Advanced Tools in Macroeconomics

Continuous time models (and methods)

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The Mortensen-Pissarides model

- Continuous time is frequently used in the theoretical labor literature.
- In this lecture I will go through the workhorse model developed by Christopher Pissarides and Dale Mortensen.
- In today's exercise you will be asked to solve this model.

The Mortensen-Pissarides model: Workers

- In the simplest case workers are risk-neutral and value a job according to

$$V_t = w_t + (1-\rho)[(1-\delta)V_{t+1} + \delta U_{t+1}]$$

- The value of being unemployed is given by

$$U_t = b + (1-\rho)[(1-f_{t+1})U_{t+1} + f_{t+1}V_{t+1}]$$

- The variables w_t and f_{t+1} will be endogenously determined, but we will, for the moment, treat them as exogenous.

The Mortensen-Pissarides model: Workers

- Let's rewrite these equations in Δ units of time

$$V_t = \Delta w_t + (1 - \Delta\rho)[(1 - \Delta\delta)V_{t+\Delta} + \Delta\delta U_{t+\Delta}]$$
$$U_t = \Delta b + (1 - \Delta\rho)[(1 - \Delta f_{t+\Delta})U_{t+1} + \Delta f_{t+\Delta} V_{t+1}]$$

- Or

$$\begin{aligned} V_t - V_{t+\Delta} &= \Delta w_t - \Delta[\delta + \rho - \Delta\delta\rho]V_{t+\Delta} \\ &\quad + (1 - \Delta\rho)\Delta\delta U_{t+\Delta} \\ U_t - U_{t+\Delta} &= \Delta b - \Delta[f_{t+\Delta} + \rho - \Delta f_{t+\Delta}\rho]U_{t+\Delta} \\ &\quad + (1 - \Delta\rho)\Delta f_{t+\Delta} V_{t+\Delta} \end{aligned}$$

The Mortensen-Pissarides model: Workers

- Dividing through by Δ gives

$$\begin{aligned}\frac{V_t - V_{t+\Delta}}{\Delta} &= w_t - [\delta + \rho - \Delta\delta\rho]V_{t+\Delta} \\ &\qquad + (1 - \Delta\rho)\delta U_{t+\Delta} \\ \frac{U_t - U_{t+\Delta}}{\Delta} &= b - [f_{t+\Delta} + \rho - \Delta f_{t+\Delta}\rho]U_{t+\Delta} \\ &\qquad + (1 - \Delta\rho)f_{t+\Delta}V_{t+\Delta}\end{aligned}$$

- And taking limits $\Delta \to 0$ yields

$$\begin{aligned}-\dot{V}_t &= w_t - (\delta + \rho)V_t + \delta U_t \\ -\dot{U}_t &= b - (f_t + \rho)U_t + f_t V_t\end{aligned}$$

The Mortensen-Pissarides model: Workers

- These equations are commonly written as

$$\rho V_t = w_t + \dot{V}_t + \delta(U_t - V_t)$$
$$\rho U_t = b + \dot{U}_t + f_t(V_t - U_t)$$

- Define the surplus of having a job as $S_t = V_t - U_t$, that is

$$(\rho + \delta + f_t)S_t = w_t - b + \dot{S}_t$$

The Mortensen-Pissarides model: Firms

- The value to a firm of having an employed worker is

$$J_t = z_t - w_t + (1-\rho)[(1-\delta)J_{t+1} + \delta W_{t+1}]$$

- We will assume free entry, such that $W_t = 0$ for all t.
- Following the same procedure as before we find that in continuous time

$$(\rho + \delta)J_t = z_t - w_t + \dot{J}_t$$

The Mortensen-Pissarides model: Wages

- Collecting equations

$$\rho S_t = w_t - b + \dot{S}_t + (\delta + f_t) S_t$$
$$\rho J_t = z_t - w_t + \dot{J}_t - \delta J_t$$

- Wages are set according to Nash bargaining, which are renegotiated period-by-period

$$w_t = \text{argmax}\{J_t^{\eta} S_t^{1-\eta}\}$$

- First order condition

$$\eta S_t = (1 - \eta) J_t$$

The Mortensen-Pissarides model: Wages

- Expanding

$$\eta(w_t - b + \dot{S}_t + (\delta + f_t)S_t) = (1-\eta)(z_t - w_t + \dot{J}_t - \delta J_t)$$

and using the fact that

$$S_t = \frac{1-\eta}{\eta} J_t, \quad \text{and} \quad \dot{S}_t = \frac{1-\eta}{\eta} \dot{J}_t$$

gives

$$w_t = \eta b + (1-\eta) z_t + f_t(1-\eta) J_t$$

- Inserting into the firm's value function gives

$$\rho J_t = \eta(z_t - b) + \dot{J}_t - (f_t(1-\eta) + \delta) J_t$$

The Mortensen-Pissarides model: Matching

- Suppose that there are v_t vacancies posted and u_t unemployed individuals. Then the measure of matches in a given period is given as

$$M_t = \Delta\psi v_t^{\omega} u_t^{1-\omega}$$

- The probability that an unemployed individual finds a job, Δf_t, is then given as

$$\Delta f_t = \frac{M_t}{u_t} = \Delta\psi\theta_t^{\omega}, \quad \text{with} \;\; \theta = \frac{v_t}{u_t}$$

The Mortensen-Pissarides model: Matching

- The probability that a vacant position is filled, Δh_t, is then given as

$$\Delta h_t = \frac{M_t}{v_t} = \Delta \psi \theta_t^{\omega - 1}$$

- Suppose the cost of posting a vacancy is given by $\Delta \kappa$. Free entry then ensures that

$$\kappa = h_t J_t$$

- To see this more clearly, a firm that is considering posting a vacancy faces the optimization problem

$$\max_{v_{t,i}} \{ -\kappa \Delta v_{t,i} + v_{t,i} \Delta h_t J_t \}$$

The Mortensen-Pissarides model: Matching

- Lastly, employment, $n_t = 1 - u_t$, satisfies the law of motion

$$n_{t+\Delta} = (1 - n_t)\Delta f_t + (1 - \Delta\delta)n_t$$

which can be rearranged to

$$\frac{n_{t+\Delta} - n_t}{\Delta} = (1 - n_t)f_t - \delta n_t$$

taking limits

$$\dot{n}_t = (1 - n_t)f_t - \delta n_t$$

The Mortensen-Pissarides model:

The standard Mortensen-Pissarides model is therefore characterized by the three equations

$$
\begin{aligned}
\rho J_t &= \eta(z_t - b) + \dot{J}_t - (f_t(1-\eta) + \delta)J_t \\
\kappa &= h_t J_t \\
\dot{n}_t &= (1 - n_t)f_t - \delta n_t
\end{aligned}
$$

in the three unknowns $J_t, \theta_t, \dot{n}_t$.