The Reiter Hybrid Approach to Solve Models with Heterogeneous Agents

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Background

Macroeconomic models with heterogeneous agents:

- Idiosyncratic uncertainty typically large
  - $\Longrightarrow$ individual problem likely to be non-linear
  - $\Longrightarrow$ perturbation probably bad idea
- Aggregate uncertainty small for regular business cycles
  - $\Longrightarrow$ perturbation could work
- Models with idiosyncratic and *without* aggregate uncertainty
  - cross-sectional distribution NOT a time-varying argument of policy functions
    $\Longrightarrow$ model (typically) not that difficult to solve even with nontrivial nonlinearities individual problem

Crucial insight of Reiter (2008)

Combining these three facts suggest that we should

- combine perturbation and projection
  - projection for individual problem
  - perturbation for aggregate dimension

Versions of this idea in the literature

- Reiter (2008)
- Bopart, Krusell, and Mitman (2018)
- Sequence-Space Jacobian (2021)
- LeGrand and Ragot (2022)

Perturbation combined with Projection

- σ_ε typically large $\Longrightarrow$ you can be far from the usual "$\sigma_z = 0, \boldsymbol{\sigma_\varepsilon = 0}$" steady state
- Reiter's idea: Focus on the "$\sigma_z = 0$ and $\boldsymbol{\sigma_\varepsilon > 0}$" steady state
- If $\sigma_z = 0 \Longrightarrow$ cross-sectional distribution doesn't change over time and the problem becomes much easier to solve
- Use perturbation to deal with $\sigma_z > 0$

Example environment for these slides

- Same as Krusell-Smith 1998 JPE paper **except transition probabilities are assumed to constant**
  $\Rightarrow$ constant unemployment rate $= u$
  - At the end of this section, it is discussed how to implement the Reiter method when transition probabilities do vary with the business cycle

- KS model: z_t can take on only two values and probability of switching is same in each state.
  - Perturbation only "cares" about variance and persistence $\Longrightarrow$ use a first-order AR(1) in Dynare program with the same variance and autocovariance.
    (When simulating you could use KS specification).

REITER METHOD

Rewrite the policy function

- Rewrite the *numerical* solution to the model as

$$k_{i,t} \;=\; P_N(\varepsilon_{i,t}, k_{i,t-1}, z_t, m_t; \boldsymbol{\lambda_k}) = P_n(\varepsilon_{i,t}, k_{i,t-1}; \boldsymbol{\lambda_{k,t}})$$

  with

$$\boldsymbol{\lambda}_{k,\boldsymbol{t}} = \lambda_k(z_t, z_{t-1}, p_{t-1}) = \lambda_k\left(s_t\right),$$

  where $s_t = \{z_t, z_{t-1}, p_{t-1}\}$.
- m_t is characterization of beginning-of-period t distribution
- p_t is characterization of end-of-period t distribution

Two key elements of Reiter procedure

1. A *numerical* solution to the model:

$$k_{i,t} \;=\; P_N(\varepsilon_{i,t}, k_{i,t-1}, \boldsymbol{\lambda_{k,t}})$$

2. Given $\lambda_{k,t}$, we must be able to write down explicit FORMULAS for

$$\begin{aligned} p_t &= \Gamma_{\lambda_{k,t}}(m_t) \\ m_t &= \widetilde{\Gamma}(z_t, z_{t-1}, p_{t-1}) \end{aligned}$$

"formula" means an exact algebraic expression (think: something that can be entered in the Dynare model block)

What does the second element require?

- An exact expression (i.e., formula) is required $\Longrightarrow \Gamma_{\lambda_{k,t}}$ cannot be such that it has to be determined with a simulation method or a subroutine
- Possibilities:
  1. $p_{t-1}\&m_t$ describe complete distributions $\Longrightarrow$
     - $p_{t-1}\&m_t$ can be histogram values at a fine grid (as in Reiter 2008).
     - $p_{t-1}\&m_t$ could be limited set of moments **IF** it is combined with a distributional assumption as in Winberry (2016)
  2. $p_{t-1}\&m_t$ are the moments of the *levels* of k_i so that explicit aggregation is possible as in XPA; see XPA slides

