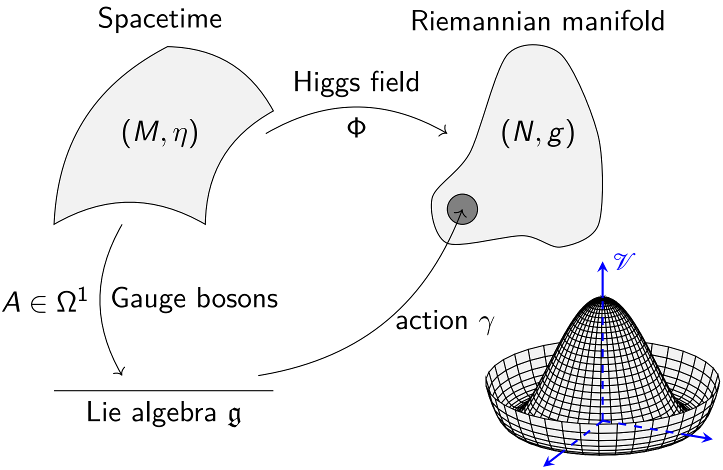
Geometry of curved Yang-Mills-Higgs gauge theories

Simon-Raphael Fischer

23 April 2022

Table of contents

Infinitesimal gauge theory



Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Definition (Lie algebroids)

Let $E \to N$ be a vector bundle. Then E is a Lie algebroid, if there is a bundle map $\rho : E \to \mathrm{T}N$, called the **anchor**, and a Lie algebra structure on $\Gamma(E)$ with Lie bracket $[\cdot,\cdot]_E$ satisfying

$$[\mu, f\nu]_E = f[\mu,\nu]_E + \mathscr{L}_{\rho(\mu)}(f)\ \nu \tag{1}$$

for all $f \in C^\infty(N)$ and $\mu, \nu \in \Gamma(E)$.

Example

- $E = \mathrm{T}N$, $\rho = \mathbb{1}_{\mathrm{T}N}$

Ana Cannas Da Silva, Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Society, 1999.

Definition (Lie algebroids)

Let $E \to N$ be a vector bundle. Then E is a Lie algebroid, if there is a bundle map $\rho : E \to \mathrm{T}N$, called the **anchor**, and a Lie algebra structure on $\Gamma(E)$ with Lie bracket $[\cdot, \cdot]_E$ satisfying

$$[\mu, f\nu]_E = f[\mu, \nu]_E + \mathscr{L}_{\rho(\mu)}(f)\ \nu \tag{1}$$

for all $f \in C^\infty(N)$ and $\mu, \nu \in \Gamma(E)$.

Example

- $E = \mathrm{T}N$, $\rho = \mathbb{1}_{\mathrm{T}N}$
- E a bundle of Lie algebras, $\rho = 0$

Ana Cannas Da Silva, Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Society, 1999.

Definition (Lie algebroids)

Let $E \to N$ be a vector bundle. Then E is a Lie algebroid, if there is a bundle map $\rho : E \to \mathrm{T}N$, called the **anchor**, and a Lie algebra structure on $\Gamma(E)$ with Lie bracket $[\cdot, \cdot]_E$ satisfying

$$[\mu, f\nu]_E = f[\mu, \nu]_E + \mathscr{L}_{\rho(\mu)}(f)\ \nu \tag{1}$$

for all $f \in C^\infty(N)$ and $\mu, \nu \in \Gamma(E)$.

Example

- $E = \mathrm{T}N$, $\rho = \mathbb{1}_{\mathrm{T}N}$
- E a bundle of Lie algebras, $\rho = 0$

Sources
Ana Cannas Da Silva, Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Society, 1999.

The classical formalism will correspond to:

Proposition (Action Lie algebroids)

Let $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}\right)$ *be a Lie algebra with a* $\mathfrak{g}$*-action* γ *on* N*. Then there is a* ***unique*** *Lie algebroid structure on* $E := N \times \mathfrak{g}$ *as a vector bundle over* N *such that*

$$\rho(\nu) = \gamma(\nu), \tag{2}$$

$$[\mu, \nu]_E = [\mu, \nu]_{\mathfrak{g}} \tag{3}$$

for all constant sections $\mu, \nu \in \Gamma(E)$.

Sources
Ana Cannas Da Silva, Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Society, 1999.

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-**connections** as lift of ρ
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

$\rightsquigarrow$ E-connections similar to vector bundle connections but Leibniz rule along ρ, *i.e.* only lifting vector fields in the image of ρ

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-**connections** as lift of ρ
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

$\rightsquigarrow$ E-connections similar to vector bundle connections but Leibniz rule along ρ, *i.e.* only lifting vector fields in the image of ρ

Example (E-connection ${}^E\nabla$ on V)

∇' a vector bundle connection on $V \to N$, then

$$ {}^E\nabla_\nu v := \nabla'_{\rho(\nu)} v \tag{4} $$

for all $\nu \in \Gamma(E)$ and $v \in \Gamma(V)$. In short denoted by ∇'_ρ.

