Undergraduate Mathematics

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Preface

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Part I

Foundation of Mathematics

Chapter 1

Logic and Philosophy

1.1 Formal and Informal Logic

There are two sorts of logic : *informal logic* and *formal logic*. Informal logic is the study of correct reasoning in our natural language. The kind of logic we are all familiar with. We use informal logic all the time. Even arguments as simple as

1. All men are mortal
2. Socrates is mortal
3. Ergo, Socrates is mortal

Is a display of informal logic.
Statement 1 and 2 are call the *premises*, and 3 is the *conclusion*. Notice that the argument feature above is *entirely independent of its context*. Meaning if we replace the word "men", "mortal", "Socrates" with other things, the argument still stands nonetheless. For example

1. All cats are cute
2. Willard is a cat
3. Therefore, Willard is cute

One would immediately recognise the two arguments are "the same". Not in terms of the meaning of each premise, but rather what made the first argument correct is quite clearly the same as the second argument. But not all arguments that follow the form "All M is P, S is M, so "S is P" would yield a true conclusion. Take, for example, that

1. All animals are four-legged
2. Chickens are animals
3. Chicken are four-legged

the conclusion is wrong. What made the conclusion wrong is not in the form of the argument, but because we included a false premise in the argument. What made an argument ”correct” requires two parts : 1.) We need the argument to have the right form. We call these arguments *valid*. A valid argument ensures that if its premises are true, then its conclusion must be true; 2.) We need the argument to have true premises. Only valid arguments with true premises have true conclusions. These arguments we call them *sound*.
What I have said here remains pretty vague. What makes an argument valid? What makes statements true? We all have a vague idea of validity and true, but not a clear-cut *definition* of them. We will clarify these ideas as we progress. We will not include a full-blown philosophical discourse. This is a book on mathematics afterall. But we will clarify these ideas ”clear enough” so that we can safely use them in constructing our mathematical universe.
Formal lo

1.2 Propositions, Facts, and Truth

Chapter 2

Naive Set Theory

Chapter 3

Propositional Logic

We have now entered the world of formal logic.

Chapter 4

Predicate Logic

Chapter 5

Axiomatic Set Theory

5.1 Zermelo-Frankel Axioms

5.1.1 Axiom of Existence, Extensionality, and Restricted Comprehension

Remark 1. *From this chapter and onward, we will no longer write proofs in pure symbols, for writing proof like this is extremely inconvenient and daunting for the readers and the authors. We will write proofs in natural language from now on. But the reader should under*

In previous chapters, we have already gone through Naive Set Theory. We have witnessed a careless use of set would yield gruesome contradictions. Therefore, until now, we have only used set very carefully. We wish to develop a system of axiomatic set theory that would put clear demarcation on what is definable and what is not. We have already constructed the ZFC model in the previous chapter. We will now enter such a model and develop of Mathematics in its entirety from here.

To start off, there will be nothing for us to work on if not a single set exist. Hence, the following axiom

Axiom 1. ***(Axiom of Existence)*** *There is a set ($\vdash \exists x(x = x)$)*

As we have discussed in naive set theory, set equality should be determined extensionally, i.e. two sets are equal when they have the same elements.

Axiom 2. ***(Axiom of Extensionality)*** *Suppose A, B are sets, then the following are equivalent*

- $A = B$
- *for any x, $x \in A$ if and only if $x \in B$.*

($\vdash \forall x \forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$)

By these two axiom along, it allows us to define subsets

Definition 1. *(Subset relation) Let A, B be sets, then A is a **subset** of B, written $A \subseteq B$, if for any x, $x \in A$ implies $y \in B$.*

Definition 2. *(Strict subset) Let A, B be sets, then A is a **strict subset** of B, written $A \subset B$, if $A \subseteq B$ and $A \neq B$.*

Proposition 1. *Let A, B be sets, then the following are equivalent*

- $A = B$
- $A \subseteq B$ *and* $B \subseteq A$

Proof.

