Curved Yang-Mills-Higgs theories

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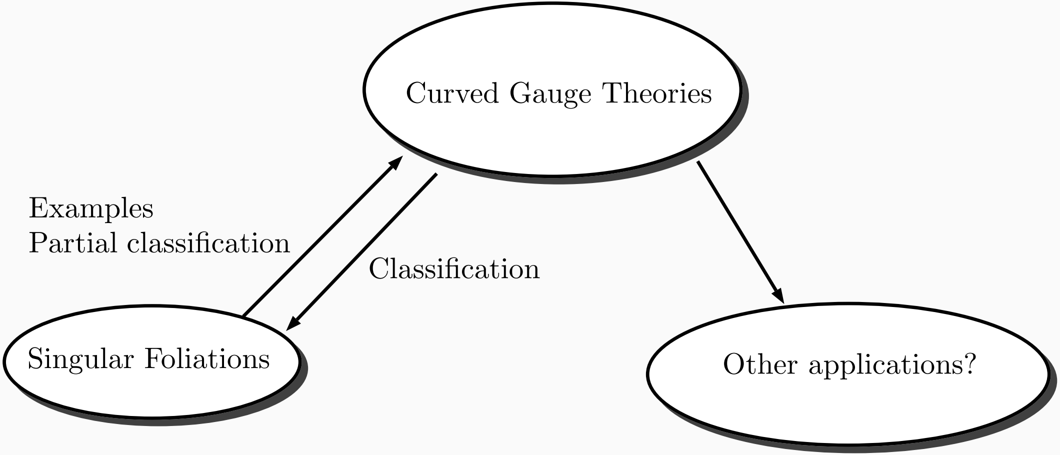
Curved Gauge Theories
Examples
Partial classification
Classification
Singular Foliations
Other applications?

Motivation

Motivation: Covariantisation of Yang-Mills(-Higgs) theory

Covariantization $\longrightarrow$

Classical theory	Covariantised flat theory	Curved Theory
Vector space V	Trivial vector bundle $M \times V$	Vector bundle $V \to M$
$\frac{\partial}{\partial x^i}$	Canonical flat connection ∇^0	Vector bundle connection ∇
Coordinate changes may lead to extra terms	Coordinate expressions form-invariant under coordinate changes	

Curved Yang-Mills gauge theory (curved YM theory)

Covariantization

YM theory	Covariantised YM theory	Curved YM theory
Lie group G	Trivial Lie group bundle $M \times G$	Lie group bundle $G \to M$
Maurer-Cartan form	Fibre-wise Maurer-Cartan	Multiplicative YM connection

Field redefinitions lead to extra terms in gauge transformations and field strength

Expressions form-invariant under field redefinitions,
but curvature transforms non-trivially

S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.
S.-R. Fischer. *Geometry of curved Yang–Mills–Higgs gauge theories*, Ph.D. thesis, Institut Camille Jordan [Villeurbanne], France, U. Geneva, Switzerland, 2021; doi: 10.13097/archive-ouverte/unige:152555

Curved Yang-Mills-Higgs theory (curved YMH theory)

Covariantization

YMH theory	Covariantised YMH theory	Curved YMH theory
Lie group G with right-action on N	Action groupoid $N \times G$	Lie groupoid $G \rightrightarrows N$
Maurer-Cartan form	Fibre-wise Maurer-Cartan	Covariant adjustments

Field redefinitions lead to extra terms in gauge transformations and field strength

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S.-R. Fischer, M. Jalali Farahani, H. Kim, and C. Saemann. *Adjusted connections I: Differential cocycles for principal groupoid bundles with connection*, arXiv: 2406.16755, 202.
S.-R. Fischer. *Geometry of curved Yang–Mills–Higgs gauge theories*, Ph.D. thesis, Institut Camille Jordan [Villeurbanne], France, U. Geneva, Switzerland, 2021; doi: 10.13097/archive-ouverte/unige:152555

