Advanced Tools in Macroeconomics

Continuous time models (and methods)

Pontus Rendahl

2023

The Ramsey growth model: Euler equation

- Let's go back to the HJB equation.

$$\rho v(k) = u(c) + v'(k)(k^\alpha - \delta k - c)$$
$$\text{with} \quad u'(c) = v'(k)$$

- Thus

$$\rho v'(k) = v''(k)(k^\alpha - \delta k - c) + v'(k)(\alpha k^{\alpha-1} - \delta)$$

- And

$$v''(k) = u''(c)c'(k)$$

The Ramsey growth model: Euler equation

- Using

$$\rho v'(k) = v''(k)(k^\alpha - \delta k - c) + v'(k)(\alpha k^{\alpha-1} - \delta)$$

Together with $v'(k) = u'(c)$ and $v''(k) = u''(c)c'(k)$ gives

$$\rho u'(c) = u''(c)c'(k)(k^\alpha - \delta k - c) + u'(c)(\alpha k^{\alpha-1} - \delta)$$

or

$$-u''(c)c'(k)(k^\alpha - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho)$$

- Suppose CRRA utility, such that $\frac{u''(c)c}{u'(c)} = -\gamma$

The Ramsey growth model: Euler equation

- Then the last equation

$$-u''(c)c'(k)(k^\alpha - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho)$$

  is equal to

$$\gamma \frac{c'(k)}{c}(k^\alpha - \delta k - c) = (\alpha k^{\alpha-1} - \delta - \rho)$$

- This is the Euler equation in continuous time.

The Ramsey growth model: Euler equation

Before we attempt to solve the Euler equation, recall that we had

$$k_{t+\Delta} + \Delta c_t = \Delta k_t^{\alpha} + (1 - \Delta\delta) k_t$$

rearrange

$$k_{t+\Delta} - k_t = \Delta(k_t^{\alpha} - \delta k_t - c_t)$$

Divide with Δ and take limit $\Delta \to 0$ to get

$$\dot{k}_t = k_t^{\alpha} - \delta k_t - c_t$$

Or dropping time notation

$$\dot{k} = k^{\alpha} - \delta k - c$$

The Ramsey growth model: Euler equation

- Our Euler equation is

$$\frac{c'(k)}{c}(k^\alpha - \delta k - c) = \frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$

  or now

$$\gamma \frac{c'(k)}{c}\dot{k} = (\alpha k^{\alpha-1} - \delta - \rho)$$

- What is $c'(k)\dot{k}$? Recall chain rule

$$\dot{c} = \frac{\partial c_t}{\partial t} = \frac{\partial c_t}{\partial k}\frac{\partial k}{\partial t} = c'(k)\dot{k}$$

- Thus

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$

The Ramsey growth model: Exploring

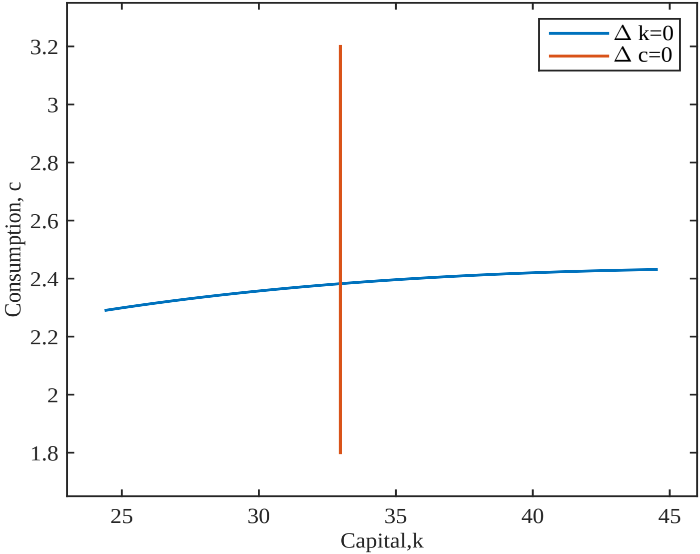
- Two equations

$$\dot{c} = \frac{c}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$
$$\dot{k} = k^{\alpha} - \delta k - c$$

