2 1.

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 = \vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \vec{y}_2 - \frac{\vec{v}_1^T \vec{y}_2}{\vec{v}_1^T \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[1\,0\,0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1\,0\,0]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{y}_3 - \frac{\vec{v}_1^T \vec{y}_3}{\vec{v}_1^T \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{v}_2^T \cdot \vec{y}_3}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[1\,0\,0]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[1\,0\,0]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{[0\,1\,0]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[0\,1\,0]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{1} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The orthogonal basis $= \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

E2.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$R^T = B^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}^{-1} = \frac{1}{|B|} \cdot B^*$$

$$|B| = -1 \times ((1 \times 0) - (-2 \times 1)) - 1 \times ((1 \times 0) - (1 \times 0)) + 1 \times ((1 \times -2) - (0 \times 1))$$

$$= -2 - 0 + (-2) = -4.$$

$$B^* = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 2 \\ -2 & -2 & -2 \end{bmatrix}$$

$$\therefore R^T = \frac{1}{-4} \cdot \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 2 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$x^v_1 = r_1^T \cdot \vec{x} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{2}$$

$$x^v_2 = r_2^T \cdot \vec{x} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1$$

$$x^v_3 = r_3^T \cdot \vec{x} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{5}{2}$$

$$\vec{x} = \frac{1}{2}\vec{v}_1 - \vec{v}_2 + \frac{5}{2}\vec{v}_3$$

Z3.

$f_1(t) = 1 \quad , x \in [0,1] \qquad f_2(t) = \begin{cases} 1 & , x \in [0,\frac{1}{4}] \\ -1 & , x \in (\frac{1}{4},1] \end{cases} \qquad f_3(t) = \begin{cases} 1 & , x \in [0,\frac{3}{4}] \\ -1 & , x \in [\frac{3}{4},1] \end{cases}$

(i)

$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t) = 0$

when $t=\frac{1}{2}$, $\alpha_1 - \alpha_2 + \alpha_3 = 0$ ① ② − ① $2\alpha_2 = 0$ $\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

$t = \frac{1}{8}$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$ ② ② + ③ $2\alpha_1 = 0$ $\therefore$ linear Independent.

$t = \frac{7}{8}$, $\alpha_1 - \alpha_2 - \alpha_3 = 0$ ③ ① − ③ $2\alpha_3 = 0$

(ii) Let $\vec{V_1} = \vec{X_1}$,

$\vec{V_2} = \vec{X_2} - \frac{\langle \vec{X_2}, \vec{V_1} \rangle}{\langle \vec{V_1}, \vec{V_1} \rangle} \vec{V_1}$

$\vec{V_3} = \vec{X_3} - \frac{\langle \vec{X_3}, \vec{V_1} \rangle}{\langle \vec{V_1}, \vec{V_1} \rangle} \vec{V_1} - \frac{\langle \vec{X_3}, \vec{V_2} \rangle}{\langle \vec{V_2}, \vec{V_2} \rangle} \vec{V_2}$

$\vec{V_1} = f_1(t) = 1 \quad \forall t \in [0,1]$.

$\langle f_2(t), f_1(t) \rangle = \int_0^1 f_1(t) f_2(t) dt = \int_0^{\frac{1}{4}} dt + \int_{\frac{1}{4}}^1 (-1) dt = \frac{1}{4} - (1-\frac{1}{4}) = -\frac{2}{4} = -\frac{1}{2}$

$\langle f_1(t), f_1(t) \rangle = \int_0^1 dt = 1$

$\therefore \vec{V_2} = f_2(t) + \frac{1}{2} f_1(t) = \begin{cases} \frac{3}{2} & t \in [0,\frac{1}{4}] \\ -\frac{1}{2} & t \in (\frac{1}{4},1] \end{cases}$

$\boxed{\begin{array}{l} \vec{V_1} = 1 \quad \forall t \in [0,1] \\ \vec{V_2} = \begin{cases} \frac{3}{2} & t \in [0,\frac{1}{4}] \\ -\frac{1}{2} & t \in (\frac{1}{4},1] \end{cases} \\ \vec{V_3} = \begin{cases} 0 & t \in [0,\frac{1}{4}] \\ \frac{2}{3} & t \in (\frac{1}{4},\frac{3}{4}] \\ -\frac{4}{3} & t \in (\frac{3}{4},1] \end{cases} \\ \vec{V_1}, \vec{V_2}, \vec{V_3} \text{ is required orthogonal set.} \end{array}}$

