

Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux

Simon-Raphael Fischer

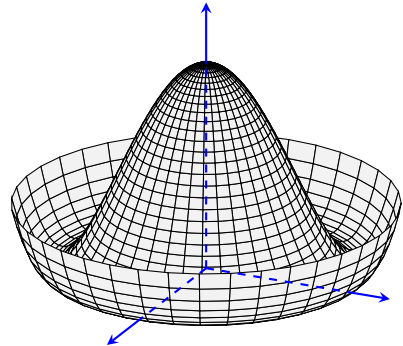
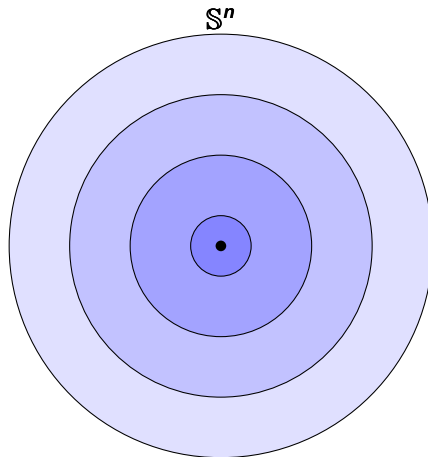
國家理論科學研究中心

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Singular Foliations



Singular Foliations:

- Gauge Theory
(Ex.: Singular foliation \leftrightarrow Symmetry breaking \rightarrow Higgs mechanism)
- Poisson Geometry
(Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** \mathcal{F} on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$ -**multiplication**,
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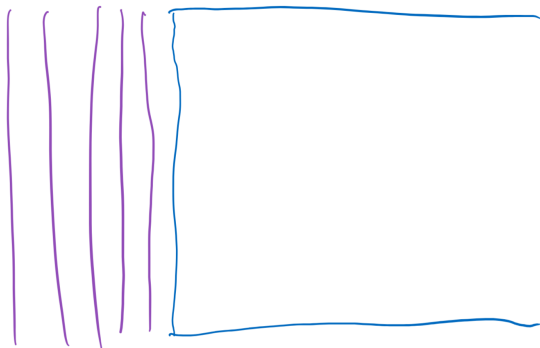
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- it is **involutive**, i.e. $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$,
- it is **stable under $C^\infty(M)$ -multiplication**, i.e. $fX \in \mathcal{F}$ for all $f \in C^\infty(M)$ and $X \in \mathcal{F}$,
- it is **locally finitely generated**, i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ ($X^i \in \mathcal{F}$) such that for all $X \in \mathcal{F}$ there are $f_i \in C^\infty(M)$ satisfying on U .

$$X = \sum_i f_i X^i.$$

Remarks (Leaves)

Following the flows in \mathcal{F} , this gives rise to a partition of connected immersed submanifolds in M .



Why finitely generated?

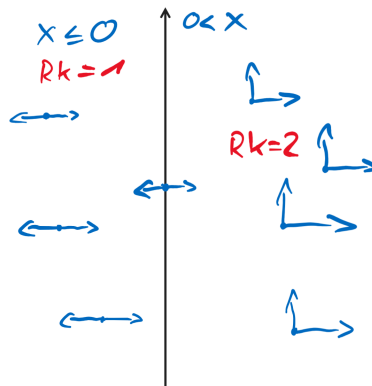
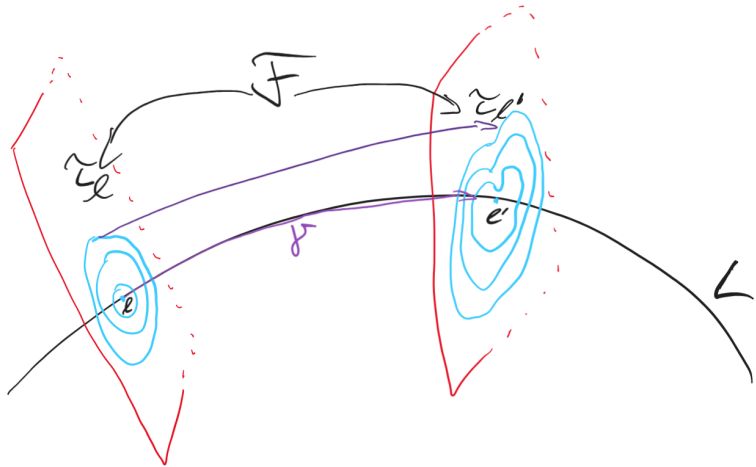
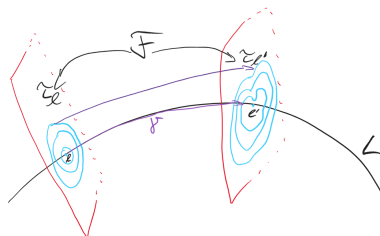


