

Geometry of curved Yang-Mills-Higgs gauge theories

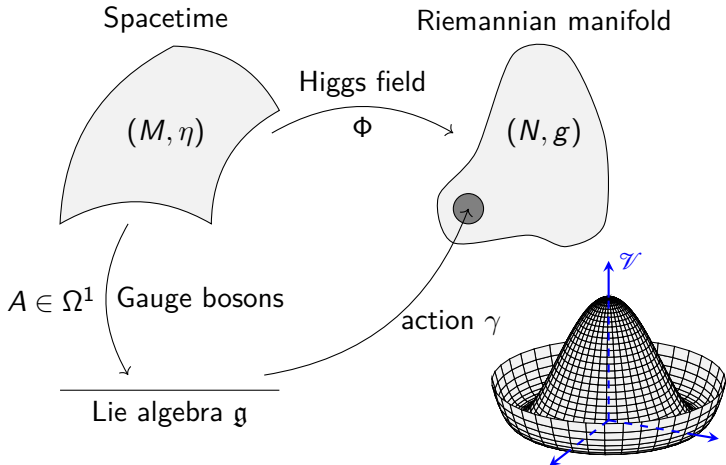
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Infinitesimal gauge theory



Guide: Curved Yang-Mills-Higgs gauge theory

Classical formalism	CYMH GT
Lie algebra \mathfrak{g} as $M \times \mathfrak{g}$	Lie algebroid $E \rightarrow N$
\mathfrak{g} -action γ	Anchor ρ of E & E -connections
Canonical flat connection ∇^0 on $M \times \mathfrak{g}$	General connection ∇ on E

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Remarks (Why a "curved theory"?)

Usually, the field strength F is given by (abelian, for simplicity)

$$F := dA = d^{\nabla^0} A.$$

\rightsquigarrow We will use a general connection ∇ instead of ∇^0 , and ∇ may not be flat.

Definition (Lie algebroids)

Let $E \rightarrow N$ be a vector bundle. Then E is a Lie algebroid, if there is a bundle map $\rho : E \rightarrow TN$, called the **anchor**, and a Lie algebra structure on $\Gamma(E)$ with Lie bracket $[\cdot, \cdot]_E$ satisfying

$$[\mu, f\nu]_E = f[\mu, \nu]_E + \mathcal{L}_{\rho(\mu)}(f) \nu \quad (1)$$

for all $f \in C^\infty(N)$ and $\mu, \nu \in \Gamma(E)$.

Example

- $E = TN, \rho = \mathbb{1}_{TN}$

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The classical formalism will correspond to:

Proposition (Action Lie algebroids)

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra with a \mathfrak{g} -action γ on N . Then there is a **unique** Lie algebroid structure on $E := N \times \mathfrak{g}$ as a vector bundle over N such that

$$\rho(\nu) = \gamma(\nu), \quad (2)$$

$$[\mu, \nu]_E = [\mu, \nu]_{\mathfrak{g}} \quad (3)$$

for all constant sections $\mu, \nu \in \Gamma(E)$.

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Example (E -connection ${}^E\nabla$ on V)

∇' a vector bundle connection on $V \rightarrow N$, then

$${}^E\nabla_\nu v := \nabla'_{\rho(\nu)} v \quad (4)$$

for all $\nu \in \Gamma(E)$ and $v \in \Gamma(V)$. In short denoted by ∇'_ρ .

Example

For ∇ a connection on E we have the **basic connection** given as a pair of E -connections on E and on $\mathbb{T}N$ by

$$\nabla_{\nu}^{\text{bas}} \mu := [\nu, \mu]_E + \nabla_{\rho(\mu)} \nu, \quad (5)$$

$$\nabla_{\nu}^{\text{bas}} X := [\rho(\nu), X] + \rho(\nabla_X \nu) \quad (6)$$

for all $X \in \mathfrak{X}(N)$ and $\nu, \mu \in \Gamma(E)$.

Remarks (Encoding of Lie algebra representations)

Test this with trivial bundles and canonical flat connection ∇^0 ,
i.e. $E = N \times \mathfrak{g}$ and $\nabla^0 \nu = 0$ for constant sections ν .

