Undergraduate Mathematics

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Preface

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Part I

Foundation of Mathematics

Chapter 1

Logic and Philosophy

1.1 Formal and Informal Logic

There are two sorts of logic : *informal logic* and *formal logic*. Informal logic is the study of correct reasoning in our natural language. The kind of logic we are all familiar with. We use informal logic all the time. Even arguments as simple as

1. All men are mortal
2. Socrates is mortal
3. Ergo, Socrates is mortal

Is a display of informal logic.
Statement 1 and 2 are call the *premises*, and 3 is the *conclusion*. Notice that the argument feature above is *entirely independent of its context*. Meaning if we replace the word ”men”, ”mortal”, ”Socrates” with other things, the argument still stands nonetheless. For example

1. All cats are cute
2. Willard is a cat
3. Therefore, Willard is cute

One would immediately recognise the two arguments are ”the same”. Not in terms of the meaning of each premise, but rather what made the first argument correct is quite clearly the same as the second argument. But not all arguments that follow the form ”All M is P, S is M, so ”S is P” would yield a true conclusion. Take, for example, that

1. All animals are four-legged
2. Chickens are animals
3. Chicken are four-legged

the conclusion is wrong. What made the conclusion wrong is not in the form of the argument, but because we included a false premise in the argument. What made an argument "correct" requires two parts : 1.) We need the argument to have the right form. We call these arguments *valid*. A valid argument ensures that if its premises are true, then its conclusion must be true; 2.) We need the argument to have true premises. Only valid arguments with true premises have true conclusions. These arguments we call them *sound*.
What I have said here remains pretty vague. What makes an argument valid? What makes statements true? We all have a vague idea of validity and true, but not a clear-cut *definition* of them. We will clarify these ideas as we progress. We will not include a full-blown philosophical discourse. This is a book on mathematics afterall. But we will clarify these ideas "clear enough" so that we can safely use them in constructing our mathematical universe.
Formal lo

1.2 Propositions, Facts, and Truth

Chapter 2

Naive Set Theory

Starting from this chapter, our goal is to develop the required logical tool to describe Mathematics. As history have went, set theory became our standard logical starting point to the Mathematical universe. But what really is a set?

2.1 Intention and Extension

"Meaning" is an ambiguous word. What does it mean by "the planets of our Solar System"? There are two possible ways to answer. You can explain the concepts of "planets" and "our solar system" and how do they relate to each other. Our discussion remains on the conceptual level and mentions no concrete object, we call this kind of meaning the *intention*. You can also explain to the person "the planets of our Solar System" as the list : Mercury, Venus, Earth, Mars, Jupiter, Saturn, Neptune, Uranus. You go beyond the concepts and mention the actual object. We are talking about the things agrees with the intention—- the *extensions*.

We always deem the intention more fundamental than the extension. This is why in Plato's *Euthyphro*, when Euthyphro defines pious as "to prosecute the wrongdoer", Socrates dismiss his definitions and says "I did not bid you tell me one or two of the many pious actions but that from itself that makes all pious actions pious." He wanted the intentional meaning of "pious", not the extensional one.

If we do not make clear when we mean something, whether we are talking about its intention or extension, ambiguity arises. Consider the proposition "All creatures with a heart is the same as all creatures with a kidney." There are two ways to interpret it. If we are talking about intention, obviously the concepts involves in "creatures with a heart" is different to those in "creatures with a kidney". So this proposition is false. To interpret it extensionally, we may go out to the world and discover all creatures that have a heart indeed also have a kidney and vice versa. Hence, it is indeed that case that the creatures with a heart is the same as the creature with a kidney. The proposition is true.

2.2 Sets and Extension

This duality of intention and extension is closely related to the notion of sets. Common wisdom would define a set as ”a collection of things”, and set elements are ”things the set contains”. These two ”definitions” are problematic. Firstly, all collections must contain things. Then the phrase ”a collection of things” makes basically no difference than ”a collection”. Thus, this definition basically equate sets with collections. What we are doing is just giving a new name to something without explaining it. Secondly, the definition of set elements only lies in the metaphoric level. Sets are, quite obviously, abstract object. A set do not really ”contains” something. It has no space, nor location, nor an inside or outside. We all have a vague idea of a set, otherwise saying a set ”contains” something is mere senseless. It is not. It is trying to communicate an idea. But we have to do better than this.

