Classification of neighbourhoods of leaves of singular foliations

joint work with Camille Laurent-Gengoux (Université de Lorraine)
Work in progress

Simon-Raphael Fischer

國家理論科學研究中心

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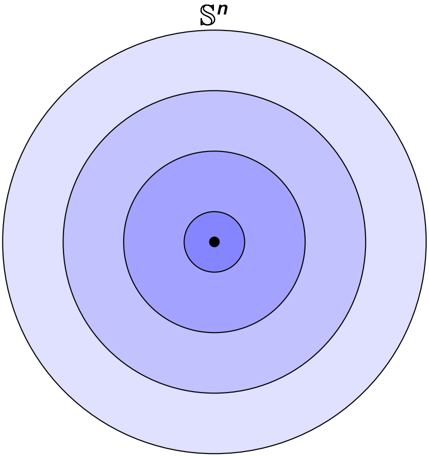
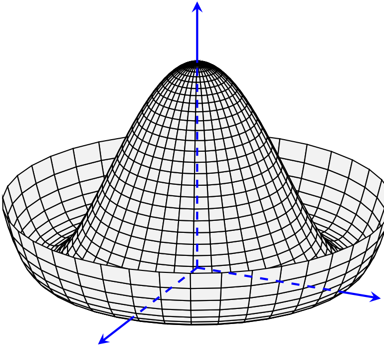
Singular Foliations





Right diagram made by Mark J.D. Hamilton.

Singular Foliations:

- Gauge Theory
  (Ex.: Singular foliation $\leftrightarrow$ Symmetry breaking $\rightarrow$ Higgs mechanism)
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

Definition (Smooth singular foliation)

A **smooth singular foliation** $\mathscr{F}$ on a smooth manifold is a subspace of $\mathfrak{X}_c(M)$ so that

- it is **involutive**,
- it is **stable under** $C^\infty(M)$**-multiplication**,
- it is **locally finitely generated**.

Sources
Peter Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.*, 29, 1974.
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- it is **stable under $C^\infty(M)$-multiplication**, *i.e. $fX \in \mathscr{F}$ for all $f \in C^\infty(M)$ and $X \in \mathscr{F}$*,
- it is **locally finitely generated**, *i.e. around each $p \in M$ there is an open neighbourhood U and a finite family $(X^i)_i^r$ $(X^i \in \mathscr{F})$ such that for all $X \in \mathscr{F}$ there are $f_i \in C^\infty(M)$ satisfying on U.*

$$X = \sum_i f_i X^i.$$

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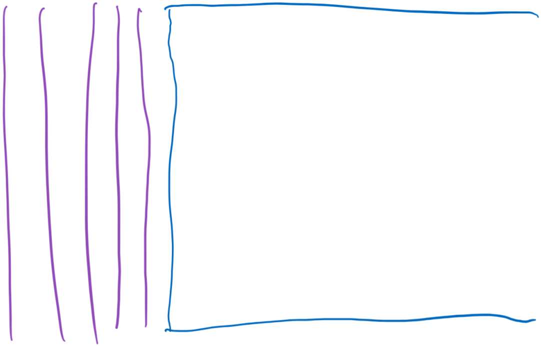
Remarks (Leaves)

Following the flows in $\mathscr{F}$, this gives rise to a partition of connected immersed submanifolds in M.



