# Euler Simulation, Python

## Contents

# 1 Introduction

The Euler Simulation project applies the methods used in Knowles and Yeh (2018) to simulate, in one dimension, the shoaling process of a shallow-water wave. The original paper uses such a simulation to predict the wave amplification process, that is, how $a$, the amplitude of the wave, relates to $h$, the local water depth.

This project uses the same algorithm, in python, to handle other cases, e.g. the wind's effect on a shoaling solitary wave.

# 2 Code Documentation

Simulation involves two python files.

## 2.1 simulator.py

*simulator.py* contains one relevant class: `Simulator1D`, which wraps necessary functionality to run a simulation.

### 2.1.1 `Simulator1D`: Constructor

```
sim = Simulator1D(bathymetry, dt, dx, eta0, phiS0)
```

Initializes a Simulator instance with the given parameters:

| argument | description |
|---|---|
| bathymetry | numpy array of the bathymetry, must have an even number of nodes. 0 is expected to be water level. |
| dt | time resolution of the simulation |
| dx | spatial resolution of the simulation (distance between points of bathymetry) |
| eta0 | initial free surface heights (with spatial resolution dx, 0 is expected for still water) expected to be a numpy array |
| phiS0 | velocity potential at the free surface, expected to be a numpy array. |

`Simulator1D` has the following keyword arguments:

| argument | description | default |
|---|---|---|
| zeta_x | First derivative (gradient) of the bathymetry, if a higher order approximation of gradient is desired. If no argument for zeta_x is passed, it is instead calculated from the finite difference $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$ where the edges are assumed to have a gradient of 0 | (see description) |
| M | Terms in the pertubation expansion (higher number is more accurate, but requires more computation) | 5 |
| v | lowpass threshold. If v is a number, any wavenumber greater than the largest wavenumber times v is clipped off each timestep. v can also be a function that takes a wavenumber magnitude and the peak wavenumber, and returns how it should be scaled | 0.7 |
| g | acceleration due to gravity. | 9.81 |
| h0 | base still water depth | bathymetry[0] |
| P | Wind pressure coefficient. Atmospheric pressure is set to be $P \cdot \eta_x$ to simulate the wind effect, unless P_atmos is over- written in the step() method. See Zdyrski and Feddersen (2021) | 0 |

In the case that $v$ is a function, the wave number is the first argument and the peak wavenumber is the second argument, so the following will result in the same low-pass filter $v = 0.7$:

```
def v(k, peak):
    if k/peak <= 0.7:
        return 1
    else:
        return 0
```

### 2.1.2 `Simulator1D`: step()

```
sim.step("RK4")
```

Steps the simulation forward using the given method. Any arguments for the method can be specified as optional arguments or keyword arguments.

| argument | description |
|---|---|
| method | The method to use. This can be a string or a function. When a string is passed the method in integrator.py of the same name is used. See section (2.2) for such methods. |

Optional arguments specific to a method (e.g. tolerance for implicit methods) can be passed into `step()`, which will be transferred to the method when called. In particular, `P_atmos` is common to all integration methods, and is shown below.

| argument | description | default |
|---|---|---|
| P_atmos | Atmospheric pressure at the surface. This can be either a function with arguments `eta,phiS,eta_x,phiS_x,w` or a numpy array of length `Nx`. See the beginning of section (2.2) for more specifics on the function. | wind_pressure() |

Note that wind_pressure() uses the simulation's `P` value, so by default, there is no atmospheric pressure when $\texttt{P} = 0$.

### 2.1.3 `Simulator1D`: calculate_time_derivatives()

```
sim.calculate_time_derivatives(eta, phiS, zeta, zeta_x, zeta_t, P_a)
```

Returns the tuple `(eta_t, phiS_t)` of time derivatives of $\eta$ and $\Phi^S$ respectively. Takes the following arguments:

| argument | description |
|---|---|
| eta | free surface height at the given time step; pass simulator.eta if you want the current time derivative. |
| phiS | free surface velocity potential at the given time step; pass simulator.phiS if you want the current time derivative |
| zeta | bathymetry ($\zeta$), with 0 corresponding with a depth of -h0 |
| zeta_x | spatial derivative of $\zeta$ |
| zeta_t | time derivative of $\zeta$ |
| P_a | atmospheric pressure at every point, should have the same samples as bathymetry. Expected to be a numpy array or a function. See the beginning of section (2.2) for more specifics on the function. |

#### 2.1.4 `Simulator1D`: volume()

`sim.volume()`

Returns the integral

$$\int \eta \, dt$$

over the bounds of the simulation, which represents the volume of water in the simulation, offset by a constant that depends only on the bathymetry. This value should be invariant in the simulation, and can give a means of measuring the accuracy of the simulation.