We must turn to analyse what we call ”a collection”. What makes a collection identifiable to us? We either enumerate all the things in that collections, or to identify some properties that are common to all members of this collection. Suppose Jack hands me a bag of ten candies and ask me to pick one. How could I possibly identify this bag of candies? I can either enumerate the candies : A strawberry-flavoured one, a lemon-flavoured one, ... Or I can identify this collection of candies as ”all candies contained in the bag Jack handed to me”. But do be careful! What am I interested is not discussing the concepts involved in ”all candies contained in the bag Jack handed to me”, but the extensions that falls under the concept, i.e. the candies in the Jack’s bag.

In logic, the concept *predicate* captures the idea of ”intention”. Then sets, being a logical tool, is exactly the *extension of the predicate*. How about enumeration? Enumeration is just a special kind of predicate. In saying that ”Jack is a member of the collection made up by Jack, Jacky, and Jackson” makes no difference to saying ”Jack belongs to the set given by the predicate ’x is Jack or x is Jacky or x is Jackson’”. Hence, we can now explicate the notion of sets as

Definition 1. ***(Sets)*** *A set are all the extensions of a predicate. x is said to be a* ***member*** *of the set, if the predicate applies to x.*

For convenience, we let φx represent a predicate, and the set corresponding to that predicate as $\{x|\varphi x\}$. If x is a member of the set A, we write $x \in A$.

An important aspect of sets we shall discuss is when should sets be equal and when they should not. We have now made clear that sets correspond to the extension of predicates. Then since sets appeal to the extensional side of meaning, the proposition

$$\{x|x \text{ is a creature with a heart}\} = \{x|x \text{ is a creature with a kidney}\}$$

should be regarded as true despite the intention is different. Hence, we have the following axiom.

Axiom 1. ***(Axiom of Extensionality)*** *Suppose A, B are sets, then the following are equivalent*

- $A = B$
- *for any x, $x \in A$ if and only if $x \in B$.*

Except talking about equality, we may also talk about subset, i.e. a part of a set.

Definition 2. *(Subset relation) Let A, B be sets, then A is a* ***subset*** *of B, written $A \subseteq B$, if for any x, $x \in A$ implies $y \in B$.*

Definition 3. *(Strict subset) Let A, B be sets, then A is a* ***strict subset*** *of B, written $A \subset B$, if $A \subseteq B$ and $A \neq B$.*

Proposition 1. *Let A, B be sets, then the following are equivalent*

- $A = B$
- *$A \subseteq B$ and $B \subseteq A$*

Proof.

- ($\Rightarrow$) Suppose $A = B$. Then for any x, $x \in A$ if and only if $x \in B$. Then we have $A \subseteq B$ and $B \subseteq A$.
- ($\Leftarrow$) Suppose $A = B$, then for any x, we have 1). if $x \in A$ then $x \in B$ and 2.) if $x \in B$ then $x \in A$. Then we have $A \subseteq B$ and $B \subseteq A$.

∎

Remark 1. *This proposition is extremely useful. Virtually all proof involving set equality uses this proposition.*

Proposition 2. ***(Transitivity of Subset Relation)*** *Let A, B, C be set, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. For any $x \in A$, since $A \subseteq B$, we have $x \in B$. Since $B \subseteq C$, we have $x \in C$. Therefore, we have $A \subseteq C$. ∎

2.3 Specification and Russell's Paradox

Chapter 3

Propositional Logic

We have now entered the world of formal logic.

Chapter 4

Predicate Logic

Chapter 5

Axiomatic Set Theory

5.1 Zermelo-Frankel Axioms

5.1.1 Axiom of Existence, Extensionality, Restricted Comprehension, and Foundation

Remark 2. *From this chapter and onward, we will no longer write proofs in pure symbols, for writing proof like this is extremely inconvenient and daunting for the readers and the authors. We will write proofs in natural language from now on. But the reader should under*

In previous chapters, we have already gone through Naive Set Theory. We have witnessed a careless use of set would yield gruesome contradictions. Therefore, until now, we have only used set very carefully. We wish to develop a system of axiomatic set theory that would put clear demarcation on what is definable and what is not. We have already constructed the ZFC model in the previous chapter. We will now enter such a model and develop of Mathematics in its entirety from here.

