ECON6096 Final Report
Shimer (2005)

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December 20, 2023

1 Model

1.1 Model Setup

In the adapted model of Shimer (2005), workers and firms are matched through a matching function.

On the unemployed worker side, the arrival rate of jobs is

$$f(\theta) = \mu\theta^\eta$$

On the firm side, the recruiting rate is

$$q(\theta) = \mu\theta^{\eta-1}$$

where θ is the market tightness, defined as $\theta = v/u$, where v is the number of vacancies and u number of unemployed workers.

The worker's Bellman equation for unemployment value reads

$$U_p = z + \delta\{f(\theta_p)\mathbb{E}_p W_{p'} + (1 - f(\theta_p))\mathbb{E}_p U_{p'}\}. \tag{1}$$

Their employment value is given by

$$W_p = w_p + \delta\{(1 - s)\mathbb{E}_p W_{p'} + s\mathbb{E}_p U_{p'}\}. \tag{2}$$

Firm's value from hiring is

$$J_p = p - w_p + \delta(1 - s)\mathbb{E}_p J_{p'}. \tag{3}$$

Its value for keeping a vacancy is

$$V_p = -c + \delta q(\theta_p)\mathbb{E}_p J_{p'}. \tag{4}$$

Note that $V_p \equiv 0$ is the free entry condition in the equilibrium.

Productivity p is the output of a firm-worker pair, where worker gets w and firm $p - w$. Chance that a worker becomes unemployed is s.

The log of productivity follows AR(1) process

$$\log(p') = \rho\log(p) + \varepsilon \tag{5}$$

where

$$\log(p) \sim N(\mu_{logp}, \sigma_{logp}^2), \ \varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$$

The bargaining process between workers and firms is

$$W_p = U_p + \beta(W_p - U_p + J_p) \tag{6}$$

which pins down the equilibrium wage w.

In the equilibrium, workers' inflow equals outflow. So unemployment u follows

$$uf(\theta_p) = s(1-u)$$

1.2   Equilibrium

Market tightness θ_p is pinned down by solving the following equation of hire rate from free entry condition

$$q(\theta_p) = \frac{c}{\delta\mathbb{E}_p J_{p'}} \tag{7}$$

Employ Rate is given by

$$f(\theta_p) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}} \tag{8}$$

And market tightness

$$\theta_p = \frac{f(\theta_p)}{q(\theta_p)} \tag{9}$$

Optimal wage at each productivity level is given by the Nash Bargaining in the algebra in Appendix A:

$$w_p = \beta p + (1 - \beta)z + \beta c\theta_p \tag{10}$$

And unemployment rate

$$u_p = \frac{s}{s + f(\theta_p)} \tag{11}$$

Definition 1. *Equilibrium* *The definition of equilibrium consists of worker's wage w_p, job arrival rate $f(\theta_p)$, recruiting rate $q(\theta_p)$ market tightness θ_p and unemployment rate u_p under productivity level p which satisfies the process by equation 5 such that*

1. *Recruiting rate satisfies equation 7.*

2. *Job arrival rate satisfies equation 8.*

3. *Market tightness satisfies equation 9.*

4. *The wage of workers satisfies equation 10.*

5. *Unemployment rate satisfies equation 11.*

2 Algorithm

2.1 Tauchen method

Tauchen (1986) provides a reliable method to simulate the conditional expectations $\mathbb{E}_p J_{p'}$, $\mathbb{E}_p U_{p'}, \mathbb{E}_p W_{p'}$ in the value functions. For AR(1) process 5, taking F as representative value function, instead of computing the integral $\mathbb{E}[F(p'|p)] = \int_{-\infty}^{\infty} F(p')f(p'|p)dp'$, we can take a finite set $\Lambda_{logp} = \{logp_1, \ldots, logp_n\}$ and derive a probability transition matrix where $p_{i,j} = \mathbf{Prob}(\tilde{p}' = p_j | \tilde{p} = p_i)$ based on the distribution of stochastic part of the process, so that the conditional expectation can be solved numerically with formula

