\big)\Big).$$

Observe that we have

$$\mathrm{pr}_2(p, g) = g = (\Phi_p)^{-1}\big(\Phi_p(g)\big) = \Phi^{-1}\big(p, \Phi(p, g)\big)$$

for all $(p, g) \in \mathscr{P} * \mathscr{G}$, where Φ denotes the right $\mathscr{G}$-action on $\mathscr{P}$. The map

$$L : \mathscr{P} * \mathscr{G} \to \pi^* \mathscr{P},$$

$$(p, g) \mapsto \big(p, \Phi(p, g)\big),$$

is a base-preserving principal bundle isomorphism (w.r.t. the LGB isomorphism given as the identity map on $\mathscr{P} * \mathscr{G}$), recall Cor. 6.12 and Remark 6.13. Thus, we have

$$\Phi^{-1} = \mathrm{pr}_2 \circ L^{-1},$$

so Φ^{-1} is smooth, and therefore σ^H is smooth, too.

- That σ^H is an element of $C^\infty(\mathscr{P}; \mathscr{G})^{\mathscr{G}}$ follows as usual:

$$(p \cdot g) \cdot \mathrm{pr}_2\big(\sigma_{p \cdot g}^H\big) = f(p \cdot g) = f(p) \cdot g = p \cdot \big(\mathrm{pr}_2\big(\sigma_p^H\big)\ g\big)$$

for all $(p, g) \in \mathscr{P} * \mathscr{G}$, so that, by the simply transitivity of the action,

$$g\ \mathrm{pr}_2\big(\sigma_{p \cdot g}^H\big) = \mathrm{pr}_2\big(\sigma_p^H\big)\ g,$$

and thus

$$\sigma_{p \cdot g}^H = \Big(p \cdot g, \mathrm{pr}_2\big(\sigma_{p \cdot g}^H\big)\Big) = \Big(p \cdot g, g^{-1}\ \mathrm{pr}_2\big(\sigma_p^H\big)\ g\Big) = c_{g^{-1}}\big(\sigma_p^H\big).$$

Thus, $\sigma^H \in C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$.

- The inverse of $H \mapsto \sigma^H$ is clearly given by

$$C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}} \to \mathscr{Aut}(\mathscr{P}),$$

$$\sigma \mapsto H^\sigma,$$

where

$$H^\sigma(p) := p \cdot \sigma_p$$

for all $p \in \mathscr{P}$. Smoothness here is now obvious, and we have

$$H^\sigma(p \cdot g) = p \cdot g \underbrace{\sigma_{p \cdot g}}_{= c_{g^{-1}}(\sigma_p)} = p \cdot g \underbrace{\mathrm{pr}_2\big(c_{g^{-1}}(\sigma_p)\big)}_{= c_{g^{-1}}(\mathrm{pr}_2(\sigma_p))} = p \cdot \mathrm{pr}_2(\sigma_p)\ g = p \cdot \sigma_p\ g = H^\sigma(p) \cdot g$$

for all $(p, g) \in \mathscr{P} * \mathscr{G}$. Thus, $H^\sigma \in \mathscr{Aut}(\mathscr{P})$, and so this finishes the proof. ■

Using this, we can finally formulate the gauge transformations of connection 1-forms; for this recall Cor. 6.9 and the Darboux derivative, Def. 6.25. Also recall the pullback Darboux derivative, Remark 6.29.

Theorem 6.59: Gauge transformations of connection 1-forms

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a horizontal distribution on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Furthermore, let $H \in \mathscr{Aut}(\mathscr{P})$. We then have that $H^! A$ is a connection 1-form on $\mathscr{P}$ and

$$H^! A = \mathscr{Ad}_{\mathrm{pr}_2 \circ (\sigma^H)^{-1}} \circ A + (\pi^*\Delta)\sigma^H,$$

where $\sigma^H \in C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$ is defined as in Prop. 6.58 and $\mathrm{pr}_2 : \pi^*\mathscr{G} \to \mathscr{G}$ is the projection onto the second component.
Similar to Def. (51) we may shortly just write

$$H^! A = \mathscr{Ad}_{(\sigma^H)^{-1}} \circ A + (\pi^*\Delta)\sigma^H.$$

Proof.
First of all observe that $H^! A \in \Omega^1(\mathscr{P}; H^*\pi^*\mathscr{g})$, and trivially $H^*\pi^*\mathscr{g} \cong (\pi \circ H)^*\mathscr{g} = \pi^*\mathscr{g}$. We will make use of this and similar isomorphisms in the following without further mentioning it. We will often not bookkeep the basepoint component in the following pairs and triples; so, everything has to be read in such a way that we have again values in the correct space. It would just be cumbersome to keep track of all these natural isomorphisms, and we decided for readability to avoid this bookkeeping most of the time.

- We then have

$$\begin{aligned}
\left(\mathscr{r}_\sigma^! H^! A\right)_p (X) &= \left(H^! A\right)_{p \cdot \sigma_x} \left(\mathscr{r}_{\sigma *}(X)\right) \\
&= \left(H^! A\right)_{p \cdot \sigma_x} \left(\mathrm{D}_p r_\sigma(X) - \left.\overbrace{\left(\pi^! \Delta\sigma\right)\Big|_p (X)}\right|_{p \cdot \sigma_x} \right) \\
&= A_{H(p \cdot \sigma_x)} \left(\mathrm{D}_{p \cdot \sigma_x} H \left(\mathrm{D}_p r_\sigma(X) - \left.\overbrace{\left(\pi^! \Delta\sigma\right)\Big|_p (X)}\right|_{p \cdot \sigma_x} \right)\right)
\end{aligned}$$

for all (local) $\sigma \in \Gamma(\mathscr{G})$, $p \in \mathscr{P}_x$ ($x \in M$) and $X \in \mathrm{T}_p\mathscr{P}$. We also get by definition of $\mathscr{Aut}(\mathscr{P})$

$$H(p \cdot \sigma_x) = H(p) \cdot \sigma_x,$$

that is,

$$H \circ r_\sigma = r_\sigma \circ H$$

and thus

$$\mathrm{D}_{p \cdot \sigma_x} H \circ \mathrm{D}_p r_\sigma = \mathrm{D}_p (H \circ r_\sigma) = \mathrm{D}_p (r_\sigma \circ H) = \mathrm{D}_{H(p)} r_\sigma \circ \mathrm{D}_p H.$$

Now also observe that

$$\mathrm{D}_p H(\widetilde{\nu}_p) = (\mathrm{D}_p H \circ \mathrm{D}_{e_x} \Phi_p)(\nu) = \mathrm{D}_{e_x} \underbrace{(H \circ \Phi_p)}_{\mathscr{G}_x \ni g \mapsto H(p \cdot g) = H(p) \cdot g}(\nu) = \mathrm{D}_{e_x} \Phi_{H(p)}(\nu) = \widetilde{\nu}_{H(p)} \tag{52}$$

for all $\nu \in \mathscr{g}_x$, where Φ_p is the orbit map through p, and due to that H is base-preserving we derive

$$\left(\pi^! \Delta\sigma\right)\Big|_{H(p)} \left(\mathrm{D}_p H(X)\right) = \left(H^! \pi^! \Delta\sigma\right)_p (X) = \left((\pi \circ H)^! \Delta\sigma\right)_p (X) = \left(\pi^! \Delta\sigma\right)_p (X).$$

In total we get

$$\begin{aligned}
\left(\mathscr{r}_\sigma^! H^! A\right)_p (X) &= A_{H(p) \cdot \sigma_x} \left(\mathrm{D}_{H(p)} r_\sigma \big(\mathrm{D}_p H(X)\big) - \left.\overbrace{\left(\pi^! \Delta\sigma\right)\Big|_{H(p)} \big(\mathrm{D}_p H(X)\big)}\right|_{H(p) \cdot \sigma_x} \right) \\
&= A_{H(p) \cdot \sigma_x} \big(\mathscr{r}_{\sigma *} \big(\mathrm{D}_p H(X)\big)\big) \\
&= \left(\mathscr{r}_\sigma^! A\right)_{H(p)} \big(\mathrm{D}_p H(X)\big) \\
&= \left(\mathscr{Ad}_{\sigma^{-1}} \circ A\right)_{H(p)} \big(\mathrm{D}_p H(X)\big) \\
&= \Big(H(p) \cdot \sigma_x,\ \mathrm{Ad}_{\sigma_x^{-1}} \Big(\widehat{A}_{H(p)} \big(\mathrm{D}_p H(X)\big)\Big)\Big)
\end{aligned}$$

$$= \mathscr{Ad}_{\sigma^{-1}}\Big(H(p),\ \widehat{A}_{H(p)}(\mathrm{D}_p H(X))\Big)$$
$$= \mathscr{Ad}_{\sigma^{-1}}\Big(\Big(H^! A\Big)_p(X)\Big),$$

where we wrote $A = \Big(p, \widehat{A}_p\Big)$ with $\widehat{A} := \pi_2 \circ A$ ($\pi_2 = \mathrm{Dpr}_2|_{\pi^*\mathscr{g}} : \pi^*\mathscr{g} \to \mathscr{g}$ the projection onto the second component), and thus

$$r_\sigma^! H^! A = \mathscr{Ad}_{\sigma^{-1}} \circ H^! A.$$

By Eq. (52) we also get $\mathrm{D}H(\widetilde{\nu}) = H^*\widetilde{\nu}$ for all (local) $\nu \in \Gamma(\mathscr{g})$ and thus

$$\Big(H^! A\Big)(\widetilde{\nu}) = (H^* A)\big(\mathrm{D}H(\widetilde{\nu})\big) = (H^* A)(H^*\widetilde{\nu}) = H^*\big(A(\widetilde{\pi})\big) = H^*\pi^*\nu = (\pi \circ H)^*\nu = \pi^*\nu.$$

Hence, $H^! A$ is a connection 1-form on $\mathscr{P}$.

- For the last part recall Prop. 6.58, especially we have a unique $\sigma^H \in C^\infty(\mathscr{P}; \mathscr{G})^{\mathscr{G}}$ such that

$$H(p) = p \cdot \sigma_p^H = p \cdot \mathrm{pr}_2\big(\sigma_p^H\big) = \Phi\big(p, \mathrm{pr}_2\big(\sigma_p^H\big)\big) = \Big(\Phi \circ \big(\mathbb{1}_{\mathscr{P}}, \mathrm{pr}_2 \circ \sigma^H\big)\Big)(p)$$

for all $p \in \mathscr{P}_x$ ($x \in M$), where Φ is the right $\mathscr{G}$-action on $\mathscr{P}$, also recall Def. 51, that is, now $\mathrm{pr}_2 : \pi^*\mathscr{G} \to \mathscr{G}$ is the projection onto the second component. Define $\widetilde{\sigma} := \mathrm{pr}_2 \circ \sigma^H$, $\mathscr{P} \ni p \mapsto \widetilde{\sigma}_p \in \mathscr{G}$; by Remark 6.22, 6.29 and Thm. 5.21, especially Remark 6.27, we can calculate

$$\begin{aligned}
\mathrm{D}_p H(X) &= \mathrm{D}_{(p,\widetilde{\sigma}_p)}\Phi\big(X, \mathrm{D}_p\widetilde{\sigma}(X)\big) \\
&= \mathrm{D}_p r_\sigma(X) - \left.\overline{\Big(\pi^!\Delta\sigma\Big)\Big|_p(X)}\right|_{p\cdot\widetilde{\sigma}_p} + \left.\overline{\big(\mu_{\mathscr{G}}^{\mathrm{tot}}\big)_{\widetilde{\sigma}_p}\big(\mathrm{D}_p\widetilde{\sigma}(X)\big)}\right|_{p\cdot\widetilde{\sigma}_p} \\
&= r_{\widetilde{\sigma}_p *}(X) + \left.\overline{\big(\mu_{\mathscr{G}}^{\mathrm{tot}}\big)_{\mathrm{pr}_2(\sigma_p^H)}\Big(\Big(\mathrm{D}_{\sigma_p^H}\mathrm{pr}_2 \circ \mathrm{D}_p\sigma^H\Big)(X)\Big)}\right|_{H(p)} \\
&= r_{\widetilde{\sigma}_p *}(X) + \left.\overline{\Big(\mathrm{pr}_2^!\mu_{\mathscr{G}}^{\mathrm{tot}}\Big)_{\sigma_p^H}\big(\mathrm{D}_p\sigma^H(X)\big)}\right|_{H(p)} \\
&= r_{\widetilde{\sigma}_p *}(X) + \left.\overline{\big(\mu_{\pi^*\mathscr{G}}^{\mathrm{tot}}\big)_{\sigma_p^H}\big(\mathrm{D}_p\sigma^H(X)\big)}\right|_{H(p)} \\
&= r_{\widetilde{\sigma}_p *}(X) + \left.\overline{\Big(\Delta^{\pi^*\mathscr{G}}\sigma^H\Big)_p(X)}\right|_{H(p)} \\
&= r_{\widetilde{\sigma}_p *}(X) + \left.\overline{\big((\pi^*\Delta)\sigma^H\big)_p(X)}\right|_{H(p)}
\end{aligned} \tag{53}$$

for all $X \in \mathrm{T}_p\mathscr{P}$, where σ is a (local) section of $\mathscr{G}$ so that $\sigma_x = \widetilde{\sigma}_p = \mathrm{pr}_2\big(\sigma_p^H\big)$.

Then

$$
\begin{aligned}
\left(H^! A\right)_p(X) &= A_{H(p)}\big(\mathrm{D}_p H(X)\big) \\
&= A_{H(p)}\left(r_{\widetilde{\sigma}_p *}(X) + \left.\overbrace{\big((\pi^*\Delta)\sigma^H\big)_p(X)}\right|_{H(p)} \right) \\
&= A_{p\cdot\widetilde{\sigma}_p}\big(r_{\widetilde{\sigma}_p *}(X)\big) + \big((\pi^*\Delta)\sigma^H\big)_p(X) \\
&= \left(r^!_{\widetilde{\sigma}_p} A\right)_p(X) + \big((\pi^*\Delta)\sigma^H\big)_p(X) \\
&= \mathscr{Ad}_{\widetilde{\sigma}_p^{-1}}\big(A_p(X)\big) + \big((\pi^*\Delta)\sigma^H\big)_p(X) \\
&= \mathscr{Ad}_{\mathrm{pr}_2\left(\left(\sigma_p^H\right)^{-1}\right)}\big(A_p(X)\big) + \big((\pi^*\Delta)\sigma^H\big)_p(X),
\end{aligned}
$$

which finishes the proof, where we used the trivial relation (alternatively recall Cor. 3.1)

$$\left(\mathrm{pr}_2\left(\sigma_p^H\right)\right)^{-1} = \mathrm{pr}_2\left(\left(\sigma_p^H\right)^{-1}\right).$$

■

Of course there is also the sense of gauge transformation with respect to gauges as in typical gauge theory (see *e.g.* [1, §5.4, page 270ff.]), recall Def. 6.15.

Definition 6.60: Local gauge field

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a horizontal distribution on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Furthermore, let $s \in \Gamma(\mathscr{P}|_U)$ be a (local) gauge over an open subset $U \subset M$. Then we define the **local connection 1-form** of **local gauge field** $A_s \in \Omega^1(U; \mathscr{g}|_U)$ **(w.r.t.** s**)** by

$$A_s := s^! A.$$

Remarks 6.61.
A has values in $\pi^*\mathscr{g}$, and thus A_s has values in $s^*\pi^*\mathscr{g} \cong (\pi \circ s)^*\mathscr{g} = \mathbb{1}_U^*\mathscr{g} \cong \mathscr{g}$.

Gauge transformations now naturally arise in a change of the gauge s. So, let U_i and U_j be two open subsets of M so that $U_i \cap U_j \neq \emptyset$. For two gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$ and $s_j \in \Gamma\big(\mathscr{P}|_{U_j}\big)$ there is then a unique $\sigma_{ji} \in \Gamma\Big(\mathscr{G}|_{U_i \cap U_j}\Big)$ such that

$$s_i = s_j \cdot \sigma_{ji} = \Phi_{s_j} \circ \sigma_{ji}$$

on $U_i \cap U_j$, where Φ_{s_j} is the orbit through s_j, especially recall Lemma 6.7. The unique existence is clear by the definition of principal bundles, while the smoothness follows by Lemma 6.7 so that we can write σ_{ji} as the composition of smooth maps

$$\sigma_{ji} = \Phi_{s_j}^{-1} \circ s_i.$$

In the following we also introduce the notation

$$\left(\Phi_{s_j}^{-1}\right)^{-1}(p) := \left(\Phi_{s_j}^{-1}(p)\right)^{-1}$$

for all $p \in \mathscr{P}_{U_j}$, that is, $\Phi_{s_j}^{-1}$ is the inverse of the orbit map Φ_{s_j}, but $(\cdot)^{-1}$ is the inverse of $\Phi_{s_j}^{-1}(p)$ as an element of $\mathscr{G}_{U_j}$.

Instead of calculating directly how A_{s_i} and A_{s_j} are related we want to use Thm. 6.59. For this it will be useful to understand that gauge transformations $\mathscr{A}ut(\mathscr{P})$ are locally isomorphic to the group of sections $\Gamma(\mathscr{G})$ via a gauge, analogously to the typical formulation of gauge theory as *e.g.* illustrated in [1, §5.3.2, page 268f.]. While the isomorphism of $\mathscr{A}ut(\mathscr{P})$ to $C^\infty(\mathscr{P}; \mathscr{G})^{\mathscr{G}} \subset \Gamma(\pi^* \mathscr{G})$ is global, the following isomorphism is in general only local.

Proposition 6.62: Gauge transformations and sections of the structural LGB

Let $\mathscr{G} \overset{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M, and $\mathscr{P} \overset{\pi}{\to} M$ a principal $\mathscr{G}$-bundle. Also let $s \in \Gamma(\mathscr{P}|_U)$ be a gauge defined over some open subset U of M. Then we have a group isomorphism given by

$$C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U} \to \Gamma(\mathscr{G}_U),$$

$$\sigma \mapsto \mathrm{pr}_2 \circ \sigma \circ s,$$

where $\mathrm{pr}_2 : \pi^* \mathscr{G} \to \mathscr{G}$ is the projection onto the second component, with inverse

$$\Gamma(\mathscr{G}_U) \to C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U},$$

$$\tau \mapsto \sigma^\tau,$$

where

$$\sigma^\tau_p := \left(p, c_{\left(\Phi_s^{-1}(p)\right)^{-1}}\left(\tau_{\pi(p)}\right)\right) = \left(p, \left(\Phi_s^{-1}(p)\right)^{-1} \tau_{\pi(p)} \, \Phi_s^{-1}(p)\right)$$

for all $p \in \mathscr{P}_U$, with Φ_s being the orbit map through s.

