Penalty Function instead of Borrowing Constraint and Monday Homework

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Notation

- Krusell-Smith and traditional notation:
  - k_t : capital available for production in period t
  - k_{t+1} : chosen in period t carried over into period $t+1$
- Dynare notation:
  - k_{t-1} : capital available for production in period t
  - k_t : chosen in period t carried over into period $t+1$

!!! These slides use Dynare notation since we will be using Dynare in homeworks

Standard borrowing constraint

$$k_{i,t} \geq \bar{k}$$

with $\bar{k} \leq 0$

Interpretation:

- You can borrow up to $-\bar{k}$ at risk-free interest rate
- Borrowing more than $-\bar{k}$ is *completely* impossible
- That is, $k_{i,t} < \bar{k}$ comes with **infinite** penalty
- Is that realistic?

Penalty function

Main idea:

- Penalty function is a more flexible approach to limit borrowing
- Penalty could appear in utility function
    - Easier because only FOC is affected
- Penalty function could also be put in interest rate charged
    - FOC and budget constraint are affected

Functional forms

Most logical specification:

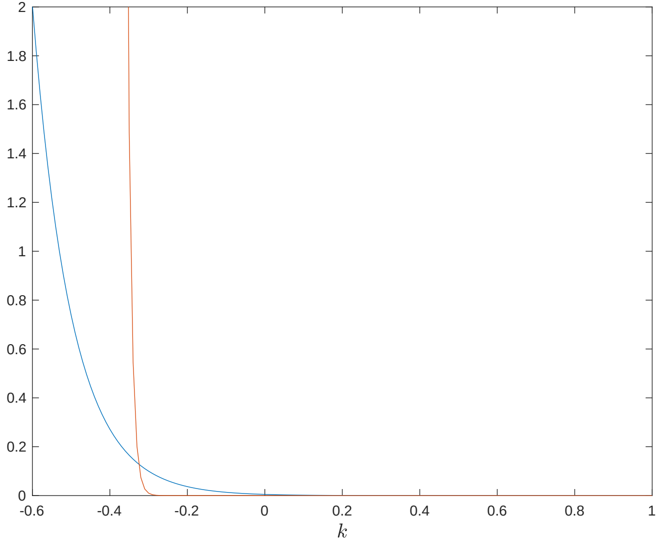
$$P(k_{i,t}) = \frac{\zeta_1}{\zeta_0} \exp(-\zeta_0(k_{i,t} - \bar{k}))$$

Small modification with practical advantage:

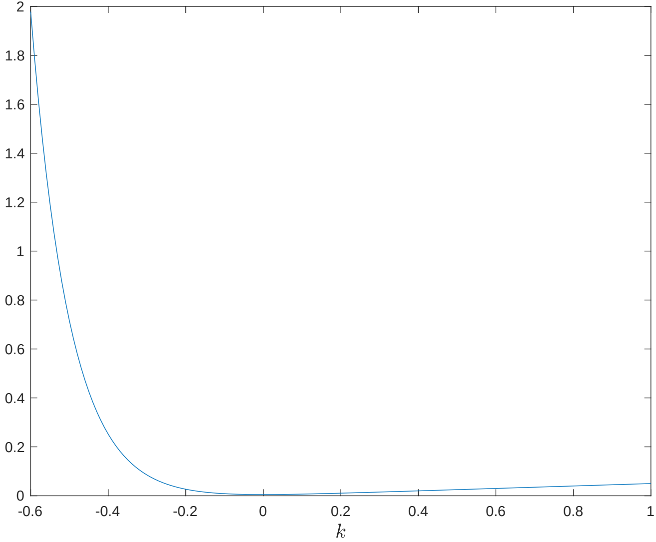
$$P(k_{i,t}) = \frac{\zeta_1}{\zeta_0} \exp(-\zeta_0 k(k_{i,t} - \bar{k})) + \zeta_2 k_{i,t}$$

