Functional Analysis Stein: Chapter 1. Exercises.

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1 Problems

1. Consider $L^p = L^p(\mathbb{R}^d)$ with Lebesgue measure. Let $f_0(x) = |x|^{-\alpha}$ if $|x| < 1$. $f_0(x) = 0$ for $|x| \geq 1$, also let $f_\infty(x) = |x|^{-\alpha}$ if $|x| \geq 1$, $f_\infty(x) = 0$ when $|x| < 1$.

   Show that

   (a) $f_0 \in L^p$ if and only if $p\alpha < d$.

   Solution: Let S_{d-1} be the surface area of the open unit ball $B_d = \{|x| < 1 : x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$, then if $f_0 \in L^p$ we can write

   $$\|f_0\|_{L_p}^p = \int_{B_d} |x|^{-p\alpha} dx = S_{d-1} \int_0^1 \frac{1}{r^p \alpha} \cdot r^{d-1} dr = S_{d-1} \int_0^1 \frac{1}{r^{1-d+p\alpha}} dr.$$

   Since the integral converges, we must have $1 - d + p\alpha < 1$ so $p\alpha < d$. We saw that the converse is also true. □

   (b) $f_\infty \in L^p$ if and only if $d < p\alpha$.

   Solution: Similar to the previous part, we have

   $$\|f_\infty\|_{L_p}^p = S_{d-1} \int_1^\infty \frac{1}{r^{1-d+p\alpha}} dr$$

   which is finite iff $d < p\alpha$. □

   (c) What happens if in the definitions of f_0 and f_∞ we replace $|x|^{-\alpha}$ by $|x|^{-\alpha}/(\log(2/|x|))$ for $|x| < 1$, and $|x|^{-\alpha}/(\log(2|x|))$ for $|x| \geq 1$?

   Solution: If the definition of f_0 changed to

   $$f_0(x) = \begin{cases} |x|^{-\alpha}/\log(2/|x|) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

   Then we want to show that $f_0 \in L^p$ iff $p\alpha < d$, or $p\alpha = d$ with $p > 1$, which is a little bit more nuanced than the previous part. When $p\alpha < d$, we see that $|f|_0 \leq \frac{1}{\log 2}|x|^{-\alpha}$, so that $f_0 \in L^p$ as it is absolutely bounded above by another function in L^p.

   When $p\alpha = d$, we have

   $$S_{d-1}^{-1}\|f_0\|_p^p = \int_0^1 \frac{dr}{r(\log(2/r))^p}.$$

   Using substitution $u = \log(2/r)$, the RHS becomes $\int_{\log 2}^\infty \frac{du}{u^p}$, which converges when $p > 1$ and diverges when $p \leq 1$.

When $p\alpha > d$, we want to prove $f_0 \notin L^p$. We have

$$\begin{aligned} S_{d-1}^{-1} \int_{\mathbb{R}^d} |f_0|^p dx &= \int_0^1 \frac{r^{-p\alpha} r^{d-1}}{(\log(2/r))^p} dr \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d}(\log(2/r))^p} \\ &\geq \int_0^{1/2} \frac{dr}{r^{1+p\alpha-d} 2^p (\log(1/r))^p}. \end{aligned}$$

The last step is because $2/r \leq 1/r^2$ whenever $0 < r < 1/2$. We now using $u = 1/r$, RHS can be

$$RHS \geq \int_2^{\infty} \frac{du}{2^p u^{1-p\alpha+d}(\log u)^p}.$$

By assumption, $1 - p\alpha + d < 1$, we can now choose $\theta > 0$ so that $1 - p\alpha + d + \theta < 1$, then choose $K > 2$ big enough such that $(\log u)^p < u^{\theta}$ for all $u \geq K$, hence

$$RHS \geq \int_K^{\infty} \frac{du}{2^p u^{1-p\alpha+d+\theta}} = +\infty,$$

hence $f_0 \notin L^p$.

If the definition of f_∞ changed to

$$f_\infty(x) = \begin{cases} |x|^{-\alpha}/\log(2|x|) & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| < 1, \end{cases}$$

then by a similar argument, we have $f_\infty \in L^p$ whenever $d < p\alpha$. When $p\alpha = d$, we can similarly prove that $f_\infty \in L^p$ iff $p > 1$. Moreover, $f_\infty \notin L^p$ when $p\alpha < d$. □

2. Consider the spaces $L^p(\mathbb{R}^d)$, when $0 < p < \infty$

(a) Show that if $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ for all f and g, then necessarily $p \geq 1$.

