Fourier Analysis Stein: Chapter 7. Problems.

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Xiamen University Malaysia, Asia Pacific University Malaysia — July 13, 2024

1 Problems

1. Let f be a function on the circle. For each $N \geq 1$ the discrete Fourier coefficients of f are defined by

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi ikn/N}, \qquad \text{for } n \in \mathbb{Z}.$$

We also let

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi nx}dx$$

denote the ordinary Fourier coefficients of f.

(a) Show that $a_N(n) = a_N(n+N)$.

Solution: The only term related to n is $e^{-2\pi ikn/N}$, so it is trivial to see this term is periodic with a period of N. □

(b) Prove that if f is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.

Solution: Let n be a fixed integer. Since f is continuous and periodic, it is bounded, we assume it is bounded above by a constant $M > 0$. Given $\varepsilon > 0$, we may choose a large enough N such that

$$|e^{-2\pi inx} - e^{-2\pi iny}| < \frac{\varepsilon}{2M} \text{ whenever } |x-y| < \frac{1}{N}.$$

Also, f is uniformly continuous because it is periodic, so we may choose another large enough N' so that

$$|f(e^{2\pi ix}) - f(e^{2\pi iy})| < \frac{\varepsilon}{2} \text{ whenever } |x-y| < \frac{1}{N'}.$$

We now simply choose $N_0 = \max(N, N')$ so the above two can be satisfied.

For $N > N_0$ we have

$$\begin{aligned}|a_N(n) - a(n)| &= \left|\sum_{k=1}^{N}\left[\frac{1}{N}f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \int_{(k-1)/N}^{k/N} f(e^{2\pi ix})e^{-2\pi inx}dx\right]\right| \\ &\leq \sum_{k=1}^{N}\int_{(k-1)/N}^{k/N}\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right|dx.\end{aligned}$$

The inner term can be estimated like so:

$$\begin{aligned}&\left|f(e^{2\pi ik/N})e^{-2\pi ikn/N} - f(e^{2\pi ix})e^{-2\pi inx}\right| \\ \leq &\left|f(e^{2\pi ik/N}) - f(e^{2\pi ix})\right| + \left|f(e^{2\pi ix})\right|\left|e^{-2\pi ikn/N} - e^{-2\pi inx}\right| \\ < &\varepsilon.\end{aligned}$$

This proves $\lim_{N\to\infty} a_N(n) = a(n)$. $\square$

2. If f is a C^1 function on the circle, prove that $|a_N(n)| \le c/|n|$ whenever $0 < |n| \le N/2$.

Solution: We have to clarify that c will be a constant that is independent from n and N, possible related only to f. We also only prove the statement for $0 < n \le N/2$, i.e., positive n, the proof for negative n should be nearly identical.

Since f is periodic and C^1, it is Lipchitz continuous. See this answer. Therefore, we let M be a positive constant such that

$$|f(x) - f(y)| \le M|x-y| \quad \text{for all } x, y \text{ on the circle.}$$

We also use the identity: $|1 - e^{i\theta}| = 2\sin(\theta/2) \le \theta$ for $\theta \in [0, \pi]$.

Following the hint from the book, we let $\ell = 1$ if $N/4 < n \le N/2$. If otherwise $n \le N/4$, we let ℓ be the largest integer satisfying $\ell n/N \le 1/2$, then since $(\ell+1)n/N > 1/2$, we have

$$\left|\frac{1}{2} - \frac{\ell n}{N}\right| = \frac{1}{2} - \frac{\ell n}{N} < \frac{n}{N} \le \frac{1}{4}.$$

This is to verify that we can always choose ℓ such that $1/4 \le \ell n/N \le 1/2$ by construction, and that we also have the following lower bound:

$$\frac{\pi}{2} \le \frac{2\pi\ell n}{N} \le \pi \implies \left|1 - e^{2\pi i\ell n/N}\right| \ge \sqrt{2}.$$

Everything is ready, we now have

$$\begin{aligned}
|a_N(n)|\sqrt{2} &\le |a_N(n)|\left|1 - e^{2\pi i\ell n/N}\right| \\
&\le \frac{1}{N}\sum_{k=1}^{N} M\left|1 - e^{2\pi i\ell/N}\right| \\
&= M\left|1 - e^{2\pi i\ell/N}\right| \\
&\le M\cdot\frac{2\pi\ell}{N} \\
\therefore |a_N(n)| &\le \frac{M\pi}{\sqrt{2}n}.
\end{aligned}$$

$\square$

3. By a similar method, show that if f is a C^2 function on the circle, then

$$|a_N(n)| \le c/|n|^2, \quad \text{whenever } 0 < |n| \le N/2.$$

As a result, prove the inversion formula for $f \in C^2$,

$$f(e^{2\pi ix}) = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi inx}$$

from its finite version.

