

# The Cyclical Behavior of Equilibrium Unemployment and Vacancies Shimer (2005)

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# Bellman Equation

## Worker

- ▶ Unemployment value

$$U_p = z + \delta \{ f(\theta_p) \mathbb{E}_p W_{p'} + (1 - f(\theta_p)) \mathbb{E}_p U_{p'} \} \quad (1)$$

- ▶ Employment value

$$W_p = w_p + \delta \{ (1 - s) \mathbb{E}_p W_{p'} + s \mathbb{E}_p U_{p'} \} \quad (2)$$

# Bellman Equation

Firm

- ▶ Hiring value

$$J_p = p - w_p + \delta(1 - s)\mathbb{E}_p J_{p'} \quad (3)$$

- ▶ Vacancy value

$$V_p = -c + \delta q(\theta_p)\mathbb{E}_p J_{p'} \equiv 0 \quad (4)$$

# Productivity

The log of productivity follows AR(1) process

$$\log(p) = \rho \log(p) + \varepsilon \quad (5)$$

where

$$\log(p) \sim N(\mu_\lambda, \sigma_\lambda^2), \quad \varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$$

# Optimal Control

## Market tightness

- ▶ Control in this problem consists of  $w_p, \theta_p, u_p$  and the state is  $p$
- ▶ Market tightness  $\theta_p$  is given by solving the following equation of hire rate from free entry condition

$$q(\theta_p) = \frac{c}{\delta \mathbb{E}_p J_{p'}} \quad (6)$$

- ▶ And market tightness

$$\theta_p = \left( \frac{q(\theta_p)}{\mu} \right)^{-\frac{1}{\eta}} \quad (7)$$

- ▶ Employ Rate is given by

$$f(\theta_p) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}} \quad (8)$$

# Optimal Control

## Continued

- ▶ Optimal wage at each productivity level is given by the Nash Bargaining:

$$W_p - U_p = \beta(W_p - U_p + J_p) \quad (9)$$

- ▶ Note Bellman Equation of  $W_p$  given by 2,  $U_p$  given by 1,  $J_p$  given by 3
- ▶ Following the algebra given in slide 17, optimal wage for each  $p$  is

$$w_p = \beta p + (1 - \beta)z + \beta c \theta_p \quad (10)$$

- ▶ And unemployment rate

$$u_p = \frac{\delta}{\delta + f(\theta_p)} \quad (11)$$

# Calibration

Parameter	Symbol	Value
Productivity std.	$\sigma_{logp}$	0.05
Productivity mean	$\mu_{logp}$	1
Stochastic std.	$\sigma_{\varepsilon}$	0.03
Stochastic mean	$\mu_{\varepsilon}$	0
Separation rate	$s$	0.1
Discount rate	$r$	0.012
Value of leisure	$z$	0.4
Matching function	$\mu$	1.355
Matching function	$\alpha$	0.72
Bargaining Power	$\beta$	0.72
Cost of vacancy	$c$	0.213

Table 1: Parameter Calibration

# Question a I

## Discretization Algorithm

Inspired by Karen A. Kopecky 2006 Lecture Note

1. Choose a relative error tolerance level  $\epsilon$ ;
2. Discretize the state space by constructing a grid for productivity

$$p = \exp\{\log p\} \text{ where } \log p = \{\log p_1, \log p_2, \dots, \log p_n\}$$

given by the Tauchen method. The  $n$  is chosen at

3. Start with an initial guess of the value function  $V^{(0)}(p)$  is a vector of length  $n$ , i.e.,  $V^{(0)}(p) = \{V_i^{(0)}\}_{i=1}^n$ , where  $V_i^{(0)} = V^{(0)}(p_i)$ .  $V$  here represents  $U, W, J$ . The initial guess is ones.



# Question a II

## Discretization Algorithm

4. Update the value function using equations 1 to 10, specifically
  - 4.1 Fix the current productivity level at one of the grid points,  $p_i$  from  $i = 1$
  - 4.2 For each possible choice of productivity next period, calculate optimal control in the following order:

$$q(\theta_{p_i}) = \frac{c}{\delta \sum_{j=1}^n p_{i,j} J^{(0)}(p_j)}$$

$$f(\theta_{p_i}) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$

$$\theta_{p_i} = \left( \frac{q(\theta_{p_i})}{\mu} \right)^{-\frac{1}{\eta}}$$

$$w_{p_i} = \beta p_i + (1 - \beta)z + \beta c \theta_{p_i}$$

- 4.3 and update the value function system with

# Question a III

## Discretization Algorithm

$$U_{p_i}^{(1)} = z + \delta \{ f(\theta_{p_i}) \sum_{j=1}^n p_{i,j} W^{(0)}(p_j) + (1 - f(\theta_{p_i})) \sum_{j=1}^n p_{i,j} U^{(0)}(p_j) \}$$

$$W_{p_i}^{(1)} = w_{p_i} + \delta \{ (1 - s) \sum_{j=1}^n p_{i,j} W^{(0)}(p_j) + s \sum_{j=1}^n p_{i,j} U^{(0)}(p_j) \}$$

$$J_{p_i}^{(1)} = p_i - w_{p_i} + \delta (1 - s) \sum_{j=1}^n p_{i,j} J^{(0)}(p_j)$$

