Welfare Metrics

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1 Introduction

Welfare and Welfare Metrics are important topics in Macroeconomics. Regimes and Policies choices are usually based on welfare. That is to say that we are going to assess the welfare implication of alternative policies or regimes and rank those policies or regimes on welfare ground. In this section, we are going to introduce how welfare is measured under the DSGE framework.

There are several ways to construct the welfare metrics. First, social welfare loss. This metric can be typically found at Woodford(2003)[1] and further explained by Gali(2008)[2]. Second, Compensation Variation (CV henceforth). This way originally purposed by Schmitt-Grohé and Uribe(2003) [3]. We will illustrate them both. Let's begin with the second way.

2 Compensation Variation Welfare Metrics

There are usually two ways in compensation variation. One is unconditional metric and the other is conditional metric. The unconditional compensating variation, which measures welfare as the unconditional expectation of the value function and thus gives a sense of the welfare difference in the long-run. The usual way is not to calculate the welfare level rather to calculate

[1]Woodford M. Interest and Prices: Foundations of a Theory of Monetary Policy[M]. Princeton University Press, 2003.

[2]Galí J. Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework[M]. Princeton University Press, 2008.

[3]Schmitt-Grohé S., Uribe M. Closing Small Open Economy Models[J]. Journal of International Economics, 2003(1):163-185.

a compensating variation when compares the welfare metric across different regimes or policies. The conditional metric will usually evaluated on some points(steady state values for example) in the state space.

2.1 Conditional vs Unconditional Welfare Metrics

We use a baseline RBC Model to illustrate how can we find out the welfare. The problem of the planner is to choose allocations of consumption, labor, and future capital to maximize the present discounted value of flow utility, subject to the constraint that consumption plus investment not exceed production. This lifetime discounted value of utility is usually be referred as Welfare Metric of households when the choice of variables are on the saddle point path(i.e., the optimal choice). The problem can be written recursively as a Bellman equation:

$$W\left(A_t, K_t\right) = \max_{C_t, N_t, K_{t+1}} U\left(C_t, N_t\right) + \beta E_t W\left(A_{t+1}, K_{t+1}\right)$$

s.t.

$$C_t + K_{t+1} - (1-\delta) K_t \leq A_t K_t^{\alpha} N_t^{1-\alpha}$$

The optimal conditional associated with above problem:

$$U_C\left(C_t, N_t\right) = \beta E_t \left(U_C\left(C_{t+1}, N_{t+1}\right) \left(\alpha A_{t+1} \left(\frac{K_{t+1}}{N_{t+1}} \right)^{\alpha-1} + (1-\delta) \right) \right) \tag{1}$$

$$U_N\left(C_t, N_t\right) = U_C\left(C_t, N_t\right)(1-\alpha) A_t \left(\frac{K_t}{N_t} \right)^{\alpha} \tag{2}$$

$$W_t = U\left(C_t, N_t\right) + \beta E_t W_{t+1} \tag{3}$$

$$K_{t+1} = A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta) K_t \tag{4}$$

$$A_t = (1-\rho) + \rho A_{t-1} + \sigma_i \epsilon_t \tag{5}$$

where Equation (3) is a representation of the value function and we treat W_t as a variable where all choice variables are optimal choices. σ_i is a parameter denotes different regimes. In this context, σ_i will take two values, one is high and the other is low. We will compare this two regimes and calculate the CV welfare metric.

Define the conditional welfare metric $W_i(A_t, K_t)$ is the expected present discounted value of flow utility evaluated at the optimal choices of consumption $\tilde{C}_t$ and labor $\tilde{N}_t$:

$$W_i(A_t, K_t) = E_t \sum_{j=0}^{\infty} \beta^j U\left(\tilde{C}_{t+j}, \tilde{N}_{t+j}\right), i = h, l \tag{6}$$

where i denotes different states of the economy. One is refer to the high volatility when $i = h$ and low volatility regime when $i = l$; We will use them in the following section without further explanation.

We define a compensation variation welfare metric for the two different volatility regimes. In particular, let λ be the fraction of consumption that the household would need each period in the high-volatility regime to yield the same welfare as would be achieved in the low- volatility regime. A positive value for λ means that the household prefers the low-volatility regime —| it would need extra consumption when volatility is high to be indifferent between the two regimes. In contrast, a negative value of λ means that the household prefers the high-volatility regime. λ is the solution to the following expression:

$$W_l(A_t, K_t) = E_t \sum_{j=0}^{\infty} \beta^j U\left((1+\lambda)\tilde{C}_{h,t+j}, \tilde{N}_{h,t+j}\right) \tag{7}$$

It is important to note that above we conditioned on the same state vector in computing the compensating variation for different volatility regimes. In other words, the value functions in both the high- and low- volatility regime are evaluated at the same point in the state space; hence the expectations operator on the right hand side of (7) is conditional on the same$(A_t; K_t)$ at which the value function in the low-volatility regime is evaluated on the left hand side. The point at which these value functions are evaluated will in general affect the magnitude of λ. We evaluate the value functions at the non-stochastic steady-state values of A_t and K_t, denoted by Z and K [4].