Sources
Mohamed Boucetta. Riemannian geometry of Lie algebroids.
Journal of the Egyptian Mathematical Society, 19(1-2):57–70, 2011.

Example

For ∇ a connection on E we have the **basic connection** given as a pair of E-connections on E and on $\mathrm{T}N$ by

$$\nabla^{\mathrm{bas}}_{\nu}\mu := [\nu, \mu]_E + \nabla_{\rho(\mu)}\nu, \tag{5}$$

$$\nabla^{\mathrm{bas}}_{\nu}X := [\rho(\nu), X] + \rho(\nabla_X \nu) \tag{6}$$

for all $X \in \mathfrak{X}(N)$ and $\nu, \mu \in \Gamma(E)$.

Remarks (Encoding of Lie algebra representations)

Test this with trivial bundles and canonical flat connection ∇^0, *i.e.* $E = N \times \mathfrak{g}$ and $\nabla^0 \nu = 0$ for constant sections ν.

Camilo Arias Abad, Marius Crainic. Representations up to homotopy of Lie algebroids. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2012(663):91–126, 2012.

Example

For ∇ a connection on E we have the **basic connection** given as a pair of E-connections on E and on $\mathrm{T}N$ by

$$\nabla^{\mathrm{bas}}_{\nu}\mu := [\nu, \mu]_E + \nabla_{\rho(\mu)}\nu, \tag{5}$$

$$\nabla^{\mathrm{bas}}_{\nu}X := [\rho(\nu), X] + \rho(\nabla_X \nu) \tag{6}$$

for all $X \in \mathfrak{X}(N)$ and $\nu, \mu \in \Gamma(E)$.

Remarks (Encoding of Lie algebra representations)

Test this with trivial bundles and canonical flat connection ∇^0, *i.e.* $E = N \times \mathfrak{g}$ and $\nabla^0 \nu = 0$ for constant sections ν.

Camilo Arias Abad, Marius Crainic. Representations up to homotopy of Lie algebroids. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2012(663):91–126, 2012.

Definition (Basic curvature)

Let ∇ be a connection on E. The **basic curvature** $R_{\nabla}^{\mathrm{bas}}$ is defined as an element of $\Gamma\left(\bigwedge^2 E^* \otimes \mathrm{T}^*N \otimes E\right)$ by

$$\begin{aligned} R_{\nabla}^{\mathrm{bas}}(\mu,\nu)X &:= \nabla_X([\mu,\nu]_E) - [\nabla_X\mu,\nu]_E - [\mu,\nabla_X\nu]_E \\ &\quad - \nabla_{\nabla_\nu^{\mathrm{bas}}X}\mu + \nabla_{\nabla_\mu^{\mathrm{bas}}X}\nu, \end{aligned} \tag{7}$$

where $\mu,\nu \in \Gamma(E)$ and $X \in \mathfrak{X}(N)$.

Proposition

We recover the curvature of the basic connection:

$$R_{\nabla^{\mathrm{bas}}} = \begin{cases} -R_{\nabla}^{\mathrm{bas}}(\cdot,\cdot)\circ\rho, & \text{on } E, \\ -\rho\circ R_{\nabla}^{\mathrm{bas}}, & \text{on } \mathrm{T}N. \end{cases} \tag{8}$$

Sources

Camilo Arias Abad, Marius Crainic. Representations up to homotopy of Lie algebroids. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2012(663):91–126, 2012.

Definition (Basic curvature)

Let ∇ be a connection on E. The **basic curvature** R_∇^{bas} is defined as an element of $\Gamma\left(\bigwedge^2 E^* \otimes \mathrm{T}^*N \otimes E\right)$ by

$$R_\nabla^{\mathrm{bas}}(\mu,\nu)X \coloneqq \nabla_X([\mu,\nu]_E) - [\nabla_X\mu,\nu]_E - [\mu,\nabla_X\nu]_E - \nabla_{\nabla_\nu^{\mathrm{bas}}X}\mu + \nabla_{\nabla_\mu^{\mathrm{bas}}X}\nu, \tag{7}$$

where $\mu,\nu \in \Gamma(E)$ and $X \in \mathfrak{X}(N)$.

Proposition

We recover the curvature of the basic connection:

$$R_{\nabla^{\mathrm{bas}}} = \begin{cases} -R_\nabla^{\mathrm{bas}}(\cdot,\cdot)\circ\rho, & \text{on } E, \\ -\rho\circ R_\nabla^{\mathrm{bas}}, & \text{on } \mathrm{T}N. \end{cases} \tag{8}$$

Camilo Arias Abad, Marius Crainic. Representations up to homotopy of Lie algebroids. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2012(663):91–126, 2012.

