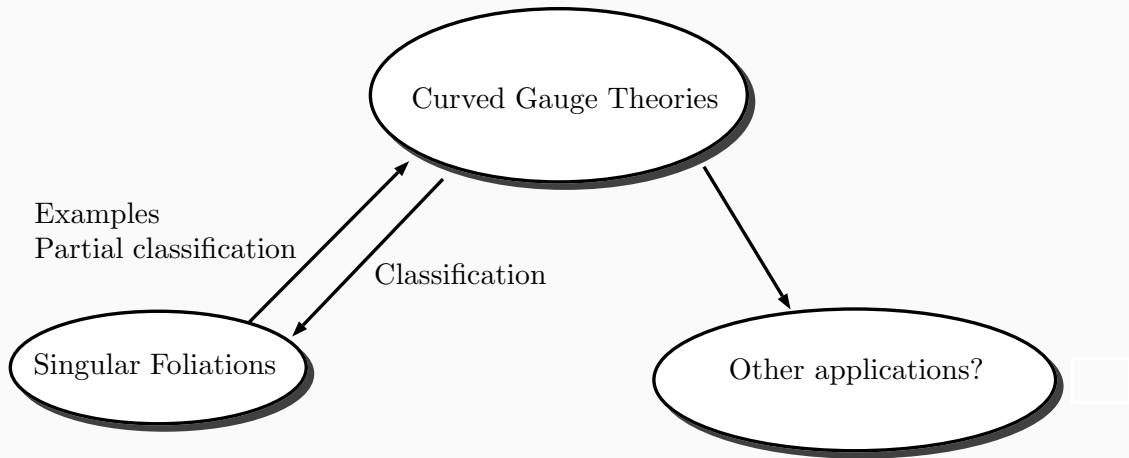


# Curved Yang-Mills-Higgs theories

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Simon-Raphael Fischer, *based on joint works with Camille Laurent-Gengoux, and with Mehran Jalali Farahani, Hyungrok Kim (金炯錄), Christian Sämann*





# Motivation

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# Motivation: Covariantisation of Yang-Mills(-Higgs) theory

Covariantization



Classical theory

Covariantised flat theory

Curved Theory

Vector space  $V$

Trivial vector bundle  $M \times V$

Vector bundle  $V \rightarrow M$

$$\frac{\partial}{\partial x^i}$$

Canonical flat connection  $\nabla^0$

Vector bundle connection  $\nabla$

Coordinate changes may lead to extra terms

Coordinate expressions form-invariant under coordinate changes



# Curved Yang-Mills gauge theory (curved YM theory)

Covariantization



YM theory

Covariantised YM theory

Curved YM theory

Lie group  $G$

Trivial Lie group bundle  $M \times G$

Lie group bundle  $G \rightarrow M$

Maurer-Cartan form

Fibre-wise Maurer-Cartan

Multiplicative YM connection

Field redefinitions lead to extra terms in gauge transformations and field strength

Expressions form-invariant under field redefinitions,  
**but curvature transforms non-trivially**



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# Curved Yang-Mills-Higgs theory (curved YMH theory)

Covariantization



YMH theory	Covariantised YMH theory	Curved YMH theory
Lie group $G$ with right-action on $N$	Action groupoid $N \times G$	Lie groupoid $G \rightrightarrows N$
Maurer-Cartan form	Fibre-wise Maurer-Cartan	Covariant adjustments

Field redefinitions lead to extra terms in gauge transformations and field strength

Expressions form-invariant under field redefinitions,  
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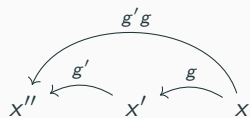


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# Lie groupoids

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$$\begin{array}{c} \mathcal{G} \\ \downarrow t \quad \downarrow s \\ \curvearrowright \\ N \end{array}$$

