Geometry of curved Yang-Mills-Higgs gauge theories

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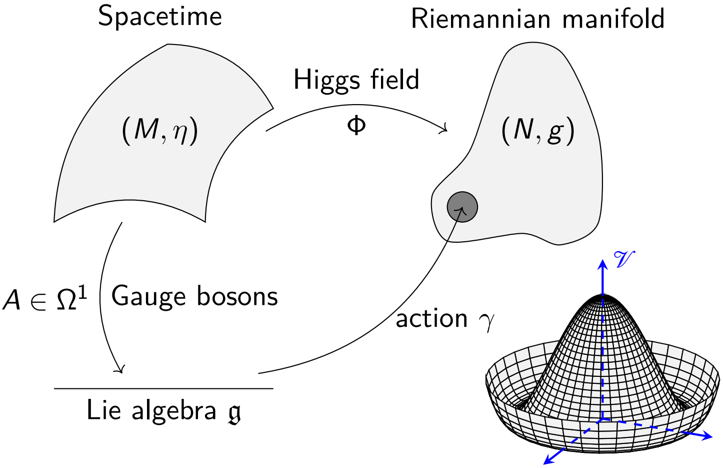
Infinitesimal gauge theory



Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra $\mathfrak{g}$ as $M \times \mathfrak{g}$	Lie algebroid $E \to N$
$\mathfrak{g}$-action γ	Anchor ρ of E & E-connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := \mathrm{d}A = \mathrm{d}^{\nabla^0} A.$$

$\rightsquigarrow$ We will use a general connection ∇ instead of ∇^0, and ∇ may not be flat.

Definition (Lie algebroids)

Let $E \to N$ be a vector bundle. Then E is a Lie algebroid, if there is a bundle map $\rho : E \to \mathrm{T}N$, called the **anchor**, and a Lie algebra structure on $\Gamma(E)$ with Lie bracket $[\cdot, \cdot]_E$ satisfying

$$[\mu, f\nu]_E = f[\mu, \nu]_E + \mathscr{L}_{\rho(\mu)}(f)\ \nu \tag{1}$$

for all $f \in C^\infty(N)$ and $\mu, \nu \in \Gamma(E)$.

Example

- $E = \mathrm{T}N$, $\rho = \mathbb{1}_{\mathrm{T}N}$

Sources: Ana Cannas Da Silva, Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10. American Mathematical Society, 1999.

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As for an action $\gamma : \mathfrak{g} \to \mathfrak{X}(N)$ we have:

Proposition (Anchor as homomorphism)

$\rho : \Gamma(E) \to \mathfrak{X}(N)$ is a homomorphism of Lie algebras.

Yvette Kosmann-Schwarzbach and Franco Magri. Poisson-Nijenhuis structures.
In Annales de l'IHP Physique théorique, volume 53, pages 35–81, 1990.

The classical formalism will correspond to:

Proposition (Action Lie algebroids)

*Let $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}\right)$ be a Lie algebra with a $\mathfrak{g}$-action γ on N. Then there is a **unique** Lie algebroid structure on $E := N \times \mathfrak{g}$ as a vector bundle over N such that*

$$\rho(\nu) = \gamma(\nu), \tag{2}$$

$$[\mu, \nu]_E = [\mu, \nu]_{\mathfrak{g}} \tag{3}$$

for all constant sections $\mu, \nu \in \Gamma(E)$.

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Example (E-connection ${}^E\nabla$ on V)

∇' a vector bundle connection on $V \to N$, then

$$^E\nabla_\nu v := \nabla'_{\rho(\nu)} v \tag{4}$$

for all $\nu \in \Gamma(E)$ and $v \in \Gamma(V)$. In short denoted by ∇'_ρ.

Sources
Mohamed Boucetta. Riemannian geometry of Lie algebroids.
Journal of the Egyptian Mathematical Society, 19(1-2):57–70, 2011.

Example

For ∇ a connection on E we have the **basic connection** given as a pair of E-connections on E and on $\mathrm{T}N$ by

$$\nabla^{\mathrm{bas}}_{\nu}\mu := [\nu, \mu]_E + \nabla_{\rho(\mu)}\nu, \tag{5}$$

$$\nabla^{\mathrm{bas}}_{\nu}X := [\rho(\nu), X] + \rho(\nabla_X \nu) \tag{6}$$

for all $X \in \mathfrak{X}(N)$ and $\nu, \mu \in \Gamma(E)$.

Remarks (Encoding of Lie algebra representations)

Test this with trivial bundles and canonical flat connection ∇^0, *i.e.* $E = N \times \mathfrak{g}$ and $\nabla^0\nu = 0$ for constant sections ν.

