Fourier Analysis Stein: Chapter 7. Problems.

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Xiamen University Malaysia, Asia Pacific University Malaysia — July 14, 2024

1 Problems

1. Let f be a function on the circle. For each $N \geq 1$ the discrete Fourier coefficients of f are defined by

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi ikn/N}, \qquad \text{for } n \in \mathbb{Z}.$$

We also let

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi nx}dx$$

denote the ordinary Fourier coefficients of f.

(a) Show that $a_N(n) = a_N(n+N)$.

Solution: The only term related to n is $e^{-2\pi ikn/N}$, so it is trivial to see this term is periodic with a period of N. $\square$

(b) Prove that if f is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.

Solution: Let n be a fixed integer. Since f is continuous and periodic, it is bounded, we assume it is bounded above by a constant $M > 0$. Given $\varepsilon > 0$, we may choose a large enough N such that

$$|e^{-2\pi inx} - e^{-2\pi iny}| < \frac{\varepsilon}{2M} \text{ whenever } |x-y| < \frac{1}{N}.$$

Also, f is uniformly continuous because it is periodic, so we may choose another large enough N' so that

$$|f(e^{2\pi ix}) - f(e^{2\pi iy})| < \frac{\varepsilon}{2} \text{ whenever } |x-y| < \frac{1}{N'}.$$

We now simply choose $N_0 = \max(N, N')$ so the above two can be satisfied.

For $N > N_0$ we have

$$\begin{aligned}|a_N(n) - a(n)| &= \left|\sum_{k=1}^{N}\left[\frac{1}{N}f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \int_{(k-1)/N}^{k/N} f(e^{2\pi ix})e^{-2\pi inx}dx\right]\right| \\ &\leq \sum_{k=1}^{N}\int_{(k-1)/N}^{k/N}\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right|dx.\end{aligned}$$

The inner term can be estimated like so:

$$\begin{aligned}&\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right| \\ \leq&\left|f(e^{2\pi ik/N}) - f(e^{2\pi ix})\right| + \left|f(e^{2\pi ix})\right|\left|e^{-2\pi ikn/N} - e^{-2\pi inx}\right| \\ <&\varepsilon.\end{aligned}$$

This proves $\lim_{N\to\infty} a_N(n) = a(n)$. □

2. If f is a C^1 function on the circle, prove that $|a_N(n)| \le c/|n|$ whenever $0 < |n| \le N/2$.

Solution: We have to clarify that c will be a constant that is independent from n and N, possible related only to f. We also only prove the statement for $0 < n \le N/2$, i.e., positive n, the proof for negative n should be nearly identical.

Since f is periodic and C^1, it is Lipchitz continuous. See this answer. Therefore, we let M be a positive constant such that

$$|f(x) - f(y)| \le M|x - y| \quad \text{for all } x, y \text{ on the circle.}$$

We also use the identity: $|1 - e^{i\theta}| = 2\sin(\theta/2) \le \theta$ for $\theta \in [0, \pi]$.

Following the hint from the book, we let $\ell = 1$ if $N/4 < n \le N/2$. If otherwise $n \le N/4$, we let ℓ be the largest integer satisfying $\ell n/N \le 1/2$, then since $(\ell+1)n/N > 1/2$, we have

$$\left|\frac{1}{2} - \frac{\ell n}{N}\right| = \frac{1}{2} - \frac{\ell n}{N} < \frac{n}{N} \le \frac{1}{4}.$$

This is to verify that we can always choose ℓ such that $1/4 \le \ell n/N \le 1/2$ by construction, and that we also have the following lower bound:

$$\frac{\pi}{2} \le \frac{2\pi\ell n}{N} \le \pi \implies \left|1 - e^{2\pi i\ell n/N}\right| \ge \sqrt{2}.$$