Lie groupoids

$$
\begin{tikzcd}
x'' & x' \arrow[l, bend right, "g'"'] & x \arrow[l, bend right, "g"'] \arrow[ll, bend right=60, "g'g"']
\end{tikzcd}
$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd} \mathscr{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ N \end{tikzcd}$$

Definition (Lie groupoids)

$\mathscr{G}$ a **Lie groupoid** if there are surjective submersions $s, t\colon \mathscr{G} \to N$, *source* and *target*, respectively, and a smooth *multiplication map* $\mathscr{G}\ {}_s\!\times_t \mathscr{G} \to \mathscr{G}$ such that

$$s(g'g) = s(g), \qquad t(g'g) = t(g')$$

for all $(g', g) \in \mathscr{G}\ {}_s\!\times_t \mathscr{G}$ (i.e. $s(g') = t(g)$), satisfying the typical expected properties, that is,

$$\begin{aligned} \text{Associativity:} &\qquad (g''g')g = g''(g'g), \\ \text{Units:} &\qquad ge_{s(g)} = g, \qquad e_{t(g)}g = g, \\ \text{Inverse:} &\qquad g^{-1}g = e_{s(g)}, \qquad gg^{-1} = e_{t(g)} \end{aligned}$$

for all $(g'', g', g) \in \mathscr{G}\ {}_s\!\times_t \mathscr{G}\ {}_s\!\times_t \mathscr{G}$, where one requires the existence of the *unit* e as a global section of both, s and t, and the *inverse* $g^{-1} \in \mathscr{G}$ of g.

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
* & * \arrow[l, bend right, "g'"'] & * \arrow[l, bend right, "g"'] \arrow[ll, bend right, "g'g"']
\end{tikzcd}
$$

Example (Lie groups)

Lie groups G

$$
\begin{tikzcd}
G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
\{*\}
\end{tikzcd}
$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd} x & x \arrow[l, bend right, "g'"'] & x \arrow[l, bend right, "g"'] \arrow[ll, bend right, "g'g"'] \end{tikzcd}$$

Example (Lie group bundles (LGBs))

LGB $\pi_{\mathscr{G}} : \mathscr{G} \to M$

$$\begin{tikzcd} \mathscr{G} \arrow[d, shift right, "\pi_G"'] \arrow[d, shift left, "\pi_G"] \\ M \end{tikzcd}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
x & x \cdot q \arrow[l, bend right, "{(x,q)}"'] & x \cdot qq' \arrow[l, bend right, "{(x \cdot q,q')}"'] \arrow[ll, bend right=50, "{(x,q)\ (x \cdot q,q')}"']
\end{tikzcd}
$$

Example (Action groupoid (trivial))

Lie group G with action $\Psi \colon N \times G \to N$, $(x, q) \mapsto x \cdot q$, on N.

$$
\begin{tikzcd}
N \times G \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\Psi"] \\
N
\end{tikzcd}
$$

$$
\begin{aligned}
(x, q)\ (x \cdot q, q') &= (x, qq')\ , \\
e_x &= (x, e)\ \ , \\
(x, q)^{-1} &= \left(x \cdot q, q^{-1}\right)
\end{aligned}
$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
P \arrow[dr, "\Phi"'] & \mathcal{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
& N
\end{tikzcd}
$$

Definition (Groupoid right-action)

A **right-action** is a smooth map $P \,{}_{\Phi}\!\times_t \mathcal{G} \to P$ such that

$$
\begin{aligned}
\Phi(p \cdot g) &= s(g), \\
(p \cdot g) \cdot g' &= p \cdot (gg'), \\
p \cdot e_{\Phi(p)} &= p
\end{aligned}
$$

for all $(p, g, g') \in P \,{}_{\Phi}\!\times_t \mathcal{G} \,{}_{s}\!\times_t \mathcal{G}$.