$\langle f_3(t), f_1(t) \rangle = \int_0^1 f_1(t) f_3(t) dt = \int_0^{\frac{3}{4}} dt + \int_{\frac{3}{4}}^1 (-1) dt = \frac{3}{4} - (1-\frac{3}{4}) = \frac{1}{2}$

$\langle f_3(t), \vec{V_2}(t) \rangle = \int_0^1 f_3(t) \vec{V_2}(t) dt = \int_0^{\frac{1}{4}} \frac{3}{2} dt + \int_{\frac{1}{4}}^{\frac{3}{4}} (-\frac{1}{2}) dt + \int_{\frac{3}{4}}^1 (-\frac{1}{2}) \cdot (-1) dt$

$= \frac{3}{2} \cdot \frac{1}{4} - \frac{1}{2}(\frac{3}{4} - \frac{1}{4}) + \frac{1}{2}(1 - \frac{3}{4}) = \frac{3}{8} - \frac{2}{8} + \frac{1}{8} = \frac{1}{4}$

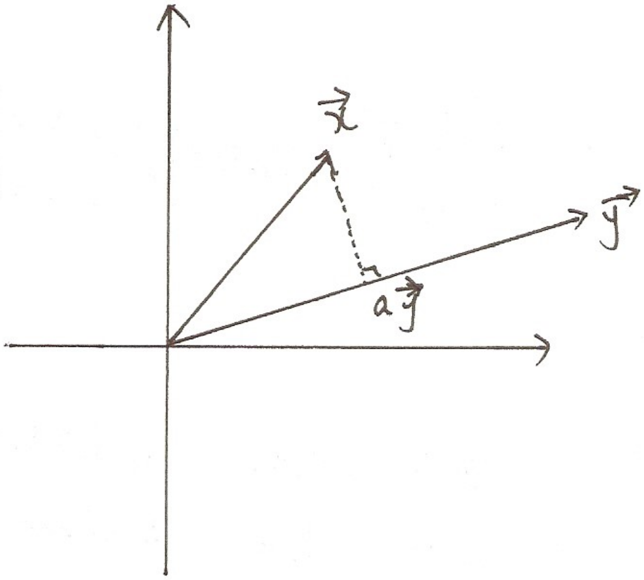
$\langle \vec{V_2}(t), \vec{V_2}(t) \rangle = \int_0^{\frac{1}{4}} \frac{9}{4} dt + \int_{\frac{1}{4}}^1 \frac{1}{4} dt = \frac{9}{16} + \frac{3}{16} = \frac{3}{4}$

$\therefore \vec{V_3} = f_3(t) - \frac{\frac{1}{2}}{1} \cdot f_1(t) - \frac{\frac{1}{4}}{\frac{3}{4}} \cdot \vec{V_2}(t) = f_3(t) - \frac{1}{2} f_1(t) - \frac{1}{3} \vec{V_2}(t)$

$\begin{cases} 1 - \frac{1}{2} - \frac{1}{2} = 0 & t \in [0,\frac{1}{4}] \\ 1 - \frac{1}{2} + \frac{1}{6} = \frac{2}{3} & t \in (\frac{1}{4},\frac{3}{4}] \\ -1 - \frac{1}{2} + \frac{1}{6} = -\frac{4}{3} & t \in (\frac{3}{4},1] \end{cases}$

$\cancel{\vec{V_3} = \begin{cases} 0 & t \in [0,\frac{1}{4}] \\ \frac{2}{3} & t \in \ldots \end{cases}} \quad \cancel{\frac{4}{3}}$ $\quad V_3 = \begin{cases} 0 & , t \in [0,\frac{1}{4}] \\ \frac{2}{3} & , t \in (\frac{1}{4},\frac{3}{4}] \\ -\frac{4}{3} & , t \in (\frac{3}{4},1] \end{cases}$

Z4.



$a\vec{y}$ should be projection of $\vec{x}$ on $\vec{y}$

projection of $\vec{x}$ on $\vec{y}$ = $\frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\vec{y}$

Hence $a = \frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2} = \frac{\langle \vec{x},\vec{y}\rangle}{\langle \vec{y},\vec{y}\rangle} = \frac{\vec{x}^T}{\vec{y}^T}$

$= \frac{\vec{x}}{\vec{y}}$

$$z = \vec{x} - a\vec{y} = \vec{x} - \frac{\langle \vec{x},\vec{y}\rangle}{\|y\|^2}\vec{y}$$

$$\langle \vec{x} - \frac{\langle \vec{x},\vec{y}\rangle}{\|y\|^2}\vec{y}, \vec{y}\rangle = \langle \vec{x},\vec{y}\rangle + \langle -\frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\vec{y}, \vec{y}\rangle$$

$$= \langle \vec{x},\vec{y}\rangle - \frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\langle \vec{y},\vec{y}\rangle$$

$$= \langle \vec{x},\vec{y}\rangle - \frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\|\vec{y}\|^2$$

$$= 0$$

Hence $z = \vec{x} - \frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\vec{y}$ is orthogonal to $\vec{y}$.

