Advanced Tools in Macroeconomics

Continuous time models (and methods)

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The Aiyagari Model in Continuous Time

- In these slides we will apply our knowledge thus far to the Aiyagari model in continuous time.
- This will also be today's exercise

1. Households' problem
2. Firms problem
3. Equilibrium

Households

- Households can be employed or unemployed
- When employed they receive income $w_t(1 - \tau_t)$
- When unemployed they receive unemployment benefits equal to μw_t
- An employed individual becomes unemployed with probability λ_e.
- An unemployed individual becomes employed with probability λ_u
- In an Aiyagari model prices are constant: $r_t = r$ and $w_t = w \ \forall t$

Households

- Dynamics of aggregate unemployment

$$
\begin{aligned}
e_{t+1} &= (1-\lambda_e)e_t + \lambda_u u_t \\
u_{t+1} &= \lambda_e e_t + (1-\lambda_u)u_t
\end{aligned}
$$

- Δ units of time

$$
\begin{aligned}
e_{t+\Delta} &= (1-\Delta\lambda_e)e_t + \Delta\lambda_u u_t \\
u_{t+\Delta} &= \Delta\lambda_e e_t + (1-\Delta\lambda_u)u_t
\end{aligned}
$$

- Rearrange and take limits

$$
\begin{aligned}
\dot{e}_t &= -\lambda_e e_t + \lambda_u u_t \\
\dot{u}_t &= \lambda_e e_t - \lambda_u u_t
\end{aligned}
$$

Households

- System

$$\dot{\mathbf{s}}_t = \mathbf{T}\mathbf{s}_t$$

with

$$\mathbf{T} = \begin{pmatrix} -\lambda_e & \lambda_u \\ \lambda_e & -\lambda_u \end{pmatrix}$$

- Stationary equilibrium

$$\mathbf{0} = \mathbf{T}\mathbf{s}$$

- Thus $\mathbf{s}$ is an eigenvector associated with a zero eigenvalue, with the eigenvector normalised to sum to one.

Households

- Can be solved as a regular eigenvalue problem
- But since the eigenvector is only defined up to a scalar we can use the following trick
  1. Create vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and matrix} \quad \hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ \lambda_e & -\lambda_u \end{pmatrix}$$

  2. Find $\hat{\mathbf{s}}$ as $\hat{\mathbf{s}} = \hat{\mathbf{T}}^{-1}\mathbf{b}$.
  3. Normalise $\hat{\mathbf{s}}$ to sum to one to find $\mathbf{s}$.
- The first element of $\mathbf{s}$ is then the stationary employment rate, and the second the stationary unemployment rate.

Households

- Government runs a balanced budget, so not deficits
- The tax rate then solves $u\mu w = e\tau w$
- Or just $\tau = \frac{u}{e}\mu$

Households

- Bellman equation for an employed agent

$$v(a_t, e) = \max_{c_t}\{u(c_t) + (1-\rho)\times$$
$$[(1-\lambda_e)v(w_t(1-\tau_t) + (1+r_t)a_t - c_t, e)$$
$$+ \lambda_e v(w_t(1-\tau_t) + (1+r_t)a_t - c_t, u)]\}$$

subject to $a_t \geq \phi \; \forall t$.

- Δ units of time

$$v(a_t, e) = \max_{c_t}\{\Delta u(c_t) + (1-\Delta\rho)\times$$
$$[(1-\Delta\lambda_e)v(\Delta(w_t(1-\tau_t) + r_t a_t - c_t) + a_t, e)$$
$$+ \Delta\lambda_e v(\Delta(w_t(1-\tau_t) + r_t a_t - c_t) + a_t, u)]\}$$

Households

- Rearrange and divide by Δ

$$
\begin{aligned}
0 = \max_{c_t} \{ & u(c_t) \\
& + \frac{v(\Delta(w_t(1-\tau_t) + r_t a_t - c_t) + a_t, e) - v(a_t, e)}{\Delta} \\
& - (\rho + \lambda_e + \Delta\rho\lambda_e) v(\Delta(w_t(1-\tau_t) + r_t a_t - c_t) + a_t, e) \\
& + \lambda_e v(\Delta(w_t(1-\tau_t) + r_t a_t - c_t) + a_t, u)]\}
\end{aligned}
$$

