Simulating Models with Heterogeneous Agents

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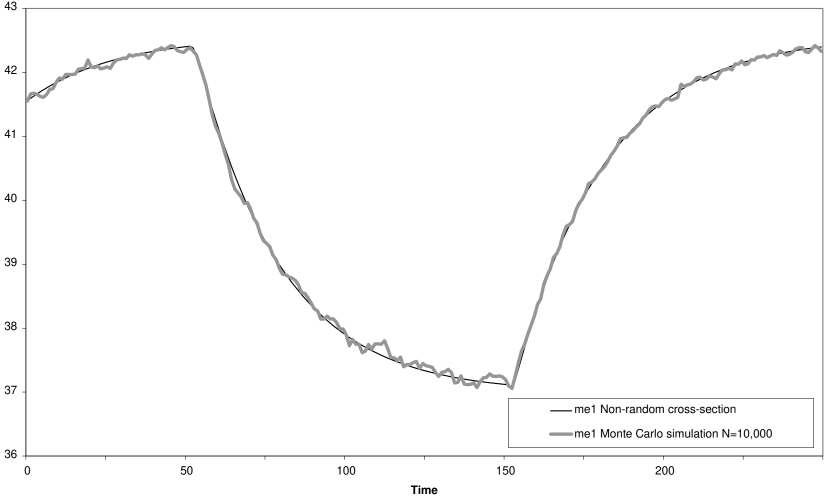
Two different ways to go

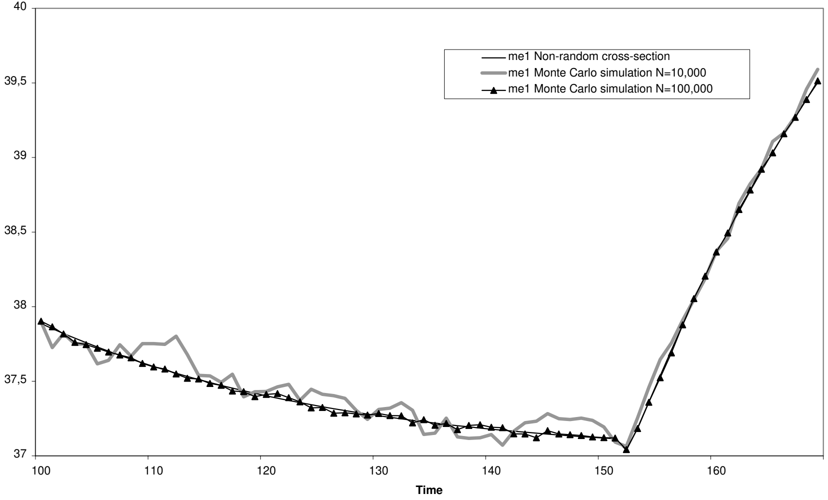
- Simulate a panel with a large number of agents
    - Monte Carlo integration to calculate cross-sectional moments
      $\Longrightarrow$ cross-sectional sampling variation
- Grid method of Young (2010)
    - faster
    - avoids cross-sectional sampling variation
      $\Longrightarrow$ much more accurate

