Homework 2: Replication Akcigit & Ates (2023)

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1 Question 1: Derivation

Consumer

- **Utility maximization problem**

$$U_t = \int_t^{\infty} \exp(-\rho(s-t)) \ln C_s ds \tag{1}$$

$$P_t \mathrm{C}_t + \dot{A}_t = w_t L_t + r_t A_t + G_t \quad \text{Where} \quad A_t = \int_{\mathcal{F}} V_{ft} df \tag{2}$$

Final Goods Production Problem:

- **Final production problem:**

$$\min_{Y_{jt}} \int_0^1 Y_{jt} P_{jt} dj \tag{3}$$

$$s.t. \quad Y_t = \exp\left(\int_0^1 \ln Y_{jt} dj\right) \tag{4}$$

  – Setup Lagrangian:

$$L = \int_0^1 Y_{j,t} P_{j,t} dj - \lambda_t \left[\exp\left(\int_0^1 \ln Y_{jt} dj\right) - Y_t\right] \tag{5}$$

  – FOC:

$$P_{jt} = \lambda_t \frac{\exp\left(\int_0^1 \ln Y_{jt} dj\right)}{Y_{jt}} = \lambda_t \frac{Y_t}{Y_{jt}} \tag{6}$$

  – Times Y_{jt}, take integration:

$$\int_0^1 P_{jt} Y_{jt} dj = \lambda_t Y_t \tag{7}$$

  – Define aggregate price P_t:

$$P_t :\equiv \frac{\int_0^1 P_{jt} Y_{jt} dj}{Y_t} = \lambda_t \tag{8}$$

– Combine first order condition and constraint:

$$Y_t = \exp \int_0^1 \ln \left(\frac{P_t}{P_{jt}} Y_t \right) dj = P_t Y_t \exp \left(- \int_0^1 \ln P_{jt} dj \right) \tag{9}$$

$$P_t = \exp \left(\int_0^1 \ln P_{jt} dj \right) \tag{10}$$

- In conclusion, demand function and aggregate price:

$$Y_{jt} = \frac{P_t}{P_{jt}} Y_t \quad \text{where} \quad P_t = \exp \left(\int_0^1 \ln P_{jt} dj \right) \tag{11}$$

Intermediary Goods Production:

- **Optimal intermediary good demand:**

$$\min_{y_{ijt}, y_{-ijt}} p_{ijt} y_{ijt} + p_{-ijt} y_{-ijt} \tag{12}$$

$$s.t. \quad Y_{jt} = \left(y_{ijt}^{\beta} + y_{-ijt}^{\beta} \right)^{1/\beta} \tag{13}$$

– Setup Lagrangian:

$$L = p_{ijt} y_{ijt} + p_{-ijt} y_{-ijt} - \mu_t \left[\left(y_{ijt}^{\beta} + y_{-ijt}^{\beta} \right)^{1/\beta} - Y_{jt} \right] \tag{14}$$

– FOC:

$$p_{ijt} = \mu_t \left(y_{ijt}^{\beta} + y_{-ijt}^{\beta} \right)^{\frac{1-\beta}{\beta}} y_{ijt}^{\beta - 1} = \mu_t \left(\frac{Y_{jt}}{y_{ijt}} \right)^{1-\beta} \tag{15}$$

$$y_{ijt} = \left(\frac{\mu_t}{p_{ijt}} \right)^{\frac{1}{1-\beta}} Y_{jt} \tag{16}$$

– Take into the constraint:

$$Y_{jt} = Y_{jt} \left[p_{ijt}^{-\frac{\beta}{1-\beta}} + p_{-ijt}^{-\frac{\beta}{1-\beta}} \right]^{\frac{1}{\beta}} \mu_t^{\frac{1}{1-\beta}} \tag{17}$$

$$\mu_t = \left[p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}} \right]^{\frac{\beta-1}{\beta}} \tag{18}$$

– Times p_{ijt} and sum up:

$$p_{ijt} y_{ijt} + p_{-ijt} y_{-ijt} = \left[p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}} \right] \mu_t^{\frac{1}{1-\beta}} y_{jt} = \mu_t Y_{jt} \tag{19}$$

– Define aggregate price p_{jt}:

$$P_{jt} :\equiv \frac{p_{ijt} y_{ijt} + p_{-ijt} y_{-ijt}}{y_{jt}} = \mu_t \tag{20}$$

– The equilibrium conditions are

$$y_{ijt} = \left(\frac{P_{jt}}{p_{ijt}}\right)^{\frac{1}{1-\beta}} y_{jt} \quad \text{where} \quad y_{jt} = \underbrace{\left(y_{ijt}^{\beta} + y_{-ijt}^{\beta}\right)^{1/\beta}}_{\text{supply}} = \underbrace{\frac{P_t Y_t}{P_{jt}}}_{\text{demand}} \tag{21}$$

$$P_{jt} = \left[p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{\beta}} \tag{22}$$

– In a Bertrand Price-war, firms compete on price, so the demand function is:

$$y_{ijt} = \left(\frac{P_{jt}}{p_{ijt}}\right)^{\frac{1}{1-\beta}} \frac{P_t}{P_{jt}} Y_t = p_{ijt}^{\frac{1}{\beta-1}} P_{jt}^{\frac{\beta}{1-\beta}} Y_t = \frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} Y_t \tag{23}$$