$\zeta_2 > 0$ but small

$\bar{k} = -0.3, \zeta_0 \in \{10, 100\}, \zeta_1 = 1, \zeta_2 = 0$



$\bar{k} = -0.3, \zeta_0 = 10, \zeta_1 = 1, \zeta_2 = 0.05$



Penalty function vs Inequality constraint

Standard inequality constraint

$$k_{i,t} \geq \bar{k}$$

corresponds to

$$P(k_{i,t}) = \begin{cases} \infty & \text{if } k_{i,t} < \bar{k} \\ 0 & \text{if } k_{i,t} \geq \bar{k} \end{cases}$$

Interpreting the penalty function

1. penalty function *implements* inequality constraint
   - η_0 must be very high
2. penalty function is alternative to penalty function
   - η_0 could be high or low

Calibrating the penalty function

- η_0, η_1, and η_2 can be chosen to match data characteristics
- Here:
  - different values for curvature parameter, η_0
  - η_1 and η_2 chosen to match mean and standard deviation of $k_{i,t}$
- many properties of this model similar to "$k_{i,t} \geq \bar{k}$" model
- specifically, in both models agents save to be shielded from being close to $\bar{k}$
  - but behavior close to $\bar{k}$ can be a bit different

Back to model with heterogeneous agents

Couple modifications to benchmark model

1. No aggregate risk $\equiv$ Aiyagari model
2. We simplify the standard setup as follows:
   - Replace borrowing constraint by penalty function
     $\Longrightarrow$ going short is possible but costly
   - workers have productivity insteady of unemployment shocks
     $\varepsilon_{i,t}$ with $\mathrm{E}[\varepsilon_{i,t}] = 1$

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathsf{E} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t}) - \left(\frac{\zeta_1}{\zeta_0}\exp(-\zeta_0 k_{i,t}) + \zeta_2 k_{i,t}\right)$$

s.t.

$$c_{i,t} + k_{i,t} = r_t k_{i,t-1} + w_t \varepsilon_{i,t} + (1-\delta) k_{i,t-1}$$

First-order condition

$$\frac{1}{c_{i,t}} = \zeta_1 \exp(-\zeta_0 k_{i,t}) - \zeta_2 + \mathsf{E}_t\left[\frac{\beta}{c_{i,t+1}}\left(r_{t+1} + 1 - \delta\right)\right]$$

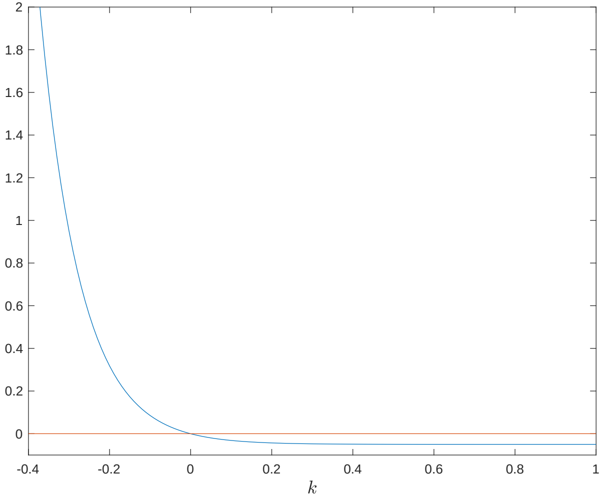
Penalty function

- advantage of ζ_2 term:
  - supppose $\bar{k}$ is the steady states of the rep agent model
  - if

$$\zeta_2 = \zeta_1 \exp(-\zeta_0 \bar{k})$$

    then steady state of this model is same

FOC Penalty; $\bar{k} = -0.3, \zeta_0 = 10, \zeta_1 = 1, \zeta_2 = 0.05$



Equilibrium

- Unit mass of workers, $L_t = 1$ since $\mathrm{E}[\varepsilon_{i,t}] = 1$
- Competitive firm $\Longrightarrow$
  - $w_t = (1-\alpha) K_t^{\alpha} L_t^{1-\alpha} = (1-\alpha) K_t^{\alpha}$
  - $r_t = \alpha K_t^{\alpha-1} L_t^{\alpha} = \alpha K_t^{\alpha-1}$
- No aggregate risk so

$$K_t = K$$

- Solve for equilibrium r to ensure demand for K implied by firm problem equals supply of K by households
- Supply of K is determined by simulating individual choices