Geometry of curved Yang-Mills-Higgs gauge theories

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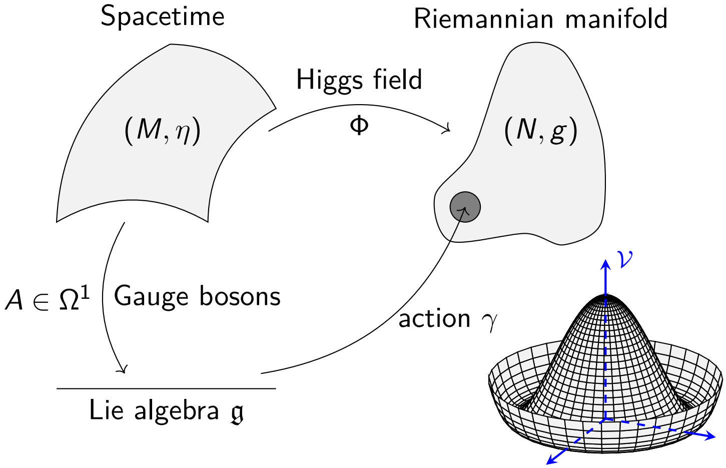
Infinitesimal gauge theory



Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Definition (Lie algebroids)

Let $E \to N$ be a vector bundle. Then E is a Lie algebroid, if there is a bundle map $\rho : E \to \mathrm{T}N$, called the **anchor**, and a Lie algebra structure on $\Gamma(E)$ with Lie bracket $[\cdot, \cdot]_E$ satisfying

$$[\mu, f\nu]_E = f[\mu, \nu]_E + \mathcal{L}_{\rho(\mu)}(f)\ \nu \tag{1}$$

for all $f \in C^\infty(N)$ and $\mu, \nu \in \Gamma(E)$.

Example

- $E = \mathrm{T}N$, $\rho = \mathbb{1}_{\mathrm{T}N}$

Ana Cannas Da Silva, Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Society, 1999.

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As for an action $\gamma : \mathfrak{g} \to \mathfrak{X}(N)$ we have:

Proposition (Anchor as homomorphism)

$\rho : \Gamma(E) \to \mathfrak{X}(N)$ *is a homomorphism of Lie algebras.*

Yvette Kosmann-Schwarzbach and Franco Magri. Poisson-Nijenhuis structures. *In Annales de l'IHP Physique théorique*, volume 53, pages 35–81, 1990.

The classical formalism will correspond to:

Proposition (Action Lie algebroids)

*Let $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}\right)$ be a Lie algebra with a $\mathfrak{g}$-action γ on N. Then there is a **unique** Lie algebroid structure on $E := N \times \mathfrak{g}$ as a vector bundle over N such that*

$$\rho(\nu) = \gamma(\nu), \tag{2}$$

$$[\mu,\nu]_E = [\mu,\nu]_{\mathfrak{g}} \tag{3}$$

for all constant sections $\mu,\nu \in \Gamma(E)$.

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Example (E-connection ${}^{E}\nabla$ on V)

∇' a vector bundle connection on $V \to N$, then

$$ {}^{E}\nabla_{\nu} v := \nabla'_{\rho(\nu)} v \tag{4} $$

for all $\nu \in \Gamma(E)$ and $v \in \Gamma(V)$. In short denoted by ∇'_{ρ}.

Mohamed Boucetta. Riemannian geometry of Lie algebroids. *Journal of the Egyptian Mathematical Society*, 19(1-2):57–70, 2011.

Example

For ∇ a connection on E we have the **basic connection** given as a pair of E-connections on E and on $\mathrm{T}N$ by

$$\nabla^{\mathrm{bas}}_{\nu}\mu := [\nu, \mu]_E + \nabla_{\rho(\mu)}\nu, \tag{5}$$

$$\nabla^{\mathrm{bas}}_{\nu}X := [\rho(\nu), X] + \rho(\nabla_X \nu) \tag{6}$$

for all $X \in \mathfrak{X}(N)$ and $\nu, \mu \in \Gamma(E)$.

Remarks (Encoding of Lie algebra representations)

Test this with trivial bundles and canonical flat connection ∇^0, *i.e.* $E = N \times \mathfrak{g}$ and $\nabla^0\nu = 0$ for constant sections ν.

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Definition (Basic curvature)

Let ∇ be a connection on E. The **basic curvature** $R_{\nabla}^{\mathrm{bas}}$ is defined as an element of $\Gamma\left(\bigwedge^2 E^* \otimes \mathrm{T}^*N \otimes E\right)$ by

$$R_{\nabla}^{\mathrm{bas}}(\mu,\nu)X := \nabla_X([\mu,\nu]_E) - [\nabla_X\mu,\nu]_E - [\mu,\nabla_X\nu]_E - \nabla_{\nabla_{\nu}^{\mathrm{bas}}X}\mu + \nabla_{\nabla_{\mu}^{\mathrm{bas}}X}\nu, \tag{7}$$

where $\mu,\nu \in \Gamma(E)$ and $X \in \mathfrak{X}(N)$.