$$\|x-ay\|^2 = \langle x-ay, x-ay\rangle = \langle x,x\rangle + \langle x,-ay\rangle + \langle -ay, x\rangle + \langle -ay,-ay\rangle$$

$$= \langle x,x\rangle + \langle x, -\frac{\langle x,y\rangle}{\|y\|^2}y\rangle + \langle -\frac{\langle x,y\rangle}{\|y\|^2}y, x\rangle + \langle \frac{\langle x,y\rangle}{\|y\|^2}y, \frac{\langle x,y\rangle}{\|y\|^2}y\rangle$$

$$= \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}}{\|y\|^2}\langle x,y\rangle - \frac{\langle x,y\rangle}{\|y\|^2}\overline{\langle x,y\rangle} + \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\|y\|^2\|y\|^2}\langle y,y\rangle$$

$$= \|x\|^2 - \frac{\overline{\langle x,y\rangle}}{\|y\|^2}\langle x,y\rangle - \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\|y\|^2} + \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\|y\|^2}$$

$$\|x-ay\|^2 = \|x\|^2 - \frac{\langle x,y\rangle}{\|y\|^2}\overline{\langle x,y\rangle}$$

$$\|x-ay\|^2 + \|ay\|^2 = \|x\|^2$$

$$\langle ay, ay\rangle = \langle \frac{\langle x,y\rangle}{\|y\|^2}y, \frac{\langle x,y\rangle}{\|y\|^2}y\rangle$$

$$\Rightarrow \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\|y\|^2}$$

Z5. $V = \{(1+\hat{j}), (1-\hat{j})\}$. $A: X \to Y$ is defined by $A(X) = (1+\hat{j})X$.

(i) $A(1+\hat{j}) = (1+\hat{j})(1+\hat{j}) = 1+2\hat{j}-1 = 2\hat{j} = 1\cdot(1+\hat{j}) + (-1)\cdot(1-\hat{j})$

$\therefore [A(1+\hat{j})]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A(1-\hat{j}) = (1+\hat{j})(1-\hat{j}) = 1+1 = 2 = 1\times(1+\hat{j}) + 1\times (1-\hat{j})$

$\therefore [A(1-\hat{j})]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ matrix of A relative to the basis set is $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(ii) $A\vec{z} = \lambda\vec{z}$. $[A-\lambda I]\vec{z} = 0 \Rightarrow |[A-\lambda I]| = 0$

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+1 = 0$ $\lambda = 1\pm\hat{j}$.

$\therefore$ $\lambda_1 = 1+\hat{j}$ and $\lambda_2 = 1-\hat{j}$ are eigenvalues of A.

$\lambda_1 = 1+\hat{j} \Rightarrow \begin{bmatrix} 1-(1+\hat{j}) & 1 \\ -1 & 1-(1+\hat{j}) \end{bmatrix}\begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{matrix} -\hat{j}\cdot z_{11} + z_{12} = 0 \\ -z_{11} - \hat{j}\cdot z_{12} = 0 \end{matrix}$ $\Rightarrow$ $\begin{matrix} z_{11} = 1 \\ z_{12} = \hat{j}. \end{matrix}$ $\vec{z}_1 = \begin{bmatrix} 1 \\ \hat{j} \end{bmatrix}$

$\lambda_2 = 1-\hat{j} \Rightarrow \begin{bmatrix} 1-(1-\hat{j}) & 1 \\ -1 & 1-(1-\hat{j}) \end{bmatrix}\begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{matrix} \hat{j}z_{21} + z_{22} = 0 \\ -z_{21} + \hat{j}\cdot z_{22} = 0 \end{matrix}$ $\Rightarrow$ $\begin{matrix} z_{21} = \hat{j} \\ z_{22} = 1 \end{matrix}$ $\vec{z}_2 = \begin{bmatrix} \hat{j} \\ 1 \end{bmatrix}$

$\therefore$ eigenvectors are $\begin{bmatrix} 1 \\ \hat{j} \end{bmatrix}, \begin{bmatrix} \hat{j} \\ 1 \end{bmatrix}$ when eigenvalues are $1+\hat{j}, 1-\hat{j}$.