Another welfare metric that one could compute is based on an unconditional value function. In this exercise one compares mean welfare across the two regimes, rather than conditioning on the same initial point in the state

[4] See more information in section 2.3.2.

space. The expected welfare of a particular regime is:

$$E\left(W_i\left(A_t, K_t\right)\right) = E\sum_{j=0}^{\infty}\beta^j U\left(\left(1+\lambda\right)\tilde{C}_{i,t+j}, \tilde{N}_{i,t+j}\right)$$

In contrast with Eq.(6), in this formulation E is an unconditional expectations operator and appears on both sides; the unconditional expectation replaces the conditional operator on the right hand side via application of the Law of Iterated Expectations. We can thus define an unconditional compensating variation, λ^u as follows:

$$E\left(W_l\left(A_t, K_t\right)\right) = E\sum_{j=0}^{\infty}\beta^j U\left(\left(1+\lambda^u\right)\tilde{C}_{h,t+j}, \tilde{N}_{h,t+j}\right)$$

The conditional and unconditional exercises will in general yield different compensating variation measures. The reason for this is that the mean of the capital stock depends on the volatility of the productivity process. For most of the exercises we consider, the mean capital stock is increasing in the innovation variance of A_t. The unconditional welfare comparison is then essentially endowing the high-volatility economy with more capital than an economy subject to the low-volatility regime, and therefore ignores the cost of the transition from a low to a high mean capital stock. Which welfare metric one prefers is contingent on the question being asked. If one wants to account for the transitional effects of changing policies, the conditional welfare metric is preferred because unconditional one ignores the sacrifices of transitioning to a higher mean capital stock. On the other hand, if one wants to know the longer term consequences of operating under different regimes, the unconditional metric is preferred because it takes into account one of the key benefits of higher volatility - a higher mean capital stock.

2.2 Welfare Metrics Compared to Steady State Welfare Metric

Above, we have discussed how we can calculate compensation variation welfare metric for different monetary policies. For the conditional welfare metric, we evaluate at the steady state of state variables under different monetary policies. But if we want to calculate compensation variation welfare metric compared to the steady states welfare, then there is a slightly different in calculation.

Elekdağ and Tchakarov(2007)[5] have used this idea. His measure of the welfare associated with a particular regime is based on Lucas (1987). More specifically, the unconditional loss is measured in terms of the fraction of additional deterministic steady-state consumption needed to equate the utility level of the unconditional expectation under uncertainty, as described as follows with the deterministic steady state:

$$\mathbb{E}\left(U\left(C_t, L_t\right)\right) = U\left(\left(1-\lambda\right)C, N\right) \tag{8}$$

where λ is the welfare metric relative to steady state consumption and labor.

The 2nd-order Taylor expansion of U_t around a steady state (C, N)yields

$$\begin{aligned}\mathbb{E}\left(U\left(C_t, N_t\right)\right) &= U\left(C, N\right) + U_c C \cdot \mathbb{E}\left(\hat{C}_t\right) + U_N N \cdot \mathbb{E}\left(\hat{N}_t\right) + \frac{U_{cc}C^2 + U_c C}{2} \cdot \mathbb{VAR}\left(\hat{C}_t\right) \\ &\quad + \frac{U_{NN}N^2 + U_N N}{2} \cdot \mathbb{VAR}\left(\hat{N}_t\right) + U_{CN}CN \cdot \mathbb{E}\left(\hat{C}_t\hat{N}_t\right)\end{aligned}$$

If we assume a simple additively separable preference like,

$$U\left(C_t, N_t\right) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi}$$

Then by the definition of λ in (8), we have

$$\lambda = 1 - \left(1 + (1-\sigma)\left(\mathbb{E}\left(\hat{C}_t\right) - \alpha\mathbb{E}\left(\hat{L}_t\right) + \frac{(1-\sigma)}{2}\mathbb{VAR}\left(\hat{C}_t\right) - \frac{\alpha\left(1+\psi\right)}{2}\mathbb{VAR}\left(\hat{L}_t\right)\right)\right)^{\frac{1}{1-\sigma}}$$

Using the technique developed below, we could easily calculate λ in Dynare.

2.3 Examples

2.3.1 Additively Separable, Log over Consumption

The utility is assumed to have the form

$$U\left(C_t, N_t\right) = \log C_t - \psi\frac{N_t^{1+\phi}}{1+\phi}, \phi, \psi \geq 0$$

[5]Elekdağ S., Tchakarov I. Balance Sheets, Exchange Rate Policy, and Welfare[J]. Journal of Economic Dynamics and Control, 2007(12):3986-4015.

The welfare or the value function evaluated at a particular point in the state space (A_t, K_t), can be defined as

$$W_i\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(\log \tilde{C}_{i,t+j} - \psi \frac{\tilde{N}_{i,t+j}^{1+\phi}}{1+\phi}\right)$$

Given the separability of the utility function, it is helpful to define two auxiliary value function W^C and W^N:

$$W_i\left(A_t, K_t\right) = W_i^C\left(A_t, K_t\right) + W_i^N\left(A_t, K_t\right)$$

$$W_i^C\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(\log \tilde{C}_{i,t+j}\right)$$

$$W_i^N\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(-\psi \frac{\tilde{N}_{i,t+j}^{1+\phi}}{1+\phi}\right)$$

The conditional compensating variation for the two volatility regimes is defined by:

$$W_l\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(\log\left(1+\lambda\right)\tilde{C}_{h,t+j} - \psi \frac{\tilde{N}_{h,t+j}^{1+\phi}}{1+\phi}\right)$$

Using the definitions above, this will reduces to:

$$\begin{aligned} W_l\left(A_t, K_t\right) &= E_t \sum_{j=0}^{\infty} \beta^j \log\left(1+\lambda\right) + W_h^C\left(A_t, K_t\right) + W_h^N\left(A_t, K_t\right) \\ &= \sum_{j=0}^{\infty} \beta^j \log\left(1+\lambda\right) + W_h\left(A_t, K_t\right) \end{aligned}$$

Solving and simplifying yields an expression for λ :

$$\lambda = \exp\left(\left(1-\beta\right)\left(W_l\left(A_t, K_t\right) - W_h\left(A_t, K_t\right)\right)\right) - 1$$

The sign of λ is determined by whether the expression inside of the exponential function is positive or negative. If $W_l\left(A_t, K_t\right) > W_h\left(A_t, K_t\right)$, so that the household will prefer to be in low-volatility regime, then $\lambda > 0$. In contrast, if the household would prefer the high-volatility regime, then $\lambda < 0$.