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Definition (Basic curvature)

Let ∇ be a connection on E . The **basic curvature** R_{∇}^{bas} is defined as an element of $\Gamma(\wedge^2 E^* \otimes T^*N \otimes E)$ by

$$R_{\nabla}^{\text{bas}}(\mu, \nu)X := \nabla_X([\mu, \nu]_E) - [\nabla_X \mu, \nu]_E - [\mu, \nabla_X \nu]_E - \nabla_{\nabla_{\nu}^{\text{bas}} X} \mu + \nabla_{\nabla_{\mu}^{\text{bas}} X} \nu, \quad (7)$$

where $\mu, \nu \in \Gamma(E)$ and $X \in \mathfrak{X}(N)$.

Proposition

We recover the curvature of the basic connection:

$$R_{\nabla^{\text{bas}}} = \begin{cases} -R_{\nabla}^{\text{bas}}(\cdot, \cdot) \circ \rho, & \text{on } E, \\ -\rho \circ R_{\nabla}^{\text{bas}}, & \text{on } TN. \end{cases} \quad (8)$$

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Definition (Space of fields)

Fields are a pair consisting of:

- Higgs field $\Phi \in C^\infty(M; N)$
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Definition (Minimal coupling)

Minimal coupling \mathfrak{D} , $(\Phi, A) \mapsto \mathfrak{D}(\Phi, A) \in \Omega^1(M; \Phi^* TN)$, by

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where $D\Phi : TM \rightarrow TN$ is the tangent map.

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Definition (Field strength)

Let ∇ be a connection on E . We define the **field strength** F , $(\Phi, A) \mapsto F(\Phi, A) \in \Omega^2(M; \Phi^* E)$, by

$$F(\Phi, A) := d^{\Phi^* \nabla} A + \frac{1}{2}(\Phi^* t_{\nabla^{\text{bas}}})(A \wedge A), \quad (10)$$

where $t_{\nabla^{\text{bas}}}$ is the torsion of ∇^{bas} on E and $d^{\Phi^* \nabla}$ the exterior covariant derivative of $\Phi^* \nabla$.

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Definition (Generalised field strength)

Let ζ be an element of $\Omega^2(N; E)$, then we define the **generalised field strength** \mathcal{F} by

$$\mathcal{F}(\Phi, A) := F(\Phi, A) + \frac{1}{2}(\Phi^*\zeta)(\mathfrak{D}^A\Phi \wedge \mathfrak{D}^A\Phi). \quad (11)$$

Definition (Curved Yang-Mills-Higgs (CYMH) Lagrangian)

Let κ be a fibre metric on E , then the **curved Yang-Mills-Higgs Lagrangian** $\mathfrak{L}_{\text{CYMH}}$, $(\Phi, A) \mapsto \mathfrak{L}_{\text{CYMH}}(\Phi, A) \in \Omega^{\dim(M)}(M)$, is defined by

$$\begin{aligned} \mathfrak{L}_{\text{CYMH}}(\Phi, A) := & -\frac{1}{2}(\Phi^*\kappa)(\mathcal{F}(\Phi, A) \wedge *\mathcal{F}(\Phi, A)) \\ & + (\Phi^*g)\left(\mathfrak{D}^A\Phi \wedge *\mathfrak{D}^A\Phi\right) - *(\Phi^*\mathcal{V}), \quad (12) \end{aligned}$$

where $*$ is the Hodge star operator related to the spacetime metric η .

Definition (CYMH GT)

Assume we have additionally the **compatibility conditions**

$$R_{\nabla} + d^{\nabla^{\text{bas}}} \zeta = 0, \quad (13)$$

$$R_{\nabla}^{\text{bas}} = 0, \quad (14)$$

$$\nabla^{\text{bas}} \kappa = 0, \quad (15)$$

$$\nabla^{\text{bas}} g = 0, \quad (16)$$

$$\mathcal{L}_{\rho} \mathcal{V} = 0, \quad (17)$$

then we say that we have a **curved Yang-Mills-Higgs gauge theory**.