Solution: We only need to show there are $f, g \in L^p$ such that $\|f + g\|_{L^p} > \|f\|_{L^p} + \|g\|_{L^p}$ when $0 < p < 1$. Let $K_1 = \{x \in \mathbb{R}^d : x_i \in (0,1) \forall 1 \leq i \leq d\}$ be a unit square in $\mathbb{R}^d$, and also $K_{-1} = \{x \in \mathbb{R}^d : x_i \in (-1,0) \forall 1 \leq i \leq d\}$ be its mirror.

We then have $\|f\|_{L^p} = \|g\|_{L^p} = 1$, but then $\|f + g\|_{L^p} = 2^{1/p} > 2 = \|f\|_{L^p} + \|g\|_{L^p}$. □

(b) Consider $L^p(\mathbb{R})$ where $0 < p < 1$. Show that there are no bounded linear functionals on this space. In other words, if ℓ is a linear functional $L^p(\mathbb{R}) \mapsto \mathbb{C}$ that satisfies

$$|\ell(f)| \leq M \|f\|_{L^p(\mathbb{R})} \quad \text{for all } f \in L^p(\mathbb{R}) \text{ and some } M > 0,$$

then $\ell = 0$.

Solution: For each $x > 0$, we let χ_x be the characteristic function of $[0, x]$ on $\mathbb{R}$, then extends it naturally to $x \leq 0$ where it is the characteristic function of $[x, 0]$. Let $F(x) = \ell(\chi_x)$.

Suppose ℓ is a bounded linear functional with the constant M as stated in the question, then for any $x, y \in \mathbb{R}$ we must have

$$|F(x) - F(y)| = |\ell(\chi_x - \chi_y)| \leq M\|\chi_x - \chi_y\|_{L^p(\mathbb{R})} = M|x - y|^{1/p}.$$

This means that F is a continuous function, but then since

$$\left|\frac{F(x) - F(y)}{x - y}\right| \leq M|x - y|^{1/p - 1},$$

F is then differentiable and has derivative 0 everywhere, hence F is a constant function, and must be zero too because $F(0) = \ell(\chi_0) = 0$. This shows ℓ can only be zero. □

3. If $f \in L^p$ and $g \in L^q$, both not identically equal to zero, show that equality holds in Hölder's inequality if and only if there exist two non-zero constants $a, b \geq 0$ such that $a|f(x)|^p = b|g(x)|^q$ for almost every x.

Solution: Since we have to prove $a|f(x)|^p = b|g(x)|^q$, we assume p, q are both finite, that means if $\theta = 1/p$, then $\theta \in (0,1)$. From the proof of Hölder's inequality, we also write $\hat{f} = f/\|f\|_p$ and $\hat{g} = g/\|g\|_q$, these are well-defined because f, g are not identically zero (we assume it means not equal to zero almost everywhere).

Again from the proof of Hölder's inequality, we have an inequality $A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B$ when A, B are non-negative numbers. Since $\theta \in (0,1)$, the inequality is strict iff $A \neq B$, thus by assumption we must have $A = B$, which means $\left|\hat{f}(x)\right|^p = |\hat{g}(x)|^q$. Since the equality $\|fg\|_1 = \|f\|_p \|g\|_q$ only holds when the above mentioned inequality holds for almost every x, we must have $\left|\hat{f}(x)\right|^p = |\hat{g}(x)|^q$ for almost every x.

Unnormalize we have $\|g\|_q^q |f(x)|^p = \|f\|_p^p |g(x)|^q$, which proves the statement because $\|f\|_p$ and $\|g\|_q$ are both positive. □

4. Suppose X is a measure space and $0 < p < 1$

   (a) Prove that $\|fg\|_{L^1} \geq \|f\|_{L^p}\|g\|_{L^q}$. Note that q, the conjugate exponent of p, is negative.

   Solution: If either $\|f\|_{L^p} = 0$, $\|g\|_{L^q} = 0$, or $fg \notin L^1$, then there is nothing to prove. Thus we may assume $fg \in L^1$ and that $\|f\|_{L^p} > 0$, $\|g\|_{L^q} > 0$, and that $\|g\|_{L^q}$ is finite, from here we note that it is easier to assume $g \neq 0$ a.e.