The calculation of the unconditional compensation variation is the same as above, but uses the unconditional expectations of the two values functions instead of the two value functions conditional on the same state vector:

$$\lambda^u = \exp\left(\left(1-\beta\right)\left(E\left(W_l\left(A_t, K_t\right)\right) - E\left(W_h\left(A_t, K_t\right)\right)\right)\right) - 1$$

2.3.2 Additively Separable

The utility is assumed to have the form

$$U\left(C_t, N_t\right) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \psi \frac{N_t^{1+\phi}}{1+\phi}, \sigma, \phi, \psi \geq 0$$

when $\gamma \rightarrowtail 1$, then the utility will be log on consumption. As in the case of log utility over consumption, we define two auxiliary value functions:

$$W_i\left(A_t, K_t\right) = W_i^C\left(A_t, K_t\right) + W_i^N\left(A_t, K_t\right)$$

$$W_i^C\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(\frac{\tilde{C}_{i,t+j}^{1-\sigma} - 1}{1-\sigma}\right)$$

$$W_i^N\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(-\psi \frac{\tilde{N}_{i,t+j}^{1+\phi}}{1+\phi}\right)$$

The conditional compensating variation for the two volatility regimes is defined by:

$$W_l\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left(\frac{\left(\left(1+\lambda\right) \tilde{C}_{h,t+j}\right)^{1-\sigma} - 1}{1-\sigma} - \psi \frac{\tilde{N}_{h,t+j}^{1+\phi}}{1+\phi}\right)$$

Using the definitions above, this will reduces to:

$$W_l\left(A_t, K_t\right) = \left(1+\lambda\right)^{(1-\sigma)} \left(W_h^C\left(A_t, K_t\right) + \frac{1}{\left(1-\sigma\right)\left(1-\beta\right)}\right) - \frac{1}{\left(1-\sigma\right)\left(1-\beta\right)} + W_h^N\left(A_t, K_t\right)$$

Solving for λ :

$$\lambda = \left(\frac{W_l\left(A_t, K_t\right) - W_h^N\left(A_t, K_t\right) + \frac{1}{(1-\sigma)(1-\beta)}}{W_h^C\left(A_t, K_t\right) + \frac{1}{(1-\sigma)(1-\beta)}}\right)^{\frac{1}{1-\sigma}} - 1$$

Generally, the conditional welfare value function is usually evaluated at the steady states of state variables. In 2nd approximation solution of Dynare, The solution takes the form of[6]

$$Y_t = Y^s + \frac{1}{2}\Delta^2 + \cdots + E\left(Y_{t-1}^h \otimes e_t\right)$$

[6]See Dynare Reference Manual version 4.4.3, Section 4.13.4 Second order approximation. This is the decision rule and all the endogenous variables are in DR order (Decision Rule order). Use the command 'M_.endo_names(oo_.dr.order_var,:)' to list variables in DR order: static, predetermined/purely backward, mixed and purely forward. Generally speaking, the DR order is different from declaration order.

where Y^s is the steady states of all endogenous variables Y_t which is stored in *oo_.steady_state* in declaration order[7]. $Y_t^h = Y_t - Y^s$, Δ^2 is the shift effect of the variance of future shocks which is stored in *oo_.dr.ghs*2 in DR order (Decision Rule order).

The conditional welfare value function $W(A_t, K_t)$ evaluated at steady states of state variables using above 2nd solution form will yield

$$W(A, K) = W^s + \frac{1}{2}\Delta^2$$

and we refer to $W(A, K)$ as the conditional welfare metric conditional on the steady states in state space.

The calculation of the unconditional compensation variation is similar, but uses the unconditional expectations of the two values functions instead of the two value functions conditional on the same state vector:

$$\lambda^u = \left(\frac{E(W_l(A_t, K_t)) - E\left(W_h^N(A_t, K_t)\right) + \frac{1}{(1-\sigma)(1-\beta)}}{E\left(W_h^C(A_t, K_t)\right) + \frac{1}{(1-\sigma)(1-\beta)}} \right)^{\frac{1}{1-\sigma}} - 1$$

In Dynare, the unconditional mean is generally believed to be the unconditional metric of that variable. For example, all the variables unconditional mean will be stored in *oo_.mean* in declaration order[8]. If the welfare value function is declared as the first variable (i.e., the first variable after var command), the we refer to *oo_.mean*(1) as the unconditional welfare metric of that regime. See the code below for more details.

2.3.3 KPR (1988) Preferences[9]

The utility takes the form of

$$U(C_t, N_t) = \left(\frac{1}{1-\gamma}\right)\left[\left(C_t \times \exp\left(-\psi \frac{N_t^{1+\phi}}{1+\phi}\right)\right)^{1-\gamma} - 1\right], \gamma, \phi \geq 0$$

[7] Y_t is a vector

[8] Most of the results are stored in declaration order except that in decision rule where DR order is used. Using command 'M_.endo_names' to list all variables in declaration order. The declaration order will be the same order in var command of mod file.

[9] King, R. G., Plosser, C. I., and Rebelo, S. T. (1988), Production, growth and business cycles : I. The basic neoclassical model, Journal of Monetary Economics, 21, 195-232.

