Curved Yang-Mills gauge theories

Infinitesimal and integrated gauge theory

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Abstract†

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†Abbreviations used in this paper: **LGB** for Lie group bundle, **LAB** for Lie algebra bundle.

Contents

1. Introduction

1.1. Basic notations

1.2. Assumed background knowledge

It is highly recommended to have basic knowledge about differential geometry and gauge theory as presented in [1, especially Chapter 1 to 5]; however, sometimes we will still give explicit references to help with more technical details. It can be useful to have knowledge about Lie algebra and Lie group bundles, and even Lie algebroids and Lie groupoids, but we will introduce their basic notions such that it is not necessarily needed to have knowledge about these upfront.

We also often give references about Lie group bundles (LGBs), but the given references are often about Lie groupoids. If the reader has no knowledge about Lie groupoids, then it is important to know that LGBs are a special example of Lie groupoids; Lie groupoids carry "two projections", called **source** and **target**. An LGB is a special example of a Lie groupoid whose source equals the target.[1] If you look into such a reference, then the source and target are often denoted by α and β, or by s and t; simply put both to be the same and identify these with our bundle projection which we often denote by π. In that way it should be possible to read the references without the need to know Lie groupoids. However, we try to re-prove the needed statements such that these types of references could be avoided by the reader.

See also the previous subsection about notions we assume to be known.

2. Basic definitions

3. Curved Yang-Mills gauge theory

Notation as in [1]

- $\widetilde{G}$ Lie group with Lie algebra $\mathfrak{g}$
- M smooth manifold (usually also a spacetime). An open subset of M is usually denoted by U; typically small enough that "everything works out" (especially without further mentioning intersections of given open sets and so on)
- $P \to M$ a principal bundle, a (local) gauge is usually denoted by s, an element of $\Gamma(P)$, sections of P
- V a vector space
- ρ a Lie group representation on V, ρ_* the induced Lie algebra representation on V
- $K := P \times_\rho V$ the associated vector bundle induced by P and ρ on V. An element Φ of K is denoted by by $[p, \phi]$ for $p \in P$ and $\phi \in V$, where $[\cdot, \cdot]$ denotes the equivalence class with respect to the equivalence

$$(p, \phi) \sim \big(pg, \rho\big(g^{-1}\big) \cdot \phi\big)$$

  for all $g \in \widetilde{G}$; pg denotes the canonical group action (from the right) $P \times \widetilde{G} \to P$ and $\cdot$ the action of $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ on V.
- Especially if fixing a local gauge $s : U \to P$ we can write for sections $\Phi \in \Gamma(K)$ locally

$$\Phi|_U = [s, \phi],$$

  where $\phi : U \to V$, *i.e.* a local section of the trivial vector bundle $M \times V \to M$.
- We especially focus on $V = \mathfrak{g}$ and $\rho = \mathrm{Ad}$ the adjoint representation of $\widetilde{G}$ on $\mathfrak{g}$.

[1] But not every Lie groupoid with equal source and target are LGBs, they're in general bundles of Lie groups which is not completely the same; this nuance will not be important here.

The field of gauge bosons A is a connection on the principal bundle, *i.e.* an element of $\Omega^1(P;\mathfrak{g})$ with

$$r_g^! A = \mathrm{Ad}_{g^{-1}}(A) \coloneqq \mathrm{Ad}_{g^{-1}} \circ A,$$
$$A\left(\widetilde{X}\right) = X$$

for all $g \in \widetilde{G}$ and $X \in \mathfrak{g}$, where $r_g^!$ is the pullback of forms via the right $\widetilde{G}$-multiplication on P, and $\widetilde{X}$ the fundamental vector field of X on P.

Typically, a lot of the formalism of gauge theory comes from how to define the minimal coupling. So, let us look at this and reinvent it a bit. Usually the covariant derivative/minimal coupling ∇^A of A and $\Phi \in \Gamma(K)$ is locally (w.r.t. to a gauge s) defined by

$$\nabla^A \Phi \coloneqq \left[s, \nabla^A \phi\right],$$

where

$$\nabla^A \phi \coloneqq \mathrm{d}\phi + \rho_*(A_s) \cdot \phi, \tag{1}$$

where $A_s \coloneqq s^! A \in \Omega^1(U;\mathfrak{g})$ (local pullback as a form of A via s) and $\mathrm{d}\phi \coloneqq \nabla^0 \phi$, ∇^0 the canonical flat connection on $M \times V$.

The explicit definition of the field strength F of A is then usually motivated by looking at the curvature R_{∇^A} of ∇^A, that is

$$R_{\nabla^A}(\cdot,\cdot)\Phi|_U = [s, \rho_*(F_s) \cdot \phi],$$

where

$$F_s \coloneqq \mathrm{d}A_s + \frac{1}{2}\left[A_s \overset{\wedge}{,} A_s\right]_{\mathfrak{g}}$$

is the typical local definition of $F_s \in \Omega^2(U;\mathfrak{g})$ with

$$\left[A_s \overset{\wedge}{,} A_s\right]_{\mathfrak{g}}(X,Y) = 2\ [A_s(X), A_s(Y)]_{\mathfrak{g}}$$

for all $X, Y \in \mathfrak{X}(U)$. (The notation F_s is of course due to the fact that $F_s = s^! F$, where F is the curvature of A. But I want to avoid that for now because of what we are about doing to do.) We shortly could denote this also as

$$R_{\nabla^A}\phi = \rho_*(F_s) \cdot \phi \tag{2}$$

Now: One could question why using $\mathrm{d}\phi = \nabla^0 \phi$ in Eq. (1). Thence, let us assume that we have a general vector bundle connection $\widehat{\nabla}$ on the trivial vector bundle $M \times V \to M$. We are going to redefine ∇^A and F locally w.r.t. a gauge s, then discuss how the gauge transformations have to look like to receive definitions independent of the chosen gauge s. This also means that the following discussion is now often local by fixing a gauge without further mentioning it.

Let us first locally redefine $\nabla^A \phi$:

$$\nabla^A \phi \coloneqq \widehat{\nabla}\phi + \rho_*(A_s) \cdot \phi. \tag{3}$$

Motivated by Eq. (2), we want to identify the field strength with the curvature of ∇^A. One can check that we have

$$R_{\nabla^A} = R_{\widehat{\nabla}} + \mathrm{d}^{\widehat{\nabla}}\big(\rho_*(A_s)\big) + \rho_*(A_s) \wedge \rho_*(A_s), \tag{4}$$

where $\mathrm{d}^{\widehat{\nabla}}$ is the exterior covariant derivative of $\widehat{\nabla}$ canonically extended to $\mathrm{End}(V)$, viewing $\rho_*(A_s)$ as an element of $\Omega^1(U; \mathrm{End}(V))$, and where $\rho_*(A_s) \wedge \rho_*(A_s)$ is an element of $\Omega^2(U; \mathrm{End}(V))$ given by

$$\begin{aligned}
\big(\rho_*(A_s) \wedge \rho_*(A_s)\big)(X,Y) &\coloneqq \rho_*\big(A_s(X)\big) \circ \rho_*\big(A_s(Y)\big) - \rho_*\big(A_s(Y)\big) \circ \rho_*\big(A_s(X)\big) \\
&= \big[\rho_*\big(A_s(X)\big), \rho_*\big(A_s(Y)\big)\big]_{\mathrm{End}(V)} \\
&= \rho_*\Big(\big[A_s(X), A_s(Y)\big]_{\mathfrak{g}}\Big) \\
&= \rho_*\left(\frac{1}{2}\big[A_s \stackrel{\wedge}{,} A_s\big]_{\mathfrak{g}}\right)(X,Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$.

In order to have a similar shape as in Eq. (2), we now assume that $\widehat{\nabla}$ satisfies the following **compatibility conditions**:

Remark 3.1: Comaptibility conditions

$$R_{\widehat{\nabla}} = \rho_*(\zeta), \tag{5}$$

$$\widehat{\nabla} \circ \rho_* = \rho_* \circ \nabla \tag{6}$$

for some $\zeta \in \Omega^2(M; \mathfrak{g})$ and ∇ a vector bundle connection on the trivial vector bundle $M \times \mathfrak{g} \to M$.