More on distribution

- Suppose m_t contains the mean and uncentered variance of capital holdings for employed and unemployed **and** we make an assumption on the functional form of the distribution
- Thus, we know the density $f\left(k_i; m_{[1],\varepsilon=0,t}, m_{[2],\varepsilon=0,t}\right)$ for unemployed and $f\left(k_i; m_{[1],\varepsilon=1,t}, m_{[2],\varepsilon=1,t}\right)$ for employed
- Notation: $m_{[q],\varepsilon,t}$ is the q^{th}-order moment for capital of workers with employment status ε in period t

More on distribution

- Expressions for end-of-period moments are easy to write down. E.g., for the second moment we get

$$\begin{aligned} p_{[2],\varepsilon,t} &= \int_{-\infty}^{+\infty} \left(P_N(\varepsilon, k_i; \boldsymbol{\lambda_{k,t}})\right)^2 f\left(k_i; m_{[1],\varepsilon,t}, m_{[2],\varepsilon,t}\right) dk_i \\ &= \Gamma_{\lambda_{k,t}}(m_t) \\ \varepsilon &\in \{0, 1\} \end{aligned}$$

- We use quadrature to turn this into a formula we can write down in say the Dynare model block ($\int$ becomes a sum)

- Getting a formula for beginning-of-next-period moments (which takes into account change in employment status), is just accounting (see simulation slides)

Notation & grid

- ε_j and κ_j: employment status and capital at grid point j
- Dimension of $\boldsymbol{\lambda}_{k,t} = n^{\#}_{\lambda_k}$
  - If $P_N(\cdot)$ is 2nd-order complete polynomial $\Longrightarrow n^{\#}_{\lambda_k} = 6$
  - number of grid points $= n^{\#}_{\text{grid}} \geq n^{\#}_{\lambda_k}$

    $\equiv$ where constraint is not binding
- no grid for s in the Reiter method !!!!!!!!!!!!!!!!

Model equation at grid points

log utility and $\delta = 1$ for simplicity $\Longrightarrow$ Euler equation becomes

$$\left(r\kappa_j + w\varepsilon_j\bar{l} - P_N(\varepsilon_j, \kappa_j; \lambda_k\right)^{-1}$$
$$= \mathbb{E}\left[\beta r' \begin{pmatrix} r' P_N(\varepsilon_j, \kappa_j; \lambda_k) + w\varepsilon'\bar{l} \\ -P_N(\varepsilon', P_N(\varepsilon_j, \kappa_j; \lambda_k); \lambda'_k) \end{pmatrix}^{-1} \middle| \varepsilon_j, s\right]$$
$$\equiv \mathbb{E}[h(\varepsilon', z') | \varepsilon_j, s]$$

Trivial to add borrowing constraints in the form of penalty functions. Inequality constraints are tricky but not impossible; specifically Dynare cannot incorporate "if-then" statements so at each grid point you either use the Euler equation or the constraint and the assumption is that this doesn't change when the aggregate state is perturbed

Additional exact expressions

1. $r = \alpha z \left(K/(\bar{l}(1-u))\right)^{\alpha-1}$
2. $w = (1-\alpha) z \left(K/(\bar{l}(1-u))\right)^{\alpha}$
3. $K = u m_{[1],\varepsilon=0} + (1-u) m_{[1],\varepsilon=1}$
4. law of motion for z
5. $p = \Gamma_{\lambda_k}(z, m)$
6. $m = \widetilde{\Gamma}(z, z_{-1}, p_{-1})$
7. $\mathbb{E}[h(\varepsilon', z')] | \varepsilon_j, s]$ needs an explicit formula for $\mathbb{E}[h(\varepsilon', \cdot)] | \varepsilon_j, s]$
    - Already the case with discrete support for ε'
    - Use quadrature approximating sum when ε' has continous support
    - Uncertainty of z' is dealt with according to standard perturbation methodology

Mental break

- Have I really done anything?