#### 2.1.5 `Simulator1D`: energy()

`sim.energy()`

Calculates the total energy in the system using the formula

$$\int \eta_t \Phi^S + g\eta^2 \, dt$$

This value should be invariant in the simulation, and can give a means of measuring the accuracy of the simulation.

#### 2.1.6 `Simulator1D`: peak_location()

`sim.peak_location()`

Returns

$$\arg\max_x \eta(x),$$

the value of $x$ that corresponds with the heighest point of the surface. This value is equivalent to the index of the highest eta, times `dx`, and provides a means of finding the approximate position of a solitary wave.

#### 2.1.7 `Simulator1D`: zeta_at()

`sim.zeta_at(x)`

Returns $\zeta(x)$, using a linear interpolation scheme for non-discrete points. calling `zeta_at(i*dx)` is equivalent to evaluating `zeta[i]`.

### 2.1.8 `Simulator1D`: run_simulation()

`sim.run_simulation(plot_dt, data_dt, directory)`

Runs a simulation, time-stepping until a stop-condition is met, and saving plots and/or data at given intervals of time.

| argument | description |
|---|---|
| saveplot_dt | the timestep between saved plots. This number is rounded to the nearest multiple of the simulation dt. If this value does not round to a positive number, plots are not saved. Plots are saved as PNG files with a name corresponding to the order it is saved in. A plot with number 'i' represents the data at time 'i*saveplot_dt' |
| savedata_dt | the timestep between saved data. This number is rounded to the nearest multiple of the simulation dt. If this value does not round to a positive number, data is not saved. Data is saved as a json file with name 'dat.json' which is created regardless if data should be saved or not. In the case that data is not saved, only the metadata of the simulation is stored. |
| directory | the directory to save the files to. This can also include a prefix to the file. If `directory="~/sim/"`, then plots are saved as "[number].png" in the ~/sim/ directory. If `directory="~/sim"`, then plots are saved as "sim[number].png" in the home directory. If `directory=None`, no files are saved. |

`run_simulation` has the following optional arguments:

| argument | description | default |
|---|---|---|
| should_continue | function that determines if a simulation should stop or not. This takes the simulation as an argument and returns a boolean. The simulation is run until should_continue returns false. By default, this is the lambda function `sim: sim.t < 500`, which stops the simulation when a time of 500 is reached | (see description) |
| integrator | function or string that timesteps the simulation. functions should only take the simulation as an argument and return nothing, modifying the passed simulation. Strings should be the name of a method in Integrator1D, which will be called by the simulation. | "RK4" |
| save_eta | Parameters for how eta should be saved when data is saved to json. This should be generated using Simulator1D.data_save_params(). If None, then eta is not saved. | None |
| save_phi | Parameters for how phiS should be saved when data is saved to json. This should be generated using Simulator1D.data_save_params(). If None, then phiS is not saved. | None |
| loop_callback | this function is called after every simulation step. It should be a void function that takes the arguments `sim, step, plot, data`, where<br>`sim` is the simulator at the step<br>`step` is the integer multiple of dt that the simulation has run<br>`plot` is a boolean representing if the plot was saved this step<br>`data` is a boolean representing if the data was saved this step<br>By default, loop_callback makes a print statement after every 100 time steps. | (see description) |
| plot_func | A function that is dedicated to plotting and saving the figure. The function is expected to be void and take the arguments (sim, filename). | (see below) |
| save_json | Whether or not to save the file to json. The metadata of the simulation is saved even if savedata_dt is not positive. Setting save_json to false prevents this. | False |
| save_netcdf | Whether or not to save the file to netcdf. The netCDF file ignores data truncation specifications of save_eta and save_phi. | True |
| save_buffer | number of datapoints to buffer in between file-writes. If 0 or 1, then every savedata_dt, the json/netCDF file is opened and written to. | 10 |
| cdf_h_invariant | Whether or not h is treated as invariant. If false, then the bathymetry is saved every frame, alongside $\eta$ and $\phi^S$. | True |
| cdf_Pderiv | The string that should populate the P_deriv field in the netcdf file. By default this will take the value "zero" if this instance's `P` value is 0 and "wind" otherwise. | (see description) |
| cdf_timeunits | The string that specifies the units of time for the simulation. | "seconds" |
| cdf_spaceunits | The string that specifies the units of x for the simulation. | "meters" |

By default, plot_func is defined as the function

```
def plot_func(sim, filename):
    plt.plot(sim.x, sim.eta, "b")
    plt.plot(sim.x, sim.zeta - sim.h0, "k")
    plt.ylabel("z")
```

```
plt.xlabel("x")
plt.title(f"dx={sim.dx},dt={sim.dt},t={round(sim.t,3)}")
plt.savefig(filename)
plt.clf()
```

where `plt` is `matplotlib.pyplot`.