If we want that Eq. (4) has a shape like Eq. (2), it is obvious why we require (5); (6) is needed for the second summand in Eq. (4). Hence, let us study (6), that is

$$\widehat{\nabla}\big(\rho_*(\nu)\big) = \rho_*(\nabla \nu)$$

for all $\nu \in \Gamma(M \times \mathfrak{g})$,[2] especially $\widehat{\nabla}$ is again extended to $\mathrm{End}(V)$ on the left hand side. With this we get

$$\mathrm{d}^{\widehat{\nabla}}\big(\rho_*(A_s)\big)(X,Y) = \widehat{\nabla}_X\big(\rho_*(A_s(Y))\big) - \widehat{\nabla}_Y\big(\rho_*(A_s(X))\big) - \rho_*\big(A_s([X,Y])\big)$$

[2] Elements of $\mathfrak{g}$ are viewed as constant sections of $M \times \mathfrak{g}$.

$$
\begin{aligned}
&= \rho_*\big(\nabla_X\big(A_s(Y)\big)\big) - \rho_*\big(\nabla_Y\big(A_s(X)\big)\big) - \rho_*\big(A_s([X,Y])\big) \\
&= \rho_*\big(\nabla_X\big(A_s(Y)\big) - \nabla_Y\big(A_s(X)\big) - A_s([X,Y])\big) \\
&= \rho_*\big(\mathrm{d}^\nabla A_s\big)(X,Y)
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(U)$. Collecting everything, Eq. (4) has now the following form

$$
R_{\nabla^A} = \rho_*\left(\mathrm{d}^\nabla A_s + \frac{1}{2}[A_s \stackrel{\wedge}{,} A_s]_{\mathfrak{g}} + \zeta\right).
$$

So, we have a new form of the field strength, assuming that ∇ and ζ satisfy the compatibility conditions in Remark 3.1. This is precisely the definition of the field strength as in the gauge theory of Thomas and Alexei, that is, we have a new field strength

$$
G \coloneqq \mathrm{d}^\nabla A_s + \frac{1}{2}[A_s \stackrel{\wedge}{,} A_s]_{\mathfrak{g}} + \zeta.
$$

Furthermore, if we are interested into Yang-Mills gauge theories, then we'd have $K = P \times_{\mathrm{Ad}} \mathfrak{g}$ (the adjoint bundle), and so also $\rho_* = \mathrm{ad}$. In this case we can put $\widehat{\nabla} = \nabla$ and then the compatibility conditions in Remark 3.1 read

$$
\begin{aligned}
R_\nabla &= \mathrm{ad}(\zeta), \\
\nabla \circ \mathrm{ad} &= \mathrm{ad} \circ \nabla.
\end{aligned}
$$

The second condition precisely gives after a short calculation

$$
\nabla\Big([\mu,\nu]_{\mathfrak{g}}\Big) = [\nabla\mu,\nu]_{\mathfrak{g}} + [\mu,\nabla\nu]_{\mathfrak{g}}
$$

for all $\mu, \nu \in \Gamma(M \times \mathfrak{g})$, so, ∇ has to be a Lie bracket derivation. So, in this case the compatibility conditions in Remark 3.1 precisely reduce to the compatibility conditions of Alexei's and Thomas's theory! (in the case of Lie algebra bundles; the general theory is more general, formulated on general Lie algebroids)

As a summary:

Remark 3.2: Summary

We have

$$
R_\nabla = \mathrm{ad}(\zeta), \tag{7}
$$

$$
\nabla \circ \mathrm{ad} = \mathrm{ad} \circ \nabla, \tag{8}
$$

$$
G = \mathrm{d}^\nabla A_s + \frac{1}{2}[A_s \stackrel{\wedge}{,} A_s]_{\mathfrak{g}} + \zeta. \tag{9}
$$

In fact, the compatibility conditions lead to a gauge invariant theory: Fix an ad-invariant scalar product κ on $\mathfrak{g}$; then define the Lagrangian by

$$\mathfrak{L}_{\mathrm{YM}} := -\frac{1}{2}\kappa\left(G \overset{\wedge}{,} *G\right) \tag{10}$$

where $*$ is the Hodge star operator w.r.t. some spacetime metric. (In short, the typical definition, but replace F with G) It is easier to look at the infinitesimal version of the gauge transformations, hence everything with respect to a gauge s now.

In order to derive a formula for these, let us again look at ∇^A. Fix an $\varepsilon \in \Gamma(M \times \mathfrak{g})$, then the infinitesimal gauge transformation $\delta_\varepsilon \phi$ of $\phi \in \Gamma(M \times \mathfrak{g})$ is usually defined by

$$\delta_\varepsilon \phi := \rho_*(\varepsilon) \cdot \phi.$$

We fix the infinitesimal gauge trafo $\delta_\varepsilon A$ of A by looking at the gauge trafo of $\nabla^A \phi$ via

$$\begin{aligned}
\delta_\varepsilon \nabla^A \phi &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\nabla^{A + t\delta_\varepsilon A}(\phi + t\delta_\varepsilon \phi)\right) \\
&= \underbrace{\widehat{\nabla}(\delta_\varepsilon \phi)}_{=\left(\widehat{\nabla}(\rho_*(\varepsilon))\right)\cdot\phi + \rho_*(\varepsilon)\cdot\widehat{\nabla}\phi} + \rho_*(\delta_\varepsilon A_s) \cdot \phi + \rho_*(A_s) \cdot \delta_\varepsilon \phi \\
&= \left(\rho_*(\nabla\varepsilon + \delta_\varepsilon A_s) + \rho_*(A_s) \cdot \rho_*(\varepsilon)\right) \cdot \phi + \rho_*(\varepsilon) \cdot \widehat{\nabla}\phi
\end{aligned}$$

using Remark 3.1. We want $\delta_\varepsilon \nabla^A \phi = \rho_*(\varepsilon) \cdot \nabla^A \phi$ which gives

$$\rho_*(\varepsilon) \cdot \nabla^A \phi = \rho_*(\varepsilon) \cdot \widehat{\nabla}\phi + \rho_*(\varepsilon) \cdot \rho_*(A_s) \cdot \phi.$$

Imposing $\delta_\varepsilon \nabla^A \phi = \rho_*(\varepsilon) \cdot \nabla^A \phi$ we get

$$\rho_*\left(\delta_\varepsilon A_s + \nabla\varepsilon + [A_s, \varepsilon]_{\mathfrak{g}}\right) = 0$$

using again that ρ_* is a Lie algebra representation. If we require that this shall work for all ρ_*, we may say

$$\delta_\varepsilon A_s := -\nabla\varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}. \tag{11}$$

This is precisely the infinitesimal gauge trafo of A as in the theory of Thomas and Alexei! Hence, we achieve infinitesimal gauge invariance of $\mathfrak{L}_{\mathrm{YM}}$. For completeness, let us check the gauge trafo of G using Def. (11) and Remark 3.2, it is very similar to the "classical" calculation due to Remark 3.2 which is why I skip some straightforward calculations to keep it short,

$$\begin{aligned}
\delta_\varepsilon G &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{d}^\nabla(A_s + t\delta_\varepsilon A_s) + \frac{1}{2}\left[A_s + t\delta_\varepsilon A_s \overset{\wedge}{,} A_s + t\delta_\varepsilon A_s\right]_{\mathfrak{g}} + \zeta\right) \\
&= \mathrm{d}^\nabla\left(-\nabla\varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}\right) + \left[A_s \overset{\wedge}{,} -\nabla\varepsilon + [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}
\end{aligned}$$

$$= \underbrace{-\left(\mathrm{d}^{\nabla}\right)^{2}\varepsilon}_{=-R_{\nabla}\varepsilon=[\varepsilon,\zeta]_{\mathfrak{g}}} + \left[\nabla\varepsilon \stackrel{\wedge}{,} A_s\right]_{\mathfrak{g}} + \left[\varepsilon, \mathrm{d}^{\nabla}A_s\right]_{\mathfrak{g}} + \underbrace{\left[A_s \stackrel{\wedge}{,} -\nabla\varepsilon\right]_{\mathfrak{g}}}_{=-\left[\nabla\varepsilon\stackrel{\wedge}{,}A_s\right]_{\mathfrak{g}}} + \left[A_s \stackrel{\wedge}{,} [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}$$

