Curved Yang-Mills-Higgs theories

Simon-Raphael Fischer, *based on joint works with Camille Laurent-Gengoux, and with Mehran Jalali Farahani, Hyungrok Kim (金炯錄), Christian Sämann*

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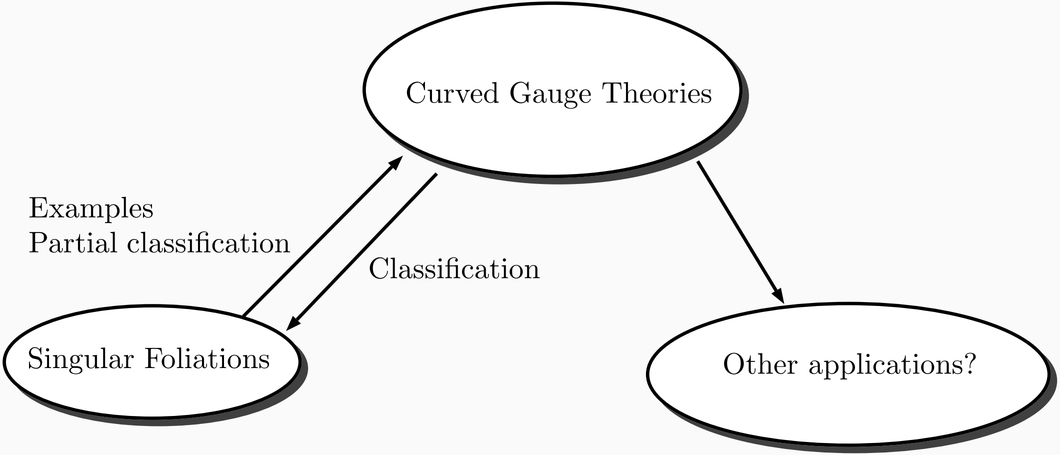
Curved Gauge Theories
Examples
Partial classification
Classification
Singular Foliations
Other applications?

Curved Yang-Mills-Higgs theory

Motivation: Covariantisation of Yang-Mills(-Higgs) theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $L \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $L \times \mathfrak{g}$	General connection ∇ on E

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

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$$\begin{tikzcd} \mathcal{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \end{tikzcd}$$

Definition
$\mathcal{G}$ a **Lie groupoid** if there are surjective submersions $s, t \colon \mathcal{G} \to M$, *source* and *target*, respectively, and a smooth *multiplication map* $\mathcal{G} \,{}_s\!\times_t \mathcal{G} \to \mathcal{G}$ such that

$$s(g'g) = s(g), \qquad t(g'g) = t(g')$$

for all $(g', g) \in \mathcal{G} \,{}_s\!\times_t \mathcal{G}$ (i.e. $s(g') = t(g)$), satisfying the typical expected properties, that is,

$$\begin{aligned} \text{Associativity:} & \qquad (g''g')g = g''(g'g), \\ \text{Units:} & \qquad ge_{s(g)} = g, \qquad e_{t(g)}g = g, \\ \text{Inverse:} & \qquad g^{-1}g = e_{s(g)}, \qquad gg^{-1} = e_{t(g)} \end{aligned}$$

for all $(g'', g', g) \in \mathcal{G} \,{}_s\!\times_t \mathcal{G} \,{}_s\!\times_t \mathcal{G}$, where one requires the existence of the *unit* e as a global section of both, s and t, and the *inverse* $g^{-1} \in \mathcal{G}$ of g.

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$\begin{tikzcd}
m'' & m' \arrow[l, bend right, "g'"'] & m \arrow[l, bend right, "g"'] \arrow[ll, bend right=60, "g'g"']
\end{tikzcd}$$

Example

Lie groups G

$$\begin{tikzcd}
G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
\{*\}
\end{tikzcd}$$

Ana Cannas Da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Soc., 1999.

$$
\begin{tikzcd}
m'' & m' \arrow[l, bend right, "g'"'] & m \arrow[l, bend right, "g"'] \arrow[ll, bend right=50, "g'g"']
\end{tikzcd}
$$

Example
Lie groups G

$$
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Example

Lie group bundles (LGBs) $\pi_G \colon G \to M$

$$\begin{tikzcd} G \arrow[d, shift right, "\pi_G"'] \arrow[d, shift left, "\pi_G"] \\ M \end{tikzcd}$$

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Motivation (sort of historical)

We will mainly focus on Yang-Mills theories:

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB[1] $\mathscr{g}$
Integrated	Lie group G	LGB[2] $\mathscr{G}$

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d] \\ & L \end{tikzcd}$$

Remarks (Why curved?)