Definition (Space of fields)

Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
- Field of gauge bosons $A \in \Omega^1(M; \Phi^* E)$

Definition (Space of fields)

Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
- Field of gauge bosons $A \in \Omega^1(M; \Phi^* E)$

Sources Alexei Kotov and Thomas Strobl. Curving Yang-Mills-Higgs gauge theories. *Physical Review D*, 92(8):085032, 2015.

Definition (Space of fields)

Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
- Field of gauge bosons $A \in \Omega^1(M; \Phi^* E)$

Definition (Minimal coupling)

Minimal coupling $\mathfrak{D}$, $(\Phi, A) \mapsto \mathfrak{D}(\Phi, A) \in \Omega^1(M; \Phi^* \mathrm{T} N)$, by

$$\mathfrak{D}(\Phi, A) := \mathfrak{D}^A \Phi := \mathrm{D}\Phi - (\Phi^* \rho)(A), \tag{9}$$

where $\mathrm{D}\Phi : \mathrm{T}M \to \mathrm{T}N$ is the tangent map.

Definition (Space of fields)

Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
- Field of gauge bosons $A \in \Omega^1(M; \Phi^* E)$

Definition (Minimal coupling)

Minimal coupling $\mathfrak{D}$, $(\Phi, A) \mapsto \mathfrak{D}(\Phi, A) \in \Omega^1(M; \Phi^* \mathrm{T}N)$, by

$$\mathfrak{D}(\Phi, A) := \mathfrak{D}^A \Phi := \mathrm{D}\Phi - (\Phi^* \rho)(A), \tag{9}$$

where $\mathrm{D}\Phi : \mathrm{T}M \to \mathrm{T}N$ is the tangent map.

Definition (Field strength)

Let ∇ be a connection on E. We define the **field strength** F, $(\Phi, A) \mapsto F(\Phi, A) \in \Omega^2(M; \Phi^* E)$, by

$$F(\Phi, A) := \mathrm{d}^{\Phi^*\nabla} A + \frac{1}{2}(\Phi^* t_{\nabla^{\mathrm{bas}}})(A \overset{\wedge}{,} A), \tag{10}$$

where $t_{\nabla^{\mathrm{bas}}}$ is the torsion of ∇^{bas} on E and $\mathrm{d}^{\Phi^*\nabla}$ the exterior covariant derivative of $\Phi^*\nabla$.

Definition (Field strength)

Let ∇ be a connection on E. We define the **field strength** F, $(\Phi, A) \mapsto F(\Phi, A) \in \Omega^2(M; \Phi^*E)$, by

$$F(\Phi, A) := \mathrm{d}^{\Phi^*\nabla} A + \frac{1}{2}(\Phi^* t_{\nabla^{\mathrm{bas}}})(A \overset{\wedge}{,} A), \tag{10}$$

where $t_{\nabla^{\mathrm{bas}}}$ is the torsion of ∇^{bas} on E and $\mathrm{d}^{\Phi^*\nabla}$ the exterior covariant derivative of $\Phi^*\nabla$.

Definition (Generalised field strength)

Let ζ be an element of $\Omega^2(N; E)$, then we define the **generalised field strength** $\mathscr{F}$ by

$$\mathscr{F}(\Phi, A) := F(\Phi, A) + \frac{1}{2}(\Phi^*\zeta)\left(\mathfrak{D}^A\Phi \overset{\wedge}{,} \mathfrak{D}^A\Phi\right). \tag{11}$$

Definition (Curved Yang-Mills-Higgs (CYMH) Lagrangian)

Let κ be a fibre metric on E, then the **curved Yang-Mills-Higgs Lagrangian** $\mathfrak{L}_{\mathrm{CYMH}}$, $(\Phi, A) \mapsto \mathfrak{L}_{\mathrm{CYMH}}(\Phi, A) \in \Omega^{\dim(M)}(M)$, is defined by

$$\mathfrak{L}_{\mathrm{CYMH}}(\Phi, A) := -\frac{1}{2}(\Phi^*\kappa)\left(\mathscr{F}(\Phi, A) \overset{\wedge}{,} *\mathscr{F}(\Phi, A)\right) + (\Phi^*g)\left(\mathfrak{D}^A\Phi \overset{\wedge}{,} *\mathfrak{D}^A\Phi\right) - *(\Phi^*\mathscr{V}), \qquad (12)$$

where $*$ is the Hodge star operator related to the spacetime metric η.

Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_\nabla + \mathrm{d}^{\nabla^{\mathrm{bas}}}\zeta = 0, \tag{13}$$

$$R_\nabla^{\mathrm{bas}} = 0, \tag{14}$$

$$\nabla^{\mathrm{bas}}\kappa = 0, \tag{15}$$

$$\nabla^{\mathrm{bas}}g = 0, \tag{16}$$

$$\mathscr{L}_\rho \mathscr{V} = 0, \tag{17}$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

We say that we have a **pre-classical gauge theory**, if ∇ is flat.