– In conclusion, the demand function that each Bertrand firm is:

$$y_{ijt} = \frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} Y_t \tag{24}$$

- **Intermediary Firm Production:**

$$\pi_{ijt} = \max_{y_{ijt}, p_{ijt}} \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right) y_{ijt} = \max_{p_{ijt}} \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right) \frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} Y_t \tag{25}$$

– FOC:

$$\frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} + \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right) \frac{\frac{1}{\beta-1} p_{ijt}^{\frac{1}{\beta-1}-1} \left(p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}\right) - \frac{\beta}{\beta-1} p_{ijt}^{\frac{\beta}{\beta-1}-1} p_{ijt}^{\frac{1}{\beta-1}}}{\left(p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}\right)^2} = 0 \tag{26}$$

$$1 + \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right) \frac{\frac{1}{\beta-1} p_{ijt}^{-1} \left(p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}\right) - \frac{\beta}{\beta-1} p_{ijt}^{\frac{\beta}{\beta-1}-1}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} = 0 \tag{27}$$

$$1 + \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right) \left[\frac{1}{\beta-1} p_{ijt}^{-1} - \frac{\beta}{\beta-1} \frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}}\right] = 0 \tag{28}$$

$$p_{ijt}(\beta-1) + \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right) \left(1 - \beta \frac{p_{ijt}^{\frac{\beta}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}}\right) = 0 \tag{29}$$

– Notice that the market share $z_{ijt} \equiv \frac{p_{ijt} y_{ijt}}{P_{jt} Y_{jt}}$:

$$z_{ijt} \equiv \frac{p_{ijt} y_{ijt}}{P_{jt} Y_{jt}} = \frac{p_{ijt} y_{ijt}}{p_{ijt} y_{ijt} + p_{-ijt} y_{-ijt}} = \frac{p_{ijt} \frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} Y_t}{p_{ijt} \frac{p_{ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} Y_t + p_{-ijt} \frac{p_{-ijt}^{\frac{1}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} Y_t} \tag{30}$$

$$z_{ijt} = \frac{p_{ijt}^{\frac{\beta}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} \tag{31}$$

– Combine it into the FOC:

$$p_{ijt}(\beta - 1) + \left(p_{ijt} - \frac{w_t}{q_{ijt}}\right)(1 - \beta z_{ijt}) = 0 \tag{32}$$

– Solve out p_{ijt}, we get the pricing rule:

$$p_{ijt} = \underbrace{\frac{1 - \beta z_{ijt}}{\beta\,(1 - z_{ijt})}}_{markup_{ijt}} \underbrace{\frac{w_t}{q_{ijt}}}_{mc_{ijt}} \quad \text{where market share:} \quad z_{ijt} = \frac{p_{ijt}^{\frac{\beta}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}} \tag{33}$$

– which equivalent to

$$\Leftrightarrow \quad p_{ijt} = \frac{1 - \beta \frac{p_{ijt}^{\frac{\beta}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}}}{\beta\left(1 - \frac{p_{ijt}^{\frac{\beta}{\beta-1}}}{p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}}\right)} \underbrace{\frac{w_t}{q_{ijt}}}_{mc_{ijt}} = \frac{(1-\beta)p_{ijt}^{\frac{\beta}{\beta-1}} + p_{-ijt}^{\frac{\beta}{\beta-1}}}{\beta p_{-ijt}^{\frac{\beta}{\beta-1}}} \underbrace{\frac{w_t}{q_{ijt}}}_{mc_{ijt}} \tag{34}$$

$$\Leftrightarrow \quad p_{ijt} = \left[\frac{1-\beta}{\beta}\left(\frac{p_{ijt}}{p_{-ijt}}\right)^{\frac{\beta}{\beta-1}} + \frac{1}{\beta}\right] \underbrace{\frac{w_t}{q_{ijt}}}_{mc_{ijt}} \tag{35}$$

– Further, notice that the other firm also have:

$$p_{-ijt} = \left[\frac{1-\beta}{\beta}\left(\frac{p_{-ijt}}{p_{ijt}}\right)^{\frac{\beta}{\beta-1}} + \frac{1}{\beta}\right] \underbrace{\frac{w_t}{q_{-ijt}}}_{mc_{-ijt}} \tag{36}$$

– Using the above two equations, we can numerically solve out p_{ijt}, p_{-ijt} as a function of q_{ijt}, q_{-ijt}, w_t, also by take the ratio of two equations, we get properties:

  * Price ratio $\frac{p_{ijt}}{p_{ijt}}$ is uniquely pinned down by productivity ratio $\frac{q_{ijt}}{q_{ijt}}$;
  * z_{ijt} is function of $\frac{p_{ijt}}{p_{ijt}} \Rightarrow z_{ijt}$ is uniquely pinned down by productivity ratio $\frac{q_{ijt}}{q_{ijt}}$;
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References

Akcigit, U., & Ates, S. T. (2023). What happened to us business dynamism? *Journal of Political Economy*, *131*(8), 2059–2124.