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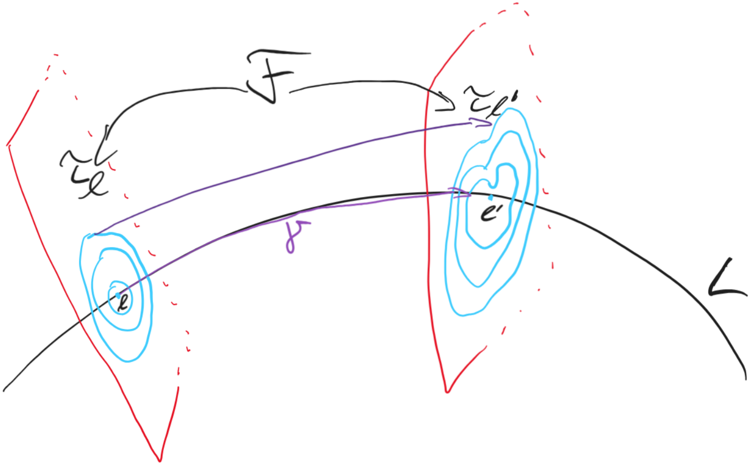
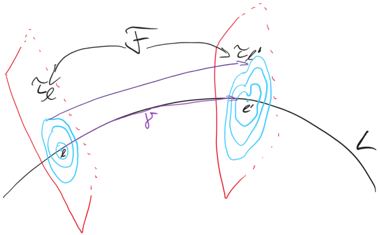
Idea: Relation to gauge theory



Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.



Theorem ($\mathcal{F}$-connections)

There is a connection on the normal bundle of a leaf L:

- *Horizontal vector fields are in* $\mathcal{F}$.
- *Parallel transport* PT_γ *has values in* $\mathrm{Sym}(\tau_l, \tau_{l'})$.
- *For a contractible loop* γ_0 *at l:* PT_{γ_0} *values in* $\mathrm{Inner}(\tau_l)$.

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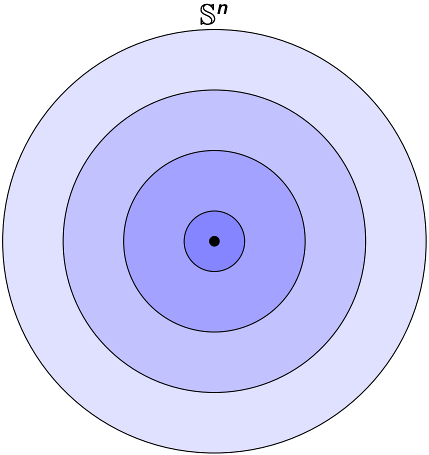
Example of a transverse foliation τ:

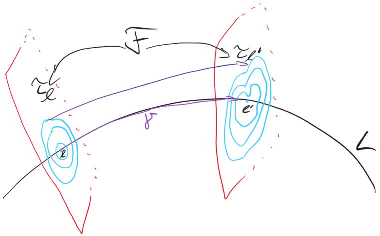


Remarks

- Inner(τ_I) maps each circle to itself
- Sym(τ_I) allows to exchange circles
- Both preserve τ_I and fix the origin

Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Idea



Idea

Generators of $\mathscr{F}$ given by $\mathscr{F}_{\text{projectable}}$:

$$X^{\uparrow} + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$, $X^{\uparrow}$ its projectable horizontal lift, $\nu \in \Gamma(\mathrm{inner}(\tau))$ and $\overline{\nu}$ its fundamental vector field.

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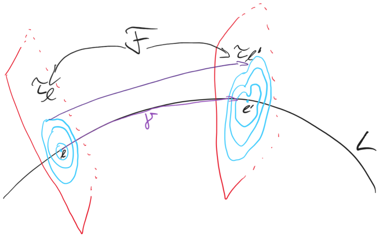

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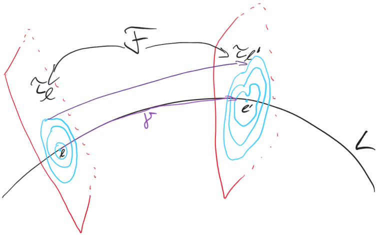

Idea

Fix l and given τ_l: Reconstruct $\mathscr{F}$.

$$
\begin{aligned}
\left[X^\uparrow + \overline{\nu}, X'^\uparrow + \overline{\mu}\right] &= [X, X']^\uparrow + \dots \\
&= \underbrace{\left[X^\uparrow, X'^\uparrow\right]}_{\rightsquigarrow \text{ curvature}} + \underbrace{\left[X^\uparrow, \overline{\mu}\right] - \left[X'^\uparrow, \overline{\nu}\right]}_{\rightsquigarrow \text{ connection}} + \overline{[\nu, \mu]}
\end{aligned}
$$