Alexei Kotov and Thomas Strobl. Curving Yang-Mills-Higgs gauge theories. *Physical Review D*, 92(8):085032, 2015.

Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_\nabla + \mathrm{d}^{\nabla^{\mathrm{bas}}}\zeta = 0, \tag{13}$$

$$R_\nabla^{\mathrm{bas}} = 0, \tag{14}$$

$$\nabla^{\mathrm{bas}}\kappa = 0, \tag{15}$$

$$\nabla^{\mathrm{bas}} g = 0, \tag{16}$$

$$\mathscr{L}_\rho \mathscr{V} = 0, \tag{17}$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

We say that we have a **pre-classical gauge theory**, if ∇ is flat.

If we have additionally $\zeta \equiv 0$, then we say that we have a **classical gauge theory**.

Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_\nabla + \mathrm{d}^{\nabla^{\mathrm{bas}}}\zeta = 0, \tag{13}$$

$$R_\nabla^{\mathrm{bas}} = 0, \tag{14}$$

$$\nabla^{\mathrm{bas}}\kappa = 0, \tag{15}$$

$$\nabla^{\mathrm{bas}} g = 0, \tag{16}$$

$$\mathscr{L}_\rho \mathscr{V} = 0, \tag{17}$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

We say that we have a **pre-classical gauge theory**, if ∇ is flat.

If we have additionally $\zeta \equiv 0$, then we say that we have a **classical gauge theory**.

Alexei Kotov and Thomas Strobl. Curving Yang-Mills-Higgs gauge theories. *Physical Review D*, 92(8):085032, 2015.

Summary

$$\begin{tikzcd}
\Phi^* E \arrow[d] & (E, [\cdot,\cdot]_E, \kappa, \nabla) \arrow[d] \arrow[dr, "\rho"] & \\
M \arrow[r, "\Phi"] & (N, g) & \mathrm{T}N \arrow[l] \arrow[ul, bend right, "\zeta\in\Omega^2"']
\end{tikzcd}$$

$\rightsquigarrow$ Together with the compatibility conditions we have gauge invariance, that is,

$$\delta_\varepsilon \mathfrak{L}_{\mathrm{CYMH}} = 0. \tag{18}$$

Remarks ([S.-R. F.])

If $\rho \equiv 0$, then $\varepsilon \in \Gamma(\Phi^* E)$ and

$$\delta_\varepsilon \Phi = 0, \qquad \delta_\varepsilon A = [\varepsilon, A]_E - \mathrm{d}^{\Phi^*\nabla} \varepsilon.$$

Summary

$$
\begin{tikzcd}
\Phi^* E \arrow[d] & (E, [\cdot,\cdot]_E, \kappa, \nabla) \arrow[d] \arrow[dr, "\rho"] & \\
M \arrow[r, "\Phi"] & (N, g) & \mathrm{T}N \arrow[l] \arrow[ul, bend right, "\zeta\in\Omega^2"']
\end{tikzcd}
$$

$\rightsquigarrow$ Together with the compatibility conditions we have gauge invariance, that is,

$$\delta_\varepsilon \mathfrak{L}_{\mathrm{CYMH}} = 0. \tag{18}$$

Remarks ([S.-R. F.])

If $\rho \equiv 0$, then $\varepsilon \in \Gamma(\Phi^* E)$ and

$$\delta_\varepsilon \Phi = 0, \qquad \delta_\varepsilon A = [\varepsilon, A]_E - \mathrm{d}^{\Phi^*\nabla} \varepsilon.$$

Sources Alexei Kotov and Thomas Strobl. Curving Yang-Mills-Higgs gauge theories. *Physical Review D*, 92(8):085032, 2015.

Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

  This was provided by Edward Witten in a private communication with Thomas Strobl about a specific example of a CYMH GT.

Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

  This was provided by Edward Witten in a private communication with Thomas Strobl about a specific example of a CYMH GT.

Definition (Field redefinition, [S.-R. F.])

Let $\lambda \in \Omega^1(N; E)$ such that $\Lambda := \mathbb{1}_E - \lambda \circ \rho$ is an automorphism of E. We then define the **field redefinitions** by

$$\widetilde{A}^\lambda := (\Phi^* \Lambda)(A) + \Phi^! \lambda, \tag{19}$$

$$\widetilde{\nabla}^\lambda := \nabla + \left(\Lambda \circ \mathrm{d}^{\nabla^{\mathrm{bas}}} \circ \Lambda^{-1} \right) \lambda, \tag{20}$$

$$\widetilde{\kappa}^\lambda := \kappa \circ \left(\Lambda^{-1}, \Lambda^{-1} \right), \tag{21}$$

