Advanced Tools in Macroeconomics

The endogenous cross-sectional distribution

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Long run distribution

- Once you have found the policy functions of a problem, you often need to calculate the long run distribution
    - We did that quite easily for the continuous time model yesterday
- It is need for market clearing in incomplete markets models
- But it can also be quite useful for representative agent models.
- This note will explain how you can do that in a discrete time setting.

Long run distribution

- These problems typically give rise to a policy function of the type $g(b, s)$.
  - where b is wealth or capital, and s is either employment status or TFP.
- This cannot be represented as a transition matrix.
- So how can we find the long run distribution of bonds (and employment status) in this case?

Long run distribution

- In general, suppose that $\psi_0(b, s)$ is a probability density function in period zero
- Then

$$\psi_1(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi_0(b, s) p(s', s)$$

- And in general

$$\psi_{t+1}(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi_t(b, s) p(s', s)$$

Complications

- A stationary cross-sectional distribution, ψ, is such that

$$\psi(b', s') = \sum_{s \in \mathcal{S}} \sum_{\{b : b' = g(b,s)\}} \psi(b, s) p(s', s)$$

- This is very tricky to compute. The best practice I am aware of is to convert $g(b, s)$ into a transition matrix!

Solutions

Consider the following policy function

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1.8 \\ 2.4 \\ 3 \\ 3.6 \\ 4.2 \end{pmatrix}$$

Solutions

Nearest neighbor interpolation

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

Solutions

Can be written as transition matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Solutions

But we can be a bit smarter. Policy function

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1.8 \\ 2.4 \\ 3 \\ 3.6 \\ 4.2 \end{pmatrix}$$

Solutions

Can be written as

$$\begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{pmatrix}$$

Solutions

But in the income fluctuation problem (and many others) we normally have two (or many) policy functions

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \underset{\text{if good state}}{\longrightarrow} \begin{pmatrix} 2.2 \\ 2.8 \\ 3.4 \\ 4 \\ 4.6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \underset{\text{if bad state}}{\longrightarrow} \begin{pmatrix} 1.4 \\ 2 \\ 2.6 \\ 3.2 \\ 3.8 \end{pmatrix}$$

With some transition matrix for good and bad states

$$T = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

Solutions

Nearest neighbor interpolation

$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix} \underset{\text{if good state}}{\longrightarrow} \begin{pmatrix}2\\3\\3\\4\\5\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix} \underset{\text{if bad state}}{\longrightarrow} \begin{pmatrix}1\\2\\3\\3\\4\end{pmatrix}$$

With some transition matrix for good and bad states

$$T = \begin{pmatrix}0.8 & 0.2\\0.3 & 0.7\end{pmatrix}$$

Solutions

Two transition matrices

$$M_g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

With some transition matrix for good and bad states

$$T = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

Solutions

Full transition matrix is given by

$$\begin{pmatrix} T(1,1)\cdot M_g & T(1,2)\cdot M_g \\ T(2,1)\cdot M_b & T(2,2)\cdot M_b \end{pmatrix}$$

Solutions

Full transition matrix is given by

$$\begin{pmatrix} 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 \\ 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 \end{pmatrix}$$

Solutions

Doing the same thing with the smarter method gives

$$\begin{pmatrix} 0 & 0.64 & 0.16 & 0 & 0 & 0 & 0.16 & 0.04 & 0 & 0 \\ 0 & 0.16 & 0.64 & 0 & 0 & 0 & 0.04 & 0.16 & 0 & 0 \\ 0 & 0 & 0.48 & 0.32 & 0 & 0 & 0 & 0.12 & 0.08 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.32 & 0.48 & 0 & 0 & 0 & 0.08 & 0.12 \\ 0.18 & 0.12 & 0 & 0 & 0 & 0.42 & 0.28 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0.12 & 0.18 & 0 & 0 & 0 & 0.28 & 0.42 & 0 & 0 \\ 0 & 0 & 0.24 & 0.06 & 0 & 0 & 0 & 0.56 & 0.14 & 0 \\ 0 & 0 & 0.06 & 0.24 & 0 & 0 & 0 & 0.14 & 0.56 & 0 \end{pmatrix}$$

