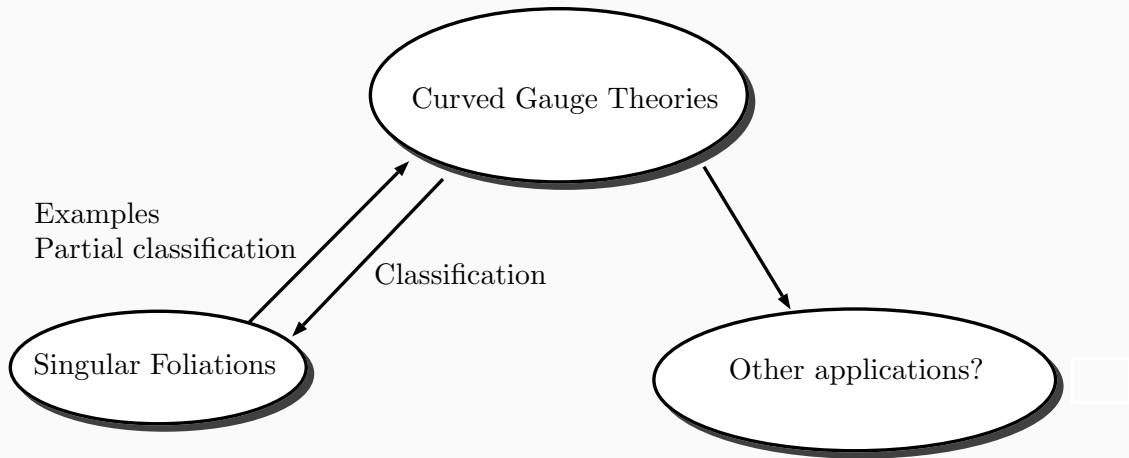


Curved Yang-Mills-Higgs theories

Simon-Raphael Fischer, *based on joint works with Camille Laurent-Gengoux, and with Mehran Jalali Farahani, Hyungrok Kim (金炯錄), Christian Sämann*





Motivation

Motivation: Covariantisation of Yang-Mills(-Higgs) theory

Covariantization



Classical theory

Covariantised flat theory

Curved Theory

Vector space V

Trivial vector bundle $M \times V$

Vector bundle $V \rightarrow M$

$$\frac{\partial}{\partial x^i}$$

Canonical flat connection ∇^0

Vector bundle connection ∇

Coordinate changes may lead to
extra terms

Coordinate expressions form-invariant under coordinate changes



Curved Yang-Mills gauge theory (curved YM theory)

Covariantization



YM theory

Covariantised YM theory

Curved YM theory

Lie group G

Trivial Lie group bundle $M \times G$

Lie group bundle $G \rightarrow M$

Maurer-Cartan form

Fibre-wise Maurer-Cartan

Multiplicative YM connection

Field redefinitions lead to extra terms in gauge transformations and field strength

Expressions form-invariant under field redefinitions,
but curvature transforms non-trivially



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Curved Yang-Mills-Higgs theory (curved YMH theory)

Covariantization



YMH theory	Covariantised YMH theory	Curved YMH theory
Lie group G with right-action on N	Action groupoid $N \times G$	Lie groupoid $G \rightrightarrows N$
Maurer-Cartan form	Fibre-wise Maurer-Cartan	Covariant adjustments

Field redefinitions lead to extra terms in gauge transformations and field strength

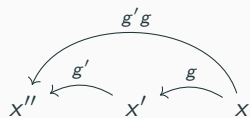
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Lie groupoids



$$\begin{array}{c} \mathcal{G} \\ \downarrow t \quad \downarrow s \\ \curvearrowright \\ N \end{array}$$

Definition (Lie groupoids)

