Models with Heterogeneous Agents: Theory

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Overview Monday and Tuesday Material

- Theory of models with heterogeneous agents
  - key to understand state varables
- Penalty function instead of borrowing constraints
  - $\Longrightarrow$ Dynare becomes a possibility
- Famous Krusell-Smith algorithm
- Simulating economies with heterogeneous agents
  - Importance of imposing equilibrium
- Famous Reiter approach
  - General idea
  - Bopart, Krusell, Mitman
  - Legrand & Ragot
- Homotopy

SIMPLE MODEL WITH HETEROGENEOUS AGENTS

First model with heterogeneous agents

- Agents are *ex ante* the same,
  but face different idiosyncratic shocks
  $\Longrightarrow$ agents are different *ex post*

- Incomplete markets
  $\Longrightarrow$ heterogeneity cannot be insured away

Individual agent

- Subject to employment shocks:
  - $\varepsilon_{i,t} \in \{0,1\}$
- Incomplete markets
  - only way to save is through holding capital
  - borrowing constraint $k_{i,t+1} \geq 0$

Aggregate shock

- $z_t \in \{z^b, z^g\}$
- z_t affects
  1. aggregate productivity
  2. probability of being employed
- transition probabilities are such that
  - unemployment rate only depends on current z_t
  - thus
    - $u_t = u^b$ if $z_t = z^b$ &
    - $u_t = u^g$ if $z_t = z^g$
    - with $u^b > u^g$.

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathrm{E} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1-\tau_t) w_t \bar{l} \varepsilon_{i,t} + \mu w_t (1-\varepsilon_{i,t}) + (1-\delta) k_{i,t}$$
$$k_{i,t+1} \geq 0$$

- this is a relatively simple problem
  if processes for r_t and w_t are given

Individual agent - foc

$$
\begin{aligned}
\frac{1}{c_{i,t}} &\geq \beta \mathsf{E}_t \left[\frac{1}{c_{i,t+1}} \left(r_{t+1} + 1 - \delta \right) \right] \\
0 &= k_{i,t+1} \left(\frac{1}{c_{i,t}} - \beta \mathsf{E}_t \left[\frac{1}{c_{i,t+1}} \left(r_{t+1} + 1 - \delta \right) \right] \right) \\
c_{i,t} + k_{i,t+1} &= r_t k_{i,t} + (1 - \tau_t) w_t \bar{l} \varepsilon_{i,t} + \mu w_t (1 - \varepsilon_{i,t}) + (1 - \delta) k_{i,t} \\
k_{i,t+1} &\geq 0
\end{aligned}
$$

Firm problem

$$
\begin{aligned}
r_t &= \alpha z_t K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t &= (1-\alpha) z_t K_t^{\alpha} L_t^{-\alpha}
\end{aligned}
$$

These are identical to those of the rep. agent version

Government

Only thing the government does is raise taxes to finance unemployment benefits.

$$
\begin{aligned}
\tau_t w_t \bar{l}(1-u(z_t)) &= \mu w_t u(z_t) \\
\tau_t &= \frac{\mu u(z_t)}{\bar{l}(1-u(z_t))}
\end{aligned}
$$

STATE VARIABLES AND EQUILIBRIUM

What aggregate info do agents care about?

- current **and** future values of r_t and w_t
- the period-t values of r_t and w_t only depend on z_t and the aggregate capital stock, K_t
  - !!! In many models, however, current-period prices also depend on other characteristics of the distribution such as the variance

What aggregate info do agents care about?

- the future values, i.e., $r_{t+\tau}$ and $w_{t+\tau}$ with $\tau > 0$ depend on
  - future values of mean capital stock, i.e. $K_{t+\tau}$, & $z_{t+\tau}$
- $\Longrightarrow$ agents are interested in all information that forecasts K_t
- $\Longrightarrow$ typically this includes the complete cross-sectional distribution of employment status and capital levels (**even when** agents only forecast future mean capital stock)

Equilibrium - first part

- Individual policy functions that solve agent's max problem
- A wage and a rental rate given by equations above
  - These are equilibrium conditions if aggregate K_t implied by the household problems is used and aggregate employment, L_t, implied by z_t

Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

  - f_t = cross-sectional distribution of beginning-of-period capital and the employment status *after* the employment status has been realized.
  - z_{t+1} does *not* affect the cross-sectional distribution of capital
  - z_{t+1} does affect the *joint* cross-sectional distribution of capital and employment status

