(i) $V = \{e^t + e^{2t}, e^t - e^{2t}\}$

$D(e^t + e^{2t}) = e^t + 2e^{2t} = a\cdot(e^t + e^{2t}) + b\cdot(e^t - e^{2t})$

$D(e^t - e^{2t}) = e^t - 2e^{2t} = c\cdot(e^t + e^{2t}) + d\cdot(e^t - e^{2t})$

$\begin{cases} a+b=1 \\ a-b=2 \end{cases} \Rightarrow \begin{cases} a=\frac{3}{2} \\ b=-\frac{1}{2} \end{cases}$ $\quad \begin{cases} c+d=1 \\ c-d=-2 \end{cases} \Rightarrow \begin{cases} c=-\frac{1}{2} \\ d=\frac{3}{2} \end{cases}$

$\therefore$ matrix is $\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

(ii) $A\vec{z} = \lambda\vec{z}$ $\quad [A - \lambda I]\vec{z} = 0$ $\quad |[A - \lambda I]| = 0.$ $\quad A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

$\begin{vmatrix} \frac{3}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2}-\lambda \end{vmatrix} = 0$ $\quad (\frac{3}{2}-\lambda)^2 - \frac{1}{4} = 0$ $\quad (\lambda-1)(\lambda-2) = 0$

$\frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda_1 = 1$

$\lambda_2 = 2.$

$\lambda_1 = 1 \Rightarrow \begin{bmatrix} \frac{3}{2}-1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2}-1 \end{bmatrix}\begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \frac{1}{2}z_{11} - \frac{1}{2}z_{12} = 0$, $-\frac{1}{2}z_{11} + \frac{1}{2}z_{12} = 0.$ $\quad \therefore z_{11} = 1$, $z_{12} = 1$ $\quad \therefore \vec{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \Rightarrow \begin{bmatrix} \frac{3}{2}-2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2}-2 \end{bmatrix}\begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -\frac{1}{2}z_{21} - \frac{1}{2}z_{22} = 0$, $-\frac{1}{2}z_{21} - \frac{1}{2}z_{22} = 0$ $\quad \therefore z_{21} = 1$, $z_{22} = -1$ $\quad \therefore \vec{z}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(iii) $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad B^{-1} = \frac{1}{\det(B)}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$A' = B^{-1}AB = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore$ required matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$