

# Geometry of curved Yang-Mills-Higgs gauge theories

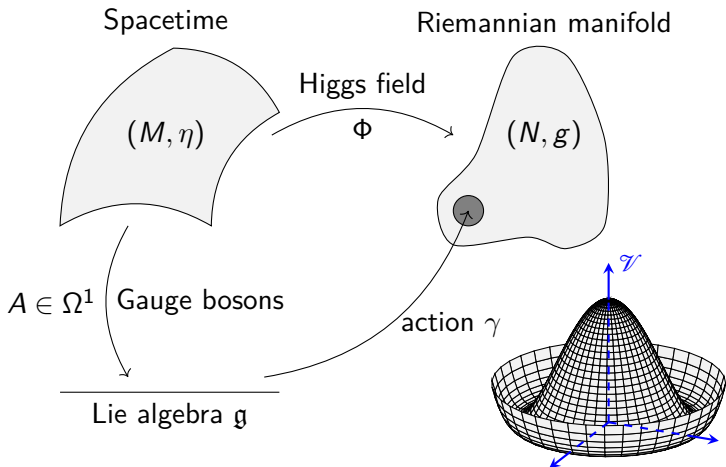
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# Table of contents

- 1 Curved Yang-Mills-Higgs gauge theory
  - Motivation
  - Mathematical basics
  - Curved Yang-Mills-Higgs gauge theory
- 2 Field Redefinition
  - General definition
  - Lie algebra bundles
- 3 Integration (WIP)
  - Principal bundles based on Lie group bundle actions
  - Connections
- 4 Conclusion

# Infinitesimal gauge theory



# Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \rightarrow N$
$\mathfrak{g}$ -action $\gamma$	Anchor $\rho$ of $E$ & $E$ -connections
Canonical flat connection $\nabla^0$ on $M \times \mathfrak{g}$	General connection $\nabla$ on $E$

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## Remarks (Why a "curved theory"?)

Usually, the field strength  $F$  is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

$\rightsquigarrow$  We will use a general connection  $\nabla$  instead of  $\nabla^0$ , and  $\nabla$  may not be flat.

## Definition (Lie algebroids)

Let  $E \rightarrow N$  be a vector bundle. Then  $E$  is a Lie algebroid, if there is a bundle map  $\rho : E \rightarrow TN$ , called the **anchor**, and a Lie algebra structure on  $\Gamma(E)$  with Lie bracket  $[\cdot, \cdot]_E$  satisfying

$$[\mu, f\nu]_E = f[\mu, \nu]_E + \mathcal{L}_{\rho(\mu)}(f) \nu \quad (1)$$

for all  $f \in C^\infty(N)$  and  $\mu, \nu \in \Gamma(E)$ .

## Example

- $E = TN, \rho = \mathbb{1}_{TN}$

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As for an action  $\gamma : \mathfrak{g} \rightarrow \mathfrak{X}(N)$  we have:

**Proposition (Anchor as homomorphism)**

$\rho : \Gamma(E) \rightarrow \mathfrak{X}(N)$  is a homomorphism of Lie algebras.

The classical formalism will correspond to:

### Proposition (Action Lie algebroids)

Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$  be a Lie algebra with a  $\mathfrak{g}$ -action  $\gamma$  on  $N$ . Then there is a **unique** Lie algebroid structure on  $E := N \times \mathfrak{g}$  as a vector bundle over  $N$  such that

$$\rho(\nu) = \gamma(\nu), \quad (2)$$

$$[\mu, \nu]_E = [\mu, \nu]_{\mathfrak{g}} \quad (3)$$

for all constant sections  $\mu, \nu \in \Gamma(E)$ .

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Canonical flat connection $\nabla^0$ on $M \times \mathfrak{g}$	General connection $\nabla$ on $E$

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### Example ( $E$ -connection ${}^E\nabla$ on $V$ )

$\nabla'$  a vector bundle connection on  $V \rightarrow N$ , then

$${}^E\nabla_\nu v := \nabla'_{\rho(\nu)} v \quad (4)$$

for all  $\nu \in \Gamma(E)$  and  $v \in \Gamma(V)$ . In short denoted by  $\nabla'_\rho$ .

