Uhlig(1999) : Undetermined Coefficients Method*

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1 Method 1: endogenous and exogenous variables

if we define x_t as a vector of endogenous variables, and z_t as exogenous variables, then we can write the log-linearized model as the following equation:

$$0 = E_t \left(F x_{t+1} + G x_t + H x_{t-1} + L z_{t+1} + M z_t \right)$$

$$z_{t+1} = N z_t + \mu_{t+1}$$

where F, G, H, L, M, N are constant coefficient matrices and $E_t \mu_{t+1} = 0$. We are looking for a solution

$$x_t = P x_{t-1} + Q z_t$$

to the problem above where P, Q are undetermined constant matrices. Uhlig(1999) tells us that if a solution exists, then the P and Q can be found by solving the following matrix equations

$$0 = FP^2 + GP + H$$

and

$$V \times vec\left(Q\right) = -vec\left(LN + M\right)$$

where $vec\left(\right)$ is column-wise vectorization and

$$V = N^T \otimes F + I_k \otimes (FP + G)$$

where k is the number of exogenous variables and $\otimes$ is the Kronecker Product. If you want to know why how this come from, you can refer to Uhlig(1999) for details.

*The note is heavily borrowed from McCandless(2008).

2 Method 2: using Jump(er) variables

Sometimes, you will find it will be much more easy to solve the system if we split the endogenous variables into state variables and other variables. If we define x_t as a vector of endogenous state variables and y_t as a vector of other endogenous variables, or called as jumper variables. Then we write the system as the following system.

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t$$

$$0 = E_t\left(Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right)$$

$$z_{t+1} = Nz_t + \mu_{t+1},\ \ E_t\left(\epsilon_{t+1}\right) = 0$$

The solution for this system is a set of matrices P, Q, R and S, that describe the equilibrium laws of motion,

$$x_t = Px_{t-1} + Qz_t$$

$$y_t = Rx_{t-1} + Sz_t$$

Uhlig(1999) tells us that if equilibrium laws of motion exist, they must fulfill

$$0 = \left(F - JC^{-1}A\right)P^2 - \left(JC^{-1}B - G + KC^{-1}A\right)P - KC^{-1}B + H$$

$$R = -C^{-1}\left(AP + B\right)$$

$$V \times vec\left(Q\right) = vec\left(\left(JC^{-1}D - L\right)N + KC^{-1}D - M\right)$$

and

$$S = -C^{-1}\left(AQ + D\right)$$

where

$$V = N^T \otimes \left(F - JC^{-1}A\right) + I_k \otimes \left(JR + FP + G - KC^{-1}A\right)$$

We are not going to show how this results come from, and please refer to Uhlig(1999) for more information.

3 The Model: Hansen(1985)

This example comes from Hansen, G. D.,1985,"Indivisible Labor and the Business Cycle",Journal of Monetary Economics,Vol.16 (3),PP309-327.

Households maximize the lifetime utility

$$max\sum_{t=0}^{\infty}\beta^t u\left(c_t, l_t\right)$$

where c_t is consumption and l_t is leisure where $l_t = 1 - h_t$ and h_t is labor. The maximization problem subject to the feasibility condition

$$y_t \geq c_t + I_t$$

The utility function we use is

$$u(c_t, 1 - h_t) = logc_t + \gamma log(1 - h_t)$$

where $\gamma > 0$ is parameter. The production function is Cobb-Douglas with a stochastic technology.

$$y_t = A_t K_t^{\alpha} h_t^{1-\alpha}$$

where A_t has a log-AR(1) process as follows:

$$logA_t = \rho logA_{t-1} + \epsilon_t$$

The law of motion for capital:

$$K_{t+1} = I_t + (1 - \delta) K_t$$

The Lagrangian is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ logc_t + \gamma log(1 - h_t) + \lambda_t \left(A_t K_t^{\alpha} h_t^{1-\alpha} - c_t + (1 - \delta) K_t - K_{t+1} \right) \right\}$$

The FOCs as follows:

$$\begin{aligned}
1 &= \beta E_t \left(\frac{C_t}{C_{t+1}} (r_{t+1} + (1 - \delta)) \right) \qquad (1) \\
\gamma C_t &= (1 - \alpha)(1 - h_t) \frac{Y_t}{h_t} \\
C_t &= Y_t + (1 - \delta) K_t - K_{t+1} \\
Y_t &= A_t K_t^{\alpha} h_t^{1-\alpha} \\
r_t &= \alpha \frac{Y_t}{K_t}
\end{aligned}$$