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Summary

$$\begin{array}{ccccc}
 \Phi^* E & & (E, [\cdot, \cdot]_E, \kappa, \nabla) & \xleftarrow{\zeta \in \Omega^2} & \\
 \downarrow & & \downarrow & \searrow \rho & \\
 M & \xrightarrow{\Phi} & (N, g) & \xleftarrow{\quad} & TN
 \end{array}$$

\leadsto Together with the compatibility conditions we have gauge invariance, that is,

$$\delta_\varepsilon \mathfrak{L}_{\text{CYMH}} = 0. \quad (18)$$

Remarks ([S.-R. F.]

If $\rho \equiv 0$, then $\varepsilon \in \Gamma(\Phi^* E)$ and

$$\delta_\varepsilon \Phi = 0, \quad \delta_\varepsilon A = [\varepsilon, A]_E - d^{\Phi^* \nabla} \varepsilon.$$

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Motivation

- Are there CYMH GTs which are neither pre-classical nor classical?
- Difficulty: There is an equivalence relation of CYMH GTs keeping the same Lagrangian and preserving the physics, possibly turning theories into (pre-)classical ones.

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Definition (Field redefinition, [S.-R. F.]

Let $\lambda \in \Omega^1(N; E)$ such that $\Lambda := \mathbb{1}_E - \lambda \circ \rho$ is an automorphism of E . We then define the **field redefinitions** by

$$\tilde{A}^\lambda := (\Phi^* \Lambda)(A) + \Phi^! \lambda, \quad (19)$$

$$\tilde{\nabla}^\lambda := \nabla + \left(\Lambda \circ d^{\nabla^{\text{bas}}} \circ \Lambda^{-1} \right) \lambda, \quad (20)$$

$$\tilde{\kappa}^\lambda := \kappa \circ \left(\Lambda^{-1}, \Lambda^{-1} \right), \quad (21)$$

$$\tilde{g}^\lambda := g \circ \left(\hat{\Lambda}^{-1}, \hat{\Lambda}^{-1} \right), \quad (22)$$

where $\hat{\Lambda} := \mathbb{1}_{TN} - \rho \circ \lambda$, and for all $X, Y \in \mathfrak{X}(N)$ we also define

$$\begin{aligned} & \tilde{\zeta}^\lambda \left(\hat{\Lambda}(X), \hat{\Lambda}(Y) \right) \\ & := \Lambda(\zeta(X, Y)) - \left(d^{\tilde{\nabla}^\lambda} \lambda \right)(X, Y) + t_{\tilde{\nabla}^\lambda \rho}(\lambda(X), \lambda(Y)). \end{aligned} \quad (23)$$

Proposition ([S.-R. F.]

- *Field redefinitions define an equivalence relation of CYMH gauge theories*
- $\tilde{\mathcal{L}}_{\text{CYMH}}^\lambda = \mathcal{L}_{\text{CYMH}}$

Let us now apply a field redefinition in order to study whether ∇ and ζ can become flat and zero, respectively.

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What happens in the case of Lie algebra bundles?

Example (Lie algebra bundles (LABs))

- $E = \mathfrak{g}$ an LAB ($\rho \equiv 0$) with a field of Lie brackets $[\cdot, \cdot]_{\mathfrak{g}} \in \Gamma(\wedge^2 \mathfrak{g}^* \otimes \mathfrak{g})$ which restricts to the bracket of a given Lie algebra \mathfrak{g}

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- We need

$$\nabla_Y([\mu, \nu]_{\mathcal{G}}) = [\nabla_Y \mu, \nu]_{\mathcal{G}} + [\mu, \nabla_Y \nu]_{\mathcal{G}}, \quad (24)$$

$$R_{\nabla}(Y, Z)\mu = [\zeta(Y, Z), \mu]_{\mathcal{G}} \quad (25)$$

for all $Y, Z \in \mathfrak{X}(N)$ and $\mu, \nu \in \Gamma(\mathcal{G})$.

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Theorem (Invariant for LABs, [S.-R. F.]

We have

$$d^{\tilde{\nabla}^\lambda} \tilde{\zeta}^\lambda = d^\nabla \zeta, \quad (26)$$

and $d^\nabla \zeta$ has values in the centre of \mathfrak{g} .

