Functionaal Analysis Stein: Chapter 1. Exercises.

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1 Problems

1. Consider $L^p = L^p(\mathbb{R}^d)$ with Lebesgue measure. Let $f_0(x) = |x|^{-\alpha}$ if $|x| < 1$. $f_0(x) = 0$ for $|x| \geq 1$, also let $f_\infty(x) = |x|^{-\alpha}$ if $|x| \geq 1$, $f_\infty(x) = 0$ when $|x| < 1$.

   Show that

   (a) $f_0 \in L^p$ if and only if $p\alpha < d$.

   Solution: Let S_{d-1} be the surface area of the open unit ball $B_d = \{|x| < 1 : x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$, then if $f_0 \in L^p$ we can write

$$\|f_0\|_{L_p}^p = \int_{B_d} |x|^{-p\alpha} dx = S_{d-1} \int_0^1 \frac{1}{r^p \alpha} \cdot r^{d-1} dr = S_{d-1} \int_0^1 \frac{1}{r^{1-d+p\alpha}} dr.$$

   Since the integral converges, we must have $1 - d + p\alpha < 1$ so $p\alpha < d$. We saw that the converse is also true. $\square$

   (b) $f_\infty \in L^p$ if and only if $d < p\alpha$.

   Solution: Similar to the previous part, we have

$$\|f_\infty\|_{L_p}^p = S_{d-1} \int_1^\infty \frac{1}{r^{1-d+p\alpha}} dr$$

   which is finite iff $d < p\alpha$. $\square$

   (c) What happens if in the definitions of f_0 and f_∞ we replace $|x|^{-\alpha}$ by $|x|^{-\alpha}/(\log(2/|x|))$ for $|x| < 1$, and $|x|^{-\alpha}/(\log(2|x|))$ for $|x| \geq 1$?

   Solution: If the definition of f_0 changed to

$$f_0(x) = \begin{cases} |x|^{-\alpha}/\log(2/|x|) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

   Then we want to show that $f_0 \in L^p$ iff $p\alpha < d$, or $p\alpha = d$ with $p > 1$, which is a little bit more nuanced than the previous part. When $p\alpha < d$, we see that $|f|_0 \leq \frac{1}{\log 2}|x|^{-\alpha}$, so that $f_0 \in L^p$ as it is absolutely bounded above by another function in L^p.

   When $p\alpha = d$, we have

$$S_{d-1}^{-1}\|f_0\|_p^p = \int_0^1 \frac{dr}{r(\log(2/r))^p}.$$

   Using substitution $u = \log(2/r)$, the RHS becomes $\int_{\log 2}^\infty \frac{du}{u^p}$, which converges when $p > 1$ and diverges when $p \leq 1$.

When $p\alpha > d$, we want to prove $f_0 \notin L^p$. We have

$$\begin{aligned} S_{d-1}^{-1} \int_{\mathbb{R}^d} |f_0|^p dx &= \int_0^1 \frac{r^{-p\alpha} r^{d-1}}{(\log(2/r))^p} dr \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d}(\log(2/r))^p} \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d} 2^p (\log(1/r))^p}. \end{aligned}$$

The last step is because $2/r \leq 1/r^2$ whenever $0 < r < 1/2$. We now using $u = 1/r$, RHS can be

$$RHS \geq \int_2^\infty \frac{du}{2^p u^{1-p\alpha+d}(\log u)^p}.$$

By assumption, $1 - p\alpha + d < 1$, we can now choose $\theta > 0$ so that $1 - p\alpha + d + \theta < 1$, then choose $K > 2$ big enough such that $(\log u)^p < u^\theta$ for all $u \geq K$, hence

$$RHS \geq \int_K^\infty \frac{du}{2^p u^{1-p\alpha+d+\theta}} = +\infty,$$

hence $f_0 \notin L^p$.

