Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux

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國家理論科學研究中心

4 September 2023

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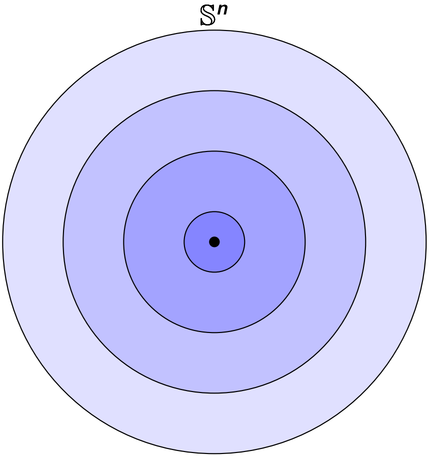

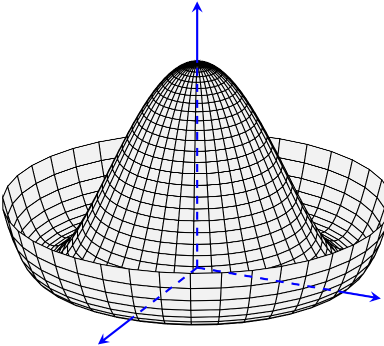

Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- . . .

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

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- it is **stable under** $C^\infty(M)$**-multiplication**, *i.e. $fX \in \mathscr{F}$ for all $f \in C^\infty(M)$ and $X \in \mathscr{F}$*,
- it is **locally finitely generated**, *i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ ($X^i \in \mathscr{F}$) such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on U.*

$$X = \sum_i f_i X^i.$$

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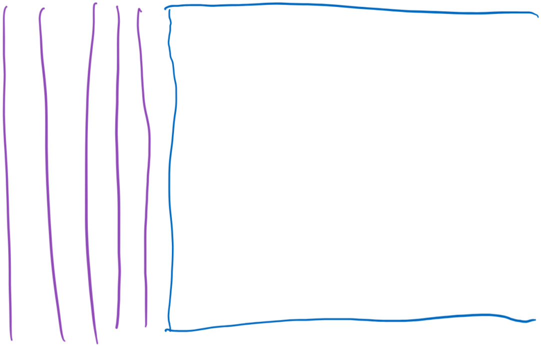
Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



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Why finitely generated?

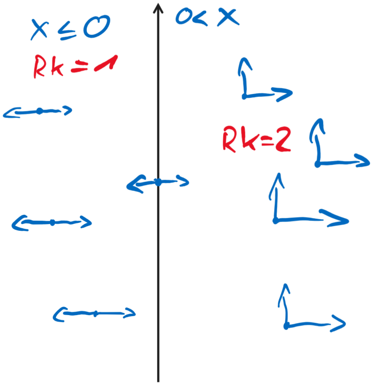
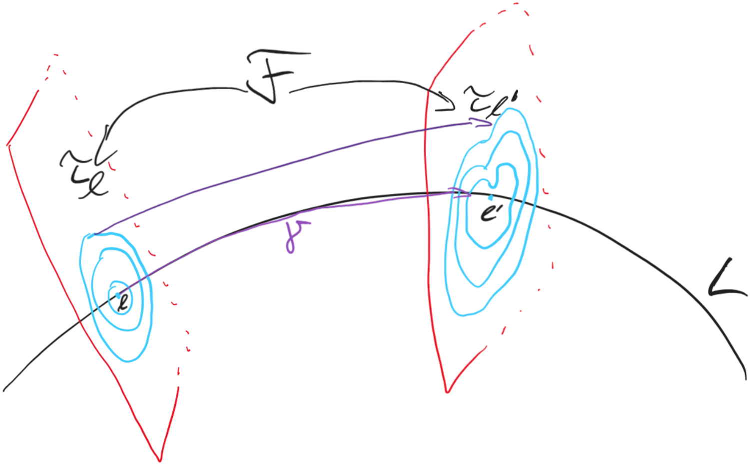
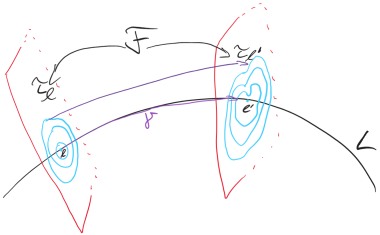


Figure: Infinite Comb

Idea: Relation to gauge theory



Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.