Everything is ready, we now have

$$\begin{aligned}
|a_N(n)|\sqrt{2} &\le |a_N(n)|\left|1 - e^{2\pi i\ell n/N}\right| \\
&\le \frac{1}{N}\sum_{k=1}^{N} M\left|1 - e^{2\pi i\ell/N}\right| \\
&= M\left|1 - e^{2\pi i\ell/N}\right| \\
&\le M \cdot \frac{2\pi\ell}{N} \\
\therefore |a_N(n)| &\le \frac{M\pi}{\sqrt{2}n}.
\end{aligned}$$

□

3. By a similar method, show that if f is a C^2 function on the circle, then

$$|a_N(n)| \le c/|n|^2, \quad \text{whenever } 0 < |n| \le N/2.$$

As a result, prove the inversion formula for $f \in C^2$,

$$f(e^{2\pi ix}) = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi inx}$$

from its finite version.

Solution: We use the second symmetric difference and the fact that f'' is bounded above, that

$$\left|f(e^{2\pi i(k+\ell)/N}) - 2f(e^{2\pi ik/N}) + f(e^{2\pi i(k-\ell)/N})\right| \le \frac{M\ell^2}{N^2}.$$

For me, I can't overcome the fact that how to apply the mean value theorem on complex domain, so I just simply let $g:[0,1]\to\mathbb{R}$ such that $g(x)=f(e^{2\pi ix})$, then it follows that g is a real-valued, periodic C^2 function on $[0,1]$. This translate to the bound above.

By this logic and choosing a suitable integer ℓ as in previous solution, we find ourselves at

$$\begin{aligned}
a_N(n)\left(1-e^{2\pi i\ell n/N}\right)^2 e^{-2\pi i\ell n/N} &= \frac{1}{N}\sum_{k=1}^{N} \\
&\qquad \left[f(e^{2\pi i(k+\ell)/N}-2f(e^{2\pi ik/N})+f(e^{2\pi i(k-\ell)/N}))e^{-2\pi ikn/N}\right] \\
|a_N(n)|\left|1-e^{2\pi i\ell n/N}\right|^2 &\le \frac{1}{N}\sum_{k=1}^{N}\frac{M\ell^2}{N^2} \\
&\le \frac{M\ell^2}{N^2}.
\end{aligned}$$

Since $\ell/N\le 1/(2n)$ and $\left|1-e^{2\pi i\ell n/N}\right|\ge\sqrt{2}$, we manage to show that

$$|a_N(n)|\le\frac{M}{8n^2}.$$

(The proof for the inversion formula takes me a few days to think about, let's see the logic.)

Based on the properties we have collected so far, we know that $\lim_{N\to\infty}a_N(n)=a(n)$, also for each positive integer n, we have $|a_N(n)|\le c/|n|^2$ when $N>2n$. This means that $a(n)$ is also bounded above by $c/|n|^2$. From this fact we know at least the summation $\sum_{n=-\infty}^{\infty}a(n)e^{2\pi inx}$ in the inversion formula is absolutely convergent.

Next, for each real number x, we let $\mathcal{Q}=\{r_1/N_1,\cdots,r_j/N_j,\cdots\}$ be a list of rational numbers that converges to x, with the additional assumption that r_j are integers and $\{N_j\}_{j\ge1}$ is a series of strictly increasing odd numbers (the text suggests we use odd N while my derivation doesn't actually use it, but there is no harm to have this condition!). Note that we do not require $\mathcal{Q}$ to be strictly increasing, nor r_j/N_j to be a fraction of simplest form. We can have $x=1/3$ and the series is $\mathcal{Q}=\{1/3,3/9,5/15,7/21,\cdots\}$, the main point is $\{N_j\}_{j\ge1}$ is strictly increasing. You get the point.

Now we define

$$S_j=\sum_{|n|<N_j/2}a(n)e^{2\pi inr_j/N_j},\quad R_j=\sum_{|n|>=N_j/2}a(n)e^{2\pi inr_j/N_j}.$$

Immediately we notice that $\lim_{j\to\infty}R_j=0$ because $|a(n)|=O(1/n^2)$.

