Financial Frictions and Risk Shocks

Xiangyang Li*

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Financial Frictions in Macroeconomics are not brand new topics. The earliest idea behinds financial frictions originated from the analysis of Townsend(1978). This idea is the so-called costly state verification (CSV). CSV is an important point in corporate finance. This involves asymmetric information in a standard debt contract between financial intermediaries and entrepreneurs. It will be costly for financial intermediaries to check the states of entrepreneurs due to the asymmetric information. In the literature, this cost is called monitor cost and assumed to be proportional to overall loan.

Bernanke, Gertler and Gilchrist (1999, BGG hereafter) integrated CSV into a full-fledged DSGE model and becomes popular among the literature ever since. The financial frictions set up in the model is called financial accelerator mechanism. The financial accelerator in macroeconomics is the idea that adverse shocks to the economy maybe amplified by worsening financial market condition. The link between the real economy and financial markets stems from firm's need for external finance to engage in investment opportunities. Firm's ability to borrow in financial market depends essentially on the market value of their net worth. The reason for this is the familiar story of asymmetric information between lenders and borrowers. Lenders are likely to have little information about the reliability of any given borrower. As such, they usually require borrowers to set forth their ability to repay, often in the form of collateralized assets. It follows that a fall in asset

*The Note is still incomplete and preliminary. Typos and Errors may still prevail. This is the first draft and improvements are expected. Any comments and suggestions are welcomed! Please send to ahnulxy#qq.com (Replace # with @ when email). This series notes dedicate to help students and researchers to understand more about Dynare and DSGE models. Special thanks goes to Prof. Christiano for his inspiring lectures and notes.

prices deteriorates the balance sheets of the firms and their net worth. The resulting deterioration of their ability to borrow has a negative impact on their investment. Decreased investment and economic activity further cuts the assets prices down, which leads to a feedback cycle of falling asset prices, deteriorating balance sheets, tightening financing conditions and declining economic activity. This vicious cycle is called a financial accelerator[1].

A lot of literature emerged after 1999 to further extend the ideas among them was the empirical analysis of Christiano, Motto and Rostagno (2003[2],2010[3],2014[4], CMR hereafter).

This CMR(2014) was published in AER[5]. The paper entitled 'Risk Shocks' has emphasized the importance of risk shocks in financial accelerator mechanism. As mentioned by Christiano in his lectures and the 2014 AER paper, risk shocks accounted for about 50% or more of the fluctuations of the economy.

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[1] It is the financial feedback loop or a loan/credit cycle, which, starting from a small change in financial markets, is in principle, able to produce a large change in economic conditions.

[2] Christiano, L. and R. Motto, et al. (2003). "The Great Depression and the Friedman-Schwartz Hypothesis." Journal of Money, Credit and Banking 35 (6): pp. 1119-1197.

[3] Christiano, L. and R. Motto, et al. (2010). "FINANCIAL FACTORS IN ECONOMIC FLUCTUATIONS." European Central Bank Working Paper Series(No.1192).

[4] Christiano, L. J. and R. Motto, et al. (2014). "Risk Shocks." American Economic Review 104 (1): 27-65.

[5] See herehttp://www.aeaweb.org/articles.php?doi=10.1257/aer.104.1.27; codes, appendix and even data can be found here.

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1 The Model

1.1 The Standard Debt Contract

There are a lot of entrepreneurs and banks out there. Let's consider the interaction between entrepreneurs with a specific amount of net worth N_t at time t with competitive banks. Households will save in the bank and entrepreneurs get loans from banks and there sign the standard debt contracts.

Each entrepreneur has access to a project with unit rate of return R^k but subject to an idiosyncratic shock ω:

$$\omega\left(1+R^k\right)$$

Here ω is a unit mean, idiosyncratic shock experienced by a individual entrepreneur after the project has been started. As a reasonable assumption, ω can not be negative. This is shock is only able to be observed by entrepreneur. And this is the source of asymmetric information and hence CSV.

Let's assume that ω has a Cumulative Distribution Function (CDF afterwards) $F(.)$:

$$\omega \sim F(\omega)$$

Hence, by definition

$$\int_0^{+\infty} \omega \mathrm{d}F(\omega) = 1$$

One of the natural choice for F is log-normal distribution[6].

Entrepreneur receives a contract from a bank which specifies a loan rate Z[7] and a loan amount B. If the contract mature, then the entrepreneur will pay back the principal and interest to end the contract. If the entrepreneur can not pay back the loan and go bankruptcy, then the bank will pay a monitoring cost and take everything left there. After got loan from bank, the total asset owned by entrepreneur denoted by A:

$$\overbrace{A}^{\text{total assets}} = \overbrace{N}^{\text{net worth}} + \overbrace{B}^{\text{loans from bank}} \tag{1}$$

[6]See Section 2 for more details.

[7]No body has even invented a symbol Z to stand for the interest rate but BGG(1999). After that, people have to follow them.

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1.2 Cutoff Value of Risk $\bar{\omega}$

Cutoff value $\bar{\omega}$ is a critical value where cut off the win and loss of the project. Below this value, the entrepreneur will lose everything since its total assets now less than the loan principal and its interest after he suffers sufficiently bad conditions[8]. If bigger than this cutoff value, the entrepreneur will be able to earn a profit after paying back the principal and interest to the bank.

The cutoff value is defined as

$$\overbrace{\bar{\omega}\left(1+R^k\right)}^{\text{unit rate of return on project}} \times \overbrace{A}^{\text{total assets owned by entrepreneur}} = \overbrace{ZB}^{\text{principal and interest owed}} \tag{2}$$

Then

$$\bar{\omega} = \frac{ZB}{\left(1+R^k\right)A} = \frac{Z}{1+R^k}\frac{\frac{B}{N}}{\frac{A}{N}} = \frac{Z}{1+R^k}\frac{\frac{A-N}{N}}{\frac{A}{N}} = \frac{Z}{1+R^k}\frac{L-1}{L} \to \frac{Z}{1+R^k},\ L \to +\infty$$

when $L \equiv \frac{A}{N}$ is the leverage ratio of the entrepreneur. This is a very interest rate result between the cutoff value and the leverage ratio. As we can see that $L \geq 1$. When $L = 1$, there will be no borrowing and $\bar{\omega} = 0$. As leverage increases, $\bar{\omega}$ is increasing too, but it does has a limit $\frac{Z}{1+R^k}$. Hence the cutoff value is a increasing function of leverage ratio. That is to say that, with the increase of leverage ratio, the cutoff value increases to counter the risk of loss. Generally speaking, $\hat{\omega}$ will not far away from one since the loan rate will not far away from the captial return. Otherwise, the enterpreneur will not loan.

Here is one more point. The overall asset A is somewhat abstract. In the literature, the overall asset is the total value of the capital owned by entrepreneur which is $Q_t K_{t+1}$ such as in CMR(2010) where Q_t is the price of the capital. See the paper for more information.

1.3 Entrepreneur Utility

Let's define the entrepreneur's utility function as the expected return over opportunity cost of funds:

$$\frac{\int_{\bar{\omega}}^{\infty}\left[\omega\left(1+R^k\right)A - ZB\right]\mathrm{d}F(\omega)}{N(1+R)}$$

[8]The more accurate expression is that entrepreneur will receive nothing if below cutoff value since it is a limited liability assumption.

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where the denominator is the opportunity cost of entrepreneur which could be payoff from depositing in bank. And the numerator is the expected payoff for entrepreneur from a probability view. R denotes the risk-free rate.

If we replace ZB with $\bar{\omega}\left(1+R^k\right)A$ as in Eq.(2), then the utility function can be rewritten as

$$\begin{aligned}
\frac{\int_{\bar{\omega}}^{\infty}\left[\omega\left(1+R^k\right)A-ZB\right]\mathrm{d}F(\omega)}{N(1+R)} &= \frac{\int_{\bar{\omega}}^{\infty}\left[\omega\left(1+R^k\right)A-\bar{\omega}\left(1+R^k\right)A\right]\mathrm{d}F(\omega)}{N(1+R)} \\
&= \int_{\bar{\omega}}^{\infty}\left[\omega-\bar{\omega}\right]\mathrm{d}F(\omega)\frac{\left(1+R^k\right)A}{(1+R)N} \\
&= \int_{\bar{\omega}}^{\infty}\left[\omega-\bar{\omega}\right]\mathrm{d}F(\omega)\frac{\left(1+R^k\right)}{(1+R)}L \qquad (3)
\end{aligned}$$

We can see that entrepreneur utility will be a increasing function of leverage L and will be eventually almost linear increasing with Big L as in Fig.1. Hence entrepreneur will borrow an infinite amount to have as much utility as possible given a fixed loan interest rate. That is to say that leverage will tend to infinity too. But in equilibrium, bank can not lend infinite amount. This is the reason why a standard debt contract must specify a interest rate Z and a loan amount B.

