Maximum Likelihood Estimation

Maximum Likelihood

Tools for Macroeconomists: The essentials

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Maximum Likelihood

Estimating structural parameters

- up to now, we assumed that model parameters are known
- we can also estimate them with Maximum Likelihood (ML)
  - i.e. given data on y_t and initial conditions
  - estimate $\mathbf{\Psi} = [H, F, Q, R]$
- the Kalman filter is particularly convenient for this task

Maximum Likelihood

Main Idea

Preliminaries

- if $\zeta_{1|0}$ is Gaussian and $\{w_t, v_t\}_{t=1}^{T}$ are Gaussian

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- $\rightarrow$ distribution of y_t conditional on $\mathcal{Y}_{t-1}$ is also Gaussian

$$
\begin{aligned}
\widetilde{y}_{t|t-1}|\mathcal{Y}_{t-1} \sim & N(0, H'P_{t|t-1}H + R) \\
y_t|\mathcal{Y}_{t-1} \sim & N(H'\widehat{\zeta}_{t|t-1}, H'P_{t|t-1}H + R)
\end{aligned}
$$

Preliminaries

- given values of $\Psi \rightarrow$ calculate mean and variance of y
- we know the distribution of y
- $\rightarrow$ calculate the probability (likelihood) of $(y_1, ..., y_T)$

Likelihood function

the likelihood of a given (Gaussian) observation is

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

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$$\begin{aligned} f(y_t|\mathcal{Y}_{t-1};\Psi) =& (2\pi)^{-1/2}(H'P_{t|t-1}H+R)^{-1/2}x \\ & \exp\left\{-1/2(y_t-\widehat{y}_{t|t-1})'(H'P_{t|t-1}H+R)^{-1}(y_t-\widehat{y}_{t|t-1})\right\} \\ & \text{for } t=1,...,T \end{aligned} \tag{1}$$

Likelihood function

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it is convenient to work with the sample log-likelihood:

$$\log \mathcal{L}(\mathcal{Y}_t|\Psi) = \log f(y_0) + \sum_{t=1}^{T} \log f(y_t|\mathcal{Y}_{t-1};\Psi)$$

Why is the Kalman filter convenient?

- the Kalman filter produces $\widehat{y}_{t|t-1}$ and $P_{t|t-1}$
- the (log)-likelihood is easy to construct with the Kalman filter
- one can then maximize it with respect to the parameters $\boldsymbol{\Psi}$
- this will be your task in the afternoon session

Maximum Likelihood

Back to DSGE Models

Neoclassical growth model

- representative household maximizing expected lifetime utility
- household owns production technology
- capital is the only factor of production
- resources spent on consumption and investment into capital
- each period existing capital depreciates at certain rate
- production subject to exogenous fluctuations in productivity

Production

$$y_t = Z_t k_t^{\alpha}$$

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$$Z_t = 1 - \rho + \rho Z_{t-1} + \epsilon_t$$

$$\mathbb{E}\epsilon_t = 0$$

$$\mathbb{E}\epsilon_t^2 = \sigma_z^2$$

Household decision

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

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$$k_0 \quad \text{given}$$

$$Z_0 \quad \text{given}$$

Equilibrium conditions

$$c_t^{-1} = \mathbb{E}_t\left[\beta c_{t+1}^{-1}(\alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta)\right]$$

$$c_t + k_t = z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1}$$

$$z_t = 1 - \rho + \rho z_{t-1} + \epsilon_t$$

$$\epsilon_t \sim N(0, \sigma^2)$$

Solution

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- a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$
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- different $k_0 \rightarrow$ optimal sequences different!
- different realizations of $Z_t \rightarrow$ optimal sequences different!

Linearized version

$$
\begin{aligned}
k_t =& \bar{k} + a_{kk}(k_{t-1} - \bar{k}) + a_{kz}(z_t - \bar{z}) \\
z_t =& 1 - \rho + \rho z_{t-1} + \epsilon_t \\
& \epsilon_t \sim N(0, \sigma^2) \\
& k_0, z_0 \text{ given}
\end{aligned}
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- a_{kk}, a_{kz} and $\bar{k}$ depend on structural parameters Ψ
- $\Psi = [\alpha, \beta, \delta, \rho, \sigma, z_0]$

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Estimating Structural Parameters

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consider estimating the structural parameters using ML

- how to write down the likelihood of the model?
  - for Kalman filter, we still need to figure out H, F
  - can we do something else (simpler) instead?