Solution: We use the second symmetric difference and the fact that f'' is bounded above, that

$$\left|f(e^{2\pi i(k+\ell)/N}) - 2f(e^{2\pi ik/N}) + f(e^{2\pi i(k-\ell)/N})\right| \le \frac{M\ell^2}{N^2}.$$

For me, I can't overcome the fact that how to apply the mean value theorem on complex domain, so I just simply let $g : [0,1] \to \mathbb{R}$ such that $g(x) = f(e^{2\pi i x})$, then it follows that g is a real-valued, periodic C^2 function on $[0,1]$. This translate to the bound above.

By this logic and choosing a suitable integer ℓ as in previous solution, we find ourselves at

$$\begin{aligned} a_N(n)\left(1 - e^{2\pi i \ell n/N}\right)^2 e^{-2\pi i \ell n/N} &= \frac{1}{N}\sum_{k=1}^{N} \\ &\quad \left[f(e^{2\pi i(k+\ell)/N} - 2f(e^{2\pi i k/N}) + f(e^{2\pi i(k-\ell)/N}))e^{-2\pi i k n/N}\right] \\ |a_N(n)|\left|1 - e^{2\pi i \ell n/N}\right|^2 &\le \frac{1}{N}\sum_{k=1}^{N}\frac{M\ell^2}{N^2} \\ &\le \frac{M\ell^2}{N^2}. \end{aligned}$$

Since $\ell/N \le 1/(2n)$ and $\left|1 - e^{2\pi i \ell n/N}\right| \ge \sqrt{2}$, we manage to show that

$$|a_N(n)| \le \frac{M}{8n^2}.$$

(Second part no solution for now.) □

4. Let e be a character on $G = \mathbb{Z}(N)$, the additive group of integers modulo N. Show that there exists a unique $0 \le \ell \le N-1$ so that

$$e(k) = e_\ell(k) = e^{2\pi i \ell k/N} \quad \text{for all } k \in \mathbb{Z}(N).$$

Conversely, every function of this type is a character on $\mathbb{Z}(N)$. Deduce that $e_\ell \mapsto \ell$ defines an isomorphism from G to G.

Solution: Let $e : G \to S^1$ be a character, if it is an identity character, then $e = e_0$. If e is non-trivial, then we let $a \in G$ so that $e(a) \ne 1$. Since $n \cdot a$ is the identity element in G, we have that

$$e(a)^n = e(n \cdot a) = 1.$$

This shows that $e(a)$ is a root of unity, and thus there is a unique $0 \le \ell < N$ such that $e = e_\ell$.

To show the mapping $M : \hat{G} \to G$ defined by $M(e_\ell) = \ell$ is an isomorphism, it is worth noting this is a well-defined mapping, i.e., for $\ell, k \in \mathbb{Z}(N)$, $M(e_\ell) = M(e_k)$ implies $\ell = k$. Moreover, this mapping is surjective and one-to-one, we only have to show it is a homomorphism:

$$M(e_\ell \circ e_k) = M(e_{\ell+k}) = \ell + k \pmod{N}.$$

□

5. Show that all characters on S^1 are given by

$$e_n(x) = e^{2\pi i n x} \text{ with } n \in \mathbb{Z},$$

and check that $e_n \mapsto n$ defines an isomorphism from $\widehat{S^1}$ to $\mathbb{Z}$.

Solution: To make the discussion less confusing, we will use f to denote a character on S^1, which is $f : S^1 \to S^1$.

We can then define $F : \mathbb{R} \to S^1$ so that $F(x) = f(e^{2\pi i x})$, it is not hard to see F is a periodic function with a period of 1, and also satisfying $F(x+y) = F(x)F(y)$.

Clearly F is a complex-valued continuous function, with $F(0) = 1$, hence there is a small enough $\delta > 0$ such that $c = \int_0^\delta F(y)dy \neq 0$. We then have

$$\begin{aligned}
cF(x) &= \int_0^\delta F(x+y)dy \\
&= \int_x^{x+\delta} F(y)dy \\
\therefore cF'(x) &= F(x+\delta) - F(x) \\
&= [F(\delta) - 1]F(x) \\
F'(x) &= \frac{F(\delta) - 1}{c} F(x).
\end{aligned}$$

Thus, $F'(x) = AF(x)$ for some constant A, combine with the condition that $F(0) = 1$, we have $F(x) = e^{Ax}$.

Since $F(x)$ have values in S^1, $A = 2\pi i n$ for some real number n.

To show that n is an integer, we have $e^{2\pi i n} = F(1) = f(e^{2\pi i}) = 1$, so $f(e^{2\pi i x}) = e_n(x)$ for $x \in \mathbb{R}$.