- Take limits and rearrange

$$
\begin{aligned}
\rho v(a, e) = \max_{c} \{ & u(c) + v_a(a, e)(w(1-\tau) + ra - c) \\
& - \lambda_e (v(a, e) - v(a, u))\}
\end{aligned}
$$

Households

So households' problem is given by the two HJB equations

$$\begin{aligned}\rho v(a,e) = \max_c \{ u(c) + v_a(a,e)(w(1-\tau) + ra - c) \\ - \lambda_e(v(a,e) - v(a,u))\}\end{aligned}$$

$$\begin{aligned}\rho v(a,u) = \max_c \{ u(c) + v_a(a,u)(w\mu + ra - c) \\ - \lambda_u(v(a,u) - v(a,e))\}\end{aligned}$$

Let's be smart in solving them!

Households

1. Start with a linearly spaced grid for assets $\mathbf{a} = [a_1, a_2, \ldots, a_N]$. Let $da = a(n+1) - a(n)$.
2. For each grid for assets guess a for $v_0(a_i, j)$, $\forall a_i \in \mathbf{a}$, and $j \in \{e, u\}$. This gives us $\mathbf{v}_{0,e}$ and $\mathbf{v}_{0,u}$
3. Call the stacked $2N \times 1$ vector $(\mathbf{v}_{0,e}, \mathbf{v}_{0,u})'$ for $\mathbf{v}_0$.

Households

4. Create two $N \times N$ difference operators as

$$\mathbf{D_f} = \begin{pmatrix} -1/da & 1/da & 0 & \dots & 0 \\ 0 & -1/da & 1/da & 0 & \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & & -1/da & 1/da \\ 0 & \dots & & 0 & -1 \end{pmatrix}$$

$$\mathbf{D_b} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1/da & 1/da & 0 & & \\ \vdots & -1/da & 1/da & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & & -1/da & 1/da \end{pmatrix}$$

Households

5. Create one $2N \times 2N$ matrix as

$$\mathbf{B} = \begin{pmatrix} -\lambda_e & 0 & \dots & 0 & \lambda_e & 0 & \dots & 0 \\ 0 & -\lambda_e & 0 & \dots & 0 & \lambda_e & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & -\lambda_e & 0 & \dots & \dots & \lambda_e \\ \lambda_u & 0 & \dots & 0 & -\lambda_u & 0 & \dots & 0 \\ 0 & \lambda_u & 0 & \dots & 0 & -\lambda_u & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_u & 0 & 0 & \dots & -\lambda_u \end{pmatrix}$$

will be used later

Households

6. Calculate the derivative of the value functions using both forward and backward differences

$$\mathbf{v}'_f(a,e) = \mathbf{D_f}\mathbf{v}_{0,e}, \quad \mathbf{v}'_b(a,e) = \mathbf{D_b}\mathbf{v}_{0,e},$$
$$\mathbf{v}'_f(a,u) = \mathbf{D_f}\mathbf{v}_{0,u}, \quad \mathbf{v}'_b(a,u) = \mathbf{D_b}\mathbf{v}_{0,u},$$

7. Set the **first** elements of $\mathbf{v}'_b(a,e) = u'(w(1-\tau) + r\phi)$ and $\mathbf{v}'_b(a,u) = u'(w\mu + r\phi)$, and the **last** elements of $\mathbf{v}'_f(a,e) = u'(w(1-\tau) + ra_N)$ and $\mathbf{v}'_f(a,u) = u'(w\mu + ra_N)$
8. Find optimal consumption through

$$u'(\mathbf{c}_{e,f}) = \mathbf{v}'_f(a,e), \quad u'(\mathbf{c}_{e,b}) = \mathbf{v}'_b(a,e),$$
$$u'(\mathbf{c}_{u,f}) = \mathbf{v}'_f(a,u), \quad u'(\mathbf{c}_{u,b}) = \mathbf{v}'_b(a,u),$$

Households

9. Find optimal savings as

$$\mathbf{s}_{e,f} = w(1-\tau) + r\mathbf{a} - \mathbf{c}_{e,f}, \quad \mathbf{s}_{e,b} = w(1-\tau) + r\mathbf{a} - \mathbf{c}_{e,b},$$
$$\mathbf{s}_{u,f} = w\mu + r\mathbf{a} - \mathbf{c}_{u,f}, \quad \mathbf{s}_{u,b} = w\mu + r\mathbf{a} - \mathbf{c}_{u,b}$$

