Fourier Analysis Stein: Chapter 8. Problems.

Kelvin Hong
kh.boon2@gmail.com

Xiamen University Malaysia, Asia Pacific University Malaysia — October 12, 2024

1 Problems

1. Prove that there are infinitely many primes by observing that if there were only finitely many, $p_1, \ldots, p_N$, then
$$\prod_{j=1}^{N} \frac{1}{1 - 1/p_j} \geq \sum_{n=1}^{\infty} \frac{1}{n}.$$

   Solution: Assumes there are only finitely many primes $p_1, \ldots, p_N$. Given a positive integer M, each positive integer $n \leq M$ can be expressed as a product of primes $p_1^{k_1} \cdots p_N^{k_N}$ for some integers $k_1, \ldots, k_N$. We let $K_1, \ldots, K_N$ be the maximum values of $k_1, \ldots, k_N$ across all $n \leq M$. Thus, we have
$$\prod_{j=1}^{N} \frac{1}{1 - 1/p_j} \geq \prod_{j=1}^{N} \left(\sum_{k=0}^{K_j} \frac{1}{p_j^k} \right) \geq \sum_{n=1}^{M} \frac{1}{n}.$$
   Taking the limit as $M \to \infty$, we see that the RHS diverges, which is a contradiction to our assumption. $\square$

2. In the text we showed that there are infinitely many primes of the form $4k + 3$ by a modification of Euclid's original argument. Adapt this technique to prove the similar result for primes of the form $3k + 2$, and for those of the form $6k + 5$.

   Solution: Assume there are only finitely many primes of the form $3k + 2$, and let $p_1, \ldots, p_N$ be all of them in increasing order and $p_1 = 5$. Consider the number $n = 3p_1 \cdots p_N + 2$. This is a number of the form $3k + 2$ and $n > p_N$, so it must be composite by our assumption. Since it is not divisible by 3, it must have prime factors of the form $3k + 1$ and $3k + 2$. If it only has prime factors of the form $3k + 1$, the product of these primes would be still of the form $3k + 1$, so it will at least has a prime factor of the form $3k + 2$. But this is a contradiction since n is not divisible by any of the primes $p_1, \ldots, p_N$.

   For another solution, we assume there are only finitely many primes of the form $6k + 5$, $q_1, \ldots, q_M$, in increasing order, where $q_1 = 11$. We can let $m = 6q_1 \cdots q_M + 5$, we can argue similarly by noting that if m is composite, it must have prime factors of the form $6k + 1$ and $6k + 5$, and also it must have at least one prime factor of the form $6k + 5$, which create a contradiction. $\square$

3. Prove that if p and q are relatively prime, then $\mathbb{Z}^*(p) \times \mathbb{Z}^*(q)$ is isomorphic to $\mathbb{Z}^*(pq)$.

   Solution: Let $\phi : \mathbb{Z}^*(p) \times \mathbb{Z}^*(q) \to \mathbb{Z}^*(pq)$ be defined by $\phi(a, b) = aq + bp \mod pq$. Obviously both sets are groups, so we only need to show the mapping is one-to-one and surjective.

   To show ϕ is one-to-one, suppose $\phi(a, b) = \phi(c, d)$, then $aq + bp \equiv cq + dp \mod pq$. There is an integer k such that $(a - c)q + (b - d)p = kpq$. Rearrange to get $(a - c)q = p(kq - b + d)$. Since p and q are relatively prime, p divides $a - c$, hence $a \equiv c \pmod p$. Similarly $b \equiv d \pmod q$.

To show ϕ is surjective, we first let x, y be integers such that $py + qx \equiv 1 \pmod{p}q$ with $x \in \mathbb{Z}^*(p)$ and $y \in \mathbb{Z}^*(q)$. For any $z \in \mathbb{Z}^*(pq)$, we can write $z = z \cdot 1 = z(py + qx) = zpy + zqx$. We find that $\phi(zx, zy) = z$ just because z is both relatively prime to p and q. $\square$

4. (Skipped)

5. If n is a positive integer, show that

$$n = \sum_{d|n} \varphi(d),$$

where φ is the Euler phi-function.