$$= \left[\varepsilon, \mathrm{d}^{\nabla}A_s + \zeta\right]_{\mathfrak{g}} + \left[A_s \stackrel{\wedge}{,} [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}$$

and, using the Jacobi identity,

$$\begin{aligned}\left[A_s \stackrel{\wedge}{,} [\varepsilon, A_s]_{\mathfrak{g}}\right]_{\mathfrak{g}}(X,Y) &= \left[A_s(X), [\varepsilon, A_s(Y)]_{\mathfrak{g}}\right]_{\mathfrak{g}} - \left[A_s(Y), [\varepsilon, A_s(X)]_{\mathfrak{g}}\right]_{\mathfrak{g}} \\ &= \left[\varepsilon, [A_s(X), A_s(Y)]_{\mathfrak{g}}\right]_{\mathfrak{g}} \\ &= \left[\varepsilon, \frac{1}{2}\left[A_s \stackrel{\wedge}{,} A_s\right]_{\mathfrak{g}}\right]_{\mathfrak{g}}(X,Y)\end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$. Altogether

$$\delta_{\varepsilon}G = \left[\varepsilon, \mathrm{d}^{\nabla}A_s + \frac{1}{2}\left[A_s \stackrel{\wedge}{,} A_s\right]_{\mathfrak{g}} + \zeta\right]_{\mathfrak{g}} = [\varepsilon, G]_{\mathfrak{g}}.$$

Hence, the field strength transforms with the adjoin of ε; since κ is ad-invariant, we can derive that $\mathfrak{L}_{\mathrm{YM}}$ is invariant under the infinitesimal gauge trafo in Def. (11)!

Observe that by Remark 3.2 that ζ can be non-trivial even if we still use $\nabla = \nabla^0$, the canonical flat connection on $M \times \mathfrak{g}$, even though this whole discussion started with allowing more general connections.

If we minimise $\mathfrak{L}_{\mathrm{YM}}$, then one obvious way would be to search solutions with $G \equiv 0$ for an absolute minimum/maximum (because of the sign), doing so would result into that the classical Yang-Mills energy would have a bound which is non-zero. May this be an explanation for the mass gap? As shown in my thesis, every classical theory has a ζ after a field redefinition. Even though field redefinitions are an equivalence for the classical theories, one may argue that it does not describe an equivalence for the quantised theory, leading to a possible explanation of the mass gap? But that is just high hope right now :)

3.1. Integration

For an integrated version of Def. (11) we need to discuss when the new "minimal coupling" of Def. (3) behaves nicely under a change of the gauge s. That is, we now want to extend the new definition of ∇^A to a well-defined connection on $K = P \times_{\rho} V$, especially on the adjoint bundle $K = P \times_{\mathrm{Ad}} \mathfrak{g}$ in our case. (and later maybe generalise this to a $\widetilde{G}$-quotient of a general Lie algebra bundle over P)

Let s' be another (local) gauge such that we have a unique smooth map $g : U \to G$ such that

$$s' = sg,$$

then we want for well-definedness

$$\nabla^A \Phi = [s, \nabla\phi + \mathrm{ad}(A_s) \cdot \phi] \stackrel{!}{=} \left[s', \nabla\phi' + \mathrm{ad}(A_{s'}) \cdot \phi'\right], \tag{12}$$

where we have $\Phi = [s, \phi] = [s', \phi']$, especially

$$\phi' = \mathrm{Ad}\big(g^{-1}\big) \cdot \phi.$$

Since the new field strength G still transforms via the adjoin under δ_ε (see above), we make the following ansatz

$$A_{s'} = \mathrm{Ad}\big(g^{-1}\big) \cdot A_s + \mu, \tag{13}$$

where $\mu \in \Omega^1(U; \mathfrak{g})$. Usually, $\mu = g^! \mu_{\widetilde{G}}$, the pullback as a form of the Maurer-Cartan-Form $\mu_{\widetilde{G}}$ on $\widetilde{G}$. One can then check with some short calculation that Eq. (12) is equivalent to

$$\nabla\big(\mathrm{Ad}\big(g^{-1}\big) \cdot \phi\big) + \mathrm{ad}(\mu) \cdot \mathrm{Ad}\big(g^{-1}\big) \cdot \phi \stackrel{!}{=} \mathrm{Ad}\big(g^{-1}\big) \cdot \nabla\phi$$

using the definition of $P \times_{\mathrm{Ad}} \mathfrak{g}$. Equivalently,

$$\mathrm{ad}(\mu) = \mathrm{Ad}(g) \circ ()$$

4. Lie group bundles (LGBs)

4.1. Definition and examples

Definition 4.1: Lie group bundle, [2, §1.1, Def. 1.1.19; p. 11]

Let $G, \mathscr{G}, M$ be smooth manifolds. A fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

is called a **Lie group bundle** if:

1. G and each fibre $\mathscr{G}_x := \pi^{-1}(\{x\})$, $x \in M$, are Lie groups;
2. there exists a bundle atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the induced maps

$$\phi_{ix} := \mathrm{pr}_2 \circ \phi_i|_{\mathscr{G}_x} : \mathscr{G}_x \to G$$

   are Lie group isomorphisms, where I is an (index) set, U_i are open sets covering M, $\phi_i : \mathscr{G}|_U \to U \times G$ subordinate trivializations, and pr_2 the projection onto the second factor. This atlas will be called **Lie group bundle atlas** or **LGB atlas**.

We also often say that $\mathscr{G}$ **is an LGB (over** M**)**, whose structural Lie group is either clear by context or not explicitly needed; and we may also denote LGBs by $G \to \mathscr{G} \stackrel{\pi}{\to} M$.

Remark 4.2: Principal and Lie group bundles

Beware, a Lie group bundle is **not** the same as a principal bundle $P \to M$ with the same fibre type G. First of all, the fibres of P are just diffeomorphic to a Lie group, a priori they carry no Lie group structure, while the fibres of $\mathscr{G}$ carry a Lie group structure.

Second, on P we have a multiplication given as an action of G on P

$$P \times G \to P,$$

preserving the fibres P_x $(x \in M)$ and simply transitive on them. Restricted on P_x we have

$$P_x \times G \to P_x.$$

For $\mathscr{G}$ we have canonically a multiplication over x given by

$$\mathscr{G}_x \times \mathscr{G}_x \to \mathscr{G}_x,$$

also clearly simply transitive. Observe, the second factor is not "constant", *i.e.* we do not have $\mathscr{G}_x \times G \to \mathscr{G}_x$ in general. Hence, there is in general no well-defined product $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$.

All of that is also resembled in the existence of sections. The existence of a section of P has a 1:1 correspondence to trivializations of P, which is why P in general only admits sections locally; see *e.g.* [1, §4.2, Thm. 4.2.19; page 219f.]. $\mathscr{G}$ clearly admits always a global section, even if $\mathscr{G}$ is non-trivial; just take the section which assigns each base point the neutral element of its fibre.

As usual, we have several trivial examples:

Example 4.3: Trivial examples

The **trivial LGB** is given as the trivial bundle $M \times G \to M$ with canonical multiplication $(x, g) \cdot (x, q) \coloneqq (x, gq)$, and we recover the notion of a Lie group in the case of $M = \{*\}$.

We are of course also interested into LGB bundle morphisms:

Definition 4.4: LGB morphism, [2, §1.2, special situation of Def. 1.2.1 & 1.2.3, page 12]

Let $\mathscr{G} \stackrel{\pi_{\mathscr{G}}}{\to} M$ and $\mathscr{H} \stackrel{\pi_{\mathscr{H}}}{\to} N$ be two LGBs over two smooth manifolds M and N. An **LGB**

morphism is a pair of smooth maps $F : \mathscr{H} \to \mathscr{G}$ and $f : N \to M$ such that

$$\pi_{\mathscr{G}} \circ F = f \circ \pi_{\mathscr{H}}, \tag{14}$$

$$F(gq) = F(g)\ F(q) \tag{15}$$

for all $g, q \in \mathscr{H}$ with $\pi_{\mathscr{H}}(g) = \pi_{\mathscr{H}}(q)$. We then say that F **is an LGB morphism over** f. If $N = M$ and $f = \mathrm{id}_M$, then we often omit mentioning f explicitly and either just write that F **is a (base-preserving) LGB morphism**.
We speak of an **LGB isomorphism (over f)** if F is a diffeomorphism.

