Fourier Analysis Stein: Chapter 7. Problems.

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1 Problems

1. Let f be a function on the circle. For each $N \geq 1$ the discrete Fourier coefficients of f are defined by

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi ikn/N}, \qquad \text{for } n \in \mathbb{Z}.$$

We also let

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi nx}dx$$

denote the ordinary Fourier coefficients of f.

(a) Show that $a_N(n) = a_N(n+N)$.

Solution: The only term related to n is $e^{-2\pi ikn/N}$, so it is trivial to see this term is periodic with a period of N. □

(b) Prove that if f is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.

Solution: Let n be a fixed integer. Since f is continuous and periodic, it is bounded, we assume it is bounded above by a constant $M > 0$. Given $\varepsilon > 0$, we may choose a large enough N such that

$$|e^{-2\pi inx} - e^{-2\pi iny}| < \frac{\varepsilon}{2M} \text{ whenever } |x-y| < \frac{1}{N}.$$

Also, f is uniformly continuous because it is periodic, so we may choose another large enough N' so that

$$|f(e^{2\pi ix}) - f(e^{2\pi iy})| < \frac{\varepsilon}{2} \text{ whenever } |x-y| < \frac{1}{N'}.$$

We now simply choose $N_0 = \max(N, N')$ so the above two can be satisfied.

For $N > N_0$ we have

$$\begin{aligned}
|a_N(n) - a(n)| &= \left|\sum_{k=1}^{N}\left[\frac{1}{N}f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \int_{(k-1)/N}^{k/N} f(e^{2\pi ix})e^{-2\pi inx}dx\right]\right| \\
&\leq \sum_{k=1}^{N}\int_{(k-1)/N}^{k/N}\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right|dx.
\end{aligned}$$

The inner term can be estimated like so:

$$\begin{aligned}
&\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right| \\
\leq &\left|f(e^{2\pi ik/N}) - f(e^{2\pi ix})\right| + \left|f(e^{2\pi ix})\right|\left|e^{-2\pi ikn/N} - e^{-2\pi inx}\right| \\
< &\varepsilon.
\end{aligned}$$

This proves $\lim_{N\to\infty} a_N(n) = a(n)$. $\square$

2. If f is a C^1 function on the circle, prove that $|a_N(n)| \leq c/|n|$ whenever $0 < |n| \leq N/2$.

Solution: We have to clarify that c will be a constant that is independent from n and N, possible related only to f. We also only prove the statement for $0 < n \leq N/2$, i.e., positive n, the proof for negative n should be nearly identical.

Since f is periodic and C^1, it is Lipchitz continuous. See this answer. Therefore, we let M be a positive constant such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \text{ on the circle.}$$

We also use the identity: $|1 - e^{i\theta}| = 2\sin(\theta/2) \leq \theta$ for $\theta \in [0, \pi]$.

Following the hint from the book, we let $\ell = 1$ if $N/4 < n \leq N/2$. If otherwise $n \leq N/4$, we let ℓ be the largest integer satisfying $\ell n/N \leq 1/2$, then since $(\ell + 1)n/N > 1/2$, we have

$$\left|\frac{1}{2} - \frac{\ell n}{N}\right| = \frac{1}{2} - \frac{\ell n}{N} < \frac{n}{N} \leq \frac{1}{4}.$$

This is to verify that we can always choose ℓ such that $1/4 \leq \ell n/N \leq 1/2$ by construction, and that we also have the following lower bound:

$$\frac{\pi}{2} \leq \frac{2\pi\ell n}{N} \leq \pi \implies \left|1 - e^{2\pi i\ell n/N}\right| \geq \sqrt{2}.$$

Everything is ready, we now have

$$\begin{aligned}
|a_N(n)|\sqrt{2} &\leq |a_N(n)|\left|1 - e^{2\pi i\ell n/N}\right| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} M\left|1 - e^{2\pi i\ell/N}\right| \\
&= M\left|1 - e^{2\pi i\ell/N}\right| \\
&\leq M \cdot \frac{2\pi\ell}{N} \\
\therefore |a_N(n)| &\leq \frac{M\pi}{\sqrt{2}n}.
\end{aligned}$$

$\square$