Simple case of evaluating the likelihood

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\begin{aligned}
k_t =& \overline{k} + a_{kk}(k_{t-1} - \overline{k}) + a_{kz}(z_t - \bar{z}) \\
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z_t =& \bar{z} + \frac{k_t - \bar{k} - a_{kk}(k_{t-1} - \bar{k})}{a_{kz}} \\
\epsilon_t =& \bar{z}(\rho - 1) + z_t - \rho z_{t-1}
\end{aligned}
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Simple case of evaluating the likelihood

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- given a guess of Ψ and given k_0, k_1 and z_0
- $\rightarrow$ calculate z_1 $\rightarrow$ calculate ϵ_1 $\rightarrow$ calculate z_2 etc.

Log-likelihood

in this case, the log-likelihood is simply

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$$\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{\epsilon_t(\Psi)^2}{2\sigma^2}$$

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Model in State-Space Form

Putting model into state-space form

- the neoclassical growth model is relatively simple
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    - the Kalman filter is convenient for likelihood construction
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Putting model into state-space form

- the neoclassical growth model is relatively simple
- for more complex models, policy function inversion is tough
- but we know that
  - the Kalman filter is convenient for likelihood construction
  - because it produces $y_t - \widehat{y}_{t|t-1}$ and $P_{t|t-1}$
- the question is how to cast DSGE model into state-space form

DSGE mode in state-space form

$$
\begin{aligned}
\zeta_{t+1} =& F\zeta_t + v_{t+1}, & \mathbb{E}(v_t, v_t') = Q \quad \forall t \\
y_t =& H'\zeta_t + w_t, & \mathbb{E}(w_t, w_t') = R \quad \forall t
\end{aligned}
$$

- what is the observable and what are the unobserved states?

DSGE mode in state-space form

DSGE MODE IN STATE-SPACE FORM

$$\begin{bmatrix} k_t - \bar{k} \\ z_{t+1} - \bar{z} \end{bmatrix} = \begin{bmatrix} a_{kk} & a_{kz} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_{t-1} - \bar{k} \\ z_t - \bar{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix}$$

$$k_{t-1} - \bar{k} = [1 \;\; 0] \begin{bmatrix} k_{t-1} - \bar{k} \\ z_t - \bar{z} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

Different observables?

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$$\begin{bmatrix} k_t - \overline{k} \\ z_{t+1} - \overline{z} \\ p_t - \overline{p} \end{bmatrix} = \begin{bmatrix} a_{kk} & a_{kz} & 0 \\ 0 & \rho & 0 \\ \alpha\overline{z}\overline{k}^{\alpha-1} & \overline{k}^{\alpha} & 0 \end{bmatrix} \begin{bmatrix} k_{t-1} - \overline{k} \\ z_t - \overline{z} \\ p_{t-1} - \overline{p} \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \\ 0 \end{bmatrix}$$

$$[p_{t-1} - \overline{p}] = [0 \;\; 0 \;\; 1] \begin{bmatrix} k_{t-1} - \overline{k} \\ z_t - \overline{z} \\ p_{t-1} - \overline{p} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

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$$\begin{bmatrix} k_t - \bar{k} \\ z_{t+1} - \bar{z} \end{bmatrix} = \begin{bmatrix} a_{kk} & a_{kz} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_{t-1} - \bar{k} \\ z_t - \bar{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix}$$

$$[p_t - \bar{p}] = \begin{bmatrix} \alpha \bar{z} \bar{k}^{\alpha-1} & \bar{k}^{\alpha} \end{bmatrix} \begin{bmatrix} k_{t-1} - \bar{k} \\ z_t - \bar{z} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

Maximum Likelihood

Observables and Singularities

What if we also observe productivity?

What if we observe capital and also productivity (z_t)?

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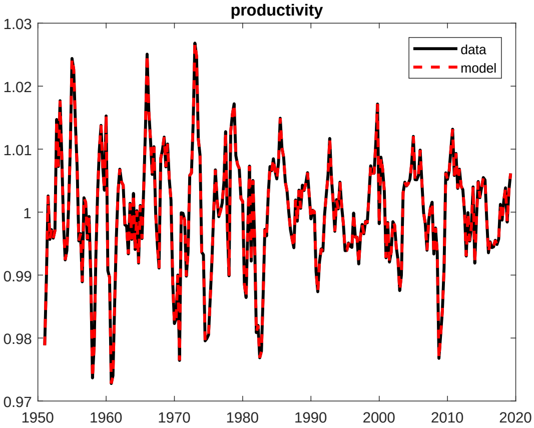
- if our model is the true data-generating process
  - $\rightarrow$ likelihood = 1 for true Ψ and 0 otherwise
- if our model is not the true data-generating process
  - $\rightarrow$ likelihood = 0 for any values of Ψ

Ways out?