To start off, there will be nothing for us to work on if not a single set exist. Hence, the following axiom

Axiom 2. ***(Axiom of Existence)*** *There is a set* ($\vdash \exists x(x = x)$)

As we have discussed in naive set theory, set equality should be determined extensionally, i.e. two sets are equal when they have the same elements.

Axiom 3. ***(Axiom of Extensionality)*** *Suppose A, B are sets, then the following are equivalent*

- $A = B$
- *for any x, $x \in A$ if and only if $x \in B$.*

($\vdash \forall x \forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$)

By these two axiom along, it allows us to define subsets

Axiom 4. ***(Axiom schema of restricted comprehension)*** *Let φ be a predicate and A be a set. Then there exist a set B such that for any x, $x \in B$ if and only if $x \in A$ and φx. ($\vdash \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi x)$)*

From this axiom, I am only picking out elements that are already pre-existing by themselves.

Remark 3. *A curious point can be pointed out here. The natural language formulation of this axiom is rather odd. In predicate logic, quantifiers do NOT range across predicates, so the writing "Let φ be a predicate" is not technically legal. In the formal language formulation of this axiom, we are not confronted with this issue, the "proposition" $\vdash \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi x)$ is really left as a schema, the readers can substitute φ with any predicates they favour. Hence, the word "schema" in the name of the axiom.*

Proposition 3. *Let A be a set and φ be a predicate. Then the set defined by specification is unique*

Proof. Let A be a set and φ is predicate, then let B, C be sets specified by the Axiom Schema of Restricted Comprehension. Then for any x, $x \in B$ if and only if $x \in A$ and φx and $x \in C$ if and only if $x \in A$ and φx. Therefore, we have $x \in B$ if and only if $x \in C$. Hence, $B = C$. ■

Remark 4. *For simplicity, we can write B as $\{x \in A | \phi x\}$*

Theorem 1. *There exist a set E, called an* ***empty set****, such that for any x, $x \notin \emptyset$.*

Proof. By the axiom of existence, there exist a set A. Let $E = \{x \in A | x \neq x\}$, then for any x, since $x = x$, then it is not the case that $x \neq x$. So, $x \notin E$. ■

Proposition 4. *Let A be a set and φ be a predicate, then $\{x \in A | \varphi x\} \subseteq A$.*

Proof. Trivial. ■

Proposition 5. *The empty set is unique.*

Proof. Let E, E' be empty sets. Then for any x, $x \notin E$. This makes the proposition "for any x, if $x \in E$ then $x \in E'$" true. Therefore, we have $E \subseteq E'$. By arguing the same way for E', we have $E' \subseteq E$. Hence, $E = E'$. ■

Remark 5. *The empty set is denoted $\emptyset$.*

Proposition 6. *For any set A, $\emptyset \subseteq A$.*

Proof. Since $x \notin \emptyset$ for any x, the proposition "for any x, if $x \in \emptyset$ then $x \in A$" is true. So $\emptyset \subseteq A$. ■

Definition 4. *(Set intersection) Let A, B be sets, the* ***intersection*** *of A and B, written $A \cap B$, is the set $A \cap B := \{x \in A | x \in B\}$*

Proposition 7. *(Commutativity of set intersection) For any set A, B, $A \cap B = B \cap A$.*

Proof.

- ($\subseteq$) For any x, suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. Then $x \in B$ and $x \in A$. So, $x \in B \cap A$.
- ($\supseteq$) By a similar argument, we can deduce that for any $x \in B \cap A$, $x \in A \cap B$. Therefore, $A \cap B = B \cap A$.

■

Proposition 8. *(Associativity of set intersection) For any set A, B, C, $(A \cap B) \cap C = A \cap (B \cap C)$*

Proof. For any x, the following are equivalent

$$\begin{aligned}
& x \in (A \cap B) \cap C \\
\Leftrightarrow & x \in A \cap B \text{ and } x \in C \\
\Leftrightarrow & (x \in A \text{ and } x \in B) and \in C \\
\Leftrightarrow & x \in A \text{ and } (x \in B \text{ and } x \in C) \\
\Leftrightarrow & x \in A \text{ and } (x \in B \cap C) \\
\Leftrightarrow & x \in A \cap (B \cap C)
\end{aligned}$$

So, $(A \cap B) \cap C = A \cap (B \cap C)$ ■

Proposition 9. *For any set A, $A \cap A = A$*

Proof.