Proposition

We recover the curvature of the basic connection:

$$R_{\nabla^{\mathrm{bas}}} = \begin{cases} -R_{\nabla}^{\mathrm{bas}}(\cdot,\cdot)\circ\rho, & \text{on } E, \\ -\rho\circ R_{\nabla}^{\mathrm{bas}}, & \text{on } \mathrm{T}N. \end{cases} \tag{8}$$

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Definition (Space of fields)

Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
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Definition (Minimal coupling)

Minimal coupling $\mathfrak{D}$, $(\Phi, A) \mapsto \mathfrak{D}(\Phi, A) \in \Omega^1(M; \Phi^* \mathrm{T}N)$, by

$$\mathfrak{D}(\Phi, A) := \mathfrak{D}^A \Phi := \mathrm{D}\Phi - (\Phi^* \rho)(A), \tag{9}$$

where $\mathrm{D}\Phi : \mathrm{T}M \to \mathrm{T}N$ is the tangent map.

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 Alexei Kotov and Thomas Strobl. Curving Yang-Mills-Higgs gauge theories. *Physical Review D*, 92(8):085032, 2015.

Definition (Field strength)

Let ∇ be a connection on E. We define the **field strength** F, $(\Phi, A) \mapsto F(\Phi, A) \in \Omega^2(M; \Phi^* E)$, by

$$F(\Phi, A) := \mathrm{d}^{\Phi^*\nabla} A + \frac{1}{2}(\Phi^* t_{\nabla^{\mathrm{bas}}})(A \stackrel{\wedge}{,} A), \tag{10}$$

where $t_{\nabla^{\mathrm{bas}}}$ is the torsion of ∇^{bas} on E and $\mathrm{d}^{\Phi^*\nabla}$ the exterior covariant derivative of $\Phi^*\nabla$.

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Definition (Generalised field strength)

Let ζ be an element of $\Omega^2(N; E)$, then we define the **generalised field strength** G by

$$G(\Phi, A) := F(\Phi, A) + \frac{1}{2}(\Phi^*\zeta)\left(\mathfrak{D}^A\Phi \overset{\wedge}{,} \mathfrak{D}^A\Phi\right). \tag{11}$$

Definition (Curved Yang-Mills-Higgs (CYMH) Lagrangian)

Let κ be a fibre metric on E, then the **curved Yang-Mills-Higgs Lagrangian** $\mathfrak{L}_{\mathrm{CYMH}}$, $(\Phi, A) \mapsto \mathfrak{L}_{\mathrm{CYMH}}(\Phi, A) \in \Omega^{\dim(M)}(M)$, is defined by

$$\begin{aligned}\mathfrak{L}_{\mathrm{CYMH}}(\Phi, A) &\coloneqq -\frac{1}{2}(\Phi^*\kappa)\left(G(\Phi, A) \stackrel{\wedge}{,} *G(\Phi, A)\right) \\ &\quad + (\Phi^*g)\left(\mathfrak{D}^A\Phi \stackrel{\wedge}{,} *\mathfrak{D}^A\Phi\right) - *(\Phi^*\mathcal{V}), \qquad (12)\end{aligned}$$

where $*$ is the Hodge star operator related to the spacetime metric η.

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Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_\nabla + \mathrm{d}^{\nabla^{\mathrm{bas}}}\zeta = 0, \tag{13}$$

$$R_\nabla^{\mathrm{bas}} = 0, \tag{14}$$

$$\nabla^{\mathrm{bas}}\kappa = 0, \tag{15}$$

$$\nabla^{\mathrm{bas}}g = 0, \tag{16}$$

$$\mathcal{L}_\rho \mathcal{V} = 0, \tag{17}$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

We say that we have a **pre-classical gauge theory**, if ∇ is flat.

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Summary

$$\begin{tikzcd}
\Phi^* E \arrow[d] & (E, [\cdot,\cdot]_E, \kappa, \nabla) \arrow[d] \arrow[dr, "\rho"] & \\
M \arrow[r, "\Phi"] & (N, g) & \mathrm{T}N \arrow[l] \arrow[ul, bend right, "\zeta\in\Omega^2"']
\end{tikzcd}$$

$\rightsquigarrow$ Together with the compatibility conditions we will have gauge invariance, that is,

$$\delta_\varepsilon \mathfrak{L}_{\mathrm{CYMH}} = 0, \tag{18}$$

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but how to define the infinitesimal gauge transformation δ_ε?