Remarks 4.5.
- The right hand side of Eq. (15) is well-defined because of Eq. (14).
- It is clear that condition 2 in Def. 4.1 is equivalent to say that $\mathscr{G}$ is locally isomorphic to a trivial LGB; as one may have expected already.
- If F is a diffeomorphism, then also f: By Eq. (14) surjectivity of f is clear; for $y \in M$ just take any $g \in \mathscr{G}_y$, and since F is a bijective, we have a $q \in \mathscr{H}_x$ for some $x \in N$ with $F(q) = g$. By Eq. (14) we have $y = \pi_{\mathscr{G}}(F(q)) \overset{(14)}{=} f(x)$, thence, surjectivity follows. For injectivity we know by Eq. (15) and (14) that $F\big(e^{\mathscr{H}}_x\big) = e^{\mathscr{G}}_{f(x)}$, where $e^{\mathscr{H}}_x$ and $e^{\mathscr{G}}_{f(x)}$ denote the unique neutral elements of $\mathscr{H}_x$ and $\mathscr{G}_{f(x)}$, respectively. Assume that there are $x, x' \in N$ with $f(x) = f(x')$, then we can derive

$$F\Big(e^{\mathscr{H}}_x\Big) = e^{\mathscr{G}}_{f(x)} = e^{\mathscr{G}}_{f(x')} = F\Big(e^{\mathscr{H}}_{x'}\Big).$$

Then we have $e^{\mathscr{H}}_x = e^{\mathscr{H}}_{x'}$ due to that F is bijective, and hence $x = x'$. Therefore f is bijective. Finally, F^{-1} is by assumption also a diffeomorphism, Eq. (15) clearly carries over, and Eq. (14) is clearly w.r.t. f^{-1}, that is

$$\pi_{\mathscr{H}} \circ F^{-1} = f^{-1} \circ \pi_{\mathscr{G}}.$$

Since $\pi_{\mathscr{H}} \circ F^{-1}$ is smooth and $\pi_{\mathscr{G}}$ is a smooth surjective submersion, it follows that f^{-1} is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We can conclude that f is a diffeomorphism. Observe that we also concluded that F^{-1} is an LGB isomorphism, too.

For another important example recall that there is the notion of associated fibre bundles; following and stating the results of [2, §1, Construction 1.3.8, page 20] and [1, §4.7, page 237ff.; see also Rem. 4.7.8, page 242f.]: Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G, a smooth manifold F and a smooth left G-action Ψ given by

$$G \times F \to F,$$

$$(g, v) \mapsto \Psi(g, v) := g \cdot v.$$

Then we have a right G-action on $P \times F$ given by

$$(P \times F) \times G \to P \times F,$$

$$(p, v, g) \mapsto \left(p \cdot g, g^{-1} \cdot v\right),$$

and one can show that the quotient under this action, $P \times_\Psi F := (P \times F) \big/ G$, yields the structure of a fibre bundle

$$\begin{tikzcd} F \arrow[r] & P \times_\Psi F \arrow[d, "\pi_{P \times_\Psi F}"] \\ & M \end{tikzcd}$$

such that the projection $P \times F \to P \times_\Psi F$ is a smooth surjective submersion, where the projection $\pi_{P \times_\Psi F} : P \times_\Psi F \to M$ is given by

$$\pi_{P \times_\Psi F}([p, v]) := \pi_P(p)$$

for all $[p, v] \in P \times_\Psi F$, denoting equivalence classes of (p, v) by square brackets. For $x \in M$, the fibre $(P \times_\Psi F)_x$ is given by $(P_x \times F) \big/ G = P_x \times_\Psi F$, and the fibre is diffeomorphic to F by $F \ni v \mapsto [p, v] \in (P \times_\Psi F)_x$ for a fixed $p \in P_x$. We will frequently use this diffeomorphism in the following without further notice.

A very important example are of course associated vector bundles, related to F being a vetor space. We need a similar concept for Lie groups.

Definition 4.6: Lie group representation on Lie groups, [2, special situation of the comment after Ex. 1.7.14, page 47]

Let G, H be Lie groups. Then a **Lie group representation of G on H** is a smooth left action ψ of G on H

$$G \times H \to H,$$

$$(g, h) \mapsto \psi_g(h) := \psi(g, h)$$

such that

$$\psi_g(hq) = \psi_g(h)\ \psi_g(q) \tag{16}$$

for all $g \in G$ and $h, q \in H$.

Remark 4.7: Note about labeling

Observe that we have by the definition of group actions

$$\psi_{gg'} = \psi_g \circ \psi_{g'}$$

for all $g, g' \in G$, viewing ψ_g as a map $H \to H$. Therefore we can view the action ψ as a homomorphism

$$G \to \mathrm{Aut}(H),$$

where $\mathrm{Aut}(H)$ is the set of Lie group automorphisms. The similarity to Lie group representations on vector spaces is obvious, thence the name.

This definition is of course also motivated by various references pointing out that Lie group representations define Lie group actions with extra properties; see for example [1, §3, Ex. 3.4.2, page 143f.]. In [2, comments after Ex. 1.7.14, page 47] this definition is also called *action by Lie group isomorphisms.*

With this we can discuss and define associated Lie group bundles.

Theorem 4.8: Associated Lie group bundle as quotient, [1, motivated by vector spaces as in §4, Thm. 4.7.2, page 239f.]

Let G, H be Lie groups, $P \overset{\pi_P}{\to} M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then $\mathscr{H} := P \times_\psi H$ is an LGB

$$\begin{array}{ccc} H & \longrightarrow & \mathscr{H} \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

with projection π given by

$$\begin{aligned} \mathscr{H} &\to M, \\ [p, h] &\mapsto \pi_P(p), \end{aligned} \tag{17}$$

and fibres

$$\mathscr{H}_x = P_x \times_\psi H \tag{18}$$

for all $x \in M$, which are isomorphic to H as Lie groups. The Lie group structure on each fibre $\mathscr{H}_x$ is defined by

$$[p, h] \cdot [p, q] := [p, hq] \tag{19}$$

for all $h, q \in H$ and $p_x \in P_x$, where $\pi_P(p) = x$.

Remark 4.9: Neutral and inverse elements

The neutral element for $\mathscr{H}_x$ ($x \in M$) is given by

$$e_x = [p, e],$$

where $p \in P_x$ is arbitrary and e is the neutral element of H. This is clearly independent of the choice of p due to

$$[p, e] = \left[p \cdot g, \psi_{g^{-1}}(e)\right] = [p \cdot g, e]$$

for all $g \in G$. Thence, the fact that e_x is the neutral element follows immediately.
The inverse of $[p, h] \in \mathscr{H}_x$ is clearly given by

$$([p, h])^{-1} = \left[p, h^{-1}\right].$$

Proof.

