Curved Yang-Mills-Higgs theories

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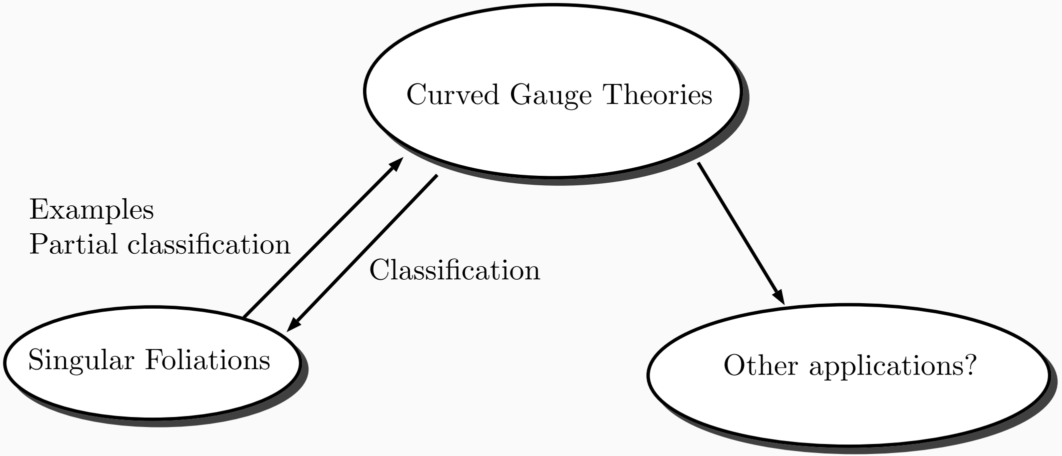
Curved Gauge Theories
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Partial classification
Classification
Singular Foliations
Other applications?

Curved Yang-Mills-Higgs theory

Motivation: Covariantisation of Yang-Mills(-Higgs) theory

Covariantization $\longrightarrow$

Classical theory	Covariantised flat theory	Curved Theory
Vector space V	Trivial vector bundle $M \times V$	Vector bundle $V \to M$
$\frac{\partial}{\partial x^i}$	Canonical flat connection ∇^0	Vector bundle connection ∇
Coordinate changes may lead to extra terms	Coordinate expressions form-invariant under coordinate changes	

Curved Yang-Mills gauge theory (curved YM theory)

Covariantization $\longrightarrow$

YM theory	Covariantised YM theory	Curved YM theory
Lie group G	Trivial Lie group bundle $M \times G$	Lie group bundle $G \to M$
Maurer Cartan form	Fibre-wise Maurer-Cartan	Multiplicative YM connection
Field redefinitions lead to extra terms in gauge transformations and field strength	Expressions form-invariant under field redefinitions, **but curvature transforms non-trivially**	

S.-R. Fischer. *Integrating curved Yang–Mills gauge theories*, arXiv: 2210.02924, 2022.

S.-R. Fischer. *Geometry of curved Yang–Mills–Higgs gauge theories*, Ph.D. thesis, Institut Camille Jordan [Villeurbanne], France, U. Geneva, Switzerland, 2021; doi: 10.13097/archive-ouverte/unige:152555

Curved Yang-Mills-Higgs theory (curved YMH theory)

Covariantization $\longrightarrow$

YMH theory	Covariantised YMH theory	Curved YMH theory
Lie group G with right-action on N	Action groupoid $N \times G$	Lie groupoid $G \rightrightarrows N$
Maurer Cartan form	Fibre-wise Maurer-Cartan	Covariant adjustments
Field redefinitions lead to extra terms in gauge transformations and field strength	Expressions form-invariant under field redefinitions, **but curvature transforms non-trivially**	

S.-R. Fischer, M. Jalali Farahani, H. Kim, and C. Saemann. *Adjusted connections I: Differential cocycles for principal groupoid bundles with connection*, arXiv: 2406.16755, 202.
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$$\begin{tikzcd} \mathcal{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \end{tikzcd}$$

Definition (Lie groupoids)

$\mathcal{G}$ a **Lie groupoid** if there are surjective submersions $s, t \colon \mathcal{G} \to M$, *source* and *target*, respectively, and a smooth *multiplication map* $\mathcal{G}\ {}_s\!\times_t \mathcal{G} \to \mathcal{G}$ such that

$$s(g'g) = s(g), \qquad t(g'g) = t(g')$$

for all $(g', g) \in \mathcal{G}\ {}_s\!\times_t \mathcal{G}$ (i.e. $s(g') = t(g)$), satisfying the typical expected properties, that is,

$$\begin{aligned} \text{Associativity:} & \qquad (g''g')g = g''(g'g), \\ \text{Units:} & \qquad g e_{s(g)} = g, \qquad e_{t(g)} g = g, \\ \text{Inverse:} & \qquad g^{-1} g = e_{s(g)}, \qquad g g^{-1} = e_{t(g)} \end{aligned}$$

for all $(g'', g', g) \in \mathcal{G}\ {}_s\!\times_t \mathcal{G}\ {}_s\!\times_t \mathcal{G}$, where one requires the existence of the *unit* e as a global section of both, s and t, and the *inverse* $g^{-1} \in \mathcal{G}$ of g.