MOTIVATIONAL PICS
TO USE GRID METHOD

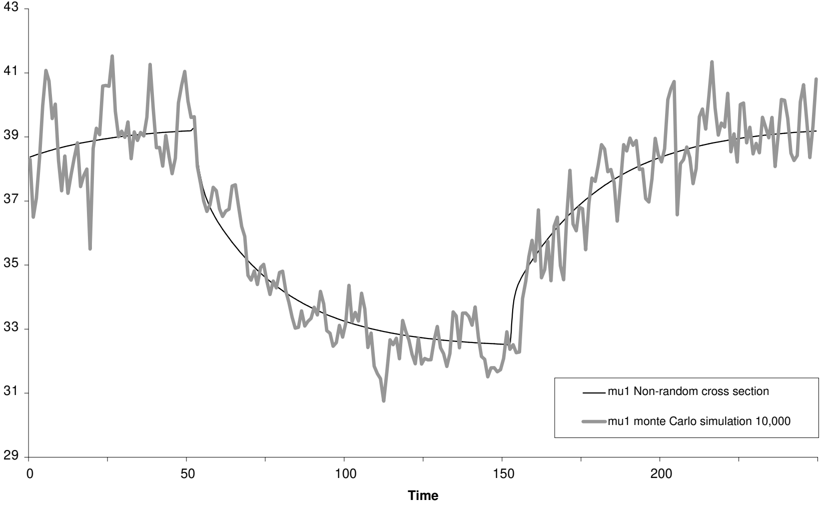
Average capital stock employed agents



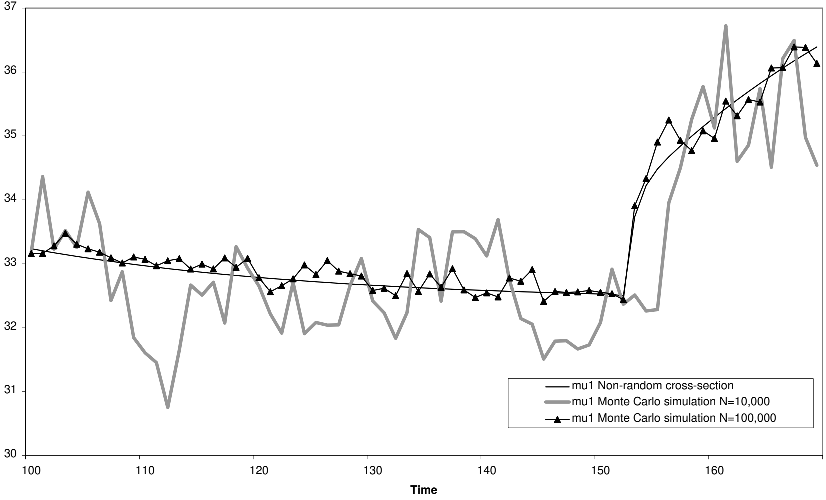
Average capital stock employed agents



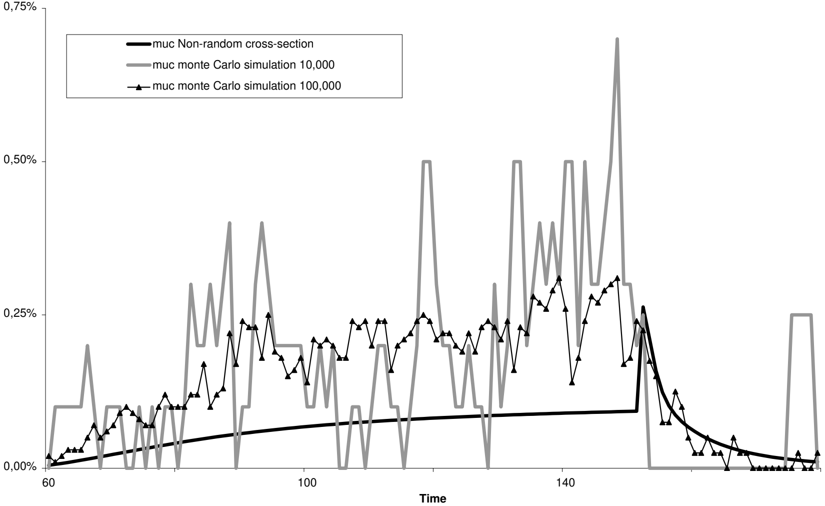
Average capital stock unemployed agents



Average capital stock unemployed agents



Fraction of agents at constraint



THE METHODS

What is given?

- A policy function for individual choice $k'(k_{i,t}, \varepsilon_{i,t}, S_t)$
    - S_t: the aggregate state variables
- initial distribution for $t = 1$
    - characterizes the density of capital holdings of the employed and unemployed.
- Approximate laws of motion for aggregate variables (e.g., for K_t or any other moment) are NOT used. If needed, for example to calculate r_t or w_t, then actual cross-sectional moments are used.

Simulation method

- Cross-section consists of I agents.

- For each t
  - Draw $\varepsilon_{i,t}$ for each i
    - Draw $\varepsilon_{i,t}$s such that unemployment rate *and* transition probabilities are exactly equal to theoretical value. (e.g., if $u_t = 0.04$ and $I = 10,000$, then 400 $\varepsilon_{i,t}$s should be equal to 0.)
    - This does NOT eliminate sampling noise, since it doesn't ensure these are imposed across capital holdings
  - Calculate $k_{i,t+1} = k'(k_{i,t}, \varepsilon_{i,t}, S_t)$ for each i.

- Cross-sectional moments follow directly from cross-section. E.g., $K_t = \sum_i^I k_{i,t}$.

Grid method

- Fine grid with nodes: κ_i, $i = 0, 1, \cdots, I$
- Only mass *AT* grid points
    - $f^{\varepsilon}_{i,t}$: mass of agents with $k^{\varepsilon}_t = \kappa_i$, $i = 0, 1, \cdots, I$
    - ε : employment status
    - no mass in between grid points
- If $k'_i \geq 0$ is binding $\Longrightarrow f^{\varepsilon}_{0,t} > 0$