- That π is the well-defined projection and that the fibres are precisely $P_x \times_\psi H$ for all $x \in M$ is well-known, see our discussion before Def. 4.6 and the references therein; it is also very straightforward to check. We also discussed that $\mathscr{H}$ is a fibre bundle with structural fibre H. Hence, if one knows that the proposed group structure in Def. (19) is well-defined, then the smoothness of the group structure is implied by the smoothness structures of H and $\mathscr{H}$. Thence, let us check whether Def. (19) is well-defined. Let $x \in M$, $p \in P_x$ and $p' := p \cdot g'$ be another element of P_x, where $g' \in G$. Also let $[p_1, h_1], [p_2, h_2] \in P_x \times_\psi H$; then we have unique elements q_i, q_i' of G such that ($i \in \{1, 2\}$)

$$p_i = p \cdot q_i, \qquad\qquad p_i = p' \cdot q_i',$$

especially, it follows $q_i = g' q_i'$. On the one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, h_1] \cdot [p_2, h_2] = [p, \psi_{q_1}(h_1)] \cdot [p, \psi_{q_2}(h_2)] = [p, \psi_{q_1}(h_1)\ \psi_{q_2}(h_2)], \tag{20}$$

on the other hand, using Def. 4.6 and $p' = p \cdot g'$ instead of p,

$$\begin{aligned}
[p_1, h_1] \cdot [p_2, h_2] &= \left[p \cdot g', \psi_{q_1'}(h_1)\ \psi_{q_2'}(h_2)\right] \\
&= \Bigg[p,\ \underbrace{\psi_{g'}\Big(\psi_{q_1'}(h_1)\ \psi_{q_2'}(h_2)\Big)}_{=\psi_{g'}\left(\psi_{q_1'}(h_1)\right)\ \psi_{g'}\left(\psi_{q_2'}(h_2)\right)}\ \Bigg] \\
&= \left[p, \psi_{g'q_1'}(h_1)\ \psi_{g'q_2'}(h_2)\right] \\
&= [p, \psi_{q_1}(h_1)\ \psi_{q_2}(h_2)],
\end{aligned}$$

which implies that Def. (19) is well-defined, and thus defines a Lie group structure on each fibre of $\mathscr{H}$.

• That the fibres $\mathscr{H}_x$ are isomorphic to H as Lie groups for all $x \in M$ also quickly follows. Recall by our discussion before Def. 4.6 that the fibres are diffeormorphic to H by $H \ni h \mapsto [p,h] \in \mathscr{H}_x$ for a fixed $p \in P_x$. By Def. (19) it is clear that this map is a Lie group homomorphism and hence a Lie group isomorphism.

• Let us now construct an LGB atlas for $\mathscr{H}$ by using a principal bundle atlas for P. That is, for some $U \subset M$ open and a trivialization $\varphi_U : P|_U \to U \times G$ we write

$$\varphi_U(p) = \big(\pi_P(p), \beta_U(p)\big)$$

for all $p \in P$, where $\beta_U : P|_U \to G$ is an equivariant map, *i.e.* $\beta_U(p \cdot g) = \beta_U(p)\, g$ for all $g \in G$. Then define ϕ_U as a map by

$$\mathscr{H}|_U \to U \times H,$$

$$[p,h] \mapsto \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big).$$

ϕ_U is well-defined: Let $[p',h'] \in \mathscr{H}|_U$ with $[p',h'] = [p,h]$. Then there is a $g \in G$ such that

$$(p',h') = \big(p \cdot g, \psi_{g^{-1}}(h)\big),$$

hence, using the equivariance of β_U and Def. 4.6,

$$\phi_U\big([p',h']\big) = \Big(\underbrace{\pi_P(p\cdot g)}_{=\pi_P(p)}, \underbrace{\big(\psi_{\beta_U(p\cdot g)} \circ \psi_{g^{-1}}\big)}_{=\psi_{\beta_U(p)}\circ\psi_g\circ\psi_{g^{-1}}}(h)\Big) = \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big) = \phi_U\big([p,h]\big),$$

which proves that ϕ_U is well-defined. Denote the projection onto equivalence classes $P \times H \to \mathscr{H}$ by ϖ, then observe

$$\phi_U \circ \varpi = L,$$

where $L_U : P|_U \times H \to U \times H$ is given by $L_U(p,h) \coloneqq \big(\pi_P(p), \psi_{\beta_U(p)}(h)\big)$ for all $(p,h) \in P|_U \times H$. L_U is clearly smooth and recall that ϖ is a smooth surjective submersion, therefore ϕ_U is smooth; this is a well-known fact for right-compositions with surjective submersions, see *e.g.* [1, §3.7.2, Lemma 3.7.5, page 153]. We define a candidate of the inverse $\phi_U^{-1} : U \times H \to \mathscr{H}|_U$ by

$$\phi_U^{-1}(x,h) = \big[\varphi_U^{-1}(x,e), h\big]$$

for all $(x,h) \in U \times H$, where e is the neutral element of G. By the definition of φ_U we immediately get

$$\big(\varphi_U \circ \varphi_U^{-1}\big)(x,e) = \Big(\pi_P\big(\varphi_U^{-1}(x,e)\big), \beta_U\big(\varphi_U^{-1}(x,e)\big)\Big) = (x,e),$$

for all $x \in U$, and, also using again the equivariance of β_U,

$$\varphi_U^{-1}(\pi_P(p), e) = \varphi_U^{-1}\Big(\pi_P\big(p \cdot \beta_U^{-1}(p)\big), \beta_U(p)\ \beta_U^{-1}(p)\Big)$$

$$
\begin{aligned}
&= \varphi_U^{-1}\Big(\pi_P\big(p \cdot \beta_U^{-1}(p)\big), \beta_U\big(p \cdot \beta_U^{-1}(p)\big)\Big) \\
&= \big(\varphi_U^{-1} \circ \varphi_U\big)\big(p \cdot \beta_U^{-1}(p)\big) \\
&= p \cdot \beta_U^{-1}(p)
\end{aligned}
$$

for all $p \in P|_U$. Then

$$
\big(\phi_U \circ \phi_U^{-1}\big)(x,h) = \Big(\pi_P\big(\varphi_U^{-1}(x,e)\big), \psi_{\beta_U\left(\varphi_U^{-1}(x,e)\right)}(h)\Big) = \big(x, \psi_e(h)\big) = (x,h),
$$

for all $(x,h) \in U \times H$, and

$$
\begin{aligned}
\big(\phi_U^{-1} \circ \phi_U\big)([p,h]) &= \big[\underbrace{\varphi_U^{-1}(\pi_P(p), e)}_{=p\cdot\beta_U^{-1}(p)}, \psi_{\beta_U(p)}(h)\big] \\
&= [p,h]
\end{aligned}
$$

for all $[p,h] \in \mathscr{H}|_U$. Thus, ϕ_U is bijective; additionally observe

$$
\phi_U^{-1}(x,h) = \varpi\big(\varphi_U^{-1}(x,e), h\big)
$$

such that ϕ_U^{-1} is clearly smooth as the composition of smooth maps, and we therefore conclude that ϕ_U is a diffeomorphism. Finally, derive with Def. 4.6 and Eq. (20) that

$$
\begin{aligned}
(\mathrm{pr}_2 \circ \phi_U)\big([p_1,h_1] \cdot [p_2,h_2]\big) &= (\mathrm{pr}_2 \circ \phi_U)\big([p, \psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)]\big) \\
&= \psi_{\beta_U(p)}\big(\psi_{q_1}(h_1) \cdot \psi_{q_2}(h_2)\big) \\
&= \underbrace{\psi_{\beta_U(p)}\big(\psi_{q_1}(h_1)\big)}_{=\psi_{\beta_U(p)\cdot q_1}(h)} \cdot \psi_{\beta_U(p)}\big(\psi_{q_2}(h_2)\big) \\
&= \psi_{\beta_U(p_1)}(h) \cdot \psi_{\beta_U(p_2)}(h) \\
&= (\mathrm{pr}_2 \circ \phi_U)\big([p_1,h_1]\big) \cdot (\mathrm{pr}_2 \circ \phi_U)\big([p_1,h_1]\big)
\end{aligned}
$$

for all $[p_1,h_1], [p_2,h_2] \in \mathscr{H}_x$, where we used again the equivariance of β_U and the same notation as introduced for Eq. (20), and pr_2 denotes the projection onto the second factor. Thence, $\mathrm{pr}_2 \circ \phi_U$ induces Lie group isomorphisms $\mathscr{H}_x \to H$ for all $x \in U$; by Def. 4.1 we can finally conclude that $\mathscr{H}$ is an LGB. ■

Hence, we define:

Definition 4.10: Associated Lie group bundle, labeling similar to [1, §4.7, Def. 4.7.3, page 240]

Let G, H be Lie groups, $P \overset{\pi_P}{\to} M$ a principal G-bundle over a smooth manifold M, and ψ a G-representation on H. Then we call the LGB

$$\mathscr{H} := P \times_\psi H = (P \times H) \Big/ G$$

the **Lie group bundle (LGB) associated** to the principal bundle P and the representation ψ on H:

$$\begin{tikzcd} H \arrow[r] & P \times_\psi H \arrow[d, "\pi_{\mathscr{H}}"] \\ & M \end{tikzcd}$$

The special situation of $H = G$ is already an important example:

Example 4.11: Inner group bundle, [2, §1, paragraph after Def. 1.1.19, page 11; comment after Construction 1.3.8, page 20]

The **inner group bundle** or **inner LGB** of a principal bundle $P \to M$, denoted by $c_G(P)$, is defined by

$$c_G(P) := P \times_{c_G} G, \tag{21}$$

where $c_G : G \times G \to G$ is the left action of G on itself given by the very well-known **conjugation**

$$c_G(g, h) := c_g(h) = \big(L_g \circ R_{g^{-1}}\big)(h) = ghg^{-1} \tag{22}$$

for all $g, h \in G$, where we also denote left- and right-multiplications (with g) by L_g and R_g, respectively; see *e.g.* [1, beginning of §1.5.2, page 40f.] for its common properties. It is well-known that c_G satsfies the properties of a Lie group representation of G on itself in the sense of Def. 4.6.
$c_G(P)$ is an LGB by Thm. 4.8.