My results and contributions, starting with the following:

Definition of infinitesimal gauge transformation $\delta_\varepsilon L$ for functionals L, where W is a vector space and $V \to N$ a vector bundle:

	Classical	CYMH GT
ε	$\varepsilon \in C^\infty(M;\mathfrak{g})$	$(\Phi, A) \mapsto \varepsilon(\Phi, A) \in \Gamma(\Phi^* E)$
$(\delta_\varepsilon \Phi, \delta_\varepsilon A)$	Vector field Ψ_ε on $\{(\Phi, A)\}$	
L	$L(\Phi, A) \in \Omega^\bullet(M; W)$	$L(\Phi, A) \in \Omega^\bullet(M; \Phi^* V)$
δ_ε	Derivation as lift of Ψ_ε with:	
	canon. flat conn. ∇^0 on $M \times W$	E-connection ${}^E\nabla$ on V

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- What about: $[\delta_\varepsilon, \delta_\vartheta] = \delta_\varepsilon \delta_\vartheta - \delta_\vartheta \delta_\varepsilon$?

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- What about: $[\delta_\varepsilon, \delta_\vartheta] = \delta_\varepsilon \delta_\vartheta - \delta_\vartheta \delta_\varepsilon$?

Theorem (Curvature of δ, [S.-R. F.])

Let ${}^{E}\nabla$ be a flat E-connection on V, and ∇ a connection on E such that $R_{\nabla}^{\mathrm{bas}} = 0$. Then

$$\left[\delta_{\varepsilon}^{{}^{E}\nabla}, \delta_{\vartheta}^{{}^{E}\nabla}\right] = -\delta_{[\![\varepsilon,\vartheta]\!]}^{{}^{E}\nabla} \tag{19}$$

where $[\![\cdot,\cdot]\!]$ is a Lie bracket given by

$$\begin{aligned} [\![\varepsilon,\vartheta]\!]|_{(\Phi,A)} :=& \left(\delta_{\vartheta}^{\nabla^{\mathrm{bas}}}\varepsilon - \delta_{\varepsilon}^{\nabla^{\mathrm{bas}}}\vartheta\right)\Big|_{(\Phi,A)} \\ &- \left(\Phi^* t_{\nabla^{\mathrm{bas}}}\right)\left(\varepsilon(\Phi,A), \vartheta(\Phi,A)\right). \end{aligned} \tag{20}$$

Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

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Definition (Field redefinition, [S.-R. F.])

Let $\lambda \in \Omega^1(N; E)$ such that $\Lambda := \mathbb{1}_E - \lambda \circ \rho$ is an automorphism of E. We then define the **field redefinitions** by

$$\widetilde{A}^\lambda := (\Phi^*\Lambda)(A) + \Phi^!\lambda, \tag{21}$$

$$\widetilde{\nabla}^\lambda := \nabla + \left(\Lambda \circ \mathrm{d}^{\nabla^{\mathrm{bas}}} \circ \Lambda^{-1}\right)\lambda, \tag{22}$$

$$\widetilde{\kappa}^\lambda := \kappa \circ \left(\Lambda^{-1}, \Lambda^{-1}\right), \tag{23}$$

$$\widetilde{g}^\lambda := g \circ \left(\widehat{\Lambda}^{-1}, \widehat{\Lambda}^{-1}\right), \tag{24}$$

where $\widehat{\Lambda} := \mathbb{1}_{\mathrm{T}N} - \rho \circ \lambda$, and for all $X, Y \in \mathfrak{X}(N)$ we have

$$\begin{aligned}&\widetilde{\zeta}^\lambda\left(\widehat{\Lambda}(X), \widehat{\Lambda}(Y)\right)\\ &= \Lambda(\zeta(X, Y)) - \left(\mathrm{d}^{\widetilde{\nabla}^\lambda}\lambda\right)(X, Y) + t_{\widetilde{\nabla}^\lambda_\rho}(\lambda(X), \lambda(Y)).\end{aligned} \tag{25}$$

Proposition ([S.-R. F.])

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\widetilde{\mathfrak{L}}^{\lambda}_{\mathrm{CYMH}} = \mathfrak{L}_{\mathrm{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

Proposition ([S.-R. F.])

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Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = K$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot,\cdot]_K \in \Gamma\left(\bigwedge^2 K^* \otimes K\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

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- We need

$$\nabla_Y([\mu, \nu]_K) = [\nabla_Y \mu, \nu]_K + [\mu, \nabla_Y \nu]_K, \tag{26}$$

$$R_\nabla(Y, Z)\mu = [\zeta(Y, Z), \mu]_K \tag{27}$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu, \nu \in \Gamma(K)$.

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Theorem (Invariant for LABs, [S.-R. F.])

We have

$$\mathrm{d}^{\widetilde{\nabla}^\lambda}\widetilde{\zeta}^\lambda = \mathrm{d}^\nabla \zeta, \tag{28}$$

and $\mathrm{d}^\nabla \zeta$ *has values in the centre of* K.