If the definition of f_∞ changed to

$$f_\infty(x) = \begin{cases} |x|^{-\alpha} / \log(2|x|) & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| < 1, \end{cases}$$

then by a similar argument, we have $f_\infty \in L^p$ whenever $d < p\alpha$. When $p\alpha = d$, we can similarly prove that $f_\infty \in L^p$ iff $p > 1$. Moreover, $f_\infty \notin L^p$ when $p\alpha < d$. □

2. Consider the spaces $L^p(\mathbb{R}^d)$, when $0 < p < \infty$

   (a) Show that if $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ for all f and g, then necessarily $p \geq 1$.

   Solution: We only need to show there are $f, g \in L^p$ such that $\|f + g\|_{L^p} > \|f\|_{L^p} + \|g\|_{L^p}$ when $0 < p < 1$. Let $K_1 = \{x \in \mathbb{R}^d : x_i \in (0,1) \forall 1 \leq i \leq d\}$ be a unit square in $\mathbb{R}^d$, and also $K_{-1} = \{x \in \mathbb{R}^d : x_i \in (-1,0) \forall 1 \leq i \leq d\}$ be its mirror.

   We then have $\|f\|_{L^p} = \|g\|_{L^p} = 1$, but then $\|f + g\|_{L^p} = 2^{1/p} > 2 = \|f\|_{L^p} + \|g\|_{L^p}$. □

   (b) Consider $L^p(\mathbb{R})$ where $0 < p < 1$. Show that there are no bounded linear functionals on this space. In other words, if ℓ is a linear functional $L^p(\mathbb{R}) \mapsto \mathbb{C}$ that satisfies

   $$|\ell(f)| \leq M \|f\|_{L^p(\mathbb{R})} \quad \text{for all } f \in L^p(\mathbb{R}) \text{ and some } M > 0,$$

   then $\ell = 0$.

   Solution: For each $x > 0$, we let χ_x be the characteristic function of $[0, x]$ on $\mathbb{R}$, then extends it naturally to $x \leq 0$ where it is the characteristic function of $[x, 0]$. Let $F(x) = \ell(\chi_x)$.

   Suppose ℓ is a bounded linear functional with the constant M as stated in the question, then for any $x, y \in \mathbb{R}$ we must have

   $$|F(x) - F(y)| = |\ell(\chi_x - \chi_y)| \leq M \|\chi_x - \chi_y\|_{L^p(\mathbb{R})} = M|x - y|^{1/p}.$$

   This means that F is a continuous function, but then since

   $$\left| \frac{F(x) - F(y)}{x - y} \right| \leq M|x - y|^{1/p - 1},$$

   F is then differentiable and has derivative 0 everywhere, hence F is a constant function, and must be zero too because $F(0) = \ell(\chi_0) = 0$. This shows ℓ can only be zero. □

3. If $f \in L^p$ and $g \in L^q$, both not identically equal to zero, show that equality holds in Hölder's inequality if and only if there exist two non-zero constants $a, b \geq 0$ such that $a|f(x)|^p = b|g(x)|^q$ for almost every x.

Solution: Since we have to prove $a|f(x)|^p = b|g(x)|^q$, we assume p, q are both finite, that means if $\theta = 1/p$, then $\theta \in (0,1)$. From the proof of Hölder's inequality, we also write $\hat{f} = f/\|f\|_p$ and $\hat{g} = g/\|g\|_q$, these are well-defined because f, g are not identically zero (we assume it means not equal to zero almost everywhere).

Again from the proof of Hölder's inequality, we have an inequality $A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B$ when A, B are non-negative numbers. Since $\theta \in (0,1)$, the inequality is strict iff $A \neq B$, thus by assumption we must have $A = B$, which means $\left|\hat{f}(x)\right|^p = |\hat{g}(x)|^q$. Since the equality $\|fg\|_1 = \|f\|_p \|g\|_q$ only holds when the above mentioned inequality holds for almost every x, we must have $\left|\hat{f}(x)\right|^p = |\hat{g}(x)|^q$ for almost every x.

Unnormalize we have $\|g\|_q^q |f(x)|^p = \|f\|_p^p |g(x)|^q$, which proves the statement because $\|f\|_p$ and $\|g\|_q$ are both positive. □

4. Suppose X is a measure space and $0 < p < 1$

   (a) Prove that $\|fg\|_{L^1} \geq \|f\|_{L^p} \|g\|_{L^q}$. Note that q, the conjugate exponent of p, is negative.