Curved Yang-Mills gauge theory

Curved Yang-Mills gauge theories:

Classical	Curved
Lie group G	Lie group bundle $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

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Definition (LGB actions)

$$\begin{tikzcd} & \mathscr{G} \arrow[dl, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathscr{T} \arrow[r, "\phi"] & L \end{tikzcd}$$

A **right-action of $\mathscr{G}$ on $\mathscr{T}$** is a smooth map $\mathscr{T} * \mathscr{G} := \mathscr{T} {}_{\phi}\times_{\pi_{\mathscr{G}}} \mathscr{G} \to \mathscr{T}$, $(t, g) \mapsto t \cdot g$, satisfying the following properties:

$$\phi(t \cdot g) = \phi(t), \tag{1}$$
$$(t \cdot g) \cdot h = t \cdot (gh), \tag{2}$$
$$t \cdot e_{\phi(t)} = p \tag{3}$$

for all $t \in \mathscr{T}$ and $g, h \in \mathscr{G}_{\phi(t)}$, where $e_{\phi(t)}$ is the neutral element of $\mathscr{G}_{\phi(t)}$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Definition (Principal bundle)

$$\begin{tikzcd} \mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ L' \arrow[r, "\Psi"] & L \end{tikzcd}$$

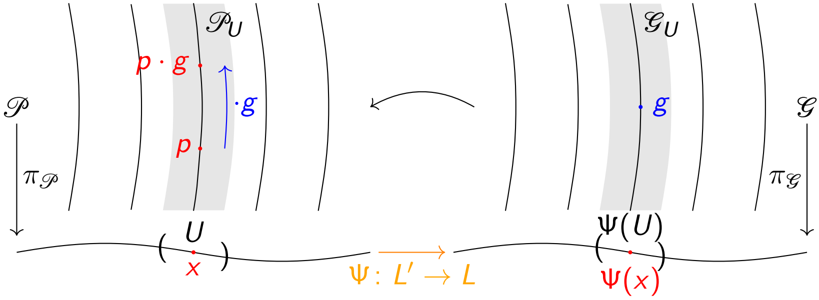
A surjective submersion $\pi_{\mathscr{P}} \colon \mathscr{P} \to L'$, with $\mathscr{G}$-action

$$\begin{array}{c} \cancel{\mathscr{P} \times G} \\ \mathscr{P} * \mathscr{G} \end{array} \to \mathscr{P}$$

simply transitive on $\pi_{\mathscr{P}}$-fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids.
Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003

Connection on $\mathscr{P}$: Idea



But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

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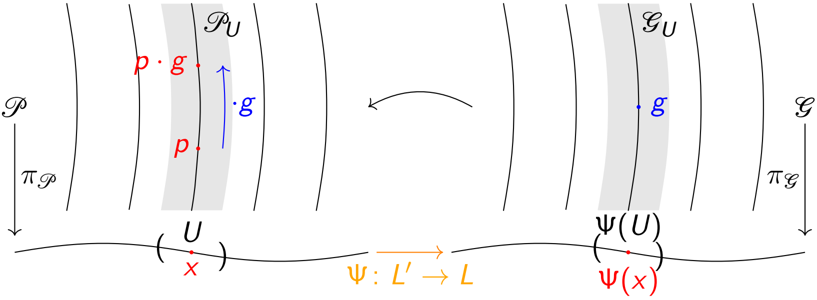