Under KPR(1988) specification, the value of being in a particular state under a particular volatility regime i is:

$$W_i\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left\{ \left(\frac{1}{1-\gamma}\right) \left[\left(\tilde{C}_{i,t+j} \times \exp\left(-\psi \frac{\tilde{N}_{i,t+j}^{1+\phi}}{1+\phi}\right)\right)^{1-\gamma} - 1 \right] \right\}$$

Given non-separability of the utility, auxiliary value functions are of no use here. The conditional compensation variation for the two volatility regimes is given by:

$$W_l\left(A_t, K_t\right) = E_t \sum_{j=0}^{\infty} \beta^j \left\{ \left(\frac{1}{1-\gamma}\right) \left[\left(\left(1+\lambda\right) \tilde{C}_{i,t+j} \times \exp\left(-\psi \frac{\tilde{N}_{i,t+j}^{1+\phi}}{1+\phi}\right)\right)^{1-\gamma} - 1 \right] \right\}$$

This could be reduced to

$$W_l\left(A_t, K_t\right) = \left(1+\lambda\right)^{1-\gamma} \left(W_h\left(A_t, K_t\right) + + \frac{1}{\left(1-\gamma\right)\left(1-\beta\right)} \right) - \frac{1}{\left(1-\gamma\right)\left(1-\beta\right)}$$

Solving for λ :

$$\lambda = \left(\frac{W_l\left(A_t, K_t\right) + \frac{1}{(1-\gamma)(1-\beta)}}{W_h\left(A_t, K_t\right) + \frac{1}{(1-\gamma)(1-\beta)}} \right)^{\frac{1}{1-\gamma}} - 1$$

The unconditional compensation variation is similar, but uses the unconditional expectations of the two values functions instead of the two value functions conditional on the same state vector:

$$\lambda^u = \left(\frac{E\left(W_l\left(A_t, K_t\right)\right) + \frac{1}{(1-\gamma)(1-\beta)}}{E\left(W_h\left(A_t, K_t\right)\right) + \frac{1}{(1-\gamma)(1-\beta)}} \right)^{\frac{1}{1-\gamma}} - 1$$

2.4 The Codes

The codes here only focus on the section 2.3.2. For the other two sections, you could follow the codes below and DIY. There are the Dynare mod file (cv.mod) and a m script file (cv_index.m). The m file will recursively invoke the mod file to compute both conditional and unconditional welfare metrics under two regimes: $\sigma_l = 0.01$ and $\sigma_h = 0.02$. So all you need do is to run the m file and you will get the two welfare metrics under two regimes. In our example, we totally have 12 endogenous variables.

The m file will look like[10]:

```
%% This is the main file that recursively invokes cv.mod file

clear all;
close all;
clc;

%the high and low volatility; we only focus on this unique parameter here;
%you could add more desired parameters if you want to study the
%sensitivity for the cv welfare metric of these parameters;
%however, the cv will generally depends on the size of sigmae, the
%volatility of exogenous shock.
sigmae_arr = [0.01, 0.02];

% unconditional and conditional welfare metric
uncond_mean = zeros(3,length(sigmae_arr));
cond = zeros(3,length(sigmae_arr));

%running the mod file and save the results
for ii=1:length(sigmae_arr)
    sigmae=sigmae_arr(ii);
    save parameterfile_cv sigmae;
    dynare cv noclearall;

    %what does this mean here, explain more;
    %variables in delcaration order in oo_.mean
    %we extract unconditional welfare metric here;
    %The welfare value function w,w_c,w_l  are at No. 1, 2,3 in declaration ord
    uncond_mean(1,ii) = oo_.mean(1); %value function w
    uncond_mean(2,ii) = oo_.mean(2);%consumption part w_c
    uncond_mean(3,ii) = oo_.mean(3); %labor part w_l

    %we extract conditional welfare metric here;
    %oo_.steady_state in declaration order
```

[10]See cv_index.m in the directory.

```
    %oo_.dr_ghs2 in DR(Decision Rule) order:w n wage y i r k a w_c w_l c Rk.;
    %The welfare value function w,w_c,w_l  are at No. 1, 9,10 in DR order
    %The welfare value function w,w_c,w_l  are at No. 1, 2,3 in declaration ord
    cond(1,ii) = oo_.steady_state(1) + oo_.dr.ghs2(1); %value function
    cond(2,ii) = oo_.steady_state(2) + oo_.dr.ghs2(9); %consumption part
    cond(3,ii) = oo_.steady_state(3) + oo_.dr.ghs2(10);%labor part
end

% calculating the unconditional compensation variation welfare metric
nomin= (uncond_mean(1,1) - uncond_mean(3,2) + cab_lab);
denomin = uncond_mean(2,2) +cab_lab;

%see the note for details, lambda_u, unconditional CV
% in section 2.2.2 Additively Separable; lambda_u is usually small,
%we multiply it by 100;
%under the current parameter setting, cv = 0.0106. This mean that low
%volatility regime will be preferred.
lambda_u = 100*((nomin/denomin)^(1/(1-sigma)) -1);

% conditional compensation variation welfare metric
nomin_c= (cond(1,1) - cond(3,2) + cab_lab);
denomin_c = cond(2,2) +cab_lab;
lambda = 100*((nomin_c/denomin_c)^(1/(1-sigma)) -1);

%after calculation, we display it.
disp('conditional   unconditional');
disp([lambda lambda_u]);
```

The DR order of our variables is w n wage y i r k a w_c w_l c Rk. The welfare value function W, W^c, W^l are at No. 1, 9, 10 in DR order. The Declaration order is: w w_c w_l k a c n wage y i Rk r. The welfare value function W, W^c, W^l are at No. 1, 2, 3 in Declaration order as in mod file. You can see that the two orders are different. To understand these points and correctly indexed the welfare variables are crucial to extract correct quantities from the oo_ object.