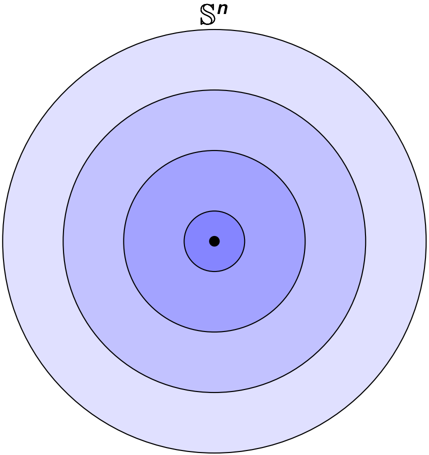
Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in* $\mathscr{F}$.
- *Parallel transport* PT_γ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop* γ_0 *at* l*:* PT_{γ_0} *values in* $\mathrm{Inner}(\tau_l)$.

Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Example of a transverse foliation τ:



Remarks

- $\mathrm{Inner}(\tau_I)$ maps each circle to itself
- $\mathrm{Sym}(\tau_I)$ allows to exchange circles
- Both preserve τ_I and fix the origin

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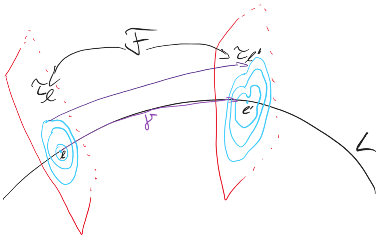
Idea



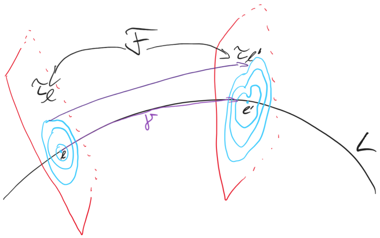
Idea

For all $Y \in \mathscr{F}$:

$$Y = X^{\uparrow} + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.

Idea



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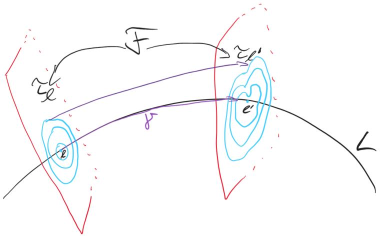

Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$\left[X^\uparrow + \overline{\nu}, X'^\uparrow + \overline{\mu}\right] = \underbrace{\left[X^\uparrow, X'^\uparrow\right]}_{\to \text{ curvature}} + \underbrace{\left[X^\uparrow, \overline{\mu}\right] - \left[X'^\uparrow, \overline{\nu}\right]}_{\to \text{ connection}} + \overline{[\nu, \mu]}$$

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Definition (LGB actions)

$$
\begin{tikzcd}
 & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\
\mathscr{T} \arrow[r, "\phi"] & L
\end{tikzcd}
$$

A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\phi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(t, g) \mapsto t \cdot g$, satisfying the following properties:

$$
\begin{aligned}
\phi(t \cdot g) &= \phi(t), && (1)\\
(t \cdot g) \cdot h &= t \cdot (gh), && (2)\\
t \cdot e_{\phi(t)} &= p && (3)
\end{aligned}
$$

for all $t \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\phi(t)}$, where $e_{\phi(t)}$ is the neutral element of $\mathscr{G}_{\phi(t)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L'$ so that one has a commuting diagram

$$
\begin{tikzcd}
\mathscr{T} \arrow[d, "\pi_{\mathscr{T}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\
L' \arrow[r, "\Psi"] & L
\end{tikzcd}
$$

1. **Ehresmann connection:** $\mathscr{G}$ preserving $\pi_{\mathscr{T}}$ and

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(t \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(t) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi \circ \gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = t \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}_{\phi(t)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = g_{\gamma_0} \cdot t \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle)

$$
\begin{tikzcd}
\mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\
L' \arrow[r, "\Psi"] & L
\end{tikzcd}
$$

A surjective submersion $\pi_{\mathscr{P}} : \mathscr{P} \to L'$, with $\mathscr{G}$-action

$$
\begin{array}{c} \cancel{\mathscr{P} \times G} \\ \mathscr{P} * \mathscr{G} \end{array} \to \mathscr{P}
$$

simply transitive on $\pi_{\mathscr{P}}$-fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids.
Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

Remarks ([S.-R. F.])

This gives rise to a generalised gauge theory by contracting the involved curvatures.

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Definition (Associated bundles, [C. L.-G., S.-R. F.])

$$\begin{tikzcd}
\mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\
L' \arrow[r, "\Psi"] & L & \mathscr{T} \arrow[l, "\pi_{\mathscr{T}}"]
\end{tikzcd}$$

Equivalence relation on $\mathscr{P}\,{}_{\phi}\!\times_{\pi_{\mathscr{T}}} \mathscr{T}$

$$(p,t) \sim \left(p \cdot g, g^{-1} \cdot t\right)$$

defines the **associated bundle** $\mathscr{P} \tilde{\times} \mathscr{T}$ over L'.