We define two spreads here. One is the interest rate spread

$$ZR=\frac{Z}{1+R}$$

The other one is the risk spread

$$R_kR=\frac{1+R_k}{1+R}$$

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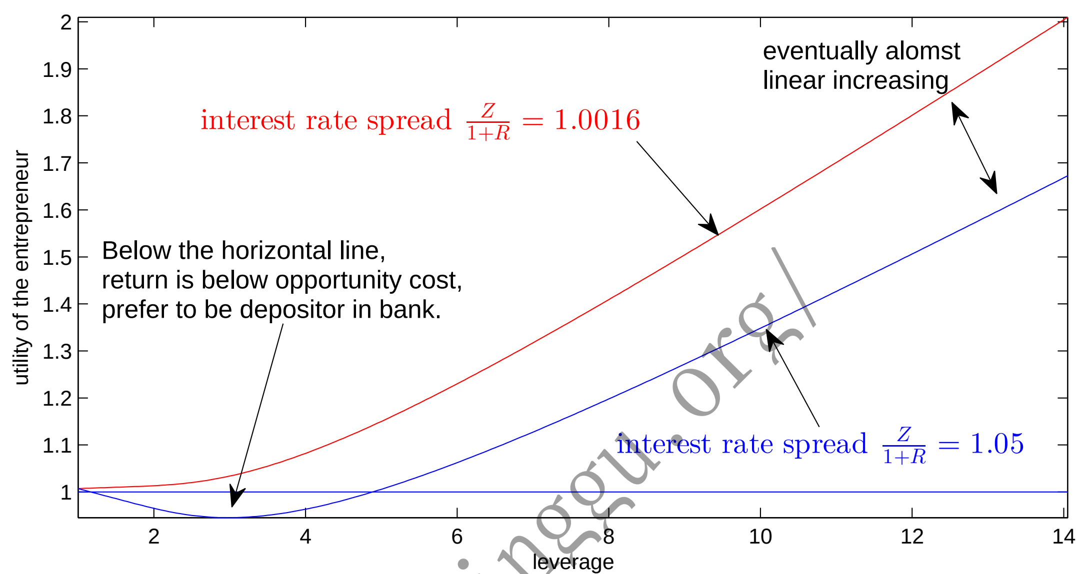
Figure 1: Entrepreneur utility over opportunity cost



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1.4 The Bank

The funds loaned to entrepreneur come from households and banks pay households a fixed interest rate R. The bank will provide entrepreneur a standard debt contract (Z, B) as mentioned above.

Once the entrepreneur experiences a idiosyncratic shock $\omega < \bar{\omega}$, and can not pay back the loan to the bank. Then the bank will pay a monitoring cost which is characterized by a cost parameter μ

$$\mu\omega\left(1 + R^k\right) A$$

Then the bank will have a zero profit condition under the assumption that full competitive market with free entry and exit. On the one side of the condition is the bank's expected return on loan and on the other side of the condition is the cost of the loan, i.e., the interest paid to households who save.

$$\overbrace{\int_{\bar{\omega}}^{\infty} \bar{\omega}\left(1+R^k\right) AdF(\omega)}^{\text{fraction of entrepreneur whose } \omega>\bar{\omega}} + \overbrace{(1-\mu)\int_0^{\bar{\omega}} \omega\left(1+R^k\right) AdF(\omega)}^{\text{fraction of entrepreneur whose } \omega<\bar{\omega}} = \overbrace{(1+R)\,B}^{\text{amount paid to the households by bank}}$$

Hence, in expectation, the overall share of the gross return on the project $\left(1+R^k\right)A$ goes to bank is

$$\int_{\bar{\omega}}^{\infty} \bar{\omega} dF(\omega) + (1-\mu)\int_0^{\bar{\omega}} \omega dF(\omega) = \bar{\omega}\left(1-F(\bar{\omega})\right) + (1-\mu)\int_0^{\bar{\omega}} \omega dF(\omega) \tag{4}$$

This zero profit condition can thus be further reduced to

$$\left(1-F(\bar{\omega})\right) ZB + (1-\mu)\int_0^{\bar{\omega}} \omega\left(1+R^k\right) AdF(\omega) = (1+R)\,B \tag{5}$$

by noticing the definition Eq.(2) and Eq.(4). Then the risk-free rate can be written as the average return of entrepreneurial projects:

$$1+R = \frac{\left(1-F(\bar{\omega})\right) ZB + (1-\mu)\int_0^{\bar{\omega}} \omega\left(1+R^k\right) AdF(\omega)}{B}$$

Noticing (4), the zero profit condition can be further written as

$$\bar{\omega}\left(1-F(\bar{\omega})\right) + (1-\mu)\int_0^{\bar{\omega}} \omega dF(\omega) = \frac{(1+R)\,B}{\left(1+R^k\right)A} = \frac{1+R}{1+R^k}\frac{L-1}{L}$$

Then leverage ratio can be written as function of $\bar{\omega}$ which can be plotted in $(\bar{\omega}, L)$ space : $\frac{1+R^k}{1+R} = 1.0073, \mu = 0.2149, \sigma = 0.2592.$

$$L = \frac{1}{1-\frac{1+R^k}{1+R}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)} \tag{6}$$

where $\Gamma(\bar{\omega}) = \bar{\omega}[1-F(\bar{\omega})] + G(\bar{\omega}), G(\bar{\omega}) \equiv \int_0^{\bar{\omega}} \omega dF(\omega)$. By the definition of Γ, it has the meaning of the share of overall return of the project $\bar{\omega}\left(1+R^k\right)A$ goes to the bank including the monitor cost.

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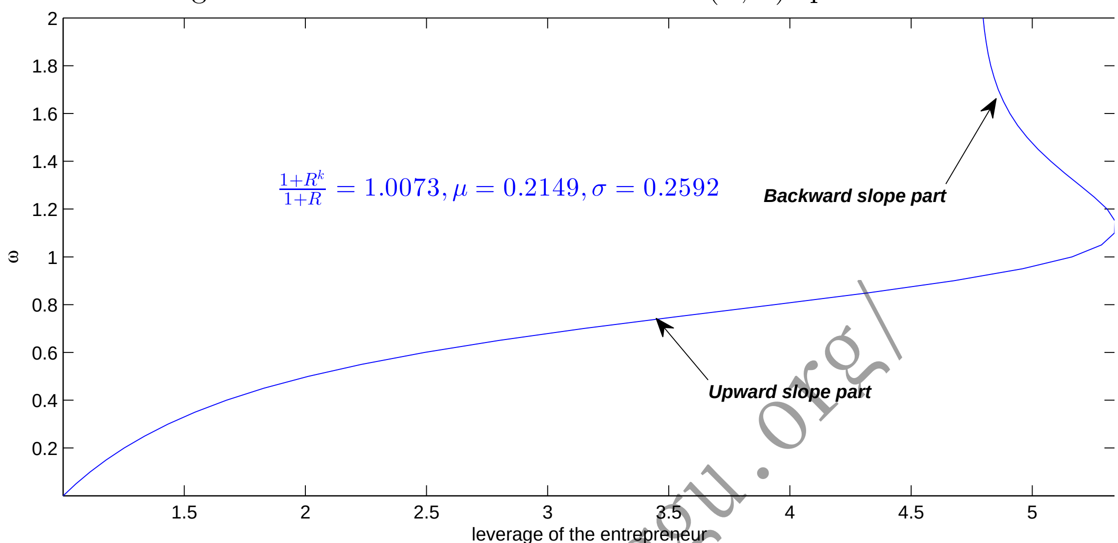
Figure 2: Bank Zero Profit Condition - $(\bar{\omega}, L)$ space



Note: We plot the leverage ratio against cutoff value and then revolve the figure 90 degree counter clockwise.

We can see the only upward slope part is relevant in this two-dimension space. The reason is that the entrepreneur will never choose a higher cutoff value at the same leverage ratio if a lower cutoff value is available. The $(\bar{\omega}, L)$ curve represents a 'menu' of contract that can be offered in equilibrium. Here menu means that you have a lot of choice to offer to the customers. Hence an optimal one is out there.

1.5 Optimal Contract

The optimal contract is a combination of $(\bar{\omega}, L)$ which maximize the entrepreneur's utility given the risk spread. Taking logarithm to entrepreneur's

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utility U

$$
\begin{aligned}
U &\equiv \int_{\bar{\omega}}^{\infty} [\omega - \bar{\omega}] \, \mathrm{d}F(\omega) \frac{(1+R^k)}{(1+R)} L \\
&= \overbrace{\int_{\bar{\omega}}^{\infty} [\omega - \bar{\omega}] \, \mathrm{d}F(\omega) \frac{(1+R^k)}{(1+R)}}^{\text{profit earned by unit leverage given } \bar{\omega}} \times \overbrace{\frac{1}{1 - \frac{1+R^k}{1+R}(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))}}^{\text{leverage offered by bank, given } \bar{\omega}} \\
&= (1 - \Gamma(\bar{\omega})) \times \frac{(1+R^k)}{(1+R)} L
\end{aligned}
$$

$$
\log U = \underbrace{\log(1 - \Gamma(\bar{\omega}))}_{\text{Larger cutoff will low the utility, bad!}} + \log\left(\frac{(1+R^k)}{(1+R)}\right) \tag{7}
$$

$$
\underbrace{-\log\left(1 - \frac{1+R^k}{1+R}(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))\right)}_{\text{Larger cutoff will improve leverage, good!}} \tag{8}
$$

The two effects will cancel with each other and finally there will be a balance. So, by intuition, there will be a unique $\bar{\omega}$ that achieves maximum.

Taking first derivative w.r.t. $\bar{\omega}$ to (7)

$$
\frac{d \log U}{d\bar{\omega}} = -\frac{1 - F(\bar{\omega})}{1 - \Gamma(\bar{\omega})} + \frac{\frac{1+R^k}{1+R}(1 - F(\bar{\omega}) - \mu\bar{\omega}F'(\bar{\omega}))}{1 - \frac{1+R^k}{1+R}(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))} = 0 \tag{9}
$$

i.e.,

$$
\frac{1 - F(\bar{\omega})}{1 - \Gamma(\bar{\omega})} = \frac{\frac{1+R^k}{1+R}(1 - F(\bar{\omega}) - \mu\bar{\omega}F'(\bar{\omega}))}{1 - \frac{1+R^k}{1+R}(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))} \tag{10}
$$

The left-hand side is greater than zero, thus the right-hand side must greater than zero too. Under the assumption that[9]

$$
\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) < \frac{1+R}{1+R^k}
$$

[9]This assumption could always hold under reasonable calibration unless risk spread is too large. The maximum of $\Gamma(\bar{\omega}) - \mu G(\bar{\omega})$ is about less than 0.82. If risk spread is less than 1.20, then the assumption holds.

