Perturbation and Dynare

Perturbation

Tools for Macroeconomists: The essentials

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Perturbation

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    - more generally, it is a way of taking derivatives
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- Perturbation is a way to approximate a function
  - more generally, it is a way of taking derivatives
  - as such it has broad applications
- it uses Taylor's theorem
- it also uses the Implicit function theorem

Perturbation

Theoretical Underpinning

Theorem *Let $k \geq 1$ be an integer and let function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at point $a \in \mathbb{R}$. Then there exists a function $h_k : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + h_k(x)(x-a)^k,$$

and $\lim_{x \to a} h_k(x) = 0$.

Theorem *Let* $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ *be a continuously differentiable function and let* $\mathbb{R}^{n+m}$ *have coordinates* (x, y). *Fix a point* $(\bar{x}, \bar{y})$ *with* $f(\bar{x}, \bar{y}) = 0$. *If the Jacobian matrix* $\mathcal{J}_{f,y}(\bar{x}, \bar{y})$ *is invertible, then there exists an open set* U *of* $\mathbb{R}^n$ *containing* $\bar{x}$ *such that there exists a unique continuously differentiable function* $g : U \to \mathbb{R}^m$ *such that*

$$g(\bar{x}) = \bar{y}$$

and

$$f(x, g(x)) = 0 \quad \textit{for all } x \in U.$$

Moreover, the partial derivatives of g *in* U *are given by the matrix product*

$$\frac{\partial g}{\partial x_j}(x) = -[\mathcal{J}_{f,y}(x, g(x))]^{-1} \left[\frac{\partial f}{\partial x_j}(x, g(x)) \right]$$

Perturbation

Details

Back to the neoclassical model

- the above is all very nice
- but at this point a bit abstract
- lets see if we can write the neoclassical growth model
- in a way that looks like the notation we just used...

Optimality conditions

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$$c_t^{-\gamma} = \beta \mathbb{E}_t c_{t+1}^{-\gamma} \alpha Z_{t+1} k_{t+1}^{\alpha-1}$$

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- σ controls the degree of uncertainty

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$$x_{t+1} = h(x_t, \sigma) + \sigma\widetilde{\epsilon}_{t+1}$$

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$$x_{t+1} = h(x_t, \sigma) + \sigma \widetilde{\epsilon}_{t+1}$$

$$c_t = g(x_t, \sigma)$$

  - notice that uncertainty (σ) explicitly enters the policy function!

Rewrite the system

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$$\mathbb{E}_t F\left(g(h(x_t,\sigma)+\sigma\widetilde{\epsilon}_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\sigma\widetilde{\epsilon}_{t+1}, x_t\right) = 0$$

Perturbation

1ST ORDER PERTURBATION AND CERTAINTY EQUIVALENCE

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\begin{aligned}
g(x,\sigma) \approx & g(\overline{x},\overline{\sigma}) + g_x(\overline{x},\overline{\sigma})(x-\overline{x}) + g_\sigma(\overline{x},\overline{\sigma})(\sigma-\overline{\sigma}) \\
& + 1/2[g_{xx}(\overline{x},\overline{\sigma})(x-\overline{x})^2 + 2g_{x\sigma}(\overline{x},\overline{\sigma})(x-\overline{x})(\sigma-\overline{\sigma}) \\
& + g_{\sigma\sigma}(\overline{x},\overline{\sigma})(\sigma-\overline{\sigma})^2] + \cdots
\end{aligned}
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h(x,\sigma) \approx & h(\overline{x},\overline{\sigma}) + h_x(\overline{x},\overline{\sigma})(x-\overline{x}) + h_\sigma(\overline{x},\overline{\sigma})(\sigma-\overline{\sigma}) \\
& + 1/2[h_{xx}(\overline{x},\overline{\sigma})(x-\overline{x})^2 + 2h_{x\sigma}(\overline{x},\overline{\sigma})(x-\overline{x})(\sigma-\overline{\sigma}) \\
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- we approximate the policy functions with a polynomial
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- how do we solve for them?
- recall that $F[x_t, \sigma] = 0$ for any value of x and σ
- $\rightarrow$ derivatives (of any order) of F also 0!

$$F_{x^k, \sigma^j}[x_t, \sigma] = 0 \quad \forall x, \sigma, j, k$$

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    - $\bar{c} = g(\bar{x}, 0)$ and $\bar{x} = h(\bar{x}, 0)$
- why is so convenient?
- in principle you can approximate around any point