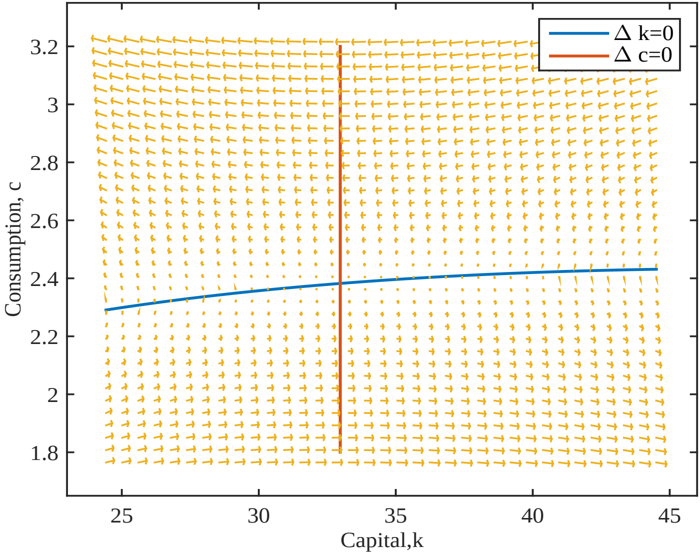
- Nullclines

$$0 = \alpha k^{\alpha-1} - \delta - \rho$$
$$0 = k^{\alpha} - \delta k - c$$

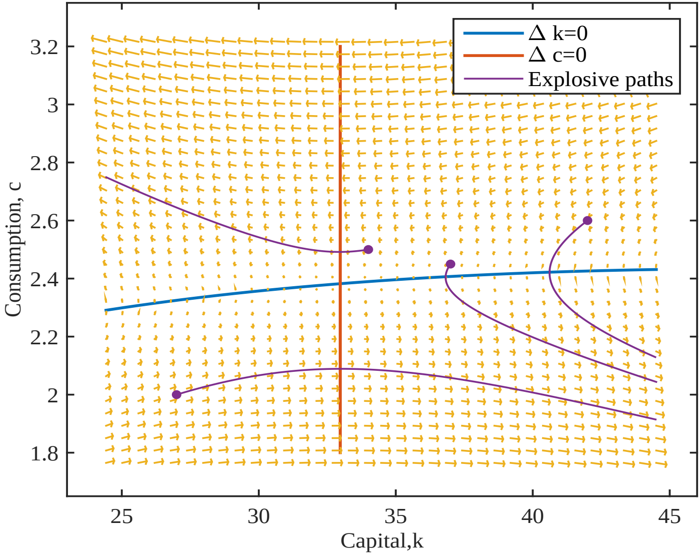
The Ramsey growth model: Exploring



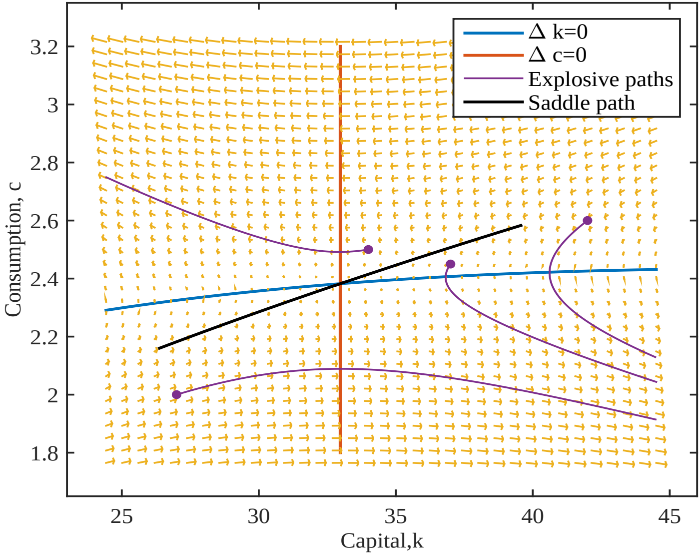
The Ramsey growth model: Exploring



The Ramsey growth model: Exploring



The Ramsey growth model: Exploring



The Ramsey growth model: Exploring

- How did I do that?
- I created a grid for k and c, and found $\dot{c}$ and $\dot{k}$ through

$$
\begin{aligned}
\dot{c} &= \frac{c}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) \\
\dot{k} &= k^{\alpha} - \delta k - c
\end{aligned}
$$

- Then I used Matlab's command `quiver(k,c,`$\dot{k}$`,`$\dot{c}$`)`
  - This creates the swarm of arrows
- I then used Matlab's command `streamline(k,c,`$\dot{k}$`,`$\dot{c}$`)` at various starting values to get the explosive paths.
- Lastly I solved for the saddle path and plotted it.