The mod file looks like as[11] :

[11] See cv.mod in the directory.

```
/*
 *This file try to implement the ideas in the Chap8 Welfare Metrics
 *The utility takes the forms as in section 2.2.2 Additively Separable;
 *Variables are in levels; we totally have 12 endogenous variables.
 */
var w  //welfare metric
 w_c   //auxiliary variables, consumption part
 w_l   //auxiliary variables, labor part
k       //capital stock
a      //technology shock
c      //consumption
n     //labor
wage    //real wage
y     //output
i      //investment
Rk    //capital rate
r     //real rate, not gross
;

varexo e;

parameters alpha omega beta delta tau rho sigmae sigma phi psi cab_lab n_ss;

alpha=1/3; % capital's share
n_ss=1/3; % target steady state labor
beta=0.995; % Household discont factor
delta=0.02; % capital dep.
rho=0.95; % persistence of TFP
%sigmae=0.01; % std of TFP
sigma=1.05; % CRRA parameter
phi=0.4; % inverse Frisch elasticity
psi=3; % disutility of labor
cab_lab = 1/(1-sigma)/(1-beta);

%mechansim on recursively running
load parameterfile_cv;
set_param_value('sigmae',sigmae);
```

```
model;
% Firm optimality conditions
%(1) marginal product of labor = real wage
wage=a*(1-alpha)*k(-1)^(alpha)*n^(-alpha);

%(2) marginal product of capital stock = capital rate
Rk=alpha*a*k(-1)^(alpha-1)*n^(1-alpha);

% (3)TFP
a=(1-rho)+rho*a(-1)+e;

% (4) labor/leisure
psi*n^phi=(wage)*(c)^(-sigma);

% (5) capital accumulation
i=k-(1-delta)*k(-1);

% (6)production function
y=a*(k(-1))^alpha*n^(1-alpha);

% (7)resource constraint
y=c+i;

% (8)Euler equation
c^(-sigma)=beta*(1+r)*c(+1)^(-sigma);

%(9)real rate
1+r = Rk(+1)+1-delta;

% value functions
% (10) consumption part
w_c = 1/(1-sigma)*(c^(1-sigma))+beta*w_c(+1);

%(11) labor part
w_l=-psi*n^(1+phi)/(1+phi)+beta*w_l(+1);

%(12)welfare metric
```

```
w=w_c+w_l;

end;

initval;
k = n_ss*(alpha/(1/beta-1+delta))^(1/(1-alpha));
n = (1/3);
Rk = alpha*k^(alpha-1)*n^(1-alpha);
r = Rk - delta  ;
wage = (1-alpha)*n^(-alpha)*k^alpha;
y = n^(1-alpha)*k^alpha;
c = y-delta*k;
i = delta*k;
a = 1;
w_c=1/(1-beta)*(c^(1-sigma)-1)/(1-sigma);
w_l=-1/(1-beta)*psi*n^(1+phi)/(1+phi);
w=1/(1-beta)*((c^(1-sigma)-1)/(1-sigma)-psi*n^(1+phi)/(1+phi));
end;

shocks;
var e = sigmae^2;
end;

steady;

stoch_simul(order=2,irf=0,noprint);
```

3 Welfare Loss

In this section, I will focus on Gali(2008, Chapter 7) and Galí and Monacelli(2005)[12]. The analysis is based on a welfare loss function developed under some special settings which may be limited when carry over to more general cases[13]. Hence, this will limit the scope where this welfare analysis applies

[12] Galí J., Monacelli T. Monetary Policy and Exchange Rate Volatility in a Small Open Economy[J]. The Review of Economic Studies, 2005(3):707-734.

[13] If there is capital stock, things will be different. That will greatly complicate the analysis.

to. But it still deserves some attentions from us. And Let's see how we can calculate the welfare loss.

3.1 The Derivation of Welfare Loss Function

This section draws directly from Galí(2008, Chapter 7) and Galí and Monacelli(2005). But I give more details.

3.1.1 Definition

The welfare loss function $\mathbb{W}$ is defined as a **FRACTION OF STEADY STATE CONSUMPTION** and up to additive terms independent of policy.

$$\mathbb{W} \equiv \sum_{t=0}^{\infty} \beta^t \left(\frac{U_t - U}{U_c C} \right)$$

where $U \equiv U(C, N), U_c \equiv U_c(C, N)$, C, N are steady states of consumption and labor and $U_t \equiv U(C_t, N_t)$. Let's focus on expected or average period welfare loss $\frac{U_t - U}{U_c C}$. The numerator is the deviation of utility away from steady state utility. Then divided by marginal utility will convert the deviation into the unit of steady state consumption. And then divided by steady state consumption will convert the deviation into the fraction of steady state consumption and hence a dimensionless quantity.

3.1.2 Derivation

In this section, we will derive the welfare loss function under the assumption that

$$U(C_t, N_t) = \frac{C_t^{1-\sigma} - 1}{1 - \sigma} - \psi \frac{N_t^{1+\phi}}{1 + \phi}, \sigma, \phi, \psi \geq 0$$

and $\sigma = \psi = 1$. Furthermore $\eta = \gamma = 1$.