Using the hint from the text, we have

$$f(e^{2\pi ir_j/N_j})=\sum_{|n|<N_j/2}a_{N_j}(n)e^{2\pi inr_j/N_j}.$$

We therefore have a bound:

$$\left|S_j-f(e^{2\pi ir_j/N_j})\right|\le\sum_{|n|<N_j/2}\left|a(n)-a_{N_j}(n)\right|.$$

Now we show that as $j\to\infty$, this difference will converge to 0. Given $\varepsilon>0$, we choose N_0 such that $\sum_{|n|\ge N_0/2}c/|n|^2<\varepsilon$. Since there are only finitely many n satisfying $|n|<N_0/2$, we can find a large enough j such that $N_j>N_0$ and $\left|a(n)-a_{N_j}(n)\right|\le\varepsilon/N_0$ for all $|n|<N_0/2$.

We now have

$$\begin{aligned}
\left|S_j-f(e^{2\pi ir_j/N_j})\right| &\le \sum_{|n|<N_0/2}\left|a(n)-a_{N_j}(n)\right|+\sum_{|n|\ge N_0/2}|a(n)|+\sum_{N_0/2\le|n|<N_j/2}\left|a_{N_j}(n)\right| \\
&\le 3\varepsilon.
\end{aligned}$$

Since $\lim_{j\to\infty} R_j = 0$ and f is continuous, which implies $\lim_{j\to\infty} f(e^{2\pi i r_j/N_j}) = f(x)$, we have proved the inversion formula is true. $\square$

4. Let e be a character on $G = \mathbb{Z}(N)$, the additive group of integers modulo N. Show that there exists a unique $0 \le \ell \le N-1$ so that

$$e(k) = e_\ell(k) = e^{2\pi i \ell k/N} \quad \text{for all } k \in \mathbb{Z}(N).$$

Conversely, every function of this type is a character on $\mathbb{Z}(N)$. Deduce that $e_\ell \mapsto \ell$ defines an isomorphism from G to G.

Solution: Let $e : G \to S^1$ be a character, if it is an identity character, then $e = e_0$. If e is non-trivial, then we let $a \in G$ so that $e(a) \neq 1$. Since $n \cdot a$ is the identity element in G, we have that

$$e(a)^n = e(n \cdot a) = 1.$$

This shows that $e(a)$ is a root of unity, and thus there is a unique $0 \le \ell < N$ such that $e = e_\ell$.

To show the mapping $M : \hat{G} \to G$ defined by $M(e_\ell) = \ell$ is an isomorphism, it is worth noting this is a well-defined mapping, i.e., for $\ell, k \in \mathbb{Z}(N)$, $M(e_\ell) = M(e_k)$ implies $\ell = k$. Moreover, this mapping is surjective and one-to-one, we only have to show it is a homomorphism:

$$M(e_\ell \circ e_k) = M(e_{\ell+k}) = \ell + k \pmod N.$$

$\square$

5. Show that all characters on S^1 are given by

$$e_n(x) = e^{2\pi i n x} \text{ with } n \in \mathbb{Z},$$

and check that $e_n \mapsto n$ defines an isomorphism from $\widehat{S^1}$ to $\mathbb{Z}$.

Solution: To make the discussion less confusing, we will use f to denote a character on S^1, which is $f : S^1 \to S^1$.

We can then define $F : \mathbb{R} \to S^1$ so that $F(x) = f(e^{2\pi i x})$, it is not hard to see F is a periodic function with a period of 1, and also satisfying $F(x+y) = F(x)F(y)$.

Clearly F is a complex-valued continuous function, with $F(0) = 1$, hence there is a small enough $\delta > 0$ such that $c = \int_0^\delta F(y)dy \neq 0$. We then have

$$\begin{aligned} cF(x) &= \int_0^\delta F(x+y)dy \\ &= \int_x^{x+\delta} F(y)dy \\ \therefore cF'(x) &= F(x+\delta) - F(x) \\ &= [F(\delta) - 1]F(x) \\ F'(x) &= \frac{F(\delta)-1}{c} F(x). \end{aligned}$$

Thus, $F'(x) = AF(x)$ for some constant A, combine with the condition that $F(0) = 1$, we have $F(x) = e^{Ax}$.