Figure: Infinite Comb





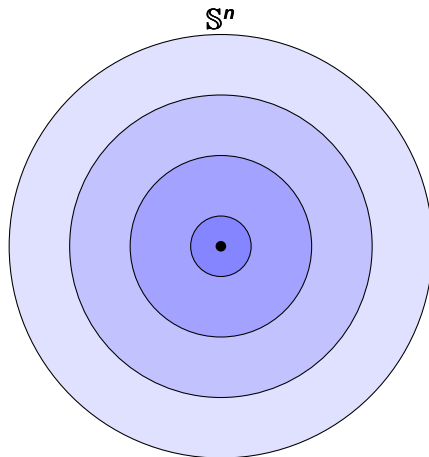
Theorem (\mathcal{F} -connections)

There is a connection on the normal bundle of a leaf L :

- *Horizontal vector fields are in \mathcal{F} .*
- *Parallel transport PT_γ has values in $\text{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l : PT_{γ_0} values in $\text{Inner}(\tau_l)$.*

Idea: Relation to gauge theory

Example of a transverse foliation τ :

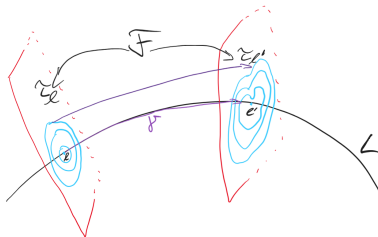


Remarks

- $\text{Inner}(\tau_I)$ maps each circle to itself
- $\text{Sym}(\tau_I)$ allows to exchange circles
- Both preserve τ_I and fix the origin

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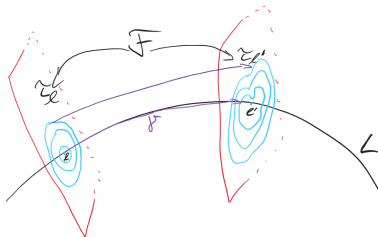
For all $Y \in \mathcal{F}$:

$$Y = X^\uparrow + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$, X^\uparrow its projectable horizontal lift, $\nu \in \Gamma(\text{inner}(\tau))$ and $\bar{\nu}$ its fundamental vector field.

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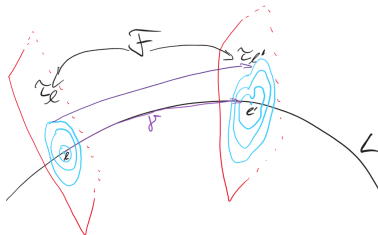


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Idea

Fix I and given τ_I : Reconstruct \mathcal{F} .

$$[X^\uparrow + \bar{\nu}, X'^\uparrow + \bar{\mu}] = \underbrace{[X^\uparrow, X'^\uparrow]}_{\rightsquigarrow \text{curvature}} + \underbrace{[X^\uparrow, \bar{\mu}] - [X'^\uparrow, \bar{\nu}]}_{\rightsquigarrow \text{connection}} + [\bar{\nu}, \bar{\mu}]$$

Yang-Mills connections

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle \mathcal{G}

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{G} \\ & & \downarrow \\ & & L \end{array}$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

\rightsquigarrow We will use a general connection ∇ instead of ∇^0 , and ∇ may not be flat.

