Advanced Tools in Macroeconomics

Continuous time models (and methods)

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Introduction

- In this lecture we will look at the Ramsey growth model
- We will show how to derive it's continuous time formulation, known as the Hamilton-Jacobi-Bellman equation
- And introduce a method on how to solve it (known as the "explicit method")

The Ramsey growth model

- Now consider the Ramsey growth model (without growth)

$$
\begin{aligned}
v(k_t) = & \max_{c_t, k_{t+1}} \{u(c_t) + (1-\rho)v(k_{t+1})\} \\
\text{s.t.} \quad & c_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t
\end{aligned}
$$

- In Δ units of time

$$
\begin{aligned}
v(k_t) = & \max_{c_t, k_{t+\Delta}} \{\Delta u(c_t) + (1-\Delta\rho)v(k_{t+\Delta})\} \\
\text{s.t.} \quad & \Delta c_t + k_{t+\Delta} = \Delta k_t^\alpha + (1-\Delta\delta)k_t
\end{aligned}
$$

The Ramsey growth model

- Notice that all flows change when the length of the time period on which they are defined changes. Stocks, k, are the same.
- I discount the future with $1 - \Delta\rho$ instead of $(1 - \rho)$ (or with $e^{-\Delta\rho}$ instead of e^{ρ}, but these are, in the limit, equivalent).
- One funny thing: Consumption, c, is still "monthly" consumption, but it now only cost Δ as much, and I only get a Δ fraction of the utility!
- These assumptions are for technical reasons, and it will (hopefully) soon be clear why they are made.

The Ramsey growth model

- Bellman equation

$$v(k_t) = \max_{c_t, k_{t+\Delta}} \{\Delta u(c_t) + (1 - \Delta\rho) v(k_{t+\Delta})\}$$

$$\text{s.t.} \quad \Delta c_t + k_{t+\Delta} = \Delta k_t^\alpha + (1 - \Delta\delta) k_t$$

- Subtract $v(k_t)$ from both sides and insert the budget constraint into $v(k_{t+\Delta})$

$$0 = \max_{c_t}\{\Delta u(c_t) + v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t) - \Delta\rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

The Ramsey growth model

- From before

$$0 = \max_{c_t}\{\Delta u(c_t) + v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t) \\ - \Delta\rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

- Divide by Δ

$$0 = \max_{c_t}\{u(c_t) + \frac{v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t)}{\Delta} \\ - \rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

The Ramsey growth model

- From before

$$0 = \max_{c_t} \{ u(c_t) + \frac{v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t)}{\Delta} - \rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) \}$$

- Take limit $\Delta \to 0$ and rearrange

$$\rho v(k_t) = \max_{c_t} \{ u(c_t) + v'(k_t)(k_t^\alpha - \delta k_t - c_t) \}$$

- This is know as the Hamilton-Jacobi-Bellman (HJB) equation.

The Ramsey growth model: Solving

- Dropping time notation we have

$$\rho v(k) = \max_c \{u(c) + v'(k)(k^\alpha - \delta k - c)\}$$

- This is simple to solve and (can be) blazing fast!

The Ramsey growth model: Solving

- Dropping time notation we have

$$\rho v(k) = \max_{c}\{u(c) + v'(k)(k^{\alpha} - \delta k - c)\}$$

- This is simple to solve and (can be) blazing fast!
- Why fast? Maximization is trivial: First order condition

$$u'(c) = v'(k)$$

- So if we know $v'(k)$ we know optimal c without searching for it!

The Ramsey growth model: Solving

- How do we find $v'(k)$?
- Suppose we have hypothetical values of $v(k)$ on a uniformly spaced grid of k, $\mathcal{K} = \{k_0, k_1, \ldots, k_N\}$ with stepsize Δk.
- We can then approximate $v'(k)$ at gridpoint k_i $(i \neq 1, N)$ as

$$\begin{aligned} v'(k_i) = 0.5(v(k_{i+1}) - v(k_i))/\Delta k \\ + 0.5(v(k_i) - v(k_{i-1}))/\Delta k \end{aligned}$$

or

$$v'(k_i) = \frac{v(k_{i+1}) - v(k_{i-1})}{2\Delta k}$$

The Ramsey growth model: Solving

- and for k_1 and k_N

$$v'(k_1) = (v(k_2) - v(k_1))/\Delta k$$

and

$$v'(k_N) = (v(k_N) - v(k_{N-1}))/\Delta k$$

- There are many ways of doing this. If you have a vector of $v(k)$ values – call it `V` – then `dV=gradient(V)/dk`.

The Ramsey growth model: Solving

- I prefer an alternative method.
- Construct the matrix D as

$$D = \begin{pmatrix} -1/dk & 1/dk & 0 & 0 & \dots & 0 \\ -0.5/dk & 0 & 0.5/dk & 0 & \dots & 0 \\ 0 & -0.5/dk & 0 & 0.5/dk & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -1/dk & 1/dk \end{pmatrix}$$

- Then

$$v'(k) \approx D \times v(k)$$

The Ramsey growth model: Solving

Algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of $\texttt{V}_0$.
3. Calculate the derivative as $\texttt{dV}_0\texttt{=D*V}_0$.
4. Find $\texttt{V}_1$ from

$$\rho V_1 = u(c_0) + dV_0(k^\alpha - \delta k - c_0),$$
$$\text{with} \quad u'(c_0) = dV_0$$

5. Back to step 3 with V_1 replacing V_0. Repeat until convergence.

The Ramsey growth model: Solving

Algorithm

1. Construct a grid for k.
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$$\begin{aligned} \rho V_1 &= u(c_0) + dV_0(k^\alpha - \delta k - c_0), \\ \text{with} \quad u'(c_0) &= dV_0 \end{aligned}$$

5. Back to step 3 with V_1 replacing V_0. Repeat until convergence.

Beware: The contraction mapping theorem does not work, so convergence is an issue. Solution: update slowly. That is, $V_1 = \gamma V_1 + (1-\gamma)V_0$, for a low value of γ.

The Ramsey growth model: Solving

Alternative algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of $\texttt{V}_0$.
3. Calculate the derivative as $\texttt{dV}_0\texttt{=D*V}_0$.
4. Find $\texttt{V}_1$ from

$$V_1 = \Gamma(u(c_0) + dV_0(k^\alpha - \delta k - c_0) - \rho V_0) + V_0,$$
$$\text{with} \quad u'(c_0) = dV_0$$

5. Back to step 3 with V_1 replacing V_0. Repeat until convergence.

We will take a look at an alternative way of doing things tomorrow.

The Ramsey growth model: Solving

Alternative vs. standard algorithm

- But to me they sort of look the same
- I.e.

$$
\begin{aligned}
V_1 &= \gamma\frac{1}{\rho}(u(c_0) + dV_0(k^\alpha - \delta k - c_0)) + (1-\gamma)V_0 \\
&= \frac{\gamma}{\rho}(u(c_0) + dV_0(k^\alpha - \delta k - c_0) - \rho V_0) + V_0
\end{aligned}
$$

- So as long as $\frac{\gamma}{\rho} = \Gamma$ they should be identical.