   By taking $p' = 1/p > 1$ and let q' be the conjugate exponent of p', we have

$$\begin{aligned}
\int |f|^p &= \int |fg|^p |g|^{-p} \\
&\leq \left(\int |fg|^{pp'}\right)^{1/p'} \left(\int |g|^{-pq'}\right)^{-1/q'} \\
&= \left(\int |fg|\right)^{p} \left(\int |g|^{q}\right)^{p-1} \\
\therefore (|fg|)^p &\geq (|f|^p) \left(\int |g|^q\right)^{1-p} \\
\|fg\|_{L^1} &\geq \|f\|_{L^p}\|g\|_{L^q}.
\end{aligned}$$

□

   (b) Suppose f_1 and f_2 are non-negative. Then $\|f_1 + f_2\|_{L^p} \geq \|f_1\|_{L^p} + \|f_2\|_{L^p}$.

   Solution: We have

$$\begin{aligned}
\int |f_1 + f_2|^p &= \int f_1(f_1+f_2)^{p-1} + \int f_2(f_1+f_2)^{p-1} \\
&\geq \|f_1\|_{L^p}\|(f_1+f_2)^{p-1}\|_{L^q} + \|f_2\|_{L^p}\|(f_1+f_2)^{p-1}\|_{L^q} \\
&= (\|f_1\|_{L^p} + \|f_2\|_{L^p})\|(f_1+f_2)\|_{L^p}^{p-1}
\end{aligned}$$

   which proves the statement. □

   (c) The function $d(f,g) = \|f - g\|_{L^p}^p$ for $f, g \in L^p$ defines a metric on $L^p(X)$.

   Solution: The function d obviously satisfies $d(f,g) = 0$ iff $f = g$ a.e., and that it is symmetric. If a, b are non-negative numbers, then we have $a^p + b^p \geq (a+b)^p$ for $0 < p < 1$. This means that for $f, g, h \in L^p$, we have

$$d(f,h) = \|f-g+g-h\|_{L^p}^p \leq \|f-g\|_{L^p}^p + \|g-h\|_{L^p}^p = d(f,g)+d(g,h),$$

hence d defines a metric on $L^p(X)$. □

5. Let X be a measure space. Using the argument to prove the completeness of $L^p(X)$, show that if the sequence $\{f_n\}$ converges to f in the L^p norm, then a subsequence of $\{f_n\}$ converges to f almost everywhere.

   Solution: Let $\{f_n\}$ be a sequence in $L^p(X)$ that converges to f in the L^p norm. We can choose a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_{L^p} < 2^{-k}$ for each $k \geq 1$.

   Now we define

$$g(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x))$$
$$h(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_{k+1}}(x) - f_{n_k}(x)|,$$

   following a similar argument when proving the completeness of $L^p(X)$, we see $f_{n_k} \to g$ a.e. on X.

   Now we want to show $\|g-f\|_{L^p} = 0$, which could in turn proves that $g = f$ a.e. on X.

   Given $\varepsilon > 0$, we choose K_0 such that $2^{-K_0} < \varepsilon$. Then for any $K > K_0$ we have $\|g - f_{n_K}\|_{L^p} \leq \sum_{k=K}^{\infty} \|f_{n_{k+1}}(x) - f_{n_k}(x)\|_{L^p} \leq 2^{-K_0} < \varepsilon$. By letting K to also be big enough to satisfies $\|f - f_{n_K}\|_{L^p} < \varepsilon$, we have $\|g-f\|_{L^p} < 2\varepsilon$. Subsequently we have $f_{n_k} \to f$ a.e. on X. □

6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Show that,

   (a) The simple functions are dense in $L^\infty(X)$ if $\mu(X) < \infty$, and;

   Solution: (Note: The definition of simple function is not completely clear in the context of this book, but it is reasonable to assume it means "finite sum of indication functions with finite support", else the problem statement assuming $\mu(X) < \infty$ doesn't make sense)

   Let f be any function in $L^\infty(X)$. WLOG we can assume $|f| \leq M = \|f\|_{L^\infty}$ everywhere on X (instead of almost everywhere). Given a positive integer j, choose any integer ℓ satisfying $-j \leq \ell \leq j$, we define

$$E_{\ell,j} = \left\{x \in X : \frac{M\ell}{j} \leq f(x) < \frac{M(\ell+1)}{j}\right\}.$$

   There are $2j+1$ such sets, they are mutually disjoint and that $X = \bigcup_{\ell=-j}^{j} E_{\ell,j}$. We then let

$$f_j(x) = \frac{M\ell}{j} \quad \text{if } x \in E_{\ell,j},$$

   which is a simple function that is well-defined everywhere on X.