Theorem (Associated connection, [C. L.-G., S.-R. F.])

Given a multiplicative Ehresmann connection on $\mathscr{G}$, and related Ehresmann connection on $\mathscr{P}$ and $\mathscr{T}$, then

$$\mathrm{PT}_{\gamma}^{\mathscr{P}\tilde{\times}\mathscr{T}}[p,t] := \left[\mathrm{PT}_{\gamma}^{\mathscr{P}}(p), \mathrm{PT}_{\Psi\circ\gamma}^{\mathscr{T}}(t)\right]$$

is a well-defined connection.

Sketch of construction

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. G a subgroup of $\mathrm{Inn}(\tau_l)$
2. P a principal G-bundle, equipped with an ordinary connection

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Remarks

Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.

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Remarks

Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.

$\mathscr{P}$ comes with two natural projections to L, denoted by t and s

$$\begin{tikzcd}
\mathscr{P} \arrow[d, "{(t,s)}"'] \arrow[dr, "s"] & \mathscr{G} \arrow[l, bend right] \arrow[d] \arrow[dr, bend left] & \\
L \times L \arrow[r, "\mathrm{pr}_2"'] & L & \mathscr{T} \arrow[l, "\pi_{\mathscr{T}}"]
\end{tikzcd}$$

This extends to the associated bundle

$$\begin{tikzcd}
\mathscr{P}\tilde{\times}\mathscr{T} \arrow[r] \arrow[d, "{(t,s)}"'] \arrow[dr, "s"] & \mathscr{T} \arrow[d, "\pi_{\mathscr{T}}"] \\
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Remarks ([C. L.-G., S.-R. F.])

Associated connection independent of the choice of $\mathscr{F}$-connection!

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Remarks ([C. L.-G., S.-R. F.])

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Explicitly, one possible way:

Remarks

Corresponding to $\mathscr{P}$ there is an Atiyah sequence:

$$\operatorname{Ad}(P) \hookrightarrow \operatorname{At}(P) \twoheadrightarrow \mathrm{T}L$$

Via pullback to $\mathscr{T}$ we have a transitive algebroid over $\mathscr{T}$:

$$\pi_{\mathscr{T}}^{!}\operatorname{Ad}(P) \hookrightarrow \pi_{\mathscr{T}}^{!}\operatorname{At}(P) \twoheadrightarrow \mathrm{T}\mathscr{T}$$

Remarks

Observe

$$\pi_{\mathscr{T}}^{!}\operatorname{At}(P) \subset \mathrm{T}(\mathscr{P}\,{}_{s}\!\times_{\pi_{\mathscr{T}}} \mathscr{T})$$

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$$\pi_{\mathscr{T}}^{!}\mathrm{At}(P) \subset \mathrm{T}(\mathscr{P}\,{}_{s}\times_{\pi_{\mathscr{T}}}\mathscr{T})$$

$\mathrm{Ad}(P)$ and $\mathrm{At}(P)$ the adjoint and Atiyah bundle of P, respectively:

$$\begin{tikzcd}
\Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi^!_{\mathscr{T}}\mathrm{Ad}(P)\right) \arrow[r, hook] \arrow[d] & \Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi^!_{\mathscr{T}}\mathrm{At}(P)\right) \arrow[r, two heads] \arrow[d] & \mathfrak{X}(L) \arrow[d, equal] \\
\tau \arrow[r, hook] & \mathscr{F}_{\text{projectable}} \arrow[r, two heads] & \mathfrak{X}(L)
\end{tikzcd}$$

Remarks

Why Yang-Mills connections?

$$\text{Involutive} \leftrightarrow \text{Connection on } \mathscr{T} \text{ and } \mathscr{G} \text{ Yang-Mills}$$

$$
\begin{tikzcd}
 & \mathscr{F} \arrow[dl] \arrow[dr] & \\
\mathrm{At}(P) & & \text{Associated connection}
\end{tikzcd}
$$

Remarks (Why curved gauge theory?)

- Associated connection invariant under choice of $\mathscr{F}$-connection $\nabla^{\mathscr{F}}$
- Associated connection has the form

$$\nabla^{\mathscr{F}} + A\cdot$$

where A is the connection 1-form on $\mathscr{P}$

Thank you!