Behaviour of the field redefinition of ζ

Theorem (Existence of non-classical theories, [S.-R. F.]

If $d^\nabla \zeta \neq 0$, then there is no field redefinition such that $\tilde{\zeta}^\lambda = 0$.

Remarks

Starting with a classical theory:

If $\dim(N) \geq 3$ and if Lie algebra \mathfrak{g} has a non-zero centre, then we can always construct a pre-classical CYMH GT which is not a classical one by adding a ζ with $d^\nabla \zeta \neq 0$.

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However, by $R_\nabla = \text{ad}_g \circ \zeta$ it may still be that ∇ becomes flat.

Turning to the field redefinition of ∇ :

Theorem (Differential on centre-valued forms, [S.-R. F.])

∇ restricts to the centre of \mathfrak{g} and induces a differential d^Ξ on centre-valued forms. Moreover, d^Ξ is independent of the field redefinitions.

Sketch of proof.

Recall

$$\begin{aligned}\nabla_Y([\mu, \nu]_{\mathfrak{g}}) &= [\nabla_Y \mu, \nu]_{\mathfrak{g}} + [\mu, \nabla_Y \nu]_{\mathfrak{g}}, \\ R_\nabla(Y, Z)\mu &= [\zeta(Y, Z), \mu]_{\mathfrak{g}}, \\ \tilde{\nabla}_Y^\lambda \mu &= \nabla_Y \mu - [\lambda(Y), \mu]_{\mathfrak{g}},\end{aligned}$$

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Theorem (Closedness of $d^\nabla \zeta$, [S.-R. F.])

We have

$$d^\Xi d^\nabla \zeta = 0. \quad (27)$$

Definition (Obstruction class, [S.-R. F.])

We define the **obstruction class** by

$$\text{Obs}(\Xi) := [d^\nabla \zeta]_{d^\Xi}. \quad (28)$$

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Theorem (Obstruction for non-pre-classical theories, [S.-R. F.]

If $\text{Obs}(\Xi) \neq 0$, then there is no field redefinition such that $\tilde{\nabla}^\lambda$ is flat.

Theorem (Locally always pre-classical)

If N is contractible, then there is a field redefinition such that $\tilde{\nabla}^\lambda$ is flat.

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Remarks

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Example (Zero obstruction class not necessarily pre-classical)

Let P be the Hopf fibration

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{S}^4 \end{array}$$

Then for the adjoint bundle

$$\mathcal{Q} := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2) := \left(\mathbb{S}^7 \times \mathfrak{su}(2) \right) / \mathrm{SU}(2)$$

we have a non-flat ∇ satisfying the compatibility conditions such that all of its field redefinitions are not flat either, but $\mathrm{Obs}(\Xi) = 0$.

Summary

Remarks

Locally, LABs are always pre-classical but not necessarily classical.
In general, $\text{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $S^7 \rightarrow S^4$?
 \rightsquigarrow Integration

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In general, $\text{Obs}(\Xi) = 0$ does not imply a flat connection.

So, what actually happens in the adjoint bundle of $S^7 \rightarrow S^4$?
 \rightsquigarrow Integration

Idea

Remarks

Observe that the Higgs field Φ doesn't really matter at all in the case of LABs

\rightsquigarrow Think in terms of bundles over M directly

	Classical	Curved
Infinitesimal	Lie algebra \mathfrak{g}	LAB \mathcal{G}
Integrated	Lie group G	LGB ¹ \mathcal{G} ?

