Advanced Tools in Macroeconomics

Continuous time models (and methods)

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Introduction

- In these lecture we will take a look at models in continuous, as opposed to discrete, time.
- There are some advantages and disadvantages
  - Advantages: Can give closed form solutions even when they do not exist for the discrete time counterpart. Can be very fast to solve. Really trendy and there's been a resurgence.
  - Disadvantages: Intuition is a bit tricky. Contraction mapping theorems / convergence results go out the window (but can be somewhat brought back). The latter can create issues for numerical computing. Difficult to deal with certain end-conditions (like finite lives etc.)

Plan for this topic

- Continuous time methods and models are not as well documented as the discrete time cases.
- Proceed through a series of examples
  1. The Solow growth model (this video)
  2. The Ramsey growth model (video #2)
  3. Euler equations and a monetary economy (video #3)
  4. Search and matching (video #4)
  5. The implicit method and heterogenous agents (remaining videos)
- How to solve (turns out to be pretty easy, and we can apply methods we know from earlier parts of the course)

Plan for this topic

- Useful papers to read (see webpage)
  1. "Finite Difference Methods for Continuous Time Dynamic Programming" by Candler 2001
  2. Online appendix of Achdou et al (2017)

The Solow growth model

- The Solow growth model is characterized by the following equations

$$
\begin{aligned}
Y_t &= K_t^{\alpha}(A_t N_t)^{1-\alpha} \\
K_{t+1} &= I_t + (1-\delta)K_t \\
S_t &= sY_t \\
I_t &= S_t \\
A_{t+1} &= (1+g)A_t \\
N_{t+1} &= (1+\eta)N_t
\end{aligned}
$$

- To solve this model we rewrite it in intensive form

$$
x_t = \frac{X_t}{A_t N_t}, \quad \text{for } X = \{Y, K, S, I\}
$$

The Solow growth model

- Using this and substituting in gives

$$\frac{K_{t+1}}{A_t N_t} = s k_t^{\alpha} + (1-\delta) k_t$$

- We can rewrite as

$$\begin{aligned}
\frac{K_{t+1}}{A_{t+1} N_{t+1}} \frac{A_{t+1} N_{t+1}}{A_t N_t} &= s k_t^{\alpha} + (1-\delta) k_t \\
k_{t+1} \frac{A_{t+1} N_{t+1}}{A_t N_t} &= s k_t^{\alpha} + (1-\delta) k_t \\
k_{t+1}(1+g)(1+\eta) &= s k_t^{\alpha} + (1-\delta) k_t
\end{aligned}$$

The Solow growth model

- Ta-daa!

$$k_{t+1} = \frac{sk_t^\alpha}{(1+g)(1+\eta)} + \frac{(1-\delta)k_t}{(1+g)(1+\eta)}$$

- Balanced growth: $k_{t+1} = k_t = k$

$$k = \left(\frac{g+\eta+g\eta+\delta}{s}\right)^{\frac{1}{\alpha-1}}$$

- This is not textbook stuff. Why? Discrete time. More elegant solution in continuous time.

The Solow growth model

- Continuous time is not a state in itself, but is the effect of a limit. A derivative is a limit, an integral is a limit, the sum to infinity is a limit, and so on.
- Continuous time is the name we use for the behavior of an economy as intervals between time periods approaches zero.

The Solow growth model

- The right approach is therefore to derive this behavior as a limit (much like you probably derived derivatives from its limit definition in high school).
- Eventually you may get so well versed in the limit behavior that you can set it up directly (like you can say that the derivative of $\ln x$ is equal to 1/x, without calculating $\lim_{\varepsilon \to 0}(\ln(x+\varepsilon) - \ln(x))/\varepsilon$)
- I'm not there yet. I have to do this the complicated way. People like Ben Moll at LSE is. Take a look at his lecture notes on continuous time stuff. They are great.