- ($\subseteq$) For all x, suppose $x \in A \cap A$. Then $x \in A$ and $x \in A$ and hence $x \in A$. Therefore, $x \in A$.
- ($\supseteq$) For all $x \in A$, we have $x \in A$ and hence $x \in A$ and $x \in A$. So, $x \in A \cap A$

■

Proposition 10. *For all set A, $A \cap \emptyset = \emptyset$*

Proof.

- ($\subseteq$) for any $x \in A \cap \emptyset$, we have $x \in \emptyset$.
- ($\supseteq$) for any x. Since $x \notin \emptyset$. Then it is not that case that $x \in A$ and $x \in \emptyset$. So, $x \notin A \cap \emptyset$ and therefore if $x \in A \cap \emptyset$, then $x \in \emptyset$.

■

Proposition 11. *For all set A, B, C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.*

Proof. For any $x \in A$, since $A \subseteq B$ and $A \subseteq C$, we have $x \in B$ and $x \in C$. So, $x \in B \cap C$. ■

Definition 5. *(Disjoint) Let A, B be sets. A, B are **disjoint** if $A \cap B = \varnothing$*

Definition 6. *(Pairwise disjoint) Let A be a set, A is **pairwise disjoint** when for any $x, y \in A$, if $x \neq y$, then $x \cap y = \varnothing$*

Definition 7. *(Set complement) Let A, B be sets. Then the **complement** of A with respect of B, denoted $A \setminus B$, is the set $A \setminus B := \{x \in A | x \notin B\}$*

Definition 8. *For any set A, $A \setminus \varnothing = A$*

Proof.

- ($\subseteq$) For any x, if $x \in A \setminus \varnothing$, then $x \in A$ and $x \notin \varnothing$. Then $x \in A$. Hence, $A \setminus \varnothing \subseteq A$.
- ($\supseteq$) For any x, if $x \in A$. By definition of empty set, $x \notin A$. So, $x \in A$ and $x \notin \varnothing$. Hence, $x \in A \setminus \varnothing$. Therefore, $A \subseteq A \setminus \varnothing$.

Therefore, $A \setminus \varnothing = A$. ∎

Proposition 12. *For any set A, $A \setminus A = \varnothing$*

Proof. For any x, $x \notin \varnothing$ by definition and it is not the case that $x \in A$ and $x \notin A$. So, $x \notin \varnothing$ and $x \notin A \setminus A$. Then if $x \in \varnothing$ then $x \in A \setminus A$, and if $x \in A \setminus A$ then $x \notin \varnothing$. Therefore, $A \setminus A = \varnothing$. ∎

Proposition 13. *For any set A, B, $A \setminus B \subseteq A$*

Proof. Pick any $x \in A \setminus B$, then we have $x \in A$ and $x \notin B$. In particular, we have $x \in A$ and hence $A \setminus B \subseteq B$. ∎

Now there is still one problem lingering. The axiom of restricted comprehension has prevented us from *constructing* sets that contains itself, i.e. sets that have the property $x \in x$. But what if such a set *already existed in the first place*? Readers might wonder why do we want to prevent any set from containing itself. Don't we already prevented Russell's paradox via restricted comprehension? Consider this : Suppose a set y with the property $y \in y$ exists. By the Axiom Schema of Restricted Comprehension, the set $x = \{zs \in y | z = y\}$

5.1.2 Axiom of Pairing, Union, and Power Set

The readers may have complainted by now that the axiom schema of specification may be too restrictive. The only set we know exist for sure is the empty set. And the axiom of comprehension only allows us to create set from an existing set. As a result, we could not make much out from these. Hence, we rely on three axioms that create a larger set from pre-existing sets.
The *Axiom of Pair* describe the existence of a doubleton, the *Axiom of Union* for a generalised union, and the *Axiom of Power Set* for the power set.

