Solving Models with Heterogeneous Agents - KS algorithm

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Overview

- Krusell-Smith algorithm to solve our benchmark heterogeneous-agent model
- Approximate aggregation
- Simulating economies with heterogeneous agents (separate slides)
- Importance of imposing equilibrium (separate slides)

STATE VARIABLES REMINDER

What matters for agents decisions?

- individual wealth, $k_{i,t}$
- employment status
- z_t :
  - affects z_{t+1} and future employment-status transition probabilities
  - affects current and future values of r_t and w_t
- current and expected future values of r_t and w_t

What matters for agents decisions?

!!!

- But state variables cannot be endogenous variables like prices; values of state variables should be known to us when we are trying to solve the period-t system of equations.
- Also, not clear how many lags of r_t and w_t we should include, since we don't know what kind of Markov process these variables are.

What are sensible state variables?

- We need *pre-determined info* that determines r_t and w_t AND is sufficient in terms of predicting future values of r_t and w_t.
- **Answer:** z_t and the cross-sectional joint distributon of individual capital holdings and employment status.
- Why does the *distribution* matter when only the aggregate capital stock, K_t, matters for r_t and w_t?
- **Answer:** Distribution of $k_{i,t}$ matters for K_{t+1} *unless* the MPC for all agents are equal.

What do we have to approximate?

- Individual policy rules.
- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

  - $f_t =$ beginning-of-period cross-sectional distribution of capital and the employment status after the employment status has been realized.
  - z_{t+1} does not affect the cross-sectional distribution of capital but does affect the joint cross-sectional distribution of capital and employment status.

Problems in approximating $\Upsilon(\cdot)$

- Functional form of $\Upsilon(\cdot)$ is not known and functions are infinite-dimensional objects
  - **Solution:** Standard solution, namely use a finite-order element from class of functions like polynomials that can approximate any function arbitrarity well.
- One of the arguments of the unknown function, namely f_t, is itself infinite-dimensional.
  - **Solution:** Dimension of f_t has to be reduced as well.

KRUSELL-SMITH ALGORITHM

Key approximating steps

1. Approximate cross-sectional distribution with limited set of "characteristics," m_t
   - Proposed in Den Haan (1996), Krusell & Smith (1997,1998), Rios-Rull (1997).
2. Solve for aggregate policy rule.
3. Solve individual policy rule for a given aggregate law of motion.
4. Make the two consistent

Krusell-Smith (1997,1998) algorithm

- Assume the following approximating aggregate law of motion

$$m_{t+1} = \Gamma(z_{t+1}, z_t, m_t; \eta_\Gamma).$$

- Start with an initial guess for its coefficients, η_Γ^0.

Krusell-Smith (1997,1998) algorithm

- Use following iteration until η_Γ^{iter} has converged:
  - Given η_Γ^{iter} solve for the individual policy rule
  - Given individual policy rule simulate economy and generate a time series for m_t
  - Run a standard projection step $\Longrightarrow \hat{\eta}_\Gamma$
  - Possibly apply some dampening

$$\eta_\Gamma^{iter+1} = \lambda\hat{\eta}_\Gamma + (1-\lambda)\eta_\Gamma^{iter}, \text{ with} 0 < \lambda \leq 1$$

- Continue until convergence, i.e., until the $\Gamma(\cdot;\eta_\Gamma)$ used when solving for individual policy rule is close to $\Gamma(\cdot;\eta_\Gamma)$ implied by individual policy rule

Problem Individual Agent

Solving for policy rules individualr

- Given aggregate law of motion $\Longrightarrow$ you can solve for policy rules individual with your favourite algorithm
- But number of state variables has increased:
  - State variables for agent: $s_{i,t} = \{\varepsilon_{i,t}, k_{i,t}, S_t\}$
  - with $S_t = \{z_t, m_t\} = \{z_t, K_t, \widetilde{m}_t\}$.