$$\mathbb{E}[F(p'|p)] = \sum_{j=1}^{n} p_{i,j} F(p').$$

We know that $logp'$ is normally distributed with mean 0 and standard deviation σ_ε, we choose the finite set Λ_{logp} to be evenly-spaced along the real line, the upper and lower bound of the set to be $\mu_{logp} \pm m\sigma_{logp}$ where $\sigma_{logp} = \sigma_\varepsilon/\sqrt{1-\rho^2}$. Let $\omega_j = logp_j - logp_{j-1}$, set the transition matrix to be

$$p_{ij} = \begin{cases} \Phi\left(\frac{logp_1 + \omega_j/2}{\sigma_\varepsilon}\right) & \text{for } j = 1 \\ \Phi\left(\frac{logp_j + \omega_j/2}{\sigma_\varepsilon}\right) - \Phi\left(\frac{logp_j - \omega_j/2}{\sigma_\varepsilon}\right) & \text{for } 1 < j < n \\ 1 - \Phi\left(\frac{logp_n - \omega_j/2}{\sigma_\varepsilon}\right) & \text{for } j = n \end{cases}$$

where $\Phi(\cdot)$ is the CDF of normal distribution. The grid is set to p with simple transformation $\Lambda_p = \exp(\Lambda_{logp})$.

We directly apply `discretizeAR1_Tauchen` function from the Matlab Toolbox of Kirkby (2023) to realize the above mentioned Tauchen method.

2.2 Discretization

The algorithm of discretization method is as follows:

1. Choose a relateive error tolerance level `tol`;

2. Discretize the state space by constructing a grid for productivity

$$p = \exp\{logp\} \text{ where } logp = \{logp_1, logp_2, \ldots, logp_n\}$$

given by the Tauchen method. The n is chosen at 5; deviation step is 3.

3. Start with an initial guess of the value function $F^{(0)}(p)$ is a vector of length n, i.e., $F^{(0)}(p) = \{F_i^{(0)}\}_{i=1}^n$, where $F_i^{(0)} = F^{(0)}(p_i)$. F here represents U, W, J. The initial guess is `tol`.

4. Update the value function using eqautions 1 to 10, specifically

   (a) Fix the current productivity level at one of the grid points, p_i from $i = 1$

   (b) For each possible choice of productivity next period, calculate optimal control in the

following order:

$$q(\theta_{p_i}) = \frac{c}{\delta \sum_{j=1}^{n} p_{i,j} J^{(0)}(p_j)}$$

$$f(\theta_{p_i}) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$

$$\theta_{p_i} = f/q$$

$$w_{p_i} = \beta p_i + (1-\beta)z + \beta c \theta_{p_i}$$

(c) and update the value function system with Jacobi method

$$U_{p_i}^{(1)} = z + \delta\{f(\theta_{p_i}) \sum_{j=1}^{n} p_{i,j} W^{(0)}(p_j) + (1 - f(\theta_{p_i})) \sum_{j=1}^{n} p_{i,j} U^{(0)}(p_j)\}$$

$$W_{p_i}^{(1)} = w_{p_i} + \delta\{(1-s) \sum_{j=1}^{n} p_{i,j} W^{(0)}(p_j) + s \sum_{j=1}^{n} p_{i,j} U^{(0)}(p_j)\}$$

$$J_{p_i}^{(1)} = p_i - w_{p_i} + \delta(1-s) \sum_{j=1}^{n} p_{i,j} J^{(0)}(p_j)$$

(d) Choose a new grid point for productivity, go through 4.1 to 4.3. Once we have done the update for all productivity grid, we have new system of value function $F_p^{(1)}$

(e) Compute distance between the two systems of value functions following the sup L-1 norm

$$d = \max_{i \in \{1,...,n\}} |F_i^{(0)} - F_i^{(1)}|$$

(f) If distance is within the error tolerance level, $d \leq \texttt{tol}$, the functions have converged and go to step 5, or else go back to step 4.