$$\widetilde{g}^\lambda := g \circ \left(\widehat{\Lambda}^{-1}, \widehat{\Lambda}^{-1} \right), \tag{22}$$

where $\widehat{\Lambda} := \mathbb{1}_{\mathrm{T}N} - \rho \circ \lambda$, and for all $X, Y \in \mathfrak{X}(N)$ we also define

$$\begin{aligned} &\widetilde{\zeta}^\lambda \left(\widehat{\Lambda}(X), \widehat{\Lambda}(Y) \right) \\ &:= \Lambda(\zeta(X, Y)) - \left(\mathrm{d}^{\widetilde{\nabla}^\lambda} \lambda \right)(X, Y) + t_{\widetilde{\nabla}^\lambda_\rho}(\lambda(X), \lambda(Y)). \end{aligned} \tag{23}$$

Proposition ([S.-R. F.])

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\widetilde{\mathfrak{L}}^{\lambda}_{\mathrm{CYMH}} = \mathfrak{L}_{\mathrm{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

Proposition ([S.-R. F.])

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\widetilde{\mathfrak{L}}^{\lambda}_{\mathrm{CYMH}} = \mathfrak{L}_{\mathrm{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathcal{g}$ an LAB $(\rho \equiv 0)$ with a field of Lie brackets $[\cdot, \cdot]_{\mathcal{g}} \in \Gamma\left(\bigwedge^2 \mathcal{g}^* \otimes \mathcal{g}\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathscr{g}$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot, \cdot]_{\mathscr{g}} \in \Gamma\left(\bigwedge^2 \mathscr{g}^* \otimes \mathscr{g}\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

Compatibilities:

- κ needs to be ad-invariant

What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathcal{g}$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot, \cdot]_{\mathcal{g}} \in \Gamma\left(\bigwedge^2 \mathcal{g}^* \otimes \mathcal{g}\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

Compatibilities:

- κ needs to be ad-invariant
- We need

$$\nabla_Y\left([\mu, \nu]_{\mathcal{g}}\right) = [\nabla_Y \mu, \nu]_{\mathcal{g}} + [\mu, \nabla_Y \nu]_{\mathcal{g}}, \tag{24}$$

$$R_\nabla(Y, Z)\mu = [\zeta(Y, Z), \mu]_{\mathcal{g}} \tag{25}$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu, \nu \in \Gamma(\mathcal{g})$.

What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathscr{g}$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot,\cdot]_{\mathscr{g}} \in \Gamma\left(\bigwedge^2 \mathscr{g}^* \otimes \mathscr{g}\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

Compatibilities:

- κ needs to be ad-invariant
- We need

$$\nabla_Y\left([\mu,\nu]_{\mathscr{g}}\right) = [\nabla_Y\mu,\nu]_{\mathscr{g}} + [\mu,\nabla_Y\nu]_{\mathscr{g}}, \tag{24}$$

$$R_\nabla(Y,Z)\mu = [\zeta(Y,Z),\mu]_{\mathscr{g}} \tag{25}$$

for all $Y,Z \in \mathfrak{X}(N)$ and $\mu,\nu \in \Gamma(\mathscr{g})$.

Theorem (Invariant for LABs, [S.-R. F.])

We have

$$\mathrm{d}^{\widetilde{\nabla}^\lambda}\widetilde{\zeta}^\lambda = \mathrm{d}^\nabla \zeta, \tag{26}$$

and $\mathrm{d}^\nabla \zeta$ *has values in the centre of* $\mathscr{g}$.

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^\nabla \zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^\lambda = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^\nabla \zeta \neq 0$.

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^{\nabla}\zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^{\lambda} = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^{\nabla}\zeta \neq 0$.

However, by $R_{\nabla} = \mathrm{ad}_{\mathscr{g}} \circ \zeta$ it may still be that ∇ becomes flat.

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^{\nabla}\zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^{\lambda} = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^{\nabla}\zeta \neq 0$.

However, by $R_{\nabla} = \mathrm{ad}_{\mathscr{g}} \circ \zeta$ it may still be that ∇ becomes flat.

Turning to the field redefinition of ∇:

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ restricts to the centre of $\mathcal{g}$ and induces a differential d^{Ξ} on centre-valued forms. Moreover, d^{Ξ} is independent of the field redefinitions.

Sketch of proof.

Recall

$$\nabla_Y\left([\mu, \nu]_{\mathcal{g}}\right) = [\nabla_Y \mu, \nu]_{\mathcal{g}} + [\mu, \nabla_Y \nu]_{\mathcal{g}},$$

$$R_{\nabla}(Y, Z)\mu = [\zeta(Y, Z), \mu]_{\mathcal{g}},$$

$$\widetilde{\nabla}^{\lambda}_Y \mu = \nabla_Y \mu - [\lambda(Y), \mu]_{\mathcal{g}},$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu, \nu \in \Gamma(\mathcal{g})$. Then insert μ with values in the centre.