The Ramsey growth model: Euler equation solution

- Back to the "recursive" Euler

$$\frac{c'(k)}{c}(k^{\alpha} - \delta k - c) = \frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$

- Solve for c

$$c = \frac{c'(k)(k^{\alpha} - \delta k)}{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) + c'(k)}$$

The Ramsey growth model: Euler equation solution

Algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of c_0.
3. Calculate the derivative as $\mathtt{dc}_0\mathtt{=D*c}_0$.
4. Find c_1 from

$$c_1 = \frac{dc_0(k^\alpha - \delta k)}{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) + dc_0}$$

5. Back to step 3 with c_1 replacing c_0. Repeat until convergence.

The Ramsey growth model: Euler equation solution

Algorithm

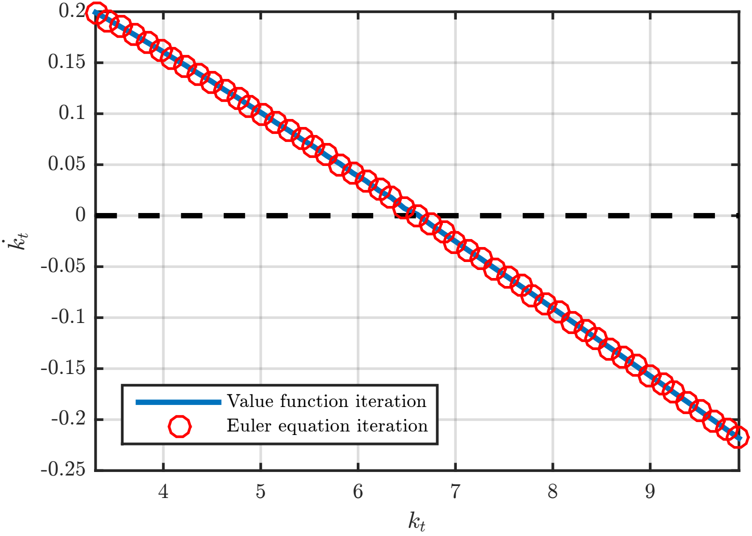
1. Construct a grid for k.
2. For each point on the grid, guess for a value of `c0`.
3. Calculate the derivative as `dc0=D*c0`.
4. Find `c1` from

$$c_1 = \frac{dc_0(k^\alpha - \delta k)}{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) + dc_0}$$

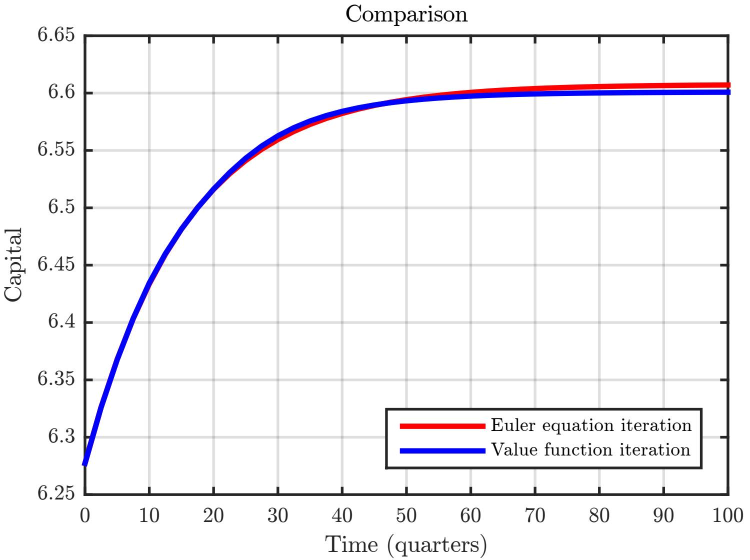
5. Back to step 3 with `c1` replacing `c0`. Repeat until convergence.