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Definition (LGB actions)

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 \swarrow & & \downarrow \pi_{\mathcal{G}} \\
 \mathcal{T} & \xrightarrow{\phi} & L
 \end{array}$$

A **right-action** of \mathcal{G} on \mathcal{T} is a smooth map

$\mathcal{T} * \mathcal{G} := \mathcal{T} \times_{\phi \times \pi_{\mathcal{G}}} \mathcal{G} \rightarrow \mathcal{T}$, $(t, g) \mapsto t \cdot g$, satisfying the following properties:

$$\phi(t \cdot g) = \phi(t), \quad (1)$$

$$(t \cdot g) \cdot h = t \cdot (gh), \quad (2)$$

$$t \cdot e_{\phi(t)} = p \quad (3)$$

for all $t \in \mathcal{T}$ and $g, h \in \mathcal{G}_{\phi(t)}$, where $e_{\phi(t)}$ is the neutral element of $\mathcal{G}_{\phi(t)}$.

Definition (Principal bundle)

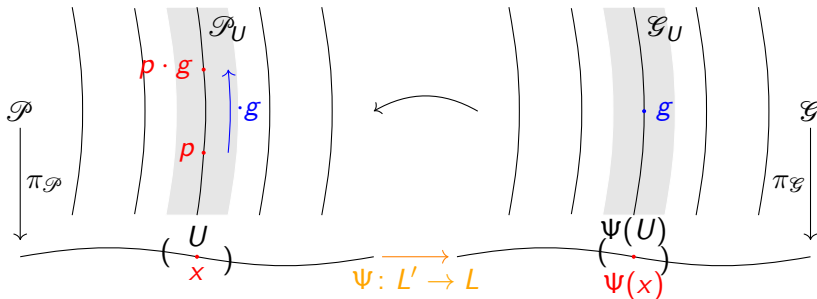
$$\begin{array}{ccc}
 \mathcal{P} & \xleftarrow{\quad} & \mathcal{G} \\
 \downarrow \pi_{\mathcal{P}} & \searrow \phi & \downarrow \pi_{\mathcal{G}} \\
 L' & \xrightarrow{\Psi} & L
 \end{array}$$

A surjective submersion $\pi_{\mathcal{P}}: \mathcal{P} \rightarrow L'$, with \mathcal{G} -action

$$\begin{array}{c}
 \cancel{\mathcal{P} \times \mathcal{G}} \\
 \mathcal{P} * \mathcal{G}
 \end{array}
 \rightarrow \mathcal{P}$$

simply transitive on $\pi_{\mathcal{P}}$ -fibres of \mathcal{P} , and "suitable" atlas.

Connections as parallel transport

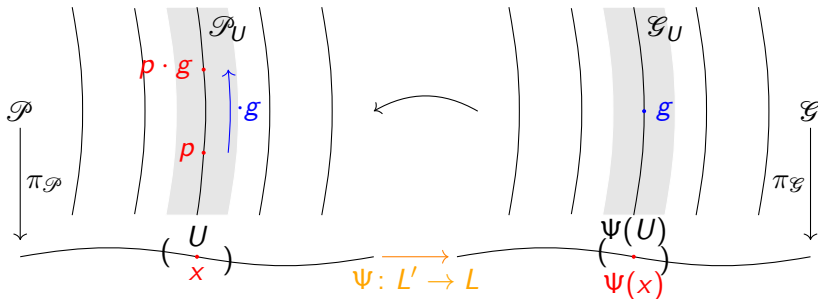
Connection on \mathcal{P} : Idea

But:

$$r_g: \mathcal{P}_x \rightarrow \mathcal{P}_x$$

 \Rightarrow $D_p r_g$ only defined on vertical structure

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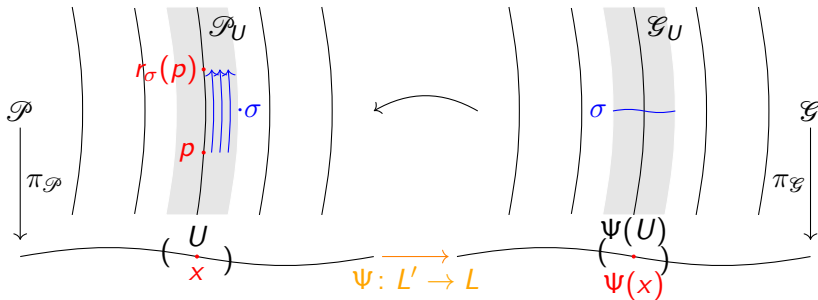
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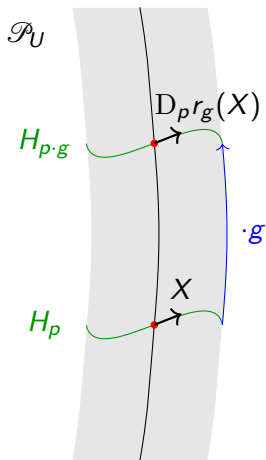
Connection on \mathcal{P} : Idea



$$\text{Use } \sigma \in \Gamma(\mathcal{G}): r_\sigma(p) := p \cdot \sigma_{\Psi(x)}$$

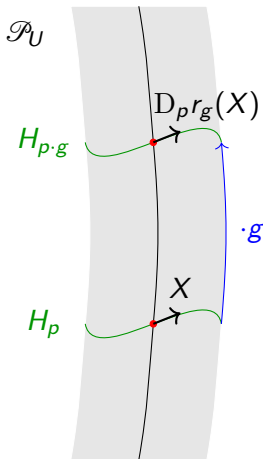
Connection on \mathcal{P} : Revisiting the classical setup

If \mathcal{P} a typical principal bundle
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Remarks (Integrated case)

Parallel transport $\text{PT}_\gamma^{\mathcal{P}}$ in \mathcal{P} :

$$\text{PT}_\gamma^{\mathcal{P}}(p \cdot g) = \text{PT}_\gamma^{\mathcal{P}}(p) \cdot g$$

where $\gamma : I \rightarrow L'$ is a base path

Connection on \mathcal{P} : General case

Remarks (Integrated case)

Ansatz: Introduce connection on \mathcal{G} ,

$$\mathrm{PT}_{\gamma}^{\mathcal{P}}(p \cdot g) = \mathrm{PT}_{\gamma}^{\mathcal{P}}(p) \cdot \mathrm{PT}_{\Psi \circ \gamma}^{\mathcal{G}}(g).$$

Back to the roots

- 1 $\mathcal{G} \cong L \times G$
- 2 Equip \mathcal{G} with canonical flat connection

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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.]

A surjective submersion $\pi_{\mathcal{T}} : \mathcal{T} \rightarrow L'$ so that one has a commuting diagram

$$\begin{array}{ccc}
 \mathcal{T} & \xleftarrow{\quad} & \mathcal{G} \\
 \pi_{\mathcal{T}} \downarrow & \searrow \phi & \downarrow \pi_{\mathcal{G}} \\
 L' & \xrightarrow{\quad \Psi \quad} & L
 \end{array}$$

- ① **Ehresmann connection:** \mathcal{G} preserving $\pi_{\mathcal{T}}$ and

$$\text{PT}_{\gamma}^{\mathcal{T}}(t \cdot g) = \text{PT}_{\gamma}^{\mathcal{T}}(t) \cdot \text{PT}_{\Psi \circ \gamma}^{\mathcal{G}}(g)$$