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then optimal $\bar{\omega}$ must satisfy

$$1 - F(\bar{\omega}) - \mu\bar{\omega}F'(\bar{\omega}) > 0$$

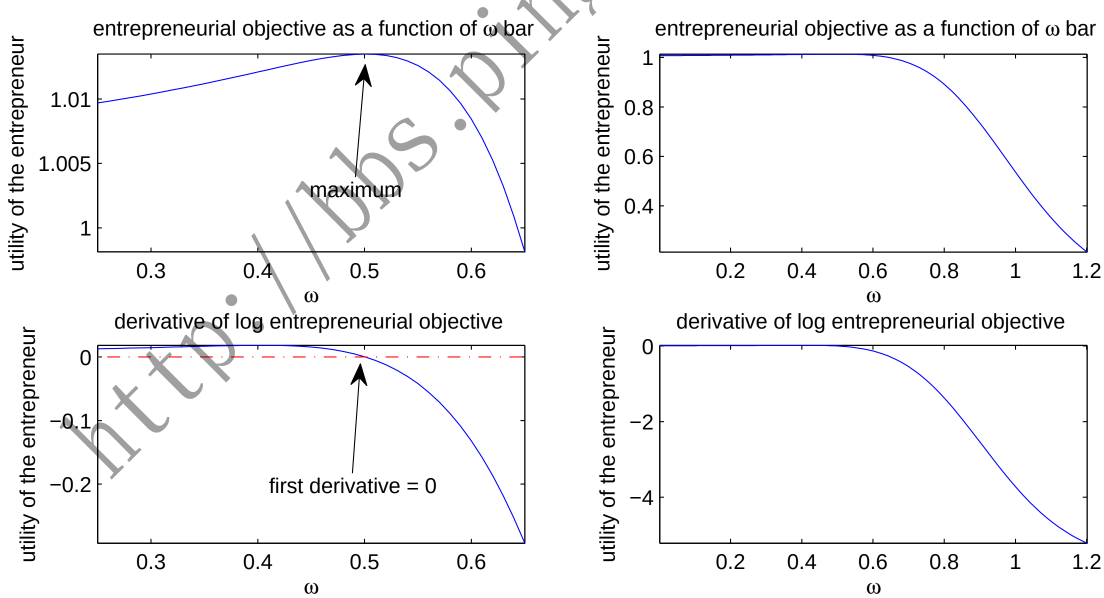
This means that with the increase of ω, leverage actually increase before ω reach the optimal. Hence larger ω , the larger leverage ratio (See eq.(6)). You can also plot it out to gain more intuition on this. If calibration of the parameters at odds with the inequality, i.e.,

$$1 - F(\bar{\omega}) - \mu\bar{\omega}F'(\bar{\omega}) \leq 0$$

then the calibration need to be reconsidered since the optimal condition holds at most time and reasonable calibration. This point will be considered in the Matlab coding.

We can also see from the figure that first order derivative of log-utility only has one zero point, hence utility has a unique solution.

Figure 3: Utility and Log-utility



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Eq.(9) can be solved for $\bar{\omega}$ uniquely. Given optimal cutoff value $\bar{\omega}$, the leverage ratio is available using

$$L = \frac{1}{1 - \frac{1+R^k}{1+R}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)}$$

Then given leverage ratio and cutoff value, you can solve the interest rate spread[10] given $\frac{1+R^k}{1+R}$:

$$\frac{Z}{1+R} = \frac{1+R^k}{1+R}\bar{\omega}\frac{L}{L-1} \tag{11}$$

So let's summarize the logic here:

1. Solve the entrepreneur's utility maximization problem and find $\bar{\omega}$;
2. Using the FOC of the maximization problem, we can find the leverage ratio L conditional on $\bar{\omega}$;
3. By the definition of cutoff value (2), we solve the interest rate risk given risk spread, leverage ratio and optimal $\bar{\omega}$.

1.6 Numerical Example

As you can see from above, calibration must be made to the following parameters before we can compute the leverage ratio and interest rate spread[11]:

$$\frac{1+R^k}{1+R} = 1.0073, \mu = 0.2149,$$

where μ is monitoring cost parameter, and σ is the standard deviation of normal distribution associated with the lognormal. Actually there are three critical parameters which are linked together:

$$F(\bar{\omega}) = \text{logncdf}\left(\bar{\omega}, -\frac{\sigma^2}{2}, \sigma\right) \tag{12}$$

Here $F(\bar{\omega}), \bar{\omega}, \sigma$ are the three parameters. $F(\bar{\omega})$ can be explained as bankruptcy rate. If we calibrate two of them, then the other one will be automatically calibrated or decided. Hence, you can not calibrate two parameters of them at the same time since there are other equilibrium conditions that give constraints to these parameters. Usually, we observe the bankruptcy rate in real

[10] If you calibrate both the interest rate spread $\frac{Z}{1+R}$ and risk spread $\frac{1+R^k}{1+R}$, by the bank zero profit condition (5) and the definition of cutoff value (2), we can determine the value of L and $\bar{\omega}$. But here we do not calibrate the interest rate spread. Hence the cutoff value definition equation (2) is used to determine the interest rate spread.

[11] I follow the calibration used by Prof. Christiano.

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economy while the other two are hard to see. So it is a natural choice to calibrate $F(\bar{\omega})$.

Once $F(\bar{\omega})$ is calibrated, then $\bar{\omega}, \sigma$ are correspondent to each other. That is to say that for every $\bar{\omega}$, there is a σ out there so that they satisfy the definition above (12). In Matlab, we need numerically solve σ for every $\bar{\omega}$.

Above, we talk about that it is natural to calibrate $F(\bar{\omega})$ which can be explained as bankruptcy rate. However, there is another way around. We can also calibrate the standard deviation of the risk shock σ. If σ is calibrated, the solution techniques are slightly different from the one above. It is a little bit more easy to implement in Matlab. The Matlab code in the following subsections locates in

1.6.1 Solution Conditional On $F(\bar{\omega})$

Here, we will solve the optimal $\bar{\omega}$ conditional on $F(\bar{\omega})$ is given. Let's recopy the two equations here:

$$F(\bar{\omega}) = \text{logncdf}\left(\bar{\omega}, -\frac{\sigma^2}{2}, \sigma\right)$$

$$\frac{1 - F(\bar{\omega})}{1 - \Gamma(\bar{\omega})} = \frac{\frac{1+R^k}{1+R}\left(1 - F(\bar{\omega}) - \mu\bar{\omega}F'(\bar{\omega})\right)}{1 - \frac{1+R^k}{1+R}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)}$$

If we calibrate $F(\bar{\omega})$, as we mentioned above, we need numerically solve σ for every $\bar{\omega}$ since σ is implicitly determined in (12).

To find the optimal $\bar{\omega}$, we need iterate on all possible values for $\bar{\omega}$. As we have seen before in (2), the cutoff value will not too far away from unity since that the loan rate will not far away from the capital reutrn rate. Without loss of generality, we assume that $\bar{\omega} \in (0, 1)$. For every $\bar{\omega}$, if we want to calculate the difference of the two sides of (9) to see that whether the difference is zero, we need to know the standard devation of the lognormal distribution, σ. Hence, before we calucate the difference, we first need to find σ from (12) given $F(\bar{\omega})$ and $\bar{\omega}$.

In code directory, there is sub directory named as **/numerical_example/conditional_on_F** contains all the m file needed to solve the optimal $\bar{\omega}$. There are 5 m files:

1. **main.m**: this file is the main file to solve the optimal $\bar{\omega}$ which will invoke the following 4 m files. Please run this file to get results.

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2. **get_omega_cond_Fomegabar.m**: given $F(\bar{\omega})$, mu μ, and risk spread, then optimal $\bar{\omega}$ is returned; σ is also returned which is associated with $\bar{\omega}$. This m file will iterate on all possible values of $\bar{\omega} \in (0,1)$, to find the optimal one that is the one satisfies (9). An indicator is also returned to indicate possible various errors associated with computation. This function will invoke **find_foc_difference.m.**

3. **find_foc_difference.m**: This function only serves to calculate some expressions in the FOC and return the difference of the two sides of FOC (efficiency condition) from utility maximization, i.e., (9). More specifically, given ω and $F(\bar{\omega})$, this function compute the Gamma Γ, Gamma - mu*G,$\Gamma - \mu G$, and most important of all, the standard deviation of the risk shock σ. An indicator will also return to indicate if there are something at odd with the efficient condition. This function will invoke the following m files.

4. **find_sigma_cond_Fomegabar.m**: Given $\bar{\omega}$ and $F(\bar{\omega})$, this routine finds the value of sigma for the lognormal distribution with mean forced to be equal to unity, such that Prob.$(\omega < \bar{\omega}) = F(\bar{\omega})$. For $F(\bar{\omega}) = .03$, this program requires $\bar{\omega}$ to live in the interval [0.000000001,0.9999]. This function will invoke **find_lognormal_difference.m**.

5. **find_lognormal_difference.m**: This function is very simple which is used as the target function of fzero in **find_sigma_cond_Fomegabar.m**. It just returns the difference of the two sides of lognormal cdf definition equation (12).