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{G} \\ & & \downarrow \pi \\ & & M \end{array}$$

¹LGB = Lie group bundle

Idea

Remarks

Observe that the Higgs field Φ doesn't really matter at all in the case of LABs

\rightsquigarrow Think in terms of bundles over M directly

	Classical	Curved
Infinitesimal	Lie algebra \mathfrak{g}	LAB \mathcal{G}
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Definition (LGB actions, simplified)

$$\begin{array}{ccc} & \mathcal{G} & \\ & \downarrow & \\ \mathcal{P} & \xrightarrow{\pi} & M \end{array}$$

$\mathcal{P} \xrightarrow{\pi} M$ a fibre bundle. A **right-action of \mathcal{G} on \mathcal{P}** is a smooth map $\mathcal{P} * \mathcal{G} := \pi^* \mathcal{G} = \mathcal{P} \times_M \mathcal{G} \rightarrow \mathcal{P}$, $(p, g) \mapsto p \cdot g$, satisfying the following properties:

$$\pi(p \cdot g) = \pi(p), \quad (29)$$

$$(p \cdot g) \cdot h = p \cdot (gh), \quad (30)$$

$$p \cdot e_{\pi(p)} = p \quad (31)$$

for all $p \in \mathcal{P}$ and $g, h \in \mathcal{G}_{\pi(p)}$, where $e_{\pi(p)}$ is the neutral element of $\mathcal{G}_{\pi(p)}$.

Examples

Example

\mathcal{G} acts canonically on itself:

$$\begin{aligned}\mathcal{G} * \mathcal{G} &\rightarrow \mathcal{G}, \\ (q, h) &\mapsto qh.\end{aligned}$$

Example (Recovering Lie group action)

- Either by $M = \{*\}$.
- Or by $\mathcal{G} \cong M \times G$, then also $\mathcal{P} * \mathcal{G} \cong \mathcal{P} \times G$, and we can define

$$\begin{aligned}\mathcal{P} \times G &\rightarrow \mathcal{P}, \\ (p, g) &\mapsto p \cdot g := p \cdot (\pi(p), g),\end{aligned}$$

which is equivalent to $\mathcal{P} * \mathcal{G} \rightarrow \mathcal{P}$.

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Definition (Principal bundle)

Still a fibre bundle

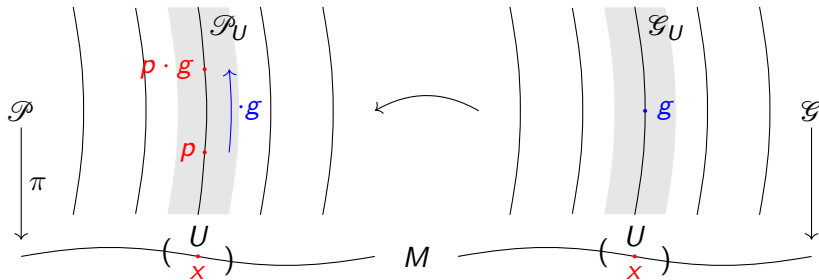
$$\begin{array}{ccc} G & \longrightarrow & \mathcal{P} \\ & & \downarrow \pi \\ & & M \end{array}$$

but with \mathcal{G} -action

$$\begin{array}{ccc} \cancel{\mathcal{P}} \times \cancel{G} & \longrightarrow & \mathcal{P} \\ \mathcal{P} * \mathcal{G} & & \end{array}$$

simply transitive on fibres of \mathcal{P} , and "suitable" atlas.

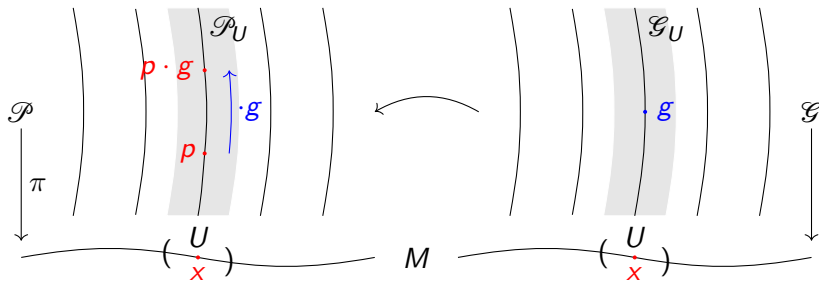
Connection on \mathcal{P} : Idea



But:

$$\begin{aligned} & r_g : \mathcal{P}_x \rightarrow \mathcal{P}_x \\ \Rightarrow & D_p r_g \text{ only defined on vertical structure} \end{aligned}$$