\mathcal{G} a **Lie groupoid** if there are surjective submersions $s, t: \mathcal{G} \rightarrow N$, *source* and *target*, respectively, and a smooth *multiplication map* $\mathcal{G}_{s \times_t \mathcal{G}} \rightarrow \mathcal{G}$ such that

$$s(g'g) = s(g), \quad t(g'g) = t(g')$$

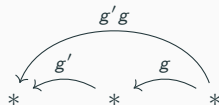
for all $(g', g) \in \mathcal{G}_{s \times_t \mathcal{G}}$ (i.e. $s(g') = t(g)$), satisfying the typical expected properties, that is,

$$\text{Associativity:} \quad (g''g')g = g''(g'g),$$

$$\text{Units:} \quad ge_{s(g)} = g, \quad e_{t(g)}g = g,$$

$$\text{Inverse:} \quad g^{-1}g = e_{s(g)}, \quad gg^{-1} = e_{t(g)}$$

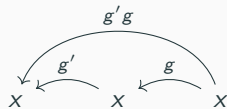
for all $(g'', g', g) \in \mathcal{G}_{s \times_t \mathcal{G}} \times_{s \times_t \mathcal{G}} \mathcal{G}$, where one requires the existence of the *unit* e as a global section of both, s and t , and the *inverse* $g^{-1} \in \mathcal{G}$ of g .



Example (Lie groups)

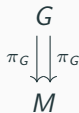
Lie groups G

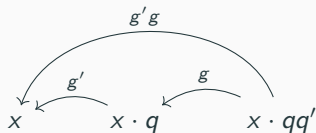




Example (Lie group bundles (LGBs))

LGB $\pi_G: G \rightarrow M$





Example (Action groupoid (trivial))

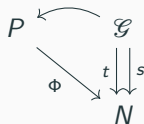
Lie group G with action $\Psi: N \times G \rightarrow N$, $(p, q) \mapsto p \cdot q$, on N .

$$\begin{array}{c}
 N \times G \\
 \text{pr}_1 \downarrow \Psi \\
 N
 \end{array}$$

$$(x, q) (x \cdot q, q') = (x, qq') ,$$

$$e_x = (x, e) ,$$

$$(x, q)^{-1} = (x \cdot q, q^{-1})$$



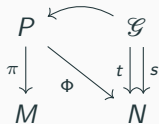
Definition (Groupoid right-action)

A **right-action** is a smooth map $P \times_t \mathcal{G} \rightarrow P$ such that

$$\begin{aligned}\Phi(p \cdot g') &= s(g'), \\ (p \cdot g') \cdot g &= p \cdot (g'g), \\ p \cdot e_{\Phi(p)} &= p\end{aligned}$$

for all $(p, g', g) \in P \times_t \mathcal{G} \times_s \mathcal{G}$.

Principal bundles and their Ehresmann connections



Definition (Principal groupoid-bundles)

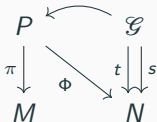
$\pi: P \rightarrow M$ surjective submersion is a **principal \mathcal{G} -bundle** if

$$\pi(p \cdot g) = \pi(p)$$

for all $(p, g) \in P \times_t \mathcal{G}$, and if

$$\begin{aligned}
 P \times_t \mathcal{G} &\rightarrow P \times_\pi P, \\
 (p, g) &\mapsto (p, p \cdot g)
 \end{aligned}$$

is a diffeomorphism.



Remarks

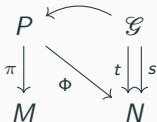
Infinitesimal action:

$$\begin{aligned}
 TP_{D\Phi \times D_t} T\mathcal{G} &\rightarrow TP, \\
 (X, Y) &\mapsto X \cdot Y.
 \end{aligned}$$

For $r_g(p) := p \cdot g$ with $\Phi(p) = t(g)$, its infinitesimal version corresponds to

$$Dr_g(X) = X \cdot 0$$

for all X with $D\Phi(X) = 0$.



