# Generalized Schur Method*

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## 1 The Method

The linear model can be written in the state space form,

$$B\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} = A\begin{pmatrix} x_t \\ y_t \end{pmatrix} + G(\epsilon_t) \tag{1}$$

where $x_t$ are predetermined or state variable. If $B$ is not invertiable, then the BK method does not work anymore. We need use generalized Schur decomposition to help us. The logic is almost the same as in BK method, but different ways to handle the matrices $A$ and $B$. A generalized Schur method is so-called QZ decomposition of the pencil $< A, B >$

$$B = QTZ^T$$

$$A = QSZ^T$$

where $Q, Z$ are orthogonal matrices and $T, S$ are upper triangular matrices. The eigenvalues of the system are given by $\lambda_{ii} = \frac{s_{ii}}{t_{ii}}$, where $s_{ii}, t_{ii}$ are the associated diagonal elements of matrics $T, S$ respectively. If $t_{ii} = 0, s_{ii} \geq 0$, then $\lambda_{ii} = +\infty$; If $t_{ii} = 0, s_{ii} \leq 0$, then $\lambda_{ii} = -\infty$; If $t_{ii} = 0, s_{ii} = 0$, then $\lambda_{ii} \in \mathbb{C}$. We look at deterministic version of the model (1):

$$QTZ^T\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} = QSZ^T\begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Eliminating $Q$ and writing out $T, Z^T, S$ so as the upper part is stable and lower part is explosive:

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Looking at the lower part,

$$T_{22}\left(Z_{21}x_{t+1} + Z_{22}E_t y_{t+1}\right) = S_{22}\left(Z_{21}x_t + Z_{22}y_t\right)$$

*The note is borrowed heavily from McCandless(2008).

If there is a stable solution or equlibrium, this requires that

$$Z_{21}x_t + Z_{22}y_t = 0$$

and $Z_{22}$ is invertible. And this in turn means that we find the solution,

$$y_t = -\left(Z_{22}\right)^{-1} Z_{21}x_t \equiv Nx_t$$

where $N = -\left(Z_{22}\right)^{-1} Z_{21}$. Going back to the original model (1), we have

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} x_{t+1} \\ -Nx_{t+1} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_t \\ -Nx_t \end{pmatrix}$$

the upper part can be written as

$$\left(B_{11} - B_{12}N\right)x_{t+1} = \left(A_{11} - A_{12}N\right)x_t$$

if $\left(B_{11} - B_{12}N\right)$ is invertiable, then we can solve

$$x_{t+1} = \left(B_{11} - B_{12}N\right)^{-1}\left(A_{11} - A_{12}N\right)x_t \equiv Cx_t$$

So the solution can be written as

$$y_t = Nx_t$$

$$x_{t+1} = Cx_t$$

# 2 Hansen's Indivisible Labor model

The model can be described as

$$max \sum_{t=0}^{\infty} \beta^t \left(logC_t + B_1H_t\right)$$

s.t.

$$C_t + K_{t+1} - (1-\delta)\,K_t = Y_t$$
$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}$$
$$log\lambda_{t+1} = \gamma log\lambda_t + \epsilon_{t+1}$$

Where $B_1$ is constant. We define $r_t \equiv \frac{\partial Y_t}{\partial K_t}, \omega_t \equiv \frac{\partial Y_t}{\partial H_t}$ as the marginal product of capital and labor respectively, i.e., the capital rate and wage rate respectively. The first order conditions can be calculated as the following:

$$1 = \beta E_t\left(\frac{C_t}{C_{t+1}}\left(r_{t+1} + (1-\delta)\right)\right)$$

$$C_t = -\frac{(1-\theta)\,Y_t}{B_1H_t}$$

We use the same notation as in McCandless(2008).

$$\bar{K}\tilde{K}_{t+1} = \bar{Y}\tilde{Y}_t - \bar{C}\tilde{C}_t + (1-\delta)\,\bar{K}\tilde{K}_t$$

$$\tilde{\lambda}_t = \gamma\tilde{\lambda}_{t-1} + \epsilon_t$$

$$0 = \tilde{\lambda}_t - \theta\tilde{Y}_t + \theta\tilde{K}_t - (1-\theta)\,\tilde{C}_t$$

$$0 = \tilde{K}_t + \tilde{r}_t - \tilde{Y}_t$$

$$E_t\tilde{C}_{t+1} - \beta\bar{r}E_t\tilde{r}_{t+1} = \tilde{C}_t$$

We define

$$\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} \equiv \begin{pmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ \tilde{Y}_t \\ E_t\tilde{C}_{t+1} \\ E_t\tilde{r}_{t+1} \end{pmatrix}$$

then write the model in state space form (1). We have

$$B = \begin{pmatrix} \bar{K} & 0 & -\bar{Y} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & \theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\bar{r}\beta \end{pmatrix}$$

$$A = \begin{pmatrix} (1-\delta)\,\bar{K} & 0 & 0 & -\bar{C} & 0 \\ 0 & \gamma & 0 & 0 & 0 \\ \theta & 0 & 0 & -(1-\theta) & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

# 3 The Steady States

Before we can calculate the steady states, we need calibrate the parameters: $\beta = 0.99, \delta = 0.025, \theta = .36, \gamma = .95$. The steady state value of technology $\bar{\lambda} = 1$. Parameter $B_1$ is chosen so as $\bar{H} = \frac{1}{3}$.

From the Euler condition, we have

$$\bar{r} = \frac{1}{\beta} - (1-\delta) = 0.0351$$

then from other FOC conditions, we have

$$\bar{C} = -\frac{(1-\theta)\,\bar{Y}}{B\bar{H}}$$

$$\bar{Y} = \bar{\lambda}\bar{K}^{\theta}\bar{H}^{1-\theta}$$

$$\bar{C} = \bar{Y} - \delta\bar{K}$$

$$\bar{r} = \theta\bar{\lambda}\bar{K}^{\theta-1}\bar{H}^{1-\theta}$$

From last equation, we can calculate the capital

$$\bar{K} = \left(\frac{\theta}{\bar{r}}\right)^{\frac{1}{1-\theta}}\bar{H} = 12.6698$$

Then from production technology, $\bar{Y} = 1.2353$, $\bar{C} = 0.9186$. After steady states are known, we could find out the Matrix $A$ and $B$.

# 4 The Code

First, we need do some preparation before we solve the system:
then find the solution: