Undergraduate Mathematics

Ho Ka Him Kelvin

October 27, 2024

Preface

Contents

Part I

Foundation of Mathematics

Chapter 1

Logic and Philosophy

1.1 Formal and Informal Logic

There are two sorts of logic : *informal logic* and *formal logic*. Informal logic is the study of correct reasoning in our natural language. The kind of logic we are all familiar with. We use informal logic all the time. Even arguments as simple as

1. All men are mortal
2. Socrates is mortal
3. Ergo, Socrates is mortal

Is a display of informal logic.
Statement 1 and 2 are call the *premises*, and 3 is the *conclusion*. Notice that the argument feature above is *entirely independent of its context*. Meaning if we replace the word "men", "mortal", "Socrates" with other things, the argument still stands nonetheless. For example

1. All cats are cute
2. Willard is a cat
3. Therefore, Willard is cute

One would immediately recognise the two arguments are "the same". Not in terms of the meaning of each premise, but rather what made the first argument correct is quite clearly the same as the second argument. But not all arguments that follow the form "All M is P, S is M, so "S is P" would yield a true conclusion. Take, for example, that

1. All animals are four-legged
2. Chickens are animals
3. Chicken are four-legged

the conclusion is wrong. What made the conclusion wrong is not in the form of the argument, but because we included a false premise in the argument. What made an argument ”correct” requires two parts : 1.) We need the argument to have the right form. We call these arguments *valid*. A valid argument ensures that if its premises are true, then its conclusion must be true; 2.) We need the argument to have true premises. Only valid arguments with true premises have true conclusions. These arguments we call them *sound*.

What I have said here remains pretty vague. What makes an argument valid? What makes statements true? We all have a vague idea of validity and true, but not a clear-cut *definition* of them. We will clarify these ideas as we progress. We will not include a full-blown philosophical discourse. This is a book on mathematics afterall. But we will clarify these ideas ”clear enough” so that we can safely use them in constructing our mathematical universe.

Formal lo

1.2 Propositions, Facts, and Truth

1.3 Schema

Chapter 2

Naive Set Theory

Starting from this chapter, our goal is to develop the required logical tool to describe Mathematics. As history have went, set theory became our standard logical starting point to the Mathematical universe. But what really is a set?

2.1 Intention and Extension

”Meaning” is an ambiguous word. What does it mean by ”the planets of our Solar System”? There are two possible ways to answer. You can explain the concepts of ”planets” and ”our solar system” and how do they relate to each other. Our discussion remains on the conceptual level and mentions no concrete object, we call this kind of meaning the *intention*. You can also explain to the person ”the planets of our Solar System” as the list : Mercury, Venus, Earth, Mars, Jupiter, Saturn, Neptune, Uranus. You go beyond the concepts and mention the actual object. We are talking about the things agrees with the intention—- the *extensions*.

We always deem the intention more fundamental than the extension. This is why in Plato’s *Euthyphro*, when Euthyphro defines pious as ”to prosecute the wrongdoer”, Socrates dismiss his definitions and says ”I did not bid you tell me one or two of the many pious actions but that from itself that makes all pious actions pious.” He wanted the intentional meaning of ”pious”, not the extensional one.

If we do not make clear when we mean something, whether we are talking about its intention or extension, ambiguity arises. Consider the proposition ”All creatures with a heart is the same as all creatures with a kidney.” There are two ways to interpret it. If we are talking about intention, obviously the concepts involves in ”creatures with a heart” is different to those in ”creatures with a kidney”. So this proposition is false. To interpret it extensionally, we may go out to the world and discover all creatures that have a heart indeed also have a kidney and vice versa. Hence, it is indeed that case that the creatures with a heart is the same as the creature with a kidney. The proposition is true.

2.2 Sets and Extension

This duality of intention and extension is closely related to the notion of sets. Common wisdom would define a set as "a collection of things", and set elements are "things the set contains". These two "definitions" are problematic. Firstly, all collections must contain things. Then the phrase "a collection of things" makes basically no difference than "a collection". Thus, this definition basically equate sets with collections. What we are doing is just giving a new name to something without explaining it. Secondly, the definition of set elements only lies in the metaphoric level. Sets are, quite obviously, abstract object. A set do not really "contains" something. It has no space, nor location, nor an inside or outside. We all have a vague idea of a set, otherwise saying a set "contains" something is mere senseless. It is not. It is trying to communicate an idea. But we have to do better than this.

We must turn to analyse what we call "a collection". What makes a collection identifiable to us? We either enumerate all the things in that collections, or to identify some properties that are common to all members of this collection. Suppose Jack hands me a bag of ten candies and ask me to pick one. How could I possibly identify this bag of candies? I can either enumerate the candies : A strawberry-flavoured one, a lemon-flavoured one, ... Or I can identify this collection of candies as "all candies contained in the bag Jack handed to me". But do be careful! What am I interested is not discussing the concepts involved in "all candies contained in the bag Jack handed to me", but the extensions that falls under the concept, i.e. the candies in Jack's bag.

In logic, the concept *predicate* captures the idea of "intention". Then sets, being a logical tool, is exactly the *extension of the predicate*. How about enumeration? Enumeration is just a special kind of predicate. In saying that "Jack is a member of the collection made up by Jack, Jacky, and Jackson" makes no difference to saying "Jack belongs to the set given by the predicate 'x is Jack or x is Jacky or x is Jackson'". Hence, we can now explicate the notion of sets as

Definition 1. ***(Sets)*** *A set is a definable totality of all extensions of a predicate. x is said to be a* ***member*** *of the set, if the predicate applies to x.*

The reason for including the word "definable" will be explained in the next section. For convenience, we let φx represent a predicate, and the set corresponding to that predicate as $\{x|\varphi x\}$. If x is a member of the set A, we write $x \in A$.

An important aspect of sets we shall discuss is when should sets be equal and when they should not. We have now made clear that sets correspond to the extension of predicates. Then since sets appeal to the extensional side of meaning, the proposition

$$\{x|x \text{ is a creature with a heart}\} = \{x|x \text{ is a creature with a kidney}\}$$

should be regarded as true despite the intention is different. Hence, we have the following axiom.

Axiom 1. ***(Axiom of Extensionality)*** *Suppose A, B are sets, then the following are equivalent*

- $A = B$
- *for any x, $x \in A$ if and only if $x \in B$.*

Except talking about equality, we may also talk about subset, i.e. a part of a set.

Definition 2. ***(Subset relation)*** *Let A, B be sets, then A is a* ***subset*** *of B, written $A \subseteq B$, if for any x, $x \in A$ implies $y \in B$.*

Definition 3. ***(Strict subset)*** *Let A, B be sets, then A is a* ***strict subset*** *of B, written $A \subset B$, if $A \subseteq B$ and $A \neq B$.*

Proposition 1. *Let A, B be sets, then the following are equivalent*

- $A = B$
- *$A \subseteq B$ and $B \subseteq A$*

Proof.

- ($\Rightarrow$) Suppose $A = B$. Then for any x, $x \in A$ if and only if $x \in B$. Then we have $A \subseteq B$ and $B \subseteq A$.
- ($\Leftarrow$) Suppose $A = B$, then for any x, we have 1). if $x \in A$ then $x \in B$ and 2.) if $x \in B$ then $x \in A$. Then we have $A \subseteq B$ and $B \subseteq A$.

∎

Remark 1. *This proposition is extremely useful. Virtually all proof involving set equality uses this proposition.*

Proposition 2. ***(Transitivity of Subset Relation)*** *Let A, B, C be set, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. For any $x \in A$, since $A \subseteq B$, we have $x \in B$. Since $B \subseteq C$, we have $x \in C$. Therefore, we have $A \subseteq C$. ∎

2.3 Specification and Russell's Paradox

Naive was once praised for its success in establishing the foundation for Mathematics. And by no doubt it still it. But there had been a period where paradoxes (or Mathematicians and Philosophers at that time would have called "antinomies") were found within set theory, and the foundation of Mathematics was in the cusp of collapsing. Here are some of them.

Paradox. ***(Russell's Paradox)*** *The set of all sets that do not contain itself, i.e.* $\{x|x \notin x\}$ *is contradictory.*

Proof. We write R as the set $R = \{x|x \notin x\}$. We will now show that $R \in R$ and $R \notin R$ simultaneously.

- Suppose $R \in R$, then it is not the case that $R \notin R$, therefore, the predicate $x \notin x$ does not apply on R. So $R \notin R$.
- But it $R \notin R$, then the predicate $x \in x$ applies on R, then we have actually $R \in R$.

Arguing from both case, we would derive $R \in R$ and $R \notin R$. A contradiction! ∎

This exploded set theory. Especially the fact that this paradox lies at the doorstep of set theory seem to render all works in set theory useless. We must investigate why the contradiction occurs.

From this paradox, we discovered the existence of R is contradictory, so the set R must not exist. Then every time when we use the writing "R" or "$\{x|x \notin x\}$", the writing is just an empty name, a name that stands for nothing. So it is meaningless to actually talk about R.

The readers should notice the said paradox only arise when we are discussing sets of sets, or equivalently, sets whose predicates apply to sets. Therefore, a specification $\{x|\varphi x\}$ where φx is a predicate only applies to non-sets is still valid. Then the problem at hand is to discuss the treatment of sets of sets, or more simply, *collections*.

Now defining a collection from a predicate, as we have seen, might lead to contradictions. But what if we take the subset of a pre-existing collection by a predicate? Since each set in the collection already exists, th

Axiom 2. ***(Axiom Schema of Restricted Comprehension)*** *Let φ be a predicate and A be a set. Then there exist a set B such that for any x, $x \in B$ if and only if $x \in A$ and φx.*

Now that we have resolve Russell's paradox. But this leads us to an unsatisfactory result — it seems that some predicates do not have a corresponding set. But this go against our intuition that all for any predicate, there is a totality of extensions of that predicate. What went wrong? Consider the "set" $R = \{x|x \notin x\}$. In normal case, we only wanted the object R to act as a "container" of all sets with the property $x \notin x$. We never intended R to be inside itself.

We have placed ourselves in a situation where, 1.) we want to preserve the existence of the totality of extension for every predicate, 2.) but we also want to prevent Russell's paradox. This forces us into believing that R is not a set. It is an object that also represents to totality of extensions of a predicate, but not a set. Hence, the following definition

Definition 4. ***(Class)***

Proposition 3. *Let A be a set and φ be a predicate. Then the set defined by specification is unique*

Proof. Let A be a set and φ is predicate, then let B, C be sets specified by the Axiom Schema of Restricted Comprehension. Then for any x, $x \in B$ if and only if $x \in A$ and φx and $x \in C$ if and only if $x \in A$ and φx. Therefore, we have $x \in B$ if and only if $x \in C$. Hence, $B = C$. ∎

Remark 2. *For simplicity, we can write B as $\{x \in A|\phi x\}$*

Theorem 1. *There exist a set E, called an **empty set**, such that for any x, $x \notin \varnothing$.*

Proof. By the axiom of existence, there exist a set A. Let $E = \{x \in A|x \neq x\}$, then for any x, since $x = x$, then it is not the case that $x \neq x$. So, $x \notin E$. ∎

Proposition 4. *Let A be a set and φ be a predicate, then $\{x \in A|\varphi x\} \subseteq A$.*

Proof. Trivial. ∎

Proposition 5. *The empty set is unique.*

Proof. Let E, E' be empty sets. Then for any x, $x \notin E$. This makes the proposition "for any x, if $x \in E$ then $x \in E'$" true. Therefore, we have $E \subseteq E'$. By arguing the same way for E', we have $E' \subseteq E$. Hence, $E = E'$. ∎

Remark 3. *The empty set is denoted $\varnothing$.*

Proposition 6. *For any set A, $\varnothing \subseteq A$.*

Proof. Since $x \notin \varnothing$ for any x, the proposition "for any x, if $x \in \varnothing$ then $x \in A$" is true. So $\varnothing \subseteq A$. ∎

Definition 5. *(Set intersection) Let A, B be sets, the **intersection** of A and B, written $A \cap B$, is the set $A \cap B := \{x \in A|x \in B\}$*

Proposition 7. *(Commutativity of set intersection) For any set A, B, $A \cap B = B \cap A$.*

Proof.

- ($\subseteq$) For any x, suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. Then $x \in B$ and $x \in A$. So, $x \in B \cap A$.
- ($\supseteq$) By a similar argument, we can deduce that for any $x \in B \cap A$, $x \in A \cap B$. Therefore, $A \cap B = B \cap A$.

∎

Proposition 8. *(Associativity of set intersection) For any set A, B, C, $(A \cap B) \cap C = A \cap (B \cap C)$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cap B) \cap C \\
\Leftrightarrow &x \in A \cap B \text{ and } x \in C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) and \in C \\
\Leftrightarrow &x \in A \text{ and } (x \in B \text{ and } x \in C) \\
\Leftrightarrow &x \in A \text{ and } (x \in B \cap C) \\
\Leftrightarrow &x \in A \cap (B \cap C)
\end{aligned}$$

So, $(A \cap B) \cap C = A \cap (B \cap C)$ ∎

Proposition 9. *For any set A, $A \cap A = A$*

Proof.

- ($\subseteq$) For all x, suppose $x \in A \cap A$. Then $x \in A$ and $x \in A$ and hence $x \in A$. Therefore, $x \in A$.
- ($\supseteq$) For all $x \in A$, we have $x \in A$ and hence $x \in A$ and $x \in A$. So, $x \in A \cap A$

∎

Proposition 10. *For all set A, $A \cap \varnothing = \varnothing$*

Proof.

- ($\subseteq$) for any $x \in A \cap \varnothing$, we have $x \in \varnothing$.
- ($\supseteq$) for any x. Since $x \notin \varnothing$. Then it is not that case that $x \in A$ and $x \in \varnothing$. So, $x \notin A \cap \varnothing$ and therefore if $x \in A \cap \varnothing$, then $x \in \varnothing$.

∎

Proposition 11. *For all set A, B, C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.*

Proof. For any $x \in A$, since $A \subseteq B$ and $A \subseteq C$, we have $x \in B$ and $x \in C$. So, $x \in B \cap C$. ∎

Definition 6. *(Disjoint) Let A, B be sets. A, B are **disjoint** if $A \cap B = \varnothing$*

Definition 7. *(Pairwise disjoint) Let A be a set, A is **pairwise disjoint** when for any $x, y \in A$, if $x \neq y$, then $x \cap y = \varnothing$*

Definition 8. *(Set complement) Let A, B be sets. Then the **complement** of A with respect of B, denoted $A \setminus B$, is the set $A \setminus B := \{x \in A | x \notin B\}$*

Definition 9. *For any set A, $A \setminus \varnothing = A$*

Proof.

- ($\subseteq$) For any x, if $x \in A \setminus \varnothing$, then $x \in A$ and $x \notin \varnothing$. Then $x \in A$. Hence, $A \setminus \varnothing \subseteq A$.

- ($\supseteq$) For any x, if $x \in A$. By definition of empty set, $x \notin A$. So, $x \in A$ and $x \notin \varnothing$. Hence, $x \in A \setminus \varnothing$. Therefore, $A \subseteq A \setminus \varnothing$.

Therefore, $A \setminus \varnothing = A$. ∎

Proposition 12. *For any set A, $A \setminus A = \varnothing$*

Proof. For any x, $x \notin \varnothing$ by definition and it is not the case that $x \in A$ and $x \notin A$. So, $x \notin \varnothing$ and $x \notin A \setminus A$. Then if $x \in \varnothing$ then $x \in A \setminus A$, and if $x \in A \setminus A$ then $x \notin \varnothing$. Therefore, $A \setminus A = \varnothing$. ∎

Proposition 13. *For any set A, B, $A \setminus B \subseteq A$*

Proof. Pick any $x \in A \setminus B$, then we have $x \in A$ and $x \notin B$. In particular, we have $x \in A$ and hence $A \setminus B \subseteq B$. ∎

2.4 Axiom of Pairing, Union, and Power Set

The readers may have complainted by now that the axiom schema of specification may be too restrictive. The only set we know exist for sure is the empty set. And the axiom of comprehension only allows us to create set from an existing set. As a result, we could not make much out from these. Hence, we rely on three axioms that create a larger set from pre-existing sets.
The *Axiom of Pair* describe the existence of a doubleton, the *Axiom of Union* for a generalised union, and the *Axiom of Power Set* for the power set.

Axiom 3. ***(Axiom of Pairing)*** *For all x, y, there is a set A such that $x \in A$ and $y \in A$. ($\vdash \forall x \forall y \exists z (x \in z \vee y \in z)$)*

Axiom 4. ***(Axiom of Union)*** *Let $\mathscr{A}$ be a set, then there exists a set B such that for any $A \in \mathscr{A}$, if $x \in A$, then $x \in B$. ($\vdash \forall x \exists y \forall z (z \in x \to z \in y)$)*

Axiom 5. ***(Axiom of Power Set)*** *Suppose A be a set, then there is a set P such that if $B \subseteq A$ is a subset, then $B \in P$. ($\vdash \forall x \exists y \forall z (z \subseteq x \to z \in y)$)*

It is actually wrong to say these axioms captures the idea of doubletons, unions, and power sets exactly. Take the Axiom of Pairing as an example, let x, y be two distinct object, then the set $\{x, y, \square, \triangle, \text{Hello}\}$ also satisfies the conditions stated in the axiom. $\square$, $\triangle$, and Hello are unwanted element that we would like to be excluded from our set. The set will be the desired doubleton only after we applied the Axiom Schema of Restricted Comprehension.

Proposition 14. *(Doubleton) For any x, y, there is a unique set A, called a **doubleton** of x, y, such that for any z, $z \in A$ if and only if $z = x$ or $z = y$.*

Proof. Let x, y be objects, then by the Axiom of Pairing, there is an A such that $x \in z$ or $y \in z$. Then let $D = \{w \in z | w = z \text{ or } w = y\}$. It follows immediately that D is the set we after. The Axiom Schema of Restricted Comprehension ensures its uniqueness. ∎

Proposition 15. *(Singleton) For any x, there is a unique set S such that for any y, $y \in S$ if and only if $y = x$.*

Proof. The set $\{x, x\}$ is our desired set. ■

The Axiom of Union and the Axiom of Power Set also faces the same problem, we also need to use Axiom Schema of Restricted Specification on these two two axioms to obtain the desired sets. The proofs go very similarly as the one we just had.

Proposition 16. *(Power Set) For any A, there exist a unique P, called the **power set** of P, such that for any B, $B \in P$ if and only if $B \subseteq A$.*

Proof. Omitted. ■

Proposition 17. *(Union) For any $\mathscr{A}$, there is a unique set U, called the **generalised union of** $\mathscr{A}$, such that for any x, $x \in U$ if and only if $x \in A$ for some $A \in \mathscr{A}$.*

Proof. Omitted. ■

Proposition 18. *(Generalised Intersection) For any set $\mathscr{A}$, if $\mathscr{A}$ is non-empty, there is a set I such that for any x, $x \in I$ if and only if $x \in A$ for any $A \in \mathscr{A}$.*

Remark 4. *We write the generalised union of $\mathscr{A}$ as $\bigcup \mathscr{A}$ or $\bigcup_{A \in \mathscr{A}} A$, the generalised intersection of $\mathscr{A}$ as $\bigcap \mathscr{A}$ or $\bigcap_{A \in \mathscr{A}} A$ and the power set of A as $\mathcal{P}(A)$. Moreover, if $\mathscr{A} = \{A, B\}$, we write $\bigcup \mathscr{A} = A \cup B = B \cup A$, and $\bigcap A \cap B = B \cap A$. We call $A \cup B$ and $A \cap B$ simply their **union** and **intersections** respectively.*

Here, we will prove some very trivial yet important properties regarding union, intersections, and power sets.

Proposition 19. *(Associativity of set union) For any set A, B, C, $(A \cup B) \cup C = A \cup (B \cup C)$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cup B) \cup C \\
\Leftrightarrow &x \in (A \cup B) \text{ or } x \in C \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ or } x \in C \\
\Leftrightarrow &x \in A \text{ or } (x \in B \text{ or } x \in C) \\
\Leftrightarrow &x \in A \text{ or } x \in B \cup C \\
\Leftrightarrow &x \in A \cup (B \cup C)
\end{aligned}$$

Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$ ■

Proposition 20. *For any set A, B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$*

Proof. For any set x, if $x \in A$, then $x \in A$ or $x \in B$. Then $x \in A \cup B$. So, $x \in A \cup B$. Similarly, $B \subseteq B \cup A$. Since $A \cup B = B \cup A$, we have $B \subseteq A \cup B$. ■

Proposition 21. *For any set A, $A \cup \varnothing = A$*

Proof. ($\subseteq$) For any x, if $x \in A \cup \varnothing$, then $x \in A$ or $x \in \varnothing$. But since $x \in \varnothing$ by definition, $x \in A$. Therefore, $A \cup \varnothing \subseteq A$
($\supseteq$) For any x, if $x \in A$, then $x \in A$ or $x \in \varnothing$. Then $x \in A \cup \varnothing$. Then $A \subseteq A \cup \varnothing$
Therefore, $A \cup \varnothing = A$ ■

Proposition 22. *For any set A, $A \cup A = A$*

Proof. For any x, the following are equivlent.

$$\begin{aligned} & x \in A \\ \Leftrightarrow & x \in A \text{ or } x \in A \\ \Leftrightarrow & x \in A \cup A \end{aligned}$$

So, $A \cup A = A$ ■

Proposition 23. *For any set A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$*

Proof. Pick any $x \in A \cup B$, then $x \in A$ or $x \in B$.

- If $x \in A$, since $A \subseteq C$, we have $x \in C$
- If $x \in B$, since $B \subseteq C$, we have $x \in C$

In any case, $x \in C$. So, $A \cup B \subseteq C$. ■

Proposition 24. *(Distributivity between set intersection and set union) For any set A, B, C, $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$ and $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$*

Proof. For any x, the following are equivalent

$$\begin{aligned} & x \in (A \cup C) \cap (B \cup C) \\ \Leftrightarrow & x \in (A \cup C) \text{ and } x \in (B \cup C) \\ \Leftrightarrow & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \\ \Leftrightarrow & (x \in A \text{ and } x \in B) \text{ or } x \in C \\ \Leftrightarrow & (x \in A \cap B) \text{ or } x \in C \\ \Leftrightarrow & x \in (A \cap B) \cup C \end{aligned}$$

$$\begin{aligned} & x \in (A \cap C) \cup (B \cap C) \\ \Leftrightarrow & x \in A \cap C \text{ or } x \in B \cap C \\ \Leftrightarrow & (x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C) \\ \Leftrightarrow & (x \in A \text{ or } x \in B) \text{ and } x \in C \\ \Leftrightarrow & x \in A \cup B \text{ and } x \in C \\ \Leftrightarrow & x \in (A \cup B) \cap C \end{aligned}$$

■

Corollary 1. *For any set A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*

Proof.

$$A \cap (B \cup C) = (B \cup C) \cap A = (B \cap A) \cup (C \cap A) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (B \cap C) \cup A = (B \cup A) \cap (C \cup A) = (A \cup B) \cap (A \cup C)$$

∎

Proposition 25. *(Distributivity between set intersection and set complement) For any set A, B, C, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cap B) \setminus C \\
\Leftrightarrow &x \in A \cap B \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } (x \notin C \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ and } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cap (B \setminus C)
\end{aligned}$$

So, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$ ∎

Proposition 26. *(Distributivity of set complement I) For any set A, B, C, $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ and $(A \cap B) \setminus C = (A \setminus C) \cup (A \setminus B)$*

Proof. For all x, the following are equivalent,

$$\begin{aligned}
&x \in (A \cup B) \setminus C \\
\Leftrightarrow &x \in A \cup B \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ or } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ or } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cup (B \setminus C)
\end{aligned}$$

So, $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

$$\begin{aligned}
&x \in (A \cap B) \setminus C \\
\Leftrightarrow &x \in (A \cap B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } (x \notin C \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ and } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cap (B \setminus C)
\end{aligned}$$

So, $(A \cap B) \setminus C = (A \setminus C) \cup (A \setminus B)$ ∎

Proposition 27. *(Distributivity of set complement II/De Morgan's law)For any set A, B, C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$*

Proof. For any x, the following are equivalent,

$$\begin{aligned}
&x \in A \setminus (B \cap C)\\
\Leftrightarrow & x \in A \text{ and } x \notin B \cap C\\
\Leftrightarrow & x \in A \text{ and } (x \notin B \text{ or } x \notin C)\\
\Leftrightarrow & (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)\\
\Leftrightarrow & x \in A \setminus B \text{ or } x \in A \setminus C\\
\Leftrightarrow & x \in (A \setminus B) \cup (A \setminus C)
\end{aligned}$$

So, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$$\begin{aligned}
&x \in A \setminus (B \cup C)\\
\Leftrightarrow & x \in A \text{ and } x \notin B \cup C\\
\Leftrightarrow & x \in A \text{ and } (x \notin B \text{ and } x \notin C)\\
\Leftrightarrow & (x \in A \text{ and } x \in A) \text{ and } (x \notin B \text{ and } x \notin C)\\
\Leftrightarrow & (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C)\\
\Leftrightarrow & x \in A \setminus B \text{ and } x \in A \setminus C\\
\Leftrightarrow & x \in (A \setminus B) \cap (A \setminus C)
\end{aligned}$$

So, $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ ■

Proposition 28. *For any $A \in \mathscr{A}$, $A \subseteq \bigcup \mathscr{A}$*

Proof. For any $A \in \mathscr{A}$, for any $x \in A$, by the definition of generalised union, we have $x \in \mathscr{A}$, then $A \subseteq \bigcup \mathscr{A}$. ■

Proposition 29. *Let $\mathscr{A}$ be a set. Suppose $A \in \mathscr{A}$. Then $\bigcap \mathscr{A} \subseteq A$.*

Proof. For any $x \in \bigcap \mathscr{A}$, we have $x \in B$ for any $B \in \mathscr{A}$. In particular, we have $x \in A$. ■

Proposition 30. *For any $\mathscr{A}, \mathscr{B}$, if $\mathscr{A} \subseteq \mathscr{B}$, then $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B}$.*

Proof. Let $\mathscr{A} \subseteq \mathscr{B}$. The for any $x \in \bigcup A$, there is some $A \in \bigcup \mathscr{A}$ such that $x \in A$. Since $\mathscr{A} \subseteq B$, then A is also a subset of B, then $x \in \bigcup B$. ■

Proposition 31. *For any $\mathscr{A}, \mathscr{B}$, if $\mathscr{A} \subseteq \mathscr{B}$, then $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$*

Proof. Let $x \in \bigcap \mathscr{B}$, then for any $A \in \mathscr{A}$, since $\mathscr{A} \subseteq \mathscr{B}$, we have $A \in \mathscr{B}$. By definition of intersection, we have $x \in A$. Therefore, by the definition of intersection again, $x \in \bigcap \mathscr{A}$. ■

Proposition 32. *For any set A, $\varnothing \in \mathcal{P}(A)$*

Proof. Since $\varnothing \subseteq P$, $\varnothing \in \mathcal{P}(A)$ ■

Proposition 33. *For any set A, $A \in \mathcal{P}(A)$.*

Proof. Since $A \subseteq A$, therefore $A \in \mathcal{P}(A)$. ■

Proposition 34. *Let A, B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof.

- ($\Rightarrow$) Let $C \in \mathcal{P}(A)$, which implies $C \subseteq B$. Then by properties of subset relation, we have $C \subseteq A$. Therefore, $C \in \mathcal{P}(B)$.
- ($\Leftarrow$) For any $x \in A$, we have $\{x\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $\{x\} \in \mathcal{P}(B)$ as well. Then $\{x\} \subseteq B$ and hence $x \in B$.

■

Proposition 35. *Let A, B be sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$*

Proof. For any C, the following are equivalent

$$\begin{aligned}
&C \in \mathcal{P}(A) \cap \mathcal{P}(B)\\
\Leftrightarrow &C \in \mathcal{P}(A) \text{ and } C \in \mathcal{P}(B)\\
\Leftrightarrow &C \subseteq A \text{ and } C \subseteq B\\
\Leftrightarrow &C \subseteq A \cap B\\
\Leftrightarrow &C \in \mathcal{P}(A \cap B)
\end{aligned}$$

■

Proposition 36. *Let A, B be sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.*

Proof. For any $C \in \mathcal{P}(A) \cup \mathcal{P}(B)$, we have $C \subseteq A$ or $C \subseteq B$. In any case, we have $C \subseteq A \cup B$. Therefore, $C \in \mathcal{P}(A \cup B)$ ■

Remark 5. *Readers may be tempted to state that $\mathcal{P}(A) \cup \mathcal{P}(A) = \mathcal{P}(A \cup B)$, so that the two preceding propositions will be parallel. But it is indeed that case that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ is not necessarily true. Consider the set $\{x\}, \{y\}$, where x, y are distinct objects. Then we have the following*

$$\begin{aligned}
\mathcal{P}(\{x\}) &= \{\varnothing, \{x\}\}\\
\mathcal{P}(\{y\}) &= \{\varnothing, \{y\}\}\\
\mathcal{P}(\{x\}) \cup \mathcal{P}(\{y\}) &= \{\varnothing, \{x\}, \{y\}\}\\
\mathcal{P}(\{x\} \cup \{y\}) &= \{\varnothing, \{x, y\}\}
\end{aligned}$$

Clearly, we cannot have $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ is not necessarily true. Consider the set $\{x\}, \{y\}$.

Proposition 37. *Let A be a set, then $\bigcup \mathcal{P}(A) = A$*

Proof. Let x be an object, the followings are equivalent

$$\begin{aligned} & x \in \bigcup \mathcal{P}(A) \\ \Leftrightarrow & x \in B \text{ for some } B \in \mathcal{P}(A) \\ \Leftrightarrow & x \in B \text{ for some } B \subseteq A \end{aligned}$$

- ($\subseteq$) For any $x \in \bigcup \mathcal{P}(A)$, we have $x \in B$ for some $B \in \mathcal{P}(A)$. It follows that $B \subseteq A$ and hence $x \in A$.
- ($\supseteq$) For any $x \in A$, we have $\{x\} \in \mathcal{P}(A)$. This implies that $\{x\} \subseteq \bigcup \mathcal{P}(A)$. Since $x \in \{x\}$, we have $x \in \bigcup \mathcal{P}(A)$.

■

2.5 Relations and Functions

Now an important concept that will be useful for us is orderliness. Now the notation $\{a, b\}$ is just a convenient notation. It does not convey any sense of order since $\{a, b\} = \{b, a\}$. We need to express order in more sophisticated way.

Definition 10. *(Ordered pair) For all x, y, the order pair (x, y) is defined as $(x, y) = \{\{x\}, \{x, y\}\}$*

Definition 11. *(Ordered triple) For all x, y, z, the ordered triple (x, y, z) is defined as $(x, y, z) = ((x, y), z)$*

Proposition 38. *For all a, b, c, d, $(a, b) = (c, d)$ if and only if $a = c$, $b = d$*

Proof.

- ($\Rightarrow$) Suppose $(a, b) = (c, d)$.
  - (Case 1) Suppose $a = b$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\}$. Also, $\{\{c\}, \{c, d\}\} = (c, d) = (a, b) = \{\{a\}\}$. Thus, $\{c\} = \{c, d\} = \{a\}$ and therefore $c = d = a$. Since $a = b$, $b = d$,
  - (Case 2) Suppose $a \neq b$. Since $(a, b) = (c, d)$, we have $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. Then $a = c$ or $a = c = d$. In any case, $a = c$. We have also $\{b\} = \{c\}$ or $\{b\} = \{c, d\}$. But since $b \neq a$, $b \neq c$. Then it is not the case that $\{b\} = \{c\}$. So, $\{b\} = \{c, d\}$. Then we have $b = c$ or $b = d$. Since $b \neq c$, we have $b = d$.
- ($\Leftarrow$) Suppose $a = c$, $b = d$. Then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$

■

Proposition 39. *For all a, b, c, d, e, f, $(a, b, c) = (d, e, f)$ if and only if $a = d, b = e, c = f$*

Proof.

- ($\Rightarrow$) Suppose $(a, b, c) = (d, e, f)$, then $\{((a, b), c)\} = ((d, e), f)$. Then $(a, b) = (d, e)$ and $c = f$. Since $(a, b) = (d, e)$, we have $a = d$ and $b = e$.
- ($\Leftarrow$) Suppose $a = d, b = e, c = f$, then $(a, b, c) = ((a, b), c) = ((d, e), f) = (d, e, f)$

■

These definitions are extremely messy. This is why we will cease at ordered triple. We will define the rest once we have developed an account of functions and natural numbers.

Proposition 40. ***(Cartesian product)*** *For any set A, B, there exist a unique set C such that for any z, $z \in C$ if and only if $z = (x, y)$ for some $x \in A, y \in B$.*

Proof. The desired C is the set $C = \{z \in \mathcal{P}(\mathcal{P}(A \cup B))|$ there exist some unique $x \in A, y \in B$ such that $(x, y) = z\}$ ■

Remark 6. *We will denote the Cartesian product of A, B as $A \times B$.*

An important concept we use in our ordinary language is the concept of *relation*.

Of course there is more to set theory than what was discussed in this chapter. But what we have now is enough for us to define the necessary logical tools that develop a language for Mathematics. We will continue our discussion of set theory in the next part of the book.

Chapter 3

Propositional Logic

We have now entered the world of formal logic.

