Curved Yang-Mills-Higgs theories

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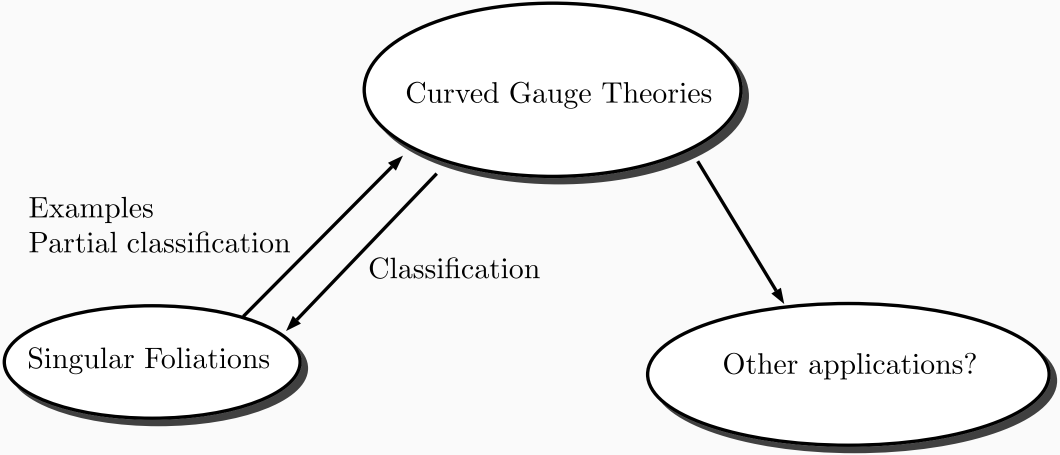
Curved Gauge Theories
Examples
Partial classification
Classification
Singular Foliations
Other applications?

Curved Yang-Mills-Higgs theory & their Ehresmann connections

Motivation: Covariantisation of Yang-Mills(-Higgs) theory

Covariantization

Classical theory	Covariantised flat theory	Curved Theory
Vector space V	Trivial vector bundle $M \times V$	Vector bundle $V \to M$
$\frac{\partial}{\partial x^i}$	Canonical flat connection ∇^0	Vector bundle connection ∇
Coordinate changes may lead to extra terms	Coordinate expressions form-invariant under coordinate changes	

Curved Yang-Mills gauge theory (curved YM theory)

Covariantization

YM theory	Covariantised YM theory	Curved YM theory
Lie group G	Trivial Lie group bundle $M \times G$	Lie group bundle $G \to M$
Maurer-Cartan form	Fibre-wise Maurer-Cartan	Multiplicative YM connection

Field redefinitions lead to extra terms in gauge transformations and field strength

Expressions form-invariant under field redefinitions, **but curvature transforms non-trivially**

S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.
S.-R. Fischer. *Geometry of curved Yang–Mills–Higgs gauge theories*, Ph.D. thesis, Institut Camille Jordan [Villeurbanne], France, U. Geneva, Switzerland, 2021; doi: 10.13097/archive-ouverte/unige:152555

Curved Yang-Mills-Higgs theory (curved YMH theory)

Covariantization $\longrightarrow$

YMH theory	Covariantised YMH theory	Curved YMH theory
Lie group G with right-action on N	Action groupoid $N \times G$	Lie groupoid $G \rightrightarrows N$
Maurer-Cartan form	Fibre-wise Maurer-Cartan	Covariant adjustments

Field redefinitions lead to extra terms in gauge transformations and field strength

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$$\begin{tikzcd} \mathscr{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ N \end{tikzcd}$$

Definition (Lie groupoids)