Solving for policy rules individual

- S_t must “reveal” K_t
  - $S_t \implies K_t \implies r_t$ and r_t
- Let
  $K_{t+1} = \Gamma_K(z_{t+1}, z_t, S_t; \eta_{\Gamma_K})$, $\widetilde{m}_{t+1} = \Gamma_{\widetilde{m}}(z_{t+1}, z_t, S_t; \eta_{\Gamma_{\widetilde{m}}})$
- If S_t includes many characteristics of the cross-sectional distribution $\implies$ high dimensional individual policy rule

Individual policy rules & projection methods

First choice to make:

- Which function to approximate?
- Here we approximate $k_i(\cdot)$

$$k_{i,t+1} = P_n(s_{i,t};\eta_{P_n})$$

- N_η : dimension η_{P_n}

Individual policy rules & projection methods

Next: Design grid

- s_κ the κ^{th} grid point
- $\{s_\kappa\}_{\kappa=1}^{\chi}$ the set with χ nodes
- $s_\kappa = \{\varepsilon_\kappa, k_\kappa, S_\kappa\}$, and $S_\kappa = \{z_\kappa, K_\kappa, \widetilde{m}_\kappa\}$

Individual policy rules & projection methods

Next: Implement projection idea

1. Substitute approximation into model equations until you get equations of only
   1. current-period state variables
   2. coefficients of approximation, η_{P_n}
2. Evaluate at χ grid points $\Longrightarrow$ χ equations to find η_{P_n}
   - $\chi = N_\eta \Longrightarrow$ use equation solver
   - $\chi > N_\eta \Longrightarrow$ use minimization routine

Individual policy rules & projection methods

First-order condition

$$
\begin{aligned}
c_t^{-\nu} &= \mathsf{E}\left[\begin{array}{c} \beta(r(z',K')+(1-\delta))\times \\ c_{t+1}^{-\nu} \end{array}\right] \\
\left(\ \text{income}_{i,t} - k_{i,t+1}\ \right)^{-\nu} &= \mathsf{E}\left[\begin{array}{c} \beta(r(z',K')+(1-\delta))\times \\ \left(\ \text{income}_{i,t+1} - k_{i,t+2}\ \right)^{-\nu} \end{array}\right]
\end{aligned}
$$

Individual policy rules & projection methods

First-order condition

$$\begin{pmatrix} (r(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1-\tau(z_\kappa))w(z_\kappa, K_\kappa)\bar{l}\varepsilon_\kappa + \mu w(z_\kappa, K_\kappa)(1-\varepsilon_\kappa) \\ -P_n(s_\kappa; \eta_{P_n}) \end{pmatrix}^{-\nu}$$

$$= \mathsf{E}\left[\begin{array}{c} \beta(r(z', K') + (1-\delta))\times \\ \begin{pmatrix} (r(z', K') + 1 - \delta)P_n(s_\kappa; \eta_{P_n}) \\ +(1-\tau(z'))w(z', K')\bar{l}\varepsilon' + \mu w(z', K')(1-\varepsilon') \\ -P_n(s'; \eta_{P_n}) \end{pmatrix}^{-\nu} \end{array} \right]$$

Individual policy rules &projection methods

Euler equation errors:

$$u_\kappa = \begin{pmatrix} (r(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1-\tau(z_\kappa))w(z_\kappa, K_\kappa)\bar{l}\varepsilon_\kappa + \mu w(z_\kappa, K_\kappa)(1-\varepsilon_\kappa) \\ -P_n(s_\kappa; \eta_{P_n}) \end{pmatrix}^{-\nu} -$$

$$\sum_{z' \in \{z^b, z^g\}} \sum_{\varepsilon' \in \{0,1\}} \left[\begin{array}{c} \beta(r(z', K') + (1-\delta)) \times \\ \begin{pmatrix} (r(z', K') + 1 - \delta)P_n(s_\kappa; \eta_{P_n}) \\ +(1-\tau(z'))w(z', K')\bar{l}\varepsilon' \\ +\mu w(z', K')(1-\varepsilon') \\ -P_n(\mathbf{s'}; \eta_{P_n}) \end{pmatrix}^{-\nu} \times \\ \pi(\varepsilon', z' \mid z_\kappa, \varepsilon_\kappa) \end{array} \right]$$