Since $F(x)$ have values in S^1, $A = 2\pi i n$ for some real number n.

To show that n is an integer, we have $e^{2\pi i n} = F(1) = f(e^{2\pi i}) = 1$, so $f(e^{2\pi i x}) = e_n(x)$ for $x \in \mathbb{R}$.

Clearly if $n \neq m$ are integers, then $e_n \neq e_m$, and for each integer n, e_n is a character on S^1. We only have to show $L : \widehat{S^1} \to \mathbb{Z}$ is a homomorphism. For all $x \in \mathbb{R}$ we have $e_n \circ e_m(x) = e^{2\pi i(n+m)x}$, hence $L(e_n \circ e_m) = n + m = L(e_n) + L(e_m)$. $\square$

6. Prove that all characters on $\mathbb{R}$ take the form

$$e_\xi(x) = e^{2\pi i \xi x} \quad \text{with } \xi \in \mathbb{R}.$$

and that $e_\xi \mapsto \xi$ defines an isomorphism from $\widehat{\mathbb{R}}$ to $\mathbb{R}$. The argument in Exercise 5 applies here as well.

Solution: We use F to denote a character on $\mathbb{R}$, which is $F : \mathbb{R} \to S^1$. Since $\mathbb{R}$ is an additive group, we have $F(x+y) = F(x)F(y)$, and also $F(0) = 1$.

Using what we proved in the previous exercise, there is a constant A such that $F(x) = e^{Ax}$, and since F maps to norm of 1, $A = 2\pi i \xi$ for some real number ξ.

Now we see that for any real ξ, F is a character on $\mathbb{R}$, and $F(x) = e_\xi(x)$.

It is clear that the mapping $e_\xi \mapsto \xi$ is surjective and one-to-one, we only have to show it is a homomorphism by noting that $e_{\xi+\eta}(x) = e^{2\pi i(\xi+\eta)x} = e^{2\pi i \xi x} e^{2\pi i \eta x} = e_\xi(x) e_\eta(x)$ for any $\xi, \eta \in \mathbb{R}$.

□

7. Let $\xi = e^{2\pi i/N}$. Define the $N \times N$ matrix $M = (a_{jk})_{1 \le j,k \le N}$ by $a_{jk} = N^{-1/2} \xi^{jk}$.

   (a) Show that M is unitary.

   Solution: Write $M = [\vec{a_1} | \cdots | \vec{a_N}]$ where each column vector is $\vec{a_k} = (a_{1k} \ldots a_{Nk})^T$. Then we have that for $1 \le k, n \le N$,

$$\vec{a_k}^* \vec{a_n} = \frac{1}{N} \sum_{j=1}^{N} \overline{\xi^{jk}} \xi^{jn} = \frac{1}{N} \sum_{j=1}^{N} \xi^{j(n-k)} = \delta_{kn}.$$

   Where it means that $\delta_{kn} = 1$ when $k = n$, and $\delta_{kn} = 0$ otherwise. This proves M to be a unitary matrix. □

   (b) Interpret the identity $(Mu, Mv) = (u, v)$ and the fact that $M^* = M^{-1}$ in terms of the Fourier series on $\mathbb{Z}(N)$.