4.2. Lie algebra bundles (LABs)

4.3. From LGBs to LABs

5. LGB actions

5.1. Definition

As for Lie groups, we are interested into their actions. The idea is the following, similar to [2, §1.6, discussion around Def. 1.6.1, page 34]: We have an LGB $\mathscr{G} \to M$ over a smooth manifold M, and we want to construct an action of $\mathscr{G}$ on another smooth manifold N. Each fibre of $\mathscr{G}$ is a Lie group, and we have a notion of Lie groups actions on manifold N. Therefore one could define an LGB action as a collection of Lie group actions, that is, only sections of $\mathscr{G}$ act on N; however, one then expects that the general outcome of a product of $\Gamma(\mathscr{G})$ on N would be smooth maps from M to N. In order to recover a typical structure of action one could instead introduce a "multiplication rule", *i.e.* each point $p \in N$ can only be multiplied with elements of a specific fibre of $\mathscr{G}$. This "multiplication rule" will be described by a smooth map $f : N \to M$ in the sense of that the fibre over $f(p)$ will act on p.

For this recall that there is the notion of pullbacks of fibre bundles, see *e.g.* [1, §4.1.4, page 203ff.; especially Thm. 4.1.17, page 204f.]. That is, if we additionally have a smooth manifold N and a smooth map $f : M \to N$, then we have the pullback $f^*\mathscr{G}$ of $\mathscr{G}$ as a fibre bundle defined as usual by

$$f^*\mathscr{G} := \{(x, g) \in N \times \mathscr{G} \mid f(x) = \pi(g)\}. \tag{23}$$

The structural fibre is the same Lie group as for $\mathscr{G}$. That is, the following diagram commutes

$$\begin{array}{ccc} f^*\mathscr{G} & \xrightarrow{\pi_2} & \mathscr{G} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}$$

where π_1 and π_2 are the projections onto the first and second factor, respectively, of $N \times \mathscr{G}$. Actually, $f^*\mathscr{G}$ carries a natural structure as an LGB.

Corollary 5.1: Pullbacks of LGBs are LGBs, [2, §2.3, simplified situation of the discussion around Prop. 2.3.1, page 63ff.]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Then $f^*\mathscr{G}$ has a unique (up to isomorphisms) LGB structure such that the projection $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$ onto the second factor is an LGB morphism over f with $\pi_2|_x : (f^*\mathscr{G})_x \to \mathscr{G}_{f(x)}$ being a Lie group isomorphism for all $x \in N$.

Remarks 5.2.

The mentioned reference, [2, §2.3, discussion around Prop. 2.3.1, page 63ff.], is rather general, formulated for Lie groupoids. If the reader is only interested into LGBs, then see *e.g.* [3, §3, Thm. 3.1].

Proof.
By construction, the structural fibre of $f^*\mathscr{G}$ is the same Lie group G as for $\mathscr{G}$, and for all $x \in N$ we have $(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)}$, thence, the fibres are Lie groups and the fibrewise group multiplication has the form

$$(x, g) \cdot (x, q) = (x, gq) \tag{24}$$

for all $x \in N$ and $g, q \in (f^*\mathscr{G})_x$. We are left to show the existence of an LGB atlas. For this fix an LGB atlas $\{(U_i, \phi_i\}_{i \in I}$ of $\mathscr{G}$, where I is an (index) set, $(U_i)_{i \in I}$ an open covering of M, and $\phi_i : \mathscr{G}|_{U_i} \to U_i \times G$ are LGB isomorphisms. Then $f^{-1}(U_i)$ gives rise to an open covering of N, and we get

$$f^*\phi_i : f^*\mathscr{G}|_{f^{-1}(U_i)} \to f^{-1}(U_i) \times G,$$

$$(x, g) \mapsto \big(x, \phi_{i,f(x)}(g)\big),$$

where $\phi_{i,f(x)} : \mathscr{G}_{f(x)} \to G$ are the Lie group isomorphisms as defined in Def. 4.1. It is immediate by construction that this gives an LGB atlas.

That this is the unique (up to isomorphisms) LGB structure such that $\pi_2 : f^*\mathscr{G} \to \mathscr{G}$ is an LGB morphism over f inducing a Lie group isomorphism on each fibre simply follows by construction; observe for all $x \in N$ that $\pi_2|_x$ is clearly bijective. Furthermore, LGB morphisms need to be homomorphisms, which means here

$$\pi_2\big((x, g) \cdot (x, q)\big) \stackrel{!}{=} \pi_2((x, g)) \cdot \pi_2((x, q)) = gq = \pi_2\big((x, gq)\big)$$

for all $x \in N$ and $g, q \in (f^*\mathscr{G})_x$. By using the bijectivity of $\pi_2|_x$, the group structure leading to this is uniquely the one provided in Eq. (24). Especially, π_2 is a homomorphism with the provided structure. Assume we have another LGB chart ψ_i on (a subset of) $f^{-1}(U_i)$, then

$$\phi_i \circ \pi_2 \circ \psi_i^{-1} = \underbrace{\phi_i \circ \pi_2 \circ (f^*\phi_i)^{-1}}_{=(f, \mathbb{1}_G)} \circ f^*\phi_i \circ \psi_i^{-1} = (f, \mathbb{1}_G) \circ f^*\phi_i \circ \psi_i^{-1},$$

If evaluating this at $x \in f^{-1}(U_i)$, then all parts are bijective, and thus the condition about $\pi_2|_x$ being a Lie group isomorphism enforces that ψ_i is an LGB atlas compatible with $f^*\phi_i$. This concludes the proof. ■

Let us now define $\mathscr{G}$-actions.

Definition 5.3: Lie group bundle actions,
[2, §1.6, special case of Def. 1.6.1, page 34]

Let M, N be smooth manifolds, $\mathscr{G} \xrightarrow{\pi} M$ an LGB over M and $f : N \to M$ a smooth map. Then a **right-action of $\mathscr{G}$ on N** is a smooth map

$$f^*\mathscr{G} \to N,$$

$$(p, g) \mapsto p \cdot g,$$

satisfying the following properties:

$$f(p \cdot g) = \pi(g), \tag{25}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{26}$$

$$p \cdot e_{f(p)} = p \tag{27}$$

for all $p \in N$ and $g, h \in \mathscr{G}_{f(p)}$, where $e_{f(p)}$ is the neutral element of $\mathscr{G}_{f(p)}$.

We similarly define left-actions, and we may sometimes write (left or right) **$\mathscr{G}$-action on N**. Furthermore, in order to increase readability as long as the dependency on f is not important, we introduce the notation

$$N * \mathscr{G} := f^*\mathscr{G}, \tag{28}$$

such that the action's notation has the typical shape $N * \mathscr{G} \to N$.