- ($\Rightarrow$) Suppose $A = B$. Then for any x, $x \in A$ if and only if $x \in B$. Then we have $A \subseteq B$ and $B \subseteq A$.
- ($\Leftarrow$) Suppose $A = B$, then for any x, we have 1). if $x \in A$ then $x \in B$ and 2.) if $x \in B$ then $x \in A$. Then we have $A \subseteq B$ and $B \subseteq A$.

∎

Remark 2. *This proposition is extremely useful. Virtually all proof involving set equality uses this proposition.*

By this point, we already know that *a set is the extension of a predicate*, expressed symbolically, $\{x|\varphi x\}$. And we have seen how this definition leads to Russell's paradox. Until now, we have been using set very carefully. But if we were to develop a reliable language for mathematics, we have to prevent Russell's paradox from ever arising, otherwise we could never be sure of the validity of our theorem.
What is really wrong with the "set of all set that does not contain itself"? For convenience, write $R = \{x|x \notin x\}$. Now we know that existence of R implies contradictions. So R cannot really exist. The writing "R" itself is merely an empty name that denotes nothing. To prevent naming a non-existent set, a strategy is to include only those elements from an already existing set.

Axiom 3. *(Axiom schema of restricted comprehension) Let φ be a predicate and A be a set. Then there exist a set B such that for any x, $x \in B$ if and only if $x \in A$ and φx. ($\vdash \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi x)$)*

From this axiom, I am only picking out elements that are already pre-existing by themselves.

Remark 3. *A curious point can be pointed out here. The natural language formulation of this axiom is rather odd. In predicate logic, quantifiers do NOT range across predicates, so the writing "Let φ be a predicate" is not technically legal. In the formal language formulation of this axiom, we are not confronted with this issue, the "proposition" $\vdash \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi x)$ is really left as a schema, the readers can substitute φ with any predicates they favour. Hence, the word "schema" in the name of the axiom.*

Proposition 2. *Let A be a set and φ be a predicate. Then the set defined by specification is unique*

Proof. Trivial. ∎

Remark 4. *For simplicity, we can write B as $\{x \in A|\phi x\}$*

Theorem 1. *There exist a set E, called an **empty set**, such that for any x, $x \notin \emptyset$.*

Proof. By the axiom of existence, there exist a set A. Let $E = \{x \in A|x \neq x\}$, then for any x, since $x = x$, then it is not the case that $x \neq x$. So, $x \notin E$. ∎

Proposition 3. *Let A be a set and φ be a predicate, then $\{x \in A|\varphi x\} \subseteq A$.*

Proof. Trivial. ∎

Proposition 4. *The empty set is unique.*

Proof. Let E, E' be empty sets. Then for any x, $x \notin E$. This makes the proposition "for any x, if $x \in E$ then $x \in E'$" true. Therefore, we have $E \subseteq E'$. By arguing the same way for E', we have $E' \subseteq E$. Hence, $E = E'$. ∎

Remark 5. *The empty set is denoted $\emptyset$.*

Proposition 5. *For any set A, $\emptyset \subseteq A$.*

Proof. Since $x \notin \emptyset$ for any x, the proposition "for any x, if $x \in \emptyset$ then $x \in A$" is true. So $\emptyset \subseteq A$. ∎

Definition 3. *(Set intersection) Let A, B be sets, the **intersection** of A and B, written $A \cap B$, is the set $A \cap B := \{x \in A|x \in B\}$*

Proposition 6. *(Commutativity of set intersection) For any set A, B, $A \cap B = B \cap A$.*

Proof.

- ($\subseteq$) For any x, suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. Then $x \in B$ and $x \in A$. So, $x \in B \cap A$.

- ($\supseteq$) By a similar argument, we can deduce that for any $x \in B \cap A$, $x \in A \cap B$. Therefore, $A \cap B = B \cap A$.