Turning to the field redefinition of ∇:

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ restricts to the centre of $\mathcal{g}$ and induces a differential d^{Ξ} on centre-valued forms. Moreover, d^{Ξ} is independent of the field redefinitions.

Sketch of proof.

Recall

$$\nabla_Y\left([\mu,\nu]_{\mathcal{g}}\right) = [\nabla_Y\mu,\nu]_{\mathcal{g}} + [\mu,\nabla_Y\nu]_{\mathcal{g}},$$
$$R_{\nabla}(Y,Z)\mu = [\zeta(Y,Z),\mu]_{\mathcal{g}},$$
$$\widetilde{\nabla}^{\lambda}_Y\mu = \nabla_Y\mu - [\lambda(Y),\mu]_{\mathcal{g}},$$

for all $Y,Z \in \mathfrak{X}(N)$ and $\mu,\nu \in \Gamma(\mathcal{g})$. Then insert μ with values in the centre.

Theorem (Closedness of $\mathrm{d}^\nabla \zeta$, [S.-R. F.])

We have

$$\mathrm{d}^\Xi \mathrm{d}^\nabla \zeta = 0. \tag{27}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^\nabla \zeta\right]_{\mathrm{d}^\Xi}. \tag{28}$$

Theorem (Closedness of $\mathrm{d}^{\nabla}\zeta$, [S.-R. F.])

We have

$$\mathrm{d}^{\Xi}\mathrm{d}^{\nabla}\zeta = 0. \tag{27}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^{\nabla}\zeta\right]_{\mathrm{d}^{\Xi}}. \tag{28}$$

Proposition ([S.-R. F.])

- *An invariant of the field redefinitions.*

Theorem (Closedness of $\mathrm{d}^\nabla\zeta$, [S.-R. F.])

We have

$$\mathrm{d}^\Xi \mathrm{d}^\nabla \zeta = 0. \tag{27}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^\nabla \zeta\right]_{\mathrm{d}^\Xi}. \tag{28}$$

Proposition ([S.-R. F.])

- *An invariant of the field redefinitions.*
- *If ∇ flat, then $\mathrm{Obs}(\Xi) = 0$.*

Theorem (Closedness of $\mathrm{d}^\nabla \zeta$, [S.-R. F.])

We have

$$\mathrm{d}^\Xi \mathrm{d}^\nabla \zeta = 0. \tag{27}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^\nabla \zeta\right]_{\mathrm{d}^\Xi}. \tag{28}$$

Proposition ([S.-R. F.])

- *An invariant of the field redefinitions.*
- *If ∇ flat, then $\mathrm{Obs}(\Xi) = 0$.*

Theorem (Obstruction for non-pre-classical theories, [S.-R. F.])

If $\mathrm{Obs}(\Xi) \neq 0$*, then there is no field redefinition such that* $\widetilde{\nabla}^\lambda$ *is flat.*

Theorem (Locally always pre-classical)

If N *is contractible, then there is a field redefinition such that* $\widetilde{\nabla}^\lambda$ *is flat.*

Theorem (Obstruction for non-pre-classical theories, [S.-R. F.])

If $\mathrm{Obs}(\Xi) \neq 0$*, then there is no field redefinition such that* $\widetilde{\nabla}^\lambda$ *is flat.*

Theorem (Locally always pre-classical)

If N *is contractible, then there is a field redefinition such that* $\widetilde{\nabla}^\lambda$ *is flat.*

Remarks

Second theorem follows as a result by K. Mackenzie (General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005). Mackenzie derived $\mathrm{Obs}(\Xi)$ in the context of extending Lie algebroids by LABs.

Theorem (Obstruction for non-pre-classical theories, [S.-R. F.])

If $\mathrm{Obs}(\Xi) \neq 0$*, then there is no field redefinition such that* $\widetilde{\nabla}^\lambda$ *is flat.*

Theorem (Locally always pre-classical)

If N *is contractible, then there is a field redefinition such that* $\widetilde{\nabla}^\lambda$ *is flat.*

Remarks

Second theorem follows as a result by K. Mackenzie (General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005). Mackenzie derived $\mathrm{Obs}(\Xi)$ in the context of extending Lie algebroids by LABs.

Example (Zero obstruction class not necessarily pre-classical)

Let P be the Hopf fibration

$$\begin{tikzcd} \mathrm{SU}(2) \arrow[r] & \mathbb{S}^7 \arrow[d] \\ & \mathbb{S}^4 \end{tikzcd}$$

Then for the adjoint bundle

$$\mathcal{g} := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left(\mathbb{S}^7 \times \mathfrak{su}(2)\right) \Big/ \mathrm{SU}(2)$$

we have a non-flat ∇ satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but $\mathrm{Obs}(\Xi) = 0$.