Axiom 5. ***(Axiom of Pairing)** For all x, y, there is a set A such that $x \in A$ and $y \in A$. ($\vdash \forall x \forall y \exists z (x \in z \lor y \in z)$)*

Axiom 6. ***(Axiom of Union)*** *Let $\mathscr{A}$ be a set, then there exists a set B such that for any $A \in \mathscr{A}$, if $x \in A$, then $x \in B$. ($\vdash \forall x \exists y \forall z(z \in x \to z \in y)$)*

Axiom 7. ***(Axiom of Power Set)*** *Suppose A be a set, then there is a set P such that if $B \subseteq A$ is a subset, then $B \in P$. ($\vdash \forall x \exists y \forall z(z \subseteq x \to z \in y)$)*

It is actually wrong to say these axioms captures the idea of doubletons, unions, and power sets exactly. Take the Axiom of Pairing as an example, let x, y be two distinct object, then the set $\{x, y, \square, \triangle, \text{Hello}\}$ also satisfies the conditions stated in the axiom. $\square, \triangle$, and Hello are unwanted element that we would like to be excluded from our set. The set will be the desired doubleton only after we applied the Axiom Schema of Restricted Comprehension.

Proposition 14. *(Doubleton) For any x, y, there is a unique set A, called a **doubleton** of x, y, such that for any z, $z \in A$ if and only if $z = x$ or $z = y$.*

Proof. Let x, y be objects, then by the Axiom of Pairing, there is an A such that $x \in z$ or $y \in z$. Then let $D = \{w \in z | w = z \text{ or } w = y\}$. It follows immediately that D is the set we after. The Axiom Schema of Restricted Comprehension ensures its uniqueness. ■

Proposition 15. *(Singleton) For any x, there is a unique set S such that for any y, $y \in S$ if and only if $y = x$.*

Proof. The set $\{x, x\}$ is our desired set. ■

The Axiom of Union and the Axiom of Power Set also faces the same problem, we also need to use Axiom Schema of Restricted Specification on these two two axioms to obtain the desired sets. The proofs go very similarly as the one we just had.

Proposition 16. *(Power Set) For any A, there exist a unique P, called the **power set** of P, such that for any B, $B \in P$ if and only if $B \subseteq A$.*

Proof. Omitted. ■

Proposition 17. *(Union) For any $\mathscr{A}$, there is a unique set U, called the **generalised union of** $\mathscr{A}$, such that for any x, $x \in U$ if and only if $x \in A$ for some $A \in \mathscr{A}$.*

Proof. Omitted. ■

Proposition 18. *(Generalised Intersection) For any set $\mathscr{A}$, if $\mathscr{A}$ is non-empty, there is a set I such that for any x, $x \in I$ if and only if $x \in A$ for any $A \in \mathscr{A}$.*

Remark 6. *We write the generalised union of $\mathscr{A}$ as $\bigcup \mathscr{A}$ or $\bigcup\limits_{A \in \mathscr{A}} A$, the generalised intersection of $\mathscr{A}$ as $\bigcap \mathscr{A}$ or $\bigcap\limits_{A \in \mathscr{A}} A$ and the power set of A as $\mathcal{P}(A)$. Moreover, if $\mathscr{A} = \{A, B\}$, we write $\bigcup \mathscr{A} = A \cup B = B \cup A$, and $\bigcap A \cap B = B \cap A$. We call $A \cup B$ and $A \cap B$ simply their **union** and **intersections** respectively.*

Here, we will prove some very trivial yet important properties regarding union, intersections, and power sets.

Proposition 19. *(Associativity of set union) For any set A, B, C,* $(A\cup B)\cup C = A\cup(B\cup C)$

Proof. For any x, the following are equivalent

$$\begin{aligned}
&x \in (A \cup B) \cup C \\
\Leftrightarrow &x \in (A \cup B) \text{ or } x \in C \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ or } x \in C \\
\Leftrightarrow &x \in A \text{ or } (x \in B \text{ or } x \in C) \\
\Leftrightarrow &x \in A \text{ or } x \in B \cup C \\
\Leftrightarrow &x \in A \cup (B \cup C)
\end{aligned}$$

Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$ ■

Proposition 20. *For any set A, B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$*

Proof. For any set x, if $x \in A$, then $x \in A$ or $x \in B$. Then $x \in A \cup B$. So, $x \in A \cup B$. Similarly, $B \subseteq B \cup A$. Since $A \cup B = B \cup A$, we have $B \subseteq A \cup B$. ■

