

Fourier Analysis Stein: Chapter 6. Problems.

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1 Problems

1. Let J_n denote the n^{th} order Bessel function, for $n \in \mathbb{Z}$. Prove that

(a) $J_n(\rho)$ is real for all real ρ .

Solution: Taking the complex conjugate of the Bessel function, we have

$$\begin{aligned} 2\pi \overline{J_n(\rho)} &= \int_0^{2\pi} e^{-i\rho \sin \theta} e^{in\theta} d\theta \\ &= \int_0^{-2\pi} e^{i\rho \sin \theta} e^{-in\theta} (-d\theta) \\ &= \int_{-2\pi}^0 e^{i\rho \sin \theta} e^{-in\theta} d\theta \\ &= \int_0^{2\pi} e^{i\rho \sin \theta} e^{-in\theta} d\theta \text{ (Periodicity)} \\ &= 2\pi J_n(\rho). \end{aligned}$$

This shows $J_n(\rho)$ is real when ρ is real. □

(b) $J_{-n}(\rho) = (-1)^n J_n(\rho)$.

Solution: The question is equivalent of showing $J_{-n}(\rho) + J_n(\rho) = 0$ when n is odd and $J_{-n}(\rho) - J_n(\rho) = 0$ when n is even.

When n is even, we have

$$J_n(\rho) - J_{-n}(\rho) = \frac{i}{\pi} \int_0^{2\pi} e^{i\rho \sin \theta} \sin(n\theta) d\theta$$

The second half of the integral can be transformed into below by substitution $u = 2\pi - \theta$:

$$\begin{aligned} \int_{\pi}^{2\pi} e^{i\rho \sin \theta} \sin(n\theta) d\theta &= \int_{\pi}^0 e^{i\rho \sin(2\pi-u)} \sin(n(2\pi-u)) (-du) \\ &= - \int_0^{\pi} e^{-i\rho \sin u} \sin(nu) du \end{aligned}$$

Notice this is true for all integer n , not just even n , but we will use it in the following derivation

with another substitution $t = \theta - \pi/2$:

$$\begin{aligned}
J_n(\rho) - J_{-n}(\rho) &= \frac{2i}{\pi} \int_0^\pi \cos(\rho \sin \theta) \sin(n\theta) d\theta \\
&= \frac{2i}{\pi} \int_{-\pi/2}^{\pi/2} \cos(\rho \sin(t + \pi/2)) \sin(n(t + \pi/2)) dt \\
&= \pm \frac{2i}{\pi} \int_{-\pi/2}^{\pi/2} \cos(\rho \cos t) \sin(nt) dt \\
&= 0
\end{aligned}$$

On the last second line, when $n \equiv 0 \pmod{4}$ it will takes the positive sign, and when $n \equiv 2 \pmod{4}$ it will takes the negative sign, but either way the integrand is an odd function on t and the value will be zero.

A similar two-times variable substitution when n is odd will give us the proof for $J_{-n}(\rho) + J_n(\rho) = 0$. □

(c) $2J'_n(\rho) = J_{n-1}(\rho) - J_{n+1}(\rho)$.

Solution: We have

$$\begin{aligned}
2J'_n(\rho) &= \frac{1}{\pi} \int_0^{2\pi} i e^{i\rho \sin \theta} e^{-in\theta} \sin \theta d\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} e^{i\rho \sin \theta} e^{-in\theta} \frac{e^{i\theta} - e^{-i\theta}}{2i} d\theta \\
&= J_{n-1}(\rho) - J_{n+1}(\rho).
\end{aligned}$$

□

(d) $\left(\frac{2n}{\rho}\right) J_n(\rho) = J_{n-1}(\rho) + J_{n+1}(\rho)$.

Solution: This identity can be proven using integration by parts. RHS is

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\rho \sin \theta} e^{-in\theta} \cdot 2 \cos \theta d\theta.$$

Consider letting $u = e^{-in\theta}$, $dv = 2e^{i\rho \sin \theta} \cos \theta d\theta$, then we have $du = -in e^{-in\theta} d\theta$, $v = \frac{2}{i\rho} e^{i\rho \sin \theta}$.