Remark 5.4: Relation to the structure of the canonical pullback Lie group bundle over N

Observe that by the definition of $f^*\mathscr{G}$ we can also write

$$f(p \cdot g) = f(p),$$

so, the $\mathscr{G}$-action is defined in such a way that f is invariant under it. Moreover, the fibre-wise group structure on $\mathscr{G}$ naturally defines a $\mathscr{G}$-action on $\mathscr{G}$; in this situation f would be π itself. This is mainly a technical condition. On one hand, having $M = \{*\}$ already recovers the notion of a Lie group action and condition (25) is then trivial, and on the other hand the mentioned reference, [2, §1.6, Def. 1.6.1, page 34], actually generalizes this condition making use of the structure of groupoids.

Furthermore, the other conditions are the typical conditions for actions, especially such that we get a $\mathscr{G}$-action on $f^*\mathscr{G}$ by

$$(p, g) \cdot q \coloneqq \left(p \cdot q, q^{-1} g\right) \tag{29}$$

for all $p \in N$ and $g, q \in \mathscr{G}_{f(p)}$.[a] As usual, this gives rise to an equivalence relation, whose set of equivalence classes $f^*\mathscr{G} \big/ G$ is isomorphic to N (as a set) by $[p, g] \mapsto p \cdot g$, where we denote equivalence classes of $(p, g) \in f^*\mathscr{G}$ by $[p, g]$. All of this is straight-forward to check. Finally, observe the similarity to associated fibre bundles.

[a] In alignment to Def. 5.3, this action is a map $(f \circ \pi_1)^*\mathscr{G} \to f^*\mathscr{G}$, where π_1 is the projection onto the first factor in $f^*\mathscr{G}$.

Remark 5.5: Left- and right-actions

In the following we usually define everything with respect to right-actions; however, one can of course define the same for left actions in a similar manner. If we ever speak of a left action, then we assume precisely this. Some subtle changes like a sign change will be pointed out though.

One can probably see that it is straightforward to extend a lot of the typical notions of Lie group actions to LGB actions; hence, we mainly focus on the definitions and properties which we need in this paper.

Definition 5.6: Left and right translations, [1, §3.2, notation similar to Def. 3.2.3, page 131]

5.2. Examples of LGB actions

We have the following examples, the second of which will be important in this paper.

Example 5.7: Trivial action, [2, §1.6, special situation of Ex. 1.6.3, page 35]

The projection π_1 onto the first factor of $f^*\mathscr{G} \xrightarrow{\pi_1} N$ satisfies the properties of a right $\mathscr{G}$-action on N, that is, the action is given by

$$N * \mathscr{G} \to N,$$

$$(p, g) \mapsto p \cdot g \coloneqq p.$$

That this action satisfies the properties of an action for all f is trivial, hence we call it the **trivial action**.

Example 5.8: Inner group bundle acting on associated fibre bundles, [2, §1.6, simplified version of Ex. 1.6.4, page 35]

Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G over a smooth manifold M, and recall Ex. 4.11. Furthermore, let F be another smooth manifold, equipped with a smooth left G-action $\Psi : G \times F \to F$. In total we have two associated bundles over M:

$$\begin{tikzcd} G \arrow[r] & c_G(P) \arrow[d, "\pi_{c_G(P)}"] & & F \arrow[r] & \mathscr{F} := P \times_\Psi F \arrow[d, "\pi_{\mathscr{F}}"] \\ & M & & & M \end{tikzcd}$$

the inner group bundle of P and an associated F-bundle, respectively.
Then we have a right $c_G(P)$-action on $\mathscr{F}$ given by

$$\mathscr{F} * c_G(P) \coloneqq \pi_{\mathscr{F}}^* c_G(P) \to \mathscr{F},$$

$$\big([p,v],[p,g]\big) \mapsto [p, \Psi(g,v)] = [p \cdot g, v]$$

for all $p \in P_x$ $(x \in M)$, $g \in G$ and $v \in F$.

Proof.

• We first check again that the action is well-defined, that is, we are going to prove that the action is independent of the choice of fixed point in P_x. Thence, let $x \in M$, $p \in P_x$ and $p' \coloneqq p \cdot g'$ be another element of P_x, where $g' \in G$. Also let $[p_1, v] \in \mathscr{F}_x$ and $[p_2, g] \in c_G(P)_x$; then we have unique elements q_i, q_i' of G such that $(i \in \{1,2\})$

$$p_i = p \cdot q_i, \qquad p_i = p' \cdot q_i',$$

especially, it follows $q_i = g' q_i'$.

On one hand, if we use p as fixed element of P_x to calculate the multiplication, we get

$$[p_1, v] \cdot [p_2, g] = [p, \Psi(q_1, v)] \cdot [p, c_{q_2}(g)] = [p \cdot c_{q_2}(g), \Psi(q_1, v)] = [p \cdot c_{q_2}(g)\ q_1, v].$$

On the other hand, using p' as a fixed element, we derive, using $q_i' = (g')^{-1} q_i$,

$$[p_1, v] \cdot [p_2, h] = \left[p' \cdot c_{q_2'}(g)\ q_1', v\right] = \left[p \cdot g' q_2' g \big(q_2'\big)^{-1} q_1', v\right] = \left[p \cdot q_2 g q_2^{-1} q_1, v\right] = [p \cdot c_{g_2}(g)\ q_1, v],$$

which finalizes the argument needed to show that the action is well-defined.

• Let us now quickly check that the conditions in Def. 5.3 are satisfied. We have

$$\pi_{\mathscr{F}}\big([p,v] \cdot [p,g]\big) = \pi_{\mathscr{F}}([p, \Psi(g,v)]) = \pi_P(p) = \pi_{c_G(P)}\big([p,g]\big)$$

for all $p \in P_x$ $(x \in M)$, $v \in F$ and $g \in G$; similarly, having additionally $h \in G$,

$$\big([p,v] \cdot [p,g]\big) \cdot [p,h] = [p \cdot g, v] \cdot [p,h] = [p \cdot gh, v] = [p,v] \cdot [p, gh] = [p,v] \cdot \big([p,g]\ [p,h]\big),$$

and

$$[p, v] \cdot [p, e] = [p \cdot e, v] = [p, v].$$

Therefore this describes an action. ■

Remark 5.9: Relation to automorphisms of principal bundles and gauge transformations

Recall that gauge transformations have a strong relation to principal bundle automorphisms f of the principal bundle P; see *e.g.* [1, §5.3, Def. 5.3.1, page 256f.] and [1, §5.4, Thm. 5.4.4, page 273]. That is, f is a diffeomorphism $P \to P$ with

$$\pi_P \circ f = \mathbb{1}_M,$$

$$f(p \cdot g) = f(p) \cdot g$$

for all $p \in P$ and $g \in G$. The group of such maps will be denoted by $\mathscr{Aut}(P)$. One can identify such automorphisms with certain G-valued maps on P, following [1, §5.3, Def. 5.3.2 & Prop. 5.3.3, page 266f.]: We define the following set of smooth maps $P \to G$ by

$$C^\infty(P; G)^G := \left\{ \sigma : P \to G \text{ smooth} \;\middle|\; \sigma(p \cdot g) = c_{g^{-1}}\big(\sigma(p)\big) \text{ for all } p \in P, g \in G \right\}.$$

It is straightforward to check that this is a group w.r.t. pointwise multiplication. Furthermore, there is a group isomorphism

$$\mathscr{Aut}(P) \to C^\infty(P; G)^G,$$

$$f \mapsto \sigma_f,$$

where σ_f is defined by

$$f(p) = p \cdot \sigma_f(p)$$

for all $p \in P$; one can prove that this is well-defined.
As argued in [1, §5.3, Thm. 5.3.8, page 269; formulated as left action there, which is why we have an inverse here], $\mathscr{Aut}(P)$ acts (on the right) on associated fibre bundles $\mathscr{F} = P \times_\Psi F$ by

$$[p, v] \cdot f := \left[f^{-1}(p), v\right] = \left[p \cdot \sigma_f(p)^{-1}, v\right]$$

for all $[p, v] \in \mathscr{F}_x$ $(x \in M)$ and $f \in \mathscr{Aut}(P)$. σ_f can also be just locally defined, therefore one could investigate whether there is also an action just with an element g of G, basically the restriction of σ_f onto the fibre P_x. However, the action given by $[p, v] \cdot g = \left[p \cdot g^{-1}, v\right]$ for $g \in G$ is in general clearly only well-defined w.r.t. a change of the representative of

$[p, v] = \big[p \cdot q, \Psi_{q^{-1}}(v)\big]$ $(q \in G)$, if G is abelian. But one can resolve this by looking at it carefully: The rough idea is that g basically comes from $\sigma_f(p)$ in this context, but

$$\sigma_f(p \cdot q) = c_{q^{-1}}\big(\sigma_f(p)\big).$$

Roughly, while p is multiplied with g^{-1}, $p \cdot q$ has to be multiplied with $q^{-1}g^{-1}q$. It is easy to check that this resolves that issue, and the result is precisely the action described in Ex. 5.8. In fact, we have the following proposition:

For the following proposition observe that the (local) sections of an LGB have a group structure given by pointwise multiplication.