Proposition 21. *For any set A, $A \cup \varnothing = A$*

Proof. ($\subseteq$) For any x, if $x \in A \cup \varnothing$, then $x \in A$ or $x \in \varnothing$. But since $x \in \varnothing$ by definition, $x \in A$. Therefore, $A \cup \varnothing \subseteq A$
($\supseteq$) For any x, if $x \in A$, then $x \in A$ or $x \in \varnothing$. Then $x \in A \cup \varnothing$. Then $A \subseteq A \cup \varnothing$
Therefore, $A \cup \varnothing = A$ ■

Proposition 22. *For any set A, $A \cup A = A$*

Proof. For any x, the following are equivlent.

$$\begin{aligned}
&x \in A \\
\Leftrightarrow &x \in A \text{ or } x \in A \\
\Leftrightarrow &x \in A \cup A
\end{aligned}$$

So, $A \cup A = A$ ■

Proposition 23. *For any set A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$*

Proof. Pick any $x \in A \cup B$, then $x \in A$ or $x \in B$.

- If $x \in A$, since $A \subseteq C$, we have $x \in C$
- If $x \in B$, since $B \subseteq C$, we have $x \in C$

In any case, $x \in C$. So, $A \cup B \subseteq C$. ■

Proposition 24. *(Distributivity between set intersection and set union) For any set A, B, C, $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$ and $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$*

Proof. For any x, the following are equivalent

$$
\begin{aligned}
&x \in (A \cup C) \cap (B \cup C) \\
\Leftrightarrow &x \in (A \cup C) \text{ and } x \in (B \cup C) \\
\Leftrightarrow &(x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ or } x \in C \\
\Leftrightarrow &(x \in A \cap B) \text{ or } x \in C \\
\Leftrightarrow &x \in (A \cap B) \cup C
\end{aligned}
$$

$$
\begin{aligned}
&x \in (A \cap C) \cup (B \cap C) \\
\Leftrightarrow &x \in A \cap C \text{ or } x \in B \cap C \\
\Leftrightarrow &(x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C) \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ and } x \in C \\
\Leftrightarrow &x \in A \cup B \text{ and } x \in C \\
\Leftrightarrow &x \in (A \cup B) \cap C
\end{aligned}
$$

■

Corollary 1. *For any set A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*

Proof.

$$
\begin{aligned}
A \cap (B \cup C) = (B \cup C) \cap A = (B \cap A) \cup (C \cap A) = (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) = (B \cap C) \cup A = (B \cup A) \cap (C \cup A) = (A \cup B) \cap (A \cup C)
\end{aligned}
$$

■

Proposition 25. *(Distributivity between set intersection and set complement) For any set A, B, C, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$*

Proof. For any x, the following are equivalent

$$
\begin{aligned}
&x \in (A \cap B) \setminus C \\
\Leftrightarrow &x \in A \cap B \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } (x \notin C \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ and } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cap (B \setminus C)
\end{aligned}
$$

So, $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$ ■

Proposition 26. *(Distributivity of set complement I) For any set A, B, C, $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ and $(A \cap B) \setminus C = (A \setminus C) \cup (A \setminus B)$*

Proof. For all x, the following are equivalent,

$$\begin{aligned}
&x \in (A \cup B) \setminus C \\
\Leftrightarrow &x \in A \cup B \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ or } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ or } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ or } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cup (B \setminus C)
\end{aligned}$$

So, $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

$$\begin{aligned}
&x \in (A \cap B) \setminus C \\
\Leftrightarrow &x \in (A \cap B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Leftrightarrow &(x \in A \text{ and } x \in B) \text{ and } (x \notin C \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus C \text{ and } x \in B \setminus C \\
\Leftrightarrow &x \in (A \setminus C) \cap (B \setminus C)
\end{aligned}$$

So, $(A \cap B) \setminus C = (A \setminus C) \cup (A \setminus B)$ ∎

Proposition 27. *(Distributivity of set complement II/De Morgan's law)For any set A, B, C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$*

Proof. For any x, the following are equivalent,

$$\begin{aligned}
&x \in A \setminus (B \cap C) \\
\Leftrightarrow &x \in A \text{ and } x \notin B \cap C \\
\Leftrightarrow &x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus B \text{ or } x \in A \setminus C \\
\Leftrightarrow &x \in (A \setminus B) \cup (A \setminus C)
\end{aligned}$$

So, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$$\begin{aligned}
&x \in A \setminus (B \cup C) \\
\Leftrightarrow &x \in A \text{ and } x \notin B \cup C \\
\Leftrightarrow &x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \in A) \text{ and } (x \notin B \text{ and } x \notin C) \\
\Leftrightarrow &(x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\
\Leftrightarrow &x \in A \setminus B \text{ and } x \in A \setminus C \\
\Leftrightarrow &x \in (A \setminus B) \cap (A \setminus C)
\end{aligned}$$

So, $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ ■

Proposition 28. *For any $A \in \mathscr{A}$, $A \subseteq \bigcup \mathscr{A}$*

Proof. For any $A \in \mathscr{A}$, for any $x \in A$, by the definition of generalised union, we have $x \in \mathscr{A}$, then $A \subseteq \bigcup \mathscr{A}$. ■

Proposition 29. *Let $\mathscr{A}$ be a set. Suppose $A \in \mathscr{A}$. Then $\bigcap \mathscr{A} \subseteq A$.*

Proof. For any $x \in \bigcap \mathscr{A}$, we have $x \in B$ for any $B \in \mathscr{A}$. In particular, we have $x \in A$. ■

Proposition 30. *For any $\mathscr{A}, \mathscr{B}$, if $\mathscr{A} \subseteq \mathscr{B}$, then $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B}$.*

Proof. Let $\mathscr{A} \subseteq \mathscr{B}$. The for any $x \in \bigcup A$, there is some $A \in \bigcup \mathscr{A}$ such that $x \in A$. Since $\mathscr{A} \subseteq B$, then A is also a subset of B, then $x \in \bigcup B$. ■

Proposition 31. *For any $\mathscr{A}, \mathscr{B}$, if $\mathscr{A} \subseteq \mathscr{B}$, then $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$*

Proof. Let $x \in \bigcap \mathscr{B}$, then for any $A \in \mathscr{A}$, since $\mathscr{A} \subseteq \mathscr{B}$, we have $A \in \mathscr{B}$. By definition of intersection, we have $x \in A$. Therefore, by the definition of intersection again, $x \in \bigcap \mathscr{A}$. ■

Proposition 32. *For any set A, $\varnothing \in \mathcal{P}(A)$*

Proof. Since $\varnothing \subseteq P$, $\varnothing \in \mathcal{P}(A)$ ■

Proposition 33. *For any set A, $A \in \mathcal{P}(A)$.*

Proof. Since $A \subseteq A$, therefore $A \in \mathcal{P}(A)$. ■

Proposition 34. *Let A, B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof.

- ($\Rightarrow$) Let $C \in \mathcal{P}(A)$, which implies $C \subseteq B$. Then by properties of subset relation, we have $C \subseteq A$. Therefore, $C \in \mathcal{P}(B)$.
- ($\Leftarrow$) For any $x \in A$, we have $\{x\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $\{x\} \in \mathcal{P}(B)$ as well. Then $\{x\} \subseteq B$ and hence $x \in B$.

■

Proposition 35. *Let A, B be sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$*

Proof. For any C, the following are equivalent

$$\begin{aligned}
&C \in \mathcal{P}(A) \cap \mathcal{P}(B)\\
\Leftrightarrow &C \in \mathcal{P}(A) \text{ and } C \in \mathcal{P}(B)\\
\Leftrightarrow &C \subseteq A \text{ and } C \subseteq B\\
\Leftrightarrow &C \subseteq A \cap B\\
\Leftrightarrow &C \in \mathcal{P}(A \cap B)
\end{aligned}$$

■

Proposition 36. *Let A, B be sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.*

Proof. For any $C \in \mathcal{P}(A) \cup \mathcal{P}(B)$, we have $C \subseteq A$ or $C \subseteq B$. In any case, we have $C \subseteq A \cup B$. Therefore, $C \in \mathcal{P}(A \cup B)$ ■

Remark 7. *Readers may be tempted to state that $\mathcal{P}(A) \cup \mathcal{P}(A) = \mathcal{P}(A \cup B)$, so that the two preceding propositions will be parallel. But it is indeed that case that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ is not necessarily true. Consider the set $\{x\}, \{y\}$, where x, y are distinct objects. Then we have the following*

$$\begin{aligned}
\mathcal{P}(\{x\}) &= \{\emptyset, \{x\}\} \\
\mathcal{P}(\{y\}) &= \{\emptyset, \{y\}\} \\
\mathcal{P}(\{x\}) \cup \mathcal{P}(\{y\}) &= \{\emptyset, \{x\}, \{y\}\} \\
\mathcal{P}(\{x\} \cup \{y\}) &= \{\emptyset, \{x, y\}\}
\end{aligned}$$

Clearly, we cannot have $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ is not necessarily true. Consider the set $\{x\}, \{y\}$.

