

# Functionaal Analysis Stein: Chapter 1. Exercises.

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## 1 Problems

1. Consider  $L^p = L^p(\mathbb{R}^d)$  with Lebesgue measure. Let  $f_0(x) = |x|^{-\alpha}$  if  $|x| < 1$ .  $f_0(x) = 0$  for  $|x| \geq 1$ , also let  $f_\infty(x) = |x|^{-\alpha}$  if  $|x| \geq 1$ ,  $f_\infty(x) = 0$  when  $|x| < 1$ .

Show that

- (a)  $f_0 \in L^p$  if and only if  $p\alpha < d$ .

**Solution:** Let  $S_{d-1}$  be the surface area of the open unit ball  $B_d = \{|x| < 1 : x \in \mathbb{R}^d\}$  in  $\mathbb{R}^d$ , then if  $f_0 \in L^p$  we can write

$$\|f_0\|_{L^p}^p = \int_{B_d} |x|^{-p\alpha} dx = S_{d-1} \int_0^1 \frac{1}{r^{p\alpha}} \cdot r^{d-1} dr = S_{d-1} \int_0^1 \frac{1}{r^{1-d+p\alpha}} dr.$$

Since the integral converges, we must have  $1 - d + p\alpha < 1$  so  $p\alpha < d$ . We saw that the converse is also true.  $\square$

- (b)  $f_\infty \in L^p$  if and only if  $d < p\alpha$ .

**Solution:** Similar to the previous part, we have

$$\|f_\infty\|_{L^p}^p = S_{d-1} \int_1^\infty \frac{1}{r^{1-d+p\alpha}} dr$$

which is finite iff  $d < p\alpha$ .  $\square$

- (c) What happens if in the definitions of  $f_0$  and  $f_\infty$  we replace  $|x|^{-\alpha}$  by  $|x|^{-\alpha}/(\log(2/|x|))$  for  $|x| < 1$ , and  $|x|^{-\alpha}/(\log(2|x|))$  for  $|x| \geq 1$ ?

**Solution:** If the definition of  $f_0$  changed to

$$f_0(x) = \begin{cases} |x|^{-\alpha}/\log(2/|x|) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

Then we want to show that  $f_0 \in L^p$  iff  $p\alpha < d$ , or  $p\alpha = d$  with  $p > 1$ , which is a little bit more nuanced than the previous part. When  $p\alpha < d$ , we see that  $|f|_0 \leq \frac{1}{\log 2} |x|^{-\alpha}$ , so that  $f_0 \in L^p$  as it is absolutely bounded above by another function in  $L^p$ .

When  $p\alpha = d$ , we have

$$S_{d-1}^{-1} \|f_0\|_p^p = \int_0^1 \frac{dr}{r(\log(2/r))^p}.$$

Using substitution  $u = \log(2/r)$ , the RHS becomes  $\int_{\log 2}^\infty \frac{du}{u^p}$ , which converges when  $p > 1$  and diverges when  $p \leq 1$ .

When  $p\alpha > d$ , we want to prove  $f_0 \notin L^p$ . We have

$$\begin{aligned} S_{d-1}^{-1} \int_{\mathbb{R}^d} |f_0|^p dx &= \int_0^1 \frac{r^{-p\alpha} r^{d-1}}{(\log(2/r))^p} dr \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d} (\log(2/r))^p} \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d} 2^p (\log(1/r))^p}. \end{aligned}$$

The last step is because  $2/r \leq 1/r^2$  whenever  $0 < r < 1/2$ . We now using  $u = 1/r$ , RHS can be

$$RHS \geq \int_2^\infty \frac{du}{2^p u^{1-p\alpha+d} (\log u)^p}.$$

By assumption,  $1 - p\alpha + d < 1$ , we can now choose  $\theta > 0$  so that  $1 - p\alpha + d + \theta < 1$ , then choose  $K > 2$  big enough such that  $(\log u)^p < u^\theta$  for all  $u \geq K$ , hence

$$RHS \geq \int_K^\infty \frac{du}{2^p u^{1-p\alpha+d+\theta}} = +\infty,$$

hence  $f_0 \notin L^p$ .

If the definition of  $f_\infty$  changed to

$$f_\infty(x) = \begin{cases} |x|^{-\alpha} / \log(2|x|) & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| < 1, \end{cases}$$

then by a similar argument, we have  $f_\infty \in L^p$  whenever  $d < p\alpha$ . When  $p\alpha = d$ , we can similarly prove that  $f_\infty \in L^p$  iff  $p > 1$ . Moreover,  $f_\infty \notin L^p$  when  $p\alpha < d$ .  $\square$

2. Consider the spaces  $L^p(\mathbb{R}^d)$ , when  $0 < p < \infty$

(a) Show that if  $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$  for all  $f$  and  $g$ , then necessarily  $p \geq 1$ .

**Solution:** We only need to show there are  $f, g \in L^p$  such that  $\|f + g\|_{L^p} > \|f\|_{L^p} + \|g\|_{L^p}$  when  $0 < p < 1$ . Let  $K_1 = \{x \in \mathbb{R}^d : x_i \in (0, 1) \forall 1 \leq i \leq d\}$  be a unit square in  $\mathbb{R}^d$ , and also  $K_{-1} = \{x \in \mathbb{R}^d : x_i \in (-1, 0) \forall 1 \leq i \leq d\}$  be its mirror.

We then have  $\|f\|_{L^p} = \|g\|_{L^p} = 1$ , but then  $\|f + g\|_{L^p} = 2^{1/p} > 2 = \|f\|_{L^p} + \|g\|_{L^p}$ .  $\square$

(b) Consider  $L^p(\mathbb{R})$  where  $0 < p < 1$ . Show that there are no bounded linear functionals on this space. In other words, if  $\ell$  is a linear functional  $L^p(\mathbb{R}) \mapsto \mathbb{C}$  that satisfies

$$|\ell(f)| \leq M \|f\|_{L^p(\mathbb{R})} \quad \text{for all } f \in L^p(\mathbb{R}) \text{ and some } M > 0,$$

then  $\ell = 0$ .

**Solution:** For each  $x > 0$ , we let  $\chi_x$  be the characteristic function of  $[0, x]$  on  $\mathbb{R}$ , then extends it naturally to  $x \leq 0$  where it is the characteristic function of  $[x, 0]$ . Let  $F(x) = \ell(\chi_x)$ .

Suppose  $\ell$  is a bounded linear functional with the constant  $M$  as stated in the question, then for any  $x, y \in \mathbb{R}$  we must have

$$|F(x) - F(y)| = |\ell(\chi_x - \chi_y)| \leq M \|\chi_x - \chi_y\|_{L^p(\mathbb{R})} = M|x - y|^{1/p}.$$

This means that  $F$  is a continuous function, but then since

$$\left| \frac{F(x) - F(y)}{x - y} \right| \leq M|x - y|^{1/p-1},$$

$F$  is then differentiable and has derivative 0 everywhere, hence  $F$  is a constant function, and must be zero too because  $F(0) = \ell(\chi_0) = 0$ . This shows  $\ell$  can only be zero.  $\square$