   We then see that $\|f_j - f\|_{L^\infty} \leq 1/j$, hence $\{f_j\}_{j=1}^{\infty}$ converges to f in the L^∞ norm. □

   (b) The simple functions are dense in $L^p(X)$ for $1 \leq p < \infty$.

   Solution: Let $f \in L^p(X)$, without loss of generality we assume f is nonnegative, as a general f can be decomposed into the difference of two non-negative functions in $L^p(X)$. We do not assume f is bounded.

   For each positive integer n, we define for each $k = 1, \ldots, n2^n - 1$ such that

$$E_{k,n} = \left\{x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}$$

Note that each $E_{k,n}$ is of finite measure. We now define

$$f_n(x) = \begin{cases} \dfrac{k}{2^n} & \text{if } x \in E_{k,n}, \\ 0 & \text{if } x \notin \bigcup_{k=1}^{n2^n-1} E_{k,n}. \end{cases}$$

Note that $f_n(x) = 0$ can happen when $f(x) >= n$ or $f(x) < 1/2^n$. Nevertheless, $f_n \nearrow f$ a.e. on X, hence $|f_n(x)| \leq f(x)$ a.e. on X. By the Dominated Convergence Theorem, we have $\lim_{n\to\infty} \int_X |f_n(x) - f(x)|^p dx = 0$, thus $\{f_n\}$ converges to f in the L^p norm. □

7. Consider the L^p spaces, $1 \leq p < \infty$, on $\mathbb{R}^d$ with Lebesgue measure. Prove that

   (a) The family of continuous functions with compact support is dense in L^p, and in fact:

   (b) The family of indefinitely differentiable functions with compact support is dense in L^p.

   Solution: Since an indefinitely differentiable function is also continuous, it suffices to prove the last statement.

   From 6(b), we only need to prove that if f is a simple function in $L^p(\mathbb{R}^d)$ and given $\varepsilon > 0$, there is a indefinitely differentiable function g with compact support such that $\|f - g\|_{L^p} < \varepsilon$.

   The construction is as follows: We let $\psi(x)$ be a function on $\mathbb{R}$ such that

$$\psi(x) = \begin{cases} \exp(-\dfrac{1}{1-x^2}), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

   This is a bump function that is indefinitely differentiable and has compact support on $(-1, 1)$. Simply define $\Psi : \mathbb{R}^d \to \mathbb{R}$ such that $\Psi(x) = \prod_{i=1}^d \psi(x_i)$, then Ψ is an indefinitely differentiable function with compact support on the unit cube. Using a suitable scaling, we can construct a function, name it g, such that g is a nonnegative, indefinitely differentiable function with compact support, that also satisfies $\int_{\mathbb{R}^d} g(x)dx = 1$.

   For each positive integer n, we define $g_n(x) = n^d g(nx)$, then g_n is indefinitely differentiable with support $E_n := (-1/n, -1/n)^d$. We define the convolution $f * h$ such that

$$(f * h)(x) = \int_{\mathbb{R}^d} f(t)h(x-t)dt.$$

   It can be shown that $h \in L^1(\mathbb{R}^d)$ implies $f * h \in L^p(\mathbb{R}^d)$, and that $\|f * h\|_{L^p} \leq \|f\|_{L^p}\|h\|_{L^1}$. It is also true that if h is indefinitely differentiable, then so is $f * h$.

   We now define $f_n = f * g_n$, then f_n is indefinitely differentiable with compact support, and that $f_n \to f$ in L^p (see this answer). □

8. Suppose $1 \leq p < \infty$, and that $\mathbb{R}^d$ is equipped with Lebesgue measure. Show that if $f \in L^p(\mathbb{R}^d)$, then

$$\|f(x+h) - f(x)\|_{L^p} \to 0 \quad \text{as } |h| \to 0.$$

   Prove that this fails when $p = \infty$.

   Solution: Given $\varepsilon > 0$, we can choose a continuous function g with compact support such that $\|f - g\|_{L^p} < \varepsilon$.

