Advanced Tools in Macroeconomics

Occasionally Binding Constraints

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Regime switching systems

- Previously we looked at models that could be written in the following way,

$$E_t[F(x_{t-1}, x_t, x_{t+1})] = 0$$

- Where x was a vector of endogenous and exogenous (possibly stochastic) variables
- Now we are going to look at models that are given by

$$E_t[F(x_{t-1}; x_t, z_t; x_{t+1}, z_{t+1})] = 0$$

- Where z_t is a discrete stochastic variable with some transition matrix T.
- (The vector x_t can still contain other stochastic variables if you'd like, but wlog, it is assumed here that it doesn't)

Regime switching systems

- Suppose z_t can take on values in $Z = \{z^1, z^2, \ldots, z^J\}$.
- We will not linearize with respect to z but only with respect to x.
- That is, *for each* $z^i \in Z$ we will linearize the system around $\bar{x}$, such that

$$E_j F(\bar{x}; \bar{x}, z^i; \bar{x}, z^j) = D^i$$

- In fact, we could linearize around a different $\bar{x}$ for each z^i if we would like to, but let's keep things simple.

Regime switching systems

- We indicate this period's state with superscript i and next-period's state with superscript j.
- The optimal choice of x_t will depend on z^i. Thus x_{t+1} will in turn depend on z^j (the exogenous state "tomorrow").
- Next period's state is not known, but it has a discrete distribution. So think of E_t as a sum and note that we have one realization of x_{t+1} for each j.

Regime switching systems

- Linearization of the system of equations gives

$$D^i + J^i_{x_{t-1}}(x_{t-1} - \bar{x}) + J^i_{x_t}(x_t - \bar{x}) \\ + E_j[J^j_{x_{t+1}}(x_{t+1}(j) - \bar{x})|i] = 0,$$

- where $J^i_{x_{t-1}}$ is the Jacobian of $E_j[F(\bar{x}, z^i; \bar{x}, z^j; \bar{x})]$ with respect to the first argument, $J^i_{x_t}$ is the Jacobian with respect to the second argument, and $J^j_{x_{t+1}}$ is the Jacobian with respect to $x_{t+1}(j)$.
- Thus there are J "last Jacobians"; one for each j.

Regime switching systems

- We can again write this as

$$A^i u_{t-1}(i) + B^i u_t(i) + \sum_{j=1}^{J} C^j u_{t+1}(j) + D^i = 0, \quad \forall i$$

Regime switching systems

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$$A^i u_{t-1}(i) + B^i u_t(i) + \sum_{j=1}^{J} C^j u_{t+1}(j) + D^i = 0, \quad \forall i$$

- Looks complicated? Let's make it more concrete.

Consumption/Savings problem with unemployment

The Euler equations for the employed and unemployed agent are

$$0 = -u'(a_{t-1}(1+r)+w-a_t)+\beta(1+r)[T_{e,e}u'(a_t(1+r)+w-a_{t+1}(e)) \\ + T_{e,u}u'(a_t(1+r) - a_{t+1}(u))]$$

$$0 = -u'(a_{t-1}(1+r)-a_t)+\beta(1+r)[T_{u,e}u'(a_t(1+r)+w-a_{t+1}(e)) \\ + T_{u,u}u'(a_t(1+r) - a_{t+1}(u))]$$

- Can take jacobian w.r.t a_{t-1}, a_t and $a_{t+1}(i)$, $i = e, u$, and evaluate around $\bar{a}$

Consumption/Savings problem with unemployment

The linearized regime switching system is given by

$$A^e u_{t-1}(e) + B^e u_t(e) + C^{e,e} u_{t+1}(e) + C^{e,u} u_{t+1}(u) + D^e = 0$$
$$A^u u_{t-1}(u) + B^u u_t(u) + C^{u,e} u_{t+1}(e) + C^{u,u} u_{t+1}(u) + D^u = 0$$

- We would look for solutions $u_t = E^i + F^i u_{t-1}$, $i = e, u$.

Regime switching systems

- Let's go back to the general formulation:

$$A^i u_{t-1}(i) + B^i u_t(i) + \sum_{j=1}^{J} C^j u_{t+1}(j) + D^i = 0, \quad \forall i$$

- We are looking for J policy functions of the type

$$u_t(i) = E^i + F^i u_{t-1}(i), \quad i = 1, 2, \ldots, J$$

Regime switching systems

- Time iteration means to find u_t as the solution to

$$A^i u_{t-1}(i) + B^i u_t(i) + \sum_{j=1}^{J} C^j (E_n^j + F_n^j u_t(i)) + D^i = 0, \quad \forall i$$

and update the coefficients E_{n+1}^i and F_{n+1}^i accordingly.

