Functionaal Analysis Stein: Chapter 1. Exercises.

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1 Problems

1. Consider $L^p = L^p(\mathbb{R}^d)$ with Lebesgue measure. Let $f_0(x) = |x|^{-\alpha}$ if $|x| < 1$. $f_0(x) = 0$ for $|x| \geq 1$, also let $f_\infty(x) = |x|^{-\alpha}$ if $|x| \geq 1$, $f_\infty(x) = 0$ when $|x| < 1$.

   Show that

   (a) $f_0 \in L^p$ if and only if $p\alpha < d$.

   Solution: Let S_{d-1} be the surface area of the open unit ball $B_d = \{|x| < 1 : x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$, then if $f_0 \in L^p$ we can write

   $$\|f_0\|_{L_p}^p = \int_{B_d} |x|^{-p\alpha} dx = S_{d-1} \int_0^1 \frac{1}{r^p \alpha} \cdot r^{d-1} dr = S_{d-1} \int_0^1 \frac{1}{r^{1-d+p\alpha}} dr.$$

   Since the integral converges, we must have $1 - d + p\alpha < 1$ so $p\alpha < d$. We saw that the converse is also true. □

   (b) $f_\infty \in L^p$ if and only if $d < p\alpha$.

   Solution: Similar to the previous part, we have

   $$\|f_\infty\|_{L_p}^p = S_{d-1} \int_1^\infty \frac{1}{r^{1-d+p\alpha}} dr$$

   which is finite iff $d < p\alpha$. □

   (c) What happens if in the definitions of f_0 and f_∞ we replace $|x|^{-\alpha}$ by $|x|^{-\alpha}/(\log(2/|x|))$ for $|x| < 1$, and $|x|^{-\alpha}/(\log(2|x|))$ for $|x| \geq 1$?

   Solution: If the definition of f_0 changed to

   $$f_0(x) = \begin{cases} |x|^{-\alpha}/\log(2/|x|) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

   Then we want to show that $f_0 \in L^p$ iff $p\alpha < d$, or $p\alpha = d$ with $p > 1$, which is a little bit more nuanced than the previous part. When $p\alpha < d$, we see that $|f|_0 \leq \frac{1}{\log 2}|x|^{-\alpha}$, so that $f_0 \in L^p$ as it is absolutely bounded above by another function in L^p.

   When $p\alpha = d$, we have

   $$S_{d-1}^{-1}\|f_0\|_p^p = \int_0^1 \frac{dr}{r(\log(2/r))^p}.$$

   Using substitution $u = \log(2/r)$, the RHS becomes $\int_{\log 2}^\infty \frac{du}{u^p}$, which converges when $p > 1$ and diverges when $p \leq 1$.

When $p\alpha > d$, we want to prove $f_0 \notin L^p$. We have

$$\begin{aligned} S_{d-1}^{-1} \int_{\mathbb{R}^d} |f_0|^p dx &= \int_0^1 \frac{r^{-p\alpha} r^{d-1}}{(\log(2/r))^p} dr \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d}(\log(2/r))^p} \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d} 2^p (\log(1/r))^p}. \end{aligned}$$

The last step is because $2/r \leq 1/r^2$ whenever $0 < r < 1/2$. We now using $u = 1/r$, RHS can be

$$RHS \geq \int_2^\infty \frac{du}{2^p u^{1-p\alpha+d} (\log u)^p}.$$

By assumption, $1 - p\alpha + d < 1$, we can now choose $\theta > 0$ so that $1 - p\alpha + d + \theta < 1$, then choose $K > 2$ big enough such that $(\log u)^p < u^\theta$ for all $u \geq K$, hence

$$RHS \geq \int_K^\infty \frac{du}{2^p u^{1-p\alpha+d+\theta}} = +\infty,$$

hence $f_0 \notin L^p$.

If the definition of f_∞ changed to

$$f_\infty(x) = \begin{cases} |x|^{-\alpha} / \log(2|x|) & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| < 1, \end{cases}$$

then by a similar argument, we have $f_\infty \in L^p$ whenever $d < p\alpha$. When $p\alpha = d$, we can similarly prove that $f_\infty \in L^p$ iff $p > 1$. Moreover, $f_\infty \notin L^p$ when $p\alpha < d$. $\square$