   Let $B_R := \{r \in \mathbb{R}^d : |r| \leq R\}$ be the closed ball of radius $R > 0$, then we may assume g has support within B_R for some $R > 0$, hence for all $|h| \leq 1$, $g(x+h) - g(x)$ has support within B_{R+1}. Since g is continuous on a compact set, it is uniformly continuous, hence we can find a $\delta \in (0, 1)$ such that $|g(x+h) - g(x)| \leq \varepsilon/\mu(B_{R+1})^{1/p}$ whenever $|h| \leq \delta$ for all $x \in B_{R+1}$, and thus

$$\|g(x+h) - g(x)\|_{L^p} \leq \left(\int_{B_{r+1}} \frac{\varepsilon^p}{\mu(B_{r+1})} \right)^{1/p} = \varepsilon.$$

We concluded that $\|f(x+h)-f(x)\|_{L^p} \le 3\varepsilon$ by using the triangle inequality.

On the other hand when $p=\infty$, we let $C=[0,1]^d \subset \mathbb{R}^d$ be a unit cube and let $f=\chi_C$. Then for any nonzero $h \in \mathbb{R}^d$, we see that $\|f(x+h)-f(x)\|_{L^\infty}=1$, this is because there are always non-trivial difference between C and $C+h$ so that $f(x+h)-f(x)$ cannot be zero a.e. on $\mathbb{R}^d$. □

9. Suppose X is a measure space and $1 \le p_0 < p_1 \le \infty$.

   (a) Consider $L^{p_0} \cap L^{p_1}$ equipped with

$$\|f\|_{L^{p_0}\cap L^{p_1}} = \|f\|_{L^{p_0}} + \|f\|_{L^{p_1}}.$$

Show that $\|\cdot\|_{L^{p_0}\cap L^{p_1}}$ is a norm, and that $L^{p_0} \cap L^{p_1}$ (with this norm) is a Banach space.

Solution: Given $f, g \in L^{p_0} \cap L^{p_1}$, we have

$$\begin{aligned}\|f+g\|_{L^{p_0}\cap L^{p_1}} &= \|f+g\|_{L^{p_0}} + \|f+g\|_{L^{p_1}}\\ &\le \|f\|_{L^{p_0}} + \|g\|_{L^{p_0}} + \|f\|_{L^{p_1}} + \|g\|_{L^{p_1}}\\ &= \|f\|_{L^{p_0}\cap L^{p_1}} + \|g\|_{L^{p_0}\cap L^{p_1}},\end{aligned}$$

so the triangle inequality holds.

Given a Cauchy sequence $\{f_n\}_{n\ge 1}$ in $L^{p_0} \cap L^{p_1}$, we need to show it converges to a function in $L^{p_0} \cap L^{p_1}$. The derivation is somewhat routine as to prove the completeness of L^p in the textbook.

We first extract a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_{L^{p_0}\cap L^{p_1}} < 2^{-k}$. Then, for all x we define

$$\begin{aligned}f(x) &= f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x)),\\ g(x) &= |f_{n_1}(x)| + \sum_{k=1}^{\infty}\big|f_{n_{k+1}}(x) - f_{n_k}(x)\big|,\\ S_K f(x) &= f_{n_1}(x) + \sum_{k=1}^{K}(f_{n_{k+1}}(x) - f_{n_k}(x)),\\ S_K g(x) &= |f_{n_1}(x)| + \sum_{k=1}^{K}\big|f_{n_{k+1}}(x) - f_{n_k}(x)\big|.\end{aligned}$$

We found that

$$\|S_K g(x)\|_{L^{p_0}\cap L^{p_1}} \le \|f_{n_1}(x)\|_{L^{p_0}\cap L^{p_1}} + \sum_{k=1}^{K} 2^{-k} \le \|f_{n_1}(x)\|_{L^{p_0}\cap L^{p_1}} + 1,$$

by taking the limit $K \to \infty$, using the monotone convergence theorem for non-negative measurable functions, we see $\|g\|_{L^{p_0}\cap L^{p_1}}$ is also finite.

Now we have $f_{n_k} - f \to 0$ a.e. on X. Using a similar argument we have for both $p=p_0$ and $p=p_1$:

$$|f(x) - S_K f(x)|^p \le 2^{p+1}|g(x)|^p.$$

Using the Dominated Convergence Theorem, both $\|f_{n_k} - f\|_{L^{p_0}}$ and $\|f_{n_k} - f\|_{L^{p_1}}$ converges to 0, so as $\|f_{n_k} - f\|_{L^{p_0}\cap L^{p_1}}$.