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd} m'' & m' \arrow[l, bend right, "g'"'] & m \arrow[l, bend right, "g"'] \arrow[ll, bend right=60, "g'g"'] \end{tikzcd}$$

Example (Lie groups)

Lie groups G

$$\begin{tikzcd} G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \{*\} \end{tikzcd}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
m'' & m' \arrow[l, "g'"'] & m \arrow[l, "g"'] \arrow[ll, bend right, "g'g"']
\end{tikzcd}
$$

Example (Lie groups)

Lie groups G

$$
\begin{tikzcd}
G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
\{*\}
\end{tikzcd}
$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
m'' & m' \arrow[l, bend right, "g'"'] & m \arrow[l, bend right, "g"'] \arrow[ll, bend right=60, "g'g"']
\end{tikzcd}
$$

Example (Lie group bundles (LGBs))

LGB $\pi_G \colon G \to M$

$$
\begin{tikzcd}
G \arrow[d, shift right, "\pi_G"'] \arrow[d, shift left, "\pi_G"] \\
M
\end{tikzcd}
$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd} m'' & m' \arrow[l, bend right, "g'"'] & m \arrow[l, bend right, "g"'] \arrow[ll, bend right, "g'g"'] \end{tikzcd}$$

Example (Action groupoid (trivial))

Lie group G with action $\Psi\colon N \times G \to N$, $(p, g) \mapsto p \cdot g$, on N.

$$\begin{tikzcd} N \times G \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\Psi"] \\ N \end{tikzcd}$$

$$\begin{aligned} (x, g)\ (x \cdot g, q) &= (x, gq)\ , \\ e_x &= (x, e)\ \ , \\ (x, g)^{-1} &= \left(x \cdot g, g^{-1}\right) \end{aligned}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

We will mainly focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB[1] $\mathscr{g}$
Integrated	Lie group G	LGB[2] $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Remarks (Why curved?)

For gauge invariance and closure of gauge transformations:

- Generalize Maurer-Cartan form $\rightarrow$ **Multiplicative Yang-Mills connection**.
- Generalize Field Strength.

[1]LAB = Lie algebra bundle
[2]LGB = Lie group bundle

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

Motivation (sort of historical)

We will mainly focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB[1] $\mathscr{g}$
Integrated	Lie group G	LGB[2] $\mathscr{G}$

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{G} \\ & & \downarrow \\ & & L \end{array}$$

Remarks (Why curved?)

For gauge invariance and closure of gauge transformations:

- Generalize Maurer-Cartan form $\to$ **Multiplicative Yang-Mills connection**.
- Generalize Field Strength.

[1] LAB = Lie algebra bundle
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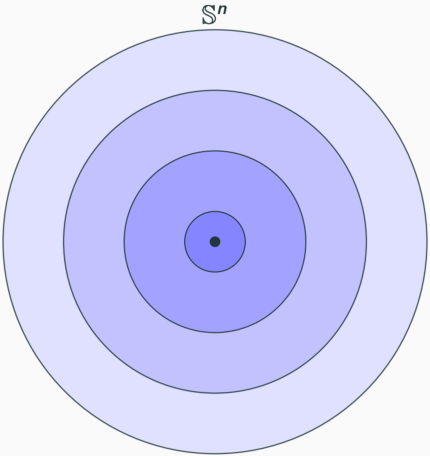
Singular Foliations

Example of a singular foliation



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Foliations are widespread

Singular Foliations:

- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- ...

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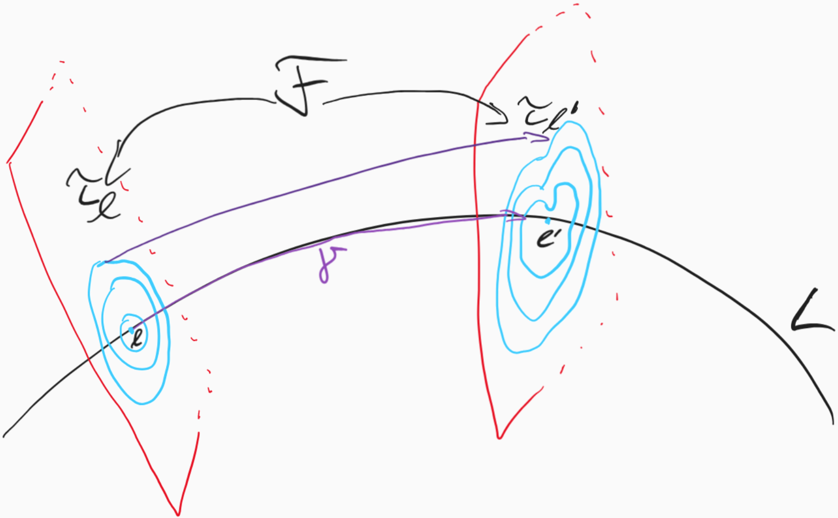
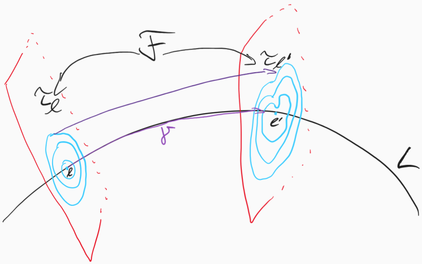
Unique transverse structure



How to classify singular foliations?



Remarks ([C. L.-G., S.-R. F.])

There is a connection on the normal bundle of a leaf L preserving the foliation!

- Transverse structure is unique: Classify singular foliation $\mathscr{F}$ like a bundle!
- Connection is a multiplicative Yang-Mills connection: Use curved gauge theory!

Source of the existence of connection on normal bundle: Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Thank you!

Classification of singular foliations

Theorem ([C. L.-G., S.-R. F.])

Formal singular foliations with leaf L and transverse model $(\mathbb{R}^d, \tau_I)$ are equivalent to:

- A Galois cover L' over L with structural group K
- A short exact sequence of groups
$$\mathrm{Inner}(\tau_I) \hookrightarrow H \twoheadrightarrow K$$
- A principal H-bundle P over L