In this case, we calibrate the bankruptcy rate $F(\bar{\omega}) = 0.0056$, i.e., 0.56%. Here is the results (the equilibrium quantities):

$$\overbrace{\bar{\omega} = 0.5010,}^{\text{cutoff value}} \quad \overbrace{\sigma = 0.2592}^{\text{standard deviation}}, \quad \overbrace{\Gamma(\bar{\omega}) = 0.5008}^{\text{fraction of gross project return goes to bank}}$$

$$\overbrace{G(\bar{\omega}) = 0.0026}^{\text{average } \omega \text{ that bankrupts}}, \overbrace{L = 2.0155}^{\text{leverage ratio}} \quad \overbrace{U = (1 - \Gamma(\bar{\omega})) \times \frac{(1 + R^k)}{(1 + R)} L = 1.0135}^{\text{average earning of entrepreneur relative to opportunity cost.}},$$

$$\overbrace{\frac{Z}{R} = 1.0015}^{\text{interest rate spread, annual percentage rate 0.62\%}}$$

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We can see from the results that, in expectation, entrepreneur will earn a profit larger than the opportunity cost. The average return rate of project is 1.0135 times than risk-free rate. Hence, the entrepreneur will better off by lending or leveraging his net worth. The leverage ratio is about 2, and this mean that the entrepreneur will lend about the amount of his net worth. The fraction of the gross entrepreneurial earning goes to lender is roughly 50%.

1.6.2 Solution Conditional on σ

Here, we will solve the optimal $\bar{\omega}$ conditional on σ is given. In code directory, there is a sub directory named as **/numerical_example/conditional_on_sigma/** contains all the m file needed to solve the optimal $\bar{\omega}$. There are 4 m files:

1. **main.m**: this file is the main file to solve the optimal $\bar{\omega}$ which will invoke the following three m files.

2. **get_omega_cond_sigma.m**: This is a function which accept three parameters: sigma, mu and spread finds the optimal omega $\bar{\omega}$ that solves the efficiency condition (9), i.e., the foc of the entrepreneurial utility maximization problem. Matlab built-in function fzero is used. This m file will invoke the following two m files.

3. **reduce_to_small_interval.m**: This function help to reduce omega interval[12] to a small one which contains the root or the optimal $\bar{\omega}$ for the efficiency condition. The interval is indicated by ix. This indicator will be returned to decide what the small interval is.

4. **find_foc_difference.m**: This function only serves to calculate some expressions in the FOC and return the difference of the two sides of FOC from utility maximization.

[12] Usually, we assume there is a relatively large interval as we can see in the figure 3 above. In the code, we assume ω lies in [.001 , 1.4]. Of course, a larger interval can be assumed.

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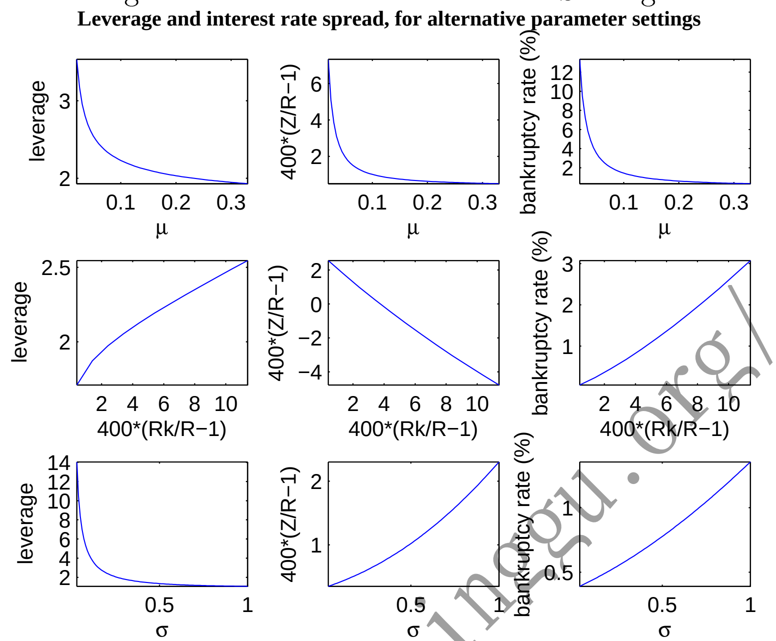
Figure 4: Alternative Parameters Settings



(a) The top row show that higher monitor cost will low leverage ratio, the interest rate spread and bankruptcy rate. It agrees with the intuition since higher monitor cost will reduce the agent problem and information asymmetry to a relatively low level. The monitor cost can be explained as the agent for the degree of asymmetric information, hence, the financial frictions. Higher monitor cost will also low the enterpreneur's utility since it is a function of leverage ratio which is lower when monitor cost is higher.

(b) The middle row shows that higher risk spread will lead higher leverage ratio, bankruptcy rate, but low interest rate spread.

(c) The bottom row shows that higher standard deviation of risk shock will result smaller leverage ratio, but higher interest rate spread and bankruptcy rate.

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In this case, we calibrate $\sigma = 0.2592$ which correspondent to the optimal solution when we conditional on $F(\bar{\omega}) = 0.0056$. The results is exactly the same.

We will also plot how leverage ratio L, bankruptcy rate $F(\bar{\omega})$ and interest rate spread $\frac{Z}{R}$ change according to different parameter settings : $\mu, \sigma, \frac{1+R^k}{1+R}$. The figures are plotted when only one parameter changes while the other two are set to original value. Please run the second cell in main file to get results, as shown in Fig.4.

1.6.3 Comparison of Two Calibrations

On solving this optimal problem, we have two ways to go: either calibrate the bankruptcy rate $F(\bar{\omega})$ or the standard deviation of risk shock σ. Which way we should choose? People usually choose the observable variable to calibrate. In this case, the bankruptcy rate is observable and standard deviation of risk shock is hard to see or figure out. Hence, $F(\bar{\omega})$ is usually the choice. But you can still calibrate σ based on some empirical analyses or estimations using related historical data.

You can verify the two ways numerically and find that they will lead to the same results if two calibrations are correspondent to each other. Specifically, if we calibrate the bankruptcy rate $F(\bar{\omega})$, then solve the model and we can find out the optimal $\bar{\omega}$ and standard deviation σ_{opt} correspondent to $\bar{\omega}$ and $F(\bar{\omega})$. If another way, you calibrate σ which happens to be σ_{opt}, then the solutions will the same for the two calibrations. That is to say that they turn out to be equivalent. But if $\sigma \neq \sigma_{\text{opt}}$, they will differ with each other. Let's show it in Matlab.

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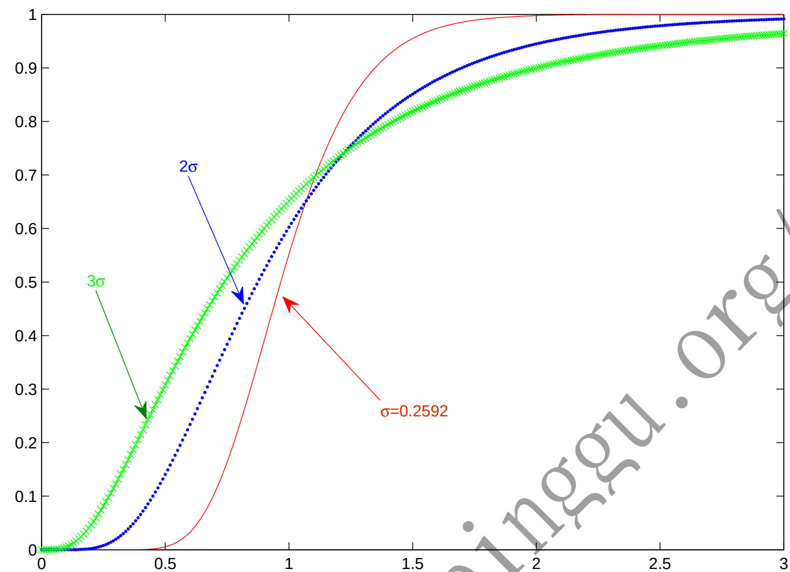
1.6.4 Consequences of Changing σ

The standard deviation of idiosyncratic shock σ usually explained as risk. The larger σ is, the larger risk is. Let's us study how the change of σ impact the overall results.

Let's first have a look of how the CDF of lognormal responds to the change of σ in Fig 5.