Regime switching systems

- Therefore we iterate on the equations

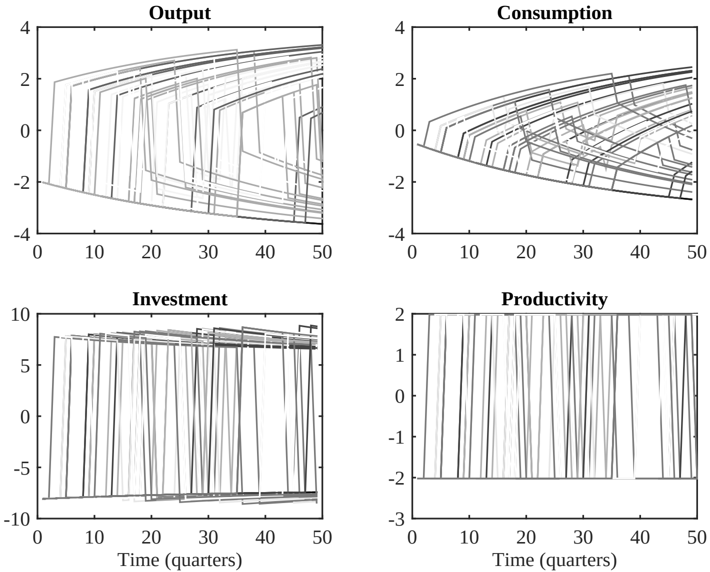
$$E^i_{n+1} = (B^i + \sum_{j=1}^{I} C^j F^j_n)^{-1}(-(D + \sum_{j=1}^{I} C^j E^j_n))$$

$$F^i_{n+1} = (B^i + \sum_{j=1}^{I} C^j F^j_n)^{-1}(-A)$$

for $i = 1, 2, \ldots, J$

- Until convergence

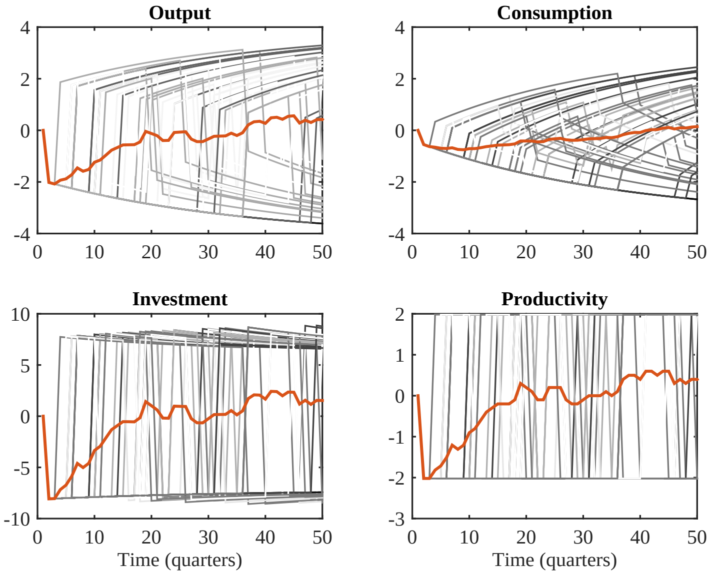
"Impulse Responses"



Regime switching systems

- Looks ok, but it's not pretty.
- Plot all possible sample paths? That would be 2^{50}. Or more generally if T is the length of the impulse response and N is the number of elements in Z, then there are N^T possible paths.
- Popular alternative: Plot the expected paths.
  - Quite good because this is what an econometrician would pick up if he had access to the data generated by the model.
  - Let's average over the samples plotted

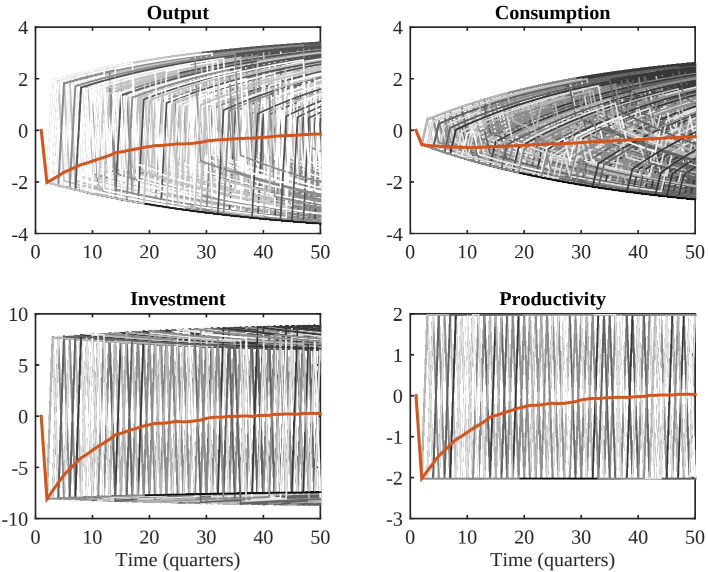
Impulse Responses



Regime switching systems

- That doesn't really look like an impulse response to me!
- Remedy: use more expected paths?
- The previous graph used 40 samples. Let's crank it up to 4,000!

