Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux

Simon-Raphael Fischer

國家理論科學研究中心

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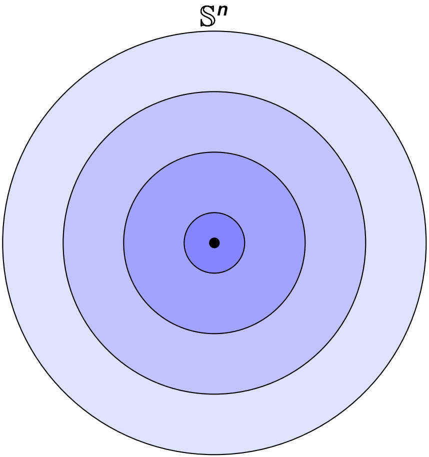
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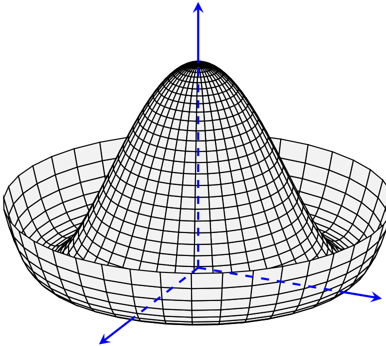

Singular Foliations:

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
Héctor J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180, 1973

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- it is **involutive**, *i.e.* $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$,
- it is **stable under $C^\infty(M)$-multiplication**, *i.e. $fX \in \mathscr{F}$ for all $f \in C^\infty(M)$ and $X \in \mathscr{F}$*,
- it is **locally finitely generated**, *i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ ($X^i \in \mathscr{F}$) such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on U.*

$$X = \sum_i f_i X^i.$$

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Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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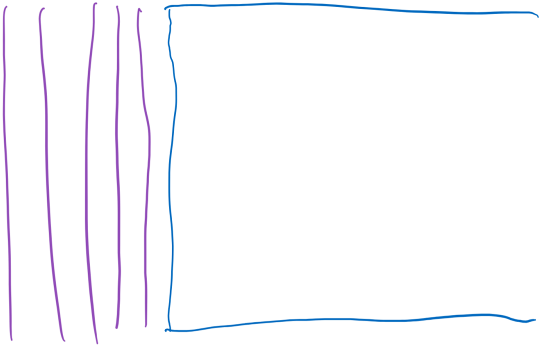
Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



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Idea: Relation to gauge theory

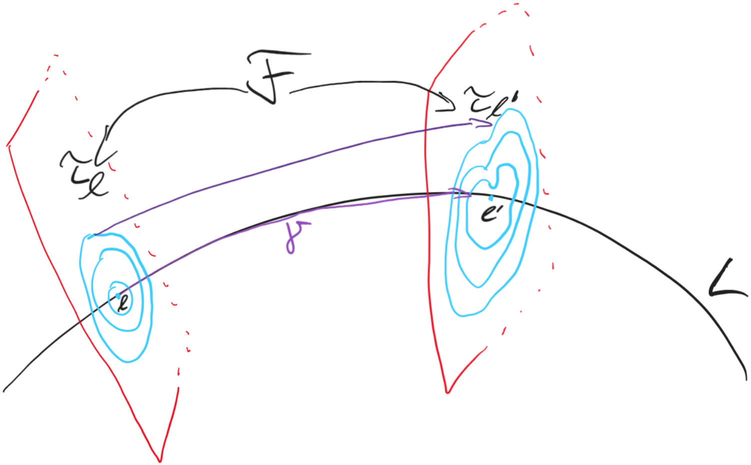

Figure: $\mathscr{F}$-connections

Camille Laurent-Gengoux and Leonid Ryvkin, The neighborhood of a singular leaf. *Journal de l'École polytechnique-Mathématiques*, 8, 2021.

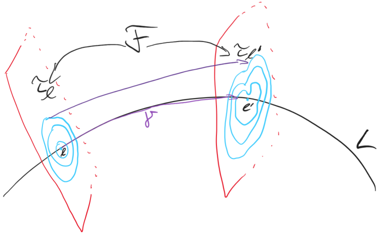

Figure: $\mathscr{F}$-connections

Theorem ($\mathscr{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in $\mathscr{F}$.*
- *Parallel transport PT_γ has values in $\mathrm{Sym}(\tau_l, \tau_{l'})$.*
- *For a contractible loop γ_0 at l: PT_{γ_0} values in $\mathrm{Inn}(\tau_l)$.*

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Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$
\begin{tikzcd}
G \arrow[r] & \mathscr{G} \arrow[d] \\
& L
\end{tikzcd}
$$

Definition (LGB actions)

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\phi"] & L \end{tikzcd}$$

A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\phi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(t, g) \mapsto t \cdot g$, satisfying the following properties:

$$\phi(t \cdot g) = \phi(t), \tag{1}$$
$$(t \cdot g) \cdot h = t \cdot (gh), \tag{2}$$
$$t \cdot e_{\phi(t)} = p \tag{3}$$

for all $t \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\phi(t)}$, where $e_{\phi(t)}$ is the neutral element of $\mathscr{G}_{\phi(t)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L'$ so that one has a commuting diagram

$$\begin{tikzcd} \mathscr{T} \arrow[d, "\pi_{\mathscr{T}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ L' \arrow[r, "\Psi"] & L \end{tikzcd}$$

1. **Ehresmann connection:** $\mathscr{G}$ preserving $\pi_{\mathscr{T}}$ and

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(t \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(t) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi \circ \gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = t \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}_{\phi(t)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = g_{\gamma_0} \cdot t \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle)

$$\begin{tikzcd} \mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ L' \arrow[r, "\Psi"] & L \end{tikzcd}$$

A surjective submersion $\pi_{\mathscr{P}} \colon \mathscr{P} \to L'$, with $\mathscr{G}$-action

$$\underset{\mathscr{P} * \mathscr{G}}{\cancel{\mathscr{P} \times G}} \to \mathscr{P}$$

simply transitive on $\pi_{\mathscr{P}}$-fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids.
Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

Remarks ([S.-R. F.])