5. Calculate the optimal control for each productivity level:

$$q(\theta_{p_i}^*) = \frac{c}{\delta \sum_{j=1}^{n} p_{i,j} J^*(p_j)}$$

$$f(\theta_{p_i}^*) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$

$$\theta_{p_i}^* = f^*/q^*$$

$$w_{p_i}^* = \beta p_i + (1-\beta)z + \beta c \theta_{p_i}^*$$

$$u_p^* = \frac{s}{s + f(\theta_p^*)}$$

where J^* is the converged value function.

2.3 Approximation

0. Since we have discovered that the value function is smooth and monotone, we simply approximate value functions with 4th-order polynomials in accordance with the number of

grids $\hat{F}(p; \mathbf{coefs})$ in the form of

$$\hat{F}(p; \mathbf{coefs}) = ap^4 + bp^3 + cp^2 + dp + e$$

to match the number of grids. The initial guess of **coefs** is 70 for U and W and `tol` for J, saved as **coefs_old**. The stopping criterion is `tol`. The approximation grid is derived from Tauchen method as done in Discretization.

1. Calculate the maximized control and value function system with the initial guess with equilibrium condition 7 to 10 as

$$q(\theta_{p_i}) = \frac{c}{\delta \sum_{j=1}^{n} p_{i,j} \hat{J}(p_j; \mathbf{coefs\_old})}$$
$$f(\theta_{p_i}) = \mu^{\frac{1}{\eta}} q^{\frac{\eta-1}{\eta}}$$
$$\theta_{p_i} = f/q$$
$$w_{p_i} = \beta p_i + (1-\beta)z + \beta c \theta_{p_i}$$

and Bellman equations 1 to 3.

$$U_{p_i} = z + \delta\{f(\theta_{p_i}) \sum_{j=1}^{n} p_{i,j} \hat{W}(p_j; \mathbf{W\_coefs\_old}) + (1 - f(\theta_{p_i})) \sum_{j=1}^{n} p_{i,j} \hat{U}(p_j; \mathbf{U\_coefs\_old})\}$$

$$W_{p_i} = w_{p_i} + \delta\{(1-s) \sum_{j=1}^{n} p_{i,j} \hat{W}(p_j; \mathbf{W\_coefs\_old}) + s \sum_{j=1}^{n} p_{i,j} \hat{U}(p_j; \mathbf{U\_coefs\_old})\}$$

$$J_{p_i} = p_i - w_{p_i} + \delta(1-s) \sum_{j=1}^{n} p_{i,j} \hat{J}(p_j; \mathbf{J\_coefs\_old})$$

Given the smooth property of value functions, we use the Gaussian-Hermite quadrature as in Tauchen's method to calculate the conditional expectation.

2. Fit for new value function system and report parameters and save as `new` in an unweighted nonlinear least squares procedure as in

$$\min_{\mathbf{coefs}} \sum_{i=1}^{n} (\hat{F}(p_i; \mathbf{coefs}) - f_i)^2$$

and save as **coefs_new** for each value function. f_i stands for the updated value functions U_{p_i}, W_{p_i} and J_{p_i} in the maximized procedure. We use Matlab code `fminsearch` to realize the fitting procedure.

3. If the largest norm of all value functions statisfies $||\hat{F}(p; \mathbf{coefs\_old}) - \hat{F}(p; \mathbf{coefs\_new})|| <$ `tol`, stop; else go to step 1, replacing **coefs_old** with **coefs_new**. We use Matlab code `norm` to return an Euclidean norm.

3 Quantitative Analysis

3.1 Calibration

Parameters calibrated are given in Table 1. In this first exercise, separation rate s is set to be constant and productivity p a stochastic process with mean 0.