Sources
K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical. In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $\mathbb{S}^7 \to \mathbb{S}^4$?

$\rightsquigarrow$ Integration

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical. In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $\mathbb{S}^7 \to \mathbb{S}^4$?
$\rightsquigarrow$ Integration

Idea

Remarks

Observe that the Higgs field Φ doesn't really matter at all in the case of LABs
$\rightsquigarrow$ Think in terms of bundles over M directly

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathscr{g}$
Integrated	Lie group G	LGB[1] $\mathscr{G}$?

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

[1] LGB = Lie group bundle

Idea

Remarks

Observe that the Higgs field Φ doesn't really matter at all in the case of LABs

$\rightsquigarrow$ Think in terms of bundles over M directly

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathcal{g}$
Integrated	Lie group G	LGB[1] $\mathscr{G}$?

$$\begin{array}{ccc} G & \longrightarrow & \mathscr{G} \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

[1] LGB = Lie group bundle

Definition (LGB actions, simplified)

$$\begin{array}{ccc} & & \mathscr{G} \\ & & \downarrow \\ \mathscr{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathscr{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of** $\mathscr{G}$ **on** $\mathscr{P}$ is a smooth map $\mathscr{P} * \mathscr{G} := \pi^*\mathscr{G} = \mathscr{P} \times_M \mathscr{G} \to \mathscr{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \tag{29}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{30}$$

$$p \cdot e_{\pi(p)} = p \tag{31}$$

for all $p \in \mathscr{P}$ and $g, h \in \mathscr{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathscr{G}_{\pi(p)}$.

Sources
K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.

$\Rightarrow$ Think of the "classical" theory as coming from a trivial LGB

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $\mathscr{P} * \mathscr{G} \cong \mathscr{P} \times G$, and we can define

$$\mathscr{P} \times G \to \mathscr{P},$$
$$(p, g) \mapsto p \cdot g := p \cdot (\pi(p), g),$$

which is equivalent to $\mathscr{P} * \mathscr{G} \to \mathscr{P}$.
$\Rightarrow$ Think of the "classical" theory as coming from a trivial LGB

Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

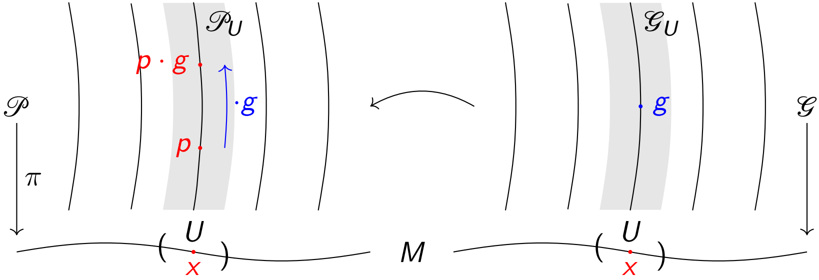
but with $\mathscr{G}$-action

$$\cancel{\mathscr{P} \times G} \to \mathscr{P}$$
$$\mathscr{P} * \mathscr{G}$$

simply transitive on fibres of $\mathscr{P}$, and "suitable" atlas.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids.
Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003

Connection on $\mathscr{P}$: Idea

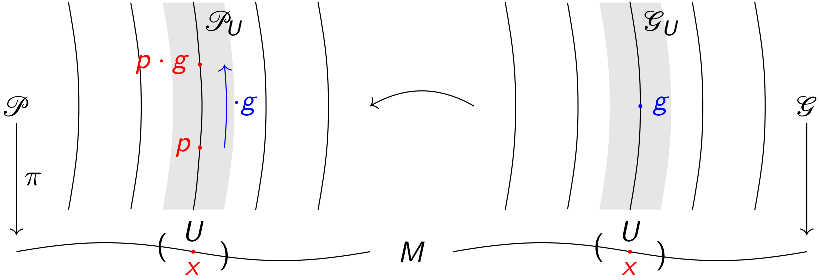


But:

$$r_g : \mathscr{P}_x \to \mathscr{P}_x$$

$\Rightarrow$ $\mathrm{D}_p r_g$ only defined on vertical structure

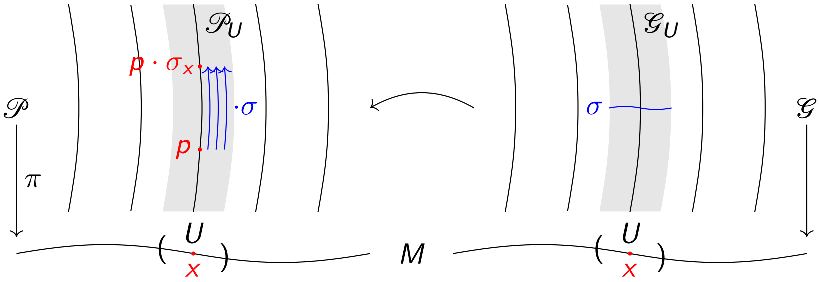
Connection on $\mathscr{P}$: Idea



But:

$$
\begin{aligned}
& r_g : \mathscr{P}_x \to \mathscr{P}_x \\
\Rightarrow \quad & \mathrm{D}_p r_g \text{ only defined on vertical structure}
\end{aligned}
$$

