Advanced Tools in Macroeconomics

Occasionally Binding Constraints

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Regular linear methods

- Advantages and disadvantages of linearized methods
  - Advantages: Fast methods that can deal with a (very) large state space. Models can be estimated.
  - Disadvantages: Uses local approximations, so accuracy only guaranteed around steady state. Certainty equivalence, so no precautionary savings. Regular perturbation cannot deal with inequality constraints unless they *always* bind.

Extensions considered here

- Regime switching, that is, exogenous *switching* between regimes when a constraint binds and a regime when it does not bind.
- Quite a few models, including ZLB models, fall in this category.

Overview

The underlying idea is simple enough.

- Consider a two state version.
- You will have one linear system in each state.
- But with some probability you will, in the next period, jump to the other state (and vice versa).
- The fact that you may jump will influence what the linear system looks like in each state.

But let's start with some basic tips and tricks about linear(ized) systems.

Overview

Before dealing with the occasionally binding constraints, we first

- Describe a simple method to find linear approximations around the steady state (without using Dynare). This is DIY linearization.
- Extend this procedure to find linear approximation around different points.

With these tools in place, we will be ready to deal with occasionally binding constraints.

DIY Linearization

- Without occasionally binding constraints, most models can be written in the following way,

$$E_t[F(x_{t-1}, x_t, x_{t+1})] = 0$$

- Where x is a vector of endogenous and exogenous (possibly stochastic) variables
- The non-stochastic steady state, x^*, satisfies

$$F(x^*, x^*, x^*) = 0$$

- In a standard neoclassical growth model this amounts to a steady state capital stock, k^*, such that

$$1 = \beta(1 + f'(k^*) - \delta)$$

DIY Linearization

- Linearisation techniques are very simple. Take a first order Taylor expansion of

$$E_t[F(x_{t-1}, x_t, x_{t+1})] = 0$$

around $x_t = x_{t+1} = x_{t+2} = x^*$

- and we get

$$F(x^*, x^*, x^*) + J_{x_{t-1}}(x_{t-1} - x^*) + J_{x_t}(x_t - x^*) \\ + J_{x_{t+1}}(E_t x_{t+1} - x^*) = 0$$

DIY Linearization

- Or simply

$$J_{x_{t-1}}(x_{t-1} - x^*) + J_{x_t}(x_t - x^*) + J_{x_{t+1}}(E_t x_{t+1} - x^*) = 0$$

  where J_{x_t} is the Jacobian of $F(x_{t-1}, x_t, x_{t+1})$ with respect to x_t evaluated at $x_{t-1} = x_t = x_{t+1} = x^*$.
- The convenient part of this is that uncertainty vanishes, and we can focus on expected variables instead (certainty equivalence).

DIY Linearization

This can be written as

$$Au_{t-1} + Bu_t + Cu_{t+1} = 0$$

with $u_t = x_t - x_t^*$, and where A, B, and C represents the three Jacobians.

DIY Linearization

$$Au_{t-1} + Bu_t + Cu_{t+1} = 0$$

- The great thing about this is that systems like these are
  1. Arbitrarily general (can be of very high dimensions)
  2. Dead-easy to solve
  3. Blazing fast
  4. Uniqueness/stability and so on can be checked by the Blanchard and Kahn's (1980) conditions.
- It's always smart to solve models using linearisation techniques first to check that you get something sensible.

DIY Linearization

$$Au_{t-1} + Bu_t + Cu_{t+1} = 0$$

- So how do we solve them?
- Let's stick to what we know: *Time Iteration*.
- We are looking for a linear solution $u_t = Fu_{t-1}$
  1. Here u_{t-1} is the state, and u_t the "choice variable".
  2. F is a matrix of the same dimensionality as the Jacobians above.

DIY Linearization

$$Au_{t-1} + Bu_t + Cu_{t+1} = 0$$

- The procedure of time iteration: Given how you act tomorrow, solve for the optimal choice today.
- If the initial guess for F is called F_0, then using this for tomorrow's behavior implies

$$Au_{t-1} + Bu_t + CF_0 u_t = 0.$$

from this, we get an update for F, that is an updated relationship between u_t and u_{t-1}.