A 2nd-order approximation of utility is derived around a given steady state allocation. Frequent use is made of the following 2nd-order approximation of relative deviations in terms of log deviations

$$\frac{Z_t - Z}{Z} = \exp\left(\log \frac{Z_t}{Z}\right) - 1 = \exp(z_t - z) - 1 \approxeq z_t - z + \frac{1}{2}(z_t - z)^2 = \tilde{z}_t + \frac{1}{2}\tilde{z}_t^2 \tag{9}$$

where Z is the steady state of Z_t and $z = \log Z$, $z_t = \log Z_t$. The approximation sign is hold by making use of the Taylor expansion of natural exponential function:

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \cdots$$

and the fact that $\exp(x) \approxeq 1 + x + \frac{1}{2}x^2$ when x is close to zero. Noticing that the terms with order 3 or higher are discarded. Generally speaking, Z_t will not far away from its steady state value Z when consider a very small exogenous shock and as a result (9) will hold.

The 2nd-order Taylor expansion of U_t around a steady state (C, N)yields

$$\begin{aligned} U(C_t, N_t) &= U(C, N) + U_C C \cdot \left(\frac{C_t - C}{C}\right) + U_N N \cdot \left(\frac{N_t - N}{N}\right) \\ &\quad + \frac{U_{CC}}{2} C^2 \cdot \left(\frac{C_t - C}{C}\right)^2 + \frac{U_{NN}}{2} N^2 \cdot \left(\frac{N_t - N}{N}\right)^2 + U_{CN} CN \cdot \left(\frac{C_t - C}{C}\frac{N_t - N}{N}\right) \end{aligned}$$

Making use of eq(9) and discarding the terms with order higher than 2, we have

$$\begin{aligned} U(C_t, N_t) &= U(C, N) + U_C C \cdot \left(\hat{c}_t + \frac{(\hat{c}_t)^2}{2}\right) + U_N N \cdot \left(\hat{n}_t + \frac{(\hat{n}_t)^2}{2}\right) \\ &\quad + \frac{U_{CC}}{2} C^2 \cdot (\hat{c}_t)^2 + \frac{U_{NN}}{2} N^2 \cdot (\hat{n}_t)^2 + U_{CN} CN \cdot (\hat{c}_t \hat{n}_t) \end{aligned}$$

which could be further simplified to

$$\begin{aligned} U_t - U &= \left(\hat{c}_t + \frac{1-\sigma}{2}(\hat{c}_t)^2\right) - N^{1+\varphi}\left(\hat{n}_t + \frac{1+\varphi}{2}(\hat{n}_t)^2\right) && (10) \\ &= \hat{c}_t - N^{1+\varphi}\left(\hat{n}_t + \frac{1+\varphi}{2}(\hat{n}_t)^2\right) && (11) \\ &= \hat{c}_t - (1-\alpha)\left(\hat{n}_t + \frac{1+\varphi}{2}(\hat{n}_t)^2\right) && (12) \end{aligned}$$

where $\sigma \equiv -\frac{U_{CC}}{U_c}C$,$\varphi = \frac{U_{NN}}{U_N}N$, $U_C = \frac{1}{C}$, $U_N = -N^{\varphi}$and $U_{CN} = 0$. The second equality holds since $\sigma = 1$. The last equality holds under the optimal subsidy scheme assumed: $N^{1+\varphi} = 1 - \alpha$ (P170 in Gali(2008)).

We could see that there are consumption and labor in the 2nd-order approximation. In order to have our welfare loss function with output gap and domestic inflation, we need some algebra.

First, let's how we can substitute the consumption $\hat{c}_t$ out. Let's recopy the involved equations here from the textbook: eq(19) in P157, eq(26) in P160 and eq(28) in P161

$$c_t = c_t^* + \frac{1-\alpha}{\sigma} s_t$$

$$Y_t = C_t S_t^{\alpha} \overset{\text{Taking logorithm}}{\rightarrow} y_t = c_t + \alpha s_t$$

$$y_t^* = c_t^*$$

then we could have $c_t = (1-\alpha)\, y_t + \alpha y_t^*$since $\sigma = 1$. If world output y_t^* taken as given[14], we have

$$\hat{c}_t = (1-\alpha)\, \hat{y}_t \tag{13}$$

Second, let's how we can substitute the labor $\hat{n}_t$ out. Let's begin by looking at the definition of aggregate employment

$$N_t \equiv \frac{Y_t}{A_t} \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\epsilon} \mathrm{d}i \tag{14}$$

and the price evolution law

$$1 = \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{1-\epsilon} \mathrm{d}i \tag{15}$$

by making use the facts that

$$Y_t(i) = A_t N_t(i)$$

$$Y_t = \left(\int_0^1 Y_t(i)^{1-\frac{1}{\epsilon}} \mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1}}$$

and

$$Y_t(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\epsilon} Y_t$$

Taking logarithm at both sides of the aggregate employment equation yields

$$\hat{n}_t \approx \hat{y}_t + d_t \tag{16}$$

[14]That is the reason why we assume home country is a small economy which could not affect the world output.

where $d_t = \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\epsilon} \mathrm{d}i$ and $\hat{y}_t = y_t - y_t^n$.[15] I will follow Appendix D in Galí and Monacelli(2005). Up to 2nd-order approximation,

$$d_t = \frac{\epsilon}{2} var\left(p_{H,t}\left(i\right)\right) \tag{17}$$

and

$$\sum_{t=0}^{\infty} \beta^t var\left(p_{H,t}\left(i\right)\right) = \frac{1}{\lambda}\sum_{t=0}^{\infty} \beta^t \pi_{H,t}^2 \tag{18}$$

where $\lambda = \frac{(1-\beta\theta)(1-\theta)}{\theta}$. By combing the equation(12,13,16) and discarding the terms with order higher than 2 (such as $d_t^2, d_t y_t$ etc):

$$U_t - U = -\left(1-\alpha\right)\left(d_t + \frac{1+\varphi}{2}\hat{y}_t^2\right)$$