### Definition (Lie groupoids)

$\mathcal{G}$  a **Lie groupoid** if there are surjective submersions  $s, t: \mathcal{G} \rightarrow N$ , *source* and *target*, respectively, and a smooth *multiplication map*  $\mathcal{G}_{s \times_t \mathcal{G}} \rightarrow \mathcal{G}$  such that

$$s(g'g) = s(g), \quad t(g'g) = t(g')$$

for all  $(g', g) \in \mathcal{G}_{s \times_t \mathcal{G}}$  (i.e.  $s(g') = t(g)$ ), satisfying the typical expected properties, that is,

$$\text{Associativity:} \quad (g''g')g = g''(g'g),$$

$$\text{Units:} \quad ge_{s(g)} = g, \quad e_{t(g)}g = g,$$

$$\text{Inverse:} \quad g^{-1}g = e_{s(g)}, \quad gg^{-1} = e_{t(g)}$$

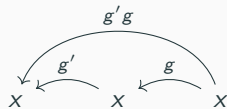
for all  $(g'', g', g) \in \mathcal{G}_{s \times_t \mathcal{G}} \times_{s \times_t \mathcal{G}} \mathcal{G}$ , where one requires the existence of the *unit*  $e$  as a global section of both,  $s$  and  $t$ , and the *inverse*  $g^{-1} \in \mathcal{G}$  of  $g$ .



## Example (Lie groups)

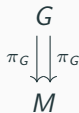
Lie groups  $G$

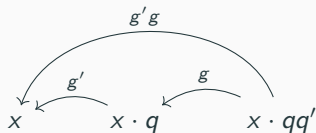




### Example (Lie group bundles (LGBs))

LGB  $\pi_G: G \rightarrow M$





### Example (Action groupoid (trivial))

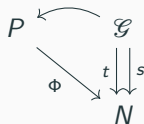
Lie group  $G$  with action  $\Psi: N \times G \rightarrow N$ ,  $(p, q) \mapsto p \cdot q$ , on  $N$ .

$$\begin{array}{c}
 N \times G \\
 \text{pr}_1 \downarrow \Psi \\
 N
 \end{array}$$

$$(x, q) (x \cdot q, q') = (x, qq') ,$$

$$e_x = (x, e) ,$$

$$(x, q)^{-1} = (x \cdot q, q^{-1})$$



### Definition (Groupoid right-action)

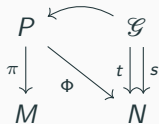
A **right-action** is a smooth map  $P \times_t \mathcal{G} \rightarrow P$  such that

$$\begin{aligned}\Phi(p \cdot g') &= s(g'), \\ (p \cdot g') \cdot g &= p \cdot (g'g), \\ p \cdot e_{\Phi(p)} &= p\end{aligned}$$

for all  $(p, g', g) \in P \times_t \mathcal{G} \times_s \mathcal{G}$ .

# Principal bundles and their Ehresmann connections

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### Definition (Principal groupoid-bundles)

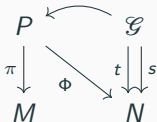
$\pi: P \rightarrow M$  surjective submersion is a **principal  $\mathcal{G}$ -bundle** if

$$\pi(p \cdot g) = \pi(p)$$

for all  $(p, g) \in P \times_t \mathcal{G}$ , and if

$$\begin{aligned} P \times_t \mathcal{G} &\rightarrow P \times_\pi P, \\ (p, g) &\mapsto (p, p \cdot g) \end{aligned}$$

is a diffeomorphism.



## Remarks

Infinitesimal action:

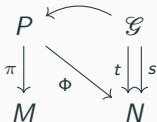
$$\begin{aligned}
 TP_{D\Phi \times_{D_t}} T\mathcal{G} &\rightarrow TP, \\
 (X, Y) &\mapsto X \cdot Y.
 \end{aligned}$$

For  $r_g(p) := p \cdot g$  with  $\Phi(p) = t(g)$ , its infinitesimal version corresponds to

$$Dr_g(X) = X \cdot 0$$

for all  $X$  with  $D\Phi(X) = 0$ .





## Remarks

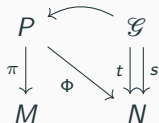
Infinitesimal action:

$$\begin{aligned}
 TP \times_{D\Phi \times Dt} T\mathcal{G} &\rightarrow TP, \\
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for all  $X$  with  $D\Phi(X) = 0$ .