#### 2.1.9 `Simulator1D`: data_save_params()

`Simulator1D.data_save_params()`

Returns a dictionary of parameters for how to save data from a simulation. The output of `data_save_params()` should be used for aguments `save_eta` and `save_phi` in `run_simulation()`.

| argument | description | default |
|---|---|---|
| dx | The spatial resolution to save with. If None, then the resolution is the same as the simulation. This value will always be rounded to a whole number multiple of the simulation dx. | None |
| point_conversion | A boolean that represents if data should be coded as a vector (array), or if the vector should be converted into a list of (x,y) points. If true, then the conversion is made. | False |
| eps | The tolerance of the save data. The data is rounded to the nearest multiple of eps. That is, with eps=0.001, the data is saved up to the 3rd decimal place. 0 corresponds with no rounding. | 0 |
| lin_tol | Only used when point_conversion is true. Specifies a tolerance for which points should not be saved when they are close enough to a linear interpolation of the data. If the points are {(0,0),(0.5,0.5),(1,1)}, any nonnegative tolerance will discard (0.5,0.5). If no points should be discarded, a negative value should be given. | -1 |
| zero_trunc | If a value is less than this distance from 0, the value is truncated to 0 before saving. | 0 |

#### 2.1.10 `Simulator1D`: init_netcdf()

`Simulator1D.init_netcdf(‘‘unforced.nc", True, ‘‘zero")`

Generates a netCDF file of the given filename and populates it with one point in time representing the simulation's current state. If the file already exists, then it is overwritten with a new file.

Returns the netCDF_File object.

| argument | description |
|---|---|
| filename | The name of the file to be saved. Overwrites existing files |

| argument | description | default |
|---|---|---|
| h_invariant | Whether the simulation should be treated as if h does not vary with time | True |
| P_deriv | Information on how pressure is obtained. Expects "zero", "wind" or "custom". By default this will take the value "zero" if this instance's `P` value is 0 and "wind" otherwise. | (see description) |
| timeunits | A string representing the units for time. | "seconds" |
| spaceunits | A string representing the units for spatial coordinates. | "meters" |
| P | If P_deriv is "zero" then this does nothing.<br>If it is "wind", then the P attribute is set to this value.<br>If it is "custom", then P is the P_a variable at time index 0. | simulator `P` |
| close | Whether or not this method should close the netcdf file resource after initialization. | True |

#### 2.1.11 `Simulator1D`: soliton()

`Simulator1D.soliton(x0,a0,h0,Nx,dx)`

Returns a tuple `(eta,phiS)` corresponding to the intial conditions of $\eta$ and $\Phi^S$ of a soliton at a given point in space.

| argument | description |
|---|---|
| x0 | The x coordinate of the soliton, where x=0 corresponds with an index of 0 in the vectorization of $\eta$ and $\Phi^S$ |
| a0 | The amplitude of the soliton |
| h0 | The water deph beneath the soliton |
| Nx | The number of points in the vectorization of $\eta$ and $\Phi^S$ |
| dx | The spatial resolution (distance between points) |

| argument | description | default |
|---|---|---|
| g | acceleration due to gravity | 9.81 |

This can be used in the constructor of `Simulator1D` through unpacking:

```
sim = Simulator1D(bathymetry, dt, dx, *Simulator1D.soliton(x0,a0,h0,Nx,dx))
```

#### 2.1.12 `Simulator1D`: KY_bathym()

`Simulator1D.KY_bathym()`

Produces a bathymetry profile similar to Knowles and Yeh's paper. Expects h0 = 1, but the result can be multiplied by the desired h0 if not 1.

| argument | description | default |
|---|---|---|
| Nx | number of points | $2^{14}$ |
| dx | spatial resolution (distance between each point) | 0.04 |
| s0 | nominal slope of the bathymetry | 0.002 |
| d0 | height of the beach plateau | 0.9 |
| gamma | smoothing parameter | 0.1 |
| X1 | position where the bathymetry should start sloping up | 4 |

