Macroeconomics Summer School
Part II: Advanced Tools
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Wednesday Assignment
Solving and simulating an Aiyagari (1994) model

For this problem set, you are asked to solve the Aiyagari (1994) model of incomplete markets. Given an interest rate, r, and a wage rate w, the households' problem is given by the Bellman equation

$$v(k,\varepsilon) = \max_{c,k'}\{u(c) + \beta \sum_{\varepsilon' \in \{0,1\}} v(q',\varepsilon')Pr(\varepsilon'|\varepsilon)\} \tag{1}$$

$$\text{subject to} \qquad c + k' = k(1+r) + \varepsilon w(1-\tau) + (1-\varepsilon)w\mu \tag{2}$$

$$a' \geq \phi \tag{3}$$

where k denotes capital holdings, μ is the replacement rate, τ is the tax rate, and $\varepsilon \in \{0,1\}$ denotes an agent's employment status. The rest is standard.

The first order condition associated with the above problem is given by

$$\begin{aligned} &u'(k(1+r) + \varepsilon w(1-\tau) + (1-\varepsilon)w\mu - g(k,\varepsilon)) \\ &= \beta(1+r) \sum_{\varepsilon' \in \{0,1\}} u'(g(k,\varepsilon)(1+r) + \varepsilon' w(1-\tau) + (1-\varepsilon')w\mu - g(g(k,\varepsilon),\varepsilon'))Pr(\varepsilon'|\varepsilon) \end{aligned} \tag{4}$$

where $k' = g(k,\varepsilon)$ denotes the policy function for capital.

Let $\psi(k,\varepsilon)$ denote the stationary probability density function for capital and employment status. Aggregate capital is then given by

$$K = \int_{k,\varepsilon} k\psi(k,\varepsilon)dkd\varepsilon \tag{5}$$

The firm's problem is given by

$$\max_{k,\ell}\{k^\alpha \ell^{1-\alpha} - w\ell - (r+\delta)k\}$$

Thus, capital and labour demand are given by

$$r = \alpha\left(\frac{k}{\ell}\right)^{\alpha-1} - \delta, \qquad w = (1-\alpha)\left(\frac{k}{\ell}\right)^{\alpha} \tag{6}$$

A stationary competitive equilibrium is then prices w and r such that,

(i) $g(k,\varepsilon)$ solves the households' problem in (1)-(3).

(ii) The stationary distribution satisfies

$$\psi(k',\varepsilon') = \sum_{\varepsilon\in\{0,1\}} \int_{\{k:k'=g(k,\varepsilon)\}} k\psi(k,\varepsilon)dkPr(\varepsilon',\varepsilon)$$

(iii) The government's budget balances: $\tau = u/(1-u)\mu$, where u denotes the stationary unemployment rate.

(iv) Markets clear,

$$r = \alpha\left(\frac{K}{1-u}\right)^{\alpha-1} - \delta, \qquad w = (1-\alpha)\left(\frac{K}{1-u}\right)^{\alpha}$$

Attached with this problem set are four files: aiyagari.m, newton.m, and transition.m. The first provides the parameterization (together with the transition matrix), sets up the grid for capital, provides some functions that will be used as arguments in newton.m, and then solves the households' problem for given price vector (r, w). The second, newton.m, takes policy functions as inputs, and provides new policy functions as outputs. This is used for the time iteration algorithm to solve the households' problem, and you will need to use it in a similar way in a later stage of the assignment. The third and last file, transition.m, accepts policy functions as inputs and provides an associated transition matrix M as output. The stationary distribution can then be found as the eigenvector of M associated with a unit eigenvalue normalized to sum to one.

Your *first task* is going to be to solve for the stationary competitive equilibrium. In particular,

(i) Guess for an interest rate, r_1. A good guess is $r_1 = 1/\beta - 1 - 1e(-6)$. The wage rate is then given by

$$w_1 = (1-\alpha)\left(\frac{r_1+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

(ii) Use newton.m together with time iteration to solve the households' problem.

(iii) Now, given your policy functions use distribution.m to find the associated stationary distribution. This will be an $N \times 2$ matrix.

(iv) Calculate aggregate capital, K_1, and the associated implied interest rate:

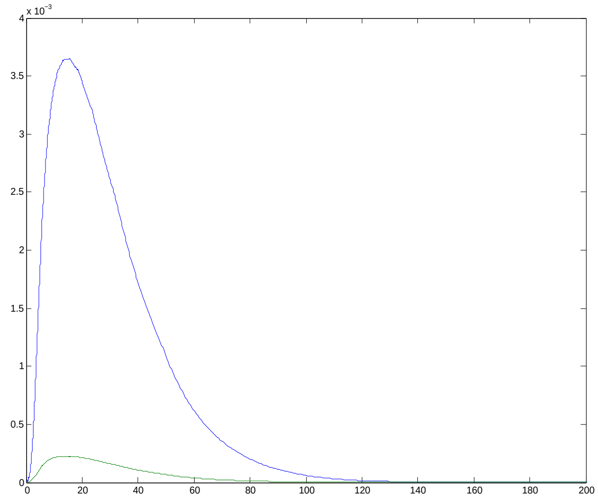
$$r_{implied} = \alpha\left(\frac{K_1}{1-u}\right)^{\alpha-1} - \delta$$

- Notice that these first steps are already coded for you in aiyagari.m. Thus if you run the code, you will receive the K_1 implied by the guess r_1.