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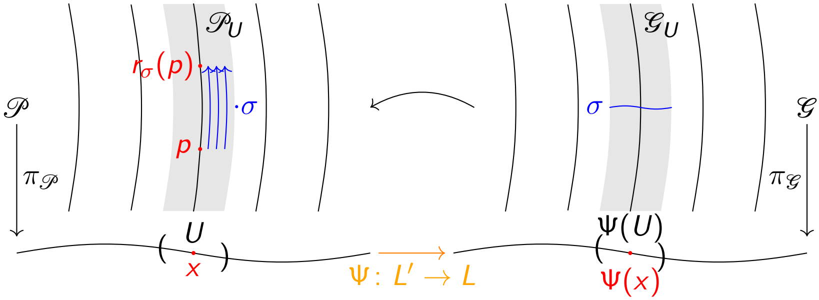
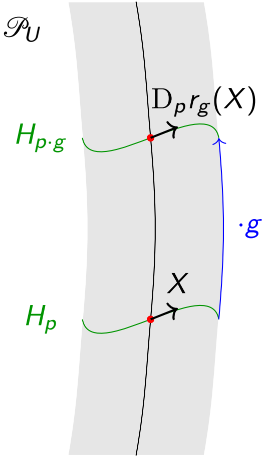


Use $\sigma \in \Gamma(\mathscr{G})$: $r_\sigma(p) := p \cdot \sigma_{\Psi(x)}$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle ($\mathscr{G}$ trivial, $\sigma \equiv g$ constant), and H a connection:



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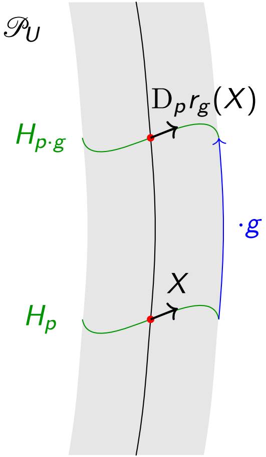

Remarks (Integrated case)

Parallel transport $\mathrm{PT}_\gamma^{\mathscr{P}}$ in $\mathscr{P}$:

$$\mathrm{PT}_\gamma^{\mathscr{P}}(p \cdot g) = \mathrm{PT}_\gamma^{\mathscr{P}}(p) \cdot g$$

where $\gamma : I \to L'$ is a base path

Connection on $\mathscr{P}$: General case

Remarks (Integrated case)

Ansatz: Introduce connection on $\mathscr{G}$,

$$\mathrm{PT}^{\mathscr{P}}_{\gamma}(p \cdot g) = \mathrm{PT}^{\mathscr{P}}_{\gamma}(p) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi\circ\gamma}(g).$$

Back to the roots

1. $\mathscr{G} \cong L \times G$
2. Equip $\mathscr{G}$ with canonical flat connection

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Definition (Ehresmann/Yang-Mills connection, [C. L.-G., S.-R. F.])

A surjective submersion $\pi_{\mathscr{T}} : \mathscr{T} \to L'$ so that one has a commuting diagram

$$\begin{tikzcd} \mathscr{T} \arrow[d, "\pi_{\mathscr{T}}"'] \arrow[dr, "\phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \\ L' \arrow[r, "\Psi"] & L \end{tikzcd}$$

1. **Ehresmann connection:** $\mathscr{G}$ preserving $\pi_{\mathscr{T}}$ and

$$\mathrm{PT}^{\mathscr{T}}_{\gamma}(t \cdot g) = \mathrm{PT}^{\mathscr{T}}_{\gamma}(t) \cdot \mathrm{PT}^{\mathscr{G}}_{\Psi\circ\gamma}(g)$$

2. **Yang-Mills connection:** Additionally

$$\mathrm{PT}^{\mathscr{T}}_{\gamma_0}(t) = t \cdot g_{\gamma_0}$$

for some $g_{\gamma_0} \in \mathscr{G}_{\phi(t)}$, where γ_0 is a contractible loop.

Definition (Multiplicative YM connection, [S.-R. F.])