#### 2.1.13 `Simulator1D`: KY_sim()

```
sim = Simulator1D.KY_sim()
```

Returns a new simulator similar to Knowles and Yeh's initial conditions. The bathymetry is produced by `KY_bathym()` and the initial $\eta$ and $\Phi^S$ values are produced by `soliton()`.

| argument | description | default |
|---|---|---|
| Nx | Number of points (nodes) in the discreteized simulation | $2^{14}$ |
| dx | Spatial resolution | 0.04 |
| dt | Temporal resolution (time step) | 0.01 |
| s0 | Slope of the bathymetry | 1/500 |
| x0 | location of the center of the starting soliton | 30 |
| a0 | amplitude of the soliton | 0.1 |
| h0 | depth of the water | 1 |

#### 2.1.14 `Simulator1D`: fields

In an instance of Simulator1D, the following fields may be of importance:

| argument | description |
|---|---|
| dt | Temporal resolution (time step) of the simulation. This variable is used by an integrator when stepping the simulation. Most methods will use this time step, but an adaptive method may use a timestep that is smaller. This can be modified externally. |
| dx | Spatial resolution. This is the distance between two points of a vectorized function of $x$. This should not be modified externally. |
| eta | Vectorized $\eta$ with a spatial resolution `dx` at the current time step. This can be modified externally, but must have the same length (Nx). |
| phiS | Vectorized $\Phi^S$ with a spatial resolution `dx` at the current time step. This can be modified externally, but must have the same length (Nx). |
| M | Terms in the pertubation expansion of $\Phi$. Calculating the vertical velocity $\Phi_x$ scales approximately $O(M^2)$. This can be modified externally. |
| g | The acceleration due to gravity in this simulation. This can be modified externally. |
| h0 | base still water depth. We approximate the bathymetry as h0 in many calculations. This can be modified externally. |
| zeta | Vectorized bathymetry ($\zeta$), offset so $\zeta = 0$ corresponds with $z = -$h0. This can be modified externally, but must have the same length (Nx). Additionally, zeta_x should also be changed to the gradient of the new bathymetry. |
| zeta_x | Vectorized bathymetry gradient ($\nabla\zeta$). This can be modified externally, but must have the same length (Nx). Additionally, zeta should also be changed to match the new bathymetry. |
| Nx | Number of points used in the discretization (vectorization) of the simulation along the $x$-axis. This should not be modified. |
| sim_length | distance in $x$ that the simulation uses. The vectorizations of $\eta, \Phi^S$, and $\zeta$ have the domain [0, sim_length). This should not be modified. |
| x | Vectorized domain. It holds that `x[i] = dx*i`. This should not be modified. |
| kxdb | The double-domain of wave number. When performing an FFT on a function $f$ with spacing $dx$, the values of the output correspond to the wavenumber by index. If $V$ is such a vectorization of the function $f$, then $$\big(\mathcal{F}(f)\big)(\texttt{kxdb[i]}) \approx \texttt{FFT}(V)\texttt{[i]}$$ where $\mathcal{F}$ is the continuous fourier transform. This should not be modified. |
| kappadb | Normalized wavenumber of kxdb. This is equivalent to `abs(kxdb)`. This should not be modified |
| chi | Vectorized low-pass filter. Any function in the wavenumber domain can apply the filter by pointwise multiplication. This can be modified externally, but kxdb_im must also be modified accordingly. |
| kxdb_im | The precomputed value $ik\chi(k)$, which is computed from `complex(0,1) * kxdb * chi`. This should only be modified when chi is modified. |
| t | The time the simulation has run. This is only ever incremented by an integration method inside a `step()` call, and can be freely modified and accessed externally. |
| P | Wind pressure coefficient. The surface pressure is found as $P_a = \texttt{P} \cdot \eta_x$. This can be modified externally. |

## 2.2 integrator.py

This python file has the class `Integrator1D` which contains only static members. Each of which is a function that takes a `Simulator1D` instance, an atmospheric pressure argument `P_atmos`, and potential optional arguments.

