Fourier Analysis Stein: Chapter 7. Problems.

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1 Problems

1. Let f be a function on the circle. For each $N \geq 1$ the discrete Fourier coefficients of f are defined by

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi ikn/N}, \qquad \text{for } n \in \mathbb{Z}.$$

We also let

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi nx}dx$$

denote the ordinary Fourier coefficients of f.

(a) Show that $a_N(n) = a_N(n+N)$.

Solution: The only term related to n is $e^{-2\pi ikn/N}$, so it is trivial to see this term is periodic with a period of N. □

(b) Prove that if f is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.

Solution: Let n be a fixed integer. Since f is continuous and periodic, it is bounded, we assume it is bounded above by a constant $M > 0$. Given $\varepsilon > 0$, we may choose a large enough N such that

$$|e^{-2\pi inx} - e^{-2\pi iny}| < \frac{\varepsilon}{2M} \text{ whenever } |x-y| < \frac{1}{N}.$$

Also, f is uniformly continuous because it is periodic, so we may choose another large enough N' so that

$$|f(e^{2\pi ix}) - f(e^{2\pi iy})| < \frac{\varepsilon}{2} \text{ whenever } |x-y| < \frac{1}{N'}.$$

We now simply choose $N_0 = \max(N, N')$ so the above two can be satisfied.

For $N > N_0$ we have

$$\begin{aligned}|a_N(n) - a(n)| &= \left|\sum_{k=1}^{N}\left[\frac{1}{N}f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \int_{(k-1)/N}^{k/N} f(e^{2\pi ix})e^{-2\pi inx}dx\right]\right| \\ &\leq \sum_{k=1}^{N}\int_{(k-1)/N}^{k/N}\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right|dx.\end{aligned}$$

The inner term can be estimated like so:

$$\begin{aligned}&\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right| \\ \leq&\left|f(e^{2\pi ik/N}) - f(e^{2\pi ix})\right| + \left|f(e^{2\pi ix})\right|\left|e^{-2\pi ikn/N} - e^{-2\pi inx}\right| \\ <&\varepsilon.\end{aligned}$$

This proves $\lim_{N\to\infty} a_N(n) = a(n)$. $\square$