Long run distribution

- Once you have the transition matrix it is quite easy to find the long run distribution, ψ
- As a linear system the law of motion is

$$\boldsymbol{\psi_{t+1}} = \mathbf{T}\boldsymbol{\psi}_{\mathbf{t}}$$

  where **T** is the transition matrix.
- Then ψ can be found as the solution to

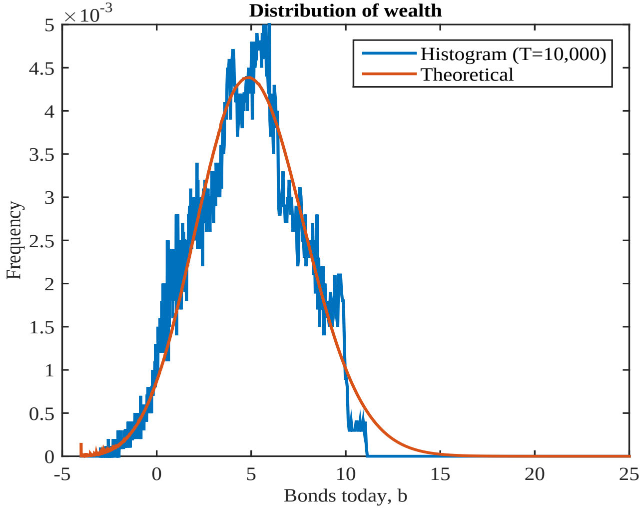
$$(\mathbf{T} - \mathbf{I})\psi = \mathbf{0}$$

- That is, the long run distribution is the eigenvector of **T** associated with a *unit* eigenvalue, and normalized to sum to one.

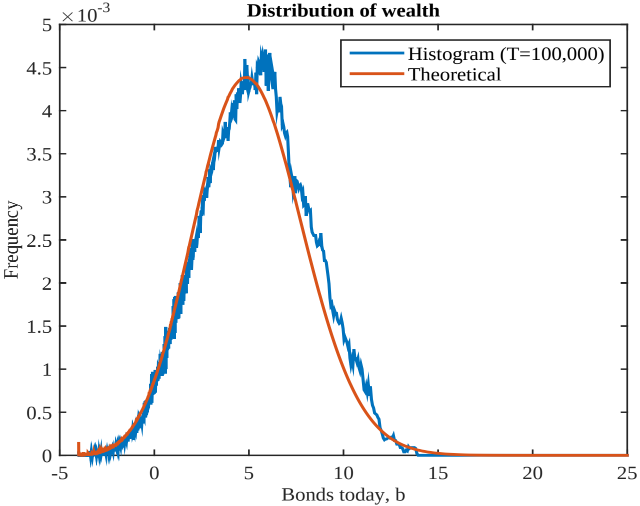
Long run distribution

- The next graphs will show this distribution for a heterogenous agent model
- I will illustrate it in two ways: The red line is the "theoretical" distribution calculated as in the previous slides
- The blue line is instead a histogram over a simulated path for one agent
- In the limit, when the simulated path is infinitely long, they will coincide
- But the theoretical distribution is much much faster to compute.

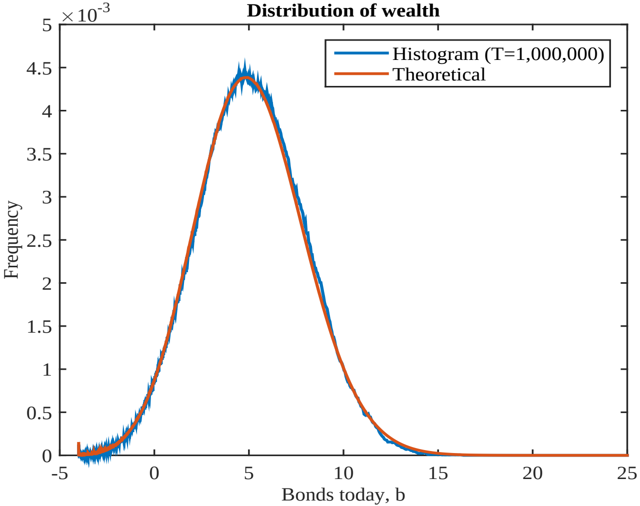
Income fluctuation problem



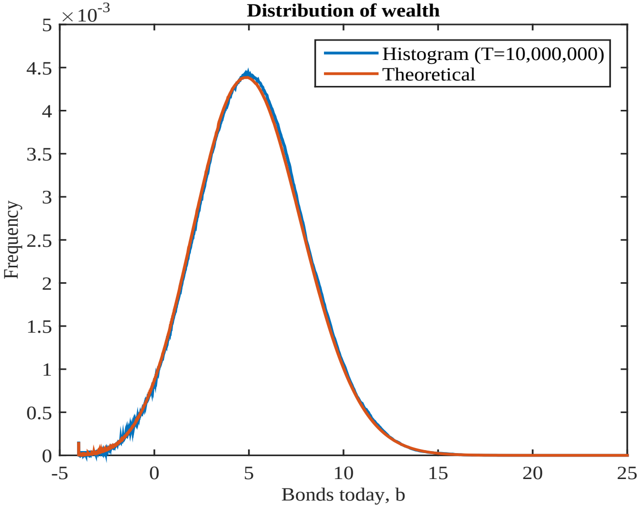
Income fluctuation problem



Income fluctuation problem



Income fluctuation problem



Long run distribution

- The next graph shows the theoretical distribution for the model with irreversible investment
- Often it's not as important in rep agent models as in het agent models
- But I just want to show that you can do it.

Irreversible investment