By substituting out eq.(17,18), we have

$$\mathbb{W} = \sum_{t=0}^{\infty} \beta^t \left(\frac{U_t - U}{U_c C}\right) = -\frac{1-\alpha}{2}\sum_{t=0}^{\infty} \beta^t \left(\frac{\epsilon}{\lambda}\pi_{H,t}^2 + \left(1+\varphi\right)\hat{y}_t^2\right) \tag{19}$$

The above loss function could be in general written in recursive form. But from statistical point, we are more interested in the 2nd moments of domestic inflation and output gap. Hence, the expected period welfare losses of any policy that deviates from strict inflation targeting can be written in terms of the variances inflation and the output gap

$$\mathbb{V} = -\frac{1-\alpha}{2}\left(\frac{\epsilon}{\lambda} var\left(\pi_{H,t}\right) + \left(1+\varphi\right) var\left(\hat{y}_t\right)\right)$$

Below, the expected welfare loss function is used to assess the welfare implications of alternative monetary policy rules.

[15] Since Natural level of employment is constant, hence natural level is one for one to technology. However, this depends on the form of production function. If $Y_t^n = A_t N_t^n$ where N_t^n is constant in flexible price equilibrium is not held, for example, this will complicate the following analysis and as a result different form of welfare loss function and expected period welfare loss function. That is one of the reason why a lot of literature adopt simple production function when welfare analysis based on loss function.

3.2 The Codes

In order to calculate the expected period welfare losses, we need to find out the numerical representation in Dynare output for $var\left(\pi_{H,t}\right)$ and $var\left(\tilde{y}_t\right)$ respectively. The struct *oo_.var* is the numerical representation of the interested variables which listed after *stoch_simul* command. The struct *oo_.var* is the theoretical moments of the listed variables. This struct is a square matrix and the diagonal correspondent to the variances of all the listed variables. For example, if $\tilde{y}_t$ is the first variable that listed after *stoch_simul* command, then $var\left(\tilde{y}_t\right) = oo\_.var(1,1)$. And if $\pi_{H,t}$ is the second variable, then $var\left(\pi_{H,t}\right) = oo\_.var(2,2)$.

There are two files in this section:

1. **WELFARE_LOSS.MOD**: The mod file to implement the model in Galí(2008, Chapter 7) as we have done in last chapter. I only focus on one monetary policy in this mod file for illustration: The PEG. You could choose whatever policy you want since we have listed out a lots.

2. **WELFARE_LOSS_INDEX.M**: the main file that will recursively invoke welfare_loss.mod and calculate the welfare loss function.

Let's list the files below:

The index m file:

```
%% This is the main file that recursively invokes welfare_loss.mod file

clear all;
close all;
clc;

%We iterate on two parameters as the textbook in P177,Table7.2:
%phi:the reciprocal of the elasticity of Frisch labor supply
%epsilon: the elasticity of substitution between varieties
% produced within any given country;
%in textbook, mu = epsilon/(epsilon-1);
epsilon_arr = [6, 11];
phi_arr =[3,10];
beta =0.995;
theta = 0.75;
lambda = (1-beta*theta)*(1-theta)/theta;
```

```
% welfare loss metric;
welfare_loss_arr = zeros(length(epsilon_arr),length(phi_arr));
var_inflation =zeros(length(epsilon_arr),length(phi_arr));
var_outputgap =zeros(length(epsilon_arr),length(phi_arr));

%running the mod file and save the desired results
for ii=1:length(epsilon_arr)
    for jj=1:length(phi_arr)
        epsilon=epsilon_arr(ii);
        phi = phi_arr(jj);
        save parameterfile_welfare epsilon phi;
        dynare welfare_loss noclearall;
        var_outputgap(ii,jj) = oo_.var(1,1);
        var_inflation(ii,jj) = oo_.var(2,2);
        welfare_loss_arr(ii,jj) =epsilon/lambda*var_inflation(ii,jj)  + ...
            (1+phi)*var_outputgap(ii,jj);
    end
end

%after calculation, we display expected period welfare loss.
disp('The welfare loss');
disp(welfare_loss_arr);
```