Chapter 4

Predicate Logic

Part II

Mathematics

Chapter 5

Axiomatic Set Theory

5.1 Zermelo-Frankel Set Theory

In the previous chapter on Naive set theory we have already developed a minimal account of sets. Now that we have transferred our domain of discussion from the meta-language to the object language, we will continue our discussion of it.

A question one might ask is : Why do we need axiomatisation? Why don't we just continue our discussion in natural language. Our preceding discussion on set theory rely on an "trial-and-error" method. We give out a first characterisation of a concept, find some problems with it, revise it with a better one. This give rise to a problem — What if we missed something? What if we were in fact mistaken with our characterisation without knowing it? To prevent our fourth crisis in Mathematics, we would like to introduce an axiomatic system. The validity of all theorems in this system rely on a small set of axioms. Then we can direct all our attention to only those axioms instead of inspecting every construction we have made. Only an axiomatised system can give us hope in devloping a valid account of Mathematics. But reader should always bear in mind an axiomatised system only ensures the validity of the language, but the language itself is not the entirety of Mathematics.

In the ontology of Zermelo-Frankel Set Theory(ZFC), everything is a set. This means every object in ZFC is either a collection or an empty set. But Mathematical objects are not themselves sets, at least we do not expect them do. Sets are merely a useful and powerful way to *represent* those objects. This is why we must not confuse the language with the subject of discourse. This point would require more discussion. But they are only possible once we have reached the construction of different kinds of number.

We will define some notations that we already semantically know their meaning. We will reiterate them in symbolic logic.

Axiom 1. ***(Zermelo-Frankel Axioms)***

- ***ZFC1*** ***(Axiom of Existence)*** *A set exists.*
  $\vdash \exists x(x = x)$
- ***ZFC2*** ***(Axiom of Foundation)*** *All sets contains a member that is disjoint to itself*
  $\vdash \forall x(\exists y(y \in x) \to \exists z(z \in x \land \neg\exists w(w \in z \land w \in x)))$
- ***ZFC3*** ***(Axiom of Extension)*** *Two sets are equal if and only if they have the same elements*
  $\vdash \forall x\forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$
- ***ZFC4*** ***(Axiom Schema of Specification)*** *Given a predicate φ. Let A be a set, then there is a set B such that for all $x \in B$ if and only if $x \in A$ and φx.*
  $\vdash \forall x\exists y\forall z(z \in y \leftrightarrow z \in x \land \varphi x)$
- ***ZFC5*** ***(Axiom of Pairing)*** *Let x, y be objects. Then there is a set containing x and y.*
  $\vdash \forall x\forall y\exists z(x \in z \land y \in z)$
- ***ZFC6*** ***(Axiom of Union)*** *For each collection, there is a set that contains all the members of each sets in the collection.*
  $\vdash \forall x\exists y\forall z\forall w(w \in y \land y \in x \to w \in y)$
- ***ZFC7*** ***(Axiom of Power)*** *There is a collection that contains all the subsets of a set.*
  $\vdash \forall x\exists y\forall z(\forall w(w \in z \to w \in x)) \to z \in y$
- ***ZFC8*** ***(Axiom Schema of Replacement)***
- ***ZFC9*** ***(Axiom of Infinity)*** *There is a set that contains the empty set and if x is a member of the set, then $x \cup \{x\}$ is also a member of the set.*
  $\vdash \exists x((\exists y(\forall z(z \notin y)) \land y \in x) \land \forall w(w \in x \to \exists u(\forall v(v \in u \leftrightarrow v \in w \lor v = w)) \land u \in x))$
- ***ZFC10*** ***(Axiom of Choice)*** *For every collection, there is a set that contains exactly one element from each set of that collection.*
  $\vdash \forall x((\exists a(a \in x) \land \forall r\forall s(r \in x \land s \in x \land r \neq s \to \forall t(t \in r \leftrightarrow t \notin s))) \to \exists y\forall z(z \in x \to \exists w(w \in z \land w \in y \land (\forall \ell(\ell \in z \land \ell \in y \to \ell = w)))))$

Remark 1. *I know, the axioms are ugly. An unconditional pursuit for rigor would result in a loss of mathematical beauty and intuition. Please bear with me, we will return to more accessible plain English formulation in a section or two.*

Remark 2. *All the relevant proof and definition related to the Axiom of Extension, Axiom Schema of Specification, Axiom of Pairing, Axiom of Union, and Axiom of Power could be rewritten into symbolic logic and inserted here. But for the sake of brevity, we will only hereby prove the major results.*

We had quite some discussion with Axiom of Extension, Axiom Schema of Specification, Axiom of Pairing, Axiom of Union, and Axiom of Power. But in addition to these, we have some new axioms as well. These axioms are all results proved to be necessary and unprovable from the rest of ZFC. We will only discuss axiom of foundation here. We delay the discussion of Axiom Schema of Replacement, Axiom of Infinity, and Axiom of Choice until later.