Remark 6.63: Other notation

Due to the fact that $s^* \pi^* \mathscr{G}_U \cong (\pi \circ s)^* \mathscr{G}_U = \mathbb{1}_U^* \mathscr{G}_U \cong \mathscr{G}_U$, and due to what we discussed in Subsection 1.1 about pullback sections, we could rewrite the first map to

$$C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U} \to \Gamma(\mathscr{G}_U),$$

$$\sigma \mapsto s^* \sigma.$$

Proof of Prop. 6.62.
For $\sigma \in C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U} \subset \Gamma(\pi^* \mathscr{G}_U)$ observe that $\mathrm{pr}_2 \circ \sigma \circ s$ is by construction a section of $\mathscr{G}_U$ due to

$$\pi_{\mathscr{G}} \circ \mathrm{pr}_2 \circ \sigma \circ s = \pi \circ \mathrm{pr}_1 \circ \sigma \circ s = \pi \circ s = \mathbb{1}_U,$$

where $\mathrm{pr}_1 : \pi^* \mathscr{G} \to \mathscr{P}$ is the projection onto the first component. For the supposed inverse we have similarly that σ^τ is a section of $\Gamma(\pi^* \mathscr{G}_U)$ by construction; then observe, using Lemma 6.7,

$$\begin{aligned}
\sigma^\tau_{p \cdot g} &= \left(p \cdot g, c_{\left(\Phi_s^{-1}(p \cdot g)\right)^{-1}}\left(\tau_{\pi(p \cdot g)}\right) \right) \\
&= \left(p \cdot g, c_{g^{-1}\ \left(\Phi_s^{-1}(p)\right)^{-1}}\left(\tau_{\pi(p)}\right) \right) \\
&= \left(p \cdot g, \left(c_{g^{-1}} \circ c_{\left(\Phi_s^{-1}(p)\right)^{-1}} \right)\left(\tau_{\pi(p)}\right) \right) \\
&= \mathscr{c}_{g^{-1}}\left(p, c_{\left(\Phi_s^{-1}(p)\right)^{-1}}\left(\tau_{\pi(p)}\right) \right) \\
&= \mathscr{c}_{g^{-1}}\left(\sigma^\tau_p\right)
\end{aligned}$$

for all $(p, g) \in \mathscr{P}_U * \mathscr{G}_U$, thus, $\sigma^\tau \in C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U}$. Recall

$$p = s_{\pi(p)} \cdot \Phi_s^{-1}(p)$$

for all $p \in \mathscr{P}_U$, so that

$$\Phi_s^{-1} \circ s = e|_U,$$

where e is the section of the neutral element of $\mathscr{G}$. Then

$$(\mathrm{pr}_2 \circ \sigma^\tau)(s_x) = \mathrm{pr}_2\left(\sigma^\tau_{s_x}\right) = c_{\left(\Phi_s^{-1}(s_x)\right)^{-1}}\left(\tau_{\pi(s_x)}\right) = c_{e_x}(\tau_x) = \tau_x$$

for all $x \in M$ and $\tau \in \Gamma(\mathscr{G}_U)$, and by Def. 6.56 and Ex. 6.40 we have

$$\begin{aligned}
\sigma^{\mathrm{pr}_2 \circ \sigma \circ s}_p &= \left(p, \left(c_{\left(\Phi_s^{-1}(p)\right)^{-1}} \circ \mathrm{pr}_2 \right)\left(\sigma_{s_{\pi(p)}} \right) \right) \\
&= \left(p, \left(c_{\left(\Phi_s^{-1}(p)\right)^{-1}} \circ \mathrm{pr}_2 \right)\left(\sigma_{p \cdot \left(\Phi_s^{-1}(p)\right)^{-1}} \right) \right) \\
&= \left(p, \left(c_{\left(\Phi_s^{-1}(p)\right)^{-1}} \circ \mathrm{pr}_2 \circ \mathscr{c}_{\Phi_s^{-1}(p)} \right)(\sigma_p) \right) \\
&= \left(p, \left(c_{\left(\Phi_s^{-1}(p)\right)^{-1}} \circ c_{\Phi_s^{-1}(p)} \circ \mathrm{pr}_2 \right)(\sigma_p) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \big(p, \mathrm{pr}_2(\sigma_p)\big) \\
&= \sigma_p
\end{aligned}
$$

for all $p \in \mathscr{P}_U$ and $\sigma \in C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U}$. Thus, $\sigma \mapsto \mathrm{pr}_2 \circ \sigma \circ s$ is bijective with inverse $\tau \mapsto \sigma^\tau$, and so it is only left to show that $\sigma \mapsto \mathrm{pr}_2 \circ \sigma \circ s$ is a group homomorphism. For $\sigma' \in C^\infty(\mathscr{P}_U; \mathscr{G}_U)^{\mathscr{G}_U}$ we have

$$
\sigma\sigma'\big|_p = \big(p, \mathrm{pr}_2(\sigma_p)\ \mathrm{pr}_2(\sigma'_p)\big),
$$

thus,

$$
\big(\mathrm{pr}_2 \circ (\sigma\sigma') \circ s\big)_x = \mathrm{pr}_2(\sigma_{s_x})\ \mathrm{pr}_2(\sigma'_{s_x}) = \big((\mathrm{pr}_2 \circ \sigma \circ s) \cdot (\mathrm{pr}_2 \circ \sigma' \circ s)\big)_x
$$

for all $x \in M$. This finishes the proof. ■

Proposition 6.64: Change of gauge as a local bundle automorphism

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle. Also let U_i and U_j be two open subsets of M so that $U_i \cap U_j \neq \emptyset$, two gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$ and $s_j \in \Gamma(\mathscr{P}|_{U_j})$, and the unique $\sigma_{ji} \in \Gamma\left(\mathscr{G}|_{U_i \cap U_j}\right)$ with $s_i = s_j \cdot \sigma_{ji}$ on $U_i \cap U_j$. Then the map H_{ji} defined by

$$
\begin{aligned}
\mathscr{P}|_{U_i \cap U_j} &\to \mathscr{P}|_{U_i \cap U_j}, \\
p &\mapsto s_i|_{\pi(p)} \cdot \Phi_{s_j}^{-1}(p),
\end{aligned}
$$

is an element of $\mathscr{A}ut\left(\mathscr{P}|_{U_i \cap U_j}\right)$, where Φ_{s_j} is the orbit map through s_j. Using the notation of Prop. 6.58, we also get that $\sigma^{H_{ji}}$ is related to σ_{ji} via s_j in sense of Prop. 6.62, that is,

$$
\sigma_p^{H_{ji}} = \left(p, c_{\left(\Phi_{s_j}^{-1}(p)\right)^{-1}}\left(\sigma_{ji}|_{\pi(p)}\right)\right)
$$

for all $p \in \mathscr{P}_{U_i \cap U_j}$, and

$$
\sigma_{ji} = \mathrm{pr}_2 \circ \sigma^{H_{ji}} \circ s_j,
$$

where $\mathrm{pr}_2 : \pi^*\mathscr{G} \to \mathscr{G}$ is the projection onto the second component.

Proof.
Recall Lemma 6.7 for the following proof.

We have $s_i|_{\pi(p)} \in \mathscr{P}_{\pi(p)}$ and Φ_{s_j} is base-preserving, and thus $\Phi_{s_j}^{-1}(p) \in \mathscr{G}_{\pi(p)}$, so that H_{ji} is well-defined and also base-preserving because the right $\mathscr{G}$-action on $\mathscr{P}$ preserves the fibres. H_{ji} is clearly smooth by construction. Finally,

$$
H_{ji}(p \cdot g) = s_i|_{\pi(p \cdot g)} \cdot \Phi_{s_j}^{-1}(p \cdot g) = s_i|_{\pi(p)} \cdot \Phi_{s_j}^{-1}(p) \cdot g = H_{ji}(p) \cdot g
$$

for all $(p,g) \in (\mathscr{P} * \mathscr{G})|_{U_i \cap U_j}$, thus, H_{ji} is a principal bundle morphism. The inverse H_{ji}^{-1} is given by

$$H_{ji}^{-1} = H_{ij}$$

due to

$$(H_{ij} \circ H_{ji})(p) = s_j|_x \cdot \Phi_{s_i}^{-1}\Big(s_i|_x \cdot \Phi_{s_j}^{-1}(p)\Big) = s_j|_x \cdot \underbrace{\Phi_{s_i}^{-1}(s_i|_x)}_{=e_x} \cdot \Phi_{s_j}^{-1}(p) = s_j|_x \cdot \Phi_{s_j}^{-1}(p) = p$$

for all $p \in \mathscr{P}_{U_i \cap U_j}$, where $x \coloneqq \pi(p)$; by symmetry of course similarly when swapping j and i. Thus, H_{ji} is bijective and its inverse smooth, so that $H_{ji} \in \mathscr{Aut}\big(\mathscr{P}_{U_i \cap U_j}\big)$.

Now write

$$\begin{aligned}
H_{ji}(p) &= s_i|_x \cdot \Phi_{s_j}^{-1}(p) \\
&= s_j|_x \cdot \sigma_{ji}|_x \cdot \Phi_{s_j}^{-1}(p) \\
&= s_j|_x \cdot \Phi_{s_j}^{-1}(p) \cdot \Big(\Phi_{s_j}^{-1}(p)\Big)^{-1} \cdot \sigma_{ji}|_x \cdot \Phi_{s_j}^{-1}(p) \\
&= p \cdot c_{\left(\Phi_{s_j}^{-1}(p)\right)^{-1}}\big(\sigma_{ji}|_x\big),
\end{aligned}$$

and due to $H_{ji}(p) = p \cdot \mathrm{pr}_2\Big(\sigma_p^{H_{ji}}\Big)$ we can immediately derive

$$\sigma_p^{H_{ji}} = \left(p, c_{\left(\Phi_{s_j}^{-1}(p)\right)^{-1}}\Big(\sigma_{ji}|_{\pi(p)}\Big)\right),$$

thus, $\sigma^{H_{ji}}$ is related to σ_{ji} w.r.t. s_j in sense of Prop. 6.62 and so the remaining part of the proposition's statement follows by Prop. 6.62. ■

Using this gauge transformation we can now relate A_{s_i} and A_{s_j}.

Theorem 6.65: Gauge transformations as a change of gauge in the local gauge field

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a horizontal distribution on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^* \mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Also let U_i and U_j be two open subsets of M so that $U_i \cap U_j \neq \emptyset$, two gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$ and $s_j \in \Gamma\big(\mathscr{P}|_{U_j}\big)$, and the unique $\sigma_{ji} \in \Gamma\Big(\mathscr{G}|_{U_i \cap U_j}\Big)$ with $s_i = s_j \cdot \sigma_{ji}$ on $U_i \cap U_j$.
Then we have

$$A_{s_i} = \mathrm{Ad}_{\sigma_{ji}^{-1}} \circ A_{s_j} + \Delta\sigma_{ji}.$$

Proof.
Define $H_{ji} \in \mathscr{Aut}\left(\mathscr{P}_{U_i \cap U_j}\right)$ as in Prop. 6.64 (also following its notation), then observe

$$\left(H_{ji} \circ s_j\right)_x = s_i|_x \cdot \underbrace{\Phi_{s_j}^{-1}\left(s_j|_x\right)}_{=e_x} = s_i|_x$$

for all $x \in U_i \cap U_j$, and thus

$$s_j^! H_{ji}^! A = \left(H_{ji} \circ s_j\right)^! A = A_{s_i}.$$

By Thm. 6.59 we also have

$$H_{ji}^! A = \mathscr{Ad}_{\mathrm{pr}_2 \circ \left(\sigma^{H_{ji}}\right)^{-1}} \circ A + (\pi^* \Delta) \sigma^{H_{ji}}.$$

We now want to apply $s_j^!$ on both hand sides. Recall Remark 6.61 and 6.63, we will now use several natural isomorphisms without further mention, especially the base-point component will be treated in a "sloppy" way; the advantage will be a cleaner notation, and this is the reason why $\mathscr{Ad}$ becomes Ad by applying $s_j^!$ on the right hand side. Using Prop. 6.64 and Remark 6.29 we can derive

$$\Delta \sigma_{ji} = \left((\pi \circ s_j)^* \Delta\right)\left(s_j^* \sigma^{H_{ji}}\right) = \left(s_j^* \pi^* \Delta\right)\left(s_j^* \sigma^{H_{ji}}\right) = s_j^!\left((\pi^* \Delta)\left(\sigma^{H_{ji}}\right)\right),$$

and

$$s_j^!\left(\mathscr{Ad}_{\mathrm{pr}_2 \circ \left(\sigma^{H_{ji}}\right)^{-1}} \circ A\right) = \mathrm{Ad}_{\mathrm{pr}_2 \circ \left(\sigma^{H_{ji}}\right)^{-1} \circ s_j} \circ s_j^! A = \mathrm{Ad}_{\left(\mathrm{pr}_2 \circ \sigma^{H_{ji}} \circ s_j\right)^{-1}} \circ s_j^! A = \mathrm{Ad}_{\sigma_{ji}^{-1}} \circ A_{s_j},$$

where we used again the trivial relation (alternatively recall Cor. 3.1)

$$\left(\mathrm{pr}_2 \circ \sigma^{H_{ji}}\right)^{-1} = \mathrm{pr}_2 \circ \left(\sigma^{H_{ji}}\right)^{-1}.$$

Altogether, concluding the proof with

$$A_{s_i} = s_j^! H_{ji}^! A = \mathrm{Ad}_{\sigma_{ji}^{-1}} \circ A_{s_j} + \Delta \sigma_{ji}.$$

■

This allows us to finally show the first relation to the infinitesimal gauge theory developed by Alexei Kotov and Thomas Strobl.

Remark 6.66: Integrating curved Yang-Mills gauge theories, part I

Let the situation be as in Thm. 6.65, then set

$$\sigma_{ji} := \mathrm{e}^{t\varepsilon}$$

for a $\varepsilon \in \Gamma\left(\mathscr{g}|_{U_i \cap U_j}\right)$ and $t \in \mathbb{R}$. Then we define the **infinitesimal gauge transforma-**

tion $\delta_\varepsilon A_{s_j}$ of A along ε (w.r.t. s_j) as an element of $\Omega^1\Big(U_i \cap U_j; \mathscr{g}|_{U_i\cap U_j}\Big)$ by

$$\delta_\varepsilon A_{s_j} \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A_{s_i}.$$

The corresponding derivative related to the adjoint representation can be calculated as usual, and for the corresponding term related to the Darboux derivative we use the vector bundle connection introduced in Def. 6.34. Thus, we get[a]

$$\delta_\varepsilon A_{s_j} = \nabla^{\mathscr{G}}\varepsilon - \left[\varepsilon, A_{s_j}\right]_{\mathscr{g}}.$$

This is *precisely* the infinitesimal gauge transformation developed by Alexei Kotov and Thomas Strobl, see [8] for a concise summary or [5] for an extended introduction. These references are for the very general situation using Lie algebroids, hence see alternatively [10] for this type of gauge theory restricted to Lie algebra bundles.
Thus, we successfully integrated this definition of infinitesimal gauge transformations.

[a] Rigorously, one has to define $\delta_\varepsilon A_{s_j}$ via the natural projection $\mathrm{V}\mathscr{g} \to \mathscr{g}$ as we did in Prop. 6.32.

Alexei's and Thomas's theory also generalizes the curvature/field strength related to gauge theory. Hence, let us now turn to this notion.

6.4. Generalized curvature/field strength

6.4.1. Yang-Mills connections on LGBs

In the following we will introduce the curvature; in order to achieve gauge invariance later in the physical gauge theory, we will now need to assume certain what we will call **compatibility conditions** on the horizontal distribution of the structural LGB $\mathscr{G}$. Here in this work we will just present these compatibility conditions, and you will be able to see in the proofs why this leads to gauge invariance later. However, if you want to understand the following conditions coming from a point of view not knowing the solution beforehand, then either see [8] or [5]; the latter also provides a physical motivation using what I called the **field redefinitions** which naturally motivate the gauge theory presented here by transforming a classical gauge theory in such a way that the physics is preserved; recall Def. 6.34 for the following.

We start with the infinitesimal version of the compatibility conditions, already known and pointed out in the previously-mentioned references.

Definition 6.67: Infinitesimal Yang-Mills connection

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$. Then we say that $\mathrm{H}\mathscr{G}$ is an **infinitesimal Yang-Mills connection (on $\mathscr{G}$)** if there is a $\zeta \in \Omega^2(M;\mathscr{g})$ such that the $\mathscr{G}$-connection $\nabla^{\mathscr{G}}$ on the associated LAB $\mathscr{g}$ with bracket $[\cdot,\cdot]_{\mathscr{g}}$ satisfies the **infinitesimal compatibility conditions**

$$\nabla^{\mathscr{G}}\left([\mu,\nu]_{\mathscr{g}}\right) = \left[\nabla^{\mathscr{G}}\mu,\nu\right]_{\mathscr{g}} + \left[\mu,\nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}}, \tag{54}$$

$$R_{\nabla^{\mathscr{G}}}(X,Y)\mu = [\zeta(X,Y),\mu]_{\mathscr{g}} \tag{55}$$

for all $\mu,\nu \in \Gamma(\mathscr{g})$ and $X,Y \in \mathfrak{X}(M)$, where $R_{\nabla^{\mathscr{G}}}$ is the curvature of $\nabla^{\mathscr{G}}$. We then also call $\nabla^{\mathscr{G}}$ an **(infinitesimal) Yang-Mills connection** and denote it by ∇^{YM}.
In the following ζ is always such a 2-form as given here, and even though it is not uniquely given we will treat it as if it is fixed for a given ∇^{YM}.

Remarks 6.68.
As illustrated in [5, §5.1] and [10] such connections also arise in [2, §7.2, page 271ff.]: There these connections are called **Lie derivation laws covering a coupling** due to the fact that these are important in the construction of "coupling" an LAB to a Lie algebroid via a Whitney sum in such a way that the outcome is again a Lie algebroid. There is a strong relationship of this study with the gauge theory we are going to define, however, we will not need this so that we will not repeat what has been discussed in [5, §5.1] and [10].

One may want to write that $\left(\nabla^{\mathrm{YM}},\zeta\right)$ is a Yang-Mills connection, giving an emphasis on the non-uniqueness of ζ. However, we are not going to do so, and the mentioned references and relations to Mackenzie's work actually show that there is just one object behind all of that describing this Yang-Mills connection and its relation to the following gauge theory. It would take too much time introducing all of this again; see the references.

The relation to Mackenzie's study can be useful for the interested reader to understand the importance of these compatibility conditions, too. In my opinion, the compatibility conditions together with a trivial what Mackenzie calls the obstruction class (and an ad-invariant fibre metric on the LAB) are needed to define Yang-Mills gauge theories from an axiomatic point of view. If the reader is interested into that, then I would appreciate an email so that I can send these details.

We want to integrate these infinitesimal compatibility conditions to conditions on $\mathrm{H}\mathscr{G}$ directly. Let us initially focus on the first infinitesmal compatibility condition (54). In the following we will also view $\mathscr{G}$ as a principal $\mathscr{G}$-bundle which is equipped with the same horizontal distribution as $\mathscr{G}$ as LGB; related notions and notations carry over. One can show that Ehresmann connections satisfy (54), and in fact Ehresmann connections on LGBs (and Lie groupoids) were already discussed [9]; and there is also a rather recent preprint related to a similar subject, see [11].

Lemma 6.69: Ehresmann connections induce Lie bracket derivations, [9, §4.5, Prop. 4.21]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ a horizontal distribution on $\mathscr{G}$. If $\mathrm{H}\mathscr{G}$ is an (Ehresmann) connection on $\mathscr{G}$ as principal bundle, then the infinitesimal compatibility condition (54) is satisfied.

Proof.
This can be quickly answered by recalling how the parallel transport behaves; recall that Def. 6.45 comes from Eq. (46) which itself stems from Eq. (45); that is, the associated parallel transport to an Ehresmann connection $\mathrm{H}\mathscr{G}$ satisfies Eq. (45). This means that the corresponding parallel transport $\mathrm{PT}^{\mathscr{G}}$ of $\mathscr{G}$ (omitting the notation of the involved curve) is a homomorphism, *i.e.*

$$\mathrm{PT}^{\mathscr{G}}(gq) = \mathrm{PT}^{\mathscr{G}}(g)\ \mathrm{PT}^{\mathscr{G}}(q)$$

for all $(g, q) \in \mathscr{G} * \mathscr{G}$. Thus, infinitesimally the associated parallel transport on $\mathscr{g}$ is a homomorphism w.r.t. the Lie bracket, and thus $\nabla^{\mathscr{G}}$ is a Lie bracket derivation. ■

Remark 6.70: Base manifold a horizontal leave

Observe that the proof's argument implies that a parallel transported neutral element is again the neutral element of the corresponding final fibre, if $\mathrm{H}\mathscr{G}$ is an Ehresmann connection. Thus, M naturally embedded via the neutral section e can then be viewed as a horizontal leave of $\mathrm{H}\mathscr{G}$.