Infinitesimal  $\mathcal{G}$ -action on  $P$ :

$$\begin{aligned} TP_{D\Phi \times_{Dt}} T\mathcal{G} &\rightarrow TP, \\ (X, Y) &\mapsto X \cdot Y. \end{aligned}$$


## Idea

Horizontal distribution  $HP$  (w.r.t.  $\pi$ ) with

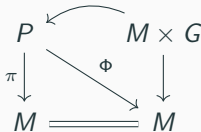
$$D\Phi(HP) = 0.$$

# We lose covariantization

YM theory	Covariantised YM theory
Lie group $G$	Trivial Lie group bundle $M \times G$
Maurer-Cartan form	Fibre-wise Maurer-Cartan



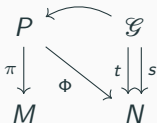
! We would lose the equivalence to the covariantised theory where we set  $\Phi = \pi$  !



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# New idea!



## Idea (Ehresmann connection on $P$ )

Equip  $\mathcal{G}$  with a horizontal distribution  $H\mathcal{G}$  (w.r.t.  $t$ ). An Ehresmann connection on  $P$  is a horizontal distribution  $HP$  (w.r.t.  $\pi$ ) so that the infinitesimal  $\mathcal{G}$ -action on  $P$  restricts

$$HP \times_{D\phi} H\mathcal{G} \rightarrow HP.$$

Invariance then via the **modified right-pushforward**  $r_{g*}$

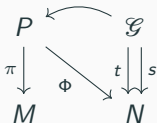
$$r_{g*}(X) := X \cdot Y$$

for all  $X \in T_p P$ , where  $Y \in H_g \mathcal{G}$  is the unique lift of  $D_p \phi(X)$ .

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## Idea

HP Ehresmann, then

$$HP_{D\Phi \times_{Dt}} H\mathcal{G} \rightarrow HP.$$

## Theorem

Given such an HP, then  $H\mathcal{G}$  is **Cartan**,

$$H\mathcal{G}_{Ds \times_{Dt}} H\mathcal{G} \rightarrow H\mathcal{G}.$$

Sketch as a proof for LGBs.

W.r.t. parallel transports we have

$$PT^P(p \cdot g) = PT^P(p) \cdot PT^{\mathcal{G}}(g).$$

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## Sketch as a proof for LGBs.

W.r.t. parallel transports we have

$$PT^P(p \cdot g) = PT^P(p) \cdot PT^{\mathcal{G}}(g),$$

then

$$\begin{array}{c} PT^P(p \cdot (gq)) \quad \quad \quad = \quad \quad \quad PT^P(p) \cdot PT^{\mathcal{G}}(gq) \\ \parallel \\ PT^P((p \cdot g) \cdot q) \quad \quad = \quad \quad PT^P(p) \cdot \left( PT^{\mathcal{G}}(g) PT^{\mathcal{G}}(q) \right). \end{array}$$



## Idea

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$$HP \times_{D\phi \times D_t} H\mathcal{G} \rightarrow HP.$$

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## Remarks

Vice versa, if  $H\mathcal{G}$  is Cartan, then such  $HP$  exist.

# What else?

## Idea

Open a textbook about ordinary gauge theory, and replace:

- Right-pushforward  $\rightsquigarrow$  Modified right-pushforward,
- Adjoint representation  $\rightsquigarrow$  Adjoint representation via Cartan connections (?)
- Generalising the concept of field strength accounting for a possible curvature coming from  $\mathcal{G}$

Too much to introduce! (Lie algebroids, Lie algebroid connections, basic curvature, basic connections, and so on...)

Thus, for today: LGBs!

# Curved Yang-Mills gauge theory

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## Remarks

The groupoid is now an LGB  $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow M$  with structural Lie group  $G$ .