∎

Proposition 7. *(Associativity of set intersection) For any set A, B, C, $(A \cap B) \cap C = A \cap (B \cap C)$*

Proof. For any x, the following are equivalent

$$
\begin{aligned}
& x \in (A \cap B) \cap C \\
\Leftrightarrow & x \in A \cap B \text{ and } x \in C \\
\Leftrightarrow & (x \in A \text{ and } x \in B) and \in C \\
\Leftrightarrow & x \in A \text{ and } (x \in B \text{ and } x \in C) \\
\Leftrightarrow & x \in A \text{ and } (x \in B \cap C) \\
\Leftrightarrow & x \in A \cap (B \cap C)
\end{aligned}
$$

So, $(A \cap B) \cap C = A \cap (B \cap C)$ ■

Proposition 8. *For any set A, $A \cap A = A$*

Proof.

- ($\subseteq$) For all x, suppose $x \in A \cap A$. Then $x \in A$ and $x \in A$ and hence $x \in A$. Therefore, $x \in A$.
- ($\supseteq$) For all $x \in A$, we have $x \in A$ and hence $x \in A$ and $x \in A$. So, $x \in A \cap A$

■

Proposition 9. *For all set A, $A \cap \varnothing = \varnothing$*

Proof.

- ($\subseteq$) for any $x \in A \cap \varnothing$, we have $x \in \varnothing$.
- ($\supseteq$) for any x. Since $x \notin \varnothing$. Then it is not that case that $x \in A$ and $x \in \varnothing$. So, $x \notin A \cap \varnothing$ and therefore if $x \in A \cap \varnothing$, then $x \in \varnothing$.

■

Proposition 10. *For all set A, B, C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.*

Proof. For any $x \in A$, since $A \subseteq B$ and $A \subseteq C$, we have $x \in B$ and $x \in C$. So, $x \in B \cap C$. ■

Definition 4. *(Disjoint) Let A, B be sets. A, B are* ***disjoint*** *if $A \cap B = \varnothing$*

Definition 5. *(Pairwise disjoint) Let A be a set, A is* ***pairwise disjoint*** *when for any $x, y \in A$, if $x \neq y$, then $x \cap y = \varnothing$*

Definition 6. *(Set complement) Let A, B be sets. Then the* ***complement*** *of A with respect of B, denoted $A \setminus B$, is the set $A \setminus B := \{x \in A | x \notin B\}$*

Definition 7. *For any set A, $A \setminus \varnothing = A$*

Proof.

- ($\subseteq$) For any x, if $x \in A \setminus \varnothing$, then $x \in A$ and $x \notin \varnothing$. Then $x \in A$. Hence, $A \setminus \varnothing \subseteq A$.

- ($\supseteq$) For any x, if $x \in A$. By definition of empty set, $x \notin A$. So, $x \in A$ and $x \notin \emptyset$. Hence, $x \in A \setminus \emptyset$. Therefore, $A \subseteq A \setminus \emptyset$.

Therefore, $A \setminus \emptyset = A$. ■

Proposition 11. *For any set A, $A \setminus A = \emptyset$*

Proof. For any x, $x \notin \emptyset$ by definition and it is not the case that $x \in A$ and $x \notin A$. So, $x \notin \emptyset$ and $x \notin A \setminus A$. Then if $x \in \emptyset$ then $x \in A \setminus A$, and if $x \in A \setminus A$ then $x \notin \emptyset$. Therefore, $A \setminus A = \emptyset$. ■

Proposition 12. *For any set A, B, $A \setminus B \subseteq A$*

Proof. Pick any $x \in A \setminus B$, then we have $x \in A$ and $x \notin B$. In particular, we have $x \in A$ and hence $A \setminus B \subseteq B$. ■

5.1.2 Axiom of Pairing, Union, and Power Set

The readers may have complainted by now that the axiom schema of specification may be too restrictive. The only set we know exist for sure is the empty set. And the axiom of comprehension only allows us to create set from an existing set. As a result, we could not make much out from these. Hence, we rely on three axioms that create a larger set from pre-existing sets.
The *Axiom of Pair* describe the existence of a doubleton, the *Axiom of Union* for a generalised union, and the *Axiom of Power Set* for the power set.