`P_atmos` can either be a numpy array or a function that takes in the arguments `eta,phiS,eta_x,phiS_x,w` and returns a numpy array. The array should have a length equal to the number of nodes `Nx` used in the simulator, which matches the length of the `bathymetry` array passed into the constructor. For example, when measuring the wind effect, one may consider

$$P_a = P\frac{d\eta}{dx}$$

for some constant $P$. Such a function can be expressed as

```
def P_atmos(eta,phiS,eta_x,phiS_x,w):
    return P * eta_x
```

### 2.2.1 `Integrator1D`: euler

Referenced by string `"euler"`. Makes one derivative calculation per step, using the method:

$$y_{n+1} = y_n + hf'(y_n)$$

### 2.2.2 `Integrator1D`: RK4

Referenced by string `"RK4"`. Makes 4 derivative calculation per step, using the classic 4 step, 4th order, Runge-Kutta method with the Butcher tableau:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

### 2.2.3 `Integrator1D`: implicit_midpoint

Referenced by string `"implicit_midpoint"`. Uses the implicit midpoint rule:

$$y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right)$$

This equation is solved using fixed point iteration after an initial guess from euler's method.

Takes additional arguments:

| argument | description | default |
|---|---|---|
| max_iters | The most iterations used to achieve the desired tolerance, after which, RK4 is defaulted to. | 100 |
| tol | The tolerance allowed for the iteration to stop. | $10^{-10}$ |

### 2.2.4 `Integrator1D`: AM1

Referenced by string `"AM1"`. Uses the one step Adams-Moulton method, the implicit trapezoidal rule:

$$y_{n+1} = y_n + h\frac{f(y_n) + f(y_{n+1})}{2}$$

This equation is solved using fixed point iteration after an initial guess from euler's method.

Takes additional arguments:

| argument | description | default |
|---|---|---|
| max_iters | The most iterations used to achieve the desired tolerance, after which, RK4 is defaulted to. | 100 |
| tol | The tolerance allowed for the iteration to stop. | $10^{-10}$ |

### 2.2.5 `Integrator1D`: DIRK3

Referenced by string `"DIRK3"`. Uses Nørsett's 3 stage, 4th order diagonally implicit Runge-Kutta method withe the Butcher tableau:

| | | | |
|---|---|---|---|
| $x$ | $x$ | $0$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2} - x$ | $x$ | $0$ |
| $1 - x$ | $2x$ | $1 - 4x$ | $x$ |
| | $\frac{1}{6(1-2x)^2}$ | $\frac{3(1-2x)^2-1}{3(1-2x)^2}$ | $\frac{1}{6(1-2x)^2}$ |

where $x = 1.06858$. This equation is solved using fixed point iteration for each stage, where each stage has an initial guess of the former stage, with the first stage's initial guess as the derivative at $y_n$. For example, if $k_1$ and $k_2$ are the results of the first and second stage respectively, the second stage solves

$$k_2 = f\left(y_n + \left(\frac{1}{2} - x\right)hk_1 + xhk_2\right)$$

with an initial guess $k_2 = k_1$.

Takes additional arguments:

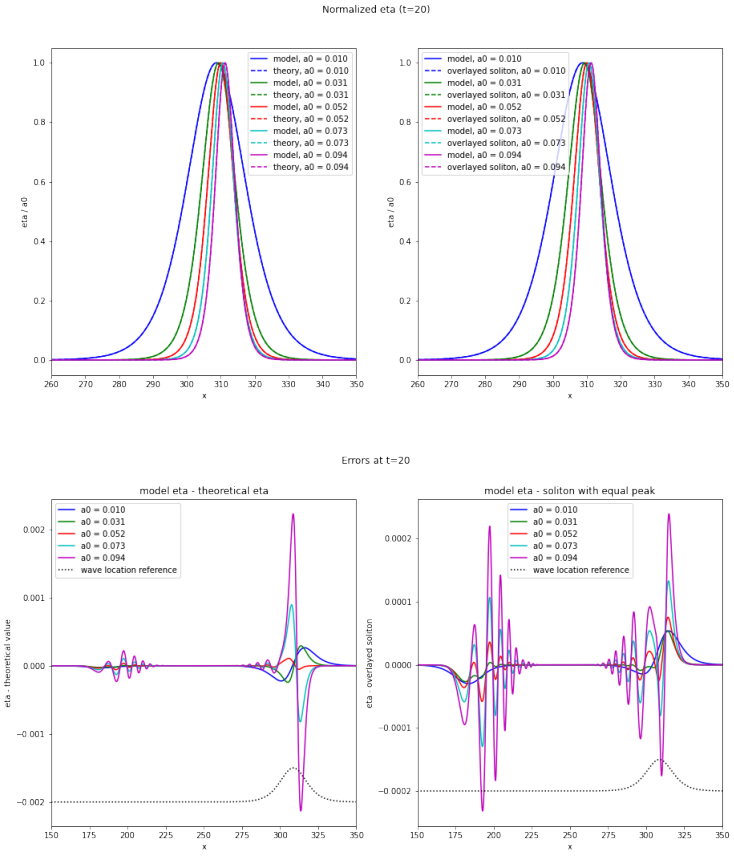
| argument | description | default |
|---|---|---|
| max_iters | The most iterations used to achieve the desired tolerance, after which, RK4 is defaulted to. | 100 |
| tol | The tolerance allowed for the iteration to stop. | $10^{-10}$ |