Principal bundles and their Ehresmann connections

$$
\begin{tikzcd}
P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
M & N
\end{tikzcd}
$$

Definition (Principal groupoid-bundles)

$\pi\colon P \to M$ surjective submersion is a **principal $\mathscr{G}$-bundle** if

$$\pi(p \cdot g) = \pi(p)$$

for all $(p, g) \in P\,{}_{\Phi}\!\times_t \mathscr{G}$, and if

$$\begin{gathered} P\,{}_{\Phi}\!\times_t \mathscr{G} \to P\,{}_{\pi}\!\times_{\pi} P, \\ (p, g) \mapsto (p, p \cdot g) \end{gathered}$$

is a diffeomorphism.

I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, Cambridge University Press, 2003.

Ehresmann connection

$$\begin{tikzcd} P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & N \end{tikzcd}$$

Remarks

Infinitesimal action:

$$\begin{aligned} \mathrm{T}P \,{}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{T}\mathscr{G} &\to \mathrm{T}P, \\ (X, Y) &\mapsto X \cdot Y. \end{aligned}$$

For $r_g(p) := p \cdot g$ with $\Phi(p) = t(g)$, its infinitesimal version corresponds to

$$\mathrm{D}r_g(X) = X \cdot 0$$

for all X with $\mathrm{D}\Phi(X) = 0$.

Ehresmann connection

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Infinitesimal $\mathscr{G}$-action on P:

$$\begin{aligned} \mathrm{T}P \,{}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{T}\mathscr{G} &\to \mathrm{T}P, \\ (X, Y) &\mapsto X \cdot Y. \end{aligned}$$

Idea

Horizontal distribution $\mathrm{H}P$ (w.r.t. π) with

$$\mathrm{D}\Phi(\mathrm{H}P) = 0.$$

I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, Cambridge University Press, 2003.
D. Signori and M. Stiénon. *On nonlinear gauge theories*, J. Geom. Phys. **59** 1063, 2009.

We lose covariantization

YM theory	Covariantised YM theory
Lie group G	Trivial Lie group bundle $M \times G$
Maurer-Cartan form	Fibre-wise Maurer-Cartan

We would lose the equivalence to the covariantised theory where we set $\Phi = \pi$

$$
\begin{tikzcd}
P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & M \times G \arrow[l, bend right] \arrow[d] \\
M \arrow[r, equal] & M
\end{tikzcd}
$$

S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.
S.-R. Fischer. *Geometry of curved Yang–Mills–Higgs gauge theories*, Ph.D. thesis, Institut Camille Jordan [Villeurbanne], France, U. Geneva, Switzerland, 2021; doi: 10.13097/archive-ouverte/unige:152555

New idea!

$$\begin{tikzcd} P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \mathscr{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & N \end{tikzcd}$$

Idea (Ehresmann connection on P)

Equip $\mathscr{G}$ with a horizontal distribution $\mathrm{H}\mathscr{G}$ (w.r.t. t). An Ehresmann connection on P is a horizontal distribution $\mathrm{H}P$ (w.r.t. π) so that the infinitesimal $\mathscr{G}$-action on P restricts

$$\mathrm{H}P\ {}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{H}\mathscr{G} \to \mathrm{H}P.$$

Invariance then via the **modified right-pushforward** $\mathscr{r}_{g*}$

$$\mathscr{r}_{g*}(X) := X \cdot Y$$

for all $X \in \mathrm{T}_p P$, where $Y \in \mathrm{H}_g\mathscr{G}$ is the unique lift of $\mathrm{D}_p\Phi(X)$.