Therefore,

$$\begin{aligned}
\text{RHS} &= \frac{1}{2\pi} \left(\left[\frac{2}{i\rho} e^{i\rho \sin \theta} e^{-in\theta} \right]_0^{2\pi} + \frac{2n}{\rho} \int_0^{2\pi} e^{i\rho \sin \theta} e^{-in\theta} d\theta \right) \\
&= \frac{1}{2\pi} \left(\frac{2}{i\rho} - \frac{2}{i\rho} \right) + \frac{2n}{\rho} J_n(\rho) \\
&= \frac{2n}{\rho} J_n(\rho).
\end{aligned}$$

□

(e) $(\rho^{-n} J_n(\rho))' = -\rho^{-n} J_{n+1}(\rho)$.

Solution: It will be easier to prove this identity by using previous identities. We have

$$\begin{aligned}
LHS &= -n\rho^{-n-1}J_n(\rho) + \rho^{-n}J_n(\rho)' \\
&= -\frac{\rho^{-n}}{2} \cdot \left(\frac{2n}{\rho}J_n(\rho)\right) + \frac{\rho^{-n}}{2}(J_{n-1}(\rho) - J_{n+1}(\rho)) \\
&= -\frac{\rho^{-n}}{2}(J_{n-1}(\rho) + J_{n+1}(\rho)) + \frac{\rho^{-n}}{2}(J_{n-1}(\rho) - J_{n+1}(\rho)) \\
&= -\rho^{-n}J_{n+1}(\rho).
\end{aligned}$$

□

(f) $(\rho^n J_n(\rho))' = \rho^n J_{n-1}(\rho).$

Solution: Similar as previous identity, we have

$$\begin{aligned}
LHS &= n\rho^{n-1}J_n(\rho) + \rho^n J_n'(\rho) \\
&= \frac{\rho^n}{2}(J_{n-1}(\rho) + J_{n+1}(\rho)) + \frac{\rho^n}{2}(J_{n-1}(\rho) - J_{n+1}(\rho)) \\
&= \rho^n J_{n-1}(\rho).
\end{aligned}$$

□

(g) $J_n(\rho)$ satisfies the second order differential equation

$$J_n''(\rho) + \rho^{-1}J_n'(\rho) + (1 - n^2/\rho^2)J_n(\rho) = 0.$$

Solution: We basically need to use the identities from (e) and (f), noticing how they can decrease the index $n \rightarrow n - 1$ and then increase it again $n \rightarrow n + 1$.

We use J_n as a shorthand notation for $J_n(\rho)$ temporarily for the derivation below.

$$\begin{aligned}
&\because (\rho^n J_n)' = \rho^n J_{n-1} \\
&n\rho^{n-1}J_n + \rho^n J_n' = \rho^n J_{n-1} \\
&n\rho^{-n}J_n + \rho^{-n+1}J_n' = \rho^{-(n-1)}J_{n-1} \\
&\therefore \rho^{-n+1}J_n'' - (n-1)\rho^{-n}J_n' + n\rho^{-n}J_n' - n^2\rho^{-n-1}J_n = (\rho^{-(n-1)}J_{n-1})' \\
&\rho^{-n+1}J_n'' + \rho^{-n}J_n' - n^2\rho^{-n-1}J_n = -\rho^{-n+1}J_n \\
&J_n'' + \rho^{-1}J_n' + (1 - n^2\rho^{-2})J_n = 0.
\end{aligned}$$

□

(h) Show that

Solution:

$$J_n(\rho) = \left(\frac{\rho}{2}\right)^n \sum_{m=0}^{\infty} (-1)^m \frac{\rho^{2m}}{2^{2m}m!(n+m)!}.$$

Notice that from (g), $J_n(\rho)$ satisfies a second order ODE in the form of $\rho^2 J_n'' + \rho J_n' + (\rho^2 - n^2)J_n = 0$, which can be solved by Frobenius method.