Axiom 4. ***(Axiom of Pairing)*** *For all x, y, there is a set A such that $x \in A$ and $y \in A$. ($\vdash \forall x \forall y \exists z (x \in z \vee y \in z)$)*

Axiom 5. ***(Axiom of Union)*** *Let $\mathscr{A}$ be a set, then there exists a set B such that for any $A \in \mathscr{A}$, if $x \in A$, then $x \in B$. ($\vdash \forall x \exists y \forall z (z \in x \to z \in y)$)*

Axiom 6. ***(Axiom of Power Set)*** *Suppose A be a set, then there is a set P such that if $B \subseteq A$ is a subset, then $B \in P$. ($\vdash \forall x \exists y \forall z (z \subseteq x \to z \in y)$)*

It is actually wrong to say these axioms captures the idea of doubletons, unions, and power sets exactly. Take the Axiom of Pairing as an example, let x, y be two distinct object, then the set $\{x, y, \square, \triangle, \text{Hello}\}$ also satisfies the conditions stated in the axiom. $\square, \triangle$, and Hello are unwanted element that we would like to be excluded from our set. The set will be the desired doubleton only after we applied the Axiom Schema of Restricted Comprehension.

Proposition 13. *(Doubleton) For any x, y, there is a unique set A, called a **doubleton** of x, y, such that for any z, $z \in A$ if and only if $z = x$ or $z = y$.*

Proof. Let x, y be objects, then by the Axiom of Pairing, there is an A such that $x \in z$ or $y \in z$. Then let $D = \{w \in z | w = z \text{ or } w = y\}$. It follows immediately that D is the set we after. The Axiom Schema of Restricted Comprehension ensures its uniqueness. ■

Proposition 14. *(Singleton) For any x, there is a unique set S such that for any y, $y \in S$ if and only if $y = x$.*

Proof. The set $\{x, x\}$ is our desired set. ■

The Axiom of Union and the Axiom of Power Set also faces the same problem, we also need to use Axiom Schema of Restricted Specification on these two two axioms to obtain the desired sets. The proofs go very similarly as the one we just had.

Proposition 15. *(Power Set) For any A, there exist a unique P, called the **power set** of P, such that for any B, $B \in P$ if and only if $B \subseteq A$.*

Proof. Omitted. ■

Proposition 16. *(Union) For any $\mathscr{A}$, there is a unique set U, called the **generalised union of** $\mathscr{A}$, such that for any x, $x \in U$ if and only if $x \in A$ for some $A \in \mathscr{A}$.*

Proof. Omitted. ■

Proposition 17. *(Generalised Intersection) For any set $\mathscr{A}$, if $\mathscr{A}$ is non-empty, there is a set I such that for any x, $x \in I$ if and only if $x \in A$ for any $A \in \mathscr{A}$.*

Remark 6. *We write the generalised union of $\mathscr{A}$ as $\bigcup \mathscr{A}$ or $\bigcup\limits_{A \in \mathscr{A}} A$, the generalised intersection of $\mathscr{A}$ as $\bigcap \mathscr{A}$ or $\bigcap\limits_{A \in \mathscr{A}} A$ and the power set of A as $\mathcal{P}(A)$. Moreover, if $\mathscr{A} = \{A, B\}$, we write $\bigcup \mathscr{A} = A \cup B = B \cup A$, and $\bigcap A \cap B = B \cap A$. We call $A \cup B$ and $A \cap B$ simply their **union** and **intersections** respectively.*

Here, we will prove some very trivial yet important properties regarding union, intersections, and power sets.

