Perturbation and Dynare

Extensions

Tools for Macroeconomists: The essentials

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Extensions

- Higher-order perturbation
- Pruning

Higher-order perturbation

Getting the policy function derivatives

$$\mathbb{E}_t F\left(g(h(x_t,\sigma) + \sigma\widetilde{\epsilon}_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \sigma\widetilde{\epsilon}_{t+1}, x_t\right) = 0$$

- under 1st order perturbation we have

$$g(x,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x-\bar{x}) + g_\sigma(\bar{x},0)\sigma$$
$$h(x,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x-\bar{x}) + h_\sigma(\bar{x},0)\sigma$$

- we also know that

$$g(\bar{x},0) = \bar{c}$$
$$h(\bar{x},0) = \bar{x}$$

- still need to solve for the derivatives

To understand - simplify further

- for now, substitute out consumption via budget constraint
- we can then write the system as

$$\mathbb{E}_t F\left(h(h(x_t,\sigma)+\sigma\widetilde{\epsilon}_{t+1},\sigma)+\sigma\widetilde{\epsilon}_{t+2}, h(x_t,\sigma)+\sigma\widetilde{\epsilon}_{t+1}, x_t\right)=0$$

Higher-order perturbation

1st & higher-order without uncertainty

GETTING FIRST-ORDER DERIVATIVE W.R.T. x_t

$$F_x = \frac{\partial F}{\partial x_{t+2}} \frac{\partial x_{t+2}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial x_t} + \frac{\partial F}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial x_t} + \frac{\partial F}{\partial x_t}$$

$$= \overline{F}_1 \frac{\partial x_{t+2}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial x_t} + \overline{F}_2 \frac{\partial x_{t+1}}{\partial x_t} + \overline{F}_3$$

$$= \overline{F}_1 h_x^2 + \overline{F}_2 h_x + \overline{F}_3 = 0$$

- $\frac{\partial F(x_{t+2}, x_{t+1}, x_t, \sigma)}{\partial x_{t+i}} \Big|_{x_{t+2}=x_{t+1}=x_t=\bar{x}, \sigma=0} = \overline{F}_{3-i}$
- $\frac{\partial h(x_t, \sigma)}{\partial x_t} \Big|_{x_t=\bar{x}, \sigma=0 \ \forall t} = h_x$

Before going on

- before we look at the derivative w.r.t. σ
- and before we generalize
- let us look at the 2-nd order derivative w.r.t. x_t

$$
\begin{aligned}
h(x,\sigma) =& h(\bar{x},\bar{\sigma}) + h_x(\bar{x},\bar{\sigma})(x-\bar{x}) + h_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma}) \\
&+ 1/2[h_{xx}(\bar{x},\bar{\sigma})(x-\bar{x})^2 + 2h_{x\sigma}(\bar{x},\bar{\sigma})(x-\bar{x})(\sigma-\bar{\sigma}) \\
&+ h_{\sigma\sigma}(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})^2]
\end{aligned}
$$

Getting 2nd-order derivative w.r.t. x_t

$$
\begin{aligned}
F_{xx} = & \frac{\partial F_x}{\partial x_t} = h_x^2(\bar{F}_{11}h_x^2 + \bar{F}_{12}h_x + \bar{F}_{13}) \\
& + \bar{F}_1 2h_x h_{xx} \\
& + h_x(\bar{F}_{21}h_x^2 + \bar{F}_{22}h_x + \bar{F}_{23}) \\
& + \bar{F}_2 h_{xx} \\
& + (\bar{F}_{31}h_x^2 + \bar{F}_{32}h_x + \bar{F}_{33}) = 0
\end{aligned}
$$

- $\frac{\partial^2 F(x_{t+2}, x_{t+1}, x_t, \sigma)}{\partial x_{t+i} \partial x_{t+j}}\big|_{x_{t+2}=x_{t+1}=x_t=\bar{x}, \sigma=0} = \bar{F}_{3-i,3-j}$
- $\frac{\partial h(x_t, \sigma)}{\partial x_t^2}\big|_{x_t=\bar{x}, \sigma=0\ \forall t} = h_{xx}$

GETTING 2ND-ORDER DERIVATIVE W.R.T. x_t

- the above is *linear* in h_{xx}
- the same holds for higher-order derivatives
- i.e. easy to solve for coefficients of approximating polynomial

Higher-order perturbation

Introducing uncertainty

Back to 1st order case

$$h(x,\sigma) = h(\bar{x},0) + h_x(\bar{x},0)(x-\bar{x}) + h_\sigma(\bar{x},0)\sigma$$

- we can find h_x from a 2nd order system
- further higher-order terms can be solved from linear systems
- but what about h_σ?