Getting the policy function derivatives

- under 1st order perturbation we have

$$g(x,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x-\bar{x}) + g_\sigma(\bar{x},0)\sigma$$

$$h(x,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x-\bar{x}) + h_\sigma(\bar{x},0)\sigma$$

- we also know that

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- solve for the derivatives (coefficients of approximating Taylor polynomial)

$$F_{x^k,\sigma^j}[x_t,\sigma] = 0 \qquad \forall x,\sigma,j,k$$

Deriving coefficients of Taylor polynomial

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$$F_x = \frac{\partial F}{\partial x_{t+2}} \frac{\partial x_{t+2}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial x_t} + \frac{\partial F}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial x_t} + \frac{\partial F}{\partial x_t}$$

$$= \overline{F}_1 \frac{\partial x_{t+2}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial x_t} + \overline{F}_2 \frac{\partial x_{t+1}}{\partial x_t} + \overline{F}_3$$

$$= \overline{F}_1 h_x^2 + \overline{F}_2 h_x + \overline{F}_3 = 0$$

- $\frac{\partial F(x_{t+2}, x_{t+1}, x_t, \sigma)}{\partial x_{t+i}} \big|_{x_{t+2}=x_{t+1}=x_t=\bar{x}, \sigma=0} = \overline{F}_{3-i}$
- $\frac{\partial h(x_t, \sigma)}{\partial x_t} \big|_{x_t=\bar{x}, \sigma=0\ \forall t} = h_x$

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- the variance of shocks does not matter for policy rules
- important limitation of 1st order approximation
  - what economic questions cannot be studied in this case?
- what about higher order approximations?

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- only $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$ matter for policy function
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- only $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$ matter for policy function
- this affects the constant in the policy rule
- can still have important implications
    - certain economic questions can be addressed
    - can have indirect effect on dynamics (how?)
- need 3rd order to capture effect of uncertainty on “slopes”

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- when does the question of accuracy arise?
- what could go wrong?

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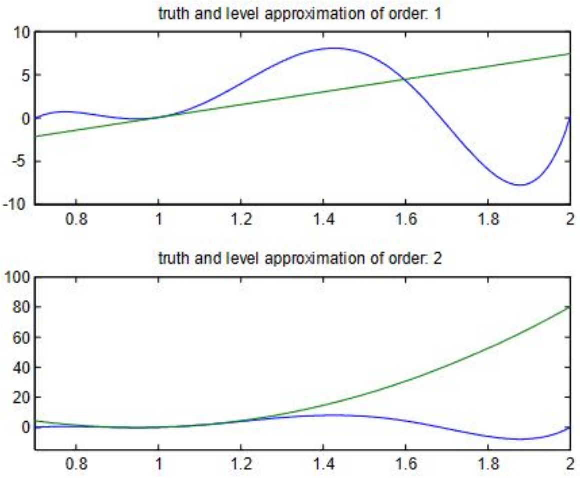
Wouter's example: Consider the true function to be defined on $x \in [0.7, 2]$ s.t.

$$f(x) = -690.59 + 3202.4x - 5739.45x^2 + 4954.2x^3 - 2053.6x^4 + 327.1x^5$$

Wouter's example: all kinds of wild things can happen



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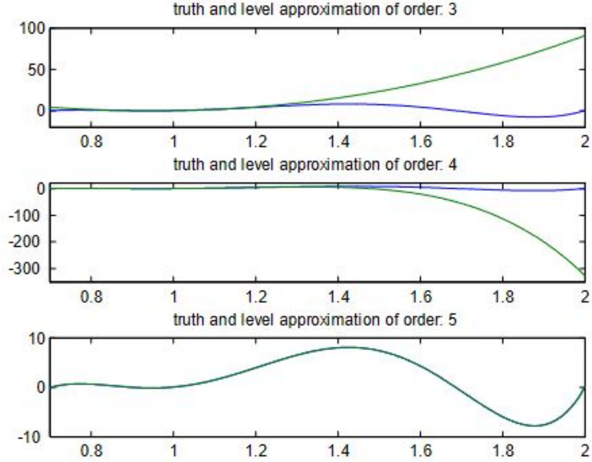

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- can't handle certain features (non-differentiabilities)
- "local" solution method


That's all Folks!