The mod file:

```
/*
 * This file tries to implements the Open Economy model of Jordi Gali(2008):
 * Monetary Policy, Inflation,
 * and the Business Cycle, Princeton University Press, Chapter 7
 *
 * This file is written by Xiangyang Li@Shanghai, 2016-3-25
 * I have tried to use the model equilibrium conditions in textbook
 * but it seems that there are colinearity or indeterminacy problem.
 * Thus, I borrow some equlibrium conditions from
 * Gali & Monacelli, NBER w.p. 8905, April 2002
 * Note that all model variables are expressed in log-deviations;
 */
var y_star  //overseas output
```

```
  a   // domestic technology shock;
 ygap //domestic output gap
rnat  //natrual rate
R      //nominal interest rate;
y      //domestic output
ynat // natural level of output
pi    // CPI inflation
pi_h //domestic home infaltion
pi_star // foreign inflation
s   //effective term of trade
q //  effective real exchange rate
e //  effective nominal exchange rate
p_h  //domestic price level
cpi_level // CPI price level
mc // marginal cost
nx // net export
r_star //foreign nominal interest rate
mc_star // foreign marginal cost
a_star // foreign technology shock;
n   //domestic labor
w  //domestic real wage
;
varexo  eps_a eps_y_star eps_a_star;

parameters beta sigma alpha eta epsilon phi theta rho_a rho_y_star
      phi_pi phi_y phi_pi_star phi_a_star rho_a_star a_shock_correl
      kappa_a omega sigma_a lambda  BigGamma_a BigGamma_star BigTheta
     ;


// Calibrations as per p.174
beta = 0.99; // Pure temporal discount factor
sigma = 1; // Intertemporal consumption elastiticy
alpha = 0.4; // Degree of 'openness' in the Home economy
eta = 1; // Elast. of sub. between Home and Foreign goods
%epsilon = 6; // Dixit-Stiglitz parameter for within-sector consumption
%phi = 3; // Labour disutility parameter
theta = 0.75; // Calvo probability
```

```
gamma =1;

%mechansim on recursively running
load parameterfile_welfare;
set_param_value('epsilon',epsilon);
set_param_value('phi',phi);

// Coefficient on marginal cost in the Phillips Curve
lambda = (1-(beta*theta))*(1-theta)/theta;

omega = sigma*gamma+(1-alpha)*(sigma*eta-1);
sigma_a = sigma/(1+alpha*(omega-1));
kappa_a = lambda*(phi +sigma_a); // Real rigidity, see eq(37) in textbook, henc
BigTheta = omega - 1;
BigGamma_a = (1+phi)/(sigma_a + phi);  //See eq(36);
BigGamma_star = -alpha*BigTheta*sigma_a/(sigma_a + phi); //See eq(36);

// Parameters of the productivity shocks (p.174 in textbook)
rho_a = 0.66;
rho_y_star = 0.86;
rho_a_star =0.9; //not present in textbook
phi_pi = 1.5;
phi_y = 0;

//// See (51) in the working paper and note 19;
phi_a_star = -(sigma*(1+phi)*(1-rho_a_star)) / (phi+sigma);
phi_pi_star = 1.01;
a_shock_correl = 0.3;



model(linear);
//(1) Home CPI inflation eq(15), P155 in textbook, the same afterward
pi = pi_h + alpha*(s - s(-1));

//(2) An identity to pin down the relative price of home goods,P155
p_h = p_h(-1) + pi_h;
```

```
//(3)An identity to pin down the consumer price level
cpi_level = cpi_level(-1) + pi;

//(4)Real exchange rate P156
q = (1-alpha)*s;

//(5) term of trade, eq(16), differenced version
s - s(-1) = e - e(-1) + pi_star - pi_h;

//(6)Market clearing eq(29)
y = y_star +s/sigma_a;

//(7) Definition of Home output (p. 164)
y = ynat + ygap;

//(8)Home's Phillips curve, eq(37)
pi_h = beta*pi_h(+1) + kappa_a*ygap;

//(9)Home's IS curve, eq(38)
ygap = ygap(+1) - (1/sigma_a)*(R - pi_h(+1) - rnat);

//(10) Home's natural level output (39)
ynat = BigGamma_a*a + BigGamma_star*y_star;

//(11)The definition of Home's Wicksellian interest rate, eq(39):
rnat = -sigma_a*(1-rho_a)*BigGamma_a*a
 - phi*BigGamma_star*(y_star(+1) - y_star);

//(12)The net export
nx = alpha*(omega/sigma-1)*s;

//(13)The home marginal cost
mc = (sigma_a + phi)*ygap;

//(14)The home monetary policy
//Home's monetary policy;
//pi = 0; // Strict inflation targeting (SCIT)
//R =  phi_pi*pi; //Domestic inflation Taylor rule (CIT)
```

```
e = 0; // Exchange rate peg (PEG)
//pi_h = 0; // Strict Domestic inflation targeting (SDIT)
//R = rnat +phi_pi*pi_h + phi_y*ygap;  //optimal policy, equi. to SDIT
// R = 0.5*R(-1) + phi_pi*pi + phi_y*ygap; // Simple Taylor rule
//R = rnat; // indeterminacy problem arises, can not run;


//(15)Home technology shock
// you can turn off the correlation by setting a_shock_correl = 0;
a = rho_a*a(-1) + eps_a +a_shock_correl*eps_a_star;

//(16)Foreign Euler condition,in working paper, eq(22):
y_star = y_star(+1) - (r_star - pi_star(+1))/sigma;

//AR(1) for y_star in textbook(P174) like this also works
//y_star = rho_y_star*y_star(-1) + eps_y_star;

//(17) Foreign's marginal cost, ,in working paper, eq(32):
mc_star = (sigma + phi)*y_star - (1+phi)*a_star;

// (18)Foreign's Phillips curve,in working paper, eq(31):
pi_star = beta*pi_star(+1) + lambda*mc_star;

//(19)foreign interest rate rule, Taylor rule in working paper eq(51)
r_star = phi_pi_star*pi_star(+1) + phi_a_star*a_star;

// (20)Foreign's technology process
a_star = rho_a_star*a_star(-1) + eps_a_star;

//(21) domestic real wage,in P154, eq(8), log-linearized:
w = sigma*y +phi*n;

//(22) domestic production technology,in P162, eq(32):
y = a+n;
end;

//you can turn on all the shocks; I only turn on the technology shock here;
//using the standard deviation in P174, it is unrealistic to have
```

```
// that sizes of IRFs in the textbook.
shocks;
var eps_a; stderr 0.01;
var eps_y_star; stderr 0;
var eps_a_star; stderr 0;
end;

stoch_simul(irf=16,nograph,noprint)  ygap pi_h R pi e q p_h cpi_level;
```