LSE Macroeconomics Summer Program
Part II: Heterogeneous Agents
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Learning Asset Prices Assignment

1 Objective

The objective of this assignment is to investigate the behavior of equity prices in models with boundedly rational agents.

2 Model and rational expectations equilibrium

Dividends are generated by the following process:

$$d_t = \mu_t + z_t. \tag{1}$$

$\mu_t \in \left\{\mu_{\text{low}}, \mu_{\text{high}}\right\}$ is a regime dependent fixed component. We assume that

$$\mu_{\text{low}} = -\mu_{\text{high}}. \tag{2}$$

Moreover, the probability of switching regimes is equal to $(1-\rho)$. The component z_t has the following *dgp*:

$$z_t = \rho_z z_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2). \tag{3}$$

The rational expectations equilibrium (REE) solution for the price level satisfies

$$p_t = \mathrm{E}_t\left[\beta\left(p_{t+1} + d_{t+1}\right)\right]. \tag{4}$$

Note that $\mathrm{E}[d_t]$ is equal to zero and dividends can be both positive and negative. $\mathrm{E}[p_t]$ is also equal to zero.

The REE solution has the following form:

$$p_t = B_{0,j} + B_{1,j} z_t, \qquad j \in \{\text{low,high}\}. \tag{5}$$

The coefficients can be solved from the following equations:

$$B_{0,\text{low}} + B_{1,\text{low}} z_t = \beta \begin{pmatrix} \rho & \left(B_{0,\text{low}} + B_{1,\text{low}}\rho_z z_t + \mu_{\text{low}} + \rho_z z_t\right) \\ +(1-\rho) & \left(B_{0,\text{high}} + B_{1,\text{high}}\rho_z z_t + \mu_{\text{high}} + \rho_z z_t\right) \end{pmatrix}, \tag{6}$$

$$B_{0,\text{high}} + B_{1,\text{high}} z_t = \beta \begin{pmatrix} (1-\rho) & \left(B_{0,\text{low}} + B_{1,\text{low}}\rho_z z_t + \mu_{\text{low}} + \rho_z z_t\right) \\ +\rho & \left(B_{0,\text{high}} + B_{1,\text{high}}\rho_z z_t + \mu_{\text{high}} + \rho_z z_t\right) \end{pmatrix}. \tag{7}$$

Comment about the model used. It would be easy to add a constant to the dividend process to avoid negative prices. This would be a scaling and not affect the results. In contrast, the assumption that the dividend level is stationary and not, for example, $I(1)$ is likely to be important for the results. If the level of dividends is $I(1)$ and learning is about the *growth rate* of dividends, then one can easily get a bit more volatility. With non-stationary variables you have to be careful about how to rescale variables. Therefore, to get started we use the simpler case in which all variables are stationary. Just be aware we may loose some of the action, but as you will see there is still plenty left.

3 Old-fashioned LS learning

In this and the subsequent sections we discuss the three types of learning considered in the assignment. With "old-fashioned" LS learning, agents understand that prices are the solution to equation (4), but they update their belief about the *dgp* of dividends using least-squares learning. The twist here is that agents use a misspecified process for dividends. What really matters is that agents do not know that there are regimes. To keep things as simple as possible, we assume that agents think that from now on dividends will be constant. Each period, agents use the last T_1 observations to estimate this "constant" level of dividends.[1] This is just done for simplicity.

4 AdamMarcetNicolini (AMN) learning

Here agents know the true *dgp* for dividends. Moreover, the current stock price is determined by a similar equation namely

$$p_t = \beta \left(\widehat{\mathrm{E}}_t \left[p_{t+1} \right] + \mathrm{E}_t \left[d_{t+1} \right] \right). \tag{8}$$

Each period, agents form a belief about the law of motion for p_{t+1} using LS learning. This laws of motion implies a value for $\widehat{\mathrm{E}}_t \left[p_{t+1} \right]$. The actual market price in period t is given by equation (8)

Agents use use the following regression model:

$$p_t = \gamma_{0,t} + \gamma_{1,t} i_t + \gamma_{2,t} z_t + \gamma_{3,t} i_t z_t, \tag{9}$$

where $i_t \in \{-1, 1\}$ is an indicator variable that identifies the current regime.[2,3].