Transition law & timing

- f_t & $z_t \Longrightarrow f_t^{\text{end-of-period}=p_t}$
- $p_t = f_t^{\text{end-of-period}}$ & z_t & $z_{t+1} \Longrightarrow f_{t+1}^{\text{beginning-of-period}} = f_{t+1}$

Transition law & timing

- Let g_t be the cross-sectional distribution of capital (so without any info on employment status)
- Why can I write

$$g_{t+1} = \Upsilon_g(z_t, f_t)?$$

Transition law & continuum of agents

$$\begin{aligned} g_{t+1} &= \Upsilon_g(z_t, f_t) \\ f_{t+1} &= \Upsilon(z_{t+1}, z_t, f_t) \end{aligned}$$

Why are these exact equations without additional noise?

- continuum of agents $\Longrightarrow$ rely on law of large numbers to average out idiosyncratic risk

Recursive equilibrium?

Questions

1. Does an equilibrium exist?
   1. If yes, is it unique?
2. Does a recursive equilibrium exist?
   1. If yes, is it unique?
   2. If yes, what are the state variables?

Recursive equilibrium?

Jianjun Miao (JET, 2006): a recursive equilibrium exist for following state variables:

- usual set of state variables, namely
  - individual shock, $\varepsilon_{i,t}$
  - individual capital holdings, $k_{i,t}$
  - aggregate productivity, z_t
  - joint distribution of income and capital holdings, f_t
- and *cross-sectional distribution of expected payoffs*

Unique?

Heterogeneity $\Longrightarrow$ more reasons to expect multiplicity

- my actions depend on what I think others will do
- heterogeneity tends to go together with frictions and multiplicity more likely with frictions
    - e.g. market externalities

Wealth-recursive (WR) equilibrium

- WR equilibrium is a recursive equilibrium with only $\varepsilon_{i,t}$, $k_{i,t}$, z_t, and f_t as state variables.
  (Also referred to as Krusell-Smith (KS) recursive equilibrium)
- Not proven that WR equilbrium exists in model discussed here (at least not without making unverifiable assumptions such as equilibrium is unique for all possible initial conditions)
- Kubler & Schmedders (2002) give examples of equilibria that are *not* recursive in wealth
  (i.e., wealth distribution by itself is not sufficient)

Wealth distribution not sufficient - Example

- Static economy
  two agents, $i = 1, 2$, two goods, $j = A, B$
- Utility: $\ln q_A + \ln q_B$
- Endowments in state I: $\omega_{1,A} = \omega_{2,A} = 1; \omega_{1,B} = \omega_{2,B} = 1$
- Endowments in state II:
  $\omega_{1,A} = \omega_{2,A} = 1; \omega_{1,B} = \omega_{2,B} = 10/9$
- Normalization: $p_A = 1$

Wealth distribution not sufficient - Example

- State I:
  - equilibrium: $p_B = 1; q_{1,A} = q_{2,A} = 1; q_{1,B} = q_{2,B} = 1$
    wealth of each agent: $= 2$
- State II:
  - equilibrium: $p_B = 0.9; q_{1,A} = q_{2,A} = 1; q_{1,B} = q_{2,B} = 10/9$
    wealth of each agent: $= 2$

- Thus: same wealth levels, but different outcome

How to proceed?

- Wealth distribution may not be sufficient!
- For numerical approximation less problematic:
  - Approximations always ignore bits
    (for example, higher-order polynomial terms)
- After obtaining solution, you should check whether the approximation is accurate or not

How to proceed?

- For now we assume that a wealth recursive equilibrium exists (or an approximation based on it is accurate)
- This is still a tough numerical problem

If a wealth recursive equilibrium exists

- Suppose that recursive RE for usual state space exists
  - $s_{i,t} = \{\varepsilon_{i,t}, k_{i,t}, S_t\} = \{\varepsilon_{i,t}, k_{i,t}, z_t, f_t\}$
- Equilibrium:
  - $c(s_{i,t})$
  - $k(s_{i,t})$
  - $r(S_t)$
  - $w(S_t)$
  - $\Upsilon(z_{t+1}, z_t, f_t)$

Alternative representation state space

- Suppose that recursive RE for usual state space exist
  - $s_{i,t} = \{\varepsilon_{i,t}, k_{i,t}, S_t\} = \{\varepsilon_{i,t}, k_{i,t}, z_t, f_t\}$
- What determines current shape f_t?
  - z_t, z_{t-1}, f_{t-1} or
  - $z_t, z_{t-1}, z_{t-2}, f_{t-2}$ or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, f_{t-3}$ or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, z_{t-4}, f_{t-4}$ or

No aggregate uncertainty

$$S_t = \lim_{n \longrightarrow \infty} \{z_t, z_{t-1}, \cdots, z_{t-n}, f_{t-n}\}$$

- Why is this useful from a numerical point of view,
  - when z_t is stochastic?
  - when z_t is not stochastic ($\equiv$ no aggregate uncertainty)?