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Definition (Basic curvature)

Let ∇ be a connection on E. The **basic curvature** R_∇^{bas} is defined as an element of $\Gamma\left(\bigwedge^2 E^* \otimes \mathrm{T}^*N \otimes E\right)$ by

$$\begin{aligned} R_\nabla^{\mathrm{bas}}(\mu,\nu)X \coloneqq{}& \nabla_X([\mu,\nu]_E) - [\nabla_X\mu,\nu]_E - [\mu,\nabla_X\nu]_E \\ &- \nabla_{\nabla_\nu^{\mathrm{bas}}X}\mu + \nabla_{\nabla_\mu^{\mathrm{bas}}X}\nu, \end{aligned} \tag{7}$$

where $\mu,\nu \in \Gamma(E)$ and $X \in \mathfrak{X}(N)$.

Proposition

We recover the curvature of the basic connection:

$$R_{\nabla^{\mathrm{bas}}} = \begin{cases} -R_\nabla^{\mathrm{bas}}(\cdot,\cdot)\circ\rho, & \text{on } E, \\ -\rho\circ R_\nabla^{\mathrm{bas}}, & \text{on } \mathrm{T}N. \end{cases} \tag{8}$$

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Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
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Definition (Minimal coupling)

Minimal coupling $\mathfrak{D}$, $(\Phi, A) \mapsto \mathfrak{D}(\Phi, A) \in \Omega^1(M; \Phi^* \mathrm{T}N)$, by

$$\mathfrak{D}(\Phi, A) := \mathfrak{D}^A \Phi := \mathrm{D}\Phi - (\Phi^* \rho)(A), \tag{9}$$

where $\mathrm{D}\Phi : \mathrm{T}M \to \mathrm{T}N$ is the tangent map.

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Definition (Field strength)

Let ∇ be a connection on E. We define the **field strength** F, $(\Phi, A) \mapsto F(\Phi, A) \in \Omega^2(M; \Phi^* E)$, by

$$F(\Phi, A) := \mathrm{d}^{\Phi^*\nabla} A + \frac{1}{2}(\Phi^* t_{\nabla^{\mathrm{bas}}})(A \overset{\wedge}{,} A), \tag{10}$$

where $t_{\nabla^{\mathrm{bas}}}$ is the torsion of ∇^{bas} on E and $\mathrm{d}^{\Phi^*\nabla}$ the exterior covariant derivative of $\Phi^*\nabla$.

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Definition (Generalised field strength)

Let ζ be an element of $\Omega^2(N; E)$, then we define the **generalised field strength** G by

$$G(\Phi, A) \coloneqq F(\Phi, A) + \frac{1}{2}(\Phi^*\zeta)\left(\mathfrak{D}^A\Phi \overset{\wedge}{,} \mathfrak{D}^A\Phi\right). \tag{11}$$

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Definition (Curved Yang-Mills-Higgs (CYMH) Lagrangian)

Let κ be a fibre metric on E, then the **curved Yang-Mills-Higgs Lagrangian** $\mathfrak{L}_{\mathrm{CYMH}}$, $(\Phi, A) \mapsto \mathfrak{L}_{\mathrm{CYMH}}(\Phi, A) \in \Omega^{\dim(M)}(M)$, is defined by

$$\mathfrak{L}_{\mathrm{CYMH}}(\Phi, A) := -\frac{1}{2}(\Phi^*\kappa)(G(\Phi, A) \overset{\wedge}{,} *G(\Phi, A)) + (\Phi^*g)\left(\mathfrak{D}^A\Phi \overset{\wedge}{,} *\mathfrak{D}^A\Phi\right) - *(\Phi^*\mathscr{V}), \quad (12)$$

where $*$ is the Hodge star operator related to the spacetime metric η.

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Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_\nabla + \mathrm{d}^{\nabla^{\mathrm{bas}}}\zeta = 0, \tag{13}$$

$$R_\nabla^{\mathrm{bas}} = 0, \tag{14}$$

$$\nabla^{\mathrm{bas}}\kappa = 0, \tag{15}$$

$$\nabla^{\mathrm{bas}}g = 0, \tag{16}$$

$$\mathscr{L}_\rho \mathscr{V} = 0, \tag{17}$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

We say that we have a **pre-classical gauge theory**, if ∇ is flat.

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Summary

$$
\begin{tikzcd}
\Phi^*E \arrow[d] & (E,[\cdot,\cdot]_E,\kappa,\nabla) \arrow[d] \arrow[dr, "\rho"] & \\
M \arrow[r, "\Phi"] & (N,g) & \mathrm{T}N \arrow[l] \arrow[ul, bend right, "\zeta\in\Omega^2"']
\end{tikzcd}
$$

$\rightsquigarrow$ Together with the compatibility conditions we will have gauge invariance, that is,

$$\delta_\varepsilon \mathfrak{L}_{\mathrm{CYMH}} = 0. \tag{18}$$

Sources Alexei Kotov and Thomas Strobl. Curving Yang-Mills-Higgs gauge theories. *Physical Review D*, 92(8):085032, 2015.

Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

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Definition (Field redefinition, [S.-R. F.])

Let $\lambda \in \Omega^1(N; E)$ such that $\Lambda := \mathbb{1}_E - \lambda \circ \rho$ is an automorphism of E. We then define the **field redefinitions** by

$$\widetilde{A}^\lambda := (\Phi^* \Lambda)(A) + \Phi^! \lambda, \tag{19}$$

$$\widetilde{\nabla}^\lambda := \nabla + \left(\Lambda \circ \mathrm{d}^{\nabla^{\mathrm{bas}}} \circ \Lambda^{-1}\right)\lambda, \tag{20}$$

$$\widetilde{\kappa}^\lambda := \kappa \circ \left(\Lambda^{-1}, \Lambda^{-1}\right), \tag{21}$$

$$\widetilde{g}^\lambda := g \circ \left(\widehat{\Lambda}^{-1}, \widehat{\Lambda}^{-1}\right), \tag{22}$$

where $\widehat{\Lambda} := \mathbb{1}_{\mathrm{T}N} - \rho \circ \lambda$, and for all $X, Y \in \mathfrak{X}(N)$ we also define

$$\widetilde{\zeta}^\lambda\left(\widehat{\Lambda}(X), \widehat{\Lambda}(Y)\right) := \Lambda(\zeta(X, Y)) - \left(\mathrm{d}^{\widetilde{\nabla}^\lambda} \lambda\right)(X, Y) + t_{\widetilde{\nabla}^\lambda_\rho}(\lambda(X), \lambda(Y)). \tag{23}$$

Proposition ([S.-R. F.])

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\widetilde{\mathfrak{L}}^{\lambda}_{\mathrm{CYMH}} = \mathfrak{L}_{\mathrm{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

Proposition ([S.-R. F.])

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What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathscr{g}$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot,\cdot]_{\mathscr{g}} \in \Gamma\left(\bigwedge^2 \mathscr{g}^* \otimes \mathscr{g}\right)$ which restricts to the bracket of a given Lie algebra $\mathfrak{g}$

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- We need

$$\nabla_Y\left([\mu,\nu]_{\mathcal{g}}\right) = [\nabla_Y\mu,\nu]_{\mathcal{g}} + [\mu,\nabla_Y\nu]_{\mathcal{g}}, \tag{24}$$

$$R_\nabla(Y,Z)\mu = [\zeta(Y,Z),\mu]_{\mathcal{g}} \tag{25}$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu,\nu \in \Gamma(\mathcal{g})$.

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Theorem (Invariant for LABs, [S.-R. F.])

We have

$$\mathrm{d}^{\widetilde{\nabla}^{\lambda}} \widetilde{\zeta}^{\lambda} = \mathrm{d}^{\nabla} \zeta, \tag{26}$$

and $\mathrm{d}^{\nabla}\zeta$ *has values in the centre of* $\mathscr{g}$.

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.])

If $\mathrm{d}^\nabla \zeta \neq 0$, then there is no field redefinition such that $\widetilde{\zeta}^\lambda = 0$.

Remarks

Starting with a classical theory:
If $\dim(N) \geq 3$ and if Lie algebra $\mathfrak{g}$ has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $\mathrm{d}^\nabla \zeta \neq 0$.

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However, by $R_{\nabla} = \mathrm{ad}_{\mathscr{g}} \circ \zeta$ it may still be that ∇ becomes flat.

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Turning to the field redefinition of ∇:

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ restricts to the centre of $\mathcal{g}$ and induces a differential d^{Ξ} on centre-valued forms. Moreover, d^{Ξ} is independent of the field redefinitions.

Sketch of proof.

Recall

$$\nabla_Y\left([\mu,\nu]_{\mathcal{g}}\right) = [\nabla_Y\mu,\nu]_{\mathcal{g}} + [\mu,\nabla_Y\nu]_{\mathcal{g}},$$
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Turning to the field redefinition of ∇:

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Sketch of proof.

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Theorem (Closedness of $\mathrm{d}^\nabla\zeta$, [S.-R. F.])

We have

$$\mathrm{d}^\Xi \mathrm{d}^\nabla \zeta = 0. \tag{27}$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\mathrm{Obs}(\Xi) := \left[\mathrm{d}^\nabla \zeta\right]_{\mathrm{d}^\Xi}. \tag{28}$$

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Theorem (Obstruction for non-pre-classical theories, [S.-R. F.])