Neoclassical growth model estimated on labor productivity



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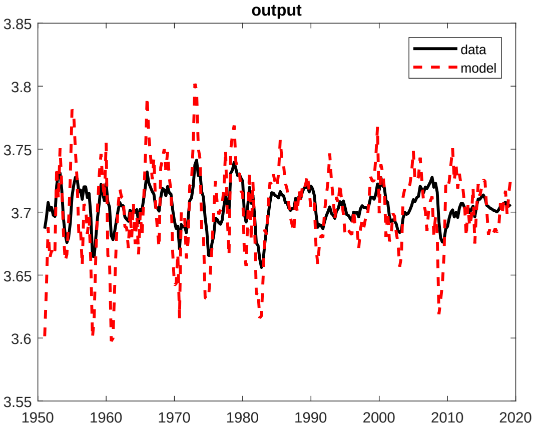

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What about the other periods?

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To understand the above notice that with productivity data

- 4 periods are enough to pin down $\bar{k}$, ρ, a_{kk}, a_{kz}

What about the other periods?

- shocks adjust such that productivity is matched perfectly

$$z_t = 1 - \rho + \rho z_{t-1} + \epsilon_t$$

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- same logic applies:
  - 4 productivity observations enough to pin down $\overline{k}$, ρ, a_{kk}, a_{kz}
  - rest of productivity time-series matched by shocks

What if we also observe productivity?

What if we now add data on capital?

- same logic applies:
    - 4 productivity observations enough to pin down $\overline{k}$, ρ, a_{kk}, a_{kz}
    - rest of productivity time-series matched by shocks
- capital data consistent with model only if model is true DGP!

$$k_t = \overline{k} + a_{kk}(k_{t-1} - \overline{k}) + a_{kz}(z_t - \bar{z})$$

Ways out?

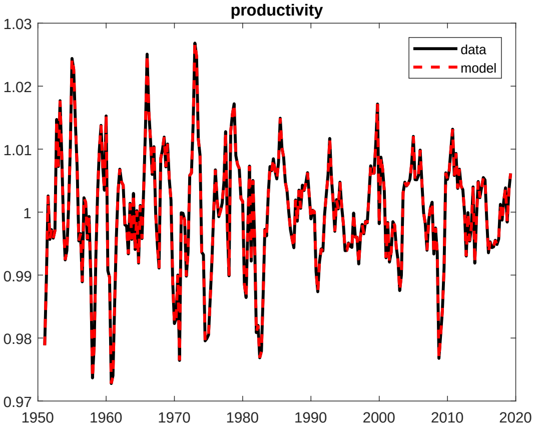
Ways out?

can we simply add an error term?

$$k_t = \overline{k} + a_{kk}(k_{t-1} - \overline{k}) + a_{kz}(z_t - \bar{z}) + u_t$$

Ways out?

Neoclassical growth model estimated on labor productivity & GDP



Ways out?

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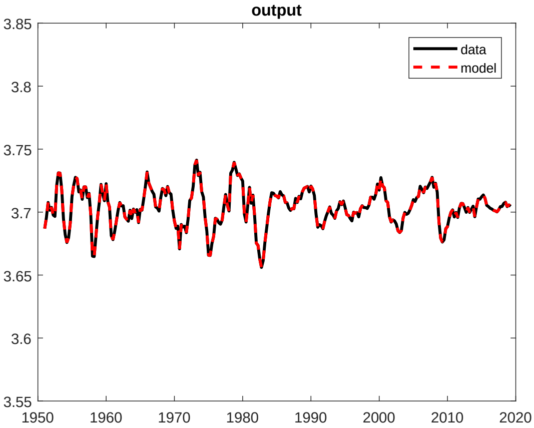

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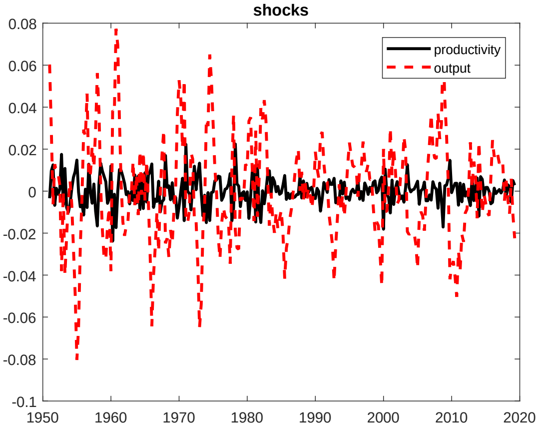

WAYS OUT?