S.-R. Fischer, M. Jalali Farahani, H. Kim, and C. Saemann. *Adjusted connections I: Differential cocycles for principal groupoid bundles with connection*, arXiv: 2406.16755, 202.
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New idea!

$$\begin{tikzcd} P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \mathcal{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & N \end{tikzcd}$$

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Idea

$\mathrm{H}P$ Ehresmann, then

$$\mathrm{H}P \;{}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{H}\mathscr{G} \to \mathrm{H}P.$$

Theorem

Given such an $\mathrm{H}P$*, then* $\mathrm{H}\mathscr{G}$ *is* ***Cartan****,*

$$\mathrm{H}\mathscr{G} \;{}_{\mathrm{D}s}\!\times_{\mathrm{D}t} \mathrm{H}\mathscr{G} \to \mathrm{H}\mathscr{G}.$$

Sketch as a proof for LGBs.

W.r.t. parallel transports we have

$$\mathrm{PT}^{P}(p \cdot g) = \mathrm{PT}^{P}(p) \cdot \mathrm{PT}^{\mathscr{G}}(g).$$

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then

$$\begin{tikzcd}
\mathrm{PT}^P\big(p \cdot (gq)\big) \arrow[r, equal] \arrow[d, equal] & \mathrm{PT}^P(p) \cdot \mathrm{PT}^{\mathscr{G}}(gq) \\
\mathrm{PT}^P\big((p \cdot g) \cdot q\big) \arrow[r, equal] & \mathrm{PT}^P(p) \cdot \Big(\mathrm{PT}^{\mathscr{G}}(g)\ \mathrm{PT}^{\mathscr{G}}(q)\Big).
\end{tikzcd}$$

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Remarks

Vice versa, if $\mathrm{H}\mathscr{G}$ is Cartan, then such $\mathrm{H}P$ exist.

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S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.

What else?

Idea

Open a textbook about ordinary gauge theory, and replace:

- Right-pushforward $\rightsquigarrow$ Modified right-pushforward,
- Adjoint representation $\rightsquigarrow$ Adjoint representation via Cartan connections (?)
- Generalising the concept of field strength accounting for a possible curvature coming from $\mathscr{G}$

Too much to introduce! (Lie algebroids, Lie algebroid connections, basic curvature, basic connections, and so on...)

Thus, for today: LGBs!

Curved Yang-Mills gauge theory

Remarks

The groupoid is now an LGB $\pi_{\mathscr{G}} \colon \mathscr{G} \to M$ with structural Lie group G.

Gauge invariance requires us to study *special* Cartan connections:

Definition (Multiplicative Yang-Mills connections)

A connection 1-form $\mu_{\mathscr{G}}$ on the LGB $\mathscr{G}$ is a **multiplicative Yang-Mills connection (w.r.t. a $\zeta \in \Omega^2(M; \mathscr{g})$)** if it is a Cartan connection satisfying the **generalised Maurer-Cartan equation**:

$$\left(\mathrm{d}^{\pi_{\mathscr{G}}^* \nabla^{\mathscr{G}}} \mu_{\mathscr{G}} + \frac{1}{2} [\mu_{\mathscr{G}} \stackrel{\wedge}{,} \mu_{\mathscr{G}}]_{\pi_{\mathscr{G}}^* \mathscr{g}} \right) \Big|_g = \mathrm{Ad}_{g^{-1}} \circ \pi_{\mathscr{G}}^! \zeta \big|_g - \pi_{\mathscr{G}}^! \zeta \big|_g$$

for all $g \in \mathscr{G}$, where $\mathscr{g}$ is the Lie algebra bundle (LAB) of $\mathscr{G}$, and $\nabla^{\mathscr{G}}$ is the naturally induced LAB connection.

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Remarks (Closed connection, exact curvature)

$$R_{\nabla^{\mathscr{G}}} = \mathrm{ad} \circ \zeta.$$

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Definition

The field strength F of an Ehresmann connection $A \in \Omega^1(P; \pi^*\mathscr{g})$ on $\pi\colon P \to M$ is defined by

$$F := \mathrm{d}^{\pi^*\nabla^{\mathscr{G}}} A + \frac{1}{2}[A \stackrel{\wedge}{,} A]_{\pi^*\mathscr{g}} + \pi^! \zeta.$$

Theorem

By contracting F with an ad*-invariant scalar product we achieve a gauge-invariant Lagrangian $\mathscr{L}[A]$, that is,*

$$\mathscr{L}\left[L^! A\right] = \mathscr{L}[A]$$

for all $L \in \mathrm{Aut}(P)$.