The log-linear version:

$$\begin{aligned}
\tilde{C}_t &= E_t \tilde{C}_{t+1} - \beta \bar{r} E_t \tilde{r}_{t+1} \\
\tilde{Y}_t &= \frac{\tilde{h}_t}{1 - \bar{h}} + \tilde{C}_t \\
0 &= \bar{Y} \tilde{Y}_t - \bar{C} \tilde{C}_t + \bar{K} \left((1 - \delta) \tilde{K}_t - \tilde{K}_{t+1} \right) \\
0 &= \tilde{A}_t + \alpha \tilde{K}_t + (1 - \alpha) \tilde{h}_t - \tilde{Y}_t \\
\tilde{Y}_t &= \tilde{K}_t + \tilde{r}_t \\
\tilde{A}_t &= \rho \tilde{A}_{t-1} + \epsilon_t
\end{aligned}$$

4 The Steady State

First, We need calculate the model's steady state values. Then we calibrate the parameters.$\alpha = 0.36, \beta = .99, \delta = 0.025$. The steady state value of technology is calibrated to unity, i.e., $\bar{A} = 1$.

Then we calculate the steady state states. From Euler Equation (1), we have

$$\bar{r} = \frac{1}{\beta} - (1 - \delta).$$

Then, from the wage rate equation, $\gamma \bar{C} = (1-\alpha)\left(1-\bar{h}\right)\frac{\bar{Y}}{\bar{h}}$. And from the resource constraint, we have $\bar{Y} = \bar{C} + \delta \bar{K}$. The production function, we have $\bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{h}^{1-\alpha}$. The capital rate equation shows that $\bar{r} = \alpha \frac{\bar{Y}}{\bar{K}}$. Then

$$\bar{h} = \frac{1}{1 + \frac{\gamma}{1-\alpha}\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)}\right)}$$

and

$$\bar{K} = \bar{h}\left(\frac{\alpha \bar{A}}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

Parameter γ is chosen so as to $\bar{h} = \frac{1}{3}$. That is to say that people spend about one-third of their daily time to work. This require that $\gamma = 1.72$. Then we can calculate $\bar{K}, \bar{C}, \bar{Y}$ and $\bar{r}$.

5 The Code

If we use method 2 to solve our system, we define

$$x_t = \tilde{K}_t$$

and

$$y_t = \left(\tilde{Y}_t, \tilde{C}_t, \tilde{h}_t, \tilde{r}_t\right)'$$

and

$$z_t = \tilde{A}_t$$

The coefficient matrices:

$$A = \left(0, -\bar{K}, 0, 0\right)'$$

$$B = \left(0, (1-\delta)\bar{K}, \alpha, -1\right)'$$

$$C = \begin{pmatrix} 1 & -1 & -\frac{1}{1-\bar{h}} & 0 \\ \bar{Y} & -\bar{C} & 0 & 0 \\ -1 & 0 & 1-\alpha & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$D = (0, 0, 1, 0)'$$

$$F = (0), G = (0), H = (0)$$

$$J = (0, -1, 0, \beta\bar{r})$$

$$K = (0, 1, 0, 0),$$

$$L = (0), M = (0), N = (\gamma).$$

The solution for this system is a set of matrices P, Q, R and S, that describe the equilibrium laws of motion,