Figure 5: Impact on lognormal CDF of Doubling and Tripling Standard Deviation



We can clearly see that the increase of σ will significantly raise the density in the tail. This means that the greater risk will cause more entrepreneurs incur lower idiosyncratic shock, hence lower overall return, hence higher bankruptcy rate. Let's have a look how the solution of the utility maximization problem changes if we doubling σ.

$$\overbrace{\bar{\omega} = 0.2505}^{\text{cutoff value}}, \overbrace{\sigma = 0.2592 * 2}^{\text{standard deviation}}, \qquad \overbrace{\Gamma\left(\bar{\omega}\right) = 0.2503}^{\text{fraction of gross project return goes to bank}}$$

$$\overbrace{G\left(\bar{\omega}\right) = 0.0017}^{\text{average } \omega \text{ that bankrupts}}, \overbrace{L = 1.3364}^{\text{leverage ratio}} \quad \overbrace{U = \left(1 - \Gamma\left(\bar{\omega}\right)\right) \times \frac{\left(1 + R^k\right)}{\left(1 + R\right)} L = 1.0092}^{\text{average earning of entrepreneur relative to opportunity cost.}},$$

$$\overbrace{\frac{Z}{1 + R} = 1.0026}^{\text{interest rate spread, annual percentage rate 1.06\%}}$$

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If we tripling σ, we could have

$$\overbrace{\bar{\omega}=0.1238}^{\text{cutoff value}},\overbrace{\sigma=0.2592*3}^{\text{standard deviation}},\qquad \overbrace{\Gamma\left(\bar{\omega}\right)=0.1235}^{\text{fraction of gross project return goes to bank}}$$

$$\overbrace{G\left(\bar{\omega}\right)=0.0011}^{\text{average }\omega\text{ that bankrupts}},\overbrace{L=1.1418}^{\text{leverage ratio}}\ \overbrace{U=\left(1-\Gamma\left(\bar{\omega}\right)\right)\times\frac{\left(1+R^{k}\right)}{\left(1+R\right)}L=1.0080}^{\text{average earning of entrepreneur relative to opportunity cost.}},$$

$$\overbrace{\frac{Z}{1+R}=1.0041}^{\text{interest rate spread, annual percentage rate 1.62\%}}$$

1.7 Including CSV into Neoclassical Model

1.7.1 Households in Standard Neoclassical Model

In this section, we will include CSV into a simple neoclassical model as in BGG(1999). We assume a very simple households utility function

$$\max_{\{c_t,B_{t+1},L_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^{t}u\left(c_t\right),\quad u\left(c_t\right)=\log\left(c_t\right)$$

s.t.

$$c_t+B_{t+1}+K_{t+1}-\left(1-\delta\right)K_t\leq w_tL_t+r_tK_t+\left(1+R_{t-1}\right)B_t$$
$$0\leq L_t\leq 1$$

The FOCs w.r.t c_t, B_{t+1}, L_t are

$$u'\left(c_t\right)=\beta u'\left(c_{t+1}\right)\left(r_{t+1}+1-\delta\right)$$
$$u'\left(c_t\right)=\beta u'\left(c_{t+1}\right)\left(1+R_t\right)$$
$$L_t=1$$

Since labor does not enter the utility function, hence there is no labor disutility. From budget constraint, the more labor, the more income, more consumption or capital accumulation, higher utility. Hence, $L_t=1$ is the optimal.

The firms and market clearing conditions:

$$c_t+I_t=K_t^{\alpha}L_t^{1-\alpha}$$

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$$B_{t+1} = 0$$
$$w_t = (1-\alpha) K_t^{\alpha}$$
$$r_t = \alpha K_t^{\alpha}$$

1.7.2 Standard Model with CSV

For a standard model without CSV, there are households, firms and two markets. One is the labor market where households supply labor to the firm. The other one is the market for physical capital where firm rent the capital from households. Firms use labor and physical capital to produce goods and pay income and capital rental to the households.

But for the model with CSV, there will be more micro entities that enter into the model. There will be entrepreneurs, financial intermediates, and capital producers. The entrepreneur will loan from financial intermediates, like bank, and purchase raw capital from capital producers. The raw capital will be subjected to idiosyncratic shock which will transform the raw capital into effective capital. This effective capital will then rent to the firms for production and the entrepreneur will receive capital rental from the firms. After production, the un-depreciated capital will be sold to the capital producer. Then entrepreneur will pay back the loan. At this time the entrepreneur's net worth will determined. Then the entrepreneur will once again loan from bank, the cycle will roll on.

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1.7.3 Entrepreneur

An entrepreneur with net worth N_{t+1}[13] goes to the bank and receives a loan B_{t+1}, and buy raw, physical capital[14]

$$K_{t+1} = N_{t+1} + B_{t+1}$$

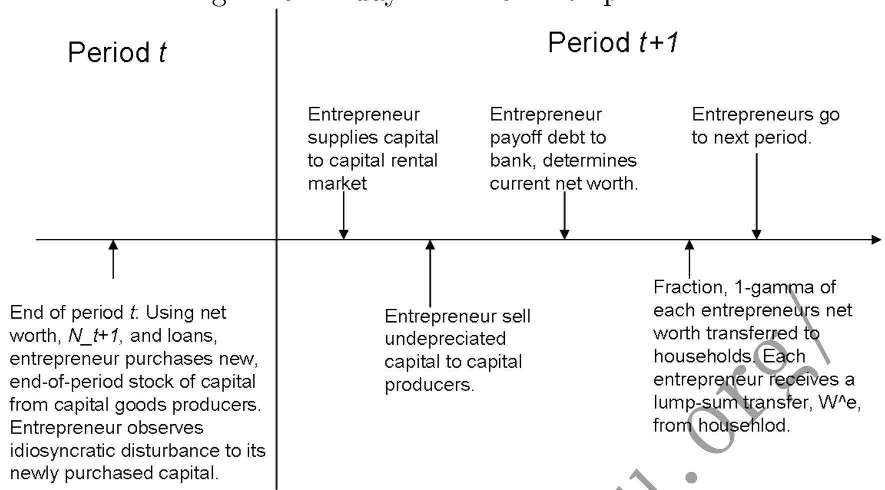
After purchasing the raw capital, the entrepreneur experiences an idiosyncratic shock ω, so that the raw capital is transformed into effective capital as follows

$$\omega K_{t+1}, \quad \omega \sim F(., \sigma_t)$$

[13] The time for net worth N_{t+1} is similar to capital which are determined at the end of period t.

[14] Here the physical capital is explained in term of consumption goods, with price unity. Usually, in the literature, the price of capital will not be unity. Sometimes, people use Q_t as the unit price of capital.

Figure 6: A day in life of entrepreneur



Then in a competitive capital rental market, the entrepreneur rents it effective capital at the rate r_{t+1} to the firm which use it as good production. After the good production, entrepreneur takes back it undepreciated capital $\omega K_{t+1}(1-\delta)$ and sell it to the capital producer at price unity. We define the return of capital for entrepreneur is $\omega\left(1+R_{t+1}^{k}\right)$ where

$$1+R_{t+1}^{k}\equiv r_{t+1}+1-\delta$$

We assume financial intermediates market are fully competitive, hence bank zero profit condition produces

$$\text{Leverage}=\frac{K_{t+1}}{N_{t+1}}=\frac{1}{1-\frac{1+R_{t+1}^{k}}{1+R_{t}}\left(\Gamma\left(\bar{\omega}_{t+1}\right)-\mu G\left(\bar{\omega}_{t+1}\right)\right)}$$

where $\left(1+R_{t+1}^{k}\right)K_{t+1}\bar{\omega}_{t+1}=Z_{t+1}B_{t+1}$.

Aggregate net worth of the entrepreneur after accounting all its income

$$\left(1-\Gamma\left(\bar{\omega}_{t+1}\right)\right)\left(1+R_{t+1}^{k}\right)K_{t+1}$$

Then a fixed fraction,$1-\gamma$ of the income goes to the households[15] as lump-

[15] There is a so-called 'large family' assumption which could abstract all complications in the details and could let us focus on the effect of financial risk. The large family assumption basically says that there are a lot of identical households which are large, each has a worker, and each has many entrepreneurs so that average net worth in the representative family is always equal to average net worth in the economy as a whole. Entrepreneur receives perfect consumption insurance from the household. Entrepreneur will consume the same amount with the worker.

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sum. The households transfers resources W_{t+1}^e as a lump sum to each entrepreneur. So the net worth of the entrepreneur at the end of period $t+1$ is:

$$N_{t+2} = \gamma \left(1 - \Gamma\left(\bar{\omega}_{t+1}\right)\right) \left(1 + R_{t+1}^k\right) K_{t+1} + W_{t+1}^e$$

Then with this net worth, the entrepreneur will go to the bank and get loans, buy capital and so on.

1.7.4 Other equations

The resource constraints, i.e., the goods market clearing condition

$$\begin{aligned} &\overbrace{c_t}^{\text{goods bought by households to feed their workers and entrepreneurs}} \\ +\ &\overbrace{i_t}^{\text{investment goods bought by capital producers}} \\ +\ &\overbrace{\mu \int_0^{\bar{\omega}_t} \omega dF\left(\omega\right) \left(1 + R_t^k\right) K_t}^{\text{goods bought by bank for monitoring cost}} = \overbrace{K_t^\alpha}^{\text{production technology}} \end{aligned}$$

The capital producer technology

$$K_{t+1} = \left(1 - \delta\right) K_t + I_t$$

which is the standard law of motion for capital. The household problem will now subject to the following constraint

$$c_t + B_{t+1} \leq w_t L_t + \overbrace{\left(1 - \gamma\right) \left(1 - \Gamma\left(\bar{\omega}_t\right)\right) \left(1 + R_t^k\right) K_t - W_t^e}^{\text{transfers from entrepreneurs}} + \left(1 + R_{t-1}\right) B_t$$

$$0 \leq L_t \leq 1$$

1.7.5 Equilibrium

The are 7 endogenous variables

$$c_t, I_t, \bar{\omega}_t, R_t^k, K_t, N_t, R_t$$

for 7 equations:

1. resource constraint: $c_t + I_t + \int_0^{\bar{\omega}_t} \omega dF\left(\omega\right) \left(1 + R_t^k\right) K_t = K_t^\alpha$

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2. household FOC: $u'(c_t) = \beta u'(c_{t+1})(1+R_t)$

3. capital accumulation: $K_{t+1} = (1-\delta) K_t + I_t$

4. capital rental rate: $1 + R_t^k \equiv \alpha K_t^{\alpha-1} + 1 - \delta$

5. entrepreneur foc:

$$\frac{1 - F(\bar{\omega}_{t+1})}{1 - \Gamma(\bar{\omega}_{t+1})} = \frac{\frac{1+R_{t+1}^k}{1+R_t}\left(1 - F(\bar{\omega}_{t+1}) - \mu\bar{\omega}F'(\bar{\omega}_{t+1})\right)}{1 - \frac{1+R_{t+1}^k}{1+R_t}\left(\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})\right)} \tag{13}$$

6. bank zero profit condition:

$$\frac{K_t}{N_t} = \frac{1}{1 - \frac{1+R_t^k}{1+R_{t-1}}\left(\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)\right)} \tag{14}$$

7. aggregate net worth: $N_{t+1} = \gamma\left(1 - \Gamma(\bar{\omega}_t)\right)\left(1 + R_t^k\right) K_t + W_t^e$. Note that we ignore the wage here since it is very small in coding.