$\mathscr{G}$ a **Lie groupoid** if there are surjective submersions $s, t\colon \mathscr{G} \to N$, *source* and *target*, respectively, and a smooth *multiplication map* $\mathscr{G} \,{}_s\!\times_t \mathscr{G} \to \mathscr{G}$ such that

$$s(g'g) = s(g), \qquad t(g'g) = t(g')$$

for all $(g', g) \in \mathscr{G} \,{}_s\!\times_t \mathscr{G}$ (i.e. $s(g') = t(g)$), satisfying the typical expected properties, that is,

$$\begin{aligned} \text{Associativity:} &\qquad (g''g')g = g''(g'g), \\ \text{Units:} &\qquad g e_{s(g)} = g, \qquad e_{t(g)} g = g, \\ \text{Inverse:} &\qquad g^{-1} g = e_{s(g)}, \qquad g g^{-1} = e_{t(g)} \end{aligned}$$

for all $(g'', g', g) \in \mathscr{G} \,{}_s\!\times_t \mathscr{G} \,{}_s\!\times_t \mathscr{G}$, where one requires the existence of the *unit* e as a global section of both, s and t, and the *inverse* $g^{-1} \in \mathscr{G}$ of g.

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd}
x'' & x' \arrow[l, bend right, "g'"'] & x \arrow[l, bend right, "g"'] \arrow[ll, bend right, "g'g"']
\end{tikzcd}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd} * & * \arrow[l, bend right, "g'"'] & * \arrow[l, bend right, "g"'] \arrow[ll, bend right=60, "g'g"'] \end{tikzcd}$$

Example (Lie groups)

Lie groups G

$$\begin{tikzcd} G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \{*\} \end{tikzcd}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd} x & x \arrow[l, bend right, "g'"'] & x \arrow[l, bend right, "g"'] \arrow[ll, bend right, "g'g"'] \end{tikzcd}$$

Example (Lie group bundles (LGBs))

LGB $\pi_G \colon G \to M$

$$\begin{tikzcd} G \arrow[d, shift right, "\pi_G"'] \arrow[d, shift left, "\pi_G"] \\ M \end{tikzcd}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd}
x & x \cdot q \arrow[l, bend right, "g'"'] & x \cdot qq' \arrow[l, bend right, "g"'] \arrow[ll, bend right=50, "g'g"']
\end{tikzcd}$$

Example (Action groupoid (trivial))

Lie group G with action $\Psi\colon N \times G \to N$, $(p, q) \mapsto p \cdot q$, on N.

$$\begin{tikzcd}
N \times G \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\Psi"] \\
N
\end{tikzcd}$$

$$\begin{aligned}
(x, q)\,(x \cdot q, q') &= (x, qq')\ , \\
e_x &= (x, e)\ , \\
(x, q)^{-1} &= \left(x \cdot q, q^{-1}\right)
\end{aligned}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
P \arrow[dr, "\Phi"'] & \mathscr{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
& N
\end{tikzcd}
$$

Definition (Groupoid right-action)

A **right-action** is a smooth map $P \;{}_{\Phi}\!\times_t \mathscr{G} \to P$ such that

$$
\begin{aligned}
\Phi(p \cdot g') &= s(g'),\\
(p \cdot g') \cdot g &= p \cdot (g' g),\\
p \cdot e_{\Phi(p)} &= p
\end{aligned}
$$

for all $(p, g', g) \in P \;{}_{\Phi}\!\times_t \mathscr{G} \;{}_{t}\!\times_s \mathscr{G}$.

Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, 213:xxxviii+501, 2005.

$$\begin{tikzcd} P \arrow[d, "\pi"'] \arrow[dr, "\Phi"'] & \mathscr{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & N \end{tikzcd}$$

Definition (Principal groupoid-bundles)

$\pi \colon P \to M$ surjective submersion is a **principal $\mathscr{G}$-bundle** if

$$\pi(p \cdot g) = \pi(p)$$

for all $(p, g) \in P \,{}_{\Phi}\!\times_t \mathscr{G}$, and if

$$\begin{aligned} P \,{}_{\Phi}\!\times_t \mathscr{G} &\to P \,{}_{\pi}\!\times_{\pi} P, \\ (p, g) &\mapsto (p, p \cdot g) \end{aligned}$$

is a diffeomorphism.

I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, Cambridge University Press, 2003.

Ehresmann connection

$$
\begin{tikzcd}
P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \mathcal{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
M & N
\end{tikzcd}
$$

Remarks

Infinitesimal action:

$$
\begin{aligned}
\mathrm{T}P \; {}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{T}\mathcal{G} &\to \mathrm{T}P, \\
(X, Y) &\mapsto X \cdot Y.
\end{aligned}
$$

For $r_g(p) := p \cdot g$ with $\Phi(p) = t(g)$, its infinitesimal version corresponds to

$$\mathrm{D}r_g(X) = X \cdot 0$$

for all X with $\mathrm{D}\Phi(X) = 0$.