$$x_t = Px_{t-1} + Qz_t$$

$$y_t = Rx_{t-1} + Sz_t.$$

If we only have one exogenous variable, then P and Q are just numbers not vectors. According to the Method 2 above, the Matlab code for solving this system is :

```
%This version includes the programs to find the stationary states,i.e., the
%steady state values of endogenous variables.
%The scripts to find the steady states of labor and capital are hbarfind.m
%and kbarfind.m respectively.
%This script was written by Xiangyang Li, 2015-7-20
%find the steady states
hbarfind; %the steady states of labor
kbarfind; %the steady states of capital
rbar=1/beta-(1-delta);
ybar=kbar^alpha*hbar^(1-alpha);
cbar=ybar-delta*kbar;

%find the solution of the system
A=[0 -kbar 0 0]';
B=[0 (1-delta)*kbar alpha -1]';
C=[1 -1 -1/(1-hbar) 0
    ybar -cbar 0 0
    -1 0 1-alpha 0
    1 0 0 -1];
D=[0 0 1 0]';
F=[0];
G=F;
H=F;
J=[0 -1 0 beta*rbar];
K=[0 1 0 0];
L=F;
M=F;
N=[.95];

a=F-J*(C\A);
```

```
b=-(J*(C\B)-G+K*(C\A));
c=-K*(C\B)+H;
P1=(-b+sqrt(b^2-4*a*c))/(2*a);
P2=(-b-sqrt(b^2-4*a*c))/(2*a);

%we need a stable P because it is the coefficient of AR(1)
if abs(P1)<1
    P=P1;
else
    P=P2;
end
R=-C\(A*P+B);
Q=(J*(C\D)-L)*N+K*(C\D)-M;
QD=kron(N',(F-J*(C\A)))+(J*R+F*P+G-K*(C\A));
Q=Q/QD;
S=-C\(A*Q+D);
```

The solution of the model is:

$$P = .9537, Q = .1132$$

$$R = \begin{pmatrix} .2045 \\ .5691 \\ -.2430 \\ -.7955 \end{pmatrix}, \quad S = \begin{pmatrix} 1.4523 \\ .3920 \\ .7067 \\ 1.4523 \end{pmatrix}$$

That state space representation of the system as follows:

$$\tilde{K}_{t+1} = .9537\tilde{K}_t + .1132\tilde{A}_t$$

$$\begin{pmatrix} \tilde{Y}_t \\ \tilde{C}_t \\ \tilde{h}_t \\ \tilde{r}_t \end{pmatrix} = \begin{pmatrix} .2045 \\ .5691 \\ -.2430 \\ -.7955 \end{pmatrix} \tilde{K}_t + \begin{pmatrix} 1.4523 \\ .3920 \\ .7067 \\ 1.4523 \end{pmatrix} \tilde{A}_t$$

After we have the solution, we can do the IRF analysis and moments analysis etc.

6 The Variances

First, we find the variances of output as the function of the variance of technology shock. From the transition equation, output equation and technology shock process,

$$\tilde{K}_{t+1} = a\tilde{K}_t + b\tilde{A}_t \tag{2}$$

$$\tilde{Y}_t = c\tilde{K}_t + d\tilde{A}_t \tag{3}$$

$$\tilde{A}_t = \rho\tilde{A}_{t-1} + \epsilon_t \tag{4}$$

where $a, b, c = .2045, d = 1.4523$ are constants and a, b can be one of the entries in R and S respectively so that we can calculate the variances of other endogenous variables.

$$\begin{aligned}
\tilde{A}_t &= \rho \tilde{A}_{t-1} + \epsilon_t \\
&= \rho \left(\rho \tilde{A}_{t-2} + \epsilon_{t-1} \right) + \epsilon_t \\
&= \rho^2 \tilde{A}_{t-2} + \rho \epsilon_{t-1} + \epsilon_t \\
&= \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i}
\end{aligned}$$

$$\begin{aligned}
\tilde{K}_{t+1} &= a\tilde{K}_t + b \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i} \\
&= a \left(a\tilde{K}_{t-1} + b \sum_{i=0}^{\infty} \rho^i \epsilon_{t-1-i} \right) + b \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i} \\
&= a^2 \tilde{K}_{t-1} + b \sum_{i=0}^{\infty} a \rho^i \epsilon_{t-1-i} + b \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i} \\
&= b \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a^j \rho^i \epsilon_{t-j-i}
\end{aligned}$$

$$\tilde{Y}_t = c\tilde{K}_t + d\tilde{A}_t = \cdots = cb \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a^j \rho^i \epsilon_{t-j-i} + d \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i}$$

Hence,

$$var \left(\tilde{Y}_t \right) = f \left(a, b, c, d, \rho \right) var \left(\epsilon_t \right)$$

since technology shock are i.i.d. and where $f(a, b, c, d, \rho)$ is the function of the known parameters a, b, c, d, ρ. This show us that if we know or calibrate the variance of output, then we can calibrate/calculate the variance of technology shock and vice versa. Subsequently, we can calculate the variance of other variables in the model. This is basics behind how Dynare calculate the theoretical variance of endogenous variables.