The Solow growth model

- Back to the model.
- Suppose that before the length of each time period was one month. Now we want to rewrite the model on a biweekly frequency.
- It seems reasonable to assume that in two weeks we produce half as much as we do in one month: $Y_t = 0.5K_t^{\alpha}(A_tN_t)^{1-\alpha}$.
- It also seems reasonable that capital depreciates slower, i.e. 0.5δ.

The Solow growth model

- Notice that we still have N_t worker and K_t units of capital: Stocks are not affected by the length of time intervals (although the accumulation of them will).
- The propensity to save is the same, but with half of the income saving is halved too (and therefore investment)
- What happens to the exogenous processes for A_t and N_t?

The Solow growth model

- Before

$$A_{t+1} = (1+g)A_t, \quad N_{t+1} = (1+\eta)N_t$$

- Now

$$A_{t+0.5} = (1+0.5g)A_t, \quad N_{t+0.5} = (1+0.5\eta)N_t$$

or

$$A_{t+0.5} = e^{0.5g}A_t, \quad N_{t+0.5} = e^{0.5\eta}N_t \ \ ?$$

The Solow growth model

- It turns out that this choice does not matter much for our purpose
- Suppose that the time period is not one month but $\Delta \times$ one month. And suppose that

$$A_{t+\Delta} = (1 + \Delta g)A_t$$

- Rearrange

$$\frac{A_{t+\Delta} - A_t}{\Delta} = gA_t.$$

- And take limit $\Delta \to 0$

$$\dot{A}_t = gA_t$$

The Solow growth model

- Suppose that the time period is not one month but $\Delta \times$ one month. And suppose that

$$A_{t+\Delta} = e^{\Delta g} A_t$$

- Rearrange

$$\frac{A_{t+\Delta} - A_t}{\Delta} = \frac{(e^{\Delta g} - 1)}{\Delta} A_t.$$

- Notice

$$\lim_{\Delta \to 0} \frac{(e^{\Delta g} - 1)}{\Delta} = \lim_{\Delta \to 0} \frac{(g e^{\Delta g})}{1} = g$$

The Solow growth model

- So

$$\lim_{\Delta \to 0} \frac{(e^{\Delta g} - 1)}{\Delta} A_t = g A_t$$

  and thus

$$\dot{A}_t = g A_t$$

- Therefore, it doesn't matter if $A_{t+\Delta} = (1 + \Delta g) A_t$ or $A_{t+\Delta} = e^{\Delta g} A_t$. The limits are the same.

The Solow growth model

- There is another useful lesson here
- Take a look again at

$$A_{t+\Delta} = (1 + \Delta g)A_t$$

- If we just take limits on both sides we trivially get $A_t = A_t$
- Or if we take limits of

$$A_{t+\Delta} - A_t = \Delta g A_t$$

we get $0 = 0$.

The Solow growth model

- To get something meaningful we need to subtract A_t from both sides, AND to divide by Δ
- This is quite common in continuous time; it is an art seeing how to manipulate the expressions in the right way to get nontrivial expressions.
- You will see this more in the next video.

The Solow growth model

- Solow growth model in Δ units of time

$$
\begin{aligned}
Y_t &= \Delta K_t^{\alpha}(A_t N_t)^{1-\alpha} \\
K_{t+\Delta} &= I_t + (1-\Delta\delta)K_t \\
S_t &= sY_t \\
I_t &= S_t \\
A_{t+\Delta} &= (1+\Delta g)A_t \\
N_{t+\Delta} &= (1+\Delta\eta)N_t
\end{aligned}
$$

The Solow growth model

- Substitute and rearrange as before

$$\frac{K_{t+\Delta}}{A_{t+\Delta}N_{t+\Delta}}\frac{A_{t+\Delta}N_{t+\Delta}}{A_tN_t} = s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t$$

$$k_{t+\Delta}\frac{A_{t+\Delta}N_{t+\Delta}}{A_tN_t} = s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t$$

$$k_{t+\Delta}(1+\Delta g)(1+\Delta\eta) = s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t$$