Grid method

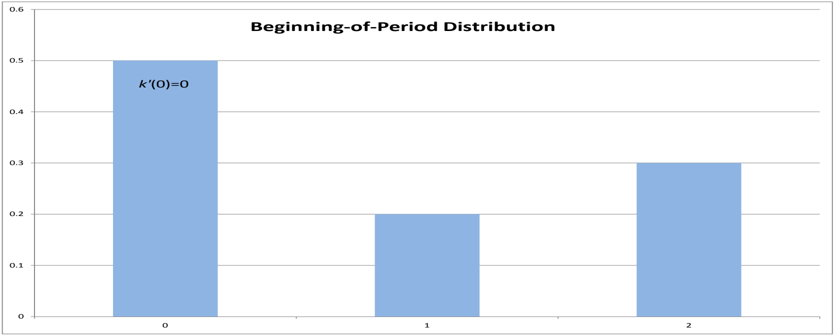
- Fix employment status (for now) to illustrate procedure
  - remain within the period t (for now), that is, go from beginning-of-period to end-of-period distribution

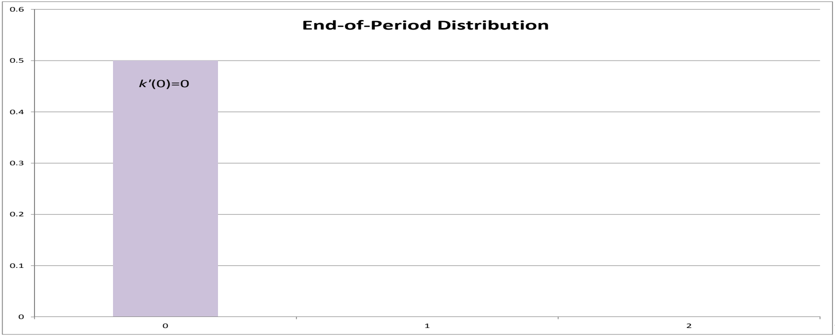
Grid method

- focus on node j with mass $f_t^{\varepsilon,j}$ and capital value κ_j
- find i such that $k'(\kappa_j, \varepsilon, \cdot)$ satisfies

$$\kappa_{i-1} < k'(\kappa_j, \varepsilon, \cdot) \leq \kappa_i$$

- if $k'(\kappa_j, \varepsilon, \cdot) > \kappa_I$, $i = I$


Beginning-of-Period Distribution
k'(0)=0
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2


End-of-Period Distribution
k'(0)=0
0.6
0.5
0.4
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0.2
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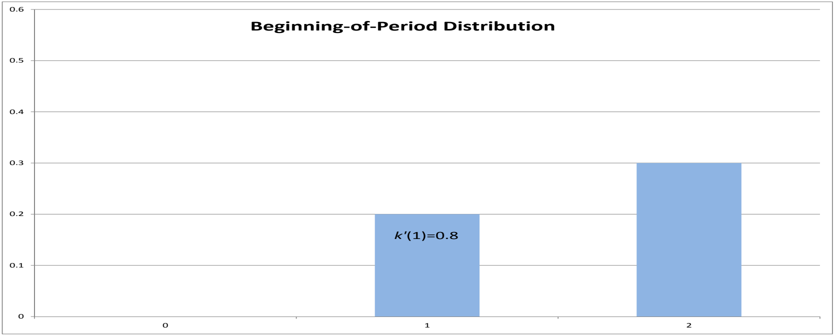
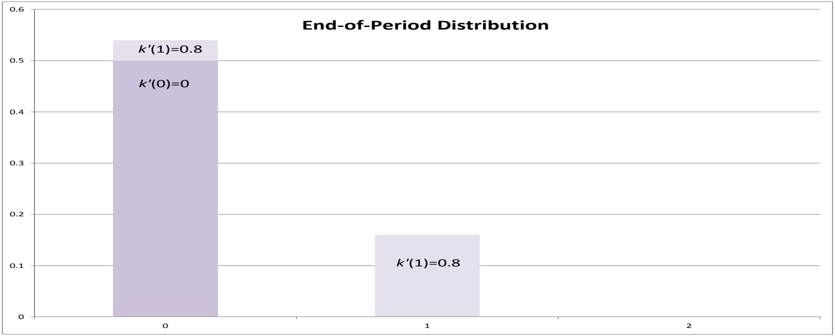
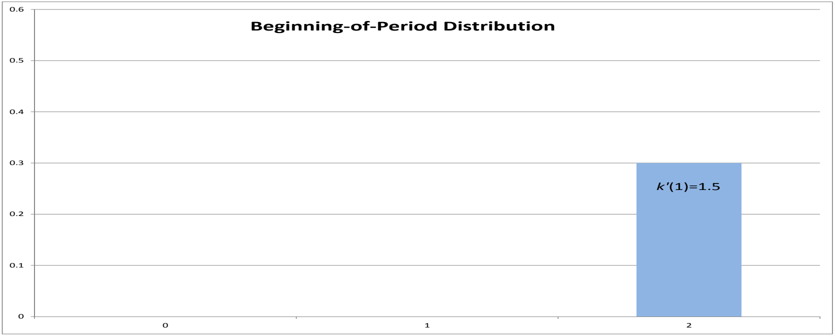
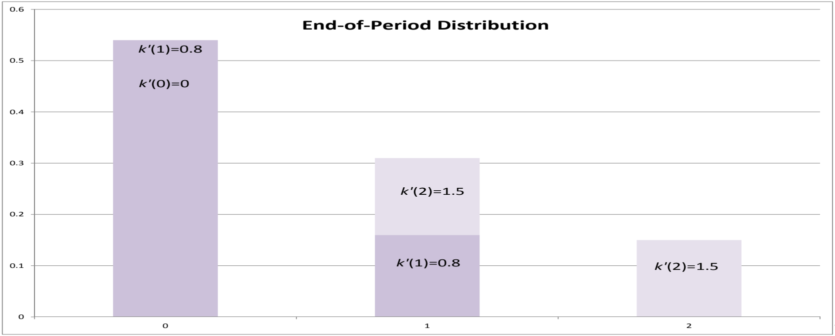
0
0
1
2