10. Create indicator vectors

$$\mathbf{I}_{e,f} = (I_{1,e,f}, I_{2,e,f}, \ldots, I_{N,e,f})', \quad \mathbf{I}_{e,b} = (I_{1,e,b}, I_{2,e,f}, \ldots, I_{N,e,b})',$$
$$\mathbf{I}_{u,f} = (I_{1,u,f}, I_{2,u,f}, \ldots, I_{N,u,f})', \quad \mathbf{I}_{u,b} = (I_{1,u,f}, I_{2,u,f}, \ldots, I_{N,u,f})',$$

where $I_{i,j,f} = 1$ **if** $s_{i,j,f} > 0$ and $I_{i,j,b} = 1$ **if** $s_{i,j,b} < 0$, for $i = 1, \ldots, N$ and $j \in \{e, u\}$.

Households

11. Find consumption as

$$\mathbf{c}_e = \mathbf{l}_{e,f} \cdot \mathbf{c}_{e,f} + \mathbf{l}_{e,b} \cdot \mathbf{c}_{e,b}$$
$$\mathbf{c}_u = \mathbf{l}_{u,f} \cdot \mathbf{c}_{u,f} + \mathbf{l}_{u,b} \cdot \mathbf{c}_{u,b}$$

12. Find savings as

$$\mathbf{s}_e = \mathbf{l}_{e,f} \cdot \mathbf{s}_{e,f} + \mathbf{l}_{e,b} \cdot \mathbf{s}_{e,b}$$
$$\mathbf{s}_u = \mathbf{l}_{u,f} \cdot \mathbf{s}_{u,f} + \mathbf{l}_{u,b} \cdot \mathbf{s}_{u,b}$$

13. And matrices $\mathbf{S}_e\mathbf{D}_e$ and $\mathbf{S}_u\mathbf{D}_u$ as

$$\mathbf{S}_e\mathbf{D}_e = diag(\mathbf{l}_{e,f} \cdot \mathbf{s}_{e,f})\mathbf{D}_\mathbf{f} + diag(\mathbf{l}_{e,b} \cdot \mathbf{s}_{e,b})\mathbf{D}_\mathbf{b}$$
$$\mathbf{S}_u\mathbf{D}_u = diag(\mathbf{l}_{u,f} \cdot \mathbf{s}_{u,f})\mathbf{D}_\mathbf{f} + diag(\mathbf{l}_{u,b} \cdot \mathbf{s}_{u,b})\mathbf{D}_\mathbf{b}$$

Households

14. Lastly find the $2N \times 2N$ matrix $\mathbf{S}_0\mathbf{D}_0$ as

$$\mathbf{S}_0\mathbf{D}_0 = \begin{pmatrix} \mathbf{S}_e\mathbf{D}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_u\mathbf{D}_u \end{pmatrix}$$

15. And the matrix $\mathbf{P}_0$ as

$$\mathbf{P}_0 = \mathbf{S}_0\mathbf{D}_0 + \mathbf{B}$$

Households

Using the implicit method the households' problem is given by the two HJB equations

$$\begin{aligned}\rho v_{n+1}(a,e) = u(c_n) + v_{a,n+1}(a,e)(w(1-\tau) + ra - c_n) \\ - \lambda_e(v_{n+1}(a,e) - v_{n+1}(a,u))\end{aligned}$$

$$\begin{aligned}\rho v_{n+1}(a,u) = u(c_n) + v_{a,n+1}(a,u)(w\mu + ra - c) \\ - \lambda_u(v_{n+1}(a,u) - v_{n+1}(a,e))\end{aligned}$$

Households

- These can now be written as

$$\rho \mathbf{v}_{n+1} = u(\mathbf{c}_n) + \mathbf{P}_n \mathbf{v}_{n+1}$$

  with $\mathbf{c}_n = (\mathbf{c}_{n,e}, \mathbf{c}_{n,u})$.