For gauge invariance and closure of gauge transformations:

- Generalize Maurer-Cartan form $\to$ **Multiplicative Yang-Mills connection**.
- Generalize Field Strength.

[1] LAB = Lie algebra bundle
[2] LGB = Lie group bundle

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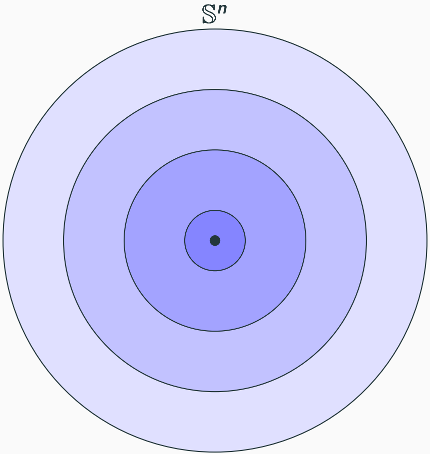
Singular Foliations

Example of a singular foliation

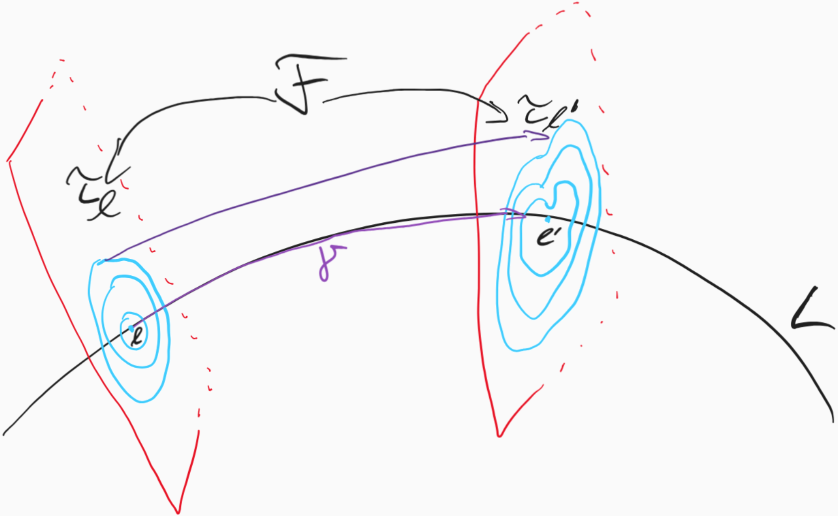


Foliations are widespread

Singular Foliations:

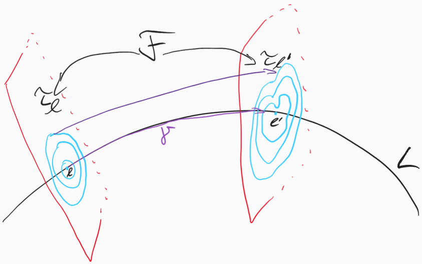
- Gauge Theory
- Poisson Geometry
  (Singular foliation of symplectic leaves)
- Lie groupoids and algebroids
- Dirac structures
- Generalised complex manifolds
- Non-commutative geometry
- . . .

Unique transverse structure



Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

How to classify singular foliations?



Remarks ([C. L.-G., S.-R. F.])

There is a connection on the normal bundle of a leaf L preserving the foliation!

- Transverse structure is unique: Classify singular foliation $\mathscr{F}$ like a bundle!
- Connection is a multiplicative Yang-Mills connection: Use curved gauge theory!

Source of the existence of connection on normal bundle: Camille Laurent-Gengoux and Leonid Ryvkin, The holonomy of a singular leaf, *Selecta Mathematica 28*, no. 2, 45, 2022.

Thank you!

Classification of singular foliations

Theorem ([C. L.-G., S.-R. F.])

Formal singular foliations with leaf L and transverse model $(\mathbb{R}^d, \tau_I)$ are equivalent to:

- A Galois cover L' over L with structural group K
- A short exact sequence of groups

$$\mathrm{Inner}(\tau_I) \hookrightarrow H \twoheadrightarrow K$$

- A principal H-bundle P over L