Proposition 18. *(Associativity of set union) For any set A, B, C, $(A \cup B) \cup C = A \cup (B \cup C)$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cup B) \cup C \\
\Leftrightarrow &x \in (A \cup B) \text{ or } x \in C \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ or } x \in C \\
\Leftrightarrow &x \in A \text{ or } (x \in B \text{ or } x \in C) \\
\Leftrightarrow &x \in A \text{ or } x \in B \cup C \\
\Leftrightarrow &x \in A \cup (B \cup C)
\end{aligned}$$

Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$ ■

Proposition 19. *For any set A, B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$*

Proof. For any set x, if $x \in A$, then $x \in A$ or $x \in B$. Then $x \in A \cup B$. So, $x \in A \cup B$. Similarly, $B \subseteq B \cup A$. Since $A \cup B = B \cup A$, we have $B \subseteq A \cup B$. ■

Proposition 20. *For any set A, $A \cup \varnothing = A$*

Proof. ($\subseteq$) For any x, if $x \in A \cup \varnothing$, then $x \in A$ or $x \in \varnothing$. But since $x \in \varnothing$ by definition, $x \in A$. Therefore, $A \cup \varnothing \subseteq A$
($\supseteq$) For any x, if $x \in A$, then $x \in A$ or $x \in \varnothing$. Then $x \in A \cup \varnothing$. Then $A \subseteq A \cup \varnothing$
Therefore, $A \cup \varnothing = A$ ■

Proposition 21. *For any set A, $A \cup A = A$*

Proof. For any x, the following are equivlent.

$$\begin{aligned}
&x \in A\\
\Leftrightarrow &x \in A \text{ or } x \in A\\
\Leftrightarrow &x \in A \cup A
\end{aligned}$$

So, $A \cup A = A$ ■

Proposition 22. *For any set A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$*

Proof. Pick any $x \in A \cup B$, then $x \in A$ or $x \in B$.

- If $x \in A$, since $A \subseteq C$, we have $x \in C$
- If $x \in B$, since $B \subseteq C$, we have $x \in C$

In any case, $x \in C$. So, $A \cup B \subseteq C$. ■

Proposition 23. *(Distributivity between set intersection and set union) For any set A, B, C, $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$ and $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cup C) \cap (B \cup C)\\
\Leftrightarrow &x \in (A \cup C) \text{ and } x \in (B \cup C)\\
\Leftrightarrow &(x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C)\\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ or } x \in C\\
\Leftrightarrow &(x \in A \cap B) \text{ or } x \in C\\
\Leftrightarrow &x \in (A \cap B) \cup C
\end{aligned}$$

$$\begin{aligned}
&x \in (A \cap C) \cup (B \cap C)\\
\Leftrightarrow &x \in A \cap C \text{ or } x \in B \cap C\\
\Leftrightarrow &(x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C)\\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ and } x \in C\\
\Leftrightarrow &x \in A \cup B \text{ and } x \in C\\
\Leftrightarrow &x \in (A \cup B) \cap C
\end{aligned}$$

■

Corollary 1. *For any set A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*

Proof.

$$A \cap (B \cup C) = (B \cup C) \cap A = (B \cap A) \cup (C \cap A) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (B \cap C) \cup A = (B \cup A) \cap (C \cup A) = (A \cup B) \cap (A \cup C)$$

■

Proposition 24. *(Distributivity between set intersection and set complement) For any set A, B, C, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cap B) \setminus C \\
\Leftrightarrow &x \in A \cap B \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } (x \notin C \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ and } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cap (B \setminus C)
\end{aligned}$$

So, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$ ■

Proposition 25. *(Distributivity of set complement I) For any set A, B, C, $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ and $(A \cap B) \setminus C = (A \setminus C) \cup (A \setminus B)$*