Ehresmann connection

$$
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Infinitesimal $\mathscr{G}$-action on P:

$$\begin{aligned} \mathrm{T}P \,{}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{T}\mathscr{G} &\to \mathrm{T}P, \\ (X, Y) &\mapsto X \cdot Y. \end{aligned}$$

Idea

Horizontal distribution $\mathrm{H}P$ (w.r.t. π) with

$$\mathrm{D}\Phi(\mathrm{H}P) = 0.$$

I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, Cambridge University Press, 2003.
D. Signori and M. Stiénon. *On nonlinear gauge theories*, J. Geom. Phys. **59** 1063, 2009.

We lose covariantization

YM theory	Covariantised YM theory
Lie group G	Trivial Lie group bundle $M \times G$
Maurer-Cartan form	Fibre-wise Maurer-Cartan

We would lose the equivalence to the covariantised theory where we set $\Phi = \pi$

$$\begin{tikzcd}
P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & M \times G \arrow[d] \arrow[l, bend right] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

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New idea!

$$\begin{tikzcd} P \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \mathcal{G} \arrow[l, bend right] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & N \end{tikzcd}$$

Idea (Ehresmann connection on P)

Equip $\mathcal{G}$ with a horizontal distribution $\mathrm{H}\mathcal{G}$ (w.r.t. t). An Ehresmann connection on P is a horizontal distribution $\mathrm{H}P$ (w.r.t. π) so that the infinitesimal $\mathcal{G}$-action on P restricts

$$\mathrm{H}P \,{}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{H}\mathcal{G} \to \mathrm{H}P.$$

Invariance then via the **modified right-pushforward** $\mathscr{r}_{g*}$

$$\mathscr{r}_{g*}(X) := X \cdot Y$$

for all $X \in \mathrm{T}_p P$, where $Y \in \mathrm{H}_g\mathcal{G}$ is the unique lift of $\mathrm{D}_p\Phi(X)$.

S.-R. Fischer, M. Jalali Farahani, H. Kim, and C. Saemann. *Adjusted connections I: Differential cocycles for principal groupoid bundles with connection*, arXiv: 2406.16755, 202.
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$$

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Idea

$\mathrm{H}P$ Ehresmann, then

$$\mathrm{H}P \;{}_{\mathrm{D}\Phi}\!\times_{\mathrm{D}t} \mathrm{H}\mathscr{G} \to \mathrm{H}P.$$

Theorem

Given such an $\mathrm{H}P$*, then* $\mathrm{H}\mathscr{G}$ *is* ***Cartan****,*

$$\mathrm{H}\mathscr{G} \;{}_{\mathrm{D}s}\!\times_{\mathrm{D}t} \mathrm{H}\mathscr{G} \to \mathrm{H}\mathscr{G}.$$

Sketch as a proof for LGBs.

W.r.t. parallel transports we have

$$\mathrm{PT}^P(p \cdot g) = \mathrm{PT}^P(p) \cdot \mathrm{PT}^{\mathscr{G}}(g).$$

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Idea

HP Ehresmann, then

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Given such an HP*, then* H$\mathscr{G}$ *is* ***Cartan****,*

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Sketch as a proof for LGBs.

W.r.t. parallel transports we have

$$\mathrm{PT}^P(p\cdot g) = \mathrm{PT}^P(p)\cdot \mathrm{PT}^{\mathscr{G}}(g),$$

then

$$\begin{tikzcd}
\mathrm{PT}^P\big(p\cdot(gq)\big) \arrow[r, equal] \arrow[d, equal] & \mathrm{PT}^P(p)\cdot \mathrm{PT}^{\mathscr{G}}(gq) \\
\mathrm{PT}^P\big((p\cdot g)\cdot q\big) \arrow[r, equal] & \mathrm{PT}^P(p)\cdot \Big(\mathrm{PT}^{\mathscr{G}}(g)\ \mathrm{PT}^{\mathscr{G}}(q)\Big).
\end{tikzcd}$$

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S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.