[1] The nice thing about this simple setup is that it makes clear the somewhat silly aspect of all least-squares learning approaches that agents update coefficients every period while market prices are determined based on the belief that coefficients are constant.

[2] In the computer program, the γ coefficients are referred to as `beta_p`.

[3] Thus, conditional on the regime the law of motion is linear. If $i_t = 1$, then the law of motion is given by $p_t = \left(\gamma_{o,t} + \gamma_{1,t} \right) + \left(\gamma_{2,t} + \gamma_{3,t} \right) z_t$ and when $i_t = -1$, then the law of motion is given by $p_t = \left(\gamma_{o,t} - \gamma_{1,t} \right) + \left(\gamma_{2,t} - \gamma_{3,t} \right) z_t$.

Each period, the $\gamma_{*,t}$ coefficients are updated using recursive least squares. Eventually, the perceived law of motion will converge towards the rational expectations equilibrium,[4] but the purpose of this assignment is to show that this can take quite a while.

5 Bayesian learning

With Bayesian learning, agents know the equation that determines equity prices (equation 4) and the laws of motion that determine dividends. However, they do not directly observe in which regime the economy is currently operating.

Let $\widetilde{\xi}_t = [\widetilde{\xi}_{t,\text{low}}, \widetilde{\xi}_{t,\text{high}}]'$ be the period t prior perceptions of agents that the economy is in the low and high-dividend-level regime respectively, *before* the level of dividends is observed. At the beginning of the period, the value of d_t is revealed. For simplicity, we also assume that the value of z_{t-1} is revealed at the beginning of the period.[5] Note that this information is not enough to identify ε_t or the current regime. But is enough to calculate ε_t conditional on being in a particular regime and thus the corresponding probability of being in a particular regime. Denote the two values for ε_t by $\varepsilon_{t,\text{low}}$ and $\varepsilon_{t,\text{high}}$. The posterior belief for period t is then given by[6]

$$\xi_t = \begin{bmatrix} \frac{\widetilde{\xi}_{t,\text{low}} \times \text{prob}(\varepsilon_{t,\text{low}})}{\widetilde{\xi}_{t,\text{low}} \times \text{prob}(\varepsilon_{t,\text{low}}) + \widetilde{\xi}_{t,\text{high}} \times \text{prob}(\varepsilon_{t,\text{high}})} \\ \frac{\widetilde{\xi}_{t,\text{high}} \times \text{prob}(\varepsilon_{t,\text{high}})}{\widetilde{\xi}_{t,\text{low}} \times \text{prob}(\varepsilon_{t,\text{low}}) + \widetilde{\xi}_{t,\text{high}} \times \text{prob}(\varepsilon_{t,\text{high}})} \end{bmatrix}. \tag{10}$$

The prior for the next period is given by

$$\widetilde{\xi}_{t+1} = \begin{bmatrix} \rho & 1-\rho \\ 1-\rho & \rho \end{bmatrix} \xi_t. \tag{11}$$

The price level is given by

$$p_t = \xi_{t,\text{low}} \left(B_{0,\text{low}} + B_{1,\text{low}} z_t\right) + \xi_{t,\text{high}} \left(B_{0,\text{high}} + B_{1,\text{high}} z_t\right) \tag{12}$$

6 Program

- There is only one program called `main_learning.m`.
- Places in the program where you got to do something are indicated with `XYZ`

[4] Convergence does not always happen. It requires for example that agents use the correct functional form.

[5] And this information does not lead to a change in priors. It would make more sense to assume that z_{t-1} is also not revealed. But then the setup is a bit more complicated and I wanted to start with the simplest setup.

[6] The notation probably could be better but with $\text{prob}(\varepsilon_{t,\text{low}})$ I mean the probability of being in the low regime which is equal to the probability of $\varepsilon_{t,\text{low}}$.

7 Exercises

1. Get an idea about the structure of the program.
2. Complete the part to determine the coefficients of the REE solution.
3. Complete the part to get the formulas for Bayesian learning. You can already run the program. How quickly do agents determine there has been a regime change? The answer of course depends on the parameters, but before analyzing sensitivity first complete the program.
4. Uncommand (ctrl t) the part on least-squares learning (don't forget the figures at the end).
5. Complete the formulas for the two types of least-squares learning.
6. Run the program and observe how slow AMN learning is.
7. Finally, play around with parameter values. For example, increase the volatility of ε_t (which was set to a low value).