## Example

For  $\nabla$  a connection on  $E$  we have the **basic connection** given as a pair of  $E$ -connections on  $E$  and on  $\mathbb{T}N$  by

$$\nabla_{\nu}^{\text{bas}} \mu := [\nu, \mu]_E + \nabla_{\rho(\mu)} \nu, \quad (5)$$

$$\nabla_{\nu}^{\text{bas}} X := [\rho(\nu), X] + \rho(\nabla_X \nu) \quad (6)$$

for all  $X \in \mathfrak{X}(N)$  and  $\nu, \mu \in \Gamma(E)$ .

## Remarks (Encoding of Lie algebra representations)

Test this with trivial bundles and canonical flat connection  $\nabla^0$ ,  
i.e.  $E = N \times \mathfrak{g}$  and  $\nabla^0 \nu = 0$  for constant sections  $\nu$ .

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$$R_{\nabla}^{\text{bas}}(\mu, \nu)X := \nabla_X([\mu, \nu]_E) - [\nabla_X \mu, \nu]_E - [\mu, \nabla_X \nu]_E - \nabla_{\nabla_{\nu}^{\text{bas}} X} \mu + \nabla_{\nabla_{\mu}^{\text{bas}} X} \nu, \quad (7)$$

where  $\mu, \nu \in \Gamma(E)$  and  $X \in \mathfrak{X}(N)$ .

## Proposition

*We recover the curvature of the basic connection:*

$$R_{\nabla^{\text{bas}}} = \begin{cases} -R_{\nabla}^{\text{bas}}(\cdot, \cdot) \circ \rho, & \text{on } E, \\ -\rho \circ R_{\nabla}^{\text{bas}}, & \text{on } TN. \end{cases} \quad (8)$$



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Fields are a pair consisting of:

- Higgs field  $\Phi \in C^\infty(M; N)$
- Field of gauge bosons  $A \in \Omega^1(M; \Phi^* E)$

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**Minimal coupling**  $\mathfrak{D}$ ,  $(\Phi, A) \mapsto \mathfrak{D}(\Phi, A) \in \Omega^1(M; \Phi^* TN)$ , by

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where  $D\Phi : TM \rightarrow TN$  is the tangent map.

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Let  $\nabla$  be a connection on  $E$ . We define the **field strength**  $F$ ,  $(\Phi, A) \mapsto F(\Phi, A) \in \Omega^2(M; \Phi^* E)$ , by

$$F(\Phi, A) := d^{\Phi^* \nabla} A + \frac{1}{2}(\Phi^* t_{\nabla^{\text{bas}}})(A \wedge A), \quad (10)$$

where  $t_{\nabla^{\text{bas}}}$  is the torsion of  $\nabla^{\text{bas}}$  on  $E$  and  $d^{\Phi^* \nabla}$  the exterior covariant derivative of  $\Phi^* \nabla$ .

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## Definition (Generalised field strength)

Let  $\zeta$  be an element of  $\Omega^2(N; E)$ , then we define the **generalised field strength**  $G$  by

$$G(\Phi, A) := F(\Phi, A) + \frac{1}{2}(\Phi^*\zeta)(\mathfrak{D}^A\Phi \wedge \mathfrak{D}^A\Phi). \quad (11)$$

## Definition (Curved Yang-Mills-Higgs (CYMH) Lagrangian)

Let  $\kappa$  be a fibre metric on  $E$ , then the **curved Yang-Mills-Higgs Lagrangian**  $\mathfrak{L}_{\text{CYMH}}$ ,  $(\Phi, A) \mapsto \mathfrak{L}_{\text{CYMH}}(\Phi, A) \in \Omega^{\dim(M)}(M)$ , is defined by

$$\begin{aligned} \mathfrak{L}_{\text{CYMH}}(\Phi, A) := & -\frac{1}{2}(\Phi^*\kappa)(G(\Phi, A) \wedge *G(\Phi, A)) \\ & + (\Phi^*g)\left(\mathfrak{D}^A\Phi \wedge *\mathfrak{D}^A\Phi\right) - *(\Phi^*\mathcal{V}), \quad (12) \end{aligned}$$

where  $*$  is the Hodge star operator related to the spacetime metric  $\eta$ .



## Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_{\nabla} + d^{\nabla^{\text{bas}}} \zeta = 0, \quad (13)$$

$$R_{\nabla}^{\text{bas}} = 0, \quad (14)$$

$$\nabla^{\text{bas}} \kappa = 0, \quad (15)$$

$$\nabla^{\text{bas}} g = 0, \quad (16)$$

$$\mathcal{L}_{\rho} \mathcal{V} = 0, \quad (17)$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

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# Summary

$$\begin{array}{ccccc}
 \Phi^* E & & (E, [\cdot, \cdot]_E, \kappa, \nabla) & \xleftarrow{\zeta \in \Omega^2} & \\
 \downarrow & & \downarrow & \searrow \rho & \\
 M & \xrightarrow{\Phi} & (N, g) & \xleftarrow{\quad} & TN
 \end{array}$$

$\rightsquigarrow$  Together with the compatibility conditions we have gauge invariance, that is,

$$\delta_\varepsilon \mathfrak{L}_{\text{CYMH}} = 0. \quad (18)$$

## Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

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## Definition (Field redefinition, [S.-R. F.]

Let  $\lambda \in \Omega^1(N; E)$  such that  $\Lambda := \mathbb{1}_E - \lambda \circ \rho$  is an automorphism of  $E$ . We then define the **field redefinitions** by

$$\tilde{A}^\lambda := (\Phi^* \Lambda)(A) + \Phi^! \lambda, \quad (19)$$

$$\tilde{\nabla}^\lambda := \nabla + \left( \Lambda \circ d^{\nabla^{\text{bas}}} \circ \Lambda^{-1} \right) \lambda, \quad (20)$$

$$\tilde{\kappa}^\lambda := \kappa \circ \left( \Lambda^{-1}, \Lambda^{-1} \right), \quad (21)$$

$$\tilde{g}^\lambda := g \circ \left( \hat{\Lambda}^{-1}, \hat{\Lambda}^{-1} \right), \quad (22)$$

where  $\hat{\Lambda} := \mathbb{1}_{TN} - \rho \circ \lambda$ , and for all  $X, Y \in \mathfrak{X}(N)$  we also define

$$\begin{aligned} & \tilde{\zeta}^\lambda \left( \hat{\Lambda}(X), \hat{\Lambda}(Y) \right) \\ & := \Lambda(\zeta(X, Y)) - \left( d^{\tilde{\nabla}^\lambda} \lambda \right)(X, Y) + t_{\tilde{\nabla}^\lambda \rho}(\lambda(X), \lambda(Y)). \end{aligned} \quad (23)$$

## Proposition ([S.-R. F.]

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\tilde{\mathcal{L}}_{\text{CYMH}}^\lambda = \mathcal{L}_{\text{CYMH}}$

Let us now apply a field redefinition in order to study whether  $\nabla$  and  $\zeta$  can become flat and zero, respectively.



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# What happens in the case of Lie algebra bundles?

## Example (Lie algebra bundles (LABs))

- $E = \mathfrak{g}$  an LAB ( $\rho \equiv 0$ ) with a field of Lie brackets  $[\cdot, \cdot]_{\mathfrak{g}} \in \Gamma(\wedge^2 \mathfrak{g}^* \otimes \mathfrak{g})$  which restricts to the bracket of a given Lie algebra  $\mathfrak{g}$

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Compatibilities:

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Compatibilities:

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- We need

$$\nabla_Y([\mu, \nu]_{\mathcal{g}}) = [\nabla_Y \mu, \nu]_{\mathcal{g}} + [\mu, \nabla_Y \nu]_{\mathcal{g}}, \quad (24)$$

$$R_{\nabla}(Y, Z)\mu = [\zeta(Y, Z), \mu]_{\mathcal{g}} \quad (25)$$

for all  $Y, Z \in \mathfrak{X}(N)$  and  $\mu, \nu \in \Gamma(\mathcal{g})$ .