- ② **Yang-Mills connection:** Additionally

$$\text{PT}_{\gamma_0}^{\mathcal{T}}(t) = t \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathcal{G}_{\phi(t)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On \mathcal{G} there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\begin{aligned} \mathrm{PT}_{\gamma}^{\mathcal{G}}(q \cdot g) &= \mathrm{PT}_{\gamma}^{\mathcal{G}}(q) \cdot \mathrm{PT}_{\gamma}^{\mathcal{G}}(g), \\ \mathrm{PT}_{\gamma_0}^{\mathcal{G}}(q) &= g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1} \end{aligned}$$

Definition (Principal bundle connection, [S.-R. F.])

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Remarks

There is a simplicial differential δ on $\mathcal{G} \xrightarrow{\pi_{\mathcal{G}}} L$ with Lie algebra bundle \mathcal{Q}

$$\delta : \Omega^{\bullet}(\underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{Q}) \rightarrow \Omega^{\bullet}(\underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathcal{Q})$$

such that the definition of the multiplicative Yang-Mills connection is equivalent to the **compatibility conditions**

- Connection closed
- Curvature exact ([S.-R. F.])

Remarks

On the Lie algebra bundle \mathcal{g} we have a connection $\nabla^{\mathcal{G}}$ with

$$\begin{aligned}\nabla^{\mathcal{G}}([\mu, \nu]_{\mathcal{g}}) &= [\nabla^{\mathcal{G}}\mu, \nu]_{\mathcal{g}} + [\mu, \nabla^{\mathcal{G}}\nu]_{\mathcal{g}}, \\ R_{\nabla^{\mathcal{G}}} &= \text{ad} \circ \zeta. \quad ([S.-R. F.])\end{aligned}$$

Example

Given a short exact sequence of algebroids

$$\mathcal{g} \hookrightarrow E \twoheadrightarrow TL$$

with splitting $\chi: TL \rightarrow E$, then

$$\begin{aligned}\nabla_X^{\mathcal{G}}\nu &= [\chi(X), \nu]_E, \\ \zeta(X, X') &= [\chi(X), \chi(X')]_E - \chi([X, X']).\end{aligned}$$

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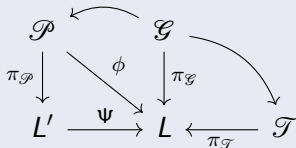
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Definition (Associated bundles, [C. L.-G., S.-R. F.]



Equivalence relation on $\mathcal{P} \times_{\phi \times \pi_{\mathcal{T}}} \mathcal{T}$

$$(p, t) \sim (p \cdot g, g^{-1} \cdot t)$$

defines the **associated bundle** $\mathcal{P} \tilde{\times} \mathcal{T}$ over L' .

Theorem (Associated connection, [C. L.-G., S.-R. F.]

Given a multiplicative Ehresmann connection on \mathcal{G} , and related Ehresmann connection on \mathcal{P} and \mathcal{T} , then

$$\mathrm{PT}_{\gamma}^{\mathcal{P} \tilde{\times} \mathcal{T}}[p, t] := \left[\mathrm{PT}_{\gamma}^{\mathcal{P}}(p), \mathrm{PT}_{\Psi \circ \gamma}^{\mathcal{T}}(t) \right]$$

is a well-defined connection.

Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on \mathcal{G} and a Yang-Mills connection on \mathcal{T} , then there is a natural foliation on \mathcal{T} generated by

$$X^\uparrow + \bar{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathfrak{g})$.

Proof.