Due to this, given compatibility condition (54), the following is clear by definition.

Corollary 6.71: The total Maurer-Cartan form as connection 1-form

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ an Ehresmann connection on $\mathscr{G}$. Then the total Maurer-Cartan form $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is the connection 1-form corresponding to $\mathrm{H}\mathscr{G}$.

Proof.
This simply follows by Def. 6.21 and Thm. 6.53. ■

Remark 6.72: Darboux derivative as local gauge field

If $\mathrm{H}\mathscr{G}$ an Ehresmann connection on $\mathscr{G}$, we know by Cor. 6.71 that the Darboux derivative $\Delta\sigma$ of a (local) section σ of $\mathscr{G}$ (Def. 6.25) is a local gauge field (w.r.t. σ) corresponding to $\mu_{\mathscr{G}}^{\mathrm{tot}}$, recall Def. 6.60. That is,

$$\Delta\sigma = \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{\sigma}.$$

Furthermore, following [9], we can derive that $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is a multiplicative form if and only if $\mathrm{H}\mathscr{G}$ is an Ehresmann connection; let us introduce multiplicative 1-forms. Recall Lemma 5.20.

Definition 6.73: Multiplicative forms, [12, §2.1, special situation of Def. 2.1]

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M. Then we call an $\omega \in \Omega^1\left(\mathscr{G}; \pi_{\mathscr{G}}^* \mathscr{g}\right)$ a **multiplicative form** if

$$\omega_{gq}\big(\mathrm{D}_{(g,q)}\Phi(X,Y)\big) = \mathscr{Ad}_{q^{-1}}\big(\omega_g(X)\big) + \big(gq, \widehat{\omega}_q(Y)\big)$$

for all $(g,q) \in \mathscr{G} * \mathscr{G}$ and $(X,Y) \in \mathrm{T}_{(g,q)}(\mathscr{G} * \mathscr{G})$, where $\Phi : \mathscr{G} * \mathscr{G} \to \mathscr{G}$ is the multiplication in $\mathscr{G}$, and $\widehat{\omega} := \mathrm{pr}_2 \circ \omega$, where $\mathrm{pr}_2 : \pi_{\mathscr{G}}^* \mathscr{g} \to \mathscr{g}$ is the natural projection onto the second component.

Remarks 6.74.
The shape of the second summand is again just because of technical reasons due to the definition of pullback bundles and their base points. If we do not care so much about the involved base points since these are clear by context, then we can also shortly write

$$\left(\Phi^! \omega\right)_{(g,q)}(X,Y) = \mathscr{Ad}_{q^{-1}}\big(\omega_g(X)\big) + \omega_q(Y).$$

Also recall what we mentioned in Subsection 1.1 about pullback sections.

Theorem 6.75: Ehresmann connection 1-forms on LGBs are multiplicative, [9, §4.4, implication of Lemma 4.14]

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ a horizontal distribution on $\mathscr{G}$ with corresponding total Maurer-Cartan form $\mu_{\mathscr{G}}^{\mathrm{tot}}$. Then $\mathrm{H}\mathscr{G}$ is an (Ehresmann) connection on $\mathscr{G}$ as principal bundle if and only if $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is multiplicative.

Proof.
The general approach in the mentioned reference is different to ours; hence, let us provide a proof suitable to our constructions. We will view $\mathscr{G}$ as a principal $\mathscr{G}$-bundle whose horizontal distribution is also $\mathrm{H}\mathscr{G}$. We will also use the same notation as in Def. 6.73.

To answer whether or not $\mathrm{H}\mathscr{G}$ is an Ehresmann connection we just need to discuss whether $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is a connection 1-form on $\mathscr{G}$ due to the fact that $\mathrm{H}\mathscr{G}$ is the kernel of $\mu_{\mathscr{G}}^{\mathrm{tot}}$ by definition (see also Cor. 6.71), also using Thm. 6.53. Also by definition, $\mu_{\mathscr{G}}^{\mathrm{tot}}$ already satisfies

$$\mu_{\mathscr{G}}^{\mathrm{tot}}(\widetilde{\nu}) = \pi_{\mathscr{G}}^* \nu.$$

By additionally using Thm. 5.21 (especially the associated Rem. 6.27) we show

$$\begin{aligned}\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{gq}\big(\mathrm{D}_{(g,q)}\Phi(X,Y)\big) &= \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{gq}\left(\mathrm{D}_g R_\sigma(X) - \left.\overline{\left(\pi_{\mathscr{G}}^! \Delta\sigma\right)\Big|_g (X)}\right|_{gq} + \left.\overline{\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_q (Y)}\right|_{gq}\right) \\ &= \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{gq}\big(r_{q*}(X)\big) + \left(gq, \left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}}\right)_q (Y)\right)\end{aligned}$$

$$= \left(\mathscr{r}_q^! \mu_{\mathscr{G}}^{\mathrm{tot}} \right)_g (X) + \left(gq, \left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}} \right)_q (Y) \right)$$

for all $(g, q) \in \mathscr{G} * \mathscr{G}$ and $(X, Y) \in \mathrm{T}_{(g,q)}(\mathscr{G} * \mathscr{G})$, where σ is a (local) section of $\mathscr{G}$ such that $\sigma_x = q$. Thus, $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is multiplicative if and only if

$$\mathscr{Ad}_{q^{-1}} \left(\left(\mu_{\mathscr{G}}^{\mathrm{tot}} \right)_g (X) \right) + \left(gq, \left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}} \right)_q (Y) \right) = \left(\mathscr{r}_q^! \mu_{\mathscr{G}}^{\mathrm{tot}} \right)_g (X) + \left(gq, \left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}} \right)_q (Y) \right)$$

if and only if

$$\mathscr{Ad}_{q^{-1}} \left(\left(\mu_{\mathscr{G}}^{\mathrm{tot}} \right)_g (X) \right) = \left(\mathscr{r}_q^! \mu_{\mathscr{G}}^{\mathrm{tot}} \right)_g (X)$$

if and only if $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is a connection 1-form. This finishes the proof. ■

This Theorem also implies the Leibniz rule of the Darboux derivative.

Proposition 6.76: Leibniz rule of the Darboux derivative

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ an Ehresmann connection on $\mathscr{G}$. Then we have

$$\Delta(\sigma\tau) = \mathrm{Ad}_{\tau^{-1}} \circ \Delta\sigma + \Delta\tau$$

and

$$\Delta\left(\sigma^{-1}\right) = -\mathrm{Ad}_{\sigma} \circ \Delta\sigma$$

for all $\sigma, \tau \in \Gamma(\mathscr{G})$.

Remarks 6.77.
These are precisely the formulas one expects, see *e.g.* [2, §5.1, Eq. (2) and (3), page 182] for the "classical" case; beware that the reference uses right- instead of left-invariant vector fields to describe the Lie algebra, so that the referenced formulas look explicitly a bit different. Nevertheless, the similarity of the structure should be obvious.

The mentioned reference for Thm. 6.75, [9, §4.4, Lemma 4.14], actually uses the Leibniz rule of the Darboux derivative as an equivalent description of Ehresmann connections.

Proof of Prop. 6.76.
For this calculation we will take care about base points of pullback bundles w.r.t. the Darboux derivative, thus recall the last part of Remark 6.26. Using the same notation as in Def. 6.73, this follows simply by Thm. 6.75, that is, set $Y := \mathrm{D}_x\sigma(X)$ and $Z := \mathrm{D}_x\tau(X)$, then

$$\begin{aligned}\left(\sigma_x\tau_x, \Delta(\sigma\tau)_x(X)\right) &= \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{\sigma_x\tau_x}\left(\mathrm{D}_x(\sigma\tau)(X)\right) \\ &= \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{\sigma_x\tau_x}\left(\mathrm{D}_{(\sigma_x,\tau_x)}\Phi(Y, Z)\right)\end{aligned}$$

$$= \mathscr{A}d_{\tau_x^{-1}}\left(\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_{\sigma_x}(Y)\right) + \left(\sigma_x\tau_x, \left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}}\right)_{\tau_x}(Z)\right)$$

$$= \left(\sigma_x\tau_x, \mathrm{Ad}_{\tau_x^{-1}}\left(\left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}}\right)_{\sigma_x}(Y)\right) + \left(\widehat{\mu_{\mathscr{G}}^{\mathrm{tot}}}\right)_{\tau_x}(Z)\right)$$

$$= \left(\sigma_x\tau_x, \mathrm{Ad}_{\tau_x^{-1}}\big((\Delta\sigma)|_x(X)\big) + (\Delta\tau)|_x(X)\right).$$

Hence,

$$\Delta(\sigma\tau) = \mathrm{Ad}_{\tau^{-1}} \circ \Delta\sigma + \Delta\tau.$$

Now recall Remark 6.70; this implies for the neutral element section e that

$$\Delta e \equiv 0,$$

and therefore

$$0 = \Delta e = \Delta\big(\sigma\sigma^{-1}\big) = \mathrm{Ad}_\sigma \circ \Delta\sigma + \Delta\big(\sigma^{-1}\big),$$

thus, concluding the proof with

$$\Delta\big(\sigma^{-1}\big) = -\mathrm{Ad}_\sigma \circ \Delta\sigma.$$

■

The Leibniz rule of the Darboux derivative extends to $\nabla^{\mathscr{G}}$.

Lemma 6.78: The conjugation of the canonical LAB connection is a Yang-Mills connection

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ an Ehresmann connection on $\mathscr{G}$. Then we have

$$\mathrm{Ad}_{\sigma^{-1}} \circ \nabla^{\mathscr{G}} \circ \mathrm{Ad}_\sigma = \nabla^{\mathscr{G}} + \mathrm{ad}_{\Delta\sigma} \tag{56}$$

for all $\sigma \in \Gamma(\mathscr{G})$, that is,

$$\left(\mathrm{Ad}_{\sigma^{-1}} \circ \nabla^{\mathscr{G}}_X \circ \mathrm{Ad}_\sigma\right)\nu = \nabla^{\mathscr{G}}_X \nu + [\Delta\sigma(X), \nu]_{\mathscr{g}}$$

for all $\nu \in \Gamma(\mathscr{g})$ and $X \in \mathfrak{X}(M)$. If $\mathrm{H}\mathscr{G}$ is an infinitesimal Yang-Mills connection, $\nabla^{\mathscr{G}} = \nabla^{\mathrm{YM}}$, especially $R_{\nabla^{\mathrm{YM}}} = \mathrm{ad}_\zeta$ for $\zeta \in \Omega^2(M;\mathscr{g})$, then $\widetilde{\nabla}^\sigma := \mathrm{Ad}_{\sigma^{-1}} \circ \nabla^{\mathrm{YM}} \circ \mathrm{Ad}_\sigma$ is also a Yang-Mills connection with

$$R_{\widetilde{\nabla}^\sigma} = \mathrm{ad}_{\mathrm{Ad}_{\sigma^{-1}}(\zeta)},$$

in particular we get

$$\left[\mathrm{d}^{\nabla^{\mathrm{YM}}}\Delta\sigma + \frac{1}{2}[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma]_{\mathscr{g}} + \zeta - \mathrm{Ad}_{\sigma^{-1}} \circ \zeta,\ \nu\right]_{\mathscr{g}} = 0 \tag{57}$$

for all $\nu \in \Gamma(\mathscr{g})$.

Proof.
By Prop. 6.76 we get

$$\begin{aligned}\Delta\left(\mathrm{e}^{\mathrm{Ad}_\sigma(\nu)}\right) &= \Delta\left(\sigma \mathrm{e}^\nu \sigma^{-1}\right) \\ &= \mathrm{Ad}_\sigma \circ \Delta(\sigma \mathrm{e}^\nu) + \Delta\left(\sigma^{-1}\right) \\ &= \mathrm{Ad}_\sigma \circ \left(\mathrm{Ad}_{\mathrm{e}^{-\nu}} \circ \Delta\sigma + \Delta \mathrm{e}^\nu\right) - \mathrm{Ad}_\sigma \circ \Delta\sigma \\ &= \mathrm{Ad}_\sigma \circ \left(\mathrm{Ad}_{\mathrm{e}^{-\nu}} \circ \Delta\sigma - \Delta\sigma + \Delta \mathrm{e}^\nu\right)\end{aligned}$$

for all $\sigma \in \Gamma(\mathscr{G})$ and $\nu \in \Gamma(\mathscr{g})$, where the first equality is a well-known fact, see *e.g.* [1, §1.7, Thm. 1.7.16, page 59]. Then

$$\begin{aligned}\nabla^{\mathscr{G}}(\mathrm{Ad}_\sigma(\nu)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\Delta\left(\mathrm{e}^{\mathrm{Ad}_\sigma(t\nu)}\right)\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{Ad}_\sigma \circ \left(\mathrm{Ad}_{\mathrm{e}^{-t\nu}} \circ \Delta\sigma - \Delta\sigma + \Delta \mathrm{e}^{t\nu}\right)\right) \\ &= \mathrm{Ad}_\sigma \circ \left(\mathrm{ad}_{-\nu} \circ \Delta\sigma + \nabla^{\mathscr{G}}\nu\right) \\ &= \mathrm{Ad}_\sigma \circ \left(\mathrm{ad}_{\Delta\sigma}(\nu) + \nabla^{\mathscr{G}}\nu\right)\end{aligned}$$

making use of that this calculation is pointwise at $x \in M$ just a standard derivative in the vector space $\mathscr{g}_x$. Thus,

$$\mathrm{Ad}_{\sigma^{-1}} \circ \nabla^{\mathscr{G}} \circ \mathrm{Ad}_\sigma = \nabla^{\mathscr{G}} + \mathrm{ad}_{\Delta\sigma}.$$

In order to show $R_{\widetilde{\nabla}^\sigma} = \mathrm{ad}_{\mathrm{Ad}_{\sigma^{-1}}(\zeta)}$ one uses the definition $\widetilde{\nabla}^\sigma := \mathrm{Ad}_{\sigma^{-1}} \circ \nabla^{\mathrm{YM}} \circ \mathrm{Ad}_\sigma$ to calculate the curvature,

$$\begin{aligned}R_{\widetilde{\nabla}^\sigma}(X,Y)\nu &= \widetilde{\nabla}^\sigma_X \widetilde{\nabla}^\sigma_Y \nu - \widetilde{\nabla}^\sigma_Y \widetilde{\nabla}^\sigma_X \nu - \widetilde{\nabla}^\sigma_{[X,Y]}\nu \\ &= \left(\mathrm{Ad}_{\sigma^{-1}} \circ \left(\nabla^{\mathrm{YM}}_X \circ \nabla^{\mathrm{YM}}_Y - \nabla^{\mathrm{YM}}_Y \circ \nabla^{\mathrm{YM}}_X - \nabla^{\mathrm{YM}}_{[X,Y]}\right) \circ \mathrm{Ad}_\sigma\right)(\nu) \\ &= \left(\mathrm{Ad}_{\sigma^{-1}} \circ R_{\nabla^{\mathscr{G}}}(X,Y) \circ \mathrm{Ad}_\sigma\right)(\nu) \\ &= \mathrm{Ad}_{\sigma^{-1}}\left(\left[\zeta(X,Y), \mathrm{Ad}_\sigma(\nu)\right]_{\mathscr{g}}\right) \\ &= \left[\mathrm{Ad}_{\sigma^{-1}}\left(\zeta(X,Y)\right), \nu\right]_{\mathscr{g}}\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\nu \in \Gamma(\mathscr{g})$. The remaining parts of the statement are trivial and straight-forward calculations, making use of Def. 6.67; in fact, we have proven all the remaining statements already in [10, §3, part of Thm. 3.6] and in a more general manner in [5, §4.6, Thm.

4.6.9]: That $\widetilde{\nabla}^\sigma$ is a Lie bracket derivation is trivial to show, and one uses identity (56) to show another identity of its curvature,

$$R_{\widetilde{\nabla}^\sigma} = \left[\mathrm{d}^{\nabla^{\mathrm{YM}}} \Delta\sigma + \frac{1}{2}[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma]_{\mathscr{g}} + \zeta, \cdot\right]_{\mathscr{g}}.$$

Combining both identities for $R_{\widetilde{\nabla}^\sigma}$ we conclude with a proof of Eq. (57). ■

We also get a generalization of statements like [1, §5.5, Lemma 5.5.5, page 276]:

Lemma 6.79: Lie bracket of horizontal and vertical vector

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a horizontal distribution on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Then we have

$$A([X, \widetilde{\nu}]) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\left(\pi^! \Delta \mathrm{e}^{t\nu}\right)(X) \circ r_{\mathrm{e}^{-t\nu}}\right)$$

for all $\nu \in \Gamma(\mathscr{g})$ and horizontal vector fields $X \in \mathfrak{X}(\mathscr{P})$. If there is an $\omega \in \mathfrak{X}(M)$ such that $\mathrm{D}\pi(X) = \pi^*\omega$, then

$$A\big([X, \widetilde{\nu}]\big) = \left(\pi^* \nabla^{\mathscr{G}}\right)_X (\pi^*\nu) = \pi^*\left(\nabla^{\mathscr{G}}_\omega \nu\right).$$

If $\mathrm{H}\mathscr{G}$ is an Ehresmann connection, then we can conclude for all horizontal vector fields X that

$$A\big([X, \widetilde{\nu}]\big) = \left(\pi^* \nabla^{\mathscr{G}}\right)_X (\pi^*\nu).$$

Remarks 6.80.
The last statement got also stated in [11, Eq. (2.8)] in the case of the LGB as principal bundle; we were unaware of this proof, which is why our proof is a bit different.