Remarks

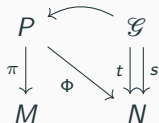
Infinitesimal action:

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For $r_g(p) := p \cdot g$ with $\Phi(p) = t(g)$, its infinitesimal version corresponds to

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Infinitesimal \mathcal{G} -action on P :

$$\begin{aligned} TP_{D\Phi \times_{Dt} T\mathcal{G}} &\rightarrow TP, \\ (X, Y) &\mapsto X \cdot Y. \end{aligned}$$


Idea

Horizontal distribution HP (w.r.t. π) with

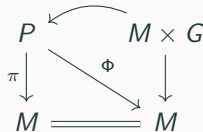
$$D\Phi(HP) = 0.$$

We lose covariantization

YM theory	Covariantised YM theory
Lie group G	Trivial Lie group bundle $M \times G$
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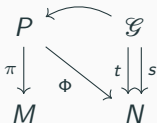
! We would lose the equivalence to the covariantised theory where we set $\Phi = \pi$!



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New idea!



Idea (Ehresmann connection on P)

Equip \mathcal{G} with a horizontal distribution $H\mathcal{G}$ (w.r.t. t). An Ehresmann connection on P is a horizontal distribution HP (w.r.t. π) so that the infinitesimal \mathcal{G} -action on P restricts

$$HP \times_{D\phi} H\mathcal{G} \rightarrow HP.$$

Invariance then via the **modified right-pushforward** r_{g*}

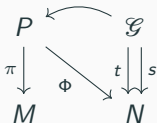
$$r_{g*}(X) := X \cdot Y$$

for all $X \in T_p P$, where $Y \in H_g \mathcal{G}$ is the unique lift of $D_p \phi(X)$.

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Idea

HP Ehresmann, then

$$HP_{D\Phi \times_{Dt}} H\mathcal{G} \rightarrow HP.$$

Theorem

Given such an HP, then $H\mathcal{G}$ is **Cartan**,

$$H\mathcal{G}_{Ds \times_{Dt}} H\mathcal{G} \rightarrow H\mathcal{G}.$$

Sketch as a proof for LGBs.

W.r.t. parallel transports we have

$$PT^P(p \cdot g) = PT^P(p) \cdot PT^{\mathcal{G}}(g).$$

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W.r.t. parallel transports we have

$$PT^P(p \cdot g) = PT^P(p) \cdot PT^{\mathcal{G}}(g),$$

then

$$\begin{array}{c} PT^P(p \cdot (gq)) \quad \quad \quad = PT^P(p) \cdot PT^{\mathcal{G}}(gq) \\ \parallel \\ PT^P((p \cdot g) \cdot q) \quad \quad = PT^P(p) \cdot \left(PT^{\mathcal{G}}(g) PT^{\mathcal{G}}(q) \right). \end{array}$$

Idea

HP Ehresmann, then

$$HP \times_{D\phi \times D_t} H\mathcal{G} \rightarrow HP.$$

Theorem

Given such an HP , then $H\mathcal{G}$ is **Cartan**,

$$H\mathcal{G} \times_{D_s \times D_t} H\mathcal{G} \rightarrow H\mathcal{G}.$$

Remarks

Vice versa, if $H\mathcal{G}$ is Cartan, then such HP exist.

What else?

Idea

Open a textbook about ordinary gauge theory, and replace:

- Right-pushforward \rightsquigarrow Modified right-pushforward,
- Adjoint representation \rightsquigarrow Adjoint representation via Cartan connections (?)
- Generalising the concept of field strength accounting for a possible curvature coming from \mathcal{G}

Too much to introduce! (Lie algebroids, Lie algebroid connections, basic curvature, basic connections, and so on...)

Thus, for today: LGBs!

Curved Yang-Mills gauge theory

Remarks

The groupoid is now an LGB $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow M$ with structural Lie group G .