Proposition 37. *Let A be a set, then $\bigcup \mathcal{P}(A) = A$*

Proof. Let x be an object, the followings are equivalent

$$\begin{aligned}
&x \in \bigcup \mathcal{P}(A) \\
\Leftrightarrow &x \in B \text{ for some } B \in \mathcal{P}(A) \\
\Leftrightarrow &x \in B \text{ for some } B \subseteq A
\end{aligned}$$

- ($\subseteq$) For any $x \in \bigcup \mathcal{P}(A)$, we have $x \in B$ for some $B \in \mathcal{P}(A)$. It follows that $B \subseteq A$ and hence $x \in A$.
- ($\supseteq$) For any $x \in A$, we have $\{x\} \in \mathcal{P}(A)$. This implies that $\{x\} \subseteq \bigcup \mathcal{P}(A)$. Since $x \in \{x\}$, we have $x \in \bigcup \mathcal{P}(A)$.

■

Definition 9. *(Ordered pair) For all x, y, the order pair (x, y) is defined as $(x, y) = \{\{x\}, \{x, y\}\}$*

Definition 10. *(Ordered triple) For all x, y, z, the ordered triple (x, y, z) is defined as $(x, y, z) = ((x, y), z)$*

Proposition 38. *For all a, b, c, d, $(a, b) = (c, d)$ if and only if $a = c$, $b = d$*

Proof.

- ($\Rightarrow$) Suppose $(a, b) = (c, d)$.
  - (Case 1) Suppose $a = b$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\}$. Also, $\{\{c\}, \{c, d\}\} = (c, d) = (a, b) = \{\{a\}\}$. Thus, $\{c\} = \{c, d\} = \{a\}$ and therefore $c = d = a$. Since $a = b$, $b = d$,

  - (Case 2) Suppose $a \neq b$. Since $(a, b) = (c, d)$, we have $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. Then $a = c$ or $a = c = d$. In any case, $a = c$. We have also $\{b\} = \{c\}$ or $\{b\} = \{c, d\}$. But since $b \neq a$, $b \neq c$. Then it is not the case that $\{b\} = \{c\}$. So, $\{b\} = \{c, d\}$. Then we have $b = c$ or $b = d$. Since $b \neq c$, we have $b = d$.

- $(\Leftarrow)$ Suppose $a = c$, $b = d$. Then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$

∎

Proposition 39. *For all a, b, c, d, e, f, $(a, b, c) = (d, e, f)$ if and only if $a = d, b = e, c = f$*

Proof. $(\Rightarrow)$ Suppose $(a, b, c) = (d, e, f)$, then $\{((a, b), c)\} = ((d, e), f)$. Then $(a, b) = (d, e)$ and $c = f$. Since $(a, b) = (d, e)$, we have $a = d$ and $b = e$.
$(\Leftarrow)$ Suppose $a = d, b = e, c = f$, then $(a, b, c) = ((a, b), c) = ((d, e), f) = (d, e, f)$ ∎

Proposition 40. ***(Cartesian product)*** *For any set A, B, there exist a unique set C such that for any z, $z \in C$ if and only if $z = (x, y)$ for some $x \in A, y \in B$.*

Proof. The desired C is the set $C = \{z \in \mathcal{P}(\mathcal{P}(A \cup B))|$ there exist some unique $x \in A, y \in B$ such that $(x, y) = z\}$ ∎

Remark 8. *We will denote the Cartesian product of A, B as $A \times B$.*

5.1.3 Axiom of Infinity and Axiom of Choice

We are left with two Axioms

Definition 11. *Let A be a set, then the **successor** of A, denoted $S(A)$, is the set $S(A) := A \cup S(A)$*

Axiom 8. ***(Axiom of Infinity)*** *There exists a set I, called the **inductive set**, such that $\varnothing \in I$ and for any x, if $x \in I$, then $S(x) \in I$*

Axiom 9. ***(Axiom of Choice)*** *Let A be non-empty and pairwise disjoint. There exists a*

Part II

Mathematics