The Cyclical Behavior of Equilibrium Unemployment and Vacancies
Shimer (2005)

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Bellman Equation

Worker

- Unempolyment value

$$U_p = z + \delta\{f(\theta_p)\mathbb{E}_p W_{p'} + (1 - f(\theta_p))\mathbb{E}_p U_{p'}\} \tag{1}$$

- Employment value

$$W_p = w_p + \delta\{(1 - s)\mathbb{E}_p W_{p'} + s\mathbb{E}_p U_{p'}\} \tag{2}$$

Bellman Equation
Firm

- Hiring value

$$J_p = p - w_p + \delta(1 - s)\mathbb{E}_p J_{p'} \tag{3}$$

- Vacancy value

$$V_p = -c + \delta q(\theta_p)\mathbb{E}_p J_{p'} \equiv 0 \tag{4}$$

Productivity

The log of productivity follows AR(1) process

$$\log(p) = \rho \log(p) + \varepsilon \tag{5}$$

where

$$\log(p) \sim N(\mu_\lambda, \sigma_\lambda^2),\ \varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$$

Optimal Control

Market tightness

- Control in this problem consists of w_p, θ_p, u_p and the state is p
- Market tightness θ_p is given by solving the following equation of hire rate from free entry condition

$$q(\theta_p) = \frac{c}{\delta \mathbb{E}_p J_{p'}} \tag{6}$$

- And market tightness

$$\theta_p = (\frac{q(\theta_p)}{\mu})^{-\frac{1}{\eta}} \tag{7}$$

- Employ Rate is given by

$$f(\theta_p) = \mu^{\frac{1}{\eta}} q^{\frac{\eta - 1}{\eta}} \tag{8}$$

Optimal Control
Continued

- Optimal wage at each productivity level is given by the Nash Bargaining:

$$W_p - U_p = \beta(W_p - U_p + J_p) \tag{9}$$

- Note Bellman Equation of W_p given by 2, U_p given by 1, J_p given by 3
- Following the algebra given in slide 13, optimal wage for each p is

$$w_p = \beta p + (1-\beta)z + \beta c\theta_p \tag{10}$$

- And unemployment rate

$$u_p = \frac{\delta}{\delta + f(\theta_p)} \tag{11}$$

Question a I

Descretization Algorithm

Inspired by Karen A. Kopecky 2006 Lecture Note

1. Choose a relateive error tolerance level `tol`;
2. Discretize the state space by constructing a grid for productivity

   $$p = \exp\{logp\} \text{ where } logp = \{logp_1, logp_2, \ldots, logp_n\}$$

   given by the Tauchen-Hussey (1991) method. The n is chosen at 100;
3. Start with an initial guess of the value function $V^{(0)}(p)$ is a vector of length n, i.e., $V^{(0)}(p) = \{V_i^{(0)}\}_{i=1}^n$, where $V_i^{(0)} = V^{(0)}(p_i)$. V here represents U, W, J. The initial guess is ones.

Question a II

Descretization Algorithm

4. Update the value function using eqautions 1 to 10, specifically
   4.1 Fix the current productivity level at one of the grid points, p_i from $i = 1$
   4.2 For each possible choice of productivity next period, calculate optimal control in the following order:

$$q(\theta_{p_i}) = \frac{c}{\delta \sum_{j=1}^{n} p_{i,j} J^{(0)}(p_j)}$$

$$f(\theta_{p_i}) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$

$$\theta_{p_i} = (\frac{q(\theta_{p_i})}{\mu})^{-\frac{1}{\eta}}$$

$$w_{p_i} = \beta p_i + (1-\beta)z + \beta c \theta_{p_i}$$

   4.3 and update the value function system with

Question a III

Descretization Algorithm

$$U^{(1)}_{p_i} = z + \delta\{f(\theta_{p_i})\sum_{j=1}^{n} p_{i,j} W^{(0)}(p_j) + (1 - f(\theta_{p_i}))\sum_{j=1}^{n} p_{i,j} U^{(0)}(p_j)\}$$

$$W^{(1)}_{p_i} = w_{p_i} + \delta\{(1 - s)\sum_{j=1}^{n} p_{i,j} W^{(0)}(p_j) + s\sum_{j=1}^{n} p_{i,j} U^{(0)}(p_j)\}$$

$$J^{(1)}_{p_i} = p_i - w_{p_i} + \delta(1 - s)\sum_{j=1}^{n} p_{i,j} J^{(0)}(p_j)$$

4.4 Choose a new grid point for productivity, go through 4.1 to 4.3. Once we have done the update for all productivity grid, we have new system of value function $V^{(1)}_p$

4.5 Compute distance between the two systems of value functions following the sup norm

$$d = \max_{i \in \{1,\ldots,n\}} |V^{(0)}_i - V^{(1)}_i|$$

Question a IV

Descretization Algorithm

4.6 If distance is within the error tolerance level, $d \leq tol * ||V_1^{(1)}||$, the functions have converged and go to step 5, or else go back to step 4.

5. Calculate the optimal control for each productivity level:

$$q(\theta^*_{p_i}) = \frac{c}{\delta \sum_{j=1}^{n} p_{i,j} J^*(p_j)}$$

$$f(\theta^*_{p_i}) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$

$$\theta^*_{p_i} = \left(\frac{q(\theta^*_{p_i})}{\mu}\right)^{-\frac{1}{\eta}}$$

$$w^*_{p_i} = \beta p_i + (1-\beta)z + \beta c \theta^*_{p_i}$$

$$u^*_p = \frac{\delta}{\delta + f(\theta^*_p)}$$

where J^* is the converged value function.

The Hermite roots and weights are produced with van Damme (2023) and checked with Salzer et al. (1952). The results comes with 4 decimals.

Calibration

Appendix A

Optimal wage

$$
\begin{aligned}
W_p - U_p &= \beta(W_p - U_p + J_p) \\
\Leftrightarrow w_p - z + \delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) &= \\
\beta(p - z + \delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) &+ \delta(1 - s)\mathbb{E}_p J_{p'}) \\
\Leftrightarrow w_p = \beta p + (1 - \beta)z + (\beta - 1)\delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) \\
+ \frac{\beta c(1 - s)}{q(\theta_p)} \\
\Leftrightarrow w_p = \beta p + (1 - \beta)z - \frac{\beta c \delta(1 - s - f(\theta_p))}{q(\theta_p)} + \frac{\beta c(1 - s)}{q(\theta_p)} \\
\Leftrightarrow w_p = \beta p + (1 - \beta)z + \beta c \theta_p
\end{aligned}
$$

where we use the fact that $\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'} = \frac{\beta}{1-\beta}\mathbb{E}_p J_{p'}$ and $f(\theta_p)/q(\theta_p) = \theta_p$

Reference I

Salzer, H. E., Zucker, R. & Capuano, R. (1952), 'Table of the zeros and weight factors of the first 20 hermite polynomials', *Journal of research of the National Bureau of Standards* **48**, 111.
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