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Examples

Idea (Cooking recipe)

1. Find $\mathscr{G}$ with curved multiplicative Yang-Mills connection, which cannot be flattened by field redefinitions.
2. Find P.

Then define the field strength F and the Lagrangian $\mathscr{L}$ as before.

$\Rightarrow$ New theory ☺

Remarks

@1: By finding $\mathscr{G}$ which do not admit flat connections at all we can avoid introducing field redefinitions

@2: By setting $P = \mathscr{G}$ ("trivial" principal bundle) we can avoid this step, too.

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Idea

Let us assume that the structural Lie group G is semisimple, then:

Theorem

There is a unique (up to isomorphism) principal G-bundle Q such that

$$\mathscr{G} = (Q \times G) \Big/ G,$$

multiplicative Yang-Mills connections on $\mathscr{G}$ are precisely associated connections, thus in 1:1 correspondence to Ehresmann connections on Q, and ζ is the corresponding curvature on Q.

Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, 213:xxxviii+501, 2005.
S.-R. Fischer. *Curved Yang–Mills–Higgs gauge theories in the case of massless gauge bosons*, Journal of Geometry and Physics, 162:104104, 2021.

Corollary

There are only curved multiplicative Yang-Mills connections on $\mathscr{G}$ if and only if Q is not flat.

Example

Hopf fibration $Q := \mathbb{S}^7 \to \mathbb{S}^4$,

$$\mathscr{G} = \left(Q \times \mathsf{SU}(2)\right) \Big/ \mathsf{SU}(2).$$

Remarks

Some technical details:

- This is stable under field redefinitions.
- This can be relaxed to $\mathscr{G}$ with a trivial centre, using a nice geometric interpretation via singular foliations. However, dynamics of gauge theory require Ad-invariance...
  See Camille's talk on Friday ☺

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About Lie groupoids...

Idea

Assume that G is semisimple **and** acts faithfully on $\mathbb{R}^d$ fixing 0. Then $\mathscr{G}$ acts faithfully on

$$N := \left(Q \times \mathbb{R}^d\right) \Big/ G,$$

giving rise to an action groupoid structure on $\phi^*\mathscr{G}$ ($\phi\colon N \to M$)

$$\begin{tikzcd} \phi^*\mathscr{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ N \end{tikzcd}$$

Theorem

The pullback of the multiplicative Yang-Mills connection on $\phi^*\mathscr{G}$ is flat up to field redefinitions if and only if Q is flat.

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S.-R. Fischer, M. Jalali Farahani, H. Kim, and C. Saemann. *Adjusted connections I: Differential cocycles for principal groupoid bundles with connection*, arXiv: 2406.16755, 202.

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Other technical details:

- Also stable under field redefinitions!
- Here semi-simplicity is rather important.

Example

Again Q the Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^4$, then naturally choose $\mathbb{R}^d = \mathbb{C}^2$.

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Outlook

- More applications? So far: Singular foliations & Symmetry Breaking
- Local examples?

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Thank you!

A quick funny example, again by $\mathbb{S}^7$

Example

The Lie groupoid $\mathscr{G}$ given as the pair groupoid of $\mathbb{S}^7$ leads to descriptions of curved Yang-Mills-Higgs theories, which cannot be described in an ordinary way due to the lack of non-associativity on $\mathbb{S}^7$.

S.-R. Fischer. *Geometry of curved Yang–Mills–Higgs gauge theories*, Ph.D. thesis, Institut Camille Jordan [Villeurbanne], France, U. Geneva, Switzerland, 2021; doi: 10.13097/archive-ouverte/unige:152555