- Not much
  - I constructed a grid
  - I constructed a system with individual choices substituted out using $P_n(\varepsilon_{i,t}, k_{i,t}; \boldsymbol{\lambda_{k,t}})$

Perturbation system

- Suppose 2nd-order polynomial is used: $n^{\#}_{\lambda_k} = 6$
- Suppose there are 6 grid points
- After taking care of $\mathbb{E}_t\left[\cdot\right]$, we get

$$\begin{array}{ccc}
\underset{(6\times 1)}{F(\lambda_k; r, w, z)} & = & \underset{(6\times 1)}{0} \\
z & = & (1-\rho)\,\bar{z} + \rho z_{-1} + \varepsilon_z \\
p & = & \Gamma_{\lambda_k}(m) \\
m & = & \widetilde{\Gamma}(z, z_{-1}, p_{-1}) \\
K & = & u m_{[1],\varepsilon=0} + (1-u) m_{[1],\varepsilon=1} \\
r & = & \alpha z\left(K/(\bar{l}(1-u))\right)^{\alpha-1} \\
w & = & (1-\alpha) z\left(K/(\bar{l}(1-u))\right)^{\alpha}
\end{array}$$

Make sure you understand

- **Question:** What does this system solve for if $\varepsilon_z \equiv 0$?
- **Anwer:**
  1. the policy rule
  2. the ergodic distribution

  when there is no aggregate uncertainty

What is known and unknown?

- $F(\cdot)$ known
- λ_k is unknown and all six elements are variables for Dynare
  - Dynare can figure out what state variables λ_k depends on by searching for lagged variables and exogenous shocks
  - Here those are z_{-1}, p_{-1}, and ε_z
- This is a standard perturbation system!!!!!!!!!

Perturbation system

- What are Dynare state variables **in this perturbation system**?
  - $s = \{z, z_{-1}, p_{-1}\} = \{\varepsilon_z, z_{-1}, p_{-1}\}$
- What are *not* Dynare state variables **in this perturbation system**?
  - usual individual state variables: ε and κ
- What is this perturbation system solving for?
  - It will give you policy functions for the elements of $\lambda_k(s)$. **These describe how the coefficients of the individual policy rules fluctuate with** s

A simple perturbation system?

- Reiter (2008) uses a fine histogram to characterize CDF
  - $\Longrightarrow$ dimension of m typically high ($> 1,000$ in Reiter (2008))
  - $\Longrightarrow$ λ_k has many inputs $\Longrightarrow$ the perturbation system has to solve for many policy functions; the perturbation system solves how *each* element of λ_k changes with *each* element of s_t!!!!!!
  - $\Longrightarrow$ higher-order perturbation becomes very tough (even first-order may be tricky)
- Winberry (2016) approach using moments makes problem more tractable

Small comment

- If number of grid points exceeds $n^{\#}_{\lambda_k}$, then you have to take a stand on how to weigh the elements of $F(\cdot)$ to get a system of $n^{\#}_{\lambda_k}$ equations.

Making unemployment rate vary

1. Let transition probabilities be linear function of z_t. Employment is then an additional aggregate state variable.
2. Adopt a trick similar to the one used by KS that works for more general fluctuations in z_t. Specifically,
   - Probability from employed to employed $= \frac{\overline{p} z_{t+1}}{z_t}$
   - Probability from unemployed to employed $= \frac{(1-\overline{p}) z_{t+1}}{(1-z_t)}$.
   - The law of motion for n_{t+1} is then given by

$$n_{t+1} = \frac{\overline{p} z_{t+1}}{z_t} n_t + \frac{(1-\overline{p}) z_{t+1}}{(1-z_t)} (1-n_t),$$

which is equal to z_{t+1} if $n_t = z_t$.

References

- Auclert, A. B. Bardoczy, M. Rognlie, and L. Straub, 2021, Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models, Econometrica 89, 2375-2408.
- Bopart, T., Krusell, P., and K. Mitman, 2018, Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative, Journal of Economic Dynamics and Control 89, 68-92.
- LeGrand, F., and X. Ragot, 2022, Managing Inequality over the Business Cycle: Optimal Policies with Heterogeneous Agents and Aggregate Shocks, International Economic Review 63, 511–540.

References

- Reiter, M., 2009, Solving heterogeneous-agent models by projection and perturbation, *Journal of Economic Dynamics and Control*, 32, 1120-1155.
- Winberry, T., 2016, A toolbox for solving and estimating heterogeneous agent macro models. Available at http://faculty.chicagobooth.edu/thomas.winberry/research/winber