   Solution: Not too sure what is the point of the question here. □

8. Suppose that $P(x) = \sum_{n=1}^{N} a_n e^{2\pi i n x}$.

   (a) Show by using the Parseval identities for the circle and $\mathbb{Z}(N)$, that

$$\int_0^1 |P(x)|^2 dx = \frac{1}{N} \sum_{j=1}^{N} |P(j/N)|^2.$$

   Solution: Notice that P looks like a function on $\mathbb{R}$, but essentially it is a function on $\mathbb{Z}(N)$. We let $e_n(x) = e^{2\pi i n x}$ to be characters on S^1, then we find that $P = \sum_{n=1}^{N} a_n e_n$. Moreover, for $x \in \mathbb{Z}(N)$ which by that we means $x = 0, 1, \ldots, N-1$, we let $F(x) = P(x/N)$, so F is a function on $\mathbb{Z}(N)$.

   We note that we can define an inner product for periodic functions f, g over $[0, 1]$ on $\mathbb{R}$ as $(f, g) = \int_0^1 f\bar{g}$, then $\{e_n\}_{1 \le n \le N}$ is orthonormal. We thus can have

$$\int_0^1 |P(x)|^2 dx = (P, P) = \sum_{j=1}^{N} \sum_{k=1}^{N} a_j \overline{a_k} (e_j, e_k) = \sum_{n=1}^{N} |a_n|^2.$$

   Apply Parseval identity to F and using the inner product for functions on $\mathbb{Z}(N)$, we have

$$\sum_{n=1}^{N} |a_n|^2 = (F, F) = \frac{1}{N} \sum_{x=1}^{N} |F(x)|^2 = \frac{1}{N} \sum_{x=1}^{N} \left| P\left(\frac{x}{N}\right) \right|^2.$$

This completes the proof. $\square$

(b) Prove the reconstruction formula

$$P(x) = \sum_{j=1}^{N} P(j/N)K(x-(j/N))$$

where

$$K(x) = \frac{e^{2\pi i x}}{N}\frac{1-e^{2\pi i N x}}{1-e^{2\pi i x}} = \frac{1}{N}(e^{2\pi i x} + e^{2\pi i 2x} + \cdots + e^{2\pi i N x}).$$

Solution: For ease of notation, we let $e_k(x) = e^{2\pi i k x}$, and also $\xi = e^{2\pi i/N}$ to be the N-th root of unity. We have

$$\text{RHS} = \sum_{j=1}^{N}\sum_{n=1}^{N} a_n \xi^{nj} \cdot \frac{1}{N}\sum_{k=1}^{N}\frac{e_k(x)}{\xi^{kj}} \tag{1}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{N}\left(\sum_{j=1}^{N}\xi^{(n-k)j}\right) a_n e_k(x) \tag{2}$$

$$= \sum_{n=1}^{k} a_n e_n(x) = P(x). \tag{3}$$

$\square$

9. (This question was skipped)

10. A group G is **cyclic** if there exists $g \in G$ that generates all of G, that is, if any element in G can be written as g^n for some $n \in \mathbb{Z}$. Prove that a finite abelian group is cyclic if and only if it is isomorphic to $\mathbb{Z}(N)$ for some N.

    Solution: Let G be a finite abelian group. Apparently if G is isomorphic to $\mathbb{Z}(N)$ then it is cyclic, since it can be generated by the element that is isomorphic to $1 \in \mathbb{Z}(N)$. Now assume G is cyclic and let g be the generator of G. Since G is finite, there is a smallest positive integer n such that $g^n = e$. We define the mapping $\varphi : \mathbb{Z}(n) \to G$ by $\varphi(k) = g^k$. This is a homomorphism because $\varphi(k+\ell) = \varphi(k)\varphi(\ell)$. It is also one-to-one and surjective, hence it is an isomorphism. $\square$

11. (This question was also skipped :p)

12. Suppose that G is a finite abelian group and $e : G \to \mathbb{C}$ is a function that satisfies $e(x \cdot y) = e(x)e(y)$ for all $x, y \in G$. Prove that either e is identically 0, or e never vanishes. In the second case, show that for each x, $e(x) = e^{2\pi i r}$ for some $r \in \mathbb{Q}$ of the form $r = p/q$, where $q = |G|$.