- if u_t is a structural shock (e.g. preferences)
  - $\rightarrow$ its law of motion influences policy function ($\bar{k}$, a_{kk}, a_{kz})
- if u_t is measurement error
  - OK from an econometric point of view
  - but is it truly measurement error?

Ways out?

what if we also observe consumption (but not productivity)?

$$k_t = \bar{k} + a_{kk}(k_{t-1} - \bar{k}) + a_{kz}(z_t - \bar{z})$$

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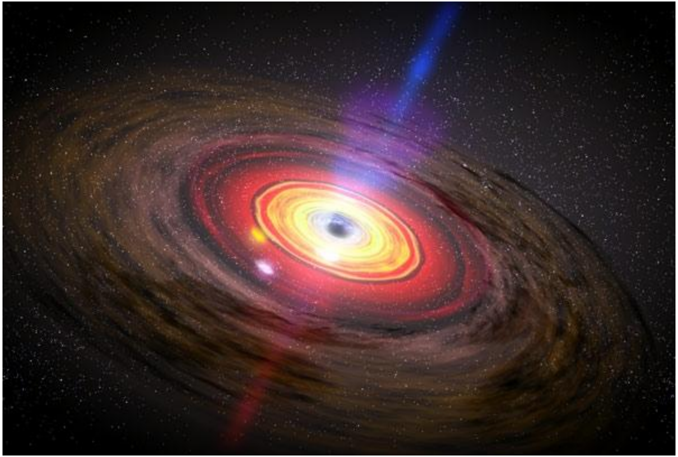
won't work!

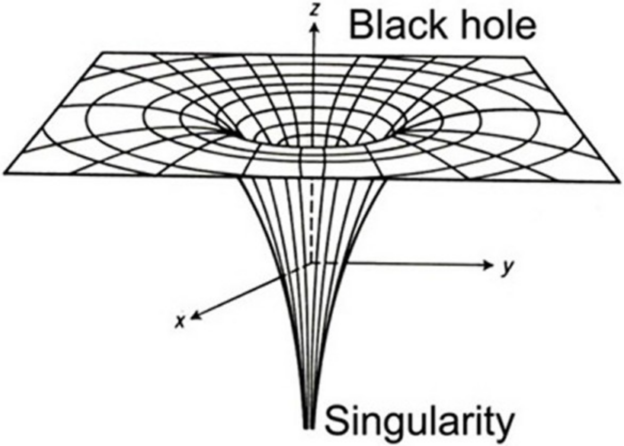
- given Ψ, you can back out z_t from both equations
- with actual data this will give inconsistent answers

Singularity problem

Singularity problem



Singularity problem



Singularity problem

- (stochastic) singularity:
  - many endogenous variables ...
  - driven by a smaller number of structural shocks

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Singularity problem

- (stochastic) singularity:
    - many endogenous variables ...
    - driven by a smaller number of structural shocks
- $\rightarrow$ some observables are linear combinations of others
- $\rightarrow$ the var-covar matrix of observables is *singular*
- what is the problem mathematically?

General rule

- for every observable, you need at least one unobservable shock
- (letting them be measurement error is hard to defend)

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Note that:

- more shocks (measurement errors) than observables is OK
- the choice of observables for estimation is not innocent
- there are ways to choose observables carefully
  - see e.g. Canova, Ferroni, Matthes (2012)

Maximum Likelihood

Taking Stock

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Estimating DSGE models with Maximum Likelihood

- Kalman filter very convenient
  - delivers objects needed to construct likelihood function
  - allows for estimation of underlying structural shocks

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Estimating DSGE models with Maximum Likelihood

- Kalman filter very convenient
  - delivers objects needed to construct likelihood function
  - allows for estimation of underlying structural shocks
- beware of stochastic singularity ("one shock per observable")


That's all Folks!