For some convenient reason, let's assume $n \geq 1$ for now. We will then prove for $n = 0$. Since there is factorial, we assume we only have to prove for $n \geq 0$.

Let $J_n(\rho)$ has the form

$$J_n(\rho) = \sum_{k=0}^{\infty} A_k \rho^{k+r}, \quad A_0 \neq 0.$$

Then we have

$$J'_n(\rho) = \sum_{k=0}^{\infty} (k+r) A_k \rho^{k+r-1},$$

$$J''_n(\rho) = \sum_{k=0}^{\infty} (k+r)(k+r-1) A_k \rho^{k+r-2}.$$

Substitutes into the second order ODE, we have

$$\sum_{k=0}^{\infty} [(k+r)(k+r-1) + k+r+\rho^2 - n^2] A_k \rho^{k+r} = 0.$$

Cancel out ρ^r , using the assumption that $A_0 \neq 0$ and substitute $\rho = 0$, we obtain the indicial equation:

$$r(r-1) + r - n^2 = 0 \implies r = \pm n.$$

Seems like we have two possibilities for r , but $r = -n$ is actually impossible because $J_n(\rho)$ is a finite function, it wouldn't have any singularity. We thus have $r = n$.

Simplify the equations, we have

$$\begin{aligned} \sum_{k=0}^{\infty} [(k+n)(k+n-1) + k+n+\rho^2 - n^2] A_k \rho^k &= 0 \\ \sum_{k=0}^{\infty} (k^2 + 2kn + \rho^2) A_k \rho^k &= 0 \\ \sum_{k=0}^{\infty} (k^2 + 2kn) A_k \rho^k + \sum_{k=0}^{\infty} A_k \rho^{k+2} &= 0 \\ (2n+1)A_1\rho + \sum_{k=2}^{\infty} [k(k+2n)A_k + A_{k-2}] \rho^k &= 0. \end{aligned}$$

The first term tells us $A_1 = 0$ since we can cancel a ρ everywhere and let $\rho = 0$. Then we also have $A_k = -\frac{1}{k(k+2n)} A_{k-2}$ for $k \geq 2$. This shows that $A_{2m+1} = 0$ for all integer $m \geq 0$, and we made some progress that proves the series only has even power terms.

Now let $k = 2m$ for $m \geq 1$, we have

$$\begin{aligned} A_{2m} &= -\frac{1}{4m(m+n)} A_{2m-2} \\ &= (-1)^2 \frac{A_{2m-4}}{4^2 m(m-1)(m+n)(m-1+n)} \\ &= (-1)^m \frac{A_0 n!}{4^m m! (m+n)!}. \end{aligned}$$

We now only have to show $A_0 = 1/(2^n n!)$ and the proof will be done for $n \geq 1$. To show this, we first notice that as $r = n$, that $\lim_{\rho \rightarrow 0} \rho^{-m} J_n(\rho) = 0$ for any integer $m < n$. Then, we prove by induction on n : We denote temporarily $A_0(n)$ to be our A_0 coefficient and want to show that $A_0(n) = \frac{1}{2^n n!}$.

First it is not hard to see $A_0(n) = \lim_{\rho \rightarrow 0} \rho^{-n} J_n(\rho)$, for $n = 0$ we have $J_0(0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1$,

and for $n = 1$ we use the identity from (d):

$$\begin{aligned}\frac{J_1(\rho)}{\rho} &= \frac{J_0(\rho) + J_2(\rho)}{2} \\ \therefore A_0(1) &= \lim_{\rho \rightarrow 0} \frac{J_1(\rho)}{\rho} \\ &= \frac{1}{2}.\end{aligned}$$

Now assumes $A_0(n) = \frac{1}{2^n n!}$ for some $n \geq 1$, then

$$\begin{aligned}\frac{J_{n+1}(\rho)}{\rho} &= \frac{J_n(\rho) + J_{n+2}(\rho)}{2n+2} \\ \frac{J_{n+1}(\rho)}{\rho^{n+1}} &= \frac{\rho^{-n} J_n(\rho) + \rho^{-n} J_{n+2}(\rho)}{2(n+1)} \\ \therefore A_0(n+1) &= \frac{A_0(n)}{2(n+1)} \\ &= \frac{1}{2^{n+1}(n+1)!}.\end{aligned}$$

This shows the series expansion for $J_n(\rho)$ is true for $n \geq 1$.