On $\mathscr{G}$ there is also the notion of **multiplicative Yang-Mills connections**, that is,

$$\mathrm{PT}^{\mathscr{G}}_{\gamma}(q \cdot g) = \mathrm{PT}^{\mathscr{G}}_{\gamma}(q) \cdot \mathrm{PT}^{\mathscr{G}}_{\gamma}(g),$$
$$\mathrm{PT}^{\mathscr{G}}_{\gamma_0}(q) = g_{\gamma_0} \cdot q \cdot g_{\gamma_0}^{-1}$$

Definition (Principal bundle connection, [S.-R. F.])

- On $\mathscr{G}$: Multiplicative Yang-Mills connection
- On $\mathscr{P}$: Ehresmann connection

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Remarks

There is a simplicial differential δ on $\mathcal{G} \overset{\pi_{\mathcal{G}}}{\to} L$ with Lie algebra bundle $\mathscr{g}$

$$\delta : \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k \text{ times}}; \pi_{\mathcal{G}}^* \mathscr{g}) \to \Omega^\bullet(\underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k+1 \text{ times}}; \pi_{\mathcal{G}}^* \mathscr{g})$$

such that the definition of the multiplicative Yang-Mills connection is equivalent to the **compatibility conditions**

- Connection closed
- Curvature exact ([S.-R. F.])

 For differential: Marius Crainic, Maria Amelia Salazar, and Ivan Struchiner. Multiplicative forms and Spencer operators. *Mathematische Zeitschrift*, 279(3):939–979, 2015.

Remarks

On the Lie algebra bundle $\mathscr{g}$ we have a connection $\nabla^{\mathscr{G}}$ with

$$\nabla^{\mathscr{G}}\left([\mu,\nu]_{\mathscr{g}}\right) = \left[\nabla^{\mathscr{G}}\mu,\nu\right]_{\mathscr{g}} + \left[\mu,\nabla^{\mathscr{G}}\nu\right]_{\mathscr{g}},$$

$$R_{\nabla^{\mathscr{G}}} = \mathrm{ad}\circ\zeta. \qquad \text{([S.-R. F.])}$$

Example

Given a short exact sequence of algebroids

$$\mathscr{g} \hookrightarrow E \twoheadrightarrow \mathrm{T}L$$

with splitting $\chi\colon \mathrm{T}L \to E$, then

$$\nabla^{\mathscr{G}}_X\nu = [\chi(X),\nu]_E,$$

$$\zeta(X,X') = \left[\chi(X),\chi(X')\right]_E - \chi([X,X']).$$

For the first equation: Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Non-abelian differentiable gerbes. *Advances in Mathematics*, 220(5):1357–1427, 2009.

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Going back to foliations

Theorem ([C. L.-G., S.-R. F.])

Given a multiplicative Yang-Mills connection on $\mathscr{G}$ and a Yang-Mills connection on $\mathscr{T}$, then there is a natural foliation on $\mathscr{T}$ generated by

$$X^{\uparrow} + \overline{\nu},$$

where $X \in \mathfrak{X}(L)$ and $\nu \in \Gamma(\mathscr{g})$.

Proof.

We have

$$\left[X^{\uparrow}, \overline{\nu}\right] = \overline{\nabla^{\mathscr{G}}_X \nu},$$
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Idea (Leaf L simply connected)

Fix a point $l \in L$ with transverse model $\left(\mathbb{R}^d, \tau_l\right)$:

1. $G = \mathrm{Inn}(\tau_l)$
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- $\mathscr{G}$ acts on $\mathscr{T}$ (canonically from the left).

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Proposition ([C. L.-G., S.-R. F.])

The associated connection on $\mathscr{G}$ is a multiplicative Yang-Mills connection and the one on $\mathscr{T}$ is a corresponding Yang-Mills connection.

Remarks

Thus, we have a singular foliation on $\mathscr{T}$, which, by construction, admits L as a leaf and τ_l as transverse data.

Lemma ([C. L.-G., S.-R. F.])