(iii) The matrix for A relative to the eigenvectors as the basis vectors is

$$\boxed{\begin{bmatrix} 1+\hat{j} & 0 \\ 0 & 1-\hat{j} \end{bmatrix}}$$

$$A' = [B^{-1}AB] = \begin{bmatrix} 1 & \hat{j} \\ \hat{j} & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \hat{j} \\ \hat{j} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\hat{j}}{2} \\ -\frac{\hat{j}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \hat{j} \\ \hat{j} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{\hat{j}}{2} & \frac{1}{2}-\frac{\hat{j}}{2} \\ -\frac{\hat{j}}{2}-\frac{1}{2} & -\frac{\hat{j}}{2}+\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & \hat{j} \\ \hat{j} & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det\begin{bmatrix} 1 & \hat{j} \\ \hat{j} & 1 \end{bmatrix}}\begin{bmatrix} 1 & -\hat{j} \\ -\hat{j} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\hat{j}}{2} \\ -\frac{\hat{j}}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\hat{j} & 0 \\ 0 & 1-\hat{j} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Z6. $V = \{\sin t, \cos t\}$.

(i) $D(\cos t) = -\sin t = -1 \cdot \sin t + 0 \cdot \cos t$

$D(\sin t) = \cos t = 0 \cdot \sin t + 1 \cdot \cos t$

$\therefore$ matrix of D is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(ii) $A\vec{z} = \lambda \vec{z}$. $[A - \lambda I]\vec{z} = 0 \Rightarrow |[A-\lambda I]| = 0$ $\quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\left|\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right| = 0 \quad (-\lambda)^2 + 1 = 0 \quad \begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}$

$\lambda_1 = i \Rightarrow \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -iz_{11} - z_{12} = 0 \\ z_{11} - iz_{12} = 0 \end{cases} \Rightarrow \begin{cases} z_{11} = 1 \\ z_{12} = -i \end{cases} \quad \vec{z}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda_2 = -i \Rightarrow \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \vec{z}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} iz_{21} - z_{22} = 0 \\ z_{21} + z_{22} i = 0 \end{matrix} \Rightarrow \begin{cases} z_{21} = 1 \\ z_{22} = i \end{cases} \quad \vec{z}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

As function of $\{\sin t, \cos t\}$: $\quad \begin{matrix} \sin t - i \cdot \cos t & \text{for } \lambda = i \\ \sin t + i \cos t & \text{for } \lambda = -i. \end{matrix}$

(iii)

$B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad B^{-1} = \dfrac{1}{\det(B)} \cdot \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \dfrac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{bmatrix}$

$$\cancel{A' = B^{-1}AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2i} \\ \frac{1}{2} & \frac{1}{2i} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} -\frac{1}{2i} & -\frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}$$

$$A' = B^{-1}AB = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

E7.

i. $\lambda_1 = 1$, $\vec{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = 2$, $\vec{z}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ Then $B^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\because |B| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$

$\therefore \vec{z}_1, \vec{z}_2$ linearly Independent.

$\therefore A' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = B^{-1} A B$ $\Rightarrow$ A is matrix where eigen values are 1, 2 and eigen vectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\Rightarrow A = B \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} B^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+2 \\ 1+0 & 0+4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2-2 & -1+2 \\ 2-4 & -1+4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

$\therefore$ matrix of the transformation A relative to the standard basis set is $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

ii. $A: R^2 \to R^2$ Then $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -2x+3y \end{bmatrix}$

$\therefore L_A(x, y) = (y, -2x+3y)$ is linear transformation of A.

$V = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

$L_A(1,1) = (1, -2\times1+3\times1) = (1, 1) = 1\times(1,1) + 0\times(-1,1)$

$L_A(-1,1) = (1, -2\times(-1)+3\times1) = (1,5) = a_1(1,1) + a_2(-1,1)$

have $\begin{cases} a_1 - a_2 = 1 \\ a_1 + a_2 = 5 \end{cases}$ $\therefore \begin{cases} a_1 = 3 \\ a_2 = 2 \end{cases}$

$\therefore L_A(-1,1) = (1,5) = 3(1,1) + 2(-1,1)$

$\therefore$ matrix representation relative to the new basis is $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

28. $V = \vec{v}_1, \vec{v}_2 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

i) $B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad R^T B = I \qquad R^T = B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$\therefore$ Reciprocal Basis Vectors: $\vec{r}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{r}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

ii) $A\vec{v}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$X_1^v = R^T(A\vec{v}_1) = B^{-1}(A\vec{v}_1) = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

iii) $A\vec{v}_2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$X_2^v = R^T(A\vec{v}_2) = B^{-1}(A\vec{v}_2) = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4+4 \\ 2-4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

iv) Matrix is $A^v = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$