Proof of Lemma 6.79.
First of all recall the very basic fact (see *e.g.* [1, §A.1, Thm. A.1.46, page 615]) that for two vector fields $X, Y \in \mathfrak{X}(\mathscr{P})$ we have[10]

$$[X, Y]_p = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{D}_{\phi_t(p)}\phi_{-t}\big(X_{\phi_t(p)}\big)$$

for all $p \in \mathscr{P}$, where ϕ_t is the local flow of Y (with parameter t in an open interval of $\mathbb{R}$ containing 0). Now let X be a horizontal vector field and $Y = \widetilde{\nu}$. For such a Y we know by Def. 5.17 that

$$\widetilde{\nu}_p = \mathrm{D}_{e_x}\Phi_p(\nu_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(p \cdot \mathrm{e}^{t\nu_x}\right)$$

[10] Rigorously, similarly to Prop. 6.32 one has a projection involved, but as explained in Remark 6.33 we will omit notating this projection; in the context of this proof we can allow ourselves to take the typical less rigorous approach for simplicity.

for all $p \in \mathscr{P}$, where Φ_p is the orbit map through p, $x \coloneqq \pi(p)$, and e_x the neutral element of $\mathscr{G}_x$. Thus,

$$\phi_t(p) = p \cdot \mathrm{e}^{t\nu_x} = r_{\mathrm{e}^{t\nu}}(p),$$

and by rewriting in sense of Def. 6.42 we achieve

$$\begin{aligned}
[X, \widetilde{\nu}]_p &= - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{D}_{p\cdot \mathrm{e}^{t\nu_x}} r_{\mathrm{e}^{-t\nu}} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right) \\
&= - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{D}_{p\cdot \mathrm{e}^{t\nu_x}} r_{\mathrm{e}^{-t\nu}} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right) - \left.\widetilde{\left(\pi^! \Delta \mathrm{e}^{-t\nu}\right)\Big|_{p\cdot \mathrm{e}^{t\nu_x}} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right)}\right|_p \right) \\
&\quad - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\left.\widetilde{\left(\pi^! \Delta \mathrm{e}^{-t\nu}\right)\Big|_{p\cdot \mathrm{e}^{t\nu_x}} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right)}\right|_p \right) \\
&= - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathscr{r}_{\mathrm{e}^{-t\nu_x} *} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right) \right) \\
&\quad - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\left.\widetilde{\left(\pi^! \Delta \mathrm{e}^{-t\nu}\right)\Big|_{p\cdot \mathrm{e}^{t\nu_x}} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right)}\right|_p \right).
\end{aligned}$$

The first summand, $t \mapsto \mathscr{r}_{\mathrm{e}^{-t\nu_x} *} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right)$, is a curve with values in $\mathrm{H}_p\mathscr{P}$ due to the fact that X is a horizontal vector field. Thus, the velocity of this curve at $t = 0$ will also be an element of $\mathrm{H}_p\mathscr{P}$, and the first summand is therefore in the kernel of A. Similarly,

$$t \mapsto \eta(t) \coloneqq \left(\pi^! \Delta \mathrm{e}^{-t\nu}\right)\Big|_{p\cdot \mathrm{e}^{t\nu_x}} \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right) = \left(\Delta \mathrm{e}^{-t\nu}\right)\big|_x \left(\mathrm{D}_{p\cdot \mathrm{e}^{t\nu_x}} \pi \left(X_{p\cdot \mathrm{e}^{t\nu_x}} \right) \right)$$

is a curve with values in $\mathscr{g}_x$, and therefore its fundamental vector field at p is a curve with values in $\mathrm{V}_p\mathscr{P}$. Hence, we can calculate in the sense of vector spaces to derive

$$- \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{\eta(t)}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(\mathrm{D}_{e_x} \Phi_p \big(\eta(-t) \big) \Big) = \mathrm{D}_{e_x} \Phi_p \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(-t) \right) = \left.\widetilde{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(-t)}\right|_p$$

where we substituted $t \leftrightarrow -t$ after the first equality sign. In total we achieve

$$A_p\Big([X, \widetilde{\nu}]_p \Big) = A_p \left(\left.\widetilde{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(-t)}\right|_p \right) = \left(p, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(\left(\Delta \mathrm{e}^{t\nu}\right)\big|_x \left(\mathrm{D}_{p\cdot \mathrm{e}^{-t\nu_x}} \pi \left(X_{p\cdot \mathrm{e}^{-t\nu_x}} \right) \right) \Big) \right),$$

and thus

$$A([X, \widetilde{\nu}]) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(\left(\pi^! \Delta \mathrm{e}^{t\nu}\right)(X) \circ r_{\mathrm{e}^{-t\nu}} \Big).$$

If there is an $\omega \in \mathfrak{X}(M)$ such that $\mathrm{D}\pi(X) = \pi^*\omega$, then

$$A_p\Big([X,\widetilde{\nu}]_p\Big) = \left(p,\, \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(\left(\Delta \mathrm{e}^{t\nu}\right)\big|_x (\omega_x) \Big) \right),$$

and therefore by Def. 6.34

$$A_p\Big([X,\widetilde{\nu}]_p\Big) = \left(p,\, \nabla^{\mathscr{G}}_{\omega}\nu\Big|_x\right) = \pi^*\left(\nabla^{\mathscr{G}}_{\omega}\nu\right)\Big|_p.$$

By the definition of pullback connections we could also write

$$A\big([X,\widetilde{\nu}]\big) = \left(\pi^*\nabla^{\mathscr{G}}\right)_X (\pi^*\nu).$$

If $\mathrm{H}\mathscr{G}$ is an Ehresmann connection, then recall Remark 6.70; this implies for the neutral element section e that

$$\Delta e \equiv 0,$$

and, again, $t \mapsto \left(\Delta \mathrm{e}^{t\nu}\right)_x$ and $t \mapsto \mathrm{D}_{p\cdot \mathrm{e}^{-t\nu_x}}\pi\left(X_{p\cdot \mathrm{e}^{-t\nu_x}}\right)$ are curves with values in the vector spaces $\mathrm{T}^*_x M \otimes \mathscr{g}_x$ and $\mathrm{T}_x M$, respectively. Thus, we can just apply typical Leibniz rules to calculate the derivative of the contraction

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(\left(\Delta \mathrm{e}^{t\nu}\right)\big|_x \left(\mathrm{D}_{p\cdot \mathrm{e}^{-t\nu_x}}\pi\left(X_{p\cdot \mathrm{e}^{-t\nu_x}}\right)\right) \Big) \\
&\quad = \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Delta \mathrm{e}^{t\nu}\right)\bigg|_x \left(\mathrm{D}_p\pi(X_p)\right) + \underbrace{(\Delta e)|_x}_{=0} \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{D}_{p\cdot \mathrm{e}^{-t\nu_x}}\pi\left(X_{p\cdot \mathrm{e}^{-t\nu_x}}\right)\right) \\
&\quad = \nabla^{\mathscr{G}}_{\mathrm{D}_p\pi(X_p)}\nu,
\end{aligned}$$

and as before we conclude

$$A\big([X,\widetilde{\nu}]\big) = \left(\pi^*\nabla^{\mathscr{G}}\right)_X (\pi^*\nu).$$

■

This concludes our discussion about the infinitesimal compatibility condition (54) and its integrated version, "$\mathrm{H}\mathscr{G}$ = Ehresmann connection"'. Via Thm. 6.75 and Lemma 6.69 we therefore concluded that compatibility condition (54) holds if the total Maurer-Cartan form is multiplicative. Let us therefore now turn to compatibility condition (55), which we can actually integrate in to an equivalent statement.

Theorem 6.81: Generalized Maurer-Cartan equation

Let $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ be an LGB over a smooth manifold M, $\mathrm{H}\mathscr{G}$ an Ehresmann connection on $\mathscr{G}$, and $\zeta \in \Omega^2(M;\mathscr{g})$. Then $\mathrm{H}\mathscr{G}$ satisfies the infinitesimal compatibility condition (55) w.r.t.

ζ if and only if the total Maurer-Cartan form $\mu_{\mathscr{G}}^{\mathrm{tot}}$ satisfies

$$\left(\mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}}^{\mathrm{tot}} + \frac{1}{2}\left[\mu_{\mathscr{G}}^{\mathrm{tot}} \overset{\wedge}{,} \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\pi_{\mathscr{G}}^* \mathscr{g}} + \pi_{\mathscr{G}}^! \zeta\right)\Bigg|_g = \left(g, \mathrm{Ad}_{g^{-1}} \circ \zeta_x \circ (\mathrm{D}_g \pi_{\mathscr{G}}, \mathrm{D}_g \pi_{\mathscr{G}})\right) \tag{58}$$

*for all $g \in \mathscr{G}_x$ ($x \in M$). We then say that $\mu_{\mathscr{G}}^{\mathrm{tot}}$ solves the **generalized Maurer-Cartan equation (w.r.t. ζ)**.*

Remarks 6.82.
The structure on the right hand side is again due to denoting the base point in pullback bundles; one may decide to drop the base point for simplicity in the notation. In that case, since the base point is clear by context, one may also just write

$$\mathscr{Ad}_{g^{-1}} \circ \pi_{\mathscr{G}}^! \zeta\Big|_g \quad \text{or} \quad \mathrm{Ad}_{g^{-1}} \circ \pi_{\mathscr{G}}^! \zeta\Big|_g$$

on the right hand side.

Remark 6.83: Classical Maurer-Cartan equation recovered

Recall Rem. 6.30. If $\mathscr{G}$ is a trivial LGB, $\mathrm{H}\mathscr{G}$ its canonical flat connection and $\zeta \equiv 0$, then $\nabla^{\mathscr{G}}$ is the canonical flat connection on the trivial LAB $\mathscr{g}$ (recall Ex. 6.36) and it clearly satisfies the compatibility conditions in Def. 6.67 w.r.t. $\zeta \equiv 0$. Furthermore, the generalized Maurer-Cartan equation then reduces to the classical Maurer-Cartan equation.

Remark 6.84: Relation to simplicial differential

In fact, Def. 6.73 comes from a simplicial differential (w.r.t. points on $\mathscr{G}$) defined in [13, beginning of §1.2]. We do not have time to introduce it completely, but we give a short sketch about its definitions, also following the style of [11, appendix]. On degree one the cited differential is a map

$$\Omega^\bullet(\mathscr{G}; \pi_{\mathscr{G}}^* \mathscr{g}) \to \Omega^\bullet(\mathscr{G} * \mathscr{G}; \pi_{\mathscr{G}}^* \mathscr{g})$$

$$\omega \mapsto \delta\omega$$

with

$$(\delta\omega)_{(g,q)} \coloneqq \mathscr{Ad}_{q^{-1}} \circ \left(\mathrm{pr}_1^! \omega\right)_{(g,q)} + \left(\mathrm{pr}_2^! \omega\right)_{(g,q)} - \left(\Phi^! \omega\right)_{(g,q)}$$

for all $(g, q) \in \mathscr{G} * \mathscr{G}$, where $\Phi : \mathscr{G} * \mathscr{G} \to \mathscr{G}$ is the $\mathscr{G}$-multiplication, and $\mathrm{pr}_i : \mathscr{G} * \mathscr{G} \to \mathscr{G}$ are the canonical projections onto the i-th component ($i \in \{1, 2\}$). Recall Def. 6.73 and its Rem. 6.74, ω can be defined to be multiplicative if $\delta\omega = 0$, *i.e.* ω is closed w.r.t. δ. On degree 0 δ is a map $\Omega^\bullet(M; \mathscr{g}) \to \Omega^\bullet\left(\mathscr{G}; \pi_{\mathscr{g}}^*\right)$ given for $\xi \in \Omega^\bullet(M; \mathscr{g})$ by

$$(\delta\xi)_g \coloneqq \Big(g, \mathrm{Ad}_{g^{-1}} \circ \xi_x \circ \underbrace{(\mathrm{D}_g \pi_{\mathscr{G}}, \dots, \mathrm{D}_g \pi_{\mathscr{G}})}_{\text{times the degree of } \xi}\Big) - \left(\pi_{\mathscr{G}}^! \xi\right)\Big|_g$$

for all $g \in \mathscr{G}$. Then observe that we can rewrite the generalized Mauer-Cartan equation to

$$F_{\mathscr{G}} = \delta\zeta,$$

where

$$F_{\mathscr{G}} = \mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}}^{\mathrm{tot}} + \frac{1}{2} \left[\mu_{\mathscr{G}}^{\mathrm{tot}} \stackrel{\wedge}{,} \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\pi_{\mathscr{G}}^* \mathscr{g}} \in \Omega^2(\mathscr{G}; \pi_{\mathscr{G}}^* \mathscr{g}).$$

It is straight-forward to check that $\delta F_{\mathscr{G}} = 0$ (see *e.g.* [9, §4.6, Thm. 4.27, Eq. (53)] or [11, §2.5, Prop. 2.22]) by making use of that $\mathrm{H}\mathscr{G}$ is an Ehresmann connection. In total we concluded that compatibility condition (54) is implied, if $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is multiplicative, which also implies that $F_{\mathscr{G}}$ is multiplicative and thus δ-closed; additionally with compatibility condition (55) we achieve δ-exactness of $F_{\mathscr{G}}$. This is no wonder, the compatibility conditions already tell us by definition that the connection-1-form of ∇ is closed w.r.t. the Chevalley-Eilenberg complex, which also implies the same for the curvature, however the curvature has to be exact by (55). We depict that as $\delta^{\mathrm{CE}} \nabla^{\mathscr{G}} = 0$ and $R_{\nabla^{\mathscr{G}}} = \delta^{\mathrm{CE}}\zeta$; a summary is shown in Table 1.

Table 1: Infinitesimal compatibility conditions on the LAB $\mathscr{g}$ and integrated compatibility conditions on the overlying LGB $\mathscr{G}$

	$\mathscr{g}$	$\mathscr{G}$
Closedness	$\delta^{\mathrm{CE}}\nabla^{\mathscr{G}} = 0 \ \left(\Rightarrow \delta^{\mathrm{CE}} R_{\nabla^{\mathscr{G}}} = 0\right)$	$\delta\mu_{\mathscr{G}}^{\mathrm{tot}} = 0 \ (\Rightarrow \delta F_{\mathscr{G}} = 0)$
Exactness	$R_{\nabla^{\mathscr{G}}} = \delta^{\mathrm{CE}}\zeta$	$F_{\mathscr{G}} = \delta\zeta$

Proof of Thm. 6.81.
The idea of the proof is similar to the proof of a statement about when two multiplicative 2-forms[11] are equal, see *e.g.* [14, §3, Cor. 3.4]. That is, we want to discuss under what conditions

$$\mathscr{G} \ni g \mapsto \omega_g := \left(\mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}}^{\mathrm{tot}} + \frac{1}{2} \left[\mu_{\mathscr{G}}^{\mathrm{tot}} \stackrel{\wedge}{,} \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\pi_{\mathscr{G}}^* \mathscr{g}} + \pi_{\mathscr{G}}^! \zeta \right) \Bigg|_g - \left(g, \mathrm{Ad}_{g^{-1}} \circ \zeta_x \circ (\mathrm{D}_g \pi_{\mathscr{G}}, \mathrm{D}_g \pi_{\mathscr{G}})\right)$$

is constant along the fibres of $\mathscr{G}$, that is, constant along the flows of vertical tangent vectors, the left-invariant vector fields (recall Cor. 4.24).

• First of all observe that ω vanishes if at least one tangent vector is vertical. Let $Y \in \mathrm{V}_g\mathscr{G}$ ($g \in \mathscr{G}_x$, $x \in M$), and write $Y = \left.\widetilde{\nu_x}\right|_g$ for a $\nu_x \in \mathscr{g}_x$. Due to $\mathrm{D}\pi_{\mathscr{G}}(Y) = 0$ it is clear that the contraction of $\pi_{\mathscr{G}}^! \zeta$ with Y is 0. Let $Z \in \mathrm{T}_g\mathscr{G}$ for which we write

$$Z := \left.\widetilde{\mu_x}\right|_g + X_g$$

[11] See Rem. 6.84; multiplicativity is related to closedness w.r.t. a differential. However, we will not need this general notion.

for a $\mu_x \in \mathscr{g}_x$ and $X_g \in \mathrm{H}_g\mathscr{G}$; then by Cor. 6.71

$$\frac{1}{2}\left[\mu_{\mathscr{G}}^{\mathrm{tot}} \stackrel{\wedge}{,} \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\pi_{\mathscr{G}}^*\mathscr{g}}\Big|_g(Y,Z) = \left[\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Y), \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Z)\right]_{\pi_{\mathscr{G}}^*\mathscr{g}} = \left(g, [\nu_x, \mu_x]_{\mathscr{g}_x}\right).$$

Extend Y and Z naturally to vector fields on $\mathscr{G}$ (for simplicity also denoted by Y and Z, respectively) by viewing ν_x and μ_x as values of sections ν and μ, respectively, of $\mathscr{g}$ at x, and extending X_g (locally) to a horizontal vector field X; thence,

$$\begin{aligned}
\left(\mathrm{d}^{\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}}\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g(Y,Z) &= \left(\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}\right)_{Y_g}\left(\mu_{\mathscr{G}}^{\mathrm{tot}}(Z)\right) - \left(\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}\right)_{Z_g}\left(\mu_{\mathscr{G}}^{\mathrm{tot}}(Y)\right) - \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g([Y,Z]) \\
&= \underbrace{\left(\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}\right)_{\widetilde{\nu}_g}\left(\pi_{\mathscr{G}}^*\mu\right)}_{=\pi_{\mathscr{G}}^*\left(\nabla^{\mathscr{g}}_{\mathrm{D}\pi_{\mathscr{G}}(\widetilde{\nu})}\mu\right)\big|_g=0} - \left(\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}\right)_{\widetilde{\mu}_g+X_g}\left(\pi_{\mathscr{G}}^*\nu\right) \\
&\quad - \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g([\widetilde{\nu},\widetilde{\mu}]) - \left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right)_g([\widetilde{\nu},X]) \\
&= -\left(\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}\right)_{X_g}\left(\pi_{\mathscr{G}}^*\nu\right) - \left(g, [\nu_x,\mu_x]_{\mathscr{g}_x}\right) + \left(\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}\right)_{X_g}\left(\pi_{\mathscr{G}}^*\nu\right) \\
&= -\left(g, [\nu_x,\mu_x]_{\mathscr{g}_x}\right)
\end{aligned}$$

using Rem. 5.19 (bracket of fundamental/left-invariant vector fields is fundamental/left-invariant) and Lemma 6.79. Thus,

$$\left(\mathrm{d}^{\pi_{\mathscr{G}}^*\nabla^{\mathscr{g}}}\mu_{\mathscr{G}}^{\mathrm{tot}} + \frac{1}{2}\left[\mu_{\mathscr{G}}^{\mathrm{tot}} \stackrel{\wedge}{,} \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\pi_{\mathscr{G}}^*\mathscr{g}}\right)\Bigg|_g(Y,Z) = 0.$$

It follows that ω vanishes if contracted with a vertical tangent vector.