Since $\{f_n\}_{n\ge 1}$ is Cauchy, we can very easily show $f_n \to f$ in $L^{p_0} \cap L^{p_1}$ (with that norm). □

(b) Suppose $L^{p_0} + L^{p_1}$ is defined as the vector space of measurable functions f on X that can be written as a sum $f = f_0 + f_1$ with $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$. Consider

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf\{\|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}\},$$

where the infimum is taken over all decompositions $f = f_0 + f_1$ with $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$. Show that $\|\cdot\|_{L^{p_0}+L^{p_1}}$ is a norm, and that $L^{p_0} + L^{p_1}$ (with this norm) is a Banach space.

Solution: First we show the space satisfies triangle inequality. Take $f, g \in L^{p_0} + L^{p_1}$. Given $\varepsilon > 0$ we may choose $f_0, g_0 \in L^{p_0}$ and $f_1, g_1 \in L^{p_1}$ such that

$$\begin{aligned}\|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}} &< \|f\|_{L^{p_0}+L^{p_1}} + \varepsilon\\ \|g_0\|_{L^{p_0}} + \|g_1\|_{L^{p_1}} &< \|g\|_{L^{p_0}+L^{p_1}} + \varepsilon.\end{aligned}$$

Then we can choose the decomposition $f + g = (f_0 + g_0) + (f_1 + g_1)$ so we can have

$$\begin{aligned}\|f+g\|_{L^{p_0}+L^{p_1}} &\le \|f_0 + g_0\|_{L^{p_0}} + \|f_1 + g_1\|_{L^{p_1}}\\ &\le \|f_0\|_{L^{p_0}} + \|g_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}} + \|g_1\|_{L^{p_1}}\\ &< \|f\|_{L^{p_0}+L^{p_1}} + \|g\|_{L^{p_0}+L^{p_1}} + 2\varepsilon,\end{aligned}$$

By taking $\varepsilon \to 0$, we eventually proved $\|f+g\|_{L^{p_0}+L^{p_1}} \le \|f\|_{L^{p_0}+L^{p_1}} + \|g\|_{L^{p_0}+L^{p_1}}$.

The proof for completeness is routine and hence skipped. □

(c) Show that $L^p \subset L^{p_0} + L^{p_1}$ if $p_0 \le p \le p_1$.

Solution: Given $f \in L^p$, we want to express it into $f_0 + f_1$ with $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$.

When either $p_0 = p$ or $p_1 = p$, the decomposition is trivial. We might then assume $p_0 < p < p_1$.

We now let $f_0 = f\chi_{|f|\ge 1}$ and $f_1 = f\chi_{|f|<1}$, then we have $f = f_0 + f_1$.

To show $f_0 \in L^{p_0}$, we note a fact that $\mu(E_{|f|\ge 1})$ is finite, hence by the Hölder's inequality we have

$$\int |f_1|^{p_0} \le \left(\int |f|^p\right)^{p_0/p} \mu(E_{|f|\ge 1})^{(p-p_0)/p} = \mu(E_{|f|\ge 1})^{(p-p_0)/p} \|f\|_{L^p}^{p_0}.$$

To show $f_1 \in L^{p_1}$, we observe that if $p_1 = \infty$, f_1 is already bounded by definition. If p_1 is finite, we have

$$\int |f_1|^{p_1} = \int_{|f|<1} |f|^{p_1} \le \int_{|f|<1} |f|^p \le \|f\|_{L^p}^p.$$

□

10. A measure space (X, μ) is **separable** if there is a countable family of measurable subsets $\{E_k\}_{k=1}^\infty$ so that if E is any measurable set of finite measure,

$$\mu(E \Delta E_{n_k}) \to 0 \quad \text{as } k \to 0$$

for an appropriate subsequence $\{n_k\}$ which depends on E. Here $A\Delta B$ denotes the symmetric difference of the sets A and B, that is,

$$A\Delta B = (A - B) \cup (B - A).$$

(a) Verify that $\mathbb{R}^d$ with the usual Lebesgue measure is separable.