The Solow growth model

- Simplify and rearrange

$$\begin{aligned} k_{t+\Delta}(1+\Delta g)(1+\Delta\eta) &= s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t \\ \Rightarrow \quad k_{t+\Delta} - k_t &= s\Delta k_t^{\alpha} - \Delta\delta k_t - \Delta(g+\eta+\Delta g\eta)k_{t+\Delta} \\ \Rightarrow \quad \frac{k_{t+\Delta} - k_t}{\Delta} &= sk_t^{\alpha} - \delta k_t - (g+\eta+\Delta g\eta)k_{t+\Delta} \end{aligned}$$

- Take limits $\Delta \to 0$

$$\dot{k}_t = sk_t^{\alpha} - (g+\eta+\delta)k_t$$

- With steady state

$$k = \left(\frac{g+\eta+\delta}{s}\right)^{\frac{1}{\alpha-1}}$$

The Solow growth model: Solution

- How do you solve this model?
- The equation

$$\dot{k}_t = sk_t^\alpha - (g + \eta + \delta)k_t$$

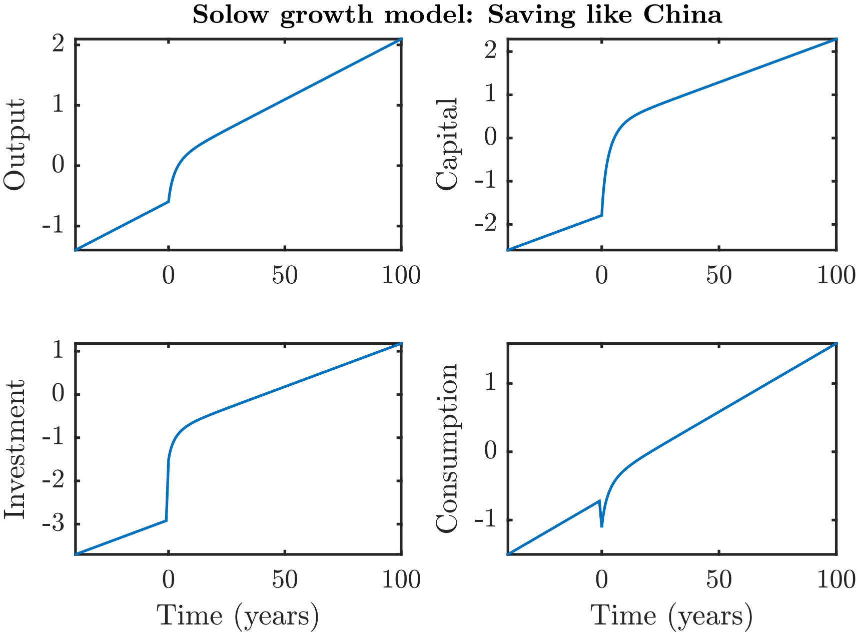
is an ODE.
- Declare it as a function with respect to time, t, and capital, k, in Matlab as

$$\texttt{solow} = \texttt{@(t,k)} \quad \texttt{sk}^\alpha - (\texttt{g} + \eta + \delta)\texttt{k};$$

- The simulate it for, say 100 units of time, with initial condition k_0 as

$$[\texttt{time}, \texttt{capital}] = \texttt{ode45}(\texttt{solow}, [0 \ \ 100], \texttt{k}_0);$$

The Solow growth model: Solution



The Solow growth model: Solution

A few pointers

- Once you got the solution of a deterministic continuous time model, the solution will always be of the form $\dot{x}_t = f(x_t)$, whether or not x_t is a vector.
- The matlab function ode45 (or other versions) can then simulate a transition (such as an impulse response).
- You could also simulate on your own through the approximation

  $$\dot{x}_t \approx \frac{x_{t+\Delta} - x_t}{\Delta}$$

  and thus find your solution as $x_{t+\Delta} = x_t + \Delta f(x_t)$.
- For this to be accurate, Δ must be small if there are a lot of nonlinearities.
- The ODE function in matlab uses so-called Runge Kutta methods to vary the step-size Δ in an optimal way.