Gauge invariance requires us to study *special* Cartan connections:

Definition (Multiplicative Yang-Mills connections)

A connection 1-form $\mu_{\mathcal{G}}$ on the LGB \mathcal{G} is a **multiplicative Yang-Mills connection (w.r.t. a $\zeta \in \Omega^2(M; \mathfrak{g})$)** if it is a Cartan connection satisfying the **generalised Maurer-Cartan equation**:

$$\left(d^{\pi_{\mathcal{G}}^* \nabla^{\mathcal{G}}} \mu_{\mathcal{G}} + \frac{1}{2} [\mu_{\mathcal{G}} \wedge \mu_{\mathcal{G}}]_{\pi_{\mathcal{G}}^* \mathfrak{g}} \right) \Big|_g = \text{Ad}_{g^{-1}} \circ \pi_{\mathcal{G}}^! \zeta \Big|_g - \pi_{\mathcal{G}}^! \zeta \Big|_g$$

for all $g \in \mathcal{G}$, where \mathfrak{g} is the Lie algebra bundle (LAB) of \mathcal{G} , and $\nabla^{\mathcal{G}}$ is the naturally induced LAB connection.

Remarks (Closed connection, exact curvature)

$$R_{\nabla^{\mathcal{G}}} = \text{ad} \circ \zeta.$$

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Definition

The field strength F of an Ehresmann connection $A \in \Omega^1(P; \pi^* \mathfrak{g})$ on $\pi: P \rightarrow M$ is defined by

$$F := d^{\pi^* \nabla^{\mathfrak{g}}} A + \frac{1}{2} [A \wedge A]_{\pi^* \mathfrak{g}} + \pi^! \zeta.$$

Theorem

By contracting F with an ad-invariant scalar product we achieve a gauge-invariant Lagrangian $\mathcal{L}[A]$, that is,

$$\mathcal{L}[L^! A] = \mathcal{L}[A]$$

for all $L \in \text{Aut}(P)$.

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Examples

Idea

Let us assume that the structural Lie group G is semisimple, then:

Theorem

There is a unique (up to isomorphism) principal G -bundle Q such that

$$\mathcal{G} = (Q \times G) / G,$$

multiplicative Yang-Mills connections on \mathcal{G} are precisely associated connections, thus in 1:1 correspondence to Ehresmann connections on Q , and ζ is the corresponding curvature on Q .

Corollary

There are only curved multiplicative Yang-Mills connections on \mathcal{G} if and only if Q is not flat.

Example

Hopf fibration $Q := \mathbb{S}^7 \rightarrow \mathbb{S}^4$,

$$\mathcal{G} = (Q \times \mathrm{SU}(2)) / \mathrm{SU}(2).$$

Remarks

Some technical details:

- This is stable under field redefinitions.
- This can be relaxed to \mathcal{G} with a trivial centre, using a nice geometric interpretation via singular foliations. However, dynamics of gauge theory require Ad-invariance...

See Camille's talk on Friday ☺

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About Lie groupoids...

Idea

Assume that G is semisimple **and** acts faithfully on \mathbb{R}^d fixing 0. Then \mathcal{G} acts faithfully on

$$N := (Q \times \mathbb{R}^d) / G,$$

giving rise to an action groupoid structure on $\phi^* \mathcal{G}$ ($\phi: N \rightarrow M$)

$$\begin{array}{c} \phi^* \mathcal{G} \\ \begin{array}{c} t \downarrow s \\ \curvearrowright \\ N \end{array} \end{array}$$

Theorem

The pullback of the multiplicative Yang-Mills connection on $\phi^ \mathcal{G}$ is flat up to field redefinitions if and only if Q is flat.*

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Other technical details:

- Also stable under field redefinitions!
- Here semi-simplicity is rather important.

Example

Again Q the Hopf fibration $\mathbb{S}^7 \rightarrow \mathbb{S}^4$, then naturally choose $\mathbb{R}^d = \mathbb{C}^2$.

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A quick funny example, again by \mathbb{S}^7

Example

The Lie groupoid \mathcal{G} given as the pair groupoid of \mathbb{S}^7 leads to descriptions of curved Yang-Mills-Higgs theories, which cannot be described in an ordinary way due to the lack of non-associativity on \mathbb{S}^7 .

Thank you!