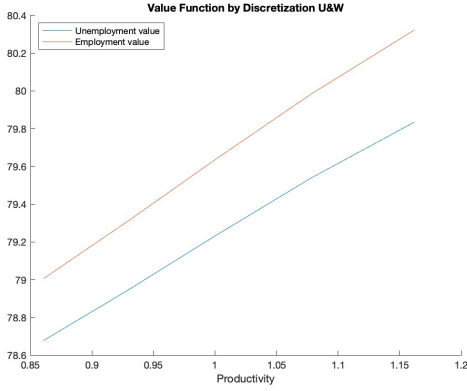| Parameter	Symbol	Value
Productivity std.	σ_λ	0.05
Productivity mean	μ_λ	0
Stochastic std.	σ_ε	0.03
Stochastic mean	μ_ε	0
Separation rate	s	0.1
Discount rate	r	0.012
Value of leisure	z	0.4
Matching function	μ	1.355
Matching function	α	0.72
Bargaining Power	β	0.72
Cost of vacancy	c	0.213

Table 1: Parameter Calibration
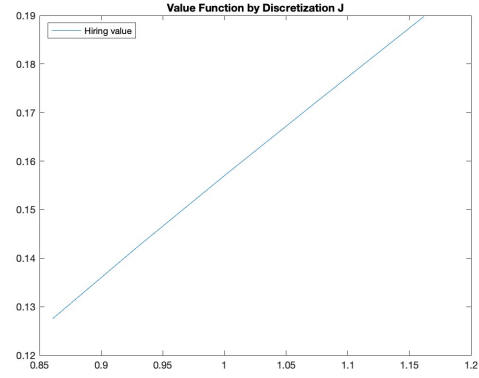

Figure 1: Employ and Unemploy Value   Figure 2: Hiring Value

Figure 3: Value Function by Discretization

3.2 Value function

The value function with respect to the productivity level is given by Discretization in Figure 3 and by Approximation in Figure 6. Coefficients of each polynomial interpolation of the value functions are reported in Table 2, in ascending order of exponent. All value functions growth in monotone with productivity. The employment value is slightly higher than unemployment value, the hiring value slightly higher than vacancy value, which shows that it is always optimal for worker to work and firm hire in equilibrium.

3.3 Control

The model fit of control with respect to the productivity level is given by Discretization in Figure 7 and by Approximation in Figure 8. The job finding rate and wage grow and unemployment rate falls in monotone to productivity level. Combined with the Value Functions, we can see that a 4-th order Lagrange Interpolation does a good job in approximating the value functions.
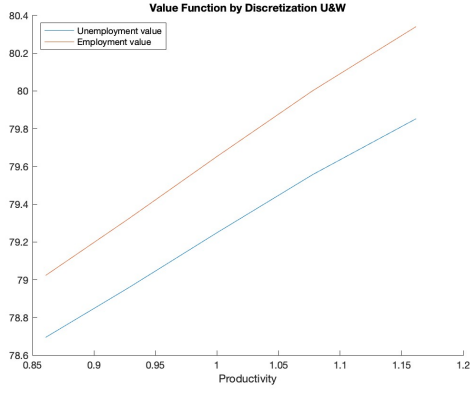
6