• Therefore we can study ω just w.r.t. tangent vectors complementary to the vertical structure, these do not necessarily need to be elements of $\mathrm{H}\mathscr{G}$. For a given $g \in \mathscr{G}_x$ ($x \in M$) define

$$\begin{aligned}
\mathbb{R} &\to \Gamma(\mathscr{G}), \\
t &\mapsto \gamma(t) := \sigma \mathrm{e}^{t\nu},
\end{aligned}$$

where ν is any section of $\mathscr{g}$ and $\sigma \in \Gamma(\mathscr{G})$ with $\sigma_x = g$; pointwise we denote $\gamma(t)$ as a map $M \ni x \mapsto \gamma_x(t)$. By Cor. 4.24 the flows of elements of $\mathrm{V}_g\mathscr{G}$ can be precisely described by curves like γ, so that the derivative w.r.t. t of pullbacks of tensors with γ characterizes any derivative along a fibre of $\mathscr{G}$. Furthermore, $\gamma(t)$ is by definition a section of $\mathscr{G}$ for all t and thence describes an embedding of M into $\mathscr{G}$. Thus, the t-derivatives of terms like $\gamma(t)^!\omega$ describe the derivatives of ω along fibres of $\mathscr{G}$ contracted with tangent vectors complementary to the vertical structure. Therefore let us calculate $\gamma(t)^!\omega$; we have

$$\gamma(t)^!\pi_{\mathscr{G}}^!\zeta = \underbrace{\left(\pi_{\mathscr{G}} \circ \gamma(t)\right)^!}_{\equiv \mathbb{1}_M^!}\zeta = \zeta$$

and under the identification $\gamma^* \pi_{\mathscr{G}}^* \mathscr{g} \cong (\pi_{\mathscr{G}} \circ \gamma)^* \mathscr{g} = \mathbb{1}_M^* \mathscr{g} \cong \mathscr{g}$ we get similarly

$$\left(\gamma(t)^! \big(g, \mathrm{Ad}_{g^{-1}} \circ \zeta_{\pi_{\mathscr{G}}(g)} \circ \mathrm{D}_g \pi_{\mathscr{G}}\big)\right)_x \cong \mathrm{Ad}_{\gamma_x^{-1}(t)} \circ \zeta_{\pi_{\mathscr{G}}(\gamma_x(t))} \circ \mathrm{D}_x\big(\pi_{\mathscr{G}} \circ \gamma(t)\big) = \mathrm{Ad}_{\gamma_x^{-1}(t)} \circ \zeta_x$$

for all $x \in M$, where $\gamma^{-1}(t)$ is the $\mathscr{G}$-inverse of $\gamma(t)$. It is also a straight-forward calculation to show that

$$\gamma(t)^! \mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}}^{\mathrm{tot}} = \mathrm{d}^{\gamma(t)^* \pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \left(\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right) = \mathrm{d}^{(\pi_{\mathscr{G}} \circ \gamma(t))^* \nabla^{\mathscr{G}}} \left(\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right) = \mathrm{d}^{\nabla^{\mathscr{G}}} \left(\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right),$$

using the definition of pullback vector bundle connections; alternatively see the explicit calulations in [10, Appendix, Prop. A.1, (A.1)] or [5, Appendix, Prop. A.1.1, (A.2)], and we have

$$\begin{aligned}
&\left(\gamma(t)^! \left(\frac{1}{2} \big[\mu_{\mathscr{G}}^{\mathrm{tot}} \overset{\wedge}{,} \mu_{\mathscr{G}}^{\mathrm{tot}}\big]_{\pi_{\mathscr{G}}^* \mathscr{g}}\right)\right)_x (X, Y) \\
&\quad = \left[\big(\mu_{\mathscr{G}}^{\mathrm{tot}}\big)_{\gamma_x(t)} \big(\mathrm{D}_x\big(\gamma(t)\big)(X)\big), \big(\mu_{\mathscr{G}}^{\mathrm{tot}}\big)_{\gamma_x(t)} \big(\mathrm{D}_x\big(\gamma(t)\big)(Y)\big)\right]_{\mathscr{g}_x} \\
&\quad = \left[\left(\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right)_x (X), \left(\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right)_x (Y)\right]_{\mathscr{g}_x} \\
&\quad = \left(\frac{1}{2} \left[\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}} \overset{\wedge}{,} \gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\mathscr{g}}\right)_x (X, Y)
\end{aligned}$$

for all $x \in M$ and $X, Y \in \mathrm{T}_x M$. Thus, we have so far

$$\begin{aligned}
\gamma(t)^! \omega &= \mathrm{d}^{\nabla^{\mathscr{G}}} \left(\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right) + \frac{1}{2} \left[\gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}} \overset{\wedge}{,} \gamma(t)^! \mu_{\mathscr{G}}^{\mathrm{tot}}\right]_{\mathscr{g}} + \zeta - \mathrm{Ad}_{\gamma^{-1}(t)} \circ \zeta \\
&= \mathrm{d}^{\nabla^{\mathscr{G}}} \big(\Delta\gamma(t)\big) + \frac{1}{2} \big[\Delta\gamma(t) \overset{\wedge}{,} \Delta\gamma(t)\big]_{\mathscr{g}} + \zeta - \mathrm{Ad}_{\gamma^{-1}(t)} \circ \zeta. \qquad (59)
\end{aligned}$$

By Prop. 6.76 we get

$$\Delta\gamma(t) = \mathrm{Ad}_{\mathrm{e}^{-t\nu}} \circ \Delta\sigma + \Delta \mathrm{e}^{t\nu},$$

and via Lemma 6.78 we can derive

$$\begin{aligned}
&\left(\mathrm{d}^{\nabla^{\mathscr{G}}} \big(\mathrm{Ad}_{\mathrm{e}^{-t\nu}} \circ \Delta\sigma\big)\right)(X, Y) \\
&\quad = \nabla_X^{\mathscr{G}} \big(\mathrm{Ad}_{\mathrm{e}^{-t\nu}} (\Delta\sigma(Y))\big) - \nabla_Y^{\mathscr{G}} \big(\mathrm{Ad}_{\mathrm{e}^{-t\nu}} (\Delta\sigma(X))\big) - \mathrm{Ad}_{\mathrm{e}^{-t\nu}} \big(\Delta\sigma([X, Y])\big) \\
&\quad = \mathrm{Ad}_{\mathrm{e}^{-t\nu}} \Big(\nabla_X^{\mathscr{G}} \big(\Delta\sigma(Y)\big) + \big[\big(\Delta \mathrm{e}^{-t\nu}\big)(X), \Delta\sigma(Y)\big]_{\mathscr{g}} \\
&\qquad\qquad - \nabla_Y^{\mathscr{G}} \big(\Delta\sigma(X)\big) - \big[\big(\Delta \mathrm{e}^{-t\nu}\big)(Y), \Delta\sigma(X)\big]_{\mathscr{g}} \\
&\qquad\qquad - \Delta\sigma([X, Y])\Big) \\
&\quad = \mathrm{Ad}_{\mathrm{e}^{-t\nu}} \left(\left(\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\sigma\right)(X, Y) + \big[\big(\Delta \mathrm{e}^{-t\nu}\big)(X), \Delta\sigma(Y)\big]_{\mathscr{g}} - \big[\big(\Delta \mathrm{e}^{-t\nu}\big)(Y), \Delta\sigma(X)\big]_{\mathscr{g}}\right)
\end{aligned}$$

$$= \left(\mathrm{Ad}_{\mathrm{e}^{-t\nu}} \circ \left(\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\sigma + \left[\Delta \mathrm{e}^{-t\nu} \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}}\right)\right)(X,Y)$$

for all $X, Y \in \mathfrak{X}(M)$, hence,

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\gamma(t)\right) &= \left[\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\sigma, \nu\right]_{\mathscr{g}} - \left[\Delta\sigma \stackrel{\wedge}{,} \nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}} + \mathrm{d}^{\nabla^{\mathscr{G}}} \nabla^{\mathscr{G}} \nu \\
&= \left[\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\sigma, \nu\right]_{\mathscr{g}} - \left[\Delta\sigma \stackrel{\wedge}{,} \nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}} + R_{\nabla^{\mathscr{g}}}(\cdot,\cdot)\nu
\end{aligned}$$

making use of that this calculation is pointwise at $x \in M$ just a standard derivative in the vector space $\mathscr{g}_x$ and that $\Delta e \equiv 0$ by Remark 6.70; we also used Prop. 1.2. In a similar fashion,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\Delta\gamma(t) \stackrel{\wedge}{,} \Delta\gamma(t)\right]_{\mathscr{g}} = 2\left[[\Delta\sigma, \nu]_{\mathscr{g}} \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}} + 2\left[\Delta\sigma \stackrel{\wedge}{,} \nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}},$$

by the Jacobi identity

$$\begin{aligned}
\left[[\Delta\sigma, \nu]_{\mathscr{g}} \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}}(X,Y) &= \left[[\Delta\sigma(X), \nu]_{\mathscr{g}}, \Delta\sigma(Y)\right]_{\mathscr{g}} - \left[[\Delta\sigma(Y), \nu]_{\mathscr{g}}, \Delta\sigma(X)\right]_{\mathscr{g}} \\
&= \left[[\Delta\sigma(X), \Delta\sigma(Y)]_{\mathscr{g}}, \nu\right]_{\mathscr{g}} \\
&= \left[\frac{1}{2}\left[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}}, \nu\right]_{\mathscr{g}}(X,Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, and trivially

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\gamma^{-1}(t)} \circ \zeta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\mathrm{e}^{-t\nu}} \circ \mathrm{Ad}_{\sigma^{-1}} \circ \zeta = [\mathrm{Ad}_{\sigma^{-1}} \circ \zeta, \nu]_{\mathscr{g}}.$$

Collecting everything we derive

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t)^! \omega &= \left[\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\sigma, \nu\right]_{\mathscr{g}} - \left[\Delta\sigma \stackrel{\wedge}{,} \nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}} + R_{\nabla^{\mathscr{g}}}(\cdot,\cdot)\nu \\
&\quad + \left[\frac{1}{2}\left[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}}, \nu\right]_{\mathscr{g}} + \left[\Delta\sigma \stackrel{\wedge}{,} \nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}} - [\mathrm{Ad}_{\sigma^{-1}} \circ \zeta, \nu]_{\mathscr{g}} \\
&= R_{\nabla^{\mathscr{g}}}(\cdot,\cdot)\nu + \left[\mathrm{d}^{\nabla^{\mathscr{G}}} \Delta\sigma + \frac{1}{2}\left[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}} - \mathrm{Ad}_{\sigma^{-1}} \circ \zeta, \nu\right]_{\mathscr{g}} \qquad (60)
\end{aligned}$$

for all $\nu \in \Gamma(\mathscr{g})$ and $\sigma \in \Gamma(\mathscr{G})$.

• On the one hand, if $\mu_{\mathscr{G}}^{\mathrm{tot}}$ satisfies Eq. (58), then $\omega \equiv 0$. As previously mentioned, we have $\Delta e = 0$, so that we then get in total for Eq. (60) w.r.t. $\sigma := e$ that

$$R_{\nabla^{\mathscr{g}}}(\cdot,\cdot)\nu = [\zeta, \nu]_{\mathscr{g}}$$

for all $\nu \in \Gamma(\mathscr{g})$, which is the infinitesimal compatibility condition (55).

• On the other hand, if $\mathrm{H}\mathscr{G}$ satisfies the infinitesimal compatibility condition (55), $R_{\nabla^{\mathscr{G}}}(\cdot,\cdot)\nu = [\zeta,\cdot]_{\mathscr{g}}$, then we know by Eq. (57) (Lemma 6.78) that

$$\left[\mathrm{d}^{\nabla^{\mathrm{YM}}}\Delta\sigma + \frac{1}{2}\left[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}} - \mathrm{Ad}_{\sigma^{-1}} \circ \zeta,\ \nu\right]_{\mathscr{g}} = -[\zeta,\nu]_{\mathscr{g}},$$

so that Eq. (60) then has the form

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t)^!\omega = R_{\nabla^{\mathscr{G}}}(\cdot,\cdot)\nu - [\zeta,\nu]_{\mathscr{g}} \stackrel{(55)}{=} 0$$

for all $\nu \in \Gamma(\mathscr{g})$ and $\sigma \in \Gamma(\mathscr{G})$. As aforementioned in the discussion around the definition of γ, also recall that ω vanishes if contracted with a vertical vector, we can conclude that ω is constant along the fibres of $\mathscr{G}$ which implies

$$\omega = \pi_{\mathscr{G}}^!\xi$$

for a $\xi \in \Omega^2(M;\mathscr{g})$, especially

$$e^!\omega = (\pi_{\mathscr{G}} \circ e)^!\xi = \xi.$$

But we already have calculated this in Eq. (59), set $t = 0$ and $\sigma := e$, then also again by $\Delta e = 0$ we have

$$\xi = e^!\omega \stackrel{(59)}{=} \zeta - \zeta = 0.$$

Finally we get

$$\omega = \pi_{\mathscr{G}}^!\xi = 0,$$

which finishes the proof. ■

Trivially, this theorem is an extension of Eq. (57) (Lemma 6.78).

Corollary 6.85: Pullback of generalized Maurer-Cartan equation

Let $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ be an LGB over a smooth manifold M, $\mathrm{H}\mathscr{G}$ an Ehresmann connection on $\mathscr{G}$ so that the total Maurer-Cartan form $\mu_{\mathscr{G}}^{\mathrm{tot}}$ satisfies the generalized Maurer-Cartan equation (58) w.r.t. a $\zeta \in \Omega^2(M;\mathscr{g})$. Then

$$\mathrm{d}^{\nabla^{\mathrm{YM}}}\Delta\sigma + \frac{1}{2}\left[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}} + \zeta = \mathrm{Ad}_{\sigma^{-1}} \circ \zeta$$

for all $\sigma \in \Gamma(\mathscr{G})$.

Proof.
Keeping the same notation as in the proof of Thm. 6.81, we have derived Eq. (59), and due to Thm. 6.81 we get $\omega = 0$; altogether Eq. (59) at $t = 0$ gives then

$$0 = \mathrm{d}^{\nabla^{\mathrm{YM}}}\Delta\sigma + \frac{1}{2}\left[\Delta\sigma \stackrel{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}} + \zeta - \mathrm{Ad}_{\sigma^{-1}} \circ \zeta.$$

■

Thus, we can finally define the connection we need on the LGB.

Definition 6.86: Yang-Mills connection

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M, and $\mathrm{H}\mathscr{G}$ be a horizontal distribution of $\mathscr{G}$. Then we say that $\mathrm{H}\mathscr{G}$ is a **Yang-Mills connection (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$)**, if it satisfies the **compatibility conditions**:

1. $\mathrm{H}\mathscr{G}$ is an Ehresmann connection,
2. the total Maurer-Cartan form $\mu_{\mathscr{G}}^{\mathrm{tot}}$ satisfies the generalized Maurer-Cartan equation (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$).

By Thm. 6.75 we can equivalently define $\mathrm{H}\mathscr{G}$ as a Yang-Mills connection if

1. $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is multiplicative,
2. $\mu_{\mathscr{G}}^{\mathrm{tot}}$ satisfies the generalized Maurer-Cartan equation (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$).

We then label $\mu_{\mathscr{G}}^{\mathrm{tot}}$ as a Yang-Mills connection, too.

Remark 6.87: Yang-Mills connections integrate infinitesimal Yang-Mills connections

By Lemma 6.69 and Thm. 6.81, every Yang-Mills connection is also an infinitesimal Yang-Mills connections, *i.e.* the infinitesimal compatibility conditions are also satisfied.

To conclude this subsubsection, let us provide canonical examples of Yang-Mills connections; we will make use of Thm. 6.81 in order to prove that the generalized Maurer-Cartan equation is satisfied.

Example 6.88: (Pre-)Classical Yang-Mills connection

Let us look at a classical principal bundle P with trivial LGB $\mathscr{G} = M \times G$, G a Lie group; recall Ex. 6.3 and 6.47. The canonical flat connection $\mathrm{H}\mathscr{G}$ on $\mathscr{G}$ induces the canonical flat connection $\nabla^{\mathscr{G}}$ on its LAB $\mathscr{g} = M \times \mathfrak{g}$, recall Ex. 6.36. Thus,

$$R_{\nabla^{\mathscr{G}}} = 0$$

and so we can set $\zeta \equiv 0$. By Ex. 6.47 (viewing $\mathscr{G}$ also as a principal bundle equipped with the same horizontal distribution) we know that $\mathrm{H}\mathscr{G}$ is an Ehresmann connection, so that its total Maurer-Cartan form is multiplicative.
We then call $\nabla^{\mathrm{cYM}} := \nabla^{\mathscr{G}}$ a **classical Yang-Mills connection**.
However, observe that ζ can be centre-valued while $\nabla^{\mathscr{G}}$ is still flat. In such an occasion we speak of a **pre-classical Yang-Mills connection**, denoted by ∇^{pYM}.

Example 6.89: Associated connections of inner group bundles are Yang-Mills connections

Recall Ex. 6.48, its setup and notation. We have discussed a canonical connection on associated LGBs $\mathscr{H} = P \times_{\psi} H$, called associated connection which is an Ehresmann connection. As it is well-known, the curvature measures the holonomy, that is, the parallel transport over closed (contractible) curves. Together with Eq. (55), the parallel transport over a closed (contractible) curve α should act as conjugation with some group element. Let us see whether this holds here; there is a unique $g \in G$ for $p \in P$ such that

$$\mathrm{PT}^{P}_{\alpha}(p) = p \cdot g,$$

and thus for $[p, h] \in \mathscr{H}$

$$\mathrm{PT}^{\mathscr{H}}_{\alpha}\big([p, h]\big) = [p \cdot g, h] = [p, \psi_g(h)].$$

Thus, if $\mathscr{H} = c_G(P)$, the inner group bundle of Ex. 2.12, we get

$$\mathrm{PT}^{\mathscr{H}}_{\alpha}\big([p, h]\big) = \left[p, ghg^{-1}\right] = [p, g] \cdot [p, h] \cdot \left[p, g^{-1}\right],$$

the conjugation of $[p, h]$ with $[p, g]$ in $c_G(P)$. It follows that there is a $\zeta \in \Omega^2(M; \mathfrak{g})$ such that Eq. (55) is satisfied. This also implies that the associated connection of an inner group bundle is a Yang-Mills connection. We will not prove this approach, instead the next approach provides an alternative proof with an explicit expression for ζ.

Example 6.90: The Yang-Mills connections on inner group bundles: Alternative point of view and explanation

Alternatively one can prove what we have argued in Ex. 6.89 by using the aforementioned relationship of the compatibility conditions with Mackenzie's study about extending Lie algebroids via LABs; recall Remark 6.68. The following is an information for the experienced reader: The LAB of $c_G(P)$ is given by the adjoint bundle $\mathrm{Ad}(P)$; recall Ex. 4.20. $\mathrm{Ad}(P)$ is the kernel of the Atiyah sequence (see *e.g.* [2, §3.2, page 90ff.]), a short exact sequence of Lie algebroids over M

$$0 \longrightarrow \mathrm{Ad}(P) \longrightarrow \mathrm{At}(P) \longrightarrow \mathrm{T}M \longrightarrow 0,$$

where $\mathrm{At}(P)$ is the Atiyah bundle of P. A connection on P in the "classical" sense (that is, related to a canonical flat connection on $M \times G$ in our sense) has a 1:1 correspondence to a splitting $\chi : \mathrm{T}M \to \mathrm{AT}(P)$ to that sequence, *i.e.* a section of the right non-trivial arrow. We can then construct a vector bundle connection ∇^{YM} on the adjoint bundle by using the Lie algebroid bracket $[\cdot, \cdot]_{\mathrm{At}(P)}$ on $\mathrm{At}(P)$, that is,

$$\nabla^{\mathrm{YM}}_X \nu = [\chi(X), \nu]_{\mathrm{At}(P)}$$

for all $X \in \mathfrak{X}(M)$ and $\nu \in \Gamma(\mathrm{Ad}(P))$. It follows by straight-forward calculations as in [2, §7.3, Prop. 7.3.2 and Lemma 7.3.3, page 278] that ∇^{YM} is indeed a Yang-Mills connection with $\zeta \in \Omega^2(M; \mathrm{Ad}(P))$ for example given by

$$\zeta(Y, Z) := [\chi(Y), \chi(Z)]_{\mathrm{At}(P)} - \chi([Y, Z])$$

for all $Y, Z \in \mathfrak{X}(M)$. In other words, ∇^{YM} is the **adjoint connection** induced by a connection on P (see for example [2, §5.3, especially Prop. 5.3.13, page 199]). Thence, by the discussion in [1, §5.9, page 289ff.] ∇^{YM} corresponds to the natural parallel transport $\mathrm{PT}^{\mathrm{Ad}(P)}$ on $\mathrm{Ad}(P)$ (omitting the notation of the corresponding curve) given by

$$\mathrm{PT}^{\mathrm{Ad}(P)}\big([p, v]\big) := \big[\mathrm{PT}^P(p), v\big]$$

for all $[p, v] \in \mathrm{Ad}(P)$. This is clearly just the infinitesimal parallel transport inherited by $\mathrm{PT}^{c_G(P)}$, so that we can conclude that ∇^{YM} corresponds to the Yang-Mills connection on $c_G(P)$ in sense of Def. 6.34 (also recall Rem. 6.35).

This proof gives a certain interpretation of ∇^{YM} and ζ: In the case of $\mathscr{G}$ being the inner group bundle of a typical gauge theory related to a principal G-bundle P, one can view ∇^{YM} as the adjoint connection on the adjoint bundle, induced by χ a classical connection on P, this implies by Eq. (55) that ζ is the field strength related to χ. Thus one may say in this case that the generalised sense of connection as in Def. 6.51 is modified by a classical connection from a classical principal bundle, thence also the label as Yang-Mills connection to reflect its relation to the classical theory.

6.4.2. Field strength related to Yang-Mills connections

Let us now introduce and discuss the field strength.