To prove the series expansion is also true when $n = 0$, we just use the identity from (f) when $n = 1$. □

(i) Show that for all integers n and all real numbers a and b we have

$$J_n(a+b) = \sum_{\ell \in \mathbb{Z}} J_\ell(a) J_{n-\ell}(b).$$

Solution: By definition of Bessel's function, the Fourier series of $e^{i\rho \sin \theta}$ is

$$e^{i\rho \sin \theta} = \sum_{n \in \mathbb{Z}} J_n(\rho) e^{in\theta}.$$

Using this, we can derive the identity.

$$\begin{aligned}J_n(a+b) &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(a+b) \sin \theta} e^{-in\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{\ell \in \mathbb{Z}} J_\ell(a) e^{i\ell\theta} \right) \left(\sum_{m \in \mathbb{Z}} J_m(b) e^{im\theta} \right) e^{-in\theta} d\theta \\ &= \sum_{\ell \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} J_\ell(a) J_m(b) \frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell+m-n)\theta} d\theta \\ &= \sum_{\ell \in \mathbb{Z}} J_\ell(a) J_{n-\ell}(b).\end{aligned}$$

The last equation is because $\int_0^{2\pi} e^{ik\theta} d\theta = 2\pi$ when $k = 0$, but become zero when k is any non-zero integer. □

2. Another formula for $J_n(\rho)$ that allows one to define Bessel functions for non-integral values of n , ($n > -1/2$) is

$$J_n(\rho) = \frac{(\rho/2)^n}{\Gamma(n+1/2)\sqrt{\pi}} \int_{-1}^1 e^{i\rho t} (1-t^2)^{n-(1/2)} dt.$$

- (a) Check that the above formula agrees with the definition of $J_n(\rho)$ for integral $n \geq 0$.

Solution: When $n = 0$, the RHS become

$$RHS = \frac{1}{\pi} \int_{-1}^1 \frac{e^{i\rho t}}{\sqrt{1-t^2}} dt.$$

Let $t = \sin \theta$, we have

$$RHS = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{i\rho \sin \theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i\rho \sin \theta} d\theta = J_0(\rho).$$

Now assume that the identity works for some non-negative integer n , using the identity from 1(e) and integration by parts, we see that

$$\begin{aligned} J_{n+1}(\rho) &= -\rho^n \frac{d}{d\rho} (\rho^{-n} J_n(\rho)) \\ &= -\rho^n \frac{d}{d\rho} \left[\frac{1}{2^n \Gamma(n+1/2) \sqrt{\pi}} \int_{-1}^1 e^{i\rho t} (1-t^2)^{n-1/2} dt \right] \\ &= -\frac{i\rho^n}{2^n \Gamma(n+1/2) \sqrt{\pi}} \int_{-1}^1 e^{i\rho t} t (1-t^2)^{n-1/2} dt \\ &= -\frac{i\rho^n}{2^n \Gamma(n+1/2) \sqrt{\pi}} \left[-\frac{e^{i\rho t} (1-t^2)^{n+1/2}}{2n+1} \Big|_{-1}^1 + \frac{i\rho}{2(n+1/2)} \int_{-1}^1 e^{i\rho t} (1-t^2)^{n+1/2} dt \right] \\ &= \left(\frac{\rho}{2}\right)^{n+1} \frac{1}{\Gamma(n+3/2) \sqrt{\pi}} \int_{-1}^1 e^{i\rho t} (1-t^2)^{n+1/2} dt. \end{aligned}$$

By induction, the identity now works for every non-negative integer n . □

- (b) Note that $J_{1/2}(\rho) = \sqrt{\frac{2}{\pi}} \rho^{-1/2} \sin \rho$.