Gauge invariance requires us to study *special* Cartan connections:

### Definition (Multiplicative Yang-Mills connections)

A connection 1-form  $\mu_{\mathcal{G}}$  on the LGB  $\mathcal{G}$  is a **multiplicative Yang-Mills connection (w.r.t. a  $\zeta \in \Omega^2(M; \mathfrak{g})$ )** if it is a Cartan connection satisfying the **generalised Maurer-Cartan equation**:

$$\left( d^{\pi_{\mathcal{G}}^* \nabla^{\mathcal{G}}} \mu_{\mathcal{G}} + \frac{1}{2} [\mu_{\mathcal{G}} \wedge \mu_{\mathcal{G}}]_{\pi_{\mathcal{G}}^* \mathfrak{g}} \right) \Big|_g = \text{Ad}_{g^{-1}} \circ \pi_{\mathcal{G}}^! \zeta|_g - \pi_{\mathcal{G}}^! \zeta|_g$$

for all  $g \in \mathcal{G}$ , where  $\mathfrak{g}$  is the Lie algebra bundle (LAB) of  $\mathcal{G}$ , and  $\nabla^{\mathcal{G}}$  is the naturally induced LAB connection.

## Remarks (Closed connection, exact curvature)

$$R_{\nabla^{\mathcal{G}}} = \text{ad} \circ \zeta.$$

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## Definition

The field strength  $F$  of an Ehresmann connection  $A \in \Omega^1(P; \pi^* \mathfrak{g})$  on  $\pi: P \rightarrow M$  is defined by

$$F := d^{\pi^* \nabla^{\mathfrak{g}}} A + \frac{1}{2} [A \wedge A]_{\pi^* \mathfrak{g}} + \pi^! \zeta.$$

## Theorem

*By contracting  $F$  with an ad-invariant scalar product we achieve a gauge-invariant Lagrangian  $\mathcal{L}[A]$ , that is,*

$$\mathcal{L}[L^! A] = \mathcal{L}[A]$$

*for all  $L \in \text{Aut}(P)$ .*

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## Examples

---



## Idea

Let us assume that the structural Lie group  $G$  is semisimple, then:

## Theorem

*There is a unique (up to isomorphism) principal  $G$ -bundle  $Q$  such that*

$$\mathcal{G} = (Q \times G) / G,$$

*multiplicative Yang-Mills connections on  $\mathcal{G}$  are precisely associated connections, thus in 1:1 correspondence to Ehresmann connections on  $Q$ , and  $\zeta$  is the corresponding curvature on  $Q$ .*

## Corollary

*There are only curved multiplicative Yang-Mills connections on  $\mathcal{G}$  if and only if  $Q$  is not flat.*

## Example

Hopf fibration  $Q := \mathbb{S}^7 \rightarrow \mathbb{S}^4$ ,

$$\mathcal{G} = (Q \times \mathrm{SU}(2)) / \mathrm{SU}(2).$$

## Remarks

Some technical details:

- This is stable under field redefinitions.
- This can be relaxed to  $\mathcal{G}$  with a trivial centre, using a nice geometric interpretation via singular foliations. However, dynamics of gauge theory require Ad-invariance...  
See Camille's talk on Friday ☺

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# About Lie groupoids...

## Idea

Assume that  $G$  is semisimple **and** acts faithfully on  $\mathbb{R}^d$  fixing 0. Then  $\mathcal{G}$  acts faithfully on

$$N := (Q \times \mathbb{R}^d) / G,$$

giving rise to an action groupoid structure on  $\phi^* \mathcal{G}$  ( $\phi: N \rightarrow M$ )

$$\begin{array}{c} \phi^* \mathcal{G} \\ \begin{array}{c} t \downarrow s \\ \curvearrowright \\ N \end{array} \end{array}$$

## Theorem

*The pullback of the multiplicative Yang-Mills connection on  $\phi^* \mathcal{G}$  is flat up to field redefinitions if and only if  $Q$  is flat.*

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Other technical details:

- Also stable under field redefinitions!
- Here semi-simplicity is rather important.

## Example

Again  $Q$  the Hopf fibration  $\mathbb{S}^7 \rightarrow \mathbb{S}^4$ , then naturally choose  $\mathbb{R}^d = \mathbb{C}^2$ .

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**Thank you!**

## A quick funny example, again by $\mathbb{S}^7$

### Example

The Lie groupoid  $\mathcal{G}$  given as the pair groupoid of  $\mathbb{S}^7$  leads to descriptions of curved Yang-Mills-Higgs theories, which cannot be described in an ordinary way due to the lack of non-associativity on  $\mathbb{S}^7$ .