Getting 1st-order derivative w.r.t. σ

$$\mathbb{E}_t F\Big(h(h(x_t,\sigma)+\sigma\widetilde{\epsilon}_{t+1},\sigma)+\sigma\widetilde{\epsilon}_{t+2}, h(x_t,\sigma)+\sigma\widetilde{\epsilon}_{t+1}, x_t \Big) =$$

$$= \mathbb{E}_t F(x'', x', x) = 0$$

$$\mathbb{E}_t F_\sigma(x'', x', x, \sigma)|_{x=\bar{x},\sigma=0} =$$

$$= \mathbb{E}_t \left[F_{x''}[h_\sigma + h_x(\widetilde{\epsilon}_{t+1} + h_\sigma) + \widetilde{\epsilon}_{t+2}] + F_{x'}(h_\sigma + \widetilde{\epsilon}_{t+1})\right]$$

$$= F_{x''} h_\sigma (1 + h_x) + F_{x'} h_\sigma = 0$$

Getting 2-order derivative w.r.t. σ

$$
\begin{aligned}
g(x,\sigma) =& g(\bar{x},\bar{\sigma}) + g_x(\bar{x},\bar{\sigma})(x-\bar{x}) + g_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma}) \\
& + 1/2[g_{xx}(\bar{x},\bar{\sigma})(x-\bar{x})^2 + 2g_{x\sigma}(\bar{x},\bar{\sigma})(x-\bar{x})(\sigma-\bar{\sigma}) \\
& + g_{\sigma\sigma}(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})^2]
\end{aligned}
$$

$$
\begin{aligned}
h(x,\sigma) =& h(\bar{x},\bar{\sigma}) + h_x(\bar{x},\bar{\sigma})(x-\bar{x}) + h_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma}) \\
& + 1/2[h_{xx}(\bar{x},\bar{\sigma})(x-\bar{x})^2 + 2h_{x\sigma}(\bar{x},\bar{\sigma})(x-\bar{x})(\sigma-\bar{\sigma}) \\
& + h_{\sigma\sigma}(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})^2]
\end{aligned}
$$

- one can show that for 2nd-order cross-terms are 0
  - $h_{x\sigma} = g_{x\sigma} = 0$

Simulating higher-order approximations and pruning

Simulating higher-order approximations and pruning

We talked about 1st-order solutions and how to simulate them

- they are by construction stationary (if Blanchard-Kahn conditions satisfied)

Higher-order approximations can be problematic (and will be so for large shocks)

- solution is to do pruning
    - ensures stability, but...
    - it's no longer a policy function of original state space

Pruning: main idea

Denote n-th order perturbation solution for k (as a function of k_{-1} and z) as $k^{(n)}(k_{-1}, z)$

Denote n-th order terms of $k^{(n)}(k_{-1}, z)$ as $\widetilde{k}^{(n)}(k_{-1}, z)$

Denote the value of k_t generated by $k^{(n)}(.,.)$ as $k_t^{(n)}$

Simulating first-order solution:

$$k_t^{(1)} - \bar{k} = a_{kk}^{(1)}(k_{t-1}^{(1)} - \bar{k}) + a_{kz}^{(1)}(z_t - \bar{z})$$

Simulating second-order solution:

$$k_t^{(2)} - \bar{k} = a_{kk}^{(2)}(k_{t-1}^{(2)} - \bar{k}) + a_{kz}^{(2)}(z_t - \bar{z}) + \widetilde{k}^{(2)}(k_{t-1}^{(2)}, z_t)$$

Simulating pruned second-order solution:

$$k_t^{(2)} - \bar{k} = a_{kk}^{(2)}(k_{t-1}^{(2)} - \bar{k}) + a_{kz}^{(2)}(z_t - \bar{z}) + \widetilde{k}^{(2)}(k_{t-1}^{(1)}, z_t)$$

- note that when using pruning, $k_t(2)$ is a function of 3 state variables!

Pruning: main idea

Simulating pruned second-order solution:

$$k_t^{(2)} - \bar{k} = a_{kk}^{(2)}(k_{t-1}^{(2)} - \bar{k}) + a_{kz}^{(2)}(z_t - \bar{z}) + \widetilde{k}^{(2)}(k_{t-1}^{(1)}, z_t)$$

- $k_t^{(1)}$ is stationary by construction (as long as Blanchard-Kahn conditions satisfied)
- $\widetilde{k}^{(2)}(k_{t-1}^{(1)}, z_t)$ is then also stationary
- $|a_{kk}^{(2)}| < 1$ then ensures that $k_t^{(2)}$ is also stationary


That's all Folks!