Solution: We simply have

$$\begin{aligned} J_{1/2}(\rho) &= \frac{\sqrt{\rho/2}}{\sqrt{\pi}} \int_{-1}^1 e^{i\rho t} dt \\ &= \sqrt{\frac{\rho}{2\pi}} \cdot \frac{e^{i\rho} - e^{-i\rho}}{i\rho} \\ &= \sqrt{\frac{2}{\pi}} \rho^{-1/2} \sin \rho. \end{aligned}$$
□

- (c) Prove that

$$\lim_{n \rightarrow -1/2} J_n(\rho) = \sqrt{\frac{2}{\pi}} \rho^{-1/2} \cos \rho.$$

Solution: Solution reference here.

Directly calculate the limit won't work, so we are looking at a way of raising the power of say $(1-t^2)$ from $n-1/2$ to $n+1/2$, which implies $K_{n+1} = K_n/(2n+1)$.

To avoid typing a lot of constants, we let $K_n = \frac{1}{2^n \Gamma(n+1/2) \sqrt{\pi}}$.

Observing the integrand, we can reflect it:

$$J_n(\rho) = 2\rho^n K_n \int_0^1 \cos(\rho t)(1-t^2)^{n-1/2} dt.$$

Then, we use a slightly non-trivial integration by parts of $u = \frac{\cos \rho t}{t}$, $dv = t(1-t^2)^{n-1/2}$ with $du = \frac{-t\rho \sin \rho t - \cos \rho t}{t^2}$, $v = \frac{1 - (1-t^2)^{n+1/2}}{2n+1}$.

The additional constant $\frac{1}{2n+1}$ in v is to make the non-integral part easy to calculate, which in turns actually make other calculations easier as well. Following this setting we have

$$J_n(\rho) = 2\rho^n K_{n+1} \cos \rho + 2\rho^n K_n \int_0^1 (t\rho \sin \rho t + \cos \rho t) \frac{1 - (1-t^2)^{n+1/2}}{t^2(2n+1)} dt.$$

We have to show the second term converges to 0 when $n \rightarrow -1/2$, although it should be, I still struggle to find ways to prove this, so I am going to leave this here for now. □

- (d) Observe that the formulas we have proved in the text giving F_0 in terms of f_0 (when describing the Fourier transform of a radial function) take the form

$$F_0(\rho) = 2\pi\rho^{-(d/2)+1} \int_0^\infty J_{(d/2)-1}(2\pi\rho r) f_0(r) r^{d/2} dr,$$

for $d = 1, 2$, and 3 , if one uses the formulas above with the understanding that $J_{-1/2}(\rho) = \lim_{n \rightarrow -1/2} J_n(\rho)$. It turns out that the relation between F_0 and f_0 given by the equation above is valid in all dimensions d .

Solution: I can't solve this bro, it is too hard... □

3. We observed that the solution $u(x, t)$ of the Cauchy problem for the wave equation given by formula (3) depends only on the initial data on the base on the backward light cone. It is natural to ask if this property is shared by *any* solution of the wave equation. An affirmative answer would imply uniqueness of the solution.

Let $B(x_0, r_0)$ denote the closed ball in the hyperplane $t = 0$ centered at x_0 and of radius r_0 . The **backward light cone** with base $B(x_0, r_0)$ is defined by

$$\mathcal{L}_{B(x_0, r_0)} = \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |x - x_0| \leq r_0 - t, 0 \leq t \leq r_0\}.$$

Theorem Suppose that $U(x, t)$ is a C^2 function on the closed upper half-plane $\{(x, t) : x \in \mathbb{R}^d, t \geq 0\}$ that solves the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u.$$

If $u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0$ for all $x \in B(x_0, r_0)$, then $u(x, t) = 0$ for all $(x, t) \in \mathcal{L}_{B(x_0, r_0)}$. theoremend.