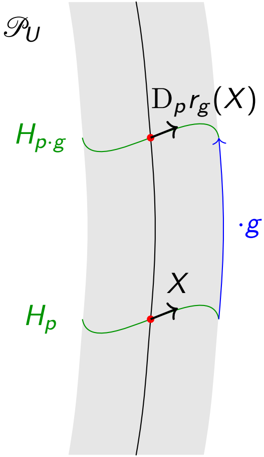
Connection on $\mathscr{P}$: Idea



Use $\sigma \in \Gamma(\mathscr{G}) : r_\sigma(p) := p \cdot \sigma_{\pi(p)}$

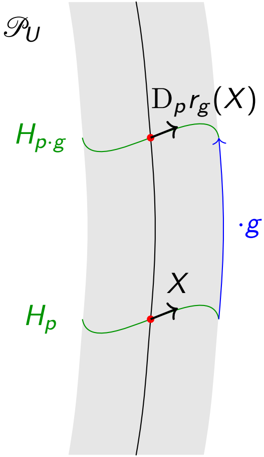
Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle
and H a connection on it:



Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle and H a connection on it:



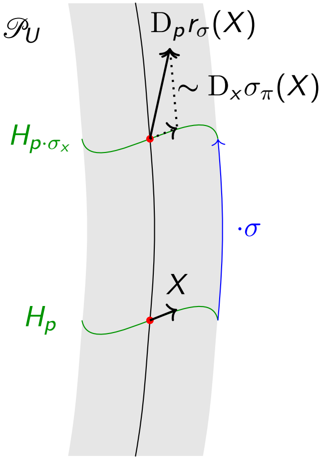
$\mathscr{G}$ trivial, $\sigma \equiv g$ constant:

$$\mathrm{D}_p r_g(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha \cdot g),$$

where α is the flow of X

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle and H a connection on it:

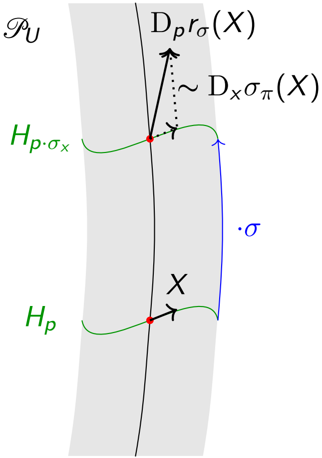


Now:

$$\mathrm{D}_p r_g(X) + \ldots = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha \cdot \sigma_{\pi\circ\alpha})$$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle and H a connection on it:



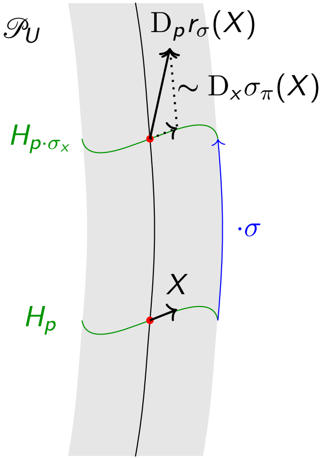
Now:

$$\mathrm{D}_p r_g(X) + \ldots = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha \cdot \sigma_{\pi\circ\alpha})$$

- Only vertical part in $\mathscr{G}$ of $\mathrm{D}\sigma$ matters
- Shift is vertical in $\mathscr{P}$

Connection on $\mathscr{P}$: Revisiting the classical setup

If $\mathscr{P}$ a typical principal bundle and H a connection on it:



Now:

$$\mathrm{D}_p r_g(X) + \ldots = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha \cdot \sigma_{\pi\circ\alpha})$$

- Only vertical part in $\mathscr{G}$ of $\mathrm{D}\sigma$ matters
- Shift is vertical in $\mathscr{P}$

Remarks (General situation)

Introduce connection on $\mathscr{G}$
$\Rightarrow \nabla$ on the LAB $\mathscr{g}$ of $\mathscr{G}$

Summary

	Locally	Globally
Curved Yang-Mills	Pre-classical	$\mathrm{ad}(\mathbb{S}^7 \to \mathbb{S}^4)$ curved

Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial
$\rightsquigarrow$ LGB action locally equivalent to a Lie group action

Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle $\mathscr{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathfrak{G}$?

$$\begin{tikzcd} & \mathfrak{G} \arrow[dl] \arrow[dr] & \\ M \arrow[rr, "\Phi"] & & N \end{tikzcd}$$

Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!