The first 4 equations are similar to the ones in standard neocalssical model. And the last three are equilibrium conditions specific to financial frictions.

Here we will introduce two more variables in the mod file in Dynare: financial friction as intertemporal wedge τ_t and variable volatility σ_t[16].

The definition for the wedge and volatility as follows:

$$\tau_{t+1} \equiv 1 - \frac{1+R_t}{1+R_{t+1}^k} = 1 - \frac{\left(1 - F(\bar{\omega}_{t+1}) - \mu\bar{\omega}F'(\bar{\omega}_{t+1})\right)}{\frac{1-F(\bar{\omega}_{t+1})}{1-\Gamma(\bar{\omega}_{t+1})}} - \Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})$$

$$\log\left(\frac{\sigma_t}{\sigma}\right) = \rho_\sigma \log\left(\frac{\sigma_{t-1}}{\sigma}\right) + \epsilon_t$$

As we will see in the IRF, the wedge will increase after the impact of positive shock of the variable volatility σ_t. If we introduce the financial wedge into the model, the household foc could modify as follows:

$$u'(c_t) = \beta u'(c_{t+1})\left(1 + R_t^k\right)(1 - \tau_t)$$

We may introduce more variables in the mod file:

[16] You could add more variables like the leverage ratio $L_t = \frac{K_t}{N_t}$.

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1. the bond B_t which is determined by $K_{t+1} = N_{t+1} + B_{t+1}$
2. the GDP $GDP = c + I$;
3. the bankruptcy rate $bankcrupt = F(\bar{\omega}_t)$
4. interest rate spread $\frac{Z}{1+R}$ which is determined by (11).

Now we totally have 13 variables in our mod file.

1.7.6 Steady States and Parameterization

We will use perturbation methods to solve the model. Before we can proceed, we need first calibrate the parameters values and then solve the model's steady states.

First, let's calibrate some parameters

$$\sigma = 0.26, \mu = 0.21, \gamma = 0.97, \alpha = 1/3, \delta = 0.02, \beta = 1.03^{-0.25}, \rho_\sigma = 0.97$$

Second, let's find the steady states. A steady state file is written to help find out the endogenous variables' steady states since there are nonlinear equilibrium conditions. We will give more details on this point. The following is the procedure to follow:

1. Use household foc and we can find the steady state value for risk-free interest rate $1 + R = \frac{1}{\beta}$
2. Fix an initial value for capital rate $R^k \geq R$
   (a) Compute optimal $\bar{\omega}$ and hence G ,F and Γ from entrepreneur's utility foc (13);
   (b) Compute capital steady states from capital rental equation;
   $$K = \left(\frac{\alpha}{1 + R^k - (1-\delta)}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha}{\frac{1}{\beta(1-\tau)} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$
   (c) Compute net worth N from net worth equation
   (d) Restart above iteration until bank zero profit condition (14) is satisfied.

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3. Finally, compute consumption from resource constraint;

Noticing that there are two constraints that we are subjected to when we calculate the steady states and that need fzero[17] to solve. The first one is the entrepreneur's utility maximization foc (13), and the other one is the bank zero profit condition (14). Hence, there should at least two iterations in our Matlab code[18]. Here is the code structure in the directory where most files name are self-explanatory:

1. **bgg_rbc.mod:** the Dynare mod file; Run this file will automatically invoke bgg_rbc_steadystate.m to compute the steady states of the endogenous variables.

2. **bgg_rbc_steadystate.m**: this is a standard file name known to Dynare to compute steady states for the mod file. This m file will invoke the following files.

3. **steadystate.m**: This file is the core for steady states computation. This file will iterate on given R^k. This is the first iteration. The following three m files will be invoked.

4. **get_bank_zero_profit_diff_cond_on_Rk.m**: This m file will return the difference of the two sides of bank zero profit condition.

5. **get_omega_cond_on_sigma.m**: This will be the 2nd iteration where we will iterate on $\bar{\omega}$ using (13). This file will find optimal $\bar{\omega}$ which will use fzero and iterate on ω in small interval with the help of the following m file.

6. **find_foc_difference.m**: this file will serve the same purpose as the one with the same name in section .

Just **run bgg_rbc.mod** to get all results below without bothering other m files.

[17] fzero is a Matlab built-in function which can be used to find root for nonlinear equations.

[18] The reason why we need iteration to find the root is that the constraints are nonlinear. You can not find the analystical solution for the optimal capital return rate R_t^k and cutoff value $\bar{\omega}$. There is one more point to mention that why we need first reduce the large interval which contains the optimal value or root to a small one. Of course, we can use fzero function in a larger interval to find the root, but it will have computation efficiency problem. To reduce to a small interval contains the roots will greatly improve the speed of calclulation.

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1.7.7 The Dynare Code

We copy the code as follows:

```
% we have 13 endogenous variables
var kbar //capital stock
 i           // investment
c           //consumption
R         // nominal rate
omegabar // cutoff value of risk
Rk          // return rate of capital
n            //net worth of enterpreneur
sigma   // variable standard devation of risk shock
spread  // interest rate spread
credit   //  loan from bank by enterpreneur
bankrupt  //rate of bankruptancy
GDP     // output
wedge   //financial wedge
;

varexo sigma_e;

parameters sigma_ss mu gamma alpha delta beta rhosigma gam;

sigma_ss=0.26; %steady state of standard dev. of risk shock
mu=0.21;          %monitoring cost parameter
gamma=0.97;    %fracation of the income goes to the households
alpha=1/3;        %capital share
delta=0.02;       %depreciation rate
beta=1.03^(-.25); %discount rate
rhosigma=0.97; %persistence
gam=1;            %log utility

model;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Auxiliary expressions.  These simplify the equations without adding
% additional variables. Noticing the timing scprits of the variables.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  # z        = (log(omegabar) + sigma(-1)^2 / 2) / sigma(-1);
```

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```
  # zplus    = (log(omegabar(+1)) + sigma^2 / 2) / sigma;
  # F           = normcdf(z);
  # G           = normcdf(z - sigma(-1));
  # d           =  mu * G   * (1 + Rk) * kbar(-1);
  # Fp1         = normcdf(zplus);
  # Gp1         = normcdf(zplus - sigma);
  # GAMMA       = omegabar*(1-F)+G;
  # GAMMAp1     = omegabar(+1)*(1-Fp1)+Gp1;
  # dFp1        = normpdf(zplus) / omegabar(+1) / sigma;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Equilibrium equations.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(1) resource constraint
    kbar(-1)^alpha = c + i + d ;

% (2)Law of motion for capital, capital producer technolgy
    kbar = (1 - delta) * kbar(-1) + i;

% (3) Household FOC w.r.t. risk-free bonds
    1 = beta * (c(+1)/c)^(-gam)*(1+R);
                              = 1

% (4) efficiency condition associated with the contract
    (1-Fp1)/(1-GAMMAp1)-((1 + Rk(+1)) / (1 + R))*(1-Fp1-mu*omegabar(+1)*dFp1)
    /(1-((1 + Rk(+1)) / (1 + R))*(GAMMAp1-mu*Gp1)); = 0

% (5) Definition of return of entrepreneurs, Rk
    1 + Rk = alpha*kbar(-1)^(alpha-1)  + 1 - delta;

%(6) Zero bank profit condition, i.e. the definition of leverage ratio
    kbar(-1)*(1 - ((1+Rk)/(1+R(-1)))*(GAMMA-mu*G)) - n(-1);

%(7) Law of motion of net worth, omitting the entrepreneurial wage
    n = gamma*(1-GAMMA)*(1+Rk)*kbar(-1);

%(8) borrowing:
    credit = kbar-n;
```

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```
%(9) interest rate spread
    spread = ((1+Rk)/(1+R(-1)))*(kbar(-1)/credit(-1))*omegabar;

%(10) Shock
   log(sigma / sigma_ss) = rhosigma * log(sigma(-1) / sigma_ss)  + sigma_e ;

%(11) the output of the economy
   GDP = c + i;

%(12) the financial wedge
   wedge = (1-(1+R)/(1+Rk(+1)));

%(13) bankruptcy rate
   bankrupt = F;
end;

shocks;
var sigma_e;
stderr .11;
end;

%this directive will invoke steady state file to have s.s.
steady;

stoch_simul(periods=1000,order=1,irf=20) kbar, i, c, R, omegabar,
                   Rk, n, sigma, spread, credit, bankrupt, GDP, wedge;
```

There are few points that need further explanation.

- First, there are model-local variables in the model block. This kind of variables are defined in model block and only visible in the model block. The only purpose of defining model-local variables is to let the coding be more easily written and let it be more readable and neatly. Dynare will only do text substitution for model-local variables in the equilibrium conditions. You can define more than one model-local variables in one mod file.

- Second, we do not directly compute the steady states for endogenous variables since it is too complicate to do in model file. A standalone m

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file is written to find out steady states.

Third, you will only need to run the mod file and get the following steady states values:

$$\bar{\omega} = 0.5308,\ G(\bar{\omega}) = 0.0051, F(\bar{\omega}) = 0.0106, \Gamma(\bar{\omega}) = 0.5303, \Gamma(\bar{\omega}) - \mu G(\bar{\omega}) = 0.5293,$$

$$N = 11.4123,\ GDP = 2.8802, c = 2.3888, K = 24.5731, i = 0.4915,$$

$$\frac{c}{GDP} = 0.8294,\ \frac{b}{GDP} = 4.5694,\ L \equiv \frac{K}{N} = 2.1532,\ \frac{K}{GDP} = 8.5317$$

$$R^k = 0.0194, R = 0.0074, 400\left(\frac{Z}{1+R} - 1\right) = 1.1640, \text{ wedge } \tau{=}0.0118.$$

1.7.8 The IRF

Here is the IRF of the endogenous variables to one standard deviation jump of shock ϵ_t. The following figure show the absolute deviation from steady state which are not percentage deviation. The jump means that risk is increasing. As we will see that with the increase of risk, the interest rate spread and risk spread will increase. From the definition of wedge, with the increase of risk spread, wedge will increase too.

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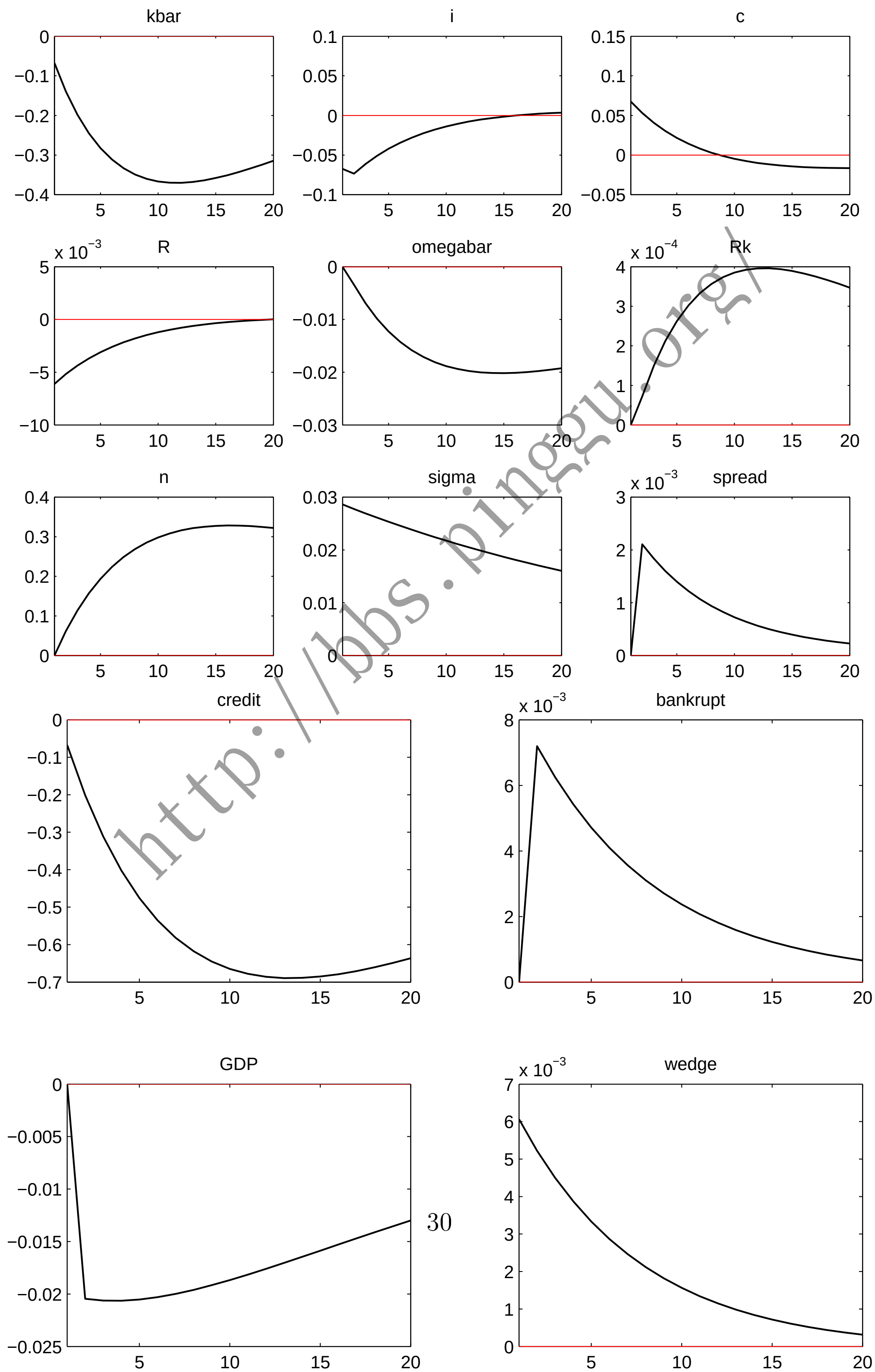
Figure 7: IRF to one standard deviation of ϵ_t



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1.7.9 Financial Distortion

We just think about the steady state value of capital stock. Without financial friction,

$$K = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} = 42.3923, c = 2.6390$$

With financial friction,

$$K = \left(\frac{\alpha}{\frac{1}{\beta(1-\tau)} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} = 24.5731, c = 2.3888$$

We can clearly see that with financial friction, both capital stock and consumption will substantially decrease in steady state. This means that financial friction is a source of inefficiency in the model.

The steady state welfare cost of the financial friction is about 8.9%=(2.64-2.39)/2.39*100%.

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2 Notes on Lognormal and Normal

2.1 Definitions

For a lognormal random variable, there is rule that can be followed:

Generally, if x has a normal distribution, then e^x has a lognormal distribution. That is to say that if $\log(x)$ has a normal distribution, then x has a lognormal distribution.

Let $F(\omega)$ be the CDF of the ω. Since ω is lognormal and hence $\log(\omega)$ has the normal distribution:

$$\text{If } \omega \sim \log Normal, \text{ then } \log(\omega) \sim Normal$$

In Short, if ω is lognormal, then its log must be normal. From this point, we can see that the domain for lognormal variables is non-negative: $[0, \infty)$.

The mean and standard deviation of $\log(\omega)$ is μ and σ. It is easy to find that the unconditional mean and standard deviation of ω[19]:

$$E(\omega) = e^{\mu + \frac{\sigma^2}{2}}, Var(\omega) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

[19] we can say it is standard result. Just use the definition of expectation and variance, you can easily get it.

Therefore we can see that it is impossible that a lognormal variable can has zero mean. In CMR(2010), μ is set to satisfy $E(\omega) = 1$. Hence, $\mu = -\frac{\sigma^2}{2}$. Then Var($\omega$)=$e^{\sigma^2}$-1. At the same time $E(\omega) = 1$ means that

$$\int_0^\infty \omega dF(\omega) = 1$$

In the following section, we only consider the case where $\mu = -\frac{\sigma^2}{2}$.

2.2 Γ and G

The functions used above is now recopied here:

$$\Gamma_t(\bar{\omega}_{t+1}) \equiv \bar{\omega}_{t+1}[1 - F_t(\bar{\omega}_{t+1})] + G_t(\bar{\omega}_{t+1})$$

$$G_t(\bar{\omega}_{t+1}) \equiv \int_0^{\bar{\omega}_{t+1}} \omega dF_t(\omega)$$

where ω is a lognormal random variable and $\bar{\omega}$ is the cutoff value for ω as mentioned in the model.

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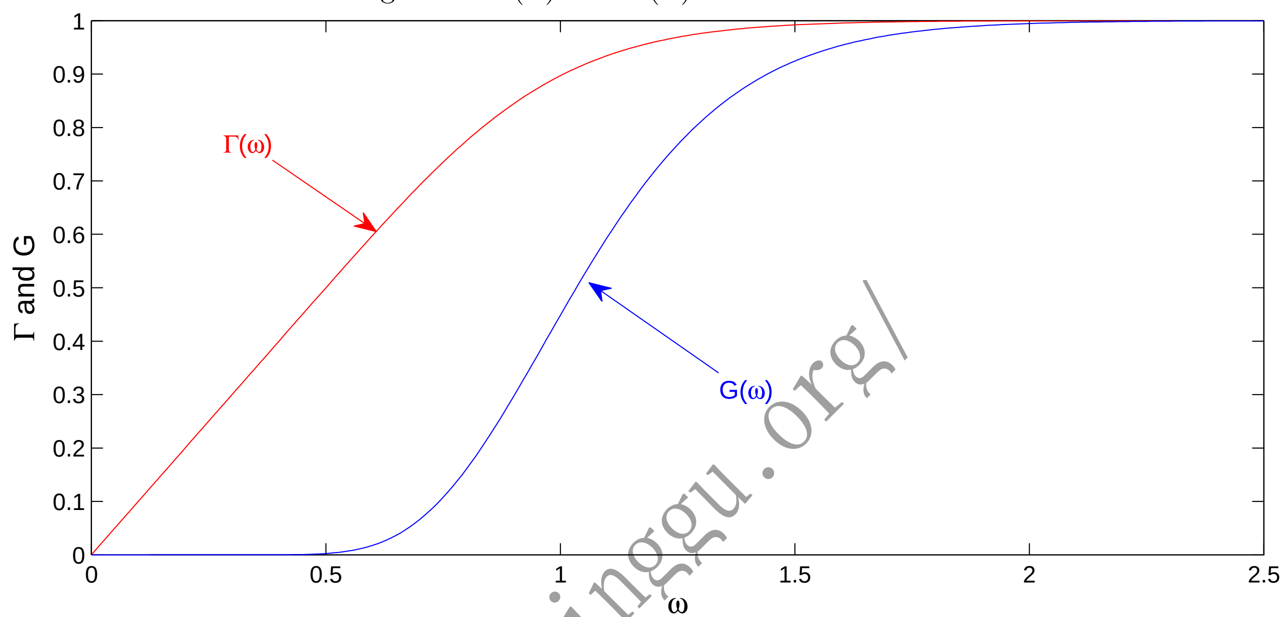
Figure 8: $\Gamma(\omega)$and $G(\omega)$



Note: $\Gamma(\omega)$ is concave function while $G(\omega)$ is not. At the first part, $G(\omega)$ is convex and second part is concave.

It is easy to verify the following results

$$0 \leq \Gamma(\omega), G(\omega) \leq 1$$

$$\lim_{\omega \to \infty} \Gamma(\omega) = 1, \ \lim_{\omega \to 0} \Gamma(\omega) = 0$$

$$\lim_{\omega \to \infty} G(\omega) = 1 \ \lim_{\omega \to 0} G(\omega) = 0$$

$$\Gamma'(\omega) = 1 - F(\omega) > 0, \ \Gamma''(\omega) = -\frac{dF(\omega)}{d\omega} < 0$$

$$G'(\omega) = \omega \frac{dF(\omega)}{d\omega} > 0$$

In the Matlab, the commands to compute $F_t(\omega)$ and $G_t(\omega)$ as follows:

$$F_t(\omega) = \text{normcdf}\left(\frac{\log(\omega) + \frac{\sigma^2}{2}}{\sigma}\right) = \text{logncdf}(\omega, \mu, \sigma)$$

Z

Zplus

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$$G_t(\omega) = \text{normcdf}\left(\frac{\log(\omega) + \frac{\sigma^2}{2}}{\sigma} - \sigma\right).$$

where μ and σ are the mean and standard deviation of associated normal distribution and **normcdf** and **logncdf** are two Matlab built-in functions. The normcdf are CDF of standard normal distribution with mean zero and variance unity if not specify the associated mean and variance. For example normcdf(0)=$\frac{1}{2}$. The logncdf stands for the CDF of lognormal. See more details for the two commands below.

For example, $\sigma = 1$,then $\mu = -0.5$. Then we have $F_t(1) = \text{normcdf}(0.5) = \text{logncdf}(1, -0.5, 1) = 0.6915$. Let's explain why is the case for both F and G:

$$\begin{aligned}
F_t(\bar{\omega}_{t+1}) &= \text{Prob.}(\omega < \bar{\omega}_{t+1}) \\
&= \text{Prob.}(\log(\omega) < \log(\bar{\omega}_{t+1})) \\
&= \text{Prob.}\left(\frac{\log(\omega) - \mu}{\sigma} < \frac{\log(\bar{\omega}_{t+1}) - \mu}{\sigma}\right) \\
&= \text{normcdf}\left(\frac{\log(\bar{\omega}_{t+1}) - \mu}{\sigma}\right) \\
&= \text{normcdf}\left(\frac{\log(\bar{\omega}_{t+1}) + \frac{\sigma^2}{2}}{\sigma}\right) \\
&= \int_{-\infty}^{\frac{\log(\bar{\omega}_{t+1}) + \frac{\sigma^2}{2}}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx
\end{aligned}$$

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$$
\begin{aligned}
G_t\left(\bar{\omega}_{t+1}\right) &\equiv \int_0^{\bar{\omega}_{t+1}} \omega dF_t\left(\omega\right) \\
&= \int_0^{\bar{\omega}_{t+1}} \omega \frac{1}{\omega\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\log(\omega)-\mu}{\sigma}\right)^2} d\omega & (15)\\
&\overset{x=\frac{\log(\omega)-\mu}{\sigma}}{=} \int_{-\infty}^{\frac{\log(\bar{\omega}_{t+1})-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{x\sigma+\mu} e^{-\frac{1}{2}x^2} dx & (16)\\
&= \int_{-\infty}^{\frac{\log(\bar{\omega}_{t+1})-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\sigma)^2} dx & (17)\\
&= \int_{-\infty}^{\frac{\log(\bar{\omega}_{t+1})-\mu}{\sigma}-\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= \text{normcdf}\left(\frac{\log\left(\bar{\omega}_{t+1}\right)+\frac{\sigma^2}{2}}{\sigma}-\sigma\right) \\
&= \text{normcdf}\left(\frac{\log\left(\bar{\omega}_{t+1}\right)-\frac{\sigma^2}{2}}{\sigma}\right) & (18)
\end{aligned}
$$

We just set $x = \frac{\log(\omega)-\mu}{\sigma}$ in Eq.(15) to get Eq.(16). Then $\omega = e^{x\sigma+\mu}$ and $d\omega = \sigma e^{x\sigma+\mu}dx$. By noticing the the PDF of normal with standard devation $\sigma > 0$ and mean μ

$$p\left(x\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

or standard normal PDF

$$\phi\left(x\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

we have

$$\frac{dF_t\left(\omega\right)}{d\omega} = \frac{1}{\omega\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\log(\omega)-\mu}{\sigma}\right)^2}$$

and

$$\frac{dG_t\left(\omega\right)}{d\omega} = \omega\frac{dF_t\left(\omega\right)}{d\omega} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\log(\omega)-\mu}{\sigma}\right)^2}$$

We summarize the related Matlab functions in Table 1.

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Table 1: Some Related Matlab Functions

Function name	Explanation
lognpdf(x, μ, σ)	PDF of lognormal
logncdf(x, μ, σ)	CDF of lognormal
normpdf(x, μ, σ)	PDF of normal
normcdf(x, μ, σ)	CDF of mormal

2.3 Simple Experiments

In order to get more intuition for lognormal distribution, you may do the following experiments in Matlab[20]:

- First, using built-in functions *logncdf* [21]and *lognpdf* [22]to plot the probability density function and cumulative probability function respectively.

- Second, you may draw samples from lognormal distribution to verify its mean and variance with associated normal distribution with built-in function *lognrnd*.

```
%plot of cdf and pdf of lognormal
omega=0:0.05:5;
y=lognpdf(omega,-0.5,1); %mu=-0.5;sigma=1;
plot(omega,y);
y2=logncdf(omega,-0.5,1);
```

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[20]I am using Matlab2012b.

[21]P = logncdf(X,MU,SIGMA) returns values at X of the lognormal cdf with distribution parameters MU and SIGMA. MU and SIGMA are the mean and standard deviation, respectively, of the associated normal distribution. Default values for MU and SIGMA are 0 and 1, respectively. At default, the expectation is $E(\omega) = e^{\mu+\frac{\sigma^2}{2}} = e^{\frac{1}{2}} \approx 1.6487$, the variance $Var(\omega) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) = e(e-1) \approx 4.6708$. If you want $E(\omega) = 1$ and $\sigma = 1$, then it must be the case $\mu = -\frac{1}{2}$. The variance at this time is $Var(\omega) = e - 1 \approx 1.718281828459$.

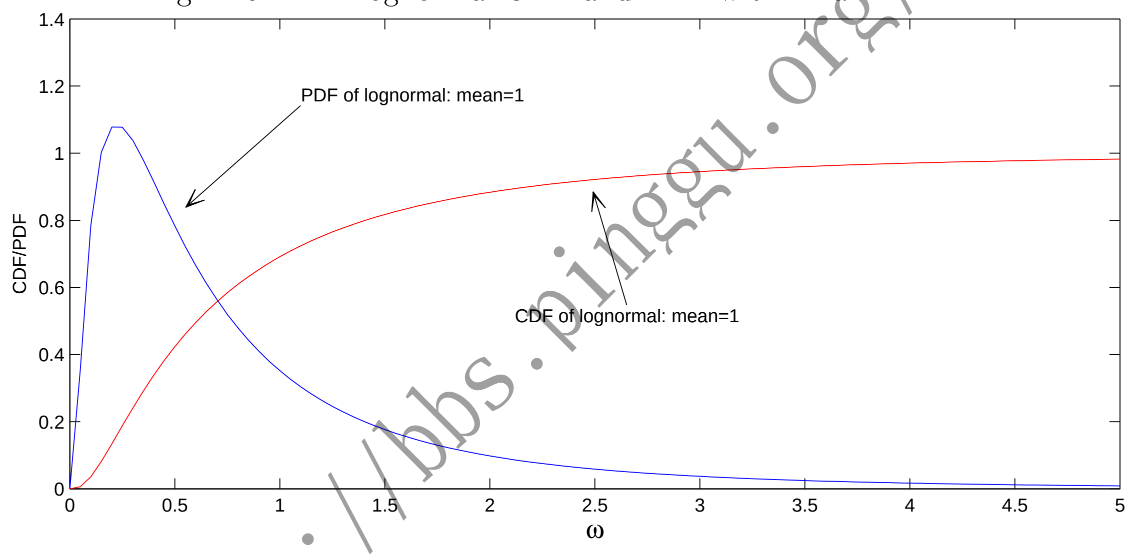
[22]lognpdf: Lognormal probability density function (pdf).

Y = lognpdf(X,MU,SIGMA) returns values at X of the lognormal pdf with distribution parameters MU and SIGMA. MU and SIGMA are the mean and standard deviation, respectively, of the associated normal distribution. The size of Y is the common size of the input arguments. A scalar input functions as a constant matrix of the same size as the other inputs. Default values for MU and SIGMA are 0 and 1 respectively.

```
plot(omega,y2);

%draw sample from lognormal
omega=lognrnd(-0.5,1,1e6,1);%one million samples
mean(omega);%should be close to unit.
std(omega);
var(omega); %should be close to e-1, which is about 1.718
```

Figure 9: The Lognormal CDF and PDF with Mean=1



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