Definition 6.91: (Generalized) Field strength

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$) and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Then we define the **(generalized) field strength** F as an element of $\Omega^2(\mathscr{P}; \pi^*\mathscr{g})$ by

$$F := \mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A \circ \left(\pi^{\mathrm{H}\mathscr{P}}, \pi^{\mathrm{H}\mathscr{P}}\right) + \pi^!\zeta,$$

where $\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}$ is the exterior covariant derivative related to the pullback connection $\pi^*\nabla^{\mathrm{YM}}$, and $\pi^{\mathrm{H}\mathscr{P}} : \mathrm{T}\mathscr{P} \to \mathrm{H}\mathscr{P}$ is the canonical projection onto the associated Ehresmann connection $\mathrm{H}\mathscr{P}$ on $\mathscr{P}$; that is,

$$F(X, Y) = \mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A\Big(\pi^{\mathrm{H}\mathscr{P}}(X), \pi^{\mathrm{H}\mathscr{P}}(Y)\Big) + (\pi^*\zeta)\big(\mathrm{D}\pi(X), \mathrm{D}\pi(Y)\big)$$

for all $X, Y \in \mathfrak{X}(\mathscr{P})$.
One may label F also as the **curvature of** A.

Of course, we expect certain properties, naturally generalized by our previous discussions:

Proposition 6.92: Properties of the generalized field strength

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$) and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Then we have the following properties of the field strength:

- ***(Form of type $\mathscr{Ad}$)***

$$r_\sigma^! F = \mathscr{Ad}_{\sigma^{-1}} \circ F$$

for all (local) $\sigma \in \Gamma(\mathscr{G})$.

- ***(Horizontal form)***
For $X, Y \in \mathrm{T}_p\mathscr{P}$ ($p \in \mathscr{P}$) we have

$$F(X, Y) = 0$$

if either of X and Y is vertical.

However, despite the easy proof in the classical theory, the proof of the first property in Prop. 6.92 will be rather involved; most of the proof is straight-forward calculation, and then use our discussion about Yang-Mills connections; the quintessence will be about using Lemma 6.79 (for first order terms w.r.t. σ) and Cor. 6.85 (for second order terms w.r.t. σ).

Proof of Prop. 6.92.
For this proof we will neglect caring for the base point information in the involved pullback bundles since the base point information is clear by context; recall Subsection 1.1. In that sense we also have $\mathscr{Ad}_\sigma \cong \pi^*\mathrm{Ad}_\sigma \cong \mathrm{Ad}_\sigma$ for all $\sigma \in \Gamma(\mathscr{G})$ *etc.*; this will simplify the following notation a bit.

• The second bullet point quickly follows by construction: The first summand of F (recall Def. 6.91) is clearly zero if either of X and Y is vertical; similar holds for the second summand because the vertical bundle is the kernel of $\mathrm{D}\pi$.

• Thence, let us focus on proving the first bullet point, and denote with $\pi_h : \mathrm{T}\mathscr{P} \to \mathrm{H}\mathscr{P}$ the canonical projection onto the associated Ehresmann connection $\mathrm{H}\mathscr{P}$ on $\mathscr{P}$, so that we write

$$F = \mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A \circ (\pi_h, \pi_h) + \pi^!\zeta.$$

We have for $\sigma \in \Gamma(\mathscr{G})$

$$\left(r_\sigma^! \pi^! \zeta\right)_p (X, Y) = \zeta_x \big(\mathrm{D}_{p\cdot\sigma_x}\pi(r_{\sigma_x *}(X)), \mathrm{D}_{p\cdot\sigma_x}\pi(r_{\sigma_x *}(Y))\big),$$

$$
\begin{aligned}
&= \zeta_x\Big(\underbrace{\mathrm{D}_{p\cdot\sigma_x}\pi(\mathrm{D}_p r_\sigma(X))}_{=\mathrm{D}_p(\pi\circ r_\sigma)=\mathrm{D}_p\pi}, \mathrm{D}_{p\cdot\sigma_x}\pi(\mathrm{D}_p r_\sigma(Y))\Big) \\
&= \left(\pi^!\zeta\right)_p(X,Y)
\end{aligned}
$$

for all $p \in \mathscr{P}_x$ ($x \in M$) and $X, Y \in \mathrm{T}_p\mathscr{P}$, making use of that fundamental vector fields are in the kernel of $\mathrm{D}\pi$ so that $\mathrm{D}\pi \circ \mathcal{r}_{\sigma_x *} = \mathrm{D}\pi \circ r_{\sigma_x *}$. For the first term in the field strength use Cor. 6.54 and Remark 6.55 so that

$$
\begin{aligned}
\mathcal{r}_\sigma^!\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A \circ (\pi_h, \pi_h)\right) &= \left(r_\sigma^*\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right) \circ (r_\sigma^*\pi_h \circ \mathcal{r}_{\sigma*}, r_\sigma^*\pi_h \circ \mathcal{r}_{\sigma*}) \\
&= \left(r_\sigma^*\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right) \circ (\mathcal{r}_{\sigma*} \circ \pi_h, \mathcal{r}_{\sigma*} \circ \pi_h) \\
&= \mathcal{r}_\sigma^!\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right) \circ (\pi_h, \pi_h).
\end{aligned}
$$

Let us denote $X^h := \pi_h(X)$ and $Y^h := \pi_h(Y)$, where X and Y are now elements of $\mathfrak{X}(\mathscr{P})$; since we are looking at tensorial equations we can w.l.o.g. assume that X^h and Y^h are the horizontal lifts of vector fields ω^X and ω^Y, respectively, on M, especially $\mathrm{D}\pi\big(X^h\big) = \mathrm{D}\pi(X) = \pi^*\omega^X$ and $\mathrm{D}\pi\big(Y^h\big) = \mathrm{D}\pi(Y) = \pi^*\omega^Y$. We also rewrite

$$
\mathcal{r}_{\sigma*}(X) = \mathrm{D}r_\sigma(X) - r_\sigma^*\left(\overparen{\left(\pi^!\Delta\sigma\right)(X)}\right) \in \Gamma(r_\sigma^*\mathrm{T}\mathscr{P}) \cong \mathfrak{X}(\mathscr{P}).
$$

In the following we will make use of the canonical identification $\mathfrak{X}(\mathscr{P}) \cong \Gamma(r_\sigma^*\mathrm{T}\mathscr{P})$, $X \mapsto r_\sigma^*X$, especially in order to calculate Lie brackets.

Then make use of Def. 6.45 and Thm. 6.53 to show

$$
\begin{aligned}
&r_\sigma^*\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right)\left(\mathcal{r}_{\sigma*}\big(X^h\big), \mathcal{r}_{\sigma*}\big(Y^h\big)\right)\right) \\
&\quad = r_\sigma^*\Bigg(\left(\pi^*\nabla^{\mathrm{YM}}\right)_{\mathcal{r}_{\sigma*}(X^h)}\underbrace{\left((A\circ\mathcal{r}_{\sigma*})\big(Y^h\big)\right)}_{=0} \\
&\qquad\quad - \left(\pi^*\nabla^{\mathrm{YM}}\right)_{\mathcal{r}_{\sigma*}(Y^h)}\left((A\circ\mathcal{r}_{\sigma*})\big(X^h\big)\right) \\
&\qquad\quad - A\left(\left[\mathcal{r}_{\sigma*}\big(X^h\big), \mathcal{r}_{\sigma*}\big(Y^h\big)\right]\right)\Bigg) \\
&\quad = r_\sigma^*\left(-A\left(\left[\mathcal{r}_{\sigma*}\big(X^h\big), \mathcal{r}_{\sigma*}\big(Y^h\big)\right]\right)\right) \\
&\quad = r_\sigma^*\Bigg(-A\left(\left[\mathrm{D}r_\sigma\big(X^h\big), \mathrm{D}r_\sigma\big(Y^h\big)\right]\right) \\
&\qquad\quad + A\left(\left[\mathrm{D}r_\sigma\big(X^h\big), r_\sigma^*\left(\overparen{\left(\pi^!\Delta\sigma\right)\big(Y^h\big)}\right)\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ A\left(\left[r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(X^h\right)}\right), \mathrm{D}r_\sigma\left(Y^h\right)\right]\right)\\
&- A\left(\left[r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(X^h\right)}\right), r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(Y^h\right)}\right)\right]\right).
\end{aligned}
\tag{61}
$$

Let us look at each summand individually; for the first summand we use the very well-known fact (see *e.g.* [1, §A.1.11, Cor. A.1.51, page 615]) that

$$
\left[\mathrm{D}r_\sigma\left(X^h\right), \mathrm{D}r_\sigma\left(Y^h\right)\right] = \mathrm{D}r_\sigma\left(\left[X^h, Y^h\right]\right),
$$

making use of that r_σ is an isomorphism, and thus we get for the first summand by using Def. 6.51

$$
\begin{aligned}
r_\sigma^*&\left(-A\left(\left[\mathrm{D}r_\sigma\left(X^h\right), \mathrm{D}r_\sigma\left(Y^h\right)\right]\right)\right)\\
&= -(r_\sigma^* A)\left(\mathscr{r}_{\sigma *}\left(\left[X^h, Y^h\right]\right)\right) - (r_\sigma^* A)\left(r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(\left[X^h, Y^h\right]\right)}\right)\right)\\
&= -\left(\mathscr{r}_\sigma^! A\right)\left(\left[X^h, Y^h\right]\right) - r_\sigma^*\left(\left(\pi^!\Delta\sigma\right)\left(\left[X^h, Y^h\right]\right)\right)\\
&= -(\mathscr{Ad}_{\sigma^{-1}} \circ A)\left(\left[X^h, Y^h\right]\right) - r_\sigma^*\left(\left(\pi^!\Delta\sigma\right)\left(\left[X^h, Y^h\right]\right)\right)\\
&= \mathscr{Ad}_{\sigma^{-1}}\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A\right)\left(X^h, Y^h\right)\right) - r_\sigma^*\left(\left(\pi^!\Delta\sigma\right)\left(\left[X^h, Y^h\right]\right)\right),
\end{aligned}
$$

where we used a similar argument for the last equality as for the beginning of the calculation for Eq. (61). We know by Remark 5.19 that the map to fundamental vector fields is a homomorphism of Lie algebras, so that

$$
\left[r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(X^h\right)}\right), r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(Y^h\right)}\right)\right] = r_\sigma^*\left(\widetilde{\left[\left(\pi^!\Delta\sigma\right)\left(X^h\right), \left(\pi^!\Delta\sigma\right)\left(Y^h\right)\right]_{\pi^*\mathscr{g}}}\right),
$$

and therefore we can derive for the fourth term in Eq. (61) that

$$
(r_\sigma^* A)\left(\left[r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(X^h\right)}\right), r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(Y^h\right)}\right)\right]\right) = r_\sigma^*\left(\left[\left(\pi^!\Delta\sigma\right)\left(X^h\right), \left(\pi^!\Delta\sigma\right)\left(Y^h\right)\right]_{\pi^*\mathscr{g}}\right).
$$

For the second and third summands in Eq. (61) we want to use Lemma 6.79 and our result of the fourth summand to show

$$
\begin{aligned}
(r_\sigma^* A)&\left(\left[\mathrm{D}r_\sigma\left(X^h\right), r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(Y^h\right)}\right)\right]\right)\\
&= (r_\sigma^* A)\left(\left[\mathscr{r}_{\sigma *}\left(X^h\right), r_\sigma^*\left(\widetilde{\left(\pi^!\Delta\sigma\right)\left(Y^h\right)}\right)\right]\right)
\end{aligned}
$$

$$+ (r_\sigma^* A)\left(\left[r_\sigma^*\left(\overline{\left(\pi^!\Delta\sigma\right)\left(X^h\right)}\right), r_\sigma^*\left(\overline{\left(\pi^!\Delta\sigma\right)\left(Y^h\right)}\right)\right]\right)$$

$$= r_\sigma^*\left(\left(\pi^*\nabla^{\mathrm{YM}}\right)_{\mathscr{r}_{\sigma*}(X^h)}\left(\left(\pi^!\Delta\sigma\right)\left(Y^h\right)\right)\right) + r_\sigma^*\left(\left[\left(\pi^!\Delta\sigma\right)\left(X^h\right), \left(\pi^!\Delta\sigma\right)\left(Y^h\right)\right]_{\pi^*\mathscr{g}}\right)$$

using that $\mathscr{r}_{\sigma*}(X^h)$ is horizontal due to the fact that X^h is horizontal, and, last but not least, we can write for terms like

$$\left(\pi^!\Delta\sigma\right)\left(Y^h\right) = (\pi^*\Delta\sigma)\left(\mathrm{D}\pi\left(Y^h\right)\right) = (\pi^*\Delta\sigma)(\pi^*\omega^Y) = \pi^*\left((\Delta\sigma)(\omega^Y)\right),$$

and

$$\mathrm{D}\pi\left(\mathscr{r}_{\sigma*}\left(X^h\right)\right) = \mathrm{D}\pi\left(\mathrm{D}r_\sigma\left(X^h\right) - r_\sigma^*\left(\overline{\left(\pi^!\Delta\sigma\right)\left(X^h\right)}\right)\right) = \mathrm{D}\underbrace{(\pi\circ r_\sigma)}_{=\pi}\left(X^h\right) = \omega^X,$$

using that fundamental vector fields are in the kernel of $\mathrm{D}\pi$, and therefore

$$\left(\pi^*\nabla^{\mathrm{YM}}\right)_{\mathscr{r}_{\sigma*}(X^h)}\left(\left(\pi^!\Delta\sigma\right)\left(Y^h\right)\right) = \pi^*\left(\nabla^{\mathrm{YM}}_{\omega^X}\left((\Delta\sigma)(\omega^Y)\right)\right).$$

Collecting all these results and using Cor. 6.85 we can write Eq. (61) as

$$\begin{aligned}
&r_\sigma^*\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right)\left(\mathscr{r}_{\sigma*}\left(X^h\right), \mathscr{r}_{\sigma*}\left(Y^h\right)\right)\right)\\
&\quad= \mathscr{Ad}_{\sigma^{-1}}\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right)\left(X^h, Y^h\right)\right)\\
&\qquad+ r_\sigma^*\Bigg(\pi^*\left(\nabla^{\mathrm{YM}}_{\omega^X}\left((\Delta\sigma)(\omega^Y)\right)\right) - \pi^*\left(\nabla^{\mathrm{YM}}_{\omega^Y}\left((\Delta\sigma)(\omega^X)\right)\right)\\
&\qquad\qquad- \underbrace{\left(\pi^!\Delta\sigma\right)\left(\left[X^h, Y^h\right]\right)}_{=\pi^*(\Delta\sigma([\omega^X,\omega^Y]))} + \left[\pi^*\left((\Delta\sigma)(\omega^X)\right), \pi^*\left((\Delta\sigma)(\omega^Y)\right)\right]_{\pi^*\mathscr{g}}\Bigg)\\
&\quad= \mathscr{Ad}_{\sigma^{-1}}\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right)\left(X^h, Y^h\right)\right) + \underbrace{r_\sigma^*\pi^*}_{=\pi^*}\left(\left(\mathrm{d}^{\nabla^{\mathrm{YM}}}\Delta\sigma + \frac{1}{2}\left[\Delta\sigma \overset{\wedge}{,} \Delta\sigma\right]_{\mathscr{g}}\right)(\omega^X, \omega^Y)\right)\\
&\quad= \mathscr{Ad}_{\sigma^{-1}}\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right)\left(X^h, Y^h\right)\right) + \pi^*\left((\mathrm{Ad}_{\sigma^{-1}}\circ\zeta - \zeta)(\mathrm{D}\pi(X), \mathrm{D}\pi(Y))\right)\\
&\quad= \mathscr{Ad}_{\sigma^{-1}}\left(\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\right)\left(X^h, Y^h\right)\right) + \left(\left(\pi^!(\mathrm{Ad}_{\sigma^{-1}}\circ\zeta - \zeta)\right)(X, Y)\right)\\
&\quad= \left(\mathscr{Ad}_{\sigma^{-1}}\circ\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}}A\circ(\pi_h, \pi_h) + \pi^!\zeta\right) - \pi^!\zeta\right)(X, Y)
\end{aligned}$$

where we used that $\mathrm{D}\pi(X) = \pi^*\omega^X$ and $\mathrm{D}\pi(Y) = \pi^*\omega^Y$, which immediately implies

$$\mathrm{D}\pi([X, Y]) = \pi^*\left(\left[\omega^X, \omega^Y\right]\right),$$

which is a well-known fact, as also given in [1, Proposition A.1.49; page 615], but also straightforward to check.

Finally, we can conclude this proof by combining the previous result with our first calculation of this proof, that is,

$$r_\sigma^! F = \mathscr{Ad}_{\sigma^{-1}} \circ \left(\mathrm{d}^{\pi^* \nabla^{\mathrm{YM}}} A \circ (\pi_h, \pi_h) + \pi^! \zeta \right) - \pi^! \zeta + \pi^! \zeta = \mathscr{Ad}_{\sigma^{-1}} \circ F.$$

■

We achieve of course also a structure equation.

Theorem 6.93: Structure equation of the generalized field strength

*Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$) and $A \in \Omega^1(\mathscr{P}; \pi^* \mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Then we have the **structure equation***

$$F = \mathrm{d}^{\pi^* \nabla^{\mathrm{YM}}} A + \frac{1}{2} [A \stackrel{\wedge}{,} A]_{\pi^* \mathscr{g}} + \pi^! \zeta.$$

Remark 6.94: The generalized field strength of the total Maurer-Cartan form

Again viewing $\mathscr{G} \xrightarrow{\pi_{\mathscr{G}}} M$ as the principal bundle itself with $\mathrm{H}\mathscr{P} = \mathrm{H}\mathscr{G}$, we know by Cor. 6.71 that the total Maurer-Cartan form $\mu_{\mathscr{G}}^{\mathrm{tot}}$ is the connection 1-form corresponding to $\mathrm{H}\mathscr{G}$. Denoting the associated generalized field strength by $F(\mu_{\mathscr{G}}^{\mathrm{tot}})$, the structure equation and the generalized Maurer-Cartan equation, Thm. 6.81 (also recall Rem. 6.82), imply

$$F_g\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right) = \mathscr{Ad}_{g^{-1}} \circ \pi_{\mathscr{G}}^! \zeta\Big|_g$$

for all $g \in \mathscr{G}_x$ $(x \in M)$. This implies that $F(\mu_{\mathscr{G}}^{\mathrm{tot}})$ is fully encoded by ζ, especially we have

$$e^! \left(F\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right) \right) = \zeta, \tag{62}$$

where e is the neutral section of $\mathscr{G}$. Thus, ζ is the curvature of $\mu_{\mathscr{G}}^{\mathrm{tot}}$ restricted on M, and

$$F = \mathrm{d}^{\pi^* \nabla^{\mathrm{YM}}} A + \frac{1}{2} [A \stackrel{\wedge}{,} A]_{\pi^* \mathscr{g}} + (e \circ \pi)^! \left(F\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right) \right).$$

Following similar calculations as in the proof of Prop. 6.92, it is trivial to show that the first two summands measure the failure of $\mathrm{H}\mathscr{P}$ being a foliation, hence this meaning carries over from the classical theory as also argued in [11, §2.5, Cor. 2.23] for LGBs. Due to the third summand, $\mathrm{H}\mathscr{P}$ is involutive if and only if

$$F = (e \circ \pi)^! \left(F\left(\mu_{\mathscr{G}}^{\mathrm{tot}}\right) \right).$$

In that sense Eq. (62) reflects the fact that M is a horizontal leave of $\mathrm{H}\mathscr{G}$. Furthermore, F can be non-zero while $\mathrm{H}\mathscr{P}$ is involutive, if $F(\mu_{\mathscr{G}}^{\mathrm{tot}})$ is non-zero along M.