- So we iterate on

$$\mathbf{v}_{n+1} = [(\rho + 1/\Gamma)\mathbf{I} - \mathbf{P}_n]^{-1}[u(\mathbf{c}_n) + \mathbf{v}_n/\Gamma]$$

  until convergence

Firms

- Firms fact the standard static optimisations problem

$$\Pi_t = \max\{K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - (r_t + \delta) K_t\}$$

- With first order conditions

$$r_t = \alpha \left(\frac{K_t}{N_t}\right)^{\alpha-1} - \delta, \quad w_t = (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha}$$

- In a stationary equilibrium this implies

$$r = \alpha \left(\frac{K}{(1-u)}\right)^{\alpha-1} - \delta, \quad w = (1-\alpha)\left(\frac{r+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

Stationary distribution

- What is the evolution of the endogenous stationary distribution of wealth and employment status?
- Denote the CDF as $G_{t+1}(a,e)$. This must satisfy

$$G_{t+1}(a,e) = (1-\lambda_e)G_t(a^e_{-1},e) + \lambda_u G_t(a^u_{-1},u),$$

  where a^j_{-1} denotes "where you came from" from optimally setting $a_{t+1} = a$ in employment status $j \in \{e,u\}$.
- In Δ units of time approximate this as $a^e_{-1} = a - \Delta s_e$ and $a - \Delta s_u$. Thus

$$G_{t+\Delta}(a,e) = (1-\Delta\lambda_e)G_t(a-\Delta s_e,e) + \Delta\lambda_u G_t(a-\Delta s_u,u),$$

Stationary distribution

$$G_{t+\Delta}(a,e) = (1-\Delta\lambda_e)G_t(a-\Delta s_e, e) + \Delta\lambda_u G_t(a-\Delta s_u, u),$$

- Subtract $G_t(a,e)$ from both sides and divide by Δ

$$\frac{G_{t+\Delta}(a,e) - G_t(a,e)}{\Delta} = \frac{G_t(a-\Delta s_e, e) - G_t(a,e)}{\Delta} - \lambda_e G_t(a-\Delta s_e, e) + \lambda_u G_t(a-\Delta s_u, u),$$

- Take limits

$$\dot{G}_t(a,e) = -g_t(a,e)s_e(a) - \lambda_e G_t(a,e) + \lambda_u G_t(a,u),$$

Stationary distribution/Kolmogorov Forward Equation

$$\dot{G}_t(a,e) = -g_t(a,e)s_e(a) - \lambda_e G_t(a,e) + \lambda_u G_t(a,u),$$

- Differentiate with respect to a

$$\dot{g}_t(a,e) = -\frac{\partial[g_t(a,e)s_e(a)]}{\partial a} - \lambda_e g_t(a,e) + \lambda_u g_t(a,u),$$

- Thus the law of motion for the endogenous distribution is

$$\dot{g}_t(a,e) = -\frac{\partial[g_t(a,e)s_e(a)]}{\partial a} - \lambda_e g_t(a,e) + \lambda_u g_t(a,u),$$
$$\dot{g}_t(a,u) = -\frac{\partial[g_t(a,u)s_u(a)]}{\partial a} - \lambda_u g_t(a,u) + \lambda_e g_t(a,e)$$

Stationary distribution/Kolmogorov Forward Equation

- Remember the matrix

$$\mathbf{P}_n = \mathbf{S}_n \mathbf{D}_n + \mathbf{B}.$$

- When converged

$$\mathbf{P} = \mathbf{S}\mathbf{D} + \mathbf{B}$$

- Turns out that

$$\dot{\mathbf{g}}_t = \mathbf{P}' \mathbf{g}_t$$

- Where $\mathbf{g}_t$ is the stacked vector $(\mathbf{g}_t(a,e), \mathbf{g}_t(a,u))'$

Solving the Aiyagari model

1. Guess for an interest rate r_n. Find w_n as

$$w_n = (1-\alpha)\left(\frac{r_n+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

2. Find $\mathbf{v}$ such that

$$\mathbf{v} = [(\rho + 1/\Gamma)\mathbf{I} - \mathbf{P}]^{-1}[u(c(\mathbf{v})) + \mathbf{v}/\Gamma]$$

3. Find $\mathbf{g}$ by solving

$$\mathbf{0} = \mathbf{P}'\mathbf{g}$$

and normalise to sum to one (remember how we found $\mathbf{s}$ above)

Solving the Aiyagari model

4. Find K_n as

$$K_n = \mathbf{g}' \begin{pmatrix} \mathbf{a} \\ \mathbf{a} \end{pmatrix}$$

5. Find $\hat{r}$ as

$$\hat{r} = \alpha \left(\frac{K_n}{(1-u)} \right)^{\alpha - 1} - \delta$$

6. If $\hat{r} > r_n$ set $r_{n+1} > r_n$, else set $r_{n+1} < r_n$.
7. Repeat until $\hat{r} \approx r_n$.