    Solution: Let's assume the non-trivial that $e \not\equiv 0$, then we must have $e(1) \neq 0$, or else for every $f \in G$, $e(f) = e(1)e(f) = 0$. Based on this we also have $e(1) = 1$. Now for each element $g \in G$, we have $g^{|G|} = 1 \in G$, hence $e(g)^{|G|} = e(g^{|G|}) = e(1) = 1$. This not only show that $e(g) \neq 0$, but also that $e(g)$ is a $|G|$-th root of unity, which means we may write $e(g) = e^{2\pi i p/|G|}$ for some integer p. $\square$

13. In analogy with ordinary Fourier series, one may interpret finite Fourier expansions using convolutions as follows. Suppose G is a finite abelian group, 1_G its unit, and V the vector space of complex-valued functions on G.

    (a) The convolution of two functions f and g in V is defined for each $a \in G$ by

    $$(f * g)(a) = \frac{1}{|G|}\sum_{b \in G} f(b)g(a \cdot b^{-1}).$$

Show that for all $e \in \hat{G}$ one has $\widehat{(f * g)}(e) = \hat{f}(e)\hat{g}(e)$.

Solution: Note that $\hat{f}(e) = (f, e) = \frac{1}{|G|} \sum_{b \in G} f(b)\overline{e(b)}$, we have

$$\begin{aligned}
\text{LHS} &= \left(\frac{1}{|G|} \sum_{b \in G} f(b) g(a \cdot b^{-1}), e \right) \\
&= \frac{1}{|G|^2} \sum_{a \in G} \left(\sum_{b \in G} f(b) g(a \cdot b^{-1}) \right) \overline{e(a)} \\
&= \frac{1}{|G|^2} \sum_{b \in G} f(b) \left(\sum_{a \in G} g(a \cdot b^{-1}) \overline{e(a)} \right) \\
&= \frac{1}{|G|^2} \sum_{b \in G} f(b) \sum_{a \in G} g(a) \overline{e(a) e(b)} \\
&= \hat{f}(e)\hat{g}(e).
\end{aligned}$$

The last 2nd transition works because we replace a by ab, since multiply by b simply become another permutation of elements in G. □

(b) Use Theorem 2.5 to show that if e is a character on G, then

$$\sum_{e \in \hat{G}} e(c) = 0 \quad \text{whenever } c \in G \text{ and } c \neq 1_G.$$

Solution: Theorem 2.5 states that the characters on G forms a basis over V. Therefore, we can conveniently define a function $I : G \to \mathbb{C}$ such that $I(c) = 1$ when $c = 1_G$ and $I(c) = 0$ otherwise. Then we have

$$I(c) = \sum_{e \in \hat{G}} \hat{I}(e) e(c) \quad \text{for all } c \in G.$$

We find out that $\hat{I}(e) = (I, e) = (1/|G|) \sum_{a \in G} I(a)\overline{e(a)} = 1/|G|$, hence for $c \neq 1_G$,

$$\sum_{e \in \hat{G}} e(c) = |G| I(c) = 0.$$

□

(c) As a result of (b), show that the Fourier series $Sf(a) = \sum_{e \in \hat{G}} \hat{f}(e) e(a)$ of a function $f \in V$ takes the form

$$Sf = f * D,$$

where D is defined by

$$D(c) = \sum_{e \in \hat{G}} e(c) = \begin{cases} |G| & \text{if } c = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f * D = f$, we recover the fact that $Sf = f$. Loosely speaking, D corresponds to a "Dirac delta functoin"; it has unit mass

$$\frac{1}{|G|} \sum_{c \in G} D(c) = 1,$$

and it has mass concentrated at the unit element in G. Thus D has the same interpretation as the "limit" of a family of good kernels.

Solution: Notice we have that $D = |G| I$, so $\hat{D}(e) = 1$ for every character e on G. We then have

$$\widehat{(f * D)}(e) = \hat{f}(e)\hat{D}(e) = \hat{f}(e),$$

so $Sf = f * D$. □