Clearly if $n \neq m$ are integers, then $e_n \neq e_m$, and for each integer n, e_n is a character on S^1. We only have to show $L : \widehat{S^1} \to \mathbb{Z}$ is a homomorphism. For all $x \in \mathbb{R}$ we have $e_n \circ e_m(x) = e^{2\pi i (n+m)x}$, hence $L(e_n \circ e_m) = n + m = L(e_n) + L(e_m)$. □

6. Prove that all characters on $\mathbb{R}$ take the form

$$e_\xi(x) = e^{2\pi i \xi x} \quad \text{with } \xi \in \mathbb{R}.$$

and that $e_\xi \mapsto \xi$ defines an isomorphism from $\widehat{\mathbb{R}}$ to $\mathbb{R}$. The argument in Exercise 5 applies here as well.

Solution: We use F to denote a character on $\mathbb{R}$, which is $F : \mathbb{R} \to S^1$. Since $\mathbb{R}$ is an additive group, we have $F(x+y) = F(x)F(y)$, and also $F(0) = 1$.

Using what we proved in the previous exercise, there is a constant A such that $F(x) = e^{Ax}$, and since F maps to norm of 1, $A = 2\pi i \xi$ for some real number ξ.

Now we see that for any real ξ, F is a character on $\mathbb{R}$, and $F(x) = e_\xi(x)$.

It is clear that the mapping $e_\xi \mapsto \xi$ is surjective and one-to-one, we only have to show it is a homomorphism by noting that $e_{\xi+\eta}(x) = e^{2\pi i(\xi+\eta)x} = e^{2\pi i \xi x} e^{2\pi i \eta x} = e_\xi(x) e_\eta(x)$ for any $\xi, \eta \in \mathbb{R}$.

□

7. Let $\xi = e^{2\pi i/N}$. Define the $N \times N$ matrix $M = (a_{jk})_{1 \leq j,k \leq N}$ by $a_{jk} = N^{-1/2} \xi^{jk}$.

   (a) Show that M is unitary.

   Solution: Write $M = [\vec{a_1} | \cdots | \vec{a_N}]$ where each column vector is $\vec{a_k} = (a_{1k} \dots a_{Nk})^T$. Then we have that for $1 \leq k, n \leq N$,

$$\vec{a_k}^* \vec{a_n} = \frac{1}{N} \sum_{j=1}^{N} \overline{\xi^{jk}} \xi^{jn} = \frac{1}{N} \sum_{j=1}^{N} \xi^{j(n-k)} = \delta_{kn}.$$

   Where it means that $\delta_{kn} = 1$ when $k = n$, and $\delta_{kn} = 0$ otherwise. This proves M to be a unitary matrix. □

   (b) Interpret the identity $(Mu, Mv) = (u, v)$ and the fact that $M^* = M^{-1}$ in terms of the Fourier series on $\mathbb{Z}(N)$.

   Solution: Not too sure what is the point of the question here. □

8. Suppose that $P(x) = \sum_{n=1}^{N} a_n e^{2\pi i n x}$.

   (a) Show by using the Parseval identities for the circle and $\mathbb{Z}(N)$, that

$$\int_0^1 |P(x)|^2 dx = \frac{1}{N} \sum_{j=1}^{N} |P(j/N)|^2.$$

Solution: Notice that P looks like a function on $\mathbb{R}$, but essentially it is a function on $\mathbb{Z}(N)$. We let $e_n(x) = e^{2\pi i n x}$ to be characters on S^1, then we find that $P = \sum_{n=1}^{N} a_n e_n$. Moreover, for $x \in \mathbb{Z}(N)$ which by that we means $x = 0, 1, \ldots, N-1$, we let $F(x) = P(x/N)$, so F is a function on $\mathbb{Z}(N)$.

We note that we can define an inner product for periodic functions f, g over $[0,1]$ on $\mathbb{R}$ as $(f, g) = \int_0^1 f\bar{g}$, then $\{e_n\}_{1 \le n \le N}$ is orthonormal. We thus can have

$$\int_0^1 |P(x)|^2 dx = (P, P) = \sum_{j=1}^{N} \sum_{k=1}^{N} a_j \overline{a_k} (e_j, e_k) = \sum_{n=1}^{N} |a_n|^2.$$

Apply Parseval identity to F and using the inner product for functions on $\mathbb{Z}(N)$, we have

$$\sum_{n=1}^{N} |a_n|^2 = (F, F) = \frac{1}{N} \sum_{x=1}^{N} |F(x)|^2 = \frac{1}{N} \sum_{x=1}^{N} \left| P\left(\frac{x}{N}\right) \right|^2.$$

This completes the proof. □