Proof. For all x, the following are equivalent,

$$\begin{aligned}
&x \in (A \cup B) \setminus C \\
\Leftrightarrow &x \in A \cup B \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ or } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ or } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cup (B \setminus C)
\end{aligned}$$

So, $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

$$\begin{aligned}
&x \in (A \cap B) \setminus C \\
\Leftrightarrow &x \in (A \cap B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } (x \notin C \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ and } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cap (B \setminus C)
\end{aligned}$$

So, $(A \cap B) \setminus C = (A \setminus C) \cup (A \setminus B)$ ■

Proposition 26. *(Distributivity of set complement II/De Morgan's law)For any set A, B, C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$*

Proof. For any x, the following are equivalent,

$$\begin{aligned} & x \in A \setminus (B \cap C) \\ \Leftrightarrow & x \in A \text{ and } x \notin B \cap C \\ \Leftrightarrow & x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ \Leftrightarrow & (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ \Leftrightarrow & x \in A \setminus B \text{ or } x \in A \setminus C \\ \Leftrightarrow & x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

So, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$$\begin{aligned} & x \in A \setminus (B \cup C) \\ \Leftrightarrow & x \in A \text{ and } x \notin B \cup C \\ \Leftrightarrow & x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ \Leftrightarrow & (x \in A \text{ and } x \in A) \text{ and } (x \notin B \text{ and } x \notin C) \\ \Leftrightarrow & (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ \Leftrightarrow & x \in A \setminus B \text{ and } x \in A \setminus C \\ \Leftrightarrow & x \in (A \setminus B) \cap (A \setminus C) \end{aligned}$$

So, $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ ■

Proposition 27. *For any $A \in \mathscr{A}$, $A \subseteq \bigcup \mathscr{A}$*

Proof. For any $A \in \mathscr{A}$, for any $x \in A$, by the definition of generalised union, we have $x \in \mathscr{A}$, then $A \subseteq \bigcup \mathscr{A}$. ■

Proposition 28. *Let $\mathscr{A}$ be a set. Suppose $A \in \mathscr{A}$. Then $\bigcap \mathscr{A} \subseteq A$.*

Proof. For any $x \in \bigcap \mathscr{A}$, we have $x \in B$ for any $B \in \mathscr{A}$. In particular, we have $x \in A$. ■

Proposition 29. *For any $\mathscr{A}, \mathscr{B}$, if $\mathscr{A} \subseteq \mathscr{B}$, then $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B}$.*

Proof. Let $\mathscr{A} \subseteq \mathscr{B}$. The for any $x \in \bigcup A$, there is some $A \in \bigcup \mathscr{A}$ such that $x \in A$. Since $\mathscr{A} \subseteq B$, then A is also a subset of B, then $x \in \bigcup B$. ■

Proposition 30. *For any $\mathscr{A}, \mathscr{B}$, if $\mathscr{A} \subseteq \mathscr{B}$, then $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$*

Proof. Let $x \in \bigcap \mathscr{B}$, then for any $A \in \mathscr{A}$, since $\mathscr{A} \subseteq \mathscr{B}$, we have $A \in \mathscr{B}$. By definition of intersection, we have $x \in A$. Therefore, by the definition of intersection again, $x \in \bigcap \mathscr{A}$. ■

Proposition 31. *For any set A, $\varnothing \in \mathcal{P}(A)$*

Proof. Since $\varnothing \subseteq P$, $\varnothing \in \mathcal{P}(A)$ ∎

Proposition 32. *For any set A, $A \in \mathcal{P}(A)$.*

Proof. Since $A \subseteq A$, therefore $A \in \mathcal{P}(A)$. ∎

Proposition 33. *Let A, B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof.

- $(\Rightarrow)$
- $(\Leftarrow)$ For any $x \in A$, we have $\{x\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $\{x\} \in \mathcal{P}(B)$ as well. Then $\{x\} \subseteq B$ and hence $x \in B$.

∎

Part II

Mathematics