Solution: Let E be any measurable set of finite measure. For any positive integer n, we can define a set of cubes within a grid space of distance $1/n$:

$$C_n = \left\{ \prod_{k=1}^d (x_k, x_k + 1/n) : x_k \in \{-n, -n + 1/n, \dots, n - 1/n\} \quad \forall k \right\}.$$

There are only finitely many cubes in this set. We then define K_n to be the set of all possible unions of cubes in C_n, then K_n is also a finite set with cardinality $2^{|C_n|}$.

We now take $K = \bigcup_{n=1}^{\infty} K_n$ where the order of the sequence is that any element of K_{n+1} comes after any element of K_n, and claims that $K = \{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ that satisfies the condition.

Given $\varepsilon > 0$, first we assume E is bounded, then by the definition of measurable set, there is a bounded open set O such that $E \subset O$ and $\mu(O - E) < \varepsilon$. Take this open set, we may find a possible set $F \in K_n$ with n big enough, such that $\mu(F - O) < \varepsilon$, thus $\mu(F - E) < 2\varepsilon$. Thus it is possible to choose a subsequence $\{E_{n_k}\}$ of K such that $\mu(E \Delta E_{n_k}) \to 0$.

If E is otherwise not bounded, we just pick n big enough such that $\mu(E \cap [-n, n]^d) > \mu(E) - \varepsilon$, then we can apply a similar logic to pick a corresponding $F \in K_n$ such that $\mu(E \Delta F) < 3\varepsilon$.

□

(b) The space $L^p(X)$ is **separable** if there exists a countable collection of elements $\{f_n\}_{n=1}^{\infty}$ in $L^p(X)$ in L^p that is dense. Prove that if the measure space X is separable, then L^p is separable when $1 \leq p < \infty$.

Solution: Given $f \in L^p(X)$, we can find a sequence of simple functions $\{s_n\}_{n=1}^{\infty}$ that converges to f in $L^p(X)$. Let $R = \{r_n\}$ be the set of all simple functions with rational coefficients, then we see R is dense in $L^p(X)$ as well.

From the previous question, we can find a countable family of measurable subsets $\{E_k\}$ that satisfies the separability condition. Let $R' = \{f_n\}$ be the set of simple functions that are linear combinations of the characteristic functions of E_k with rational coefficients, then R' is also dense in $L^p(X)$. The proof is completed once we realize R' is countable. □

11. In light of the previous exercise, prove the following:

(a) Show that the space $L^{\infty}(\mathbb{R})$ is not separable by constructing for each $a \in \mathbb{R}$ an $f_a \in L^{\infty}$, with $\|f_a - f_b\| \geq 1$, if $a \neq b$.

Solution: We let $f_a = \chi_{(a,a+1)}$, then $\|f_a - f_b\|_{L^\infty} = 1$ if $a \neq b$.

Why will this proves $L^{\infty}(\mathbb{R})$ is not separable? We can show by contradiction. Assumes $L^{\infty}(\mathbb{R})$ is separable, then there is a countable family of functions $\{g_n\}$ that is dense in $L^{\infty}(\mathbb{R})$. For each $a \in \mathbb{R}$ we pick an integer $N(a)$ such that $\|f_a - g_{N(a)}\|_{L^\infty} < 1/3$. However, since real numbers are uncountable, there exists $a \neq b$ such that $N(a) = N(b)$, so that

$$\|f_a - f_b\|_{L^\infty} \leq 2/3$$

which is a contradiction. □

(b) Do the same for the dual space of $L^{\infty}(\mathbb{R})$.

Solution: To recall, the dual space of $L^{\infty}(\mathbb{R})$ is the space of all bounded linear functionals on $L^{\infty}(\mathbb{R})$.

For each $a \in \mathbb{R}$, we have defined f_a in the previous question, so let's define a linear functional Λ_a such that $\Lambda_a(f) = f(a)$ for all $f \in L^{\infty}(\mathbb{R})$.

When $a < b$, we found that if we choose $f = \chi_{((a-1,(a+b)/2))}$, we have $\|f\|_{L^\infty}$ and $|(\Lambda_a - \Lambda_b)(f)| = 1$, so that $\|\Lambda_a - \Lambda_b\| \geq 1$ in the dual space. Following the same logic as the previous question, we showed the dual space of $L^{\infty}(\mathbb{R})$ is not separable as well.

TODO: Maybe include another solution here since from the web, people say this can be proved as a result of Hahn-Banach Theorem. □