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### Theorem (Invariant for LABs, [S.-R. F.]

*We have*

$$d^{\tilde{\nabla}^\lambda} \tilde{\zeta}^\lambda = d^\nabla \zeta, \quad (26)$$

*and  $d^\nabla \zeta$  has values in the centre of  $\mathfrak{g}$ .*

# Behaviour of the field redefinition of $\zeta$

Theorem (Existence of non-classical theories, [S.-R. F.])

*If  $d^\nabla \zeta \neq 0$ , then there is no field redefinition such that  $\tilde{\zeta}^\lambda = 0$ .*

## Remarks

Starting with a classical theory:

If  $\dim(N) \geq 3$  and if Lie algebra  $\mathfrak{g}$  has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a  $\zeta$  with  $d^\nabla \zeta \neq 0$ .

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# Turning to the field redefinition of $\nabla$ :

Theorem (Differential on centre-valued forms, [S.-R. F.])

$\nabla$  restricts to the centre of  $\mathfrak{g}$  and induces a differential  $d^\Xi$  on centre-valued forms. Moreover,  $d^\Xi$  is independent of the field redefinitions.

Sketch of proof.

Recall

$$\begin{aligned}\nabla_Y([\mu, \nu]_{\mathfrak{g}}) &= [\nabla_Y \mu, \nu]_{\mathfrak{g}} + [\mu, \nabla_Y \nu]_{\mathfrak{g}}, \\ R_\nabla(Y, Z)\mu &= [\zeta(Y, Z), \mu]_{\mathfrak{g}}, \\ \tilde{\nabla}_Y^\lambda \mu &= \nabla_Y \mu - [\lambda(Y), \mu]_{\mathfrak{g}},\end{aligned}$$

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## Theorem (Closedness of $d^\nabla \zeta$ , [S.-R. F.])

*We have*

$$d^\Xi d^\nabla \zeta = 0. \quad (27)$$

## Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\text{Obs}(\Xi) := [d^\nabla \zeta]_{d^\Xi}. \quad (28)$$

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*If  $\text{Obs}(\Xi) \neq 0$ , then there is no field redefinition such that  $\tilde{\nabla}^\lambda$  is flat.*

**Theorem (Locally always pre-classical)**

*If  $N$  is contractible, then there is a field redefinition such that  $\tilde{\nabla}^\lambda$  is flat.*



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### Remarks

Second theorem follows as a result by K. Mackenzie (General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005). Mackenzie derived  $\text{Obs}(\Xi)$  in the context of extending Lie algebroids by LABs.

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*If  $N$  is contractible, then there is a field redefinition such that  $\tilde{\nabla}^\lambda$  is flat.*

### Remarks

Second theorem follows as a result by K. Mackenzie (General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005). Mackenzie derived  $\text{Obs}(\Xi)$  in the context of extending Lie algebroids by LABs.

## Example (Zero obstruction class not necessarily pre-classical)

Let  $P$  be the Hopf fibration

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{S}^4 \end{array}$$

Then for the adjoint bundle

$$\mathcal{Q} := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left( \mathbb{S}^7 \times \mathfrak{su}(2) \right) / \mathrm{SU}(2)$$

we have a non-flat  $\nabla$  satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but  $\mathrm{Obs}(\Xi) = 0$ .

# Summary

## Remarks

Locally, LABs are always pre-classical but not necessarily classical.  
In general,  $\text{Obs}(\Xi) = 0$  does not imply a flat connection.

So, what actually happens in the adjoint bundle of  $S^7 \rightarrow S^4$ ?  
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# Idea

## Remarks

Observe that the Higgs field  $\Phi$  doesn't really matter at all in the case of LABs

$\rightsquigarrow$  Think in terms of bundles over  $M$  directly

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathcal{G}$
Integrated	Lie group $G$	LGB <sup>1</sup> $\mathcal{G}$ ?