Proof of Thm. 6.93.
The idea of the proof is similar to the proof of the "classical" statement; following the structure of [1, §5.5, Lemma 5.5.5, page 276], see also works like [9, §4.6, Lemma 4.25] and [11, §2.5, Prop. 2.24] where something similar is shown on Lie groupoids instead of principal bundles, but with no ζ. That is, we will look at vertical and horizontal tangent vectors and their mixed terms; recall Def. 6.51 and Thm. 6.53.

- We have by Prop. 6.92

$$F(\widetilde{\nu}, \widetilde{\mu}) = 0$$

for all $\mu, \nu \in \Gamma(\mathscr{g})$. Regarding the right hand side of the structure equation, this trivially also holds for the third summand $\left(\pi^! \zeta\right)(\widetilde{\nu}, \widetilde{\mu}) = 0$. We also have

$$\left(\frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}}\right)(\widetilde{\nu}, \widetilde{\mu}) = [A(\widetilde{\nu}), A(\widetilde{\mu})]_{\pi^*\mathscr{g}} = [\pi^*\nu, \pi^*\mu]_{\pi^*\mathscr{g}} = \pi^*\left([\nu, \mu]_{\mathscr{g}}\right),$$

and

$$\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A\right)(\widetilde{\nu}, \widetilde{\mu}) = \underbrace{\left(\pi^*\nabla^{\mathrm{YM}}\right)_{\widetilde{\nu}}(A(\widetilde{\mu}))}_{=\pi^*\left(\nabla^{\mathrm{YM}}_{\mathrm{D}\pi(\widetilde{\nu})}\mu\right)=0} - \left(\pi^*\nabla^{\mathrm{YM}}\right)_{\widetilde{\mu}}(A(\widetilde{\nu})) - A\big(\underbrace{[\widetilde{\nu}, \widetilde{\mu}]}_{=\widetilde{[\nu,\mu]}}\big) = -\pi^*\left([\nu, \mu]_{\mathscr{g}}\right).$$

Thus,

$$\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A + \frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}} + \pi^! \zeta\right)(\widetilde{\nu}, \widetilde{\mu}) = 0 = F(\widetilde{\nu}, \widetilde{\mu}).$$

- Let X and Y be horizontal vector fields of $\mathscr{P}$, then clearly

$$\left(\frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}}\right)(X, Y) = \big[A(X), A(Y)\big]_{\pi^*\mathscr{g}} = 0,$$

then clearly

$$\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A + \frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}} + \pi^! \zeta\right)(X, Y) = \left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A \circ (\pi_h, \pi_h) + \pi^! \zeta\right)(X, Y) = F(X, Y),$$

where $\pi_h : \mathrm{T}\mathscr{P} \to \mathrm{H}\mathscr{P}$ denotes the canonical projection onto the horizontal bundle $\mathrm{H}\mathscr{P}$ (especially, $\pi_h(X) = X$, $\pi_h(Y) = Y$).

- Now let $X \in \mathfrak{X}(\mathscr{P})$ be again horizontal and $\nu \in \Gamma(\mathscr{g})$; then again by Prop. 6.92

$$F(X, \widetilde{\nu}) = 0,$$

as also clearly $\left(\pi^! \zeta\right)(X, \widetilde{\nu}) = 0$, furthermore

$$\left(\frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}}\right)(X, \widetilde{\nu}) = [A(X), A(\widetilde{\nu})]_{\pi^*\mathscr{g}} = 0,$$

and by Lemma 6.79

$$\left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A\right)(X, \widetilde{\nu}) = \left(\pi^*\nabla^{\mathrm{YM}}\right)_X (\pi^*\nu) - A\big([X, \widetilde{\nu}]\big) \stackrel{6.79}{=} 0,$$

which concludes the proof with

$$F(X,\widetilde{\nu}) = \left(\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} A + \frac{1}{2}[A \,\overset{\wedge}{,}\, A]_{\pi^*\mathscr{g}} + \pi^!\zeta\right)(X,\widetilde{\nu}).$$

■

Concluding this subsubsection, we also achieve a generalized Bianchi identity, which we already have proven in [10, §7, Thm. 7.3] and [5, §5, Thm. 5.1.42]; the proof is straightforward to show, making use of the infinitesimal compatibility conditions, Def. 6.67.

Theorem 6.95: Generalized Bianchi identity, [5, §5, Thm. 5.1.42]

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ (w.r.t. a $\zeta \in \Omega^2(M;\mathscr{g})$) and $A \in \Omega^1(\mathscr{P};\pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Then we have the ***(generalized) Bianchi identity***

$$\mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} F + \left[A \,\overset{\wedge}{,}\, F\right]_{\pi^*\mathscr{g}} = \pi^! \mathrm{d}^{\nabla^{\mathrm{YM}}}\zeta.$$

6.4.3. Gauge transformation of the generalized field strength

Similar to Subsection 6.3 we will now discuss the gauge transformation of the generalized field strength F; recall Thm. 6.59.

Theorem 6.96: Gauge transformation of the generalized field strength

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P};\pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Furthermore, let $H \in \mathscr{Aut}(\mathscr{P})$. We then have that $H^!F$ is the field strength related to $H^!A$ and

$$H^!F = \mathscr{Ad}_{\mathrm{pr}_2 \circ (\sigma^H)^{-1}} \circ F,$$

where $\sigma^H \in C^\infty(\mathscr{P};\mathscr{G})^{\mathscr{G}}$ is defined as in Prop. 6.58 and $\mathrm{pr}_2 : \pi^*\mathscr{G} \to \mathscr{G}$ is the projection onto the second component.
Similar to Def. (51) we may shortly just write

$$H^!F = \mathscr{Ad}_{(\sigma^H)^{-1}} \circ F.$$

Proof.
That $H^!F$ is the field strength related to $H^!A$ follows quickly by the same calculation as for proving Eq. (59), using $\pi \circ H = \pi$,

$$H^!F = \mathrm{d}^{(\pi\circ H)^*\nabla^{\mathrm{YM}}}\left(H^!A\right) + \frac{1}{2}\left[H^!A \,\overset{\wedge}{,}\, H^!A\right]_{\pi^*\mathscr{g}} + (\pi \circ H)^!\zeta$$

$$= \mathrm{d}^{\pi^*\nabla^{\mathrm{YM}}} \left(H^! A \right) + \frac{1}{2} \left[H^! A \stackrel{\wedge}{,} H^! A \right]_{\pi^* \mathscr{g}} + \pi^! \zeta,$$

which is the field strength of $H^! A$. Now recall Eq. (53), that is,

$$\mathrm{D}_p H(X) = r_{\widetilde{\sigma}_p *}(X) + \left. \overbrace{\left((\pi^* \Delta) \sigma^H \right)_p (X)} \right|_{H(p)}$$

for all $X \in \mathrm{T}_p \mathscr{P}$ ($p \in \mathscr{P}$), where $\widetilde{\sigma}_p \coloneqq \mathrm{pr}_2\left(\sigma_p^H \right)$. Thus, by Prop. 6.92,

$$\begin{aligned}
\left(H^! F \right)_p (X, Y) &= F_{H(p)} \left(\mathrm{D}_p H(X), \mathrm{D}_p H(Y) \right) \\
&= F_{p \cdot \widetilde{\sigma}_p} \left(r_{\widetilde{\sigma}_p *}(X), r_{\widetilde{\sigma}_p *}(Y) \right) \\
&= \left(r_{\widetilde{\sigma}}^! F \right)_p (X, Y) \\
&= \left(\mathscr{Ad}_{\widetilde{\sigma}_p^{-1}} \circ F \right)_p (X, Y)
\end{aligned}$$

for all $X, Y \in \mathrm{T}_p \mathscr{P}$; this finishes the proof. ■

We will now extend this again to a local change of gauges (sections of $\mathscr{P}$).

Definition 6.97: Local field strength

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^* \mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Furthermore, let $s \in \Gamma(\mathscr{P}|_U)$ be a (local) gauge over an open subset $U \subset M$. Then we define the **local curvature** of **local field strength** $F_s \in \Omega^2(U; \mathscr{g}|_U)$ **(w.r.t.** s**)** by

$$F_s \coloneqq s^! F.$$

In previous calculations for the total Maurer-Cartan form and the generalized Maurer-Cartan equation we have basically already shown the pullback of the structure equation.

Corollary 6.98: Pullback of the structure equation

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^* \mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Furthermore, let $s \in \Gamma(\mathscr{P}|_U)$ be a (local) gauge over an open subset $U \subset M$. Then we can express the local field strength as the pullback of the structure equation, that is,

$$F_s = \mathrm{d}^{\nabla^{\mathrm{YM}}} A_s + \frac{1}{2} \left[A_s \stackrel{\wedge}{,} A_s \right]_{\mathscr{g}} + \zeta.$$

Proof.
This follows quickly by the same calculation as for proving Eq. (59). ■

The gauge transformation of the field strength F describes the behaviour of F under a change of gauge.

Theorem 6.99: Gauge transformations again as a change of gauge

Let $\mathscr{G} \to M$ be an LGB over a smooth manifold M and $\mathscr{P} \xrightarrow{\pi} M$ a principal $\mathscr{G}$-bundle, also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Also let U_i and U_j be two open subsets of M so that $U_i \cap U_j \neq \emptyset$, two gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$ and $s_j \in \Gamma(\mathscr{P}|_{U_j})$, and the unique $\sigma_{ji} \in \Gamma\left(\mathscr{G}|_{U_i \cap U_j}\right)$ with $s_i = s_j \cdot \sigma_{ji}$ on $U_i \cap U_j$.
Then we have for the fields strength of A that

$$F_{s_i} = \mathrm{Ad}_{\sigma_{ji}^{-1}} \circ F_{s_j}.$$

Proof.
The proof is precisely as for Thm. 6.65, just without catering for the Darboux derivative in the gauge transformation due to that one uses Thm. 6.96 instead of Thm. 6.59. ■

Remark 6.100: Integrating curved Yang-Mills gauge theories, part II

Similar to Rem. 6.66, Cor. 6.98 and Thm. 6.99 show that we recover the field strength and its gauge transformation as developed by Alexei Kotov and Thomas Strobl, see [8] for a concise summary or [5] for an extended introduction. These references are for the very general situation using Lie algebroids, hence see alternatively [10] for this type of gauge theory restricted to Lie algebra bundles.

7. Curved Yang-Mills gauge theory

We eventually are able to conclude this paper with the definition of what we will call **curved Yang-Mills gauge theory**, highlighting that this theory is an integral of the infinitesimal gauge theory originally developed by Alexei Kotov and Thomas Strobl. For the following we say that a fibre metric κ of $\mathscr{g}$ is Ad-**invariant** if

$$\kappa\left(\mathrm{Ad}_g(\nu), \mathrm{Ad}_g(\mu)\right) = \kappa(\nu, \mu)$$

for all $g \in \mathscr{G}_x$ ($x \in M$) and $\mu, \nu \in \mathscr{g}_x$.

Corollary 7.1: Contraction of local field strength with Ad-invariant fibre metric is well-defined

Let $\mathscr{G} \to M$ be an LGB over a spacetime M so that its LAB $\mathscr{g}$ admits an Ad-invariant fibre metric κ, and let $\mathscr{P} \xrightarrow{\pi} M$ be a principal $\mathscr{G}$-bundle; also let $\mathrm{H}\mathscr{G}$ be a Yang-Mills connection on $\mathscr{G}$ and $A \in \Omega^1(\mathscr{P}; \pi^*\mathscr{g})$ be a connection 1-form on $\mathscr{P}$. Furthermore, let

$(U_i)_i$ be an open covering of M so that there are subordinate gauges $s_i \in \Gamma(\mathscr{P}|_{U_i})$. Then the map $\mathfrak{L}_{\mathrm{CYM}}[A] : M \to \mathbb{R}$, defined locally by

$$\left(\mathfrak{L}_{\mathrm{CYM}}[A]\right)\big|_{U_i} := -\frac{1}{2}\kappa\left(F_{s_i} \stackrel{\wedge}{,} *F_{s_i}\right),$$

is well-defined and independent of the choice of gauge, where $*$ is the Hodge star operator w.r.t. the spacetime metric of M.

Proof.
Let s_j be another gauge corresponding to U_j so that $U_i \cap U_j \neq \emptyset$ and the unique $\sigma_{ji} \in \Gamma\left(\mathscr{G}|_{U_i \cap U_j}\right)$ with $s_i = s_j \cdot \sigma_{ji}$ on $U_i \cap U_j$. Then it follows by Thm. 6.99 that

$$\kappa\left(F_{s_i} \stackrel{\wedge}{,} *F_{s_i}\right) = \kappa\left(\left(\mathrm{Ad}_{\sigma_{ji}^{-1}} \circ F_{s_j}\right) \stackrel{\wedge}{,} *\left(\mathrm{Ad}_{\sigma_{ji}^{-1}} \circ F_{s_j}\right)\right)$$

■

8. Conclusion

Also mention that the new connection is not necessarily equivalent to a classical connection even though the related gauge theories are. What does it imply on theories based on the sense of connections like general relativity?

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A. Double tangent bundle and its canonical flip map

We will follow [2, §9.6, page 363]. For a smooth manifold M we denote the projection of its tangent bundle by $\pi_{\mathrm{T}M} : \mathrm{T}M \to M$; similarly we have the projection of the **double tangent bundle** $\pi_{\mathrm{TT}M} : \mathrm{TT}M \to \mathrm{T}M$, the tangent bundle of $\mathrm{T}M$. However, there is also $\mathrm{D}\pi_{\mathrm{T}M}$:

$\mathrm{TT}M \to \mathrm{T}M$, and in fact there is another vector bundle structure on $\mathrm{TT}M$ rendering $\mathrm{D}\pi_{\mathrm{T}M}$ a projection; see *e.g.* [2, §3.4 *et seq.*; page 110ff.]. Let us give a very rough sketch:

Let $\xi, \eta \in \mathrm{TT}M$ with

$$\mathrm{D}_{X_0}\pi_{\mathrm{T}M}(\xi) = \mathrm{D}_{Y_0}\pi_{\mathrm{T}M}(\eta) =: \omega,$$

where $X_0 \coloneqq \pi_{\mathrm{TT}M}(\xi)$ and $Y_0 \coloneqq \pi_{\mathrm{TT}M}(\eta)$, and due to the fact that $\mathrm{D}\pi_{\mathrm{T}M}$ is a vector bundle morphism over $\pi_{\mathrm{T}M}$ we get

$$p = \pi_{\mathrm{T}M}(X_0) = \pi_{\mathrm{T}M}(Y_0),$$

where $\pi_{\mathrm{T}M}(\omega) =: p$. Thus, one can take curves $X, Y : I \to \mathrm{T}M$ ($I \subset \mathbb{R}$ an open interval around 0) with

$$X(0) = X_0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} X = \xi,$$

$$Y(0) = Y_0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Y = \eta,$$

such that

$$\pi_{\mathrm{T}M} \circ X = \pi_{\mathrm{T}M} \circ Y,$$

because the condition on ξ and η imply on the base paths $\pi_{\mathrm{T}M} \circ X, \pi_{\mathrm{T}M} \circ Y : I \to M$ that

$$(\pi_{\mathrm{T}M} \circ X)(0) = p = (\pi_{\mathrm{T}M} \circ Y)(0),$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\pi_{\mathrm{T}M} \circ X\big) = \mathrm{D}_{X_0}\pi_{\mathrm{T}M}(\xi) = \omega = \mathrm{D}_{Y_0}\pi_{\mathrm{T}M}(\eta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\pi_{\mathrm{T}M} \circ Y\big).$$

Then the addition and scalar multiplication with $\lambda \in \mathbb{R}$ for $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} \mathrm{T}M$ is defined by

$$\xi \,\boldsymbol{+}\, \eta \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (X + Y),$$

$$\lambda \boldsymbol{\cdot} \xi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\lambda X),$$

where the addition of curves is well-defined because of $\pi_{\mathrm{T}M} \circ X = \pi_{\mathrm{T}M} \circ Y$ which implies $\pi_{\mathrm{T}M} \circ (X + Y) = \pi_{\mathrm{T}M} \circ X = \pi_{\mathrm{T}M} \circ Y$; so, one can take the sum of the curves and

$$\mathrm{D}\pi_{\mathrm{T}M}(\xi \,\boldsymbol{+}\, \eta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \underbrace{\big(\pi_{\mathrm{T}M} \circ (X + Y)\big)}_{= \pi_{\mathrm{T}M} \circ X} = \mathrm{D}_{X_0}\pi_{\mathrm{T}M}(\xi) = \omega.$$

The operations of the linear structure in $\mathrm{TT}M \overset{\pi_{\mathrm{TT}M}}{\to} \mathrm{T}M$ is still denoted in the same manner as usual, and by definition one also gets

$$\pi_{\mathrm{TT}M}(\xi \,\boldsymbol{+}\, \eta) = \pi_{\mathrm{TT}M}(\xi) + \pi_{\mathrm{TT}M}(\eta),$$

$$\pi_{\mathrm{TT}M}(\lambda \cdot \xi) = \lambda\ \pi_{\mathrm{TT}M}(\xi).$$

In total, we have a double vector bundle given by the following commuting diagram

$$\begin{array}{ccc} \mathrm{TT}M & \xrightarrow{\mathrm{D}\pi_{\mathrm{T}M}} & \mathrm{T}M \\ \downarrow{\scriptstyle \pi_{\mathrm{TT}M}} & & \downarrow{\scriptstyle \pi_{\mathrm{T}M}} \\ \mathrm{T}M & \xrightarrow{\pi_{\mathrm{T}M}} & M \end{array} \tag{A.1}$$

i.e. each horizontal and vertical line is a vector bundle so that the horizontal and vertical scalar multiplications on $\mathrm{TT}M$ commute; see *e.g.* [15, §3ff.] or [2, §9.1, page 340ff.] for a definition on double vector bundles in general.

Now observe that the flow X of ξ has values in $\mathrm{T}M$, that is, for all $t \in I$ we have a curve $\alpha_t : J \to M$ (J another open interval containing 0), $s \mapsto \alpha_t(s)$, so that

$$\alpha_t(0) = \pi_{\mathrm{T}M}\big(X(t)\big),$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha_t = X(t),$$

and the first equation implies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha_t(0) = \mathrm{D}_{X_0}\pi_{\mathrm{T}M}(\xi) = \omega.$$

So, in total we have for all $\xi \in \mathrm{TT}M$ a smooth map $\alpha : I \times J \to M$, $(t,s) \mapsto \alpha(t,s) = \alpha_t(s)$, such that

$$\alpha(0,0) = p = (\pi_{\mathrm{T}M} \circ \pi_{\mathrm{TT}M})(\xi) = (\pi_{\mathrm{T}M} \circ \mathrm{D}\pi_{\mathrm{T}M})(\xi),$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha(t,0) = \omega = \mathrm{D}_{X_0}\pi_{\mathrm{T}M}(\xi), \qquad \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha(0,s) = X_0 = \pi_{\mathrm{TT}M}(\xi),$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha = \xi.$$

As for tangent vectors, the class $[\alpha]$ of all such α uniquely describes ξ, in fact giving rise to an equivalence relation so that $[\alpha]$ is an equivalence class.