- 4.4 Choose a new grid point for productivity, go through 4.1 to 4.3. Once we have done the update for all productivity grid, we have new system of value function  $V_p^{(1)}$
- 4.5 Compute distance between the two systems of value functions following the sup norm

$$d = \max_{i \in \{1, \dots, n\}} |V_i^{(0)} - V_i^{(1)}|$$

# Question a IV

## Discretization Algorithm

4.6 If distance is within the error tolerance level,  $d \leq tol * ||V_1^{(1)}||$ , the functions have converged and go to step 5, or else go back to step 4.

5. Calculate the optimal control for each productivity level:

$$q(\theta_{p_i}^*) = \frac{c}{\delta \sum_{j=1}^n p_{i,j} J^*(p_j)}$$

$$f(\theta_{p_i}^*) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$

$$\theta_{p_i}^* = \left( \frac{q(\theta_{p_i}^*)}{\mu} \right)^{-\frac{1}{\eta}}$$

$$w_{p_i}^* = \beta p_i + (1 - \beta)z + \beta c \theta_{p_i}^*$$

$$u_p^* = \frac{\delta}{\delta + f(\theta_p^*)}$$

where  $J^*$  is the converged value function.

# Tauchen Method

Use `discretizeAR1_Tauchen` function from the Matlab Toolbox of Kirkby (2023) .

# Question a

## Parametric Approximation

0. Choose Hermite interpolation polynomials to approximate  $\hat{V}(p; \mathbf{coefs})$  in the form of  $f(x) = a(x - x_1)^3 + b(x - x_1)^2 + c(x - x_1) + d$  with Matlab code `pchip`. Report initial parameters with `pp.coefs` for each value function, save as `old`
1. Maximize control and calculate value function at each productivity level as done in Discretization method
2. Fit for new value function system and report parameters and save as `new`
3. If  $\|\hat{V}(p; \mathbf{coef\_sold}) - \hat{V}(p; \mathbf{coefs\_new})\| < tol$ , stop; else go to step 1.

# Optimal Controls from two methods

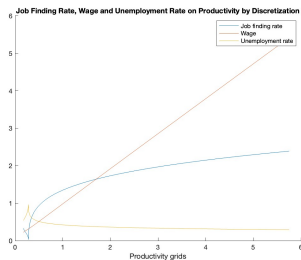


Figure 1: Discretization

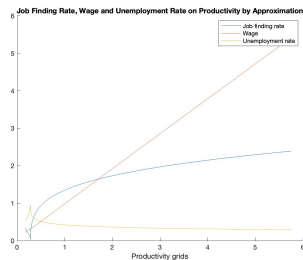


Figure 2: Approximation

# Two Results

## Question b

To use the polynomial interpolation function from course material directly

$\log(p)$	w	u	f
0.4	1.4446	0.3865	1.5683
0.7	1.9336	0.3627	1.7362
1	2.5922	0.3409	1.9104
1.3	3.4792	0.3206	2.0936
1.6	4.6740	0.3016	2.2879

Table 2: Wage, Unemployment rate and Job Finding Rate

# Two Results

## Question c

	u	f	p
Data Std.	0.190	0.118	0.020
Approximation Model Std.	0.175	0.697	1.491
Discretization Model Std.	0.175	0.697	

Table 3: Model fit on Unemployment, Job finding rate and productivity



# Appendix A

## Optimal wage

$$\begin{aligned}W_p - U_p &= \beta(W_p - U_p + J_p) \\ \Leftrightarrow w_p - z + \delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) &= \\ \beta(p - z + \delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) + \delta(1 - s)\mathbb{E}_p J_{p'}) & \\ \Leftrightarrow w_p = \beta p + (1 - \beta)z + (\beta - 1)\delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) & \\ + \frac{\beta c(1 - s)}{q(\theta_p)} & \\ \Leftrightarrow w_p = \beta p + (1 - \beta)z - \frac{\beta c \delta(1 - s - f(\theta_p))}{q(\theta_p)} + \frac{\beta c(1 - s)}{q(\theta_p)} & \\ \Leftrightarrow w_p = \beta p + (1 - \beta)z + \beta c \theta_p &\end{aligned}$$

where we use the fact that  $\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'} = \frac{\beta}{1-\beta} \mathbb{E}_p J_{p'}$  and  $f(\theta_p)/q(\theta_p) = \theta_p$

# Reference I

Kirkby, R. (2023), 'Value function iteration (vfi) toolkit for matlab',  
<https://github.com/vfitoolkit/VFIToolkit-matlab>. Github.

Shimer, R. (2005), 'The cyclical behavior of equilibrium unemployment and vacancies', *American Economic Review* **95**(1), 25–49.

**URL:** <https://www.aeaweb.org/articles?id=10.1257/0002828053828572>