Solution: For any $1 \le d \le n$, if d is a divisor of n, then $\varphi(n/d)$ is the number of integers $1 \le k \le n/d$ such that $\gcd(k, n/d) = 1$. Let $m = kd$, then we have $\gcd(m, n) = d$. We see there are exactly $\varphi(n/d)$ integers m satisfying $\gcd(m, n) = d$.

Obviously for each $1 \le m \le n$, $\gcd(m, n)$ is a divisor of d, so we actually have

$$n = \sum_{d|n} \varphi(n/d) = \sum_{d|n} \varphi(d).$$

$\square$

6. (Skipped)

7. Recall that for $|z| < 1$,

$$\log_1\left(\frac{1}{1-z}\right) = \sum_{k\ge 1} \frac{z^k}{k}.$$

We have seen that

$$e^{\log_1\left(\frac{1}{1-z}\right)} = \frac{1}{1-z}.$$

(a) Show that if $w = 1/(1-z)$, then $|z| < 1$ if and only if $\Re(w) > 1/2$.

Solution: We rearrange to see $z = 1 - 1/w$, so $|z| < 1$ iff $1 - \frac{1}{w} - \frac{1}{\bar{w}} + \frac{1}{|w|^2} < 1$, we then have

$$\frac{1}{|w|^2} < \frac{1}{w} + \frac{1}{\bar{w}} = \frac{w + \bar{w}}{|w|^2} = \frac{2\Re(w)}{|w|^2}.$$

Since the above derivation are all equivalent, we have $|z| < 1$ iff $\Re(w) > 1/2$. $\square$

(b) Show that if $\Re(w) > 1/2$ and $w = \rho e^{i\varphi}$ with $\rho > 0, |\varphi| < \pi$, then

$$\log_1 w = \log \rho + i\varphi.$$

Solution: We have $e^{\log_1 w} = w$, so there is an integer k such that $\log_1 w = \log \rho + i(2\pi + \varphi)$. Now view $\log_1 w$ as a function of φ, when $\varphi = 0$ we have that $\log_1 w$ is real, and is exactly $\log \rho$, since k is an integer, it must be zero. $\square$

8. Let ζ denote the zeta function defined for $s > 1$.

(a) Compare $\zeta(s)$ with $\int_1^\infty x^{-s} dx$ to show that

$$\zeta(s) = \frac{1}{s-1} + O(1) \quad \text{as } s \to 1^+.$$

Solution: For each integer $N \geq 2$ we let $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$, we have

$$\begin{aligned}
\zeta_N(s) &\leq 1 + \sum_{n=2}^{N} \int_{n-1}^{n} x^{-s} dx \\
&= 1 + \int_1^N x^{-s} dx \\
&= 1 + \frac{1}{s-1} - \frac{1}{(s-1)N^{s-1}} \\
&< 1 + \frac{1}{s-1}.
\end{aligned}$$

By letting $N \to \infty$, we see $\zeta(s) \leq \frac{1}{s-1} + 1$ for every $s > 1$. Similarly, if we fixed $s > 1$, we have

$$\zeta_N(s) \geq \int_1^{N+1} x^{-s} dx = \frac{1}{s-1} - \frac{1}{(s-1)(N+1)^{s-1}} \implies \zeta(s) \geq \frac{1}{s-1}.$$

Hence this also hold for every $s > 1$. Note that we actually proved something stronger:

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{1}{s-1} + 1.$$

□

(b) Prove as a consequence that

$$\sum_p \frac{1}{p^s} = \log\left(\frac{1}{s-1}\right) + O(1) \quad \text{as } s \to 1^+.$$

Solution: Note that $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$, taking logarithm both sides give us

$$\begin{aligned}
\log \zeta(s) &= -\sum_p \log\left(1 - \frac{1}{p^s}\right) \\
&= \sum_p \left(\frac{1}{p^s} + \sum_{k\geq 2} \frac{1}{kp^{ks}}\right)
\end{aligned}$$

The second term can be bounded as below:

$$0 \leq \sum_{k\geq 2} \frac{1}{kp^{ks}} \leq \frac{1}{2} \sum_{k\geq 2} \frac{1}{p^{ks}} = \frac{1}{2} \cdot \frac{p^{-2s}}{1-p^{-s}} \leq p^{-2s}$$

which gives us

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \sum_p O\left(\frac{1}{p^{2s}}\right) = \sum_p \frac{1}{p^s} + O(1).$$