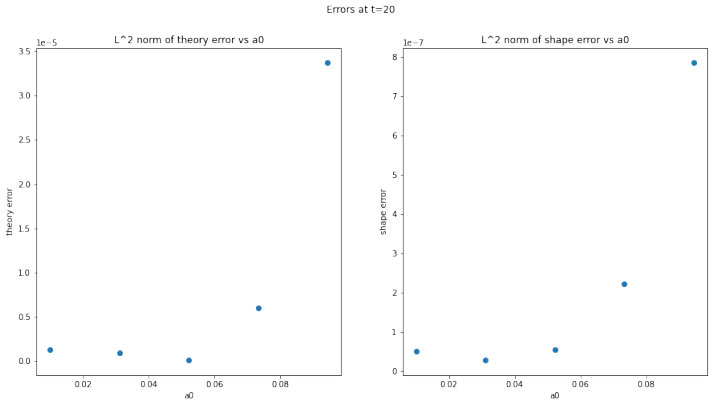
# 3 Validation

We use the properties of the soliton wave to validate our model. In the flat, unforced case, we run the model up to 20 seconds with $dx = 0.03$, $dt = 0.0025$ using the classic $4^{th}$ order Runge-Kutta scheme and compare the results with both a soliton shifted by

$$\Delta x = tv_p = 20\sqrt{g(h_0 - a_0)}$$

and a soliton overlayed on the model such that the peaks match.




Errors at t=20
L^2 norm of theory error vs a0
1e−5
theory error
a0
L^2 norm of shape error vs a0
1e−7
shape error
a0

# 4 Examples

## 4.1 `KY.py`

This file defines one function, `sim_KY`, which runs a simulation very similar to that of the Knowles and Yeh (2018) simulation. Run using `python KY.py`. When run in this manner, `sim_KY` is called with the default arguments. When imported in python using `import KY`, the function is not called, but can freely be called by the user. `sim_KY` has the following optional arguments:

| argument | description | default |
|---|---|---|
| Nx | number of points (nodes) in the simulation. | $2^{14}$ |
| a0 | amplitude of the soliton initial condition | 0.1 |
| h0 | base water depth. (depth at zeta = 0) | 1.0 |
| X0 | position (not vector index) of the soliton relative to the center of the simulation space. Negative values correspond to the left half of the simulation space. | -280 |
| d0 | height of the beach plateau (on the right side) | 0.9 |
| s0 | nominal slope of the bathymetry. | $\frac{1}{500}$ |
| dx | spatial resolution (distance between points/nodes in simulation space) | 0.04 |
| dt | Time step of the simulation | 0.01 |
| Xt | Distance from the soliton wave crest to the beach toe. Positive values mean that the beach toe is in front of the wave | 50 |
| mass_err_crit | Largest deviation in mass (calculated from `volume()`) allowed before the simulation is terminated | 0.5 |
| energy_err_crit | Largest deviation in energy (calculated from `energy()`) allowed before the simulation is terminated | 0.5 |
| tmax | Largest time in simulation allowed before termination. | 500 |
| v | The low-pass filter. Takes a number to cut off all wavenumbers greater than v*max(k), or takes a function that maps k and max(k) to how much the corresponding amplitude should be scaled. See the constructor of Simulator1D. | 0.7 |
| M | The number of terms in the pertubation expansion to simulate | 5 |
| g | Acceleration due to gravity | 9.81 |
| tsave | time in between plot saves. Zero or negative corresponds to no saving. This value is rounded to a multiple of dt. | 1 |
| save_dir | the directory/file prefix for the saved plots. By default, this is "./KY_dx[dx]_dt[dt]_s[s0]_a[a0]_plot", where each bracketed value is replaced by the relevant quantity. | (see description) |
| gui | Whether or not to run this program with matplotlib's plotting features. Use `gui=False` when running with no GUI access. | True |