Beginning-of-Period Distribution
0.6
0.5
0.4
0.3
0.2
0.1
0
$k'(1)$=0.8
0
1
2


End-of-Period Distribution
0.6
0.5
0.4
0.3
0.2
0.1
0
$k'(1)$=0.8
$k'(0)$=0
$k'(1)$=0.8
0
1
2


Beginning-of-Period Distribution
k′(1)=1.5
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2


End-of-Period Distribution
k′(1)=0.8
k′(0)=0
k′(2)=1.5
k′(1)=0.8
k′(2)=1.5
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2

From beginning to end-of-period distribution

- Set all end-of-period fractions equal to zero:

$$p_t^{\varepsilon,i} = 0 \quad \forall i$$

- For each j allocate beginning-of-period $f_t^{\varepsilon,j}$ to relevant end-of-period $p_t^{\varepsilon,i}$s:
- if $k'(\kappa_j, \varepsilon, \cdot) < \kappa_I$ then

$$\begin{aligned}
\omega_t^{i,j} &= \frac{k'(\kappa_j, \varepsilon, \cdot) - \kappa_{i-1}}{\kappa_i - \kappa_{i-1}} \\
p_t^{\varepsilon,i-1} &= p_t^{\varepsilon,i-1} + f_t^{\varepsilon,j}\left(1 - \omega_t^{i,j}\right) \\
p_t^{\varepsilon,i} &= p_t^{\varepsilon,i} + f_t^{\varepsilon,j}\omega_t^{i,j}
\end{aligned}$$

- if $k'(\kappa_j, \varepsilon, \cdot) \geq \kappa_I$ then $p_t^{\varepsilon,I} = p_t^{\varepsilon,I} + f_t^{\varepsilon,j}$

Next period's beginning-of-period distribution

- Use transition laws to go from end-of-period t distribution to beginning-of-period $t+1$ distribution

- $\phi_{00z_t z_{t+1}}$: population transition probability of remaining unemployed for current and next period's value of z
- $\phi_{11z_t z_{t+1}}$: population transition probability of remaining employed for current and next period's value of z

Next period's beginning-of-period distribution

Use **population** transition probabilities; these are exogenous and should be exactly identical at each node.

$$\varepsilon_{t+1} = 0 : f_{t+1}^{0,i} = \phi_{00z_t z_{t+1}} p_t^{0,i} + (1 - \phi_{11z_t z_{t+1}}) p_t^{1,i}$$

$$\varepsilon_{t+1} = 1 : f_{t+1}^{1,i} = (1 - \phi_{00z_t z_{t+1}}) p_t^{0,i} + \phi_{11z_t z_{t+1}} p_t^{1,i}$$

In contrast to simulate method with I agents, the grid method imposes that transition probabilities are exactly identical across the whole distribution of capital holdings (and equal to correct theoretical values).

Praise for grid method

- Simulating using a histogram with I nodes is roughly as expensive as simulating with I agents.
- But with say 1,000 nodes you get much more accurate answer than with say 100,000 agents.
- **Why?** Because you avoid sampling noise

Note: Before Young (2010), the literature used a more complicated grid method. Instead of going through each beginning-of-period node and allocating the associated mass, $f^{\varepsilon,j}$, to $p^{\varepsilon,i}$s, it considered each end-of-period node and then *using the inverse of the policy function* determined which elements of $f^{\varepsilon,j}$ would get to that node. More complicated because you had to calculate the inverse.

References

- Young, E. R., 2010, Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm and non-stochastic simulations,Journal of Economic Dynamics and Control
    - This paper introduces the grid method explained in these slides