In the context of Schwarz's Theorem one may find it natural to define the **canonical involution (or flip) on** $\mathrm{TT}M$ as a map $S : \mathrm{TT}M \to \mathrm{TT}M$ by

$$S(\xi) \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha. \tag{A.2}$$

By [2, §9.6, Thm. 9.6.1, page 363; but without proof] one has that S is an isomorphism of double vector bundles with certain "special" properties; we will not need the general definition of these. What we need of this, is that S is a base-preserving vector bundle isomorphism as both maps, from $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} \mathrm{T}M$ to $\mathrm{TT}M \overset{\pi_{\mathrm{TT}M}}{\to} \mathrm{T}M$ and vice versa. Let us prove this, by starting with showing the well-definedness of S:

W.l.o.g. we can put $J = I$ for simplicity. By construction, the equivalence class $[\beta]$ of $S(\xi)$ is represented by $\beta : I^2 \to M$ given by $\beta(t,s) \coloneqq \alpha(s,t)$ with

$$\begin{aligned}
\beta(0,0) &= p, \\
\mathrm{D}\pi_{\mathrm{T}M}\big(S(\xi)\big) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \beta(t,0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha(0,t) = \pi_{\mathrm{TT}M}(\xi) = X_0, \\
\pi_{\mathrm{TT}M}\big(S(\xi)\big) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \beta(0,s) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha(s,0) = \mathrm{D}\pi_{\mathrm{T}M}(\xi) = \omega, \\
S(\xi) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \beta.
\end{aligned}$$

With this, we can express $S(\xi)$ in coordinates: Fix local coordinates $\left(x^i\right)_i$ on M, then $\left(\pi_{\mathrm{T}M}^* x^i\right)_i$ and $\left(\mathrm{d}x^i\right)_i$ are local coordinates on $\mathrm{T}M$. Then by chain rule

$$\xi\big(\pi_{\mathrm{T}M}^* x^i\big) = \xi\big(x^i \circ \pi_{\mathrm{T}M}\big) = \mathrm{d}_p x^i\big(\mathrm{D}\pi_{\mathrm{T}M}(\xi)\big) = \omega^i,$$

and

$$\xi\big(\mathrm{d}x^i\big) = \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha\right)\big(\mathrm{d}x^i\big) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \underbrace{\left(\mathrm{d}x^i \circ \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha\right)}_{=\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\alpha\right)(x^i)} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha^i,$$

thus we have in total

$$\xi = \omega^i \left.\frac{\partial}{\partial\big(\pi_{\mathrm{T}M}^* x^i\big)}\right|_{X_0} + \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha^i\right) \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{X_0}. \tag{A.3}$$

Similarly we can proceed with $S(\xi)$ and then apply the "classical" Schwarz' Theorem on the derivatives of α^i (recall the properties of β), so that we get

$$\begin{aligned}
S(\xi) &= X_0^i \left.\frac{\partial}{\partial\big(\pi_{\mathrm{T}M}^* x^i\big)}\right|_{\omega} + \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \beta^i\right) \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{\omega} \\
&= X_0^i \left.\frac{\partial}{\partial\big(\pi_{\mathrm{T}M}^* x^i\big)}\right|_{\omega} + \left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha^i\right) \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{\omega} \\
&= X_0^i \left.\frac{\partial}{\partial\big(\pi_{\mathrm{T}M}^* x^i\big)}\right|_{\omega} + \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha^i\right) \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{\omega}.
\end{aligned} \tag{A.4}$$

It is now clear that $S(\xi)$ is independent of the choice of α and just depends on $[\alpha]$. Furthermore, due to what we have shown for β, it is also clear that S is a base-preserving map either from $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} \mathrm{T}M$ to $\mathrm{TT}M \overset{\pi_{\mathrm{TT}M}}{\to} \mathrm{T}M$ or vice versa. Let $[\gamma] \in \mathrm{TT}M$, then by the same calculation as for the properties of β we have $S([\tau]) = [\gamma]$, where $\tau(t,s) := \gamma(s,t)$; thence, surjectivity of S follows. Similarly, again by the properties of $\beta(t,s)$ above, if we have $S([\alpha]) = S([\alpha'])$, then we clearly get $[\alpha] = [\alpha']$, and hence injectivity. Alternatively, bijectivity follows by $S \circ S = \mathbb{1}_{\mathrm{TT}M}$.

It is only left to show that we have linearity for S as a map from $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} \mathrm{T}M$ to $\mathrm{TT}M \overset{\pi_{\mathrm{TT}M}}{\to} \mathrm{T}M$ and vice versa. For this we use the derived coordinate expressions. Let $\eta \in \mathrm{TT}M$ be defined as before with associated area function γ such that $\eta = [\gamma]$. Then one can derive by using what we have shown in the discussion about $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} \mathrm{T}M$ as vector bundle that

$$\lambda \cdot \xi \boldsymbol{+} \kappa \cdot \eta = \omega^i \left. \frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)} \right|_{\lambda X_0 + \kappa Y_0} + \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \left(\lambda \alpha^i + \kappa \gamma^i\right) \right) \left. \frac{\partial}{\partial(\mathrm{d}x^i)} \right|_{\lambda X_0 + \kappa Y_0} \tag{A.5}$$

for all $\lambda, \kappa \in \mathbb{R}$, and so

$$\begin{aligned} S(\lambda \cdot \xi \boldsymbol{+} \kappa \cdot \eta) &= \left(\lambda X^i_0 + \kappa Y^i_0\right) \left. \frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)} \right|_{\omega} + \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \left(\lambda \alpha^i + \kappa \gamma^i\right) \right) \left. \frac{\partial}{\partial(\mathrm{d}x^i)} \right|_{\omega} \\ &= \lambda S(\xi) + \kappa S(\eta). \end{aligned}$$

In the same fashion, let $\zeta = [\delta] \in \mathrm{T}_{X_0}\mathrm{T}M$ with $\mathrm{D}_{X_0}\pi_{\mathrm{T}M}(\zeta) = \varphi$, then $\lambda\xi + \kappa\zeta$ is just the typical sum of tangent vectors and we get

$$\begin{aligned} S(\lambda\xi + \kappa\zeta) &= X^i_0 \left. \frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)} \right|_{\lambda\omega + \kappa\varphi} + \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \left(\lambda \alpha^i + \kappa \delta^i\right) \right) \left. \frac{\partial}{\partial(\mathrm{d}x^i)} \right|_{\lambda\omega + \kappa\varphi} \\ &= \lambda \cdot \left(X^i_0 \left. \frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)} \right|_{\omega} + \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \alpha^i \right) \left. \frac{\partial}{\partial(\mathrm{d}x^i)} \right|_{\omega} \right) \\ &\qquad \boldsymbol{+} \kappa \cdot \left(X^i_0 \left. \frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)} \right|_{\varphi} + \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \delta^i \right) \left. \frac{\partial}{\partial(\mathrm{d}x^i)} \right|_{\varphi} \right) \\ &= \lambda \cdot S(\xi) \boldsymbol{+} \kappa \cdot S(\zeta). \end{aligned}$$

This finishes the proof, so we have:

Remark A.1: The canonical involution/flip map an isomorphism, [2, §9.6, Thm. 9.6.1, page 363; but without proof]

The canonical involution/flip S is a base-preserving vector bundle isomorphism from $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} \mathrm{T}M$ to $\mathrm{TT}M \overset{\pi_{\mathrm{TT}M}}{\to} \mathrm{T}M$, and similarly vice versa. We also write $S =: S_M$ to give an accentuation on M.

This isomorphism is basically now Schwarz's Theorem:

Remark A.2: Revisit: Schwarz's Theorem

By definition we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\alpha = S_M\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\alpha\right)$$

for all smooth $\alpha : I \times J \to M$, where I and J are open intervals containing 0. Similarly this extends to smooths maps $F : M \to N$, where N is another smooth manifold. So let $\xi = [\alpha] \in \mathrm{TT}M$, and then we calculate for $\mathrm{DD}F : \mathrm{TT}M \to \mathrm{TT}N$ that

$$\begin{aligned}
\mathrm{DD}F(\xi) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}(F\circ\alpha)\\
&= S_N\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(F\circ\alpha)\right)\\
&= S_N\big(\mathrm{DD}F\big(S_M(\xi)\big)\big)\\
&= (S_N\circ\mathrm{DD}F\circ S_M)(\xi).
\end{aligned}$$

Since the canonical involutions are clearly self-inverse, we can also write

$$S_N\circ\mathrm{DD}F = \mathrm{DD}F\circ S_M.$$

For a notation with base points in M, consider a special case with $M = M_1 \times M_2$, where M_i ($i \in \{1,2\}$) are smooth manifolds. Thus, $\mathrm{T}M \cong \pi_1^*\mathrm{T}M_1 \oplus \pi_2^*\mathrm{T}M_2$, where $\pi_i : M_1 \times M_2 \to M_i$ are the projections onto the i-th component. Then let $p_i \in M_i$, $Y_i \in \mathrm{T}_{p_i}M_i$ and $\xi \coloneqq (Y_1, Y_2)$. Denoting with $\gamma_1 : I \to M_1$ and $\gamma_2 : J \to M_2$ the curves with velocities Y_1 and Y_2 at 0, respectively, and so $\alpha(t,s) = (\gamma_1(s), \gamma_2(t))$; then we define $\mathrm{D}_{p_1}F(Y_1)$ as a map

$$\begin{aligned}
M_2 &\to \mathrm{T}N,\\
p_2 &\mapsto \mathrm{D}_{p_1}F(Y_1)\big|_{p_2} = \mathrm{D}_{(p_1,p_2)}F(Y_1, 0_{p_2}),
\end{aligned}$$

where 0_{p_2} is the zero tangent vector at $\mathrm{T}_{p_2}M_2$, in a similar fashion for $\mathrm{D}_{p_2}F(Y_2)$. Then

$$\begin{aligned}
\mathrm{D}_{p_2}\big(\mathrm{D}_{p_1}F(Y_1)\big)(Y_2) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{D}_{(p_1,\gamma_2(t))}F\big(Y_1, 0_{\gamma_2(t)}\big)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\big(F\circ\alpha(t,s)\big)\\
&= \mathrm{DD}F(\xi),
\end{aligned}$$

similarly

$$\mathrm{D}_{p_1}\big(\mathrm{D}_{p_2}F(Y_2)\big)(Y_1) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\mathrm{D}_{(\gamma_1(s),p_2)}F\big(0_{\gamma_1(s)}, Y_2\big)$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(F \circ \alpha(t,s)\big)$$

$$= S_N\big(\mathrm{DD}F(\xi)\big),$$

in total

$$\mathrm{D}_{p_1}\big(\mathrm{D}_{p_2}F(Y_2)\big)(Y_1) = S_N\Big(\mathrm{D}_{p_2}\big(\mathrm{D}_{p_1}F(Y_1)\big)(Y_2)\Big).$$

In fact, using the canonical flip/involution, we can construct and state other properties which can be useful for calculations related to second derivatives.

Remark A.3: Total derivatives of tangent bundle morphisms linear with respect to prolonged vertical structure

Consider a vector bundle morphism $L : \mathrm{T}M \to \mathrm{T}N$, where N is another smooth manifold. Then for $\xi, \eta \in \mathrm{TT}M$ with their approximating curves X and Y, respectively, as previously, then

$$\begin{aligned}
\mathrm{D}L(\lambda \cdot \xi \,\boldsymbol{+}\, \kappa \cdot \eta) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(L(\lambda X + \kappa Y)\big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\lambda L(X) + \kappa L(Y)\big) \\
&= \lambda \cdot \mathrm{D}L(\xi) \,\boldsymbol{+}\, \kappa \cdot \mathrm{D}L(\eta)
\end{aligned}$$

for all $\lambda, \kappa \in \mathbb{R}$. Hence we achieve linearity with respect to both vector bundle structures of $\mathrm{TT}M$. If we denote the base points as before, then this reads

$$\mathrm{D}_{\lambda X_0 + \kappa Y_0} L(\lambda \cdot \xi \,\boldsymbol{+}\, \kappa \cdot \eta) = \lambda \cdot \mathrm{D}_{X_0} L(\xi) \,\boldsymbol{+}\, \kappa \cdot \mathrm{D}_{Y_0} L(\eta).$$

Remark A.4: Alignment of both vector bundle structures on $\mathrm{TT}M$ on the restricted vertical bundle

By definition, the vertical bundle $\mathrm{VT}M$ of $\mathrm{T}M$ is a subbundle of $\mathrm{TT}M$. The zero section of $\mathrm{T}M$ is a natural embedding of M into $\mathrm{T}M$; this embedded submanifold will be denoted by $\widetilde{M}$. Then Diagram (A.1) restricts onto

$$\begin{tikzcd}
\mathrm{VT}M|_{\widetilde{M}} \arrow[r, "\mathrm{D}\pi_{\mathrm{T}M}"] \arrow[d, "\pi_{\mathrm{TT}M}"'] & \widetilde{M} \arrow[d, "\cong"] \\
\widetilde{M} \arrow[r, "\cong"] & M
\end{tikzcd}$$

and $S = S_M$ does not only restrict onto that, S is actually the identity on $\mathrm{VT}M|_{\widetilde{M}}$, as also stated in [2, §9.6, Thm. 9.6.1, page 363; but without proof]. This follows simply by the coordinate expressions Eq. (A.3) and (A.4) (recall the involved notation). $\xi \in \mathrm{VT}M|_{\widetilde{M}}$ is in the kernel of both projections, $\mathrm{D}\pi_{\mathrm{T}M}$ and $\pi_{\mathrm{TT}M}$, thus $\omega = X_0 = 0 \in \mathrm{T}_pM$, therefore $S(\xi) = \xi$ by Eq. (A.3) and (A.4). By Remark A.1 the addition $\boldsymbol{+}$ and scalar multiplication $\bullet$ align with the typical addition $+$ and scalar multiplication $\cdot$, respectively, of $\mathrm{TT}M$ as tangent bundle.

Motivated by this, we can actually recover something similar for $\mathrm{VT}M$. In this case

$$\begin{tikzcd} \mathrm{VT}M \arrow[r, "\mathrm{D}\pi_{\mathrm{T}M}"] \arrow[d, "\pi_{\mathrm{TT}M}"] & \widetilde{M} \arrow[d, "\cong"] \\ \mathrm{T}M \arrow[r, "\pi_{\mathrm{T}M}"] & M \end{tikzcd}$$

and, as already mentioned earlier, the vertical bundle of vector bundles is trivially the pullback of the bundle along itself, here $\mathrm{VT}M \cong \pi^*_{\mathrm{T}M}\mathrm{T}M$. Thus, we have a canonical projection $\mathrm{pr}_2 : \mathrm{VT}M \to \mathrm{T}M$ onto the second component, and we can write for $\xi = [\alpha] \in \mathrm{V}_{X_0}\mathrm{T}M$

$$\xi = \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha^i \right) \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{X_0} \cong \left(X_0, \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha^i \right) \left.\frac{\partial}{\partial x^i}\right|_p \right)$$

using the notation of Eq. (A.3) (especially $X_0 \in \mathrm{T}_pM$, $p \in M$). Similarly for $\eta = [\beta] \in \mathrm{V}_{Y_0}\mathrm{T}M$ (same notation as previously in this appendix, *i.e.* $Y_0 \in \mathrm{T}_pM$). Then by Eq. (A.5)

$$\lambda \bullet \xi \boldsymbol{+} \kappa \bullet \eta \cong \left(\lambda X_0 + \kappa Y_0, \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\lambda\alpha^i + \kappa\beta^i\right) \right) \left.\frac{\partial}{\partial x^i}\right|_p \right)$$

for all $\lambda, \kappa \in \mathbb{R}$. Therefore

$$\mathrm{pr}_2(\lambda \bullet \xi \boldsymbol{+} \kappa \bullet \eta) = \lambda\ \mathrm{pr}_2(\xi) + \kappa\ \mathrm{pr}_2(\eta).$$

For $\zeta \in \mathrm{V}_{X_0}\mathrm{T}M$ we clearly get $\mathrm{pr}_2(\lambda\xi + \kappa\zeta) = \lambda\ \mathrm{pr}_2(\xi) + \kappa\ \mathrm{pr}_2(\zeta)$, and so both linear structures on $\mathrm{VT}M$ align under pr_2.

Remark A.5: Tangent lift, [16, §2.2, last parapgraph in Subsection 2.2]

Let $X \in \mathfrak{X}(M)$, then its total derivative is a map $\mathrm{D}X : \mathrm{T}M \to \mathrm{TT}M$ satisfying

$$\mathrm{D}\pi_{\mathrm{T}M} \circ \mathrm{D}X = \mathrm{D}\underbrace{(\pi_{\mathrm{T}M} \circ X)}_{=\mathbb{1}_M} = \mathbb{1}_{\mathrm{T}M},$$

so that $\mathrm{D}X$ is a section of $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} M$. Due to the fact that $S = S_M$ is a base-preserving vector bundle isomorphism from $\mathrm{TT}M \overset{\mathrm{D}\pi_{\mathrm{T}M}}{\to} M$ to $\mathrm{TT}M \overset{\pi_{\mathrm{T}M}}{\to} M$ (and vice versa), we have a vector field $X_T \in \mathfrak{X}(\mathrm{T}M)$ given by

$$X_T := S \circ \mathrm{D}X,$$

called the **tangent lift of** X. We then also have

$$\mathrm{D}\pi_{\mathrm{T}M} \circ X_T = \pi_{\mathrm{TT}M} \circ \mathrm{D}X = X,$$

thus the label as lift. We also have linearity properties (sort of) and a Leibniz rule, that is, we clearly get

$$X_T(\lambda Y + \kappa Z) = \lambda \cdot X_T(Y) \boldsymbol{+} \kappa \cdot X_T(Z).$$

for all $Y, Z \in \mathrm{T}_p M$ and $\kappa, \lambda \in \mathbb{R}$, and we also have something similar w.r.t. X: Recall Eq. (A.3) and (A.4), including their notation w.r.t. to local coordinates $\left(x^i\right)_i$. That is, by definition of $\mathrm{D}X$ we can derive

$$\mathrm{D}_p X(Y) = Y^i \left.\frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)}\right|_{X_p} + Y^j \left.\frac{\partial X^i}{\partial x^j}\right|_p \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{X_p},$$

thus,

$$\begin{aligned}\mathrm{D}_p(\kappa X + \lambda W)(Y) &= Y^i \left.\frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)}\right|_{\lambda X_p + \kappa W_p} + Y^j \left.\frac{\partial\left(\lambda X^i + \kappa W^i\right)}{\partial x^j}\right|_p \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{\lambda X_p + \kappa W_p} \\ &= \lambda \cdot \mathrm{D}_p X(Y) \boldsymbol{+} \kappa \cdot \mathrm{D}_p W(Y)\end{aligned}$$

for all $X, W \in \mathfrak{X}(M)$ and $\lambda, \kappa \in \mathbb{R}$, and

$$\begin{aligned}&\mathrm{D}_p(fX)(Y) \\ &= Y^i \left.\frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)}\right|_{f(p)X_p} + f(p)\ Y^j \left.\frac{\partial X^i}{\partial x^j}\right|_p \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{f(p)X_p} + Y^j X^i(p) \left.\frac{\partial f}{\partial x^j}\right|_p \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_{f(p)X_p}\end{aligned} \tag{A.6}$$

for all $f \in C^\infty(M)$. Therefore

$$(\lambda X + \kappa W)_T = \lambda X_T + \kappa W_T,$$

and by Eq. (A.4)

$$(fX)_T(Y) = f(p)\ X^i_p \left.\frac{\partial}{\partial\left(\pi^*_{\mathrm{T}M} x^i\right)}\right|_Y + f(p)\ Y^j \left.\frac{\partial X^i}{\partial x^j}\right|_p \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_Y + Y(f)\ X^i(p) \left.\frac{\partial}{\partial(\mathrm{d}x^i)}\right|_Y$$

so

$$(fX)_T = f\ X_T + \mathrm{d}f \otimes X^i \frac{\partial}{\partial(\mathrm{d}x^i)}.$$

Similar to Remark A.4, the vertical bundle of $\mathrm{T}M$ is canonically isomorphic to $\pi^*_{\mathrm{T}M}\mathrm{T}M$, one can think of the second term as $\mathrm{d}f \otimes \pi^*_{\mathrm{T}M} X$.