Figure 4: Employ and Unemploy Value
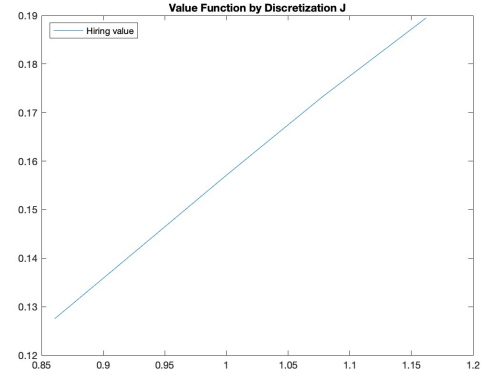

Figure 5: Hiring Value

Figure 6: Value Function by Approximation
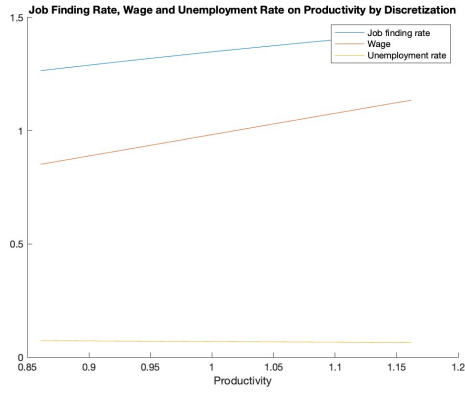
Figure 7: Control by Discretization

Figure 8: Control by Approximation
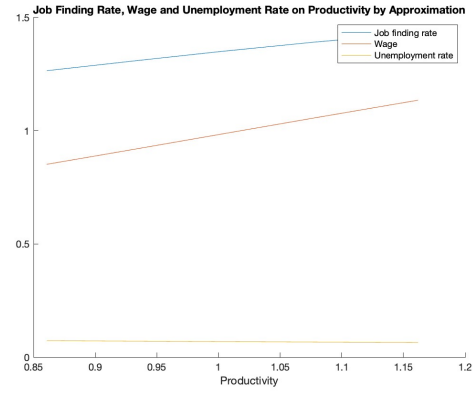
	U	W	J
e	82.567	83.090	-2.42E-04
d	-18.161	-20.700	0.134
c	21.774	26.984	-0.125
b	-6.395	-10.203	0.267
a	-0.536	0.482	-0.118

Table 2: Approximation coefficients

log(p)	w	u	f
-0.1	0.8930	0.0718	1.2925
-0.05	0.9367	0.0704	1.3203
0	0.9827	0.0690	1.3484
0.05	1.0311	0.0677	1.3764
0.1	1.0818	0.0665	1.4039

Table 3: Wage, Unemployment rate and Job Finding Rate

	u	f	p
Data Std.	1.90E-01	1.18E-01	0.020
Approximation Model Std.	3.16E-03	6.59E-02	0.119
Discretization Model Std.	3.14E-03	6.56E-02	

Table 4: Model fit on Unemployment, Job finding rate and productivity

3.4 Results at typical productivity value

The wage, unemployment rate and job finding rate at typical productivity level is given in Table 3

3.5 Extended Exercise

In this subsection, we calculate the standard deviation of productivity, unemployment rate and job finding rate. The data and model fit standard deviation of productivity, unemployment rate and job finding rate is given in Table 4. Although the model captures a higher deviation in productivity level, the variation in unemployment and job finding rate is far undermined.

We also report the correlation matrix of the three variables in Table 5, correlation between u and f are -0.9994, successfully replicating the empirical result of -0.949, but unemployment correlation with productivity -0.9967, is twice as large as the empirical result (-0.408). And the job finding rate correlation with productivity is 0.9989, off the target from the data level of 0.396. Correlation between u and v is -0.9980, well capturing the data -0.894. This shows that the baseline model successfully capture the downward slope of the Beveridge curve.

3.5.1 Tune productivity variation

In this subsection, we check if tuning σ_ε helps to improve model performance. In a Tauchen method, the standard deviation of $logp$ is defined as $\sigma_{logp} = \sigma_\varepsilon/\sqrt{1-\rho^2}$ and the lower and upper bound of grid is defined as $\mu_{logp} \pm m\sigma_{logp}$, thus deciding the grid and transition matrix. A higher σ_ε is supposed to increase the volatility of optimal control.

	u	v	f	p
u	1.0000	-0.9980	-0.9994	-0.9967
v	-	1.0000	0.9996	0.9998
f	-	-	1.0000	0.9989
p	-	-	-	1.0000