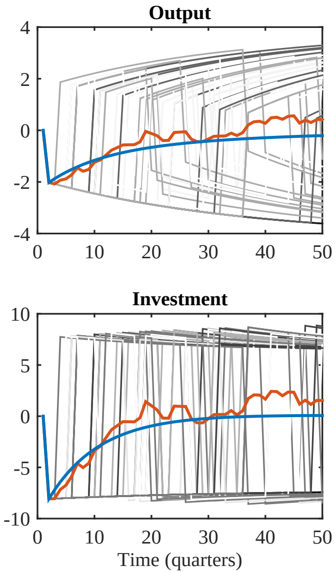
Impulse Responses



Regime switching systems

- Yep, much better!
- But it's time consuming.
- The calculation took several minutes on my desktop. And this is a simple model.
- It turns out that we can do this theoretically instead!

Impulse Responses



Regime switching systems

- Even better! All wiggles are gone.
- The theoretical paths are equal to the averaged sample paths as the sample goes to infinity.
- How is this done?

Regime switching systems

- Denote the expected value of u_{t+s} conditional on information available at time t, *and conditional on being in state* $z_{t+s} = z_j$ as $E_t[u_{t+s}|z_{t+s} = z_j]$.
- Because of the linearities of the policy functions, this can be written as

$$E_t[u_{t+s}|z_{t+s} = z_j] = \sum_{i=1}^{I} Pr(z_{t+s-1} = z_i|z_{t+s} = z_j) \times \left(E^j + F^j E_t[u_{t+s-1}|z_{t+s-1} = z_i]\right)$$

Regime switching systems

$$E_t[u_{t+s}|z_{t+s} = z_j] = \sum_{i=1}^{I} Pr(z_{t+s-1} = z_i|z_{t+s} = z_j) \\ \times (E^j + F^j E_t[u_{t+s-1}|z_{t+s-1} = z_i])$$

- Bayes' rule states that

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

Regime switching systems

$$E_t[u_{t+s}|z_{t+s} = z_j] = \sum_{i=1}^{I} Pr(z_{t+s-1} = z_i|z_{t+s} = z_j)$$
$$\times \left(E^j + F^j E_t[u_{t+s-1}|z_{t+s-1} = z_i]\right)$$

- Thus

$$Pr(z_{t+s-1} = z_i|z_{t+s} = z_j) = Pr(z_{t+s} = z_j|z_{t+s-1} = z_i)$$
$$\times \frac{Pr(z_{t+s-1} = z_i)}{Pr(z_{t+s} = z_j)}$$

Regime switching systems

- If z follows transition matrix T, this can be written as

$$\begin{aligned} Pr(z_{t+s-1} = z_i | z_{t+s} = z_j) &= Pr(z_{t+s} = z_j | z_{t+s-1} = z_i) \\ &\times \frac{Pr(z_{t+s-1} = z_i)}{Pr(z_{t+s} = z_j)} \\ &= T_{ij} \frac{v_{t+s-1,i}}{v_{t+s,j}} \end{aligned}$$

- Where T_{ij} is the (i,j)th element of transition matrix T, and $v_{t+s,j}$ is the jth element of the vector

$$v_{t+s} = v_{t+s-1} \times T$$

for some initial v_t.

Regime switching systems

- Thus our nasty equation

$$E_t[u_{t+s}|z_{t+s} = z_j] = \sum_{i=1}^{I} Pr(z_{t+s-1} = z_i|z_{t+s} = z_j) \\ \times (E^i + F^i E_t[u_{t+s-1}|z_{t+s-1} = z_i])$$

turns into something more pleasant

$$E_t[u_{t+s}|z_{t+s} = z_j] = \sum_{i=1}^{I} T_{ij} \frac{v_{t+s-1,i}}{v_{t+s,j}} \\ \times (E^i + F^i E_t[u_{t+s-1}|z_{t+s-1} = z_i])$$

- And

$$E_t[u_{t+s}] = \sum_{j=1}^{I} v_{t+s,j} E_t[u_{t+s}|z_{t+s} = z_j]$$

Regime switching systems

- To implement this procedure, we still need to answer the following:
- What is the initial condition, u_{t-1}?
- What is v_t?

Regime switching systems

What is u_{t-1}?

- This is somewhat arbitrary, but a good start is to assume that the economy is at it's long run expected value in period t; u_{ss}.
- Given a long-run distribution v, this is given by

$$u_{ss} = \sum_{j=1}^{I} u_{j,ss} v_j$$

- Where $u_{j,ss}$ solves

$$u_{j,ss} = \sum_{i=1}^{I} T_{ij} \frac{v_i}{v_j} \times (E^i + F^i u_{i,ss}), \quad j = 1, \ldots, I$$

- We can either iterate to find $u_{j,ss}$, or to set it up as a linear system of equations.

Regime switching systems

- The nice thing about this starting value is that the expected value

$$E_t[u_{t+s}] = \sum_{j=1}^{I} v_{t+s,j} E_t[u_{t+s}|z_{t+s} = z_j]$$

will converge to u_{ss} as s goes to infinity.

- That is

$$\lim_{s \to \infty} E_t[u_{t+s}] = u_{ss}$$

Regime switching systems

What is v_t?

- This is entirely up to you, and forms the basis of your impulse response.
- Setting $v_t = [0, 0, 1, 0, 0, \ldots]$ means that you know with certainty that you are in state 3 in period t.
- In the Ramsey, the starting condition was $v_t = [0, 1]$.