*$\mathscr{P} := (P \times P)\big/G$, the **Atiyah groupoid**, is a principal $\mathscr{G}$-bundle*

$$
\begin{tikzcd}
\mathscr{P} \arrow[d, "{(t,s)}"'] \arrow[dr, "s"] & \mathscr{G} \arrow[l, bend right] \arrow[d] \\
L \times L \arrow[r, "\mathrm{pr}_2"'] & L
\end{tikzcd}
$$

where t and s are the target and source arrows, respectively. A connection on P induces an Ehresmann connection on $\mathscr{P}$.

Definition (Associated bundles, [C. L.-G., S.-R. F.])

$$
\begin{tikzcd}
\mathscr{P} \arrow[d, "{(t,s)}"'] \arrow[dr, "s"] & \mathscr{G} \arrow[l, bend right] \arrow[d, "\pi_{\mathscr{G}}"] \arrow[dr, bend left] & \\
L' \arrow[r, "\mathrm{pr}_2"] & L & \mathscr{T} \arrow[l, "\pi_{\mathscr{T}}"]
\end{tikzcd}
$$

Equivalence relation on $\mathscr{P}\ {}_s\!\times_{\pi_{\mathscr{T}}} \mathscr{T}$

$$(p, t) \sim \left(p \cdot g, g^{-1} \cdot t\right)$$

defines the **associated bundle** $\mathscr{P} \tilde{\times} \mathscr{T}$ over L'.

Theorem (Associated connection, [C. L.-G., S.-R. F.])

$$\mathrm{PT}_{\gamma}^{\mathscr{P}\tilde{\times}\mathscr{T}}[p,t] := \left[\mathrm{PT}_{\gamma}^{\mathscr{P}}(p), \mathrm{PT}_{\mathrm{pr}_2\circ\gamma}^{\mathscr{T}}(t)\right]$$

is a well-defined connection.

Remarks ([C. L.-G., S.-R. F.])

Associated connection independent of the choice of connection on P!

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Associated connection independent of the choice of connection on P!

Explicitly, one possible way:

Remarks

Corresponding to $\mathscr{P}$ there is an Atiyah sequence:

$$\mathrm{Ad}(P) \hookrightarrow \mathrm{At}(P) \twoheadrightarrow \mathrm{T}L$$

Via pullback to $\mathscr{T}$ we have a transitive algebroid over $\mathscr{T}$:

$$\pi^!_{\mathscr{T}}\mathrm{Ad}(P) \hookrightarrow \pi^!_{\mathscr{T}}\mathrm{At}(P) \twoheadrightarrow \mathrm{T}\mathscr{T}$$

Remarks

Observe

$$\pi^!_{\mathscr{T}}\mathrm{At}(P) \subset \mathrm{T}(\mathscr{P}\ {}_s\!\times_{\pi_{\mathscr{T}}} \mathscr{T})$$

Explicitly, one possible way:

Remarks

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$\mathrm{Ad}(P)$ and $\mathrm{At}(P)$ the adjoint and Atiyah bundle of P, respectively:

$$
\begin{tikzcd}
\Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi^!_{\mathscr{T}}\mathrm{Ad}(P)\right) \arrow[r, hook] \arrow[d] & \Gamma^{\text{symmetric}}_{\text{parallel}}\left(\pi^!_{\mathscr{T}}\mathrm{At}(P)\right) \arrow[r, two heads] \arrow[d] & \mathfrak{X}(L) \arrow[d, equal] \\
\tau \arrow[r, hook] & \mathscr{F}_{\text{projectable}} \arrow[r, two heads] & \mathfrak{X}(L)
\end{tikzcd}
$$

$$\begin{tikzcd} & \mathscr{F} \arrow[dl] \arrow[dr] & \\ \mathrm{At}(P) & & \text{Associated connection} \end{tikzcd}$$

Remarks (Why curved gauge theory?)

- Associated connection invariant under choice of $\mathscr{F}$-connection $\nabla^{\mathscr{F}}$
- Associated connection has the form

$$\nabla^{\mathscr{F}} + A\cdot$$

where A is the connection 1-form on $\mathscr{P}$

Thank you!