We have

$$\begin{aligned} [X^\uparrow, \bar{\nu}] &= \overline{\nabla_X^\mathcal{G} \nu}, \\ [X^\uparrow, X'^\uparrow] &= [X, X']^\uparrow + \overline{\zeta(X, X')}, \end{aligned}$$

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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model (\mathbb{R}^d, τ_l) :

- 1 G a subgroup of $\text{Inn}(\tau_l)$
- 2 P a principal G -bundle, equipped with an ordinary connection

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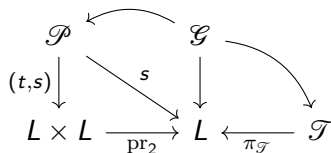
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Proposition ([C. L.-G., S.-R. F.])

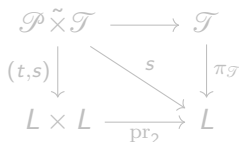
The associated connection on \mathcal{G} is a multiplicative Yang-Mills connection and the one on \mathcal{T} is a corresponding Yang-Mills connection.

Thus, we have a singular foliation on \mathcal{F} , which, by construction, admits L as a leaf and τ_l as transverse data.

\mathcal{P} comes with two natural projections to L , denoted by t and s



This extends to the associated bundle



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$$\begin{array}{ccccc}
 & & \mathcal{P} & \xleftarrow{\quad} & \mathcal{G} & & \\
 & & \downarrow (t,s) & \searrow s & \downarrow & \swarrow & \\
 L \times L & \xrightarrow{\text{pr}_2} & L & \xleftarrow{\pi_{\mathcal{F}}} & \mathcal{T} & &
 \end{array}$$

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$$\begin{array}{ccc}
 \mathcal{P} \tilde{\times} \mathcal{T} & \longrightarrow & \mathcal{T} \\
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Remarks ([C. L.-G., S.-R. F.]

Associated connection independent of the choice of \mathcal{F} -connection!

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Explicitly, one possible way:

Remarks

Corresponding to \mathcal{P} there is an Atiyah sequence:

$$\mathrm{Ad}(P) \hookrightarrow \mathrm{At}(P) \twoheadrightarrow TL$$

Via pullback to \mathcal{T} we have a transitive algebroid over \mathcal{T} :

$$\pi_{\mathcal{T}}^! \mathrm{Ad}(P) \hookrightarrow \pi_{\mathcal{T}}^! \mathrm{At}(P) \twoheadrightarrow T\mathcal{T}$$

Remarks

Observe

$$\pi_{\mathcal{T}}^! \mathrm{At}(P) \subset T(\mathcal{P} \times_{\pi_{\mathcal{T}}} \mathcal{T})$$

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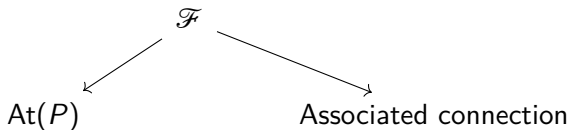
Remarks

Observe

$$\pi_{\mathcal{T}}^! \mathrm{At}(P) \subset T(\mathcal{P} \times_{\pi_{\mathcal{T}}} \mathcal{T})$$

$\text{Ad}(P)$ and $\text{At}(P)$ the adjoint and Atiyah bundle of P , respectively:

$$\begin{array}{ccccc}
 \Gamma_{\text{parallel}}^{\text{symmetric}}\left(\pi_{\mathcal{F}}^! \text{Ad}(P)\right) & \hookrightarrow & \Gamma_{\text{parallel}}^{\text{symmetric}}\left(\pi_{\mathcal{F}}^! \text{At}(P)\right) & \twoheadrightarrow & \mathfrak{X}(L) \\
 \downarrow & & \downarrow & & \parallel \\
 \tau & \hookrightarrow & \mathcal{F}_{\text{projectable}} & \twoheadrightarrow & \mathfrak{X}(L)
 \end{array}$$



Remarks (Why **curved** gauge theory?)

- Associated connection invariant under choice of \mathcal{F} -connection $\nabla^{\mathcal{F}}$

- Associated connection has the form

$$\nabla^{\mathcal{F}} + A.$$

where A is the connection 1-form on \mathcal{P}

Thank you!