Idea

$\mathrm{H}P$ Ehresmann, then

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Given such an $\mathrm{H}P$*, then* $\mathrm{H}\mathscr{G}$ *is* ***Cartan****,*

$$\mathrm{H}\mathscr{G} \,{}_{\mathrm{D}s}\!\times_{\mathrm{D}t} \mathrm{H}\mathscr{G} \to \mathrm{H}\mathscr{G}.$$

Remarks

Vice versa, if $\mathrm{H}\mathscr{G}$ is Cartan, then such $\mathrm{H}P$ exist.

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S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.

What else?

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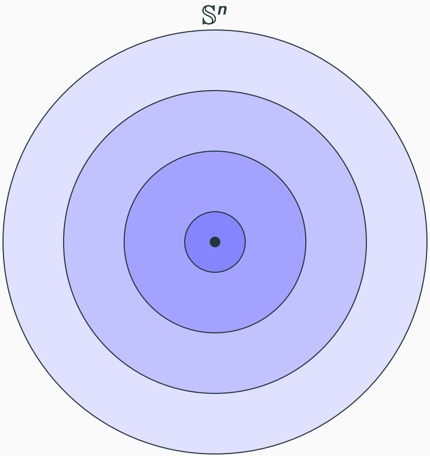
Singular Foliations

Example of a singular foliation



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S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.

Foliations are widespread

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

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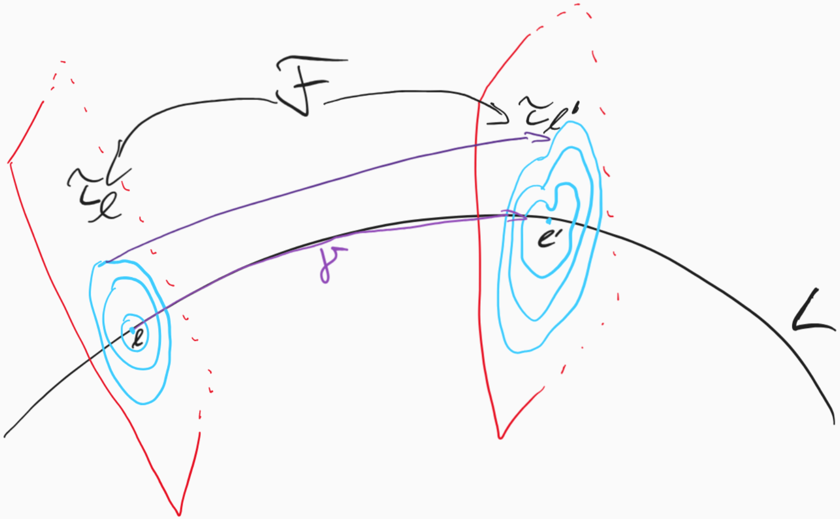
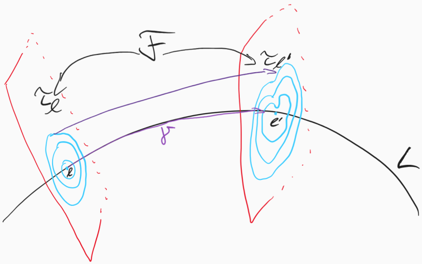
Unique transverse structure



How to classify singular foliations?



Remarks ([C. L.-G., S.-R. F.])

There is a connection on the normal bundle of a leaf L preserving the foliation!

- Transverse structure is unique: Classify singular foliation $\mathscr{F}$ like a bundle!
- Connection is a multiplicative Yang-Mills connection: Use curved gauge theory!

Source of the existence of connection on normal bundle: Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Thank you!

Classification of singular foliations

Theorem ([C. L.-G., S.-R. F.])

Formal singular foliations with leaf L and transverse model $(\mathbb{R}^d, \tau_l)$ are equivalent to:

- A Galois cover L' over L with structural group K
- A short exact sequence of groups

$$\mathrm{Inner}(\tau_l) \hookrightarrow H \twoheadrightarrow K$$

- A principal H-bundle P over L