Importance of Imposing Equilibrium

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Key message

Approximation error CANNOT be in equilibrium conditions

In KS-type algorithm this is important in simulation phase

Some errors are NOT acceptable!!!

- Numerical approximation $\Longrightarrow$ some approximation error
- But equilibrium conditions must hold exactly. Why?
  - Consider an endowment economy with heterogeneous agents in which bonds are in zero net-supply
  - If aggregate bond demand $\neq 0 \Longrightarrow$ you create/destroy real resources
  - Such "disequilibrium" errors are unlikely to have an exact mean of zero
    $\Longrightarrow$ resources available in the economy change over time (especially problematic since we rely on long time series to accurately characterize economy properties)
- Algorithm imposes equilibrium *exactly* in the capital-only model (explanation given below)

Economy with bonds

- Suppose we add one-period bonds
  - $\Longrightarrow$ we also have to solve for
    - individual demand for bonds, $b_{i,t+1} = b(s_{i,t})$
    - bond price, $q(S_t)$
  - Simulated aggregate demand for bonds not necessarily $= 0$
  - $\Longrightarrow$ resources are either added or taken out of this economy

Bonds and ensuring equilibrium #1

Possible solution $\#1$

- Add the bond price as a state variable in individual problem
- For each period, solve for the bond price that sets aggregate bond demand equal to zero
- a bit unusual to make an endogenous variable a state variable
- risky in terms of getting convergence (in my experience)

Reminder about function approximation

You usually have a lot of choices:

- Approximating the consumption function instead of the capital choice may at times be better/worse.
- Approximating log level may at times be better than the level.
- So instead of approximating $b(s_{i,t})$ you could approximate
  - $d(s_{i,t}) = \log(b(s_{i,t}))$ and solve $b_{i,t}$ from $b_{i,t+1} = \exp(d(s_{i,t}))$
  - $d(s_{i,t}) = b(s_{i,t}) + 5$ and solve $b_{i,t}$ from $b_{i,t+1} = d(s_{i,t}) - 5$
  - $d(s_{i,t}) = b(s_{i,t}) + z_t$ and solve $b_{i,t}$ from $b_{i,t+1} = d(s_{i,t}) - z_t$

For an *accurate* solution, the implied $b(s_{i,t})$ would be virtually the same for all three cases above

Bonds and ensuring equilibrium #2

1. Instead of approximating $b_i(s_{i,t})$, approximate $d_i(s_{i,t})$ where

$$d(s_{i,t}) = b(s_{i,t}) + \zeta q(S_t)$$

2. Imposing equilibrium gives

$$\begin{aligned}
0 &= \left(\sum_i b_{i,t+1}\right) / I \Longrightarrow \\
0 &= \left(\sum_i (d_{i,t+1} - \zeta q_t)\right) / I \Longrightarrow \\
q_t &= \left(\left(\sum_i d_i(s_{i,t})\right) / I\right) / \zeta
\end{aligned}$$

3. The numerical bond demand approximation now depends on q_t :

$$b_{i,t+1} = b(q_t, s_{i,t}) = d(s_{i,t}) - \zeta q_t$$

Bonds and ensuring equilibrium #2

- Does any $d(s_{i,t})$ work?
- For stability, the implied $b_{i,t}$ needs to be *like* a demand equation, that is
$$\frac{\partial b_i(q_t, s_{i,t})}{\partial q_t} < 0,$$
which is equal to $-\zeta$ here $\Longrightarrow$ we want $\zeta > 0$.
- We are not looking for some true complete derivative!
- $d(s_{i,t})$ implicitly also captures some of the relationship between $b(s_{i,t})$ and q_t (obvious when $\zeta = 0$).
- Approximating $d(s_{i,t})$ just adds some flexibility to impose equilibriume exactly.
- For an accurate solution, the implied $b(s_{i,t})$ would be the same, independent of the alternative $d(s_{i,t})$ chosen.

Bonds and ensuring equilibrium #2

Many ways to implement above idea:

- $d(s_{i,t}) = b(s_{i,t}) + \zeta q(S_t)$ is a bit ad hoc
- Alternative:
    - solve for $c(s_{i,t})$
    - get $b_{i,t}$ from budget constraint which contains q_t
    - You get $b_i(q_t, s_{i,t})$ with

$$\frac{\partial b_i(q_t, s_{i,t})}{\partial q_t} < 0$$

Imposing equilibrium

- Capital market equilibrium is automatically imposed in simulation. Why?
  - Supply of capital comes from $K_t = \sum_i k_{i,t}$
  - This value of K_t is used in firm FOC
  - r_t adjusts so that firm demand for capital equals supply
- Where is the approximation error here?
  - FOC individual contains r_{t+1} and when solving for $k_{i,t+1}$ we use $\Gamma(\cdot;\eta)$ to determine K_{t+1} which in turn determines this r_{t+1} (together with z_{t+1})
  - But simulated $K_t = \sum_i k_{i,t}$ could be a bit different
- So agents' perception about future r may have an approximation error, but equilibrium is imposed