When taking $s \to 1^+$, we have proved the result. □

9. Let χ_0 denote that trivial Dirichlet character mod q, and $p_1, \ldots, p_k$ the distinct prime divisors of q. Recall that $L(s, \chi_0) = (1 - p_1^{-s}) \cdots (1 - p_k^{-s})\zeta(s)$, and show as a consequence

$$L(s, \chi_0) = \frac{\varphi(q)}{q} \frac{1}{s-1} + O(1) \quad \text{as } s \to 1^+.$$

Solution: We let $f(s) = (1 - p_1^{-s}) \cdots (1 - p_k^{-s})$, then $f(1) = \varphi(q)/q$. Notice that f is continuous at $s = 1$ and has bounded derivative around $s = 1$, which means

$$\lim_{s\to 1+} \frac{f(s) - f(1)}{s-1} \quad \text{is bounded.}$$

By that and the previous exercise we have

$$L(s,\chi_0) = \frac{f(s)}{s-1} + O(f(s)) = \frac{\varphi(q)}{q}\frac{1}{s-1} + \frac{f(s)-f(1)}{s-1} + O(1) = \frac{\varphi(q)}{q}\frac{1}{s-1} + O(1)$$

as $s \to 1^+$. □

10. Show that if ℓ is relatively prime to q, then

$$\sum_{p\equiv \ell} \frac{1}{p^s} = \frac{1}{\varphi(q)} \log\left(\frac{1}{s-1}\right) + O(1) \quad \text{as } s \to 1^+.$$

This is a quantitative version of Dirichlet's theorem.

Solution: Using formula (4) in this chapter, we already know the second term on the RHS is $O(1)$, and since there are only finitely many primes as divisor of q, together with previous exercise it naturally gives us the result here. □

11. Use the characters for $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5)$, and $\mathbb{Z}^*(6)$ to verify directly that $L(1,\chi) \neq 0$ for all non-trivial Dirichlet characters modulo q when $q = 3, 4, 5$, and 6.

Solution: In $\mathbb{Z}^*(3)$, the only non-trivial character satisfies $\chi(3k) = 0, \chi(3k+1) = 1, \chi(3k+2) = -1$, so

$$L(1,\chi) = \sum_{k=0}^{\infty}\left(\frac{1}{3k+1} - \frac{1}{3k+2}\right) = \sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+2)}.$$

This is a positive sum and also $L(1,\chi) \neq 0$. The same argument can be applied to $\mathbb{Z}^*(4), \mathbb{Z}^*(6)$ as we can get $L(1,\chi) = \sum_{k=0}^{\infty} \frac{2}{(4k+1)(4k+3)}$ and $\sum_{k=0}^{\infty}\frac{4}{(6k+1)(6k+5)}$ respectively.

For $\mathbb{Z}^*(5)$, there are three non-trivial characters, solely depends on whether $\chi(5k+2) = -1, i, -i$. We can see that

$$\chi(5k+2) = -1 \implies L(1,\chi) = \sum_{k=0}^{\infty}\left(\frac{1}{(5k+1)(5k+2)} - \frac{1}{(5k+3)(5k+4)}\right) > 0;$$

$$\chi(5k+2) = i \implies L(1,\chi) = \sum_{k=0}^{\infty}\frac{3}{(5k+1)(5k+4)} + i\sum_{k=0}^{\infty}\frac{1}{(5k+2)(5k+3)} \neq 0;$$

$$\chi(5k+2) = -i \implies L(1,\chi) = \sum_{k=0}^{\infty}\frac{3}{(5k+1)(5k+4)} - i\sum_{k=0}^{\infty}\frac{1}{(5k+2)(5k+3)} \neq 0.$$

□

12. Suppose χ is real and non-trivial; assuming the theorem that $L(1,\chi) \neq 0$, show directly that $L(1,\chi) > 0$.

Solution: By the product formula $L(1,\chi) = \prod_p \frac{1}{(1-\chi(p)p^{-1})}$, since χ is real and $|\chi| \leq 1$, we have that $1 - \chi(p)p^{-1} > 0$, hence $L(1,\chi) > 0$. □