Connection on \mathcal{P} : Idea



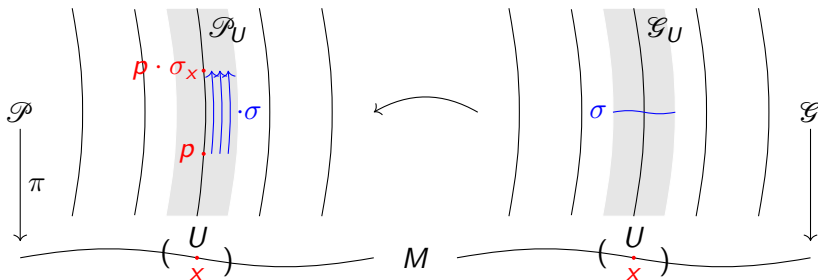
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$$r_g : \mathcal{P}_x \rightarrow \mathcal{P}_x$$

\Rightarrow

$D_p r_g$ only defined on vertical structure

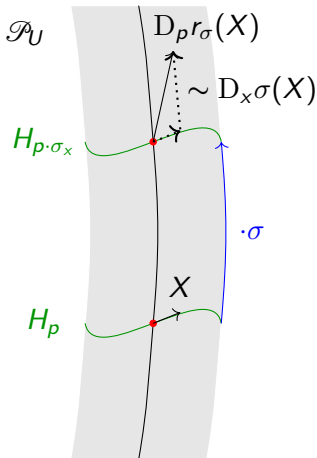
Connection on \mathcal{P} : Idea



$$\text{Use } \sigma \in \Gamma(\mathcal{G}) : r_\sigma(p) := p \cdot \sigma_{\pi(p)}$$

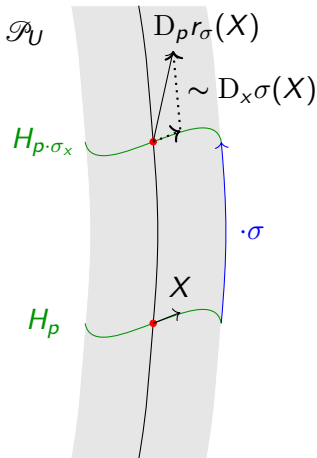
Connection on \mathcal{P} : Idea

H a horizontal bundle of \mathcal{P} :



Connection on \mathcal{P} : Idea

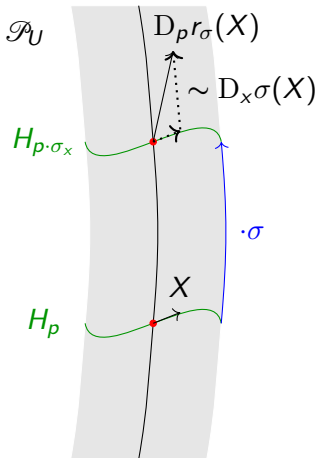
H a horizontal bundle of \mathcal{P} :



- Only vertical part in \mathcal{G} of $D\sigma$ matters
- Shift is vertical in \mathcal{P}

Connection on \mathcal{P} : Idea

H a horizontal bundle of \mathcal{P} :



- Only vertical part in \mathcal{G} of $D\sigma$ matters
- Shift is vertical in \mathcal{P}

\Rightarrow Introduce connection on \mathcal{G}

$\Rightarrow \nabla$ on the LAB \mathcal{L} of \mathcal{G}

Summary

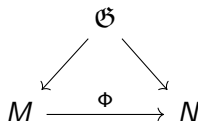
	Locally	Globally
Curved Yang-Mills	Pre-classical	$\text{ad}(\mathbb{S}^7 \rightarrow \mathbb{S}^4)$ curved

Remarks (Integrated point of view)

This is probably linked to that an LGB is locally trivial
 \rightsquigarrow LGB action locally equivalent to a Lie group action

Hope: Structural Lie groupoids

Gauge theory	Structure
Yang-Mills	Lie group G
Curved Yang-Mills	Lie group bundle \mathcal{G}
Curved Yang-Mills-Higgs	Lie groupoid \mathfrak{G} ?



Remarks

- Richer set of principal bundles, containing LGBs.
- May result into obstruction statements for curved Yang-Mills-Higgs gauge theories.

Thank you!