NO AGGREGATE UNCERTAINTY

No aggregate uncertainty

Aggregate state variables

$$\lim_{n \longrightarrow \infty} \{z_t, z_{t-1}, \cdots, z_{t-n}, f_{t-n}\}$$

- **If**
  1. $z_t = z \; \forall t$ and
  2. effect of initial distribution dies out
- **then** S_t constant
  - distribution still matters!
  - but it is no longer a *time-varying* argument

Algorithm to solve for aggregate capital, K

- Guess a value for r
- z implies value for L (these remain constant across iterations)
- firm FOC for $K: z, L$ and r imply value for K^{demand}
- firm FOC for $L: z, L$ and K^{demand} imply value for w
- Solve the individual problem with these values for r & w
- Simulate economy & calculate the supply of capital, K^{supply}
- If $K^{\text{supply}} < K^{\text{demand}}$ then r too low so raise r, say

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

- Iterate until convergence

Algorithm to solve for aggregate capital, K

Using

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

to solve

$$K^{\text{demand}}(r) = K^{\text{supply}}(r)$$

not very efficient

- Value of λ may have to be very low
- More efficient to use equation solver to solve for r

COMPLETE MARKETS
AGGREGATION TO REP AGENT

Aggregation

Statement:

The representative agent model is silly,
because there is no trade in this model,
while there is lots of trade in financial assets in reality

Problem with statement:

RA is justified by complete markets
which relies on lots of trade

Complete markets & exact aggregation

- economy with ex ante identical agents
- J different states
- complete markets $\Longrightarrow J$ contingent claims

Complete markets & exact aggregation

$$\max_{c_{i,t}, b^1_{i,t+1}, \cdots, b^J_{i,t+1}} \frac{\left(c_{i,t}\right)^{1-\gamma}}{1-\gamma} + \beta \mathsf{E}_t \left[v(b^1_{i,t+1}, \cdots, b^J_{i,t+1}) \right]$$

$$\text{s.t.} \quad c_{i,t} + \sum_{j=1}^{J} q^j b^j_{i,t+1} = y_{i,t} + \sum_{j=1}^{J} I_t(j) b^j_{i,t}$$

$$b^j_{i,t+1} > \overline{b} \;\; \text{with } \overline{b} < 0$$

$$I_t(j) = 1 \text{ if current state } = j \text{ o.w. } 0$$

Euler equations individual

$$q^j \left(c_{i,t}\right)^{-\gamma} = \beta \left(c_{i,t+1}^j\right)^{-\gamma} \text{prob}(j) \qquad \forall j$$

This can be written as follows:

$$c_{i,t} = \left(\frac{\beta \text{prob}(j)}{q^j}\right)^{-1/\gamma} c_{i,t+1}^j \qquad \forall j$$

Aggregation

Aggregation across individual i of

$$c_{i,t} = \left(\frac{\beta \text{prob}(j)}{q^j} \right)^{-1/\gamma} c^j_{i,+1} \quad \forall j$$

gives

$$C_t = \left(\frac{\beta \text{prob}(j)}{q^j} \right)^{-1/\gamma} C^j_{t+1} \quad \forall j,$$

which can be rewritten as

$$q^j \left(C_t \right)^{-\gamma} = \beta \left(C^j_{t+1} \right)^{-\gamma} \text{prob}(j) \quad \forall j$$

Back to represenative agent model

- Idential FOCs come out of this RA model:

$$\max_{C_t, B^1_{t+1}, \cdots, B_t+1^J} \frac{(C_t)^{1-\gamma}}{1-\gamma} + \beta \mathsf{E}_t \left[v(B^1_{t+1}, \cdots, B^J_{t+1}) \right]$$

$$s.t. C_t + \sum_{j=1}^{J} q^j B^j_{t+1} = Y_t + \sum_{j=1}^{J} I_t(j) B^j_t$$

$$B^j_{t+1} > \bar{b} \text{ with } \bar{b} < 0$$