(v) If $r_{implied}$ is "close to" r, then stop. Otherwise update your guess, r_2, and start over. It's important to make sure that the difference between $r_{implied}$ and r_n is very small once you have convergence.

(vi) When the program has converged, plot the stationary distribution.

It should look something like this with an equilibrium interest rate of $r = 0.0122458$.[1]



Your *second task* is going to be to calculate the transition of the competitive equilibrium when we expose it to a temporary technology shock. That is, the firm's problem is now

$$\max_{k_t, \ell_t} \{exp(z_t) k_t^{\alpha} \ell_t^{1-\alpha} - w_t \ell_t - (r_t + \delta) k_t\}$$

where z_t represents total factor productivity in period t and follows the law of motion $z_{t+1} = \rho z_t$. Factor demand are given as

$$r_t = exp(z_t)\alpha \left(\frac{k_t}{\ell_t}\right)^{\alpha-1} - \delta, \qquad w_t = exp(z_t)(1-\alpha)\left(\frac{k_t}{\ell_t}\right)^{\alpha} \tag{7}$$

The households' problem is summarized by the Euler equations

$$\begin{aligned} &u'(k(1+r_t) + \varepsilon w_t(1-\tau) + (1-\varepsilon) w_t \mu - g_t(k,\varepsilon)) = \beta(1 + r_{t+1}) \\ &\times \sum_{\varepsilon' \in \{0,1\}} u'(g_t(k,\varepsilon)(1+r_{t+1}) + \varepsilon' w_{t+1}(1-\tau) + (1-\varepsilon') w_{t+1}\mu - g_{t+1}(g_t(k,\varepsilon),\varepsilon')) Pr(\varepsilon'|\varepsilon) \end{aligned}$$

Assume that the economy is in the steady state in period t, with $z_t = 0$. Then unexpectedly, z_t falls to -0.01. In order to calculate the ensuing transition (or impulse response) of the economy, you should follow the following steps:

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[1] My distribution may look prettier than the one you got as I used 800 nodes to generate it.

(i) Set a time period in which the economy is assumed to have converged back to it's stationary distribution. In particular, the economy is assumed to have converged in period $t + T$ with $T = 300$. Thus $z_{t+T} = \rho^T \times (-0.01)$, which is virtually zero.

(ii) Guess for a sequence of aggregate capital from period t to $t + T + 1$, $\{K^1_{t+s}\}_{s=0}^{T+1}$, with $K^1_{t+T+1} = K$; the steady state level of capital. As an initial guess $K^1_{t+s} = K$ for all s is not bad.

(iii) Given your guess and the law of motion for z_t calculate $\{w^1_{t+s}\}_{s=0}^{T+1}$ and $\{r^1_{t+s}\}_{s=0}^{T+1}$. Notice that $w_{t+T+s} = w$ and $r_{t+T+1} = r$.

(iv) Now, given that you "know" that the economy has converged back to its steady state in period $t + T + 1$, we also know the households' policy functions at that date. We can use these to solve the household's problem backwards: That is, solve for the policy functions in period $t + T$ using the steady state policy functions in period $t + T + 1$; for the policy functions in period $t + T - 1$ using the policy functions in period $t + T$, and so on. The program newton.m will come in handy here. Store these policy functions.

(v) Now, given the initial distribution $\psi_t(k, \varepsilon) = \psi(k, \varepsilon)$ and an initial policy function $g_t(k, z)$, use transition.m in order to find $\psi_{t+1}(k, \varepsilon)$. Update aggregate capital in period $t+1$ according to this distribution. Keep repeating this for each time period until $t + T$.

(vi) Update your guess as $\{K^2_{t+s}\}_{s=0}^{T} = 0.1 \times \{K_{t+s}\}_{s=0}^{T} + (1 - 0.1) \times \{K^1_{t+s}\}_{s=0}^{T}$, and set $K^2_{t+T+1} = K$.

(vii) Repeat until $\|\{K^{n+1}_{t+s}\}_{s=0}^{T+1} - \{K^n_{t+s}\}_{s=0}^{T+1}\| < 1e(-6)$.

Hopefully, your solution will look like mine:

Impulse Response Functions