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## Definition (LGB actions)

$$\begin{array}{ccc}
 & \mathcal{G} & \\
 & \downarrow \pi & \\
 N & \xrightarrow{f} & M
 \end{array}$$

A **right-action of  $\mathcal{G}$  on  $N$**  is a smooth map  $N * \mathcal{G} := f^* \mathcal{G} \rightarrow N$ ,  $(p, g) \mapsto p \cdot g$ , satisfying the following properties:

$$f(p \cdot g) = \pi(g), \quad (29)$$

$$(p \cdot g) \cdot h = p \cdot (gh), \quad (30)$$

$$p \cdot e_{f(p)} = p \quad (31)$$

for all  $p \in N$  and  $g, h \in \mathcal{G}_{f(p)}$ , where  $e_{f(p)}$  is the neutral element of  $\mathcal{G}_{f(p)}$ .



# Examples

## Example

$\mathcal{G}$  acts canonically on itself:

$$\begin{aligned}\mathcal{G} * \mathcal{G} &\rightarrow \mathcal{G}, \\ (q, h) &\mapsto qh.\end{aligned}$$

## Example (Recovering Lie group action)

- Either by  $M = \{*\}$ .
- Or by  $\mathcal{G} \cong M \times G$ , then also  $N * \mathcal{G} \cong N \times G$ , and we can define

$$\begin{aligned}N \times G &\rightarrow N, \\ (p, g) &\mapsto p \cdot g := p \cdot (f(p), g),\end{aligned}$$

which is equivalent to  $N * \mathcal{G} \rightarrow N$ .

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## Definition (Principal bundle)

Still a fibre bundle

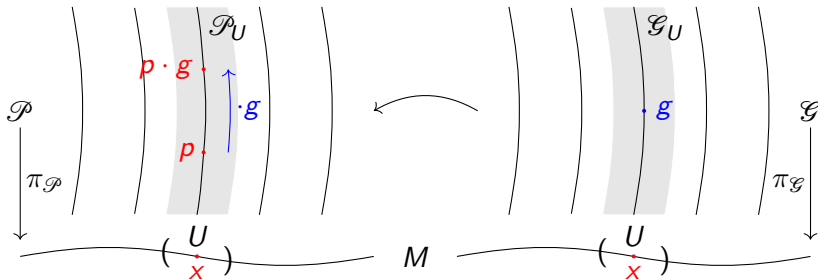
$$\begin{array}{ccc} G & \longrightarrow & \mathcal{P} \\ & & \downarrow \pi_{\mathcal{P}} \\ & & M \end{array}$$

but with  $\mathcal{G}$ -action

$$\begin{array}{ccc} \cancel{\mathcal{P}} \times \cancel{G} & \longrightarrow & \mathcal{P} \\ \mathcal{P} * \mathcal{G} & & \end{array}$$

simply transitive on fibres of  $\mathcal{P}$ , and "suitable" atlas.

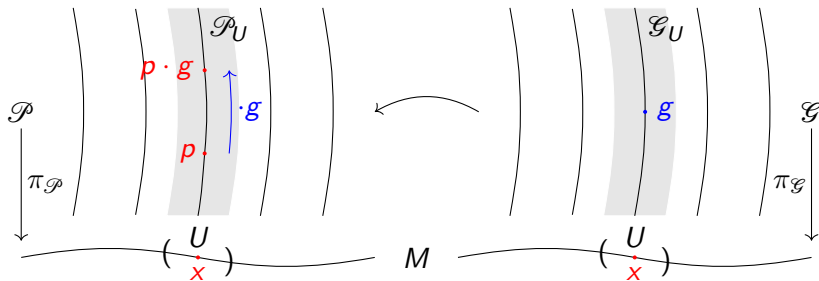
# Connection on $\mathcal{P}$ : Idea



But:

$$\begin{aligned} & r_g : \mathcal{P}_x \rightarrow \mathcal{P}_x \\ \Rightarrow & D_p r_g \text{ only defined on vertical structure} \end{aligned}$$