Error depends on known "stuff" and η_{P_n} when using

$$\begin{aligned} r(z_\kappa, K_\kappa) &= \alpha z_\kappa (K_\kappa / L(z_\kappa))^{\alpha-1} \\ w(z_\kappa, K_\kappa) &= (1-\alpha) z_\kappa (K_\kappa / L(z_\kappa))^{\alpha} \end{aligned}$$

$$\begin{aligned} r(z', K') &= \alpha z' (K' / L(z'))^{\alpha-1} \\ &= \alpha z' (\Gamma_K(z', z_\kappa, S_\kappa; \eta_\Gamma) / L(z'))^{\alpha-1} \\ w(z', K') &= (1-\alpha) z' (K' / L(z'))^{\alpha} \\ &= (1-\alpha) z' (\Gamma_K(z', z_\kappa, S_\kappa; \eta_\Gamma) / L(z'))^{\alpha} \end{aligned}$$

$$\tau(z) = \mu(1 - L\left(z\right)) / \bar{l} L(z))$$

$$s' = \left\{ \begin{array}{c} P_n(s_\kappa; \eta_{P_n}), \varepsilon', z', \\ \Gamma_K(z', z_\kappa, S_\kappa; \eta_\Gamma), \Gamma_{\widetilde{m}}(z', z_\kappa, S_\kappa; \eta_{\bar{\Gamma}}) \end{array} \right\}$$

Again standard projection problem

- Find η_{P_n} by minimizing $\sum_{\kappa=1}^{\chi} u_{\kappa}^2$

UPDATE LAW OF MOTION AGGREGATE STATE VARIABLES

Update coefficients $\Gamma(z', z, S; \eta_\Gamma)$

- Policy rules individual imply law of motion S_t
- Update η_Γ as follows
  1. Simulate timeseries for S_t using an economy with a cross-section of individual agents. That is,
     - Apply policy rules for each individual
     - Explicitly aggregate to get K_t and the other elements of S_t
  2. Run a standard projection step $\Longrightarrow \hat{\eta}_\Gamma$
  3. Possibly apply some dampening

$$\eta_\Gamma^{iter+1} = \lambda \hat{\eta}_\Gamma + (1-\lambda)\eta_\Gamma^{iter}, \text{ with} 0 < \lambda \leq 1$$

- Continue until convergence

How to simulate

Two possibilities:

- A cross-section with a large but finite number of agents
- A histogram

In the "simulation" slides, we show that using a histogram is (i) faster and (ii) much preferred, because it avoids sampling variation for cross-sectional distribution which can be substantial for some of its characteristics

APPROXIMATE AGGREGATiON

Approximate aggregation

- The mean is often sufficient $\not\Rightarrow$close to complete markets
- Why does only the mean matter?

Approximate aggregation

- Approximate aggregation $\equiv$
  - Next period's prices can be described quite well using
    - exogenous driving processes
    - means of current-period distribution
- Approximate aggregation
  - $\neq$ aggregates behave as in RA economy
    - with *same* preferences
    - with any preferences
  - $\neq$ individual consumption behaves as aggregate consumption

Why approximate aggregation

- If policy function *exactly* linear in levels (so also not loglinear)
  $\Longrightarrow$ redistributions of wealth don't matter at all
  $\Longrightarrow$ Only mean needed for calculating next period's mean

- *Approximate* aggregation still possible with non-linear policy functions
  - but policy functions must be sufficiently linear where it matters

SUMMING UP

KS algorithm: Advantages & Disadvantages

- simple
- MC integration to calculate cross-sectional means
  - can easily be avoided (see simulation slides)
- Points used in projection step are clustered around the mean
  - Theory suggests this would be bad (recall that even equidistant nodes does not ensure uniform convergence; Chebyshev nodes do)
  - At least for the model in KS (1998) this is a non-issue; in JEDC comparison project the aggregate law of motion for K obtained this way is the most accurate

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