In words, if the initial data of the Cauchy problem for the wave equation vanishes on a ball B , then *any* solution u of the problem vanishes in the backward light cone with base B . The following steps outline a proof of the theorem.

- (a) Assume that u is real. For $0 \leq t \leq r_0$ let $B_t(x_0, r_0) = \{x : |x - x_0| \leq r_0 - t\}$, and also define

$$\nabla u(x, t) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \frac{\partial u}{\partial t} \right).$$

Now consider the energy integral

$$\begin{aligned} E(t) &= \frac{1}{2} \int_{B_t(x_0, r_0)} |\nabla u|^2 dx \\ &= \frac{1}{2} \int_{B_t(x_0, r_0)} \left(\frac{\partial u}{\partial t} \right)^2 + \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j} \right)^2 dx. \end{aligned}$$

Observe that $E(t) \geq 0$ and $E(0) = 0$. Prove that

$$E'(t) = \int_{B_t(x_0, r_0)} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial t} dx - \frac{1}{2} \int_{\partial B_t(x_0, r_0)} |\nabla u|^2 d\sigma(\gamma).$$

Solution: First I was naively differentiating the integrand and only get the first term, but after that realize the domain of integral depends on t , which prevents me from doing so! Let's see the proper solution.

This solution uses Leibniz Integral Rule. We can split the integral like so:

$$E(t) = \frac{1}{2} \int_t^{r_0} \left(\int_{\partial B_r(x_0, r_0)} |\nabla u|^2 d\sigma(\gamma) \right) dr$$

Note that now only the outer integral depends on t , we can thus apply Leibniz integral rule and obtain

$$E'(t) = \frac{1}{2} \int_t^{r_0} \left(\int_{\partial B_r(x_0, r_0)} \frac{\partial}{\partial t} |\nabla u|^2 d\sigma(\gamma) \right) dr - \frac{1}{2} \int_{\partial B_t(x_0, r_0)} |\nabla u|^2 d\sigma(\gamma).$$

The term related to r_0 is differentiated to none as it is only a constant, now it should be obvious the identity is true. □

(b) Show that

$$\frac{\partial}{\partial x_j} \left[\frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \right] = \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial t} + \frac{\partial^2 u}{\partial x_j^2} \frac{\partial u}{\partial t}.$$

Solution: This is a standard product rule. □

(c) Use the last identity, the divergence theorem, and the fact that u solves the wave equation to prove that

$$E'(t) = \int_{\partial B_t(x_0, r_0)} \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} v_j d\sigma(\gamma) - \frac{1}{2} \int_{\partial B_t(x_0, r_0)} |\nabla u|^2 d\sigma(\gamma),$$

where v_j denotes the j^{th} coordinate of the outward normal to $B_t(x_0, r_0)$.

Solution: Compare to (a), we only need to show

$$\int_{B_t(x_0, r_0)} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial t} dx = \int_{\partial B_t(x_0, r_0)} \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} v_j d\sigma(\gamma).$$

With the suggested strategies, we notice the left integrand can be written as

$$\begin{aligned}\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial t} &= \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2} \frac{\partial u}{\partial t} + \sum_{j=1}^d \frac{\partial}{\partial x_j} \left[\frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \right] \\ &= \frac{\partial u}{\partial t} \left[\frac{\partial^2 u}{\partial t^2} - \Delta u \right] + \sum_{j=1}^d \frac{\partial}{\partial x_j} \left[\frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \right] \\ &= \sum_{j=1}^d \frac{\partial}{\partial x_j} \left[\frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \right].\end{aligned}$$

Therefore, by letting $F = F_t(x_1, \dots, x_d)$ to be a vector field:

$$F_t = \frac{\partial u}{\partial t} \left\langle \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d} \right\rangle,$$

we may apply the Divergence theorem so that

$$\int_{B_t(x_0, r_0)} (\nabla \cdot F_t) dx = \int_{\partial B_t(x_0, r_0)} (F \cdot \vec{n}) d\sigma(\gamma),$$

and we have proved the identity. □

(d) Use the Cauchy-Schwarz inequality to conclude that

$$\sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} v_j \leq \frac{1}{2} |\nabla u|^2,$$

and as a result, $E'(t) \leq 0$. Deduce from this that $E(t) = 0$ and $u = 0$.