If $\mathrm{Obs}(\Xi) \neq 0$, then there is no field redefinition such that $\widetilde{\nabla}^{\lambda}$ is flat.

Theorem (Locally always pre-classical)

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Example (Zero obstruction class not necessarily pre-classical)

Let P be the Hopf fibration

$$
\begin{array}{ccc}
\mathrm{SU}(2) & \longrightarrow & \mathbb{S}^7 \\
& & \big\downarrow \\
& & \mathbb{S}^4
\end{array}
$$

Then for the adjoint bundle

$$\mathcal{g} := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left(\mathbb{S}^7 \times \mathfrak{su}(2)\right) \Big/ \mathrm{SU}(2)$$

we have a non-flat ∇ satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but $\mathrm{Obs}(\Xi) = 0$.

K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical.
In general, $\mathrm{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $\mathbb{S}^7 \to \mathbb{S}^4$?
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Idea

Remarks

Observe that the Higgs field Φ doesn't really matter at all in the case of LABs
$\rightsquigarrow$ Think in terms of bundles over M directly

	Classical	Curved
Infinitesimal	Lie algebra $\mathfrak{g}$	LAB $\mathcal{g}$
Integrated	Lie group G	LGB[1] $\mathscr{G}$?

$$\begin{tikzcd} G \arrow[r] & \mathscr{G} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

[1]LGB = Lie group bundle

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Definition (LGB actions)

$$\begin{array}{ccc} & & \mathscr{G} \\ & & \downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

A **right-action of $\mathscr{G}$ on N** is a smooth map $N * \mathscr{G} := f^*\mathscr{G} \to N$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$f(p \cdot g) = \pi(g), \tag{29}$$

$$(p \cdot g) \cdot h = p \cdot (gh), \tag{30}$$

$$p \cdot e_{f(p)} = p \tag{31}$$

for all $p \in N$ and $g, h \in \mathscr{G}_{f(p)}$, where $e_{f(p)}$ is the neutral element of $\mathscr{G}_{f(p)}$.

Sources
K. Mackenzie. General Theory of Lie Groupoids and Algebroids. *London Mathematical Society Lecture Note Series*, 213, 2005.

Examples

Example

$\mathscr{G}$ acts canonically on itself:

$$\mathscr{G} * \mathscr{G} \to \mathscr{G},$$
$$(q, h) \mapsto qh.$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathscr{G} \cong M \times G$, then also $N * \mathscr{G} \cong N \times G$, and we can define

$$N \times G \to N,$$
$$(p, g) \mapsto p \cdot g := p \cdot (f(p), g),$$

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Definition (Principal bundle)

Still a fibre bundle

$$\begin{tikzcd} G \arrow[r] & \mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \\ & M \end{tikzcd}$$

but with $\mathscr{G}$-action

$$\begin{array}{l} \cancel{\mathscr{P} \times G} \quad \to \mathscr{P} \\ \mathscr{P} * \mathscr{G} \end{array}$$

simply transitive on fibres of $\mathscr{P}$.

Ieke Moerdijk, Janez Mrcun. Introduction to Foliations and Lie Groupoids. *Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge*, 2003

Horizontal distribution, idea

Horizontal distribution, definition

Definition (Connection, [S.-R. F.])

Let H be a horizontal bundle of $\mathscr{P}$. Then H is called a **connection on** $\mathscr{P}$ if

$$\left(\mathrm{D}_p r_\sigma - \left(\pi_{\mathscr{P}}^! \widetilde{\Delta\sigma}\right)\Big|_{p\cdot\sigma_{\pi_{\mathscr{P}}(p)}}\right)(H_p) = H_{p\cdot\sigma_{\pi_{\mathscr{P}}(p)}} \tag{32}$$

for all $\sigma \in \Gamma(\mathscr{G})$, where $\widetilde{\Delta\sigma}$ is the fundamental vector field of $\Delta\sigma$ which is the **Darboux derivative of** σ given by

$$\Delta\sigma := \sigma^! \mu_{\mathscr{G}}. \tag{33}$$

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Summary

	Locally	Globally
Curved Yang-Mills	Pre-classical	$\mathrm{ad}(\mathbb{S}^7 \to \mathbb{S}^4)$ curved

Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial
$\rightsquigarrow$ LGB action locally equivalent to a Lie group action

Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle $\mathscr{G}$
Curved Yang-Mills-Higgs	Lie groupoid $\mathfrak{G}$?

$$\begin{tikzcd} & \mathfrak{G} \arrow[dl] \arrow[dr] & \\ M \arrow[rr, "\Phi"] & & N \end{tikzcd}$$

Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs.

Thank you!