Beware: No guaranteed convergence. Update slowly. Fewer gridpoints appears to provide some stability.

The Ramsey growth model: Solution



The Ramsey growth model: Solution



More on continuous time Euler equations

- We derived the Euler equation in a slightly roundabout way
  1. Discrete time Bellman equation
  2. To continuous time HJB equation
  3. To continuous time Euler equation using the envelope condition.
- This can be done more directly from the discrete time Euler equation.

More on continuous time Euler equations

- The discrete time Euler equation is given by

$$u'(c_t) = (1-\rho)(1+\alpha k_{t+1}^{\alpha-1} - \delta)u'(c_{t+1})$$

- In Δ units of time

$$u'(c_t) = (1-\Delta\rho)(1+\Delta(\alpha k_{t+\Delta}^{\alpha-1} - \delta))u'(c_{t+\Delta})$$

- Use the approximation $u'(c_{t+\Delta}) \approx u'(c_t) + u''(c_t)\dot{c}_t\Delta$ to get

$$u'(c_t) = (1-\Delta\rho)(1+\Delta(\alpha k_{t+\Delta}^{\alpha-1} - \delta))(u'(c_t) + u''(c_t)\dot{c}_t\Delta)$$

More on continuous time Euler equations

$$u'(c_t) = (1 - \Delta\rho)(1 + \Delta(\alpha k_{t+\Delta}^{\alpha-1} - \delta))(u'(c_t) + u''(c_t)\dot{c}_t\Delta)$$

- Move the $u'(c_t + \dot{c}_t\Delta)$ term to the left-hand side and expand

$$\begin{aligned} & - u''(c_t)\dot{c}_t\Delta \\ = {} & \Delta[\alpha k_{t+\Delta}^{\alpha-1} - \delta - \rho - \rho\Delta(\alpha k_{t+\Delta}^{\alpha-1} - \delta)](u'(c_t) + u''(c_t)\dot{c}_t\Delta) \end{aligned}$$

- Divide by Δ and take limits $\Delta \to 0$

$$-u''(c_t)\dot{c}_t = [\alpha k_t^{\alpha-1} - \delta - \rho]u'(c_t)$$

More on continuous time Euler equations

$$-u''(c_t)\dot{c}_t = [\alpha k_t^{\alpha-1} - \delta - \rho]u'(c_t)$$

- Lastly, use the CRRA property to get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}[\alpha k_t^{\alpha-1} - \delta - \rho]$$

More on continuous time Euler equations

- Now consider a stochastic model with a "good", g, and a "bad", b, state

$$u'(c_t^g) = (1-\rho)[(1-p)(1+z_{t+1}^g\alpha(k_{t+1}^g)^{\alpha-1}-\delta)u'(c_{t+1}^g) \\ + p(1+z_{t+1}^b\alpha(k_{t+1}^b)^{\alpha-1}-\delta)u'(c_{t+1}^b)]$$

and

$$u'(c_t^b) = (1-\rho)[(1-q)(1+z_{t+1}^g\alpha(k_{t+1}^g)^{\alpha-1}-\delta)u'(c_{t+1}^g) \\ + q(1+z_{t+1}^b\alpha(k_{t+1}^b)^{\alpha-1}-\delta)u'(c_{t+1}^b)]$$

- We will focus on the good state (the treatment of the bad state is symmetric)

More on continuous time Euler equations

- Good state Euler equation in Δ units of time

$$u'(c_t^g) = (1-\Delta\rho)[(1-\Delta p)(1+\Delta(z_{t+\Delta}^g \alpha(k_{t+\Delta}^g)^{\alpha-1} - \delta))$$
$$\times u'(c_{t+\Delta}^g) + \Delta p(1+\Delta(z_{t+\Delta}^b \alpha(k_{t+\Delta}^b)^{\alpha-1} - \delta))u'(c_{t+\Delta}^b)]$$