DIY Linearization

- More generally, for some $n \geq 0$ we find u_t as

$$Au_{t-1} + Bu_t + CF_n u_t = 0$$

  and update F_n to F_{n+1} until convergence.

DIY Linearization

- More generally, for some $n \geq 0$ we find u_t as

$$Au_{t-1} + Bu_t + CF_n u_t = 0$$

  and update F_n to F_{n+1} until convergence.
- Solving for u_t

$$u_t = \underbrace{(B + CF_n)^{-1}(-A)}_{F_{n+1}} u_{t-1}$$

DIY Linearization

- Thus iterate on

$$F_{n+1} = (B + CF_n)^{-1}(-A),$$

  until

$$\|A + BF_{n+1} + CF_{n+1}^2\| \approx 0$$

- Since this goes fast, you can/should use a tight convergence criterion, like 1e(-12).

DIY Linearization

- Suppose you had shocks in the system such that

$$Au_{t-1} + Bu_t + CE[u_{t+1}] + \varepsilon_t = 0$$

- Then $u_t = Fu_{t-1} + Q\varepsilon_t$, and,

$$Au_{t-1} + Bu_t + CE_t[Fu_t + Q\varepsilon_{t+1}] + \varepsilon_t = 0$$

or

$$Au_{t-1} + Bu_t + CFu_t + \varepsilon_t = 0$$

DIY Linearization

$$Au_{t-1} + Bu_t + CFu_t + \varepsilon_t = 0$$

- Solving

$$u_t = \underbrace{-(B + CF)^{-1} A}_{F} \, u_{t-1} \underbrace{-(B + CF)^{-1}}_{Q} \varepsilon$$

- Thus F is the same as before.
- Q is given by

$$Q = -(B + CF)^{-1}$$

which can be found after we have found F

DIY Linearization

- Is the solution stable?
- If the eigenvalues of F are less than one in absolute value it is.

DIY Linearization

- Is the solution stable?
- If the eigenvalues of F are less than one in absolute value it is.
- Are there other stable solutions too?
- Iterate on

$$S_{n+1} = (B + AS_n)^{-1}(-C),$$

- And if the eigenvalues of S are less than one in absolute value there are no other stable solutions.

Linearization around an arbitrary point

- Before introducing regime switching, we generalize the procedure to allow expansion around an arbitrary point.
- The model is again

$$E_t[F(x_{t-1}, x_t, x_{t+1})] = 0$$

- Now suppose we take a first-order Taylor expansion around $\bar{x} \neq x^*$, and that

$$F(\bar{x}, \bar{x}, \bar{x}) = D$$

Linearization around an arbitrary point

- We then get

$$D + J_{x_{t-1}}(x_{t-1} - \bar{x}) + J_{x_t}(x_t - \bar{x}) \\ + J_{x_{t+1}}(E_t x_{t+1} - \bar{x}) = 0$$

- where J_{x_t} is the Jacobian of $F(x_{t-1}, x_t, x_{t+1})$ with respect to x_t evaluated at $x_{t-1} = x_t = x_{t+1} = \bar{x}$.

Linearization around an arbitrary point

- Or simply

$$Au_{t-1} + Bu_t + Cu_{t+1} + D = 0$$

with $x_t - \bar{x} = u_t$

Linearization around an arbitrary point

- Now, our solution is not of the type

$$u_t = Fu_{t-1}$$

but instead

$$u_t = E + Fu_{t-1}$$

Linearization around an arbitrary point

- With time iteration we are searching for a u_t such that

$$Au_{t-1} + Bu_t + C(E_n + F_n u_t) + D = 0$$

- Thus,

$$u_t = \underbrace{(B + CF_n)^{-1}(-(D + CE_n))}_{E_{n+1}} + \underbrace{(B + CF_n)^{-1}(-A)}_{F_{n+1}}u_{t-1}$$

- Notice that F_n can be updated without information of E_n or E_{n+1}.

Linearization around an arbitrary point

- Therefore we iterate as usual

$$F_{n+1} = (B + CF_n)^{-1}(-A)$$

- Until

$$\|A + BF_{n+1} + CF_{n+1}^2\| \approx 0$$

- And once F_n has converged, we find E as the solution to

$$E = (B + CF)^{-1}(-(D + CE))$$

or simply

$$E = (B + C + CF)^{-1}(-D)$$