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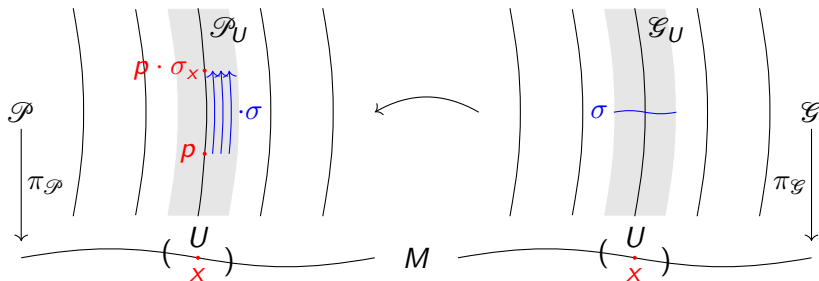


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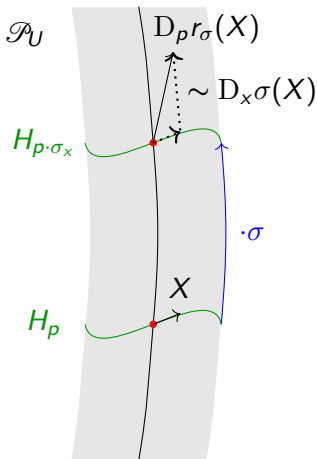
# Connection on $\mathcal{P}$ : Idea



Use  $\sigma \in \Gamma(\mathcal{G}) : r_{\sigma}(p) := p \cdot \sigma_{\pi_{\mathcal{P}}(p)}$

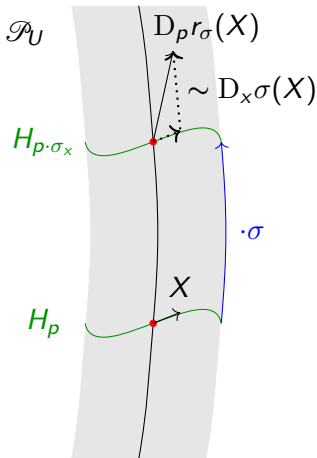
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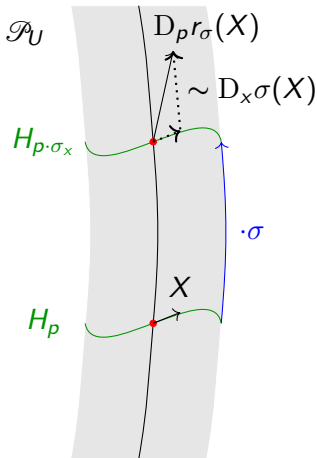
Properties of the shift:

- Vertical
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Properties of the shift:

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$\Rightarrow$  Introduce connection on  $\mathcal{G}$

# Horizontal distribution, definition

## Definition (Connection, [S.-R. F.])

Let  $H$  be a horizontal bundle of  $\mathcal{P}$ . Then  $H$  is called a **connection on  $\mathcal{P}$**  if

$$\left( D_p r_\sigma - \left( \pi_{\mathcal{P}}^! \widetilde{\Delta\sigma} \right) \Big|_{p \cdot \sigma_{\pi_{\mathcal{P}}(p)}} \right) (H_p) = H_{p \cdot \sigma_{\pi_{\mathcal{P}}(p)}} \quad (32)$$

for all  $\sigma \in \Gamma(\mathcal{G})$ , where  $\widetilde{\Delta\sigma}$  is the fundamental vector field of  $\Delta\sigma$  which is the **Darboux derivative of  $\sigma$**  given by

$$\Delta\sigma := \sigma^! \mu_{\mathcal{G}}. \quad (33)$$

## Example

Let  $P$  be a  $G$ -principal bundle, equipped with a "classical" connection.

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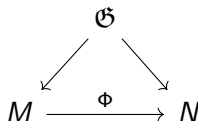
	Locally	Globally
Curved Yang-Mills	Pre-classical	$\text{ad}(\mathbb{S}^7 \rightarrow \mathbb{S}^4)$ curved

## Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial  
 $\rightsquigarrow$  LGB action locally equivalent to a Lie group action

# Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group $G$
Curved Yang-Mills	Lie group bundle $\mathcal{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathfrak{G}$ ?



## Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.



**Thank you!**