Table 5: Correlation Matrix Baseline

sigma_epsilon	u	f	p
0.01	1.04E-03	2.18E-02	0.040
0.02	2.09E-03	4.37E-02	0.079
0.03	3.14E-03	6.56E-02	0.119
0.04	4.20E-03	8.75E-02	0.159
0.05	5.28E-03	1.10E-01	0.200

Table 6: Tuning variation of productivity grid

	u	v	f	p
u	1.0000	-0.9945	-0.9984	-0.9909
v	-	1.0000	0.9988	0.9995
f	-	-	1.0000	0.9969
p	-	-	-	1.0000

Table 7: Correlation Matrix Productivity Tuning

We calculate the standard deviations for 5 values of σ_ε, *ceteris paribus*, and derive the standard deviation, the result is reported in Table 6. Higher productivity variation does lead to higher optimal control volatility as expected. We can conclude that compared with Data, a more volatile productivity shock leads to more volatile job-finding rate and unemployment rate, but is still underestimating the latter two. Even in the case where job-finding rate is about well simulated with $\sigma_\varepsilon = 0.005$, the cost is ten times higher productivity variation and the unemployment rate deviation is not yet matched. This means that productivity shock is not the only source that creates volatility in unemployment rate and job finding rate.

Similarly, the Table 7 reports the correlation matrix of variables with $\sigma_\varepsilon = 0.05$. There is only marginal improvement compared to the baseline model. This means that productivity shocks can only induce very small movements in unemployment and job finding rate. A lion's share is yet captured.

3.5.2 Set Stochastic Separation Rate

We try to improve model fit following Shimer (2005)'s suggestion of setting the separation rate as a stochastic process. Suppose that the separation rate follows i.i.d. normal distribution with mean 0.1 to 0.8 with step 0.1 and standard deviation 0.1, *ceteris paribus*, draw $N = 100$ random numbers and set lower bound 0, apply into value function iteration and use Monte-Carlo integration to evaluate value function W and J in forms of

$$W_{p_i}^{(1)} = \frac{1}{N} \sum_{n=1}^{N} (w_{p_i} + \delta\{(1 - s(n)) \sum_{j=1}^{n} p_{i,j} W^{(0)}(p_j) + s(n) \sum_{j=1}^{n} p_{i,j} U^{(0)}(p_j)\})$$

$$J_{p_i}^{(1)} = \frac{1}{N} \sum_{n=1}^{N} (p_i - w_{p_i} + \delta(1 - s(n)) \sum_{j=1}^{n} p_{i,j} J^{(0)}(p_j))$$

	u	f	p	s
Data Std.	1.90E-01	1.18E-01	0.020	0.075
Model Std. (s=0.1)	3.14E-03	6.56E-02	0.119	NaN
Model Std (mu_s=0.1)	2.99E-03	6.56E-02	0.119	0.086
Model Std (mu_s=0.2)	5.39E-03	6.58E-02	0.119	0.098
Model Std (mu_s=0.3)	8.02E-03	6.59E-02	0.119	0.088
Model Std (mu_s=0.4)	9.49E-03	6.60E-02	0.119	0.096
Model Std (mu_s=0.5)	1.10E-02	6.59E-02	0.119	0.122
Model Std (mu_s=0.6)	1.21E-02	6.58E-02	0.119	0.094
Model Std (mu_s=0.7)	1.30E-02	6.56E-02	0.119	0.102
Model Std (mu_s=0.8)	1.38E-02	6.54E-02	0.119	0.099

Table 8: Setting Stochastic Separation Rate

	u	v	f	p	s
u	1.0000	-0.9984	-0.9998	-0.9979	0.1099
v	-	1.0000	0.9994	1.0000	-0.1056
f	-	-	1.0000	0.9991	-0.1083
p	-	-	-	1.0000	-0.1050
s	-	-	-	-	1.0000

Table 9: Correlation Matrix Stochastic Separation Rate

and unemployment rate

$$u_p^* = \frac{1}{N} \sum_{n=1}^{N} \frac{s(n)}{s(n) + f(\theta_p^*)}$$

The result is reported in Table 8, we can find a growing goodness-of-fit for unemployment with higher separation rate shock which means that a higher separation rate shock induces a higher unemployment shock. However, growing mean of s only causes a very marginal change in the std of job-finding rate, remaining at around 55% of their empirical values. This exercise is in line with the paper.