Solution: Let's write $\vec{n} = (v_1, \dots, v_d)$, so that $\sum_{j=1}^d v_j^2 = 1$. Then by Cauchy-Schwarz inequality,

$$\sum_{j=1}^d \left(\frac{\partial u}{\partial x_j} \right)^2 = \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j} \right)^2 \sum_{j=1}^d v_j^2 \geq \left(\sum_{j=1}^d \frac{\partial u}{\partial x_j} v_j \right)^2.$$

Now we adding the partial derivative of t into the terms, giving us

$$\begin{aligned}\frac{1}{2} |\nabla u|^2 &\geq \frac{1}{2} \left[\left(\sum_{j=1}^d \frac{\partial u}{\partial x_j} v_j \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] \\ &\geq \frac{\partial u}{\partial t} \sum_{j=1}^d \frac{\partial u}{\partial x_j} v_j.\end{aligned}$$

We have proved that $E'(t) \leq 0$ for all $t \in [0, r_0]$. To show that the energy $E(t)$ is zero on $[0, r_0]$ and thus subsequently $u = 0$ too, we take any $t \in (0, r_0]$, then by mean value theorem there is $c \in [0, t]$ such that

$$\frac{E(t)}{t} = \frac{E(t) - E(0)}{t} = E'(c) \leq 0 \implies E(t) \leq 0.$$

Since E is nonnegative, we can conclude $E(t) = 0$. □

7. For every real number $a > 0$, define the operator $(-\Delta)^a$ by the formula

$$(-\Delta)^a f(x) = \int_{\mathbb{R}^d} (2\pi|\xi|)^{2a} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

whenever $f \in \mathcal{S}(\mathbb{R}^d)$.

- (a) Check that $(-\Delta)^a$ agrees with the usual definition of the a^{th} power of $-\Delta$ (that is, a compositions of minus the Laplacian) when a is a positive integer.

Solution: First we have

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ (-\Delta)f(x) &= - \int_{\mathbb{R}^d} (2\pi i)^2 |\xi|^2 \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= \int_{\mathbb{R}^d} (2\pi|\xi|)^2 \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi. \end{aligned}$$

This proves the definition aligns with the usual Laplacian operator at $a = 1$. Now assume it is true for some positive integer a , then

$$\begin{aligned} (-\Delta)(-\Delta)^a f(x) &= (-\Delta) \int_{\mathbb{R}^d} (2\pi|\xi|)^{2a} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= - \int_{\mathbb{R}^d} (2\pi|\xi|)^{2a} (2\pi i)^2 |\xi|^2 \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= \int_{\mathbb{R}^d} (2\pi|\xi|)^{2(a+1)} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= (-\Delta)^{a+1} f(x), \end{aligned}$$

thus $(-\Delta)^a$ agrees with the a^{th} power of negative Laplacian for all positive integer a by induction. □

- (b) Verify that $(-\Delta)^a(f)$ is indefinitely differentiable.

Solution: For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ where $|\alpha| = \sum_{1 \leq i \leq d} \alpha_i$ (I forgot whether this is the original definition, it probably is), we note that

$$\left(\frac{\partial}{\partial x} \right)^\alpha (-\Delta)^a f(x) = \int_{\mathbb{R}^d} (2\pi|\xi|)^{2a} \cdot (2\pi i)^{|\alpha|} \xi^\alpha \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_d^{\alpha_d}$. Since $f \in \mathcal{S}(\mathbb{R}^d)$, the integrand is rapidly decreasing, hence the integral value is finite. □

- (c) Prove that if a is not an integer, then in general $(-\Delta)^a(f)$ is not rapidly decreasing. **Solution:**

□