- Use $u'(c_{t+\Delta}^g) \approx u'(c_t^g) + u''(c_t^g)\dot{c}_t^g \Delta$ again, move to the left-hand side, divide by Δ and take limits

$$\begin{aligned} -u''(c_t^g)\dot{c}_t^g &= (z_t^g \alpha(k_t^g)^{\alpha-1} - \delta - \rho))u'(c_t^g) \\ &\quad + p(u'(c_t^b) - u'(c_t^g)) \end{aligned}$$

- Or

$$\frac{\dot{c}_t^g}{c_t^g} = \frac{1}{\gamma}(z_t^g \alpha(k_t^g)^{\alpha-1} - \delta - \rho)) + p\frac{(u'(c_t^b) - u'(c_t^g))}{u'(c_t^g)}$$

More on continuous time Euler equations

- For the bad state

$$\frac{\dot{c}_t^b}{c_t^b} = \frac{1}{\gamma}(z_t^b \alpha (k_t^b)^{\alpha-1} - \delta - \rho)) + q\frac{(u'(c_t^b) - u'(c_t^g))}{u'(c_t^b)}$$

- These can be solved using the previous methods. The only difference is that we now iterate on two equations instead of one. But the procedure is the same.

More on continuous time Euler equations

- As a last step, I just want to give you a hint on how these ideas can be applied in different settings.
- For instance, the Euler equation for a standard deterministic monetary model is given by

$$u'(c_t) = (1-\rho)(1+i_{t+1})\frac{p_t}{p_{t+1}}u'(c_{t+1})$$

- In Δ units of time

$$u'(c_t) = (1-\Delta\rho)(1+\Delta i_{t+1})\frac{p_t}{p_{t+\Delta}}u'(c_{t+\Delta})$$

More on continuous time Euler equations

- Use the approximations $u'(c_{t+\Delta}) \approx u'(c_t) + u''(c_t)\dot{c}_t\Delta$, and $p_t \approx p_{t+\Delta} - \dot{p}_t\Delta$ and rewrite

$$u'(c_t) = (1-\Delta\rho)(1+\Delta i_{t+\Delta})\frac{p_{t+\Delta} - \dot{p}_t\Delta}{p_{t+\Delta}}(u'(c_t) + u''(c_t)\dot{c}_t\Delta)$$

- Expand

$$u'(c_t) = (1-\Delta\rho + \Delta i_{t+\Delta} - \Delta^2 i_{t+\Delta}\rho)(1 - \frac{\dot{p}_t\Delta}{p_{t+\Delta}})(u'(c_t) + u''(c_t)\dot{c}_t\Delta)$$

- Thus

$$-u''(c_t)\dot{c}_t\Delta = (-\Delta\rho + \Delta i_{t+\Delta} - \Delta^2 i_{t+\Delta}\rho)$$
$$\times(1-\frac{\dot{p}_t\Delta}{p_{t+\Delta}})(u'(c_t)+u''(c_t)\dot{c}_t\Delta) - \frac{\dot{p}_t\Delta}{p_{t+\Delta}}(u'(c_t)+u''(c_t)\dot{c}_t\Delta)$$

More on continuous time Euler equations

- Previous equation

$$
\begin{gathered}
-u''(c_t)\dot{c}_t\Delta = (-\Delta\rho + \Delta i_{t+\Delta} - \Delta^2 i_{t+\Delta}\rho) \\
\times(1-\frac{\dot{p}_t\Delta}{p_{t+\Delta}})(u'(c_t)+u''(c_t)\dot{c}_t\Delta)-\frac{\dot{p}_t\Delta}{p_{t+\Delta}}(u'(c_t)+u''(c_t)\dot{c}_t\Delta)
\end{gathered}
$$

- Divide by Δ

$$
\begin{gathered}
-u''(c_t)\dot{c}_t = (-\rho + i_{t+\Delta} - \Delta i_{t+\Delta}\rho) \\
\times(1-\frac{\dot{p}_t\Delta}{p_{t+\Delta}})(u'(c_t)+u''(c_t)\dot{c}_t\Delta)-\frac{\dot{p}_t}{p_{t+\Delta}}(u'(c_t)+u''(c_t)\dot{c}_t\Delta)
\end{gathered}
$$

- And take limit $\Delta \to 0$

$$
\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}(i_t - \frac{\dot{p}_t}{p_t} - \rho)
$$