Mean separation rate being 0.8, the correlation between variables is reported in Table 9. To derive the correlation, we set grid size to be 100 and derive another standard deviation matrix reported in Appendix B. More grids lead to a peripheral improvement in model fit of standard deviation Correlation between u and f is -0.9998, u and p is -0.9979, u and s is 0.1099, whereas data value is 0.709. That between f and p is 0.9991, f and s is -0.1083 while data is -0.574. Correlation between v and s is -0.1056 while in data it is -0.684. There is a marginal improvement in model fit compared to the baseline model with respect to standard deviation, but fails to replicate the correlation matrix, which shows a simple IID process does not well simulate the separation rate process. Yet on the whole it successfully captures the qualitative nature of the correlation, though there is important statistical difference. The negative correlation between u and v is again well captured (-0.9984), which is different from the paper's conclusion of a strong positive correlation between u and v induced by separation rate shock. This may be explained by the fact that the separation shock process is not correlated to productivity shock process.

3.5.3 Conclusion

After tuning productivity volatility and separation rate process, we can conclude that the adapted Shimer (2005) model confirms successfully that labor productivity shocks are qualitatively consistent with a downward-sloping Beveridge curve, that is the negative correlation between unemployment and vacancies, but it fails to account for the volatility of job finding rate. What's more, productivity shock only induce very small movement along the Beveridge curve, not even amended with stochastic separation rate. An IID distribution does not well simulate the process of separation shock especially when it comes to replicating its correlation between unemployment and vacancy and job finding rate, nor does it capture the volatility feature of job finding rate.

Bibliography

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A Optimal wage

$$W_p - U_p = \beta(W_p - U_p + J_p)$$
$$\Leftrightarrow w_p - z + \delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) =$$
$$\beta(p - z + \delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'}) + \delta(1 - s)\mathbb{E}_p J_{p'})$$
$$\Leftrightarrow w_p = \beta p + (1 - \beta)z + (\beta - 1)\delta(1 - s - f(\theta_p))(\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'})$$
$$+\frac{\beta c(1 - s)}{q(\theta_p)}$$
$$\Leftrightarrow w_p = \beta p + (1 - \beta)z - \frac{\beta c \delta(1 - s - f(\theta_p))}{q(\theta_p)} + \frac{\beta c(1 - s)}{q(\theta_p)}$$
$$\Leftrightarrow w_p = \beta p + (1 - \beta)z + \beta c \theta_p$$

where we use the fact that $\mathbb{E}_p W_{p'} - \mathbb{E}_p U_{p'} = \frac{\beta}{1-\beta}\mathbb{E}_p J_{p'}$ and $f(\theta_p)/q(\theta_p) = \theta_p$

B Stochastic Separation

	u	f	p	s
Data Std.	1.90E-01	1.18E-01	0.020	0.075
Model Std (Mu_s=0.1)	2.29E-03	4.83E-02	0.088	0.082
Model Std (Mu_s=0.2)	3.69E-03	4.85E-02	0.088	0.096
Model Std (Mu_s=0.3)	5.65E-03	4.86E-02	0.088	0.106
Model Std (Mu_s=0.4)	7.12E-03	4.86E-02	0.088	0.107
Model Std (Mu_s=0.5)	8.17E-03	4.86E-02	0.088	0.095
Model Std (Mu_s=0.6)	9.06E-03	4.85E-02	0.088	0.118
Model Std (Mu_s=0.7)	9.52E-03	4.84E-02	0.088	0.105
Model Std (Mu_s=0.8)	1.02E-02	4.82E-02	0.088	0.106

Table 10: Setting Stochastic Separation Rate: n=100