Maximum Likelihood Estimation

Extensions

Tools for Macroeconomists: The essentials

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Extensions

Extensions

- Kalman smoother
- non-linear filter
- missing observations
- allowing for exogenous regressors
- allowing for covariance between w_t and v_{t+1}

Extensions

The Kalman Smoother

Kalman smoother

- main idea: one can use more information for forecasting states
- Kalman filter uses information up until the current period
- one can also use information beyond the period of the state
- objective is to calculate $\widehat{\zeta}_{t|T} = \widehat{E}[\zeta_t|\mathcal{Y}^T]$
- where $\mathcal{Y}^T = (y_1, ..., y_{t-1}, y_t, ..., y_T)$
- again uses linear projections

Smoothing recursions

- first run the Kalman filter and obtain $\widehat{\zeta}_{t|t}$, $P_{t|t-1}$ and $P_{t|t}$
- start from the end of the sample
- the smoothing recursions are:

$$\widehat{\zeta}_{t|T} = \widehat{\zeta}_{t|t} + J_t(\widehat{\zeta}_{t+1|T} - \widehat{\zeta}_{t+1|t})$$

$$P_{t|T} = P_{t|t} + J_t(P_{t+1|T} - P_{t+1|t})$$

$$J_t = P_{t|t} F' P_{t+1|t}^{-1}$$

Extensions

Nonlinear filter

A nonlinear state-space

Up until now, we assumed a linear state-space

$$\begin{aligned} y_t =& H'\zeta_t + w_t, & \mathbb{E}(w_t, w_t') = R \quad \forall t \\ \zeta_{t+1} =& F\zeta_t + v_{t+1}, & \mathbb{E}(v_t, v_t') = Q \quad \forall t \end{aligned}$$

However, non-linear forms can easily arise:

- higher-order solutions to DSGE models
- also in reduced-form empirical work

$$\begin{aligned} y_t =& h(\zeta_t) + w_t, & \mathbb{E}(w_t, w_t') = R \quad \forall t \\ \zeta_{t+1} =& f(\zeta_t) + v_{t+1}, & \mathbb{E}(v_t, v_t') = Q \quad \forall t \end{aligned}$$

Extended Kalman filter

- the idea behind the Extended Kalman filter is simple
- use a 1st-order Taylor expansion at each point in time

Extended Kalman filter recursions

update:

$$\begin{aligned}\widehat{\zeta}_{t|t} =& \widehat{\zeta}_{t|t-1} + P_{t|t-1}H_t(H_t'P_{t|t-1}H_t + R)^{-1}(y_t - h(\widehat{\zeta}_{t|t-1}))\\ P_{t|t} =& P_{t|t-1} - P_{t|t-1}H_t(H_t'P_{t|t-1}H_t + R)^{-1}H_t'P_{t|t-1}'\end{aligned}$$

forecast:

$$\begin{aligned}\widehat{\zeta}_{t+1|t} =& f(\widehat{\zeta}_{t|t})\\ P_{t+1|t} =& F_tP_{t|t}F_t' + Q\end{aligned}$$

where F_t and H_t are Jacobian matrices:

$$\begin{aligned}F_t =& \frac{\partial f}{\partial \widehat{\zeta}}|_{\widehat{\zeta}_{t|t}}\\ H_t =& \frac{\partial h}{\partial \widehat{\zeta}}|_{\widehat{\zeta}_{t|t-1}}\end{aligned}$$

Extensions

Missing observations

Missing observations

- the Kalman filter also conveniently handles
  - missing observations
  - mixed-frequency data
- the idea is that in periods of no observations
  - the Kalman gain $K_t = 0$
  - the “prediction error” $y_t - \widehat{y}_{t|t-1} = 0$
- careful with mixed-frequency data
  - average?
  - sum?

Extensions

Allowing for regressors

Allowing for regressors

- up until now we assumed that observations and states depend
  - only on the states themselves
- however, they may depend on other observables

Time-series model with exogenous regressors

$$
\begin{aligned}
y_t =& H'\zeta_t + Ax_t + w_t, & \mathbb{E}(w_t, w_t') = R \quad \forall t \\
\zeta_{t+1} =& F\zeta_t + Gx_t + v_{t+1}, & \mathbb{E}(v_t, v_t') = Q \quad \forall t
\end{aligned}
$$

- where x_t are observable (explanatory) variables

The combined Kalman filter recursions become:

$$\widehat{\zeta}_{t+1|t} = F\widehat{\zeta}_{t|t-1} + Gx_t + K_t(y_t - Ax_t - H'\widehat{\zeta}_{t|t-1})$$

$$K_t = FP_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}$$

$$\begin{aligned} P_{t+1|t} =& FP_{t|t-1}F' + Q \\ & - FP_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}F' \end{aligned}$$

Extensions

Covariance between innovations

Covariance between innovations

- up until now we assumed that $cov(w_t, v_{t+1}) = 0$
- i.e. that innovations to the states ...
- are independent of observation equation innovations
- here we allow them to covary: $\mathbb{E}[w_t, v_{t+1}] = C$
- in other words $\mathbb{E}[(w_t, v_{t+1})(w_t, v_{t+1})'] = \begin{pmatrix} R & C \\ C' & Q \end{pmatrix}$

Kalman recursions with $C \neq 0$

The combined Kalman filter recursions become:

$$\widehat{\zeta}_{t+1|t} = F\widehat{\zeta}_{t|t-1} + K_t(y_t - H'\widehat{\zeta}_{t|t-1})$$

$$K_t = (FP_{t|t-1}H + C)(H'P_{t|t-1}H + R)^{-1}$$

$$\begin{aligned} P_{t+1|t} = &FP_{t|t-1}F' + Q \\ &- (FP_{t|t-1}H + C)(H'P_{t|t-1}H + R)^{-1}(FP_{t|t-1}H + C)' \end{aligned}$$