   Solution: If either $\|f\|_{L^p} = 0$, $\|g\|_{L^q} = 0$, or $fg \notin L^1$, then there is nothing to prove. Thus we may assume $fg \in L^1$ and that $\|f\|_{L^p} > 0$, $\|g\|_{L^q} > 0$, and that $\|g\|_{L^q}$ is finite, from here we note that it is easier to assume $g \neq 0$ a.e.

   By taking $p' = 1/p > 1$ and let q' be the conjugate exponent of p', we have

$$
\begin{aligned}
\int |f|^p &= \int |fg|^p |g|^{-p} \\
&\leq \left(\int |fg|^{pp'} \right)^{1/p'} \left(\int |g|^{-pq'} \right)^{-1/q'} \\
&= \left(\int |fg| \right)^p \left(\int |g|^q \right)^{p-1} \\
\therefore (|fg|)^p &\geq (|f|^p) \left(\int |g|^q \right)^{1-p} \\
\|fg\|_{L^1} &\geq \|f\|_{L^p} \|g\|_{L^q}.
\end{aligned}
$$

□

   (b) Suppose f_1 and f_2 are non-negative. Then $\|f_1 + f_2\|_{L^p} \geq \|f_1\|_{L^p} + \|f_2\|_{L^p}$.

   Solution: We have

$$
\begin{aligned}
\int |f_1 + f_2|^p &= \int f_1 (f_1 + f_2)^{p-1} + \int f_2 (f_1 + f_2)^{p-1} \\
&\geq \|f_1\|_{L^p} \|(f_1 + f_2)^{p-1}\|_{L^q} + \|f_2\|_{L^p} \|(f_1 + f_2)^{p-1}\|_{L^q} \\
&= (\|f_1\|_{L^p} + \|f_2\|_{L^p}) \|(f_1 + f_2)\|_{L^p}^{p-1}
\end{aligned}
$$

   which proves the statement. □

   (c) The function $d(f, g) = \|f - g\|_{L^p}^p$ for $f, g \in L^p$ defines a metric on $L^p(X)$.

   Solution: The function d obviously satisfies $d(f, g) = 0$ iff $f = g$ a.e., and that it is symmetric. If a, b are non-negative numbers, then we have $a^p + b^p \geq (a+b)^p$ for $0 < p < 1$. This means that for $f, g, h \in L^p$, we have

$$d(f,h) = \|f-g+g-h\|_{L^p}^p \leq \|f-g\|_{L^p}^p + \|g-h\|_{L^p}^p = d(f,g) + d(g,h),$$

hence d defines a metric on $L^p(X)$. $\square$

5. Let X be a measure space. Using the argument to prove the completeness of $L^p(X)$, show that if the sequence $\{f_n\}$ converges to f in the L^p norm, then a subsequence of $\{f_n\}$ converges to f almost everywhere.

Solution: Let $\{f_n\}$ be a sequence in $L^p(X)$ that converges to f in the L^p norm. We can choose a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_{L^p} < 2^{-k}$ for each $k \geq 1$.

Now we define

$$g(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$
$$h(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|,$$

following a similar argument when proving the completeness of $L^p(X)$, we see $f_{n_k} \to g$ a.e. on X.

Now we want to show $\|g-f\|_{L^p} = 0$, which could in turn proves that $g = f$ a.e. on X.

Given $\varepsilon > 0$, we choose K_0 such that $2^{-K_0} < \varepsilon$. Then for any $K > K_0$ we have $\|g - f_{n_K}\|_{L^p} \leq \sum_{k=K}^{\infty} \|f_{n_{k+1}}(x) - f_{n_k}(x)\|_{L^p} \leq 2^{-K_0} < \varepsilon$. By letting K to also be big enough to satisfies $\|f - f_{n_K}\|_{L^p} < \varepsilon$, we have $\|g - f\|_{L^p} < 2\varepsilon$. Subsequently we have $f_{n_k} \to f$ a.e. on X. $\square$