This gives rise to a generalised gauge theory by contracting the involved curvatures.

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Definition (Associated bundles, [C. L.-G., S.-R. F.])

$$\begin{tikzcd}
\mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\
L' \arrow[r, "\Psi"] & L & \mathscr{T} \arrow[l, "\pi_{\mathscr{T}}"]
\end{tikzcd}$$

Equivalence relation on $\mathscr{P}\ {}_{\phi}\!\times_{\pi_{\mathscr{T}}} \mathscr{T}$

$$(p, t) \sim \left(p \cdot g, g^{-1} \cdot t\right)$$

defines the **associated bundle** $\mathscr{P} \tilde{\times} \mathscr{T}$ over L'.

Theorem (Associated connection, [C. L.-G., S.-R. F.])

Given a multiplicative Ehresmann connection on $\mathscr{G}$, and related Ehresmann connection on $\mathscr{P}$ and $\mathscr{T}$, then

$$\mathrm{PT}_\gamma^{\mathscr{P}\tilde{\times}\mathscr{T}}[p,t] := \left[\mathrm{PT}_\gamma^{\mathscr{P}}(p), \mathrm{PT}_{\Psi\circ\gamma}^{\mathscr{T}}(t)\right]$$

is a well-defined connection.

Sketch of construction

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. G a subgroup of $\mathrm{Inn}(\tau_l)$
2. P a principal G-bundle, equipped with an ordinary connection

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Remarks

Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.

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Remarks

Think of the induced connection on $\mathscr{T}$ as the $\mathscr{F}$-connection.

$\mathscr{P}$ comes with two natural projections to L, denoted by t and s

$$
\begin{tikzcd}
\mathscr{P} \arrow[d, "(t,s)"'] \arrow[dr, "s"] & \mathscr{G} \arrow[l, bend right] \arrow[d] \arrow[dr, bend left] & \\
L \times L \arrow[r, "\mathrm{pr}_2"'] & L & \mathscr{T} \arrow[l, "\pi_{\mathscr{T}}"']
\end{tikzcd}
$$

This extends to the associated bundle

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Remarks ([C. L.-G., S.-R. F.])

Associated connection independent of the choice of $\mathscr{F}$-connection!

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Remarks ([C. L.-G., S.-R. F.])

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Explicitly, one possible way:

Remarks

Corresponding to $\mathscr{P}$ there is an Atiyah sequence:

$$\mathrm{Ad}(P) \hookrightarrow \mathrm{At}(P) \twoheadrightarrow \mathrm{T}L$$

Via pullback to $\mathscr{T}$ we have a transitive algebroid over $\mathscr{T}$:

$$\pi^!_{\mathscr{T}}\mathrm{Ad}(P) \hookrightarrow \pi^!_{\mathscr{T}}\mathrm{At}(P) \twoheadrightarrow \mathrm{T}\mathscr{T}$$

Remarks

Observe

$$\pi^!_{\mathscr{T}}\mathrm{At}(P) \subset \mathrm{T}(\mathscr{P}\,{}_s\!\times_{\pi_{\mathscr{T}}} \mathscr{T})$$

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$$\pi_{\mathscr{T}}^{!}\mathrm{At}(P) \subset \mathrm{T}(\mathscr{P}\,{}_{s}\!\times_{\pi_{\mathscr{T}}} \mathscr{T})$$

$\mathrm{Ad}(P)$ and $\mathrm{At}(P)$ the adjoint and Atiyah bundle of P, respectively:

$$\begin{tikzcd}
\Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi_{\mathscr{T}}^{!}\mathrm{Ad}(P)\right) \arrow[r, hook] \arrow[d] & \Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi_{\mathscr{T}}^{!}\mathrm{At}(P)\right) \arrow[r, two heads] \arrow[d] & \mathfrak{X}(L) \arrow[d, equal] \\
\tau \arrow[r, hook] & \mathscr{F}_{\text{projectable}} \arrow[r, two heads] & \mathfrak{X}(L)
\end{tikzcd}$$

Remarks

Why Yang-Mills connections?

Involutive $\leftrightarrow$ Connection on $\mathscr{T}$ and $\mathscr{G}$ Yang-Mills

$$
\begin{tikzcd}
& \mathscr{F} \arrow[dl] \arrow[dr] & \\
\mathrm{At}(P) & & \text{Associated connection}
\end{tikzcd}
$$

Remarks (Why curved gauge theory?)

- Associated connection invariant under choice of $\mathscr{F}$-connection $\nabla^{\mathscr{F}}$
- Associated connection has the form

$$\nabla^{\mathscr{F}} + A \cdot$$

where A is the connection 1-form on $\mathscr{P}$

Thank you!