Proposition 5.10: Gauge transformations as sections of the inner LGB, [2, §1.4, (the last sentence of) Ex. 1.4.7, page 25]

Let $P \overset{\pi_P}{\to} M$ be a principal bundle with structural Lie group G over a smooth manifold M. Then there is a group isomorphism

$$\mathscr{Aut}(P) \to \Gamma\big(c_G(P)\big),$$

$$f \mapsto q_f$$

where $g_f \in \Gamma\big(c_G(P)\big)$ is defined by

$$q_f|_x := \big[p, \sigma_f(p)\big]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$, and σ_f is the element of $C^\infty(P; G)^G$ corresponding to f as introduced in Rem. 5.9.

Remarks 5.11.

As one may guess, $\Gamma\big(c_G(P)\big)$ is the analogue of $C^\infty(M; G)^G$ such that one could ask for a more direct analogue to $\mathscr{Aut}(P)$. Indeed, as argued in [2, §1.3, Prop. 1.3.9, page 20], $c_G(P)$ is actually isomorphic to $(P \times_M P)\big/G$, where $P \times_M P := \pi_P^* P$, and the G-action is the diagonal action on $P \times P$. One can prove that an isomorphism is given by

$$c_G(P) \to (P \times_M P)\big/G,$$

$$[p, g] \mapsto [p, p \cdot g].$$

It is also argued in [2, §1.4, Ex. 1.4.7, page 25] that $\mathscr{Aut}(P)$ is then directly isomorphic to $\Gamma\Big((P \times_M P)\big/G\Big)$ by

$$\mathscr{Aut}(P) \to \Gamma\Big((P \times_M P)\big/G\Big),$$

$$f \mapsto L_f,$$

where $L_f \in \Gamma\Big((P \times_M P)\Big/G\Big)$ is given by

$$L_f|_x := [p, f(p)] = [p, p \cdot \sigma_f(p)]$$

for all $x \in M$, where p is any element of P such that $\pi_P(p) = x$. This is clearly well-defined, and, so, while $c_G(P)$ is the bundle-analogue of $C^\infty(P;G)^G$ one can think of $(P \times_M P)\Big/G$ as the bundle-analogue of $\mathscr{A}ut(P)$.

However, this description often arises if one wants to use the formalism of groupoids and algebroids, here especially using the **gauge groupoid** and **Atiyah algebroid** induced by P. These would allow an even more elegant version of the gauge transformations, however, we intend to write this paper in such a way that there is no need that the reader has knowledge about those bundle structures. See the cited references for more details in that regard.

Proof of Prop. 5.10.

• Let us first quickly check whether $g_f \in \Gamma\big(c_G(P)\big)$ is well-defined for all $f \in \mathscr{A}ut(P)$. For $p \in P_x$ $(x \in M)$ we have

$$q_f|_x = [p, \sigma_f(p)],$$

If $p' = p \cdot g$ $(g \in G)$ is another element of P_x, then, using p' to define q_f,

$$q_f|_x = [p \cdot g, \sigma_f(p \cdot g)] = \big[p \cdot g, c_{g^{-1}}\big(\sigma_f(p)\big)\big] = [p, \sigma_f(p)],$$

also using the definition of $c_G(P)$, recall Ex. 4.11. It follows that q_f is well-defined, and it is clear that q_f is smooth.

• We want to show that $f \mapsto q_f$ is a group isomorphism by using that it is a composition of the group isomorphisms $\mathscr{A}ut \to C^\infty(P;G)^G$ as in Rem. 5.9 and

$$C^\infty(P;G)^G \to \Gamma\big(c_G(P)\big),$$

$$\sigma \mapsto q_\sigma, \tag{30}$$

where q_σ is effectively the same definition as q_f, that is $q_\sigma|_x = [p, \sigma(p)]$ which is well-defined by the very same reasons as before. It is only left to show that $C^\infty(P;G)^G \to \Gamma\big(c_G(P)\big)$ is a group isomorphism. For injectivity let σ' be another element of $C^\infty(P;G)^G$ and assume $[p, \sigma(p)] = [p, \sigma'(p)]$. Then

$$e_x = [p, e] = [p, \sigma(p)] \cdot \underbrace{\big([p, \sigma'(p)]\big)^{-1}}_{=\left[p, (\sigma'(p))^{-1}\right]} = \Big[p, \sigma(p)\big(\sigma'(p)\big)^{-1}\Big],$$

such that

$$\sigma(p)\big(\sigma'(p)\big)^{-1} = e,$$

so $\sigma = \sigma'$ and hence injectivity follows. For surjectivity observe that for a section $q \in \Gamma\big(c_G(P)\big)$ we can define a map $\sigma : P \to G$ by

$$q_x = [p, \sigma(p)].$$

This map satisfies

$$[p, \sigma(p)] = \big[p \cdot g, c_{g^{-1}}\big(\sigma(p)\big)\big] = [p \cdot g, \sigma(p \cdot g)]$$

for all $g \in G$; the last equality implies $\sigma(p \cdot g) = c_{g^{-1}}\big(\sigma(p)\big)$, which is precisely what we need for $C^\infty(P; G)^G$. It is only left to show smoothness of σ. For an open neighbourhood $U \subset M$ of x fix a trivialization $\varphi_U : P|_U \to U \times G$, and we denote

$$\varphi_U(p') = \big(\pi_P(p'), \beta_U(p')\big)$$

for all $p' \in P$, where $\beta_U : P|_U \to G$ is an equivariant map, *i.e.* $\beta_U(p' \cdot g) = \beta_U(p')\ g$ for all $g \in G$. As shown in the proof of Thm. 4.8, we have a trivialization of $c_G(P)$ given by

$$\begin{aligned} c_G(P)|_U &\to U \times G, \\ \big[p', g\big] &\mapsto \big(\pi_P(p'), \psi_{\beta_U(p')}(g)\big). \end{aligned}$$

Applying that trivialization to q we derive that

$$\big[p' \mapsto \psi_{\beta_U(p')}\big(\sigma(p')\big)\big]$$

is smooth, because q is smooth. Since $\psi_{\beta_U(p')}$ is smooth and bijective, we conclude that σ is smooth. Hence, $\sigma \in C^\infty(P; G)^G$, so, Def. (30) is also surjective and thence bijective.

Finally let us show that Def. (30) is a group isomorphism. Let σ, σ' be elements of $C^\infty(P; G)^G$, then use Def. (30) to derive

$$\sigma\sigma' \mapsto q_{\sigma\sigma'}$$

with

$$q_{\sigma\sigma'}|_x = \big[p, \sigma(p)\ \sigma'(p)\big] = [p, \sigma(p)] \cdot \big[p, \sigma'(p)\big] = q_\sigma|_x \cdot q_{\sigma'}|_x,$$

such that Def. (30) satisfies

$$\sigma\sigma' \mapsto q_\sigma \cdot q_{\sigma'}.$$

This concludes the proof. ∎

Associated fibre bundles are motivated by making the invariance of gauge theory under local gauge transformations (that is, the change of gauge) an inherent part of the bundle, while the action of the global and other transformations remain. This procedure of "reducing" the action onto the these is reflected in the quotient behind $c_G(P)$.

5.3. Toy model

We want to use LGBs in the context in the context of gauge theory, somewhat as a replacement of the structural Lie group.

6. Conclusion

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A. Axiomatic Yang-Mills gauge theories

Let us discuss where the compatibility conditions may arise from a certain axiomatic point of view.