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^\nabla \zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^\lambda = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^\nabla \zeta \neq 0$.

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However, by $R_{\nabla} = \mathrm{ad}_K \circ \zeta$ it may still be that ∇ becomes flat.

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^{\nabla}\zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^{\lambda} = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^{\nabla}\zeta \neq 0$.

However, by $R_{\nabla} = \mathrm{ad}_K \circ \zeta$ it may still be that ∇ becomes flat.

Turning to the field redefinition of ∇:

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ *restricts to the centre of* K *and induces a differential* d^{Ξ} *on centre-valued forms. Moreover,* d^{Ξ} *is independent of the field redefinitions.*

Sketch of proof.

Recall

$$\begin{aligned}\nabla_Y([\mu,\nu]_K) &= [\nabla_Y\mu,\nu]_K + [\mu,\nabla_Y\nu]_K,\\ R_{\nabla}(Y,Z)\mu &= [\zeta(Y,Z),\mu]_K,\\ \widetilde{\nabla}^{\lambda}_Y\mu &= \nabla_Y\mu - [\lambda(Y),\mu]_K,\end{aligned}$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu,\nu \in \Gamma(K)$. Then insert μ with values in the centre.

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Theorem (Closedness of $\mathrm{d}^{\nabla}\zeta$, [S.-R. F.])

We have

$$\mathrm{d}^{\Xi}\mathrm{d}^{\nabla}\zeta = 0. \tag{29}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^{\nabla}\zeta\right]_{\mathrm{d}^{\Xi}}. \tag{30}$$

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If $\mathrm{Obs}(\Xi) \neq 0$, then there is no field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

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If N is contractible, then there is a field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

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Second theorem follows as a result by K. Mackenzie (General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005). Mackenzie derived $\mathrm{Obs}(\Xi)$ in the context of extending Lie algebroids by LABs.

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Example (Zero obstruction class not necessarily pre-classical)

Let P be the Hopf fibration

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{S}^4 \end{array}$$

Then for the adjoint bundle

$$K := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left(\mathbb{S}^7 \times \mathfrak{su}(2)\right) \Big/ \mathrm{SU}(2)$$

we have a non-flat ∇ satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but $\mathrm{Obs}(\Xi) = 0$.

Sources
K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical.
In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

What happens in the case of tangent bundles?

Example (Tangent bundles)

- $E = \mathrm{T}N$, $\rho = \mathbb{1}_E$, $\kappa = g$
- $R_\nabla = \nabla^{\mathrm{bas}} t_{\nabla^{\mathrm{bas}}} \Rightarrow$ Canonically $\zeta := -t_{\nabla^{\mathrm{bas}}}$

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Theorem (Tangent bundles are locally pre-classical, [S.-R. F.])

Let $N = \mathbb{R}^n$ ($n \in \mathbb{N}_0$) be an Euclidean space as smooth manifold. Then there is a field redefinition such that $\widetilde{\nabla}^\lambda$ is flat.

What about global examples?

Theorem

Let N be a smooth compact and simply connected manifold, and assume we have a connection ∇ on $E := \mathrm{T}N$ such that ∇ and ∇^{bas} are flat. Then N is diffeomorphic to a Lie group.

Sketch of the proof.

1. Flat ∇ & simply connected: Global trivialization of $\mathrm{T}N$

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Theorem (Global example: Unit octonions, [S.-R. F.])

$\mathbb{S}^7$ admits a CYMH gauge theory such that the related connection ∇ on $E := \mathrm{T}\mathbb{S}^7$ is not flat. Moreover, there is no field redefinition $\widetilde{\nabla}^\lambda$ of ∇ such that $\widetilde{\nabla}^\lambda$ is flat.

Sketch of the proof.

Use the canonical trivialization $(Y_i)_{i=1}^7$ of $\mathrm{T}\mathbb{S}^7$: $\mathbb{S}^7$ are the unit octonions, and we have seven imaginary numbers $(e_i)_{i=1}^7$, $e_i^2 = -1$. Then

$$\left. Y_i \right|_z := e_i \cdot z \tag{31}$$

for all $z \in \mathbb{S}^7$.

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Summary: When are theories flat?

	Locally	Globally
LABs	Pre-classical	$\mathrm{ad}(\mathbb{S}^7 \to \mathbb{S}^4)$ curved
Tangent bundles	Pre-classical	$\mathrm{T}\mathbb{S}^7$ curved

Remarks

What about general Lie algebroids? Find an invariant like $\mathrm{d}^\nabla \zeta$
$\rightsquigarrow$ $\mathrm{d}^\nabla \